Uncertainty Quantification

and Reliability Analysis

Henrique Kroetz

December 14, 2025

Outline

- Uncertainty Quantification and Reliability Analysis?
- Organization of the course
- Introduction to Uncertainty Quantification,

Introduction

- What is Uncerteinty Quantification?
- What is Reliability Analysis?

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  - Science of characterizing and propagating input uncertainties through a computational model to assess the reliability and likelihood of its predictions.
- What is Reliability Analysis?

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  - Science of characterizing and propagating input uncertainties through a computational model to assess the reliability and likelihood of its predictions.
- What is Reliability Analysis?
  - probabilistic assessment of a system's ability to perform its intended function without failure for a specified period under stated conditions

Introduction

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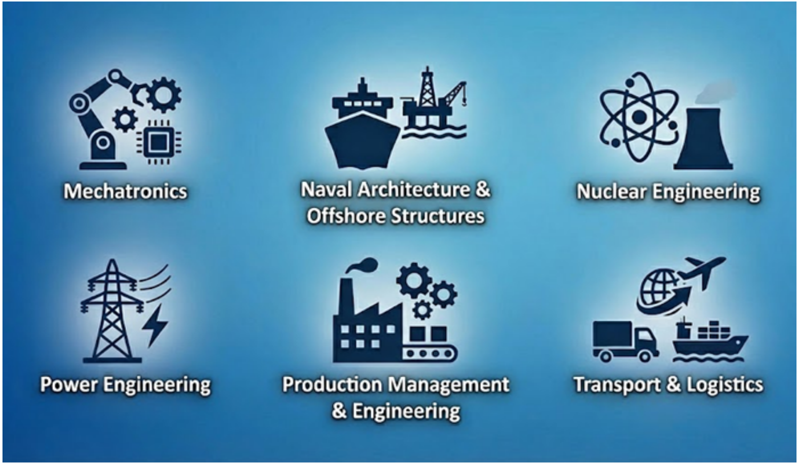
  - Science of characterizing and propagating input **uncertainties** through a **computational model** to assess the **reliability** and **likelihood** of its predictions.

- What is Reliability Analysis?
  - probabilistic assessment of a system's ability to perform its intended function without **failure** for a specified period under stated conditions

Who attends this course?

- 126 registered students
- At least 6 different fields of study!



Course Instructor

- Prof. Henrique Kroetz
- Federal University of Paraná - Center for Marine Studies
- Civil Engineer, PhD. in Structural Engeineering
- Researching structural reliability and optimization under uncertainties

Tentative Course Schedule

#	Topic	Key Concepts
1	Introduction to UQ	Motivation, Aleatory vs. Epistemic
2	Probability Theory	Review of Fundamentals
3	Probabilistic Modeling I	Random Variables, PDFs, CDFs
4	Probabilistic Modeling II	Input Characterization, Correlation
5	Uncertainty Propagation I	Direct Monte Carlo
6	Uncertainty Propagation II	Advanced Sampling (LHS)
7	Reliability Analysis I	Limit States, FORM
8	Reliability Analysis II	SORM, Simulation Methods
9	Polynomial Chaos Exp. I	Spectral Methods, Basis Functions
10	Polynomial Chaos Exp. II	Sparse Grids, Surrogate Modeling
11	Sensitivity Analysis	Local vs Global Methods
12	Application Examples	Case Studies & Course Wrap-up

Performance Assessment

Performance Assessment

Project

Development of a full technical report on an engineering system of interest to your field. The report must address:

- Quantifying relevant sources of uncertainty
- Propagating them through the system
- Evaluating results
- Performing sensitivity analysis

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Requirements for Approval

Students must meet **both** of the following criteria:

- **Component Threshold:** Achieve a score $> 0\%$ in all mandatory project components.
- **Overall Grade:** Achieve a minimal total aggregated grade of 50% for the entire project.

Grading Criteria

Component	Weight	Criteria Details
1. Problem Definition	10%	System description, physical models, deterministic solution, and definition of Quantity of Interest.
2. Uncert. Modeling	20%	Identification of sources. Justification of probability distributions and correlations for inputs.
3. Propagation	20%	Application of methods (e.g., Monte Carlo). Output analysis: histograms, moments, confidence intervals.
4. Reliability Analysis	20%	Limit State Function definition. Calculation of Probability of Failure and Reliability Index.
5. Sensitivity Analysis	20%	Computation of sensitivity indices to rank parameter importance.
6. Reporting	10%	Report structure, writing clarity, quality of figures/plots, and references.
TOTAL	**100%**	

Questions or Comments?

