PARAMETRIC MODELLING AND CATENARY ARCH ANALYSIS REPORT

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ABSTRACT

This report intends to record and analyze how strength/ultimate stress of catenary arches varies corresponding the change of rise-to-span and thickness-to-rise ratios. 32 tests are carried out on arches with four rise-to-span ratios and four thickness-to-rise ratios under both uniformly distributed loads and point loads. The impact of these ratios on arches' ability to bear distributed load and point load is then discussed and analyzed. Hand calculation is used to validify software outcomes, and the differences between hand calculate results and software-generated results are accounted for. Conclusions and limitations of this experiment are also discussed in the end of this report.

1. INTRODUCTION

The curvature of the catenary arch is assumed to be the shape formed when a chain/string is suspended from two points (but inverted) at equilibrium state[1]. The shape of the arch works well with materials that behaves stronger in compression, such as rock and concrete, which usually is weak when working with tension. The catenary curve is also related to a parabola - the curve traced in the plane by the focus of a parabola as it rolls along a straight line is a catenary[2].

1.1. Theory

The arch is assumed to be perfectly linearly elastic, and the self-weight of the arch is neglected. Buckling, cracking, block integration and other non-linear effects are neglected. For a perfect inverted catenary shape, tensile stresses in the lintel are zero and all stresses are compressive[3].

The mathematical formula and differential equation simulating the catenary correspond to the following principles of statics[4]:

I. The point loads acting on any two points of the catenary can be derived by simulating the interposed part of the catenary by a force of the same mass, as graphic example (a) and (b).

II. When the catenary is in equilibrium state, the sum of horizontal and vertical forces respectively is zero.

As required by the brief, the conditions of this experiment are *Arch being analyzed spans 3 m, is three pinned, made from concrete (C12/15) and has a square cross-section.* These requirements are met by setting up correct parameters in software.

The equation used by this experiment to express the catenary curves is:

$$y = \frac{e^{ax}+e^{-ax}}{2a} \quad \text{(eq. 1)}$$

where a is a constant determining the shapes of the curves. This equation is used both in the software setup and hand calculation.

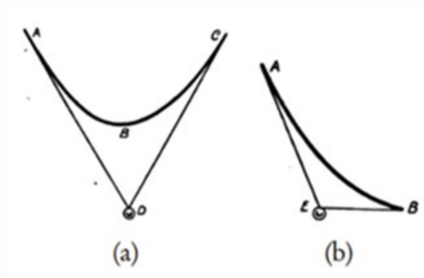


Figure 1. Point loads acting on two points on a catenary curve

1.2. Method

BHoM and Autodesk Robot are used to conduct our analysis. BHoM is developed by Buro Happold and is the abbreviation for 'Buildings and Habitats object Model'. It is a collaborative computational development project that uses a single common language between software, which enables codes to be created and shared easily. 'Adapters' are used to convert objects between the BHoM and the external software, and they can be integrated in the interface of Rhino grasshopper[6]. Autodesk Robot Structural Analysis (referred to as Robot in this report) is structural load analysis software which helps to create more resilient, constructible designs that are accurate, coordinated and connected to BIM[7]. After BHoM is installed in Rhino grasshopper, components, materials and loads can be defined and can then be 'pushed' into Robot using adapters. Robot analyzes the structural information and produces results including reaction forces, stresses, deformation, etc. The results can then be 'pulled' back to grasshopper for post processing. The workflow is shown in Figure 2.

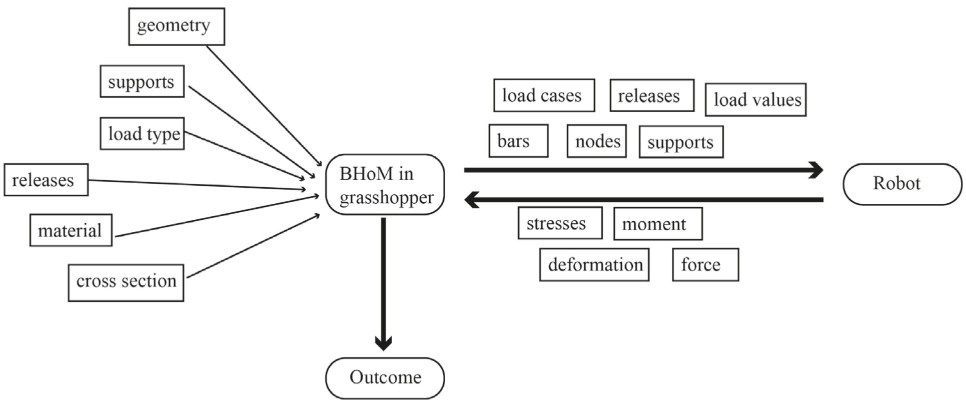


Figure 2. Workflow of this experiment.

The experiment started by obtaining the expressions for the catenary curves. Desmos calculator (https://www.desmos.com/calculator) is used to test the values of 'a' until the curves fit the ideal geometries. The results of 'a' for each case of this experiment are shown in Table 1.

