Essentials of Bessel Functions

These very useful special functions are good friends in many investigations

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Professor Relton wrote a pleasant short book on Bessel functions (see Reference) that is a model of how to present mathematical results for the practical user, combining gentleness with rigour. Lamb referred to these functions as Bessel's functions, and the possessive is probably more genteel, but everyone uses the name as an adjective, and says Bessel functions. J. D. Jackson, in his text on Electrodynamics, thought Green's function should be Green function for consistency, but here common practice differs. Whatever they are called, these functions are among the most useful special functions, and are essential in problems possessing cylindrical symmetry. This is a review of their most important properties for use in physical applications.

In the 1999 University Challenge final, none of the six members of the Oriel College or the Durham University teams knew that Bessel had any functions, much less knew anything about them.

An important class of functions can be formed using only the four processes of arithmetic, called the rational functions, and these constitute a connection between the general hazy concept of function, and the concrete realization of functions in terms of numbers. Such functions can be extended to include infinite series, especially power series, and now the scope is immeasurably greater, but the mathematical difficulties loom large. The functions defined by infinite series are called *transcendental*.

One infinite power series, $1 + x + x^2/2! + x^3/3! + \ldots$ is of breathtaking might. This series converges for any value of x, real or complex, to a function $y = e^x$ that is its own derivative (take the derivative of the series to see this). The inverse of this function is $x = \log y$. Also, $e^{ix} = \cos x + i \sin x$. The exponential, logarithmic, trigonometrical and hyperbolic functions all flow from this source, and are called the *elementary* transcendental functions. We know how to work with them because we are familiar with many of their properties, and they can be easily calculated numerically (or their values are tabulated, in an older way of speaking).

In Physics, the methods of studying a problem mathematically usually involve infinitesimal analysis, which results in a differential equation for an unknown function, instead of a specification of the function itself in terms of values alone. It is very easy to write simple differential equations using only the operations of arithmetic that have solutions that cannot be so expressed, and that certainly are not rational or elementary functions in any wise. Differential equations effectively *define* new functions, called their solutions, and it is the task of the mathematician to elucidate their properties so that they can be used with effect and confidence. Bessel functions are of this type, solutions of certain differential equations that arise in many different connections.

While the physicist generally mounts a frontal attack on the differential equation by substituting an infinite series with unknown coefficients in it, and then finding the coefficients by equating the factors multiplying each power of x to zero, the mathematician relies more on crafty procedures and cunning plans to find the properties of the functions, since the differential equation reveals its secrets grudgingly and with much tedious labour. One method is to find an integral that yields the function; Bessel used this method for his functions. Another is to use *recurrence formulas* that connect functions belonging to different parameters. Even the differential equation can reveal important information without actually being solved. Professor Relton uses all these procedures.

The functions $C_n(x)$, which we shall write suppressing the argument x as simply C_n, may be defined for any real n by the recurrence relations

$$C_{n-1} + C_{n+1} = (2n/x)\, C_n$$

$$C_{n-1} - C_{n+1} = 2(dC_n/dx)$$

From these relations, we can find that C_n satisfies the differential equation:

$$x^2C_n'' + xC_n' + (x^2 - n^2)C_n = 0,$$

which is Bessel's Equation. The primes stand for differentiation with respect to x. Relton calls the C_n Cylinder Functions, and they turn out to be the Bessel Functions because the satisfy the differential equation, but he carefully points out that the converse must be proved, and cannot be assumed from one particular case. There are, in fact, Bessel functions, solutions of the differential equation, that do not satisfy these recurrence relations. Professor Relton points out that the coefficient of C_n'' shows that the function can touch (i.e., be tangent to) the x-axis only at x = 0, since this is the only zero of the coefficient of the second derivative.

The differential equation is the same for -n as for n, so C_{-n} is also a solution, and is in general different from C_n, so we have a general solution, with two arbitrary constants, $y = A\ C_n(x) + B\ C_{-n}(x)$. However, when n is integral, it can be shown from the recurrence relations that $C_{-n} = (-1)^nC_n$, so we have only one solution of this kind, and must seek another.