Motivation: Structural Failures

Working with models

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Model

A simplified mathematical or physical representation of a real-world system, constructed to capture essential behaviors and relationships for the purpose of analysis, prediction, or control.

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Computational Model

The algorithmic implementation of a model, utilizing numerical methods and software to approximate system responses based on specific input parameters.

What does a computer model consist of?

- **Mathematical description of the governing laws**
- **Discretization of the quantities of interest**
  - Grid/mesh for spatial variables, discrete time steps for time-dependent variables
  - Trade-off between accuracy and computational cost
- **Algorithm to solve the discretized equations**
  - Solution of linear and nonlinear systems of equations
  - Numerical integration
- **Post-processing, visualization**

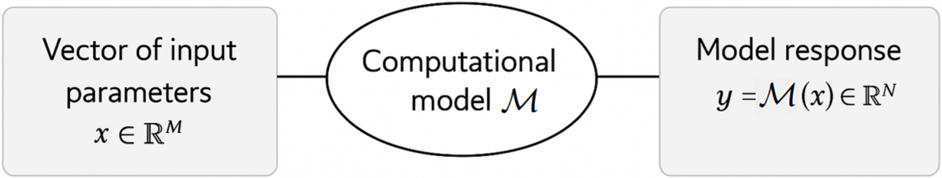
Why might Simulation not coincide with Reality?

Why might Simulation not coincide with Reality?

- **Modelling errors**
  Simplifying Assumptions, Numerical limitations (discretization, stopping criteria, etc.)

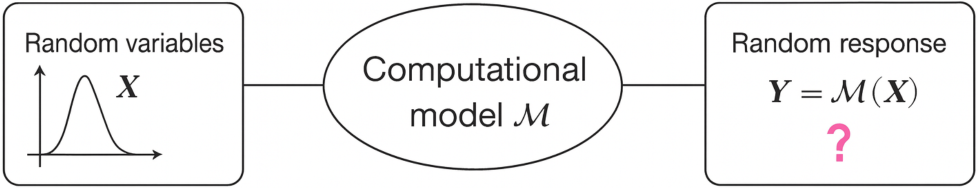
- **Parameter Uncertainty**

Computational Model for UQ



It is not always easy to determine input values!

Computational Model for UQ

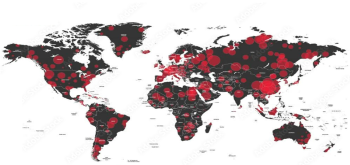


Parametric uncertainty is not an error, it is part of the analysis!

*"All models are wrong,
but some are useful."*

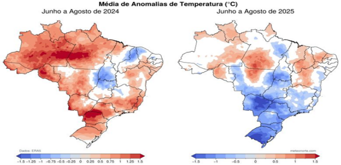
— **George Box**
(Statistician)

Some examples

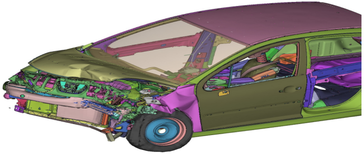


Spreading of an infectious disease

Weather prediction



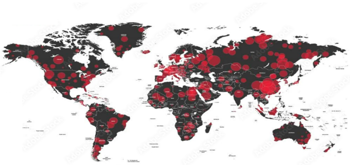
Climate model



Car crash

Colliding planets

Some examples



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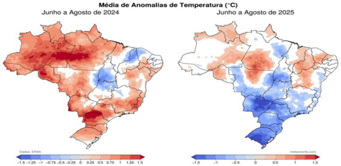

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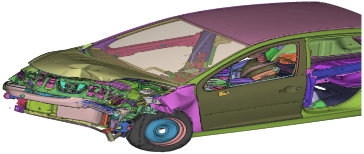

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Do your research involve computational models? What models could be interesting for your work?

