

UQ & RA: Fundamentals Summary

Fundamental Probability and Random Variables

PROBABILITY THEORY

Axioms (Kolmogorov)

- 1. $P[A] \geq 0$
- 2. $P[\Omega] = 1$ (Certain Event)
- 3. $P[A \cup B] = P[A] + P[B]$ (Mutually Exclusive)

Set Operations

Union ($A \cup B$): Occurs if A OR B occur.

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Intersection ($A \cap B$): Occurs if A AND B occur.

Independence: Events A and B are independent iff:

$$P[A \cap B] = P[A] \cdot P[B]$$

De Morgan's Laws:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} ; \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Conditional Probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Total Probability Theorem: If A_i form a partition of Ω :

$$P[B] = \sum_i P[B|A_i]P[A_i]$$

Bayes' Theorem:

$$P[A_i|B] = \frac{P[B|A_i]P[A_i]}{\sum_j P[B|A_j]P[A_j]}$$

RANDOM VARIABLES (RV)

CDF: $F_X(x) = P[X \leq x]$

PDF: $f_X(x) = \frac{dF_X(x)}{dx}$

Properties:

- $0 \leq F_X(x) \leq 1$ (Non-decreasing)
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
- $P[a < X \leq b] = F_X(b) - F_X(a)$

Moments

Mean (Expected Value):

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} xf_X(x)dx$$

Variance (Dispersion):

$$\sigma_X^2 = Var[X] = E[(X - \mu)^2] = E[X^2] - \mu^2$$

Standard Deviation: $\sigma_X = \sqrt{Var[X]}$

Coefficient of Variation (CoV):

$$\delta_X = \frac{\sigma_X}{\mu_X}$$

Skewness (Asymmetry): $\gamma_1 = \frac{E[(X - \mu)^3]}{\sigma^3}$

Kurtosis (Peak/Tails): $\gamma_2 = \frac{E[(X - \mu)^4]}{\sigma^4}$

Inequalities

Chebyshev: $P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$

KEY DISTRIBUTIONS

Normal (Gaussian) $N(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]$$

Standard Normal ($Z \sim N(0, 1)$): Transformation: $Z = \frac{X - \mu}{\sigma}$
CDF $\Phi(z)$ is tabulated.

Lognormal $LN(\lambda, \zeta)$

If $Y = \ln(X)$ is Normal, X is Lognormal.

$$\lambda = E[\ln X], \quad \zeta = \sigma_{\ln X}$$

Conversion from μ_X, σ_X :

$$\zeta^2 = \ln(1 + \delta_X^2) \quad (\text{where } \delta = \sigma/\mu)$$

$$\lambda = \ln(\mu_X) - 0.5\zeta^2$$

JOINT PROBABILITY

Joint CDF: $F_{XY}(x, y) = P[X \leq x \cap Y \leq y]$

Joint PDF: $f_{XY}(x, y) = \frac{\partial^2 F}{\partial x \partial y}$

Marginal PDF: $f_X(x) = \int f_{XY}(x, y)dy$

Conditional PDF: $f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$

Covariance & Correlation

Covariance:

$$Cov[X, Y] = E[XY] - \mu_X \mu_Y$$

Correlation Coefficient ($-1 \leq \rho \leq 1$):

$$\rho_{XY} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y}$$

$\rho = 0$ implies uncorrelation (not necessarily independence).

EXPECTATION OF FUNCTIONS

Single RV

For $Y = g(X)$:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$$

Multiple RVs

For $Y = g(X_1, \dots, X_n)$:

$$E[Y] = \int g(\mathbf{x})f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

Variance of a Function

$$\text{Var}[Y] = E[Y^2] - E[Y]^2$$

Linear Case

If $Y = \sum a_i X_i$:

$$\mu_Y = \sum a_i \mu_i$$

$$\sigma_Y^2 = \sum a_i^2 \sigma_i^2 + 2 \sum_{i < j} a_i a_j \text{Cov}[X_i, X_j]$$