

# UQ & RA: Fundamentals Summary

Fundamental Probability and Random Variables

## PROBABILITY THEORY

### Axioms (Kolmogorov)

1.  $P[A] \geq 0$  2.  $P[\Omega] = 1$  (Certain Event) 3.  $P[A \cup B] = P[A] + P[B]$  (Mutually Exclusive)

### Set Operations

**Union** ( $A \cup B$ ): Occurs if A OR B occur.

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

**Intersection** ( $A \cap B$ ): Occurs if A AND B occur.

**Independence:** Events A and B are independent iff:

$$P[A \cap B] = P[A] \cdot P[B]$$

### De Morgan's Laws:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad ; \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

### Conditional Probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

**Total Probability Theorem:** If  $A_i$  form a partition of  $\Omega$ :

$$P[B] = \sum_i P[B|A_i]P[A_i]$$

### Bayes' Theorem:

$$P[A_i|B] = \frac{P[B|A_i]P[A_i]}{\sum_j P[B|A_j]P[A_j]}$$

## RANDOM VARIABLES (RV)

**CDF:**  $F_X(x) = P[X \leq x]$

**PDF:**  $f_X(x) = \frac{dF_X(x)}{dx}$

### Properties:

- $0 \leq F_X(x) \leq 1$  (Non-decreasing)
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
- $P[a < X \leq b] = F_X(b) - F_X(a)$

### Moments

#### Mean (Expected Value):

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

#### Variance (Dispersion):

$$\sigma_X^2 = Var[X] = E[(X - \mu)^2] = E[X^2] - \mu^2$$

**Standard Deviation:**  $\sigma_X = \sqrt{Var[X]}$

#### Coefficient of Variation (CoV):

$$\delta_X = \frac{\sigma_X}{\mu_X}$$

**Skewness** (Asymmetry):  $\gamma_1 = \frac{E[(X-\mu)^3]}{\sigma^3}$

**Kurtosis** (Peak/Tails):  $\gamma_2 = \frac{E[(X-\mu)^4]}{\sigma^4}$

### Inequalities

**Chebyshev:**  $P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$

## KEY DISTRIBUTIONS

**Normal (Gaussian)**  $N(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

**Standard Normal** ( $Z \sim N(0, 1)$ ): Transformation:  $Z = \frac{X-\mu}{\sigma}$

CDF  $\Phi(z)$  is tabulated.

**Lognormal**  $LN(\lambda, \zeta)$

If  $Y = \ln(X)$  is Normal,  $X$  is Lognormal.

$$\lambda = E[\ln X], \quad \zeta = \sigma_{\ln X}$$

**Conversion from  $\mu_X, \sigma_X$ :**

$$\zeta^2 = \ln(1 + \delta_X^2) \quad (\text{where } \delta = \sigma/\mu)$$

$$\lambda = \ln(\mu_X) - 0.5\zeta^2$$

## JOINT PROBABILITY

**Joint CDF:**  $F_{XY}(x, y) = P[X \leq x \cap Y \leq y]$

**Joint PDF:**  $f_{XY}(x, y) = \frac{\partial^2 F}{\partial x \partial y}$

**Marginal PDF:**  $f_X(x) = \int f_{XY}(x, y) dy$

**Conditional PDF:**  $f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$

### Covariance & Correlation

#### Covariance:

$$Cov[X, Y] = E[XY] - \mu_X \mu_Y$$

**Correlation Coefficient** ( $-1 \leq \rho \leq 1$ ):

$$\rho_{XY} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y}$$

$\rho = 0$  implies uncorrelation (not necessarily independence).

## EXPECTATION OF FUNCTIONS

### Single RV

For  $Y = g(X)$ :

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

### Multiple RVs

For  $Y = g(X_1, \dots, X_n)$ :

$$E[Y] = \int g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

### Variance of a Function

$$Var[Y] = E[Y^2] - E[Y]^2$$

### Linear Case

If  $Y = \sum a_i X_i$ :

$$\mu_Y = \sum a_i \mu_i$$

$$\sigma_Y^2 = \sum a_i^2 \sigma_i^2 + 2 \sum_{i < j} a_i a_j Cov[X_i, X_j]$$