Historical Context

- $\sim$**12,000 BC:** Builder's Experience / Intuition.
- $\sim$**6,000 BC:** Codification of best practices based on successful designs.
- **19th Century:** Consideration of uncertainties and the Allowable Stress Design (ASD) method.
- **Late 20th Century:** Limit State Design (LSD) method.
- **State of the Art:** Explicit consideration of uncertainties.

Classification: Aleatory (Intrinsic) Uncertainty

1. **Physical Uncertainty**
   - Inherent to the nature of the system being studied.
   - Can be reduced by obtaining more information, but never fully eliminated.
   - *Examples:* Spatial variations in material strength, wind load intensity.

2. **Prediction Uncertainty**
   - Related to predicting future behavior of complex systems.
   - *Examples:* Maximum snow load over a lifespan.

Classification: Epistemic Uncertainty

1. **Statistical Uncertainty**
   - Derived from inferring characteristics from a finite sample size.
2. **Decision Uncertainty**
   - Difficulty in defining whether a specific event occurred; limits of binary definitions.
3. **Model Uncertainty**
   - Discrepancies between real system behavior and the mathematical model.
4. **Phenomenological Uncertainty**
   - Limitation of knowledge regarding unknown phenomena.

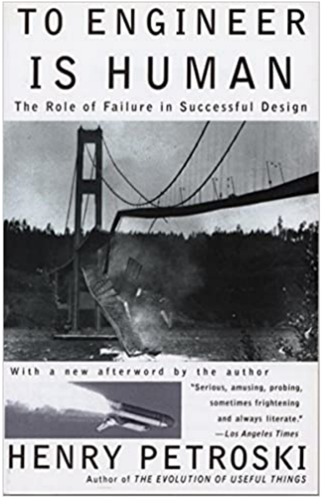
Aleatory X Epistemic Uncertainty

- Classification may change depending on the description of the system!

- Both will be modelled similarly for the purposes of our analyses!

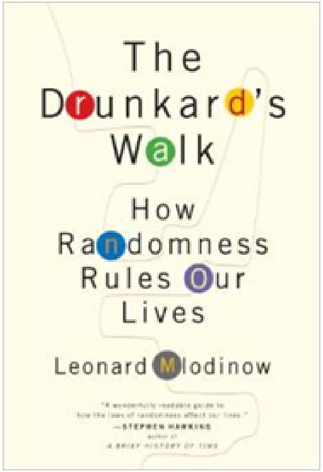
Armen Der Kiureghian, Ove Ditlevsen, Aleatory or epistemic? Does it matter?, Structural Safety, Volume 31, Issue 2, 2009.

Book Recommendation



To Engineer is Human
The Role of Failure in Successful Design
Author: Henry Petroski

Book Recommendation

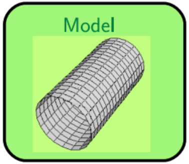


The Drunkard's Walk
How Randomness Rules Our Lives **Author:**

Leonard Mlodinow

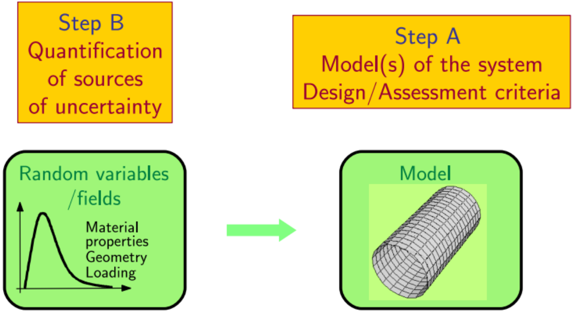
Global Framework

Step A
Model(s) of the system
Design/Assessment criteria



Source: Sudret, Bruno. Habilitationa diriger des recherches, Université Blaise Pascal, Clermont-Ferrand, France 147 (2007): 53.