A second-order differential equation can be reduced to *normal form* by the substitution y = uv, choosing u(x) so that the y' term disappears. If we start from y'' +f(x)y' + g(x)y = 0, we get v'' + F(x)v = 0, where u = exp{-(1/2)I[f(x)]} (I is the indefinite integral), and $F(x) = g(x) - (1/2)df/dx - (f/2)^2$. Applying this to Bessel's equation, we get:

$$v'' + [1 + (1 - 4n^2)/4x^2]v = 0,$$

where $C_n(x) = v(x)/x^{1/2}$. This brings in the dependence on $x^{-1/2}$ that is so characteristic of cylindrical problems, such as the amplitude of a cylindrical wave. v(x) is like a sine or cosine, with a period that slowly shortens, eventually becoming 2π. Except for behaviour near the origin, this gives a good picture of how the Bessel function behaves. When n = 1/2, we get the familiar differential equation satisfied by the sine and cosine.

The series solution will give information on behaviour near the origin, and since it will give actual values, it can be used to specify a standard function for purposes of tabulation. The familiar result of this tedious, but straightforward work is:

$$J_n = (x/2)^n/2^n\Gamma(n+1)\{1 - (x/2)^2/2(2n+2) + (x/2)^4/2.4(2n+2)(2n+4) - ...\}.$$

A function so defined is called a Bessel function of the first kind and order n. From this, we see that when n is a positive integer, $J_n(x)$ starts off as x^n. When n = 0, $J_0(0) = 1$. When n is integral, $J_n(0) = 0$. In all other cases, J_n is infinite at the origin. In many physical problems, the solution must be defined and well-behaved at the origin, which rules out all solutions except for those with integral n. We can also show that J_n satisfies the same recurrence relations as C_n, confirming our suspicions that the functions are really identical.

When n = 1/2, we obtain the surprising result that $J_{1/2}(x) = (2/\pi x)^{1/2}\sin x$. From the recurrence relations we can then find that $J_{-1/2} = (2/\pi x)^{1/2}\cos x$. The recurrence relation $J_{n+1} = (2n/x)J_n - J_{n-1}$ can now be used to find all the other functions of half-integral index. Numerical calculations using recurrence relations are easily vitiated by roundoff error, since the error can increase exponentially with successive recurrences. These functions are useful in spherical wave solutions of the wave equation.

The zeros of the Bessel functions play a dominant role in applications, so Professor Relton is constantly talking about them. He shows that the Bessel functions have an infinity of zeros, that the zeros of J_n and J_{n+1} interlace, and that the maxima and minima steadily decrease in absolute value as x increases. The first five zeros of J_0 are 2.4048, 5.5201, 8.6537, 11.7915, 14.9309. The interval between the last two is 3.1394, already close to π. The larger roots are approximately (m - 1/4)π, where m is the number of the root. For n > 1/2, the roots approach π from above instead of from below. The first positive zero of J_n is greater than n, and increase steadily with n. The first zeros are 2.405, 3.832, 5.136, 6.380, 7.588, 8.771, for n = 0 to 5. The zeros must be determined by rugged calculation, and a great deal of effort has been exerted in this direction.

The functions $J_n(ax)$ and $J_n(bx)$ are *orthogonal* to each other over the interval x = 0,1 with weight function x, when a and b are two different roots of J_n. This means that when their product is multiplied by x and integrated from 0 to 1, the result is zero. Should b = a, the result is not zero, but $J_n'^2(a)/2 = J_{n+1}^2(a)/2$. In the case of n = 1/2, this gives the orthogonality of the functions sin (nπx) over (0,1), for example. A function can be expanded in a series of $J_n(a_i x)$ corresponding to the zeros of J_n in exactly the same way as a Fourier series is formed, using orthogonality to determine the coefficients individually.

