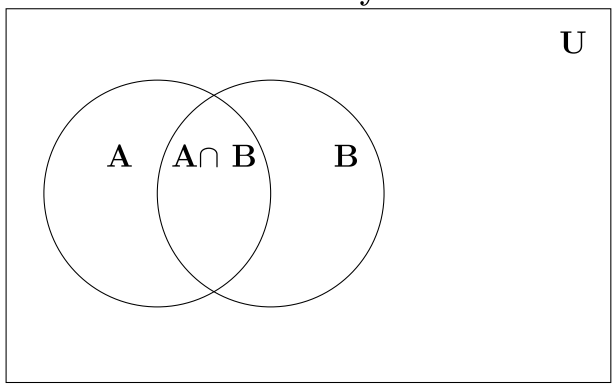
Bayes' Theorem



The probability of finding B is

$$P(B) = \frac{|B|}{|U|}, \tag{1}$$

where $|U|$ is the area of U. Similarly,

$$P(A) = \frac{|A|}{|U|}, \tag{2}$$

while the probability of both A and B is

$$P(AB) = \frac{|A \cap B|}{|U|}, \tag{3}$$

What is the probability of B, given that we know we are in A?

$$P(B|A) = \frac{|B \cap A|}{|A|}. \tag{4}$$

For future reference

$$P(B|A) = \frac{|B \cap A|}{|A|} = \frac{|B \cap A|}{|U|}\frac{|U|}{|A|} = P(AB)\Big/P(A). \tag{5}$$

Conversely, the probability of A, given that we know we are in B is

$$P(A|B) = \frac{|A \cap B|}{|B|} = P(AB)\Big/P(B). \tag{6}$$

It follows that

$$P(A|B)P(B) = P(B|A)P(A). \tag{7}$$

This is known as Bayes' Theorem.

Bayes's Theorem is often written

$$P(A|B) = P(B|A)P(A)/P(B). \tag{8}$$

Why is this useful? First, it helps counter our usual intuition about probabilities, which are frequently in error. For example, at the start of the pandemic in 2020, about 1 percent of people in Ontario have or had Covid-19. Tests for Covid-19 are fairly accurate, around 97%, meaning that if they are given to 100 people, only three test will give a positive result in error. Lets also assume that if someone has Covid-19, that the test always returns a positive result (this is not true, but it is not so bad an approximation).

So suppose Ontario starts a random testing program, and tests 1000 people. The tests of some people will be positive. Given that someone has a positive test, what are the odds that they have Covid-19?

Ok, how might this be useful for you while doing your research this summer? Many of you will be using data to decide if a theory is correct or not. The theory will have a number of parameters; for example, in the theory of star formation, relevant parameters likely include the virial parameter, or ratio of kinetic to gravitational potential energy α of the gas that stars form out of, the Mach number $\mathcal{M}$ of the flow, and the average gas density $\bar{\rho}$. The parameters of a theory are conventionally denoted by θ. In the star formation case, $\theta = (\alpha, \mathcal{M}, \bar{\rho})$. The data is denoted by d, and has errors σ_d. The theory makes prediction for the values y_i corresponding each of the components of d. One measure of the fit between theory and data is

$$\chi^2 = \sum_i \frac{(y_i - y_{i,\,obs})^2}{\sigma_i^2}. \tag{9}$$

If the measurement errors are random, they might be described by a Gaussian. In that case the probability that we have the correct value is proportional to

$$L = \prod_i \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(y_i - y_{i,\,obs})^2/2\sigma_i^2}, \tag{10}$$

if we also assume that all the measurements are independent.

Now we can use Bayes' Theorem to estimate the probability that our theory is true, given that we have seen the data d:

$$P(\theta|d) = P(\theta)P(d|\theta)/P(d). \tag{11}$$

In practice, the jargon is as follows:

$$P(\theta|d) = \Pi(\theta)L(d|\theta)/Z(d), \tag{12}$$

where $P(\theta|d)$ is called the "posterior probability", Π is called the "prior", L is the "likelihood", and Z is called the "evidence".