Uncertainty Quantification

Reliability Analysis

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January 14, 2026

Introduction to Reliability Analysis

Reliability: The probability that a system does not fail within a specified design life, respecting the operating and design conditions of the system in question.

Probability of Failure: This is the complement of reliability. It is the probability of the system failing—not meeting design specifications—within a specified design life, even when operating and design conditions are respected.

$$P_f = 1 - \mathcal{R}$$

Introduction to Reliability Analysis

The fundamental problem of reliability can be stated as: determining the probability that the demand (D) on a system is greater than its capacity (C), known as the probability of failure (P_f). This problem is posed in a completely generic way: the system in question can be a product, a component, a structural element, a structure, or even a production, service, or environmental system.

The failure of the considered system is described the limit state equation, which takes negative values when the system fails:

$$g(D, C) = D - C$$

Introduction to Reliability Analysis

It is usual to write the problem with the following notation: a Resistance (the Capacity) (R) and a Solicitation (the Demand) (S), so that the fundamental problem is written as:

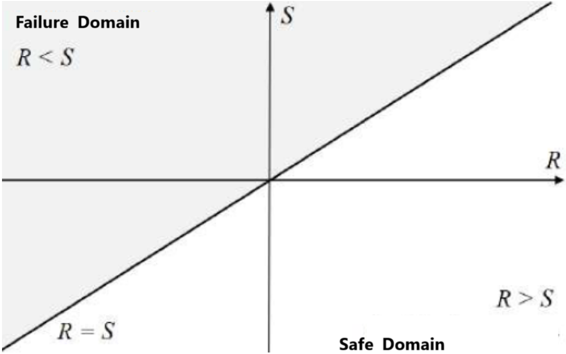
$$g(\mathbf{X}) = R - S,$$

where $\mathbf{X}$ is the random vector. For the fundamental problem, $\mathbf{X} = [R, S]^T$. For more complex problems, where $\mathbf{X}$ gathers more radnom variables, the same definition is valid: the failure of the system occurs when $g(\mathbf{X}) \leq 0$.

A limit state equation divides the random variables space in two domains:

- $\Omega_f = [\mathbf{x}|g(\mathbf{x}) \leq 0]$ - is the failure domain
- $\Omega_s = [\mathbf{x}|g(\mathbf{x}) > 0]$ - is the safe (or survival) domain

For the fundamental problem, The failure domain Ω_f is bounded by the equation $r = s$:



Introduction to Reliability Analysis

The probability of failure of a component or element is given by the probability that the solicitation (S) is greater than the resistance (R). If f_{RS} is the joint CDF of $\mathbf{X} = [R, S]^T$:

$$P_f = P[S \geq R] = \iint_{\Omega_f} f_{RS}(r, s)\, dr\, ds$$

where Ω_f is the failure domain:

$$\Omega_f = \{(r, s) \mid r \leq s\}$$

When the variables R and S are statistically independent, the joint probability density function is given by the product of the marginal functions:

$$f_{RS}(r, s) = f_R(r) f_S(s).$$

In this case, the probability of failure becomes:

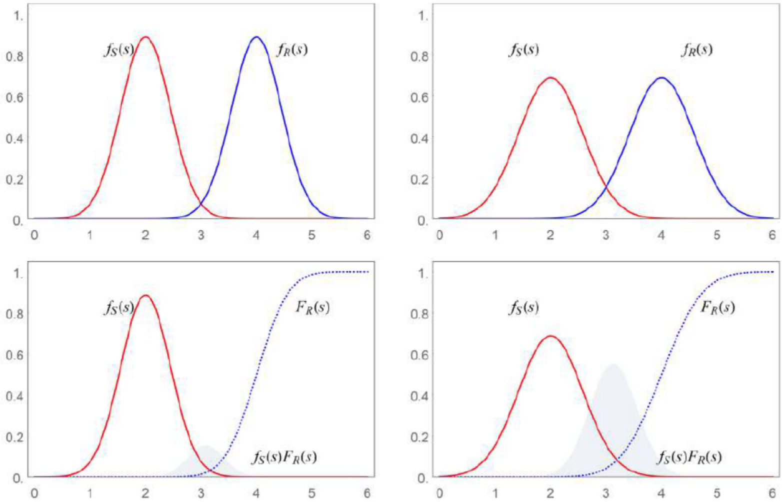
$$p_f = \int_{-\infty}^{+\infty} f_S(s) \left[\int_{-\infty}^{s} f_R(r)\, dr \right] ds = \int_{-\infty}^{\infty} f_S(s) F_R(s)\, ds,$$

where:

$f_S(s)$ is the marginal probability density function of the solicitation;

$F_R(r)$ is the marginal cumulative probability distribution function of the resistance.