Two more ways to get Bessel functions are shown in the Figure. The generating function is a surprising connection with the exponential, and Bessel's integral connects Bessel and trigonometric function. Bessel himself used the integral, which first arose in a problem in celestial mechanics, to investigate his functions.

$$\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{-\infty}^{\infty} t^n J_n(x)$$

Generating Function

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$$

Bessel's Integral

In case n is integral, we must search for a second solution linearly independent of J_n. For n = 0, such a function is Neumann's, $Y_0(x) = J_0(x) \log x + \{(x/2)^2 - (3/2)(x/2)^2/(2!)^2 + \ldots\}$. This function is tabulated, just like $J_0(x)$, and has zero that interlace with those of $J_0(x)$. Of course, it goes to negative infinity at x = 0. The general solution of order zero is then $y = A J_0(x) + B Y_0(x)$. If A = -(2/π)(log 2 - γ) and B = 2/π, with γ = Euler's constant, about 0.5772, then the corresponding function is Weber's second solution. All that is important in practice is that there are two independent solutions, J and Y, and Y is infinite at the origin. Weber's second solution can be defined for all orders, and it has the advantage of satisfying the same recurrence relations as J.

Bessel functions can also have an imaginary argument. In this case, they become the *modified* functions I and K. This substitution changes them from oscillatory to monotonic, as in the analogous case of the trigonometric functions. The function of the first kind is defined as $I_n(x) = (i)_{-n} J_n(ix)$. These functions obey recurrence relations similar to those for J_n, but slightly different due to the factor of i. $I_0(0) = 1$, and $I_n(0) = 0$ for n>0, and then the curves rise like exponentials. The second solution K is difficult to investigate, and the usual definition does not have it obeying the same recurrence relations as I. Macdonald's definition is $K_n = (\pi/2\sin n\pi)(I_{-n} - I_n)$. K(x) is infinite at x = 0, and decreases like a rectangular hyperbola, approaching the x-axis as asymptote. Indeed, $K_{1/2}(x) = K_{-1/2}(x) = (\pi/2x)^2 e^{-x}$. The corresponding relations for I, incidentally, give the sinh for n = 1/2, and cosh for n = -1/2. The I_n are the coefficients in the Fourier expansion of $e^{-kr\cos\theta}$, which is $\Sigma\, I_n(kr) \cos n\theta$, where the sum on n is from minus infinity to infinity.

The function $y = x^\alpha J_n(\beta x^\gamma)$ is a solution of the equation $y'' + [(1 - 2\alpha)/2]y' + [(\beta\gamma x^{\gamma-1})^2 + (\alpha^2 - n^2\gamma^2)/x^2]y = 0$. If the first term in the square brackets is negative, the solution has I_n instead of J_n. This will help you identify equations whose solution can be expressed in terms of Bessel functions that you may meet in applications, without having to search for the proper substitutions.

Finally, the surprising versatility of Bessel functions is shown in Weber's discontinuous integrals. These integrals can be used to find the potential of a charged disc, an interesting problem. The Figure shows these integrals, together with Lipschitz's Integral, from which they can be

$$\int_0^{\infty} J_0(bx)\, e^{-ax}\, dx = \frac{1}{\sqrt{a^2 + b^2}}$$

Lipschitz's Integral

derived by making a imaginary. It is remarkable that these integrals are discontinuous functions of their parameters. There are, of course, many others.

In this paper, I have reviewed many of the properties of Bessel functions that are required in the familiar physical applications. Professor Relton gives many applications in his book, and supplies much detail that I have omitted here. There is, of course, a voluminous literature on Bessel functions, and tables and graphs of their values and properties, such as Jahnke and Emde, which is an indispensable resource. There are also several good small books giving the essentials of Bessel functions for scientists and engineers. Every text on hydrodynamics, elasticity, electromagnetism and vibrations will contain examples. Rayleigh, Lamb, and Jackson are specially recommended. Complex variables provides further connections between Bessel functions. Bessel functions appear in frequency modulation, and as Kelvin's ber and bei functions in the electrodynamics of a conducting cylinder, so engineering also offers many applications.

Zeros of $J_n(x)$

s	n=0	n=1	n=2	n=3	n=4	n=5
1	2.405	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.147	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813
9	27.494	29.047	30.571	32.050	33.512	34.983

References

1. F. E. Relton, *Applied Bessel Functions* (London: Blackie and Son, 1946).
2. F. Bowman, *Introduction to Bessel Functions* (New York: Dover, 1958). Remarkably clear and concise introduction, with well-selected applications.

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