Global Framework



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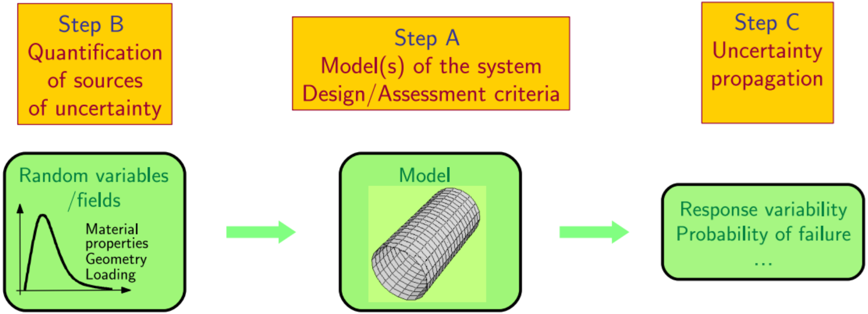

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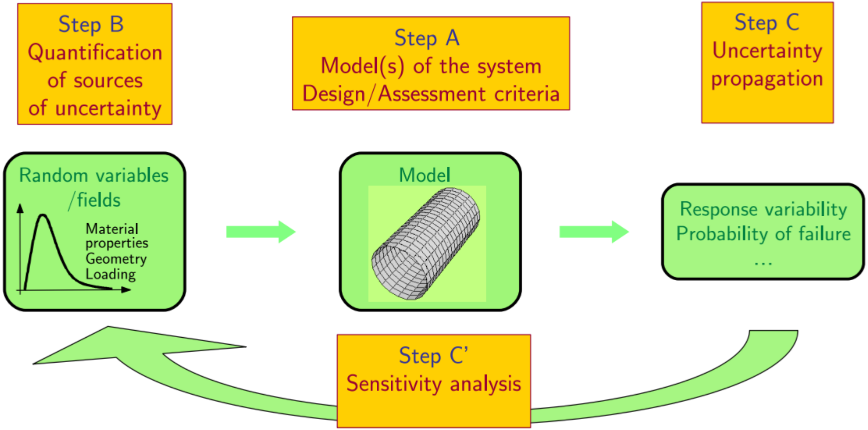

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Probability Theory

Fundamentals

1. **Random Experiment ($\mathcal{E}$)**
   - A process whose outcome cannot be predicted with certainty, even under identical conditions.
   - *Examples:* Rolling a die, structural load testing, wind speed measurement.

2. **Sample Space (Ω)**
   - The set of **all possible distinct outcomes** of the experiment.
   - The nature of Ω determines the math we use (Sum vs. Integral):

Discrete Ω

Countable outcomes.

$$\Omega = \{H, T\} \text{ or } \{1, 2, 3, ...\}$$

Ex: Number of failed components.

Continuous Ω

Uncountable continuum.

$$\Omega = \mathbb{R} \text{ or } [0, \infty)$$

Ex: Time to failure, Material stress.

Fundamentals

Symbol	Formal Name	Read As...	Intuition
$\in$	**Element of**	"belongs to"	The item is inside the bag.
$\notin$	Not element of	"does not belong to"	The item is outside.
$\subseteq$	**Subset**	"is contained in"	The small bag is inside the big bag.
$\subset$	Proper Subset	"is strictly inside"	Inside, but smaller than the container.
Ω	Sample Space	"the universe"	The set of all possibilities.
$\emptyset$	Empty Set	"the null set"	A bag with nothing in it.
$\cup$	**Union**	"A or B"	Combined contents (Add logic).
$\cap$	**Intersection**	"A and B"	Overlap (Common ground).
$\setminus$	**Difference**	"A minus B"	Remove B items from A.
A^c	**Complement**	"Not A"	Everything in Ω except A.

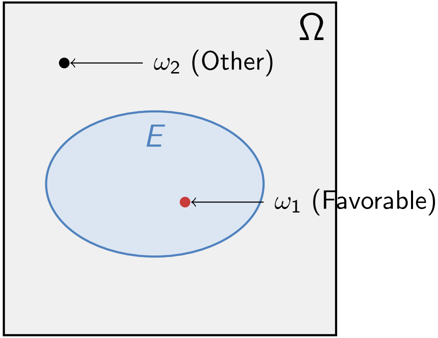
Fundamentals

3. Sample Point (ω)

- An individual element of the sample space.
- Notation: $\omega \in \Omega$
- Also called: *Elementary Outcome*.

4. Event (E)

- A subset of the sample space.
- Notation: $E \subseteq \Omega$
- We say "Event E occurred" if the realized outcome $\omega_{actual} \in E$.



$\omega_1 \in E$
$\omega_2 \notin E$
$E \subset \Omega$

Definition 1: Classical Probability

Concept: Based on counting physical symmetries.