Therefore, the probability of failure is the area under the curve $f_S(s)F_R(s)$:



Introduction to Reliability Analysis

To reduce P_f, we can separate the means or decrease the standard deviation of variables R and S:

- **Separating means:** Corresponds to increasing safety factors.
- **Reducing σ_R:** Increasing quality and dimensional control of materials and structural elements.
- **Reducing σ_S:** More difficult; involves imposing strict load limits (e.g., for bridges or elevators).

Solution for Normal Variables R and S

The problem can be solved using the **Safety Margin** variable (M):

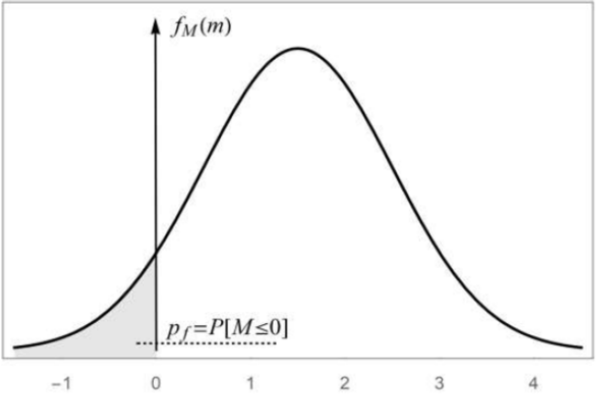
$$M = R - S$$

If R and S are Normal and independent, M is also Normal:

$$\mu_M = \mu_R - \mu_S$$

$$\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$$

Probability of Failure and Safety Margin



The probability of failure is the probability that the safety margin is less than or equal to zero.

Reliability Analysis

The variable M can be transformed into a standard normal variable Y (dimensionless, with zero mean and unit standard deviation) through the Hasofer-Lind transformation:

$$Y = \frac{M - \mu_M}{\sigma_M}$$

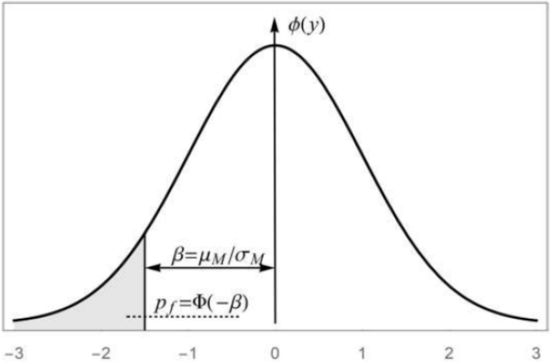
This transformation allows evaluating probabilities associated with variable M through the standard normal cumulative distribution function $\Phi(\cdot)$:

$$P_f = P[M \leq 0] = P\left[Y \leq -\frac{\mu_M}{\sigma_M}\right] = \Phi\left(-\frac{\mu_M}{\sigma_M}\right)$$

In the dimensionless variable Y, we obtain a geometric measure of the probability of failure, which corresponds to the distance between the point $m = 0$ and the origin (mean) of the distribution of Y.

This measure is called the **reliability index**, represented by the Greek letter β:

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$



Reliability Analysis Summary

Incorporating the previous results, we obtain the relationship between the probability of failure and the reliability index:

$$P_f = P[M \leq 0] = \Phi\left(-\frac{\mu_M}{\sigma_M}\right) = \Phi(-\beta)$$

Where:

- $\mu_M = \mu_R - \mu_S$
- $\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$

Log-normal Distribution Case

A similar analytical solution is obtained for the case where independent random variables R and S follow log-normal distributions.

Let $R \sim LN(\lambda_R, \zeta_R)$ and $S \sim LN(\lambda_S, \zeta_S)$. It is convenient to use the safety margin variable Z:

$$Z = \frac{R}{S}$$

Taking the logarithm:

$$\ln(Z) = M = \ln(R) - \ln(S)$$

Since R and S are log-normal, $\ln(R)$ and $\ln(S)$ are normally distributed.