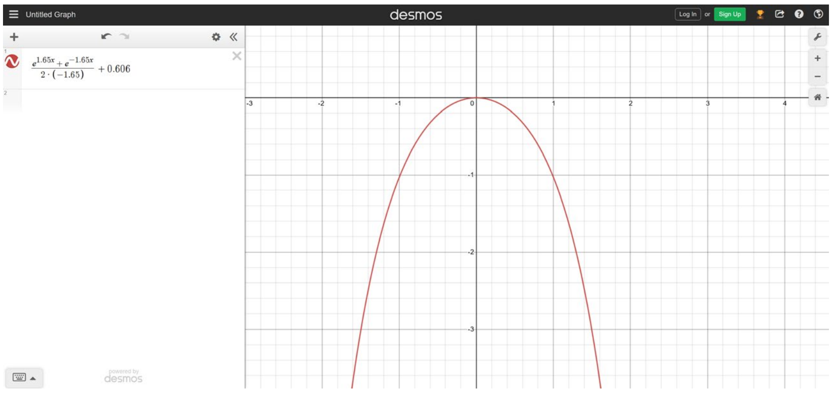


Figure 3. Using Desmos to obtain the value of 'a' for catenary curves.

The values of 'a' are then put in the grasshopper code to set up the geometries of the catenary curves (Figure 4). As required by the brief, the nodes at ¼ and middle of the curves are found (Figure 5). In order to be analyzed in Robot, the curves must be split in segments, and the length of the segments are determined by a balance of accuracy and computational power, which will be analyzed in Section 1.3.

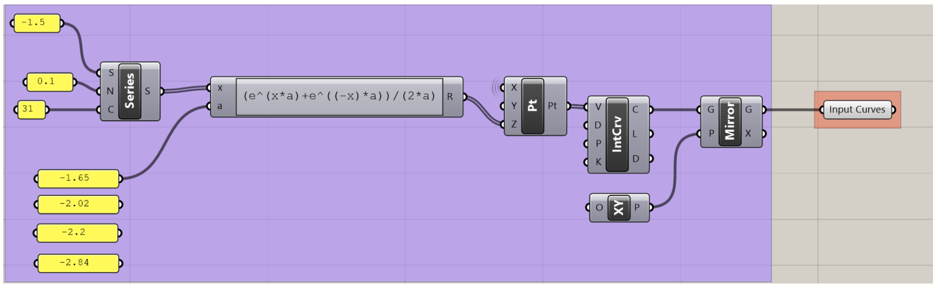


Figure 4 . Grasshopper code for setting up the catenary curves.

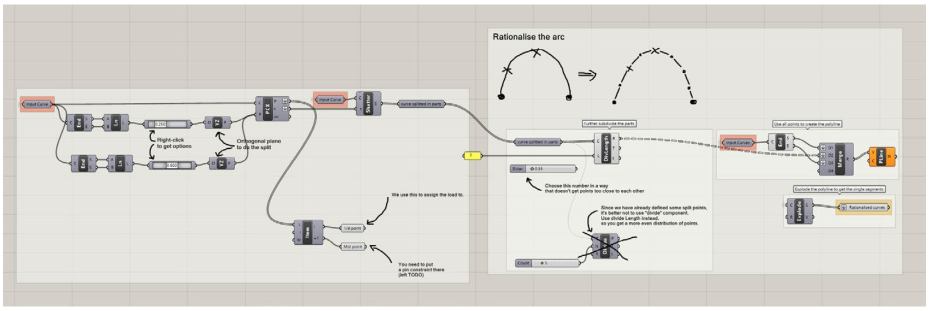


Figure 5. Defining the ¼ point and middle point, and split the curve into segments.

Pin nodes are defined at the supports of the arches by releasing the y-y rotation (Figure 6), and a bar release is defined in the middle of the arch as required (Figure 7). The material of the arch is C12/15, and the dimension of the square cross section can be adjusted (Figure 8). All nodes and bars can then be pushed into Robot.