If a random experiment has N possible outcomes that are **mutually exclusive** and **equally likely**, and if event A corresponds to N_A of these outcomes:

$$P[A] = \frac{N_A}{N} = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

- Requires a finite sample space.
- Relies on the assumption that outcomes are "equally likely" (which is itself a probability concept).
- Example: Rolling a fair die ($P = 1/6$), drawing a card.

Definition 2: Frequentist Probability

Concept: Based on long-run relative frequency.

If we repeat an experiment n times under identical conditions, and event A occurs n_A times:

$$P[A] = \lim_{n\to\infty} \frac{n_A}{n}$$

- It is a property of nature/physics, independent of the observer.
- The experiment must be repeatable (in theory) an infinite number of times.
- **Limitation:** Hard to apply to unique, one-time events (e.g., "Probability of a specific candidate winning an election").

Definition 3: Bayesian Probability

Concept: Based on Degree of Belief.

Probability is a measure of the state of knowledge or confidence an observer has that a proposition is true.

- **Subjective:** Different observers may assign different probabilities based on the information they possess.
- **Updating:** As new data (D) becomes available, the belief is updated (Prior $\rightarrow$ Posterior).

Why for Uncertainty Quantification?

In engineering and physics, we often deal with "epistemic uncertainty" (lack of knowledge) rather than pure randomness. The Bayesian framework allows us to quantify this uncertainty effectively.

Probability Theory: Axiomatic Definition

To unify the previous definitions, A. N. Kolmogorov (1933) established the axiomatic framework. Probability is a function P that assigns a number to an event $A \subseteq \Omega$.

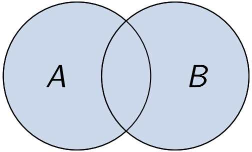
The Three Axioms:

1. **Non-negativity:**
$$P[A] \geq 0$$
2. **Normalization:**
$$P[\Omega] = 1$$
3. **Additivity (for mutually exclusive events):** If $A \cap B = \emptyset$, then:
$$P[A \cup B] = P[A] + P[B]$$

Note: All other rules (e.g., $P[\emptyset] = 0$, $P[A^c] = 1 - P[A]$) are derived from these three axioms.

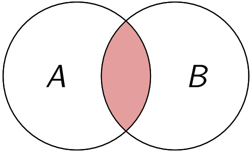
Set Operations and Venn Diagrams

Union $A \cup B$



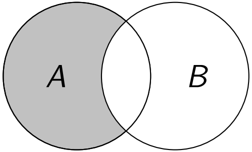
Elements in A OR B

Intersection $A \cap B$



Elements in A AND B

Difference $A \setminus B$



In A but NOT in B

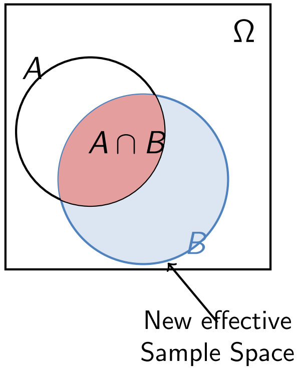
Conditional Probability

Given $P[B] > 0$, the probability of A occurring given that B has essentially occurred is:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Intuition:

- The sample space Ω collapses to B.
- We look for the fraction of B that is also covered by A.
- If A and B are independent, knowing B gives no info: $P[A|B] = P[A]$.



Independence of Events

Two events A and B are independent if and only if:

$$P(A \cap B) = P(A)P(B)$$

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Equivalent Formulation via Conditional Probability:

$$P(A|B) = P(A)$$

"The probability of A remains unchanged even if we know B happened."

Common Misconception: Independent $\neq$ Disjoint

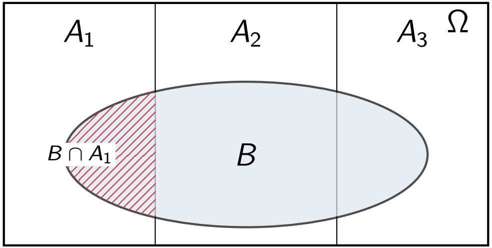
- **Disjoint (Mutually Exclusive):** $A \cap B = \emptyset$. If B happens, A *cannot* happen.
- **Independent:** They can happen together, but their causes are unrelated.