Reliability Index for Log-normal Variables

M follows a normal distribution, leading to the reliability index:

$$\beta = \frac{\ln\left(\frac{\mu_R}{\mu_S}\sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}}\right)}{\sqrt{\ln((1+\delta_R^2)(1+\delta_S^2))}}$$

where $\delta = \sigma/\mu$ is the coefficient of variation.

When the coefficients of variation are small ($\delta_R, \delta_S < 0.3$), the result simplifies to:

$$\beta \approx \frac{\ln(\mu_R/\mu_S)}{\sqrt{\delta_R^2 + \delta_S^2}}$$

Design Requirements

Structures and elements are designed, built, and maintained to fulfill a specific function:

- a) during a certain period, called **design life**;
- b) with an adequate level of **safety**;
- c) in an **economically viable** manner;
- d) without exposing the public to **unacceptable risks**.

Technical Requirements

1. **Serviceability:** The structure must remain in appropriate conditions for its intended function during its life.
2. **Safety:** The structure must withstand extreme and repetitive loads without collapse or severe permanent damage.
3. **Robustness:** The structure must not be damaged by accidental events (fire, explosions, impact) in a manner disproportionate to the event.

Economic and Social Requirements

4. **Economic requirement:** Meeting technical requirements without compromising the project's economic viability.
5. **Social requirement:** Meeting the previous requirements with risk levels that are acceptable to society and the user.

Limit States

Technical requirements can be expressed as **limit state equations**.

- Violating a service or safety requirement is an undesirable state.
- Each distinct failure mode defines a limit state equation.

Main categories:

- **Ultimate Limit States (ULS):** Related to safety and collapse.
- **Serviceability Limit States (SLS):** Related to functionality and comfort.

Ultimate Limit States Examples:

- **Loss of static equilibrium:** Overturning, capsizing, or sliding of the global system.
- **Loss of containment:** Breach of pressure boundaries, hulls, or primary cooling loops.
- **Supply chain exhaustion:** Depletion of critical inventory or energy reserves below the minimum safety threshold.
- **Thermal collapse:** Failure of material strength or integrity due to extreme temperature gradients.
- **Capacity-Demand deficit:** Failure of a production line or logistics network to meet peak throughput requirements.
- **Resonance and dynamic instability:** Failure driven by self-excited oscillations or vibrations.

Serviceability Limit States (SLS)

- **Excessive deformation:** Deflections that cause misalignment of machinery or interfere with non-structural components.
- **Localized cracking:** Surface fissures that do not threaten stability but permit the ingress of corrosive agents.
- **Service delays:** Temporary accumulation of queues or increased lead times that exceed contractual service level agreements.
- **Power quality fluctuations:** Deviations in voltage or frequency that lead to the malfunctioning of synchronized industrial systems.
- **Surface degradation:** Corrosion, erosion, or cavitation that requires premature maintenance or reduces aesthetic value.
- **Inventory imbalance:** Buffer fluctuations that cause intermittent idling of production stages without a total system stop.

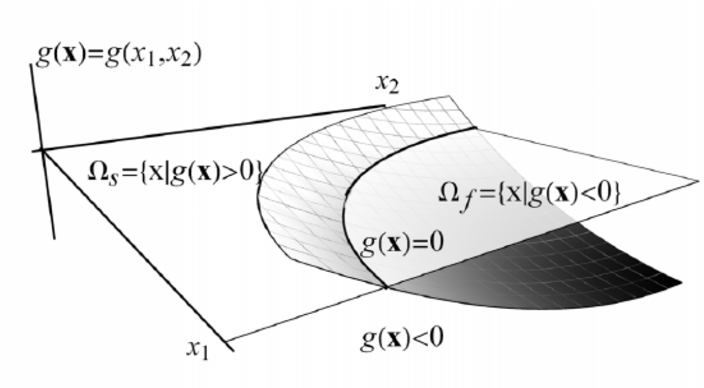
General Limit State Equations

Let X be a vector containing all random variables relevant to the studied reliability problem. For each "i_{th}" failure mode, we define:

$$g_i(X) = g_i(X_1, X_2, \ldots, X_n) = 0$$

- **Failure domain (Ω_f):** $\{\mathsf{x} \mid g(\mathsf{x}) \leq 0\}$
- **Survival domain (Ω_s):** $\{\mathsf{x} \mid g(\mathsf{x}) > 0\}$

Limit State Equations Visualization



Probability of Failure

Also in more general problems, the probability of failure measures the likelihood of violating limit states:

$$P_f = P[X \in \Omega_f] = P[g(X) \leq 0]$$

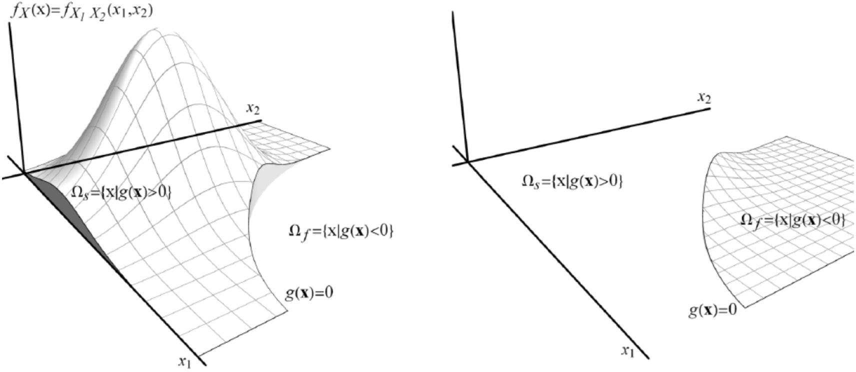
Evaluated using the joint density $f_X(x)$:

$$P_f = \int_{\Omega_f} f_X(x)\, dx$$

Hence, the reliability ($\mathcal{R}$) is the integral over the survival domain:

$$\mathcal{R} = 1 - P_f = \int_{\Omega_s} f_X(x)\, dx$$

Joint PDFs over each domain:



Limitations of Simple Analytical Solutions

In real engineering practice, the fundamental $R - S$ case is rarely enough:

- **Complex Functions:** $g(X)$ is often non-linear (e.g., buckling equations, finite element analysis outputs, or dynamic responses).
- **Non-Normal Variables:** Physical variables often follow different distributions (e.g., Gumbel for wind loads, Weibull for material strength, or Lognormal for soil properties).
- **High Dimensionality:** Real problems involve dozens of random variables, making direct integration of $f_X(x)$ mathematically impossible.
- **Correlation:** Variables are often statistically dependent (e.g., the strength of different beams from the same batch).

General Reliability Assessment

1. **FORM (First-Order Reliability Method)**
   - **Transformation:** Maps non-normal, correlated variables X into the **Standard Normal Space** U ($\mathrm{U} \sim \mathcal{N}(0, \mathrm{I})$).
   - **Optimization:** Finds the **Design Point** (u^*)—the point on the limit state $g(\mathrm{u}) = 0$ closest to the origin.
   - **Reliability Index:** $\beta = \|\mathrm{u}^*\|$, where $P_f \approx \Phi(-\beta)$.
2. **Monte Carlo Simulation (MCS)**
   - **Numerical Integration:** Estimates P_f by generating N random samples and counting failures ($g(\mathrm{x}) \leq 0$).
   - **The Benchmark:** Used to validate FORM results, especially when the limit state is highly non-linear or discontinuous.

FORM

In the physical world ($\mathbf{X}$-space), the limit state $g(\mathrm{x}) = 0$ is often a complex, curved surface. Calculating the failure probability P_f requires integrating over this messy region:

$$P_f = \int_{g(\mathrm{x}) \leq 0} f_{\mathrm{X}}(x_1, x_2, ..., x_n) d\mathrm{x}$$

The Obstacles:

- The joint PDF f_{X} is rarely known for non-normal variables.
- Correlation between variables.
- Integration in high dimensions is computationally expensive.
- Accounting for more than one limit state.