Total Probability Theorem

The Partition: Let $A_1, A_2, ..., A_N$ be a set of events that are mutually exclusive ($A_i \cap A_j = \emptyset$) and collectively exhaustive ($\cup A_i = \Omega$).

For any event B, we can calculate $P[B]$ by summing the probability of B occurring within each slice of the partition:

$$P[B] = \sum_{i=1}^{N} P[B \cap A_i] = \sum_{i=1}^{N} P[B|A_i]P[A_i]$$



Bayes' Theorem

Origin: The theorem arises from the symmetry of the joint probability $P(A \cap B)$.

1. **Recall Conditional Probability:** For any two events A and B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \Longrightarrow \quad P(A \cap B) = P(A|B)P(B)$$

2. **Symmetry:** The intersection is commutative ($A \cap B = B \cap A$). Thus:

$$P(A|B)P(B) = P(B|A)P(A)$$

3. **The Solution:** Rearranging to solve for the "reverse" conditional probability $P(A|B)$:

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

Bayes' Theorem

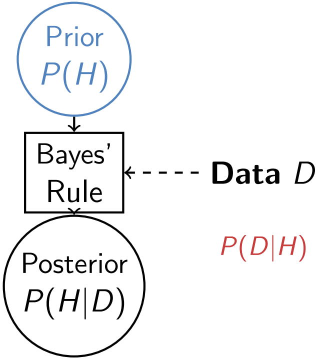
In Uncertainty Quantification, we use this theorem to **update belief** about a model given new data.

Let H be a **Hypothesis** (or model parameters) and D be observed **Data**.

$$P(H|D) = \frac{P(D|H) \cdot P(H)}{P(D)}$$

Bayes' Theorem

- **Prior** $P(H)$:
  Knowledge about H *before* seeing data (Subjective/Historical).
- **Likelihood** $P(D|H)$:
  The probability of seeing this data *if* the hypothesis were true. (The forward model).
- **Posterior** $P(H|D)$:
  The updated state of knowledge.
- **Evidence** $P(D)$:
  Normalization constant (Total Probability).



Exercise 1: The "Arithmetic" Trap

Problem: Simplify the expression:

$$E = (A \cup A) \setminus A$$

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In Real Algebra ($+$ and $-$):

$$(x + x) - x = 2x - x = x$$

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Problem: Simplify the expression:

$$E = (A \cup A) \setminus A$$

In Real Algebra ($+$ and $-$):

$$(x + x) - x = 2x - x = x$$

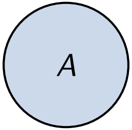
In Set Theory ($\cup$ and $\setminus$):

1. Idempotency: $A \cup A = A$
2. Difference: $A \setminus A = \emptyset$

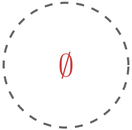
$$\therefore (A \cup A) \setminus A = \emptyset$$

Unlike numbers, "adding" a set to itself doesn't double it!

Step 1: $A \cup A$



Step 2: Remove A



(Empty Set)

Exercise 2: Subtraction is distinct from Inverse

Problem: Simplify the expression:

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Exercise 2: Subtraction is distinct from Inverse

Problem: Simplify the expression:

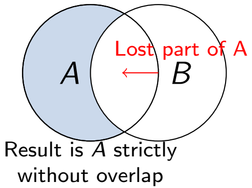
$$E = (A \cup B) \setminus B$$

In Real Algebra:

$$(a + b) - b = a$$

In Set Theory: Does $(A \cup B) \setminus B = A$?
NO! (Unless $A \cap B = \emptyset$)

Removing B removes **everything** inside B, including the part that overlaps with A.



$$\begin{aligned}(A \cup B) \setminus B &= A \setminus B \\ &= A - (A \cap B)\end{aligned}$$

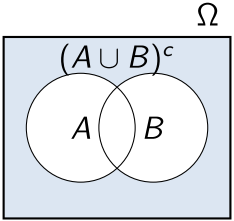
Exercise 3: De Morgan's Law

Problem: Visualize the complement of a Union.

$$E = (A \cup B)^c$$

Logic:

- "Not (A or B)" means you cannot be in A, AND you cannot be in B.
- Therefore: $(A \cup B)^c = A^c \cap B^c$



Application: Useful for calculating "None of the above" probabilities:

$$P[\text{Neither}] = 1 - P[A \cup B]$$