Standard Normal Space (U)

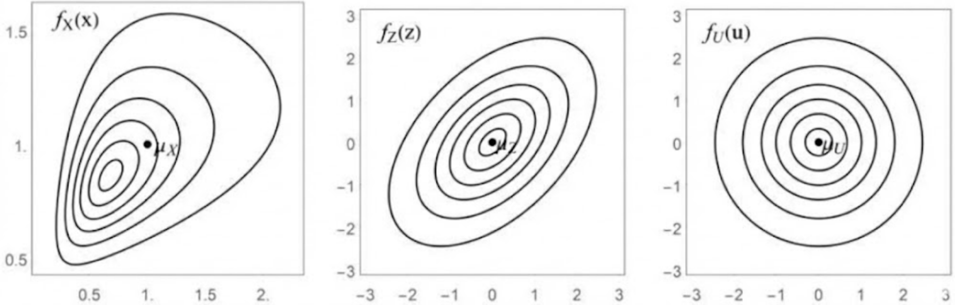
FORM works by mapping the problem into the Standard Normal Space (U-space).

Properties of U-space:

- Every variable U_i has a mean of 0 and a standard deviation of 1.
- All variables are **statistically independent** (no correlation).
- The PDF is rotationally symmetric—it only depends on the distance from the origin.

In this space, the "most likely" failure point is simply the point on the failure surface closest to the origin.

FORM Transformation Steps



Step 1: Marginal Transformation (Normal Tail Equivalence)

To approximate a non-normal variable X_i at a specific point x_i^*, we seek an Equivalent Normal Distribution $N(\mu_i^N, \sigma_i^N)$.

We impose two conditions to be met at the point x_i^*:

1. **CDF Equivalence:** The probability of being below x_i^* must match.
$$F_{X_i}(x_i^*) = \Phi\left(\frac{x_i^* - \mu_i^N}{\sigma_i^N}\right)$$
2. **PDF Equivalence:** The "likelihood" or slope at x_i^* must match.
$$f_{X_i}(x_i^*) = \frac{1}{\sigma_i^N}\phi\left(\frac{x_i^* - \mu_i^N}{\sigma_i^N}\right)$$

- Φ, ϕ: Standard Normal CDF and PDF respectively.
- μ_i^N, σ_i^N: Parameters of the locally equivalent normal distribution.

Step 1: Calculating μ_i^N and σ_i^N

By solving the two equivalence equations simultaneously, we derive the parameters for the local normal approximation:

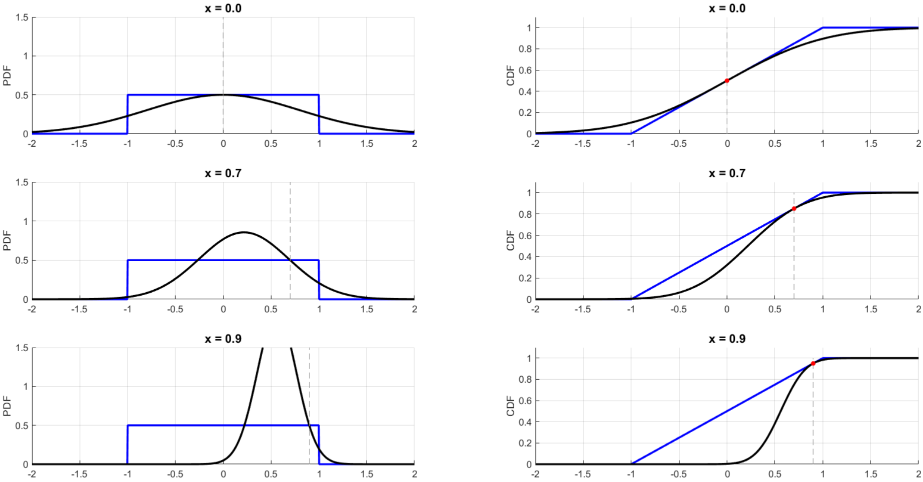
$$\sigma_i^N = \frac{\phi\left(\Phi^{-1}[F_{X_i}(x_i^*)]\right)}{f_{X_i}(x_i^*)}$$

$$\mu_i^N = x_i^* - \Phi^{-1}[F_{X_i}(x_i^*)]\sigma_i^N$$

Intuition

This transformation "stretches" and "shifts" a Normal distribution so that it looks exactly like the physical distribution in the neighborhood of x_i^*.

Equivalent Normal Distribution



Step 2: Correlation in the Normal Space

After transforming $X_i \to Z_i$, we must preserve the joint dependency between variables.

The non-linear mapping $Z_i = \Phi^{-1}[F_{X_i}(x_i)]$ distorts the linear correlation coefficient. If ρ_X is the correlation between physical variables X_i and X_j:

- The "target" correlation in the normal space, ρ_{ij}, is **not** equal to $\rho_{X,ij}$.
- We must find an **Equivalent Correlation** ρ_{ij} such that:

$$\rho_{X,ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{x_i - \mu_i}{\sigma_i} \right) \left(\frac{x_j - \mu_j}{\sigma_j} \right) \phi_2(z_i, z_j; \rho_{ij}) \, dz_i \, dz_j$$

Where ϕ_2 is the bivariate standard normal PDF with correlation ρ_{ij}.

Step 2: Practical Correction (Nataf)

In practice, we do not solve the double integral every time. The equation in solved by "trial-and-error", or an approximate correction factor is employed.

Resulting Normal Space: Once we have the matrix of ρ_{ij}, we have a vector of standard normal variables Z with a correlation matrix Σ_Z. This allows us to proceed to Step 3.

Step 3: Decorrelation (Transformation to U-Space)

The variables in Z-space are normal but still correlated by R_Z. To calculate the reliability index β, we need **Independent Standard Normal** variables U.

The Orthogonal Transformation: We use the Cholesky Decomposition of the equivalent correlation matrix: $\mathrm{R}_Z = \mathrm{L}\mathrm{L}^T$.

$$\mathrm{Z} = \mathrm{L}\mathrm{U} \quad \Longrightarrow \quad \mathrm{U} = \mathrm{L}^{-1}\mathrm{Z}$$

Why we do this:

- In the U-space, the joint PDF is rotationally symmetric.
- The distance from the origin to any point is simply the number of standard deviations.
- The Reliability Index β is the shortest distance from the origin to the limit state surface $g(\mathrm{u}) = 0$.

FORM: Finding the Design Point

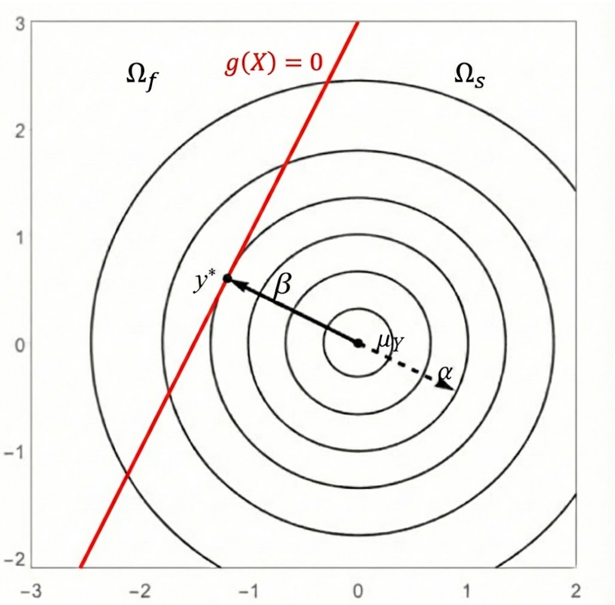
Once in U-space, the optimization problem is solved (*e.g.* iHLRF):

$$\beta = \min \sqrt{\mathrm{u}^T \mathrm{u}} \quad \text{subject to} \quad g(\mathrm{u}) \leq 0$$

The Design Point is the point on the limit state boundary $g(\mathrm{u}) = 0$ that is closest to the origin.

- In normal space, the probability density drops off exponentially with distance from the origin.
- Therefore, the point closest to the origin is the **Point of Maximum Likeliness** of failure.
- The distance to this point is the **Reliability Index** $\beta = \|\mathrm{u}^*\|$.
- The point is searched using an optimization method, usually iHRLF.
- $P_f \approx \Phi(-\beta)$

Reliability Index in U-Space



Sensitivity Analysis: The α Vector

Once the Design Point u^* is found, it is easy to calculate the unit vector pointing from the origin to that point:

$$\boldsymbol{\alpha} = \frac{u^*}{\|u^*\|} = \frac{u^*}{\beta}$$

The Importance Factors (α_i):

- Each component α_i represents the sensitivity of the reliability index β to the random variable U_i.
- **Geometric Meaning:** α_i is the direction cosine of the failure surface normal at the design point.
- **The Rule of Thumb:** A higher $|\alpha_i|$ means that variable X_i has a greater influence on the probability of failure.

- Since $\|\alpha\| = 1$, it follows that $\sum_{i=1}^{n} \alpha_i^2 = 1$.
- α_i^2 represents the approximate proportion of the total uncertainty contributed by variable X_i.
- **Positive** α_i**:** Usually associated with "Loads." Increasing the mean increases failure risk.
- **Negative** α_i**:** Usually associated with "Resistances." Increasing the mean decreases failure risk.

This allows engineers to optimize designs by focusing only on the variables that actually matter.

Beyond FORM

FORM assumes the limit state surface $g(\mathrm{u}) = 0$ is a straight line (or flat plane) at the Design Point.

When does FORM fail?

- If the failure surface is highly curved (concave or convex).
- If there are multiple variables with high non-linearity.

The Resulting Error:

- **Concave surfaces:** FORM *underestimates* the probability of failure.
- **Convex surfaces:** FORM *overestimates* the probability of failure.

We need a method that "sees" the curvature of the boundary.

SORM: Second-Order Reliability Method

Instead of a straight line, SORM approximates the failure surface using a **Paraboloid** (a second-order expansion).

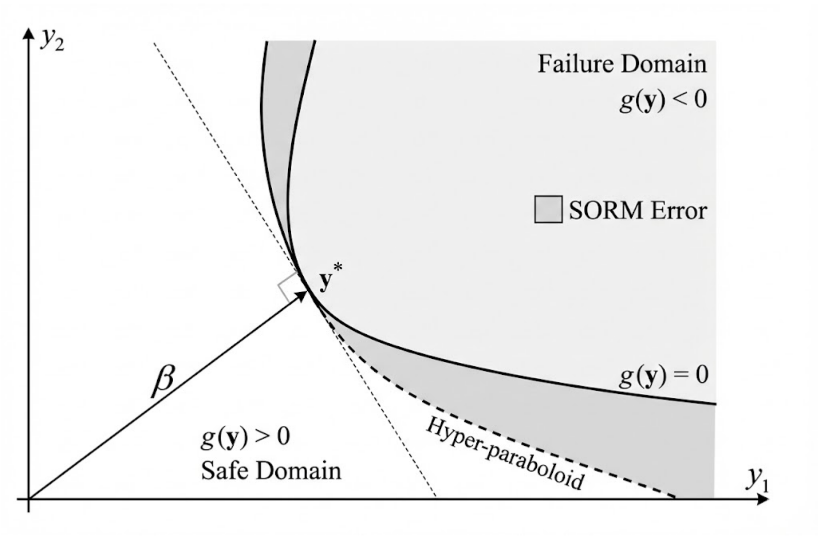
The SORM Correction (Breitung's Formula):

$$P_{f,SORM} \approx \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\beta\kappa_i}},$$

where:

- β: The **Reliability Index** calculated from FORM (the distance from the origin to the design point u^*).
- κ_i: The **Principal Curvatures** of the limit state surface at the design point.
- n: The number of random variables. (We use $n-1$ curvatures because one direction is normal to the surface).

SORM Approximation



Example 1: Fundamental $R - S$ Gaussian Case

Problem Definition:

- Resistance: $R \sim N(150, 20)$
- Load: $S \sim N(100, 10)$
- Limit State Function: $g(X) = R - S$

Analytical Solution: The safety margin $M = R - S$ is Gaussian $N(\mu_M, \sigma_M)$:

- $\mu_M = \mu_R - \mu_S = 150 - 100 = 50$
- $\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2} = \sqrt{20^2 + 10^2} = 22.36$

Reliability Index (β):

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{50}{22.36} \approx 2.236$$

$$P_f = \Phi(-\beta) \approx 0.0127$$

Example 2: Cantilever Beam Deflection (FORM/SORM)

Model: Tip deflection $\delta = \frac{PL^3}{3EI}$

Random Variables (Gaussian):

- $P \sim N(5000, 500)$ N, $L \sim N(2, 0.05)$ m
- $E \sim N(210 \times 10^9, 10 \times 10^9)$ Pa, $I \sim N(10^{-5}, 5 \times 10^{-7})$ m^4

Failure Criterion: Failure occurs if $\delta > 0.006$ m (0.9 cm).

$$g(X) = 0.009 - \frac{PL^3}{3EI}$$