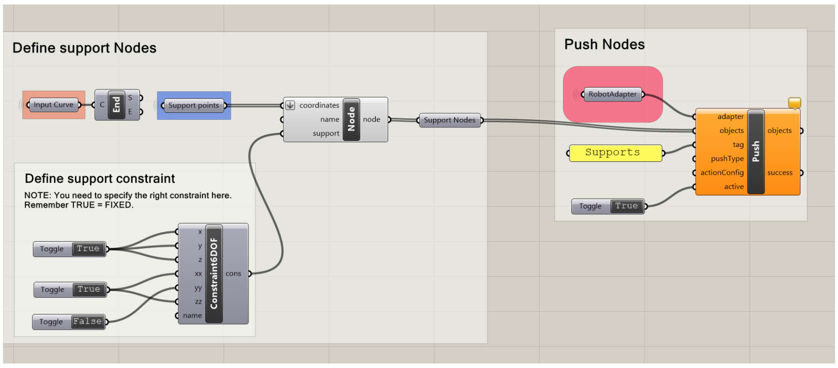


Figure 6.Defining and pushing support nodes

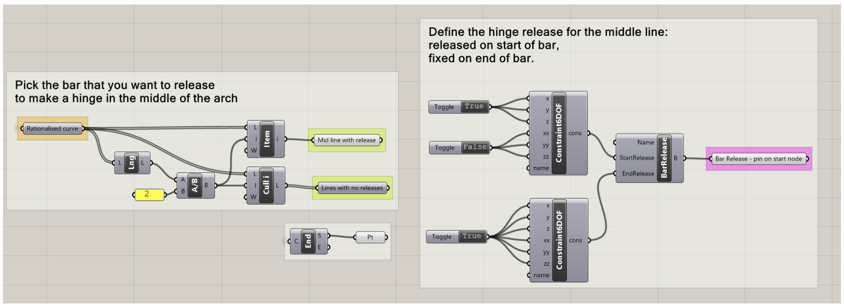


Figure 7. Creating bar release in the middle of the arch

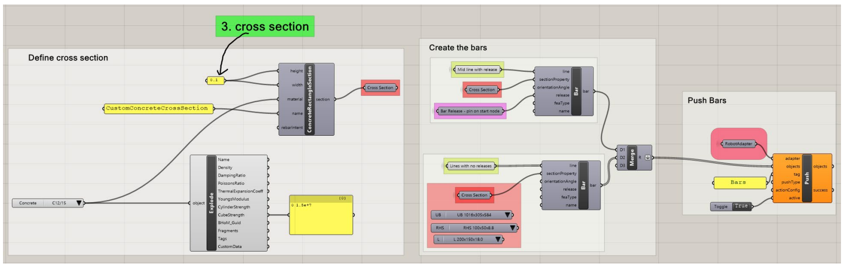


Figure 8. Defining the material of bars and sizes of cross section, and pushing the bars

Load cases are then defined, containing two types of loads that are required for this report. For the uniformly distributed load applied throughout the structure, values of loads can be adjusted and pushed into Robot (Figure 9). For the point load, nodes have to be pulled from Robot, and the node at ¼ arch can be selected by picking the nearest node there ¼ point of the curve (Figure 10). The value of point load is then imposed on this node. By activating the adapter, grasshopper pushes all the information and pulls analysis results from Robot (Figure 11), and structural data can be post processed.

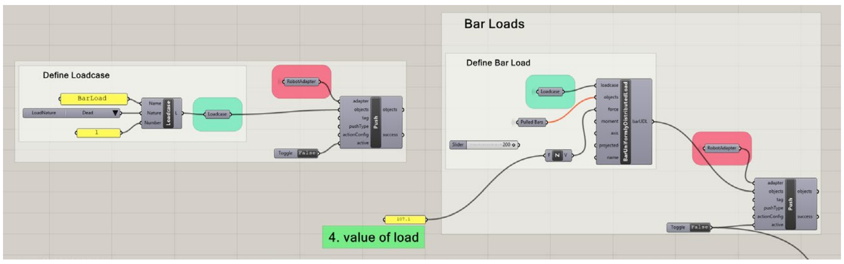


Figure 9. Defining load cases and applying distributed loads on all bars

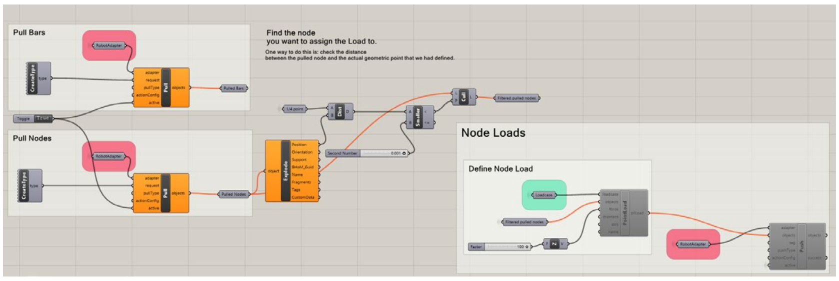


Figure 10. Selecting the ¼ node and pushing the value of point load

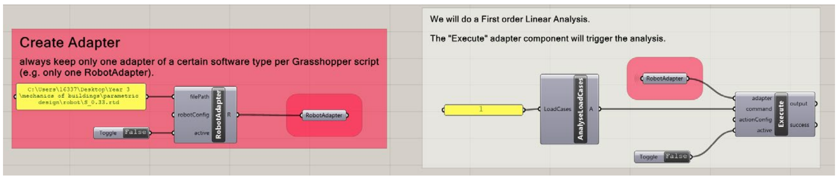


Figure 11. Activating the adapter and starting linear analysis for the structure

After stress and deformation diagrams are plotted, the maximum stress σ_{max} of the arch can be obtained and compared with σ= 1.57 x 10^6 kPa, which is the tensile strength of C12/15 concrete. If σ_{max} exceeds 1.57 x 10^6 kPa, it means the value of loads imposed on the structure is too large and the structure will fail. The value of load should thus be adjusted to a smaller value. If σ_{max} is smaller than 1.57 x 10^6 kPa, increase the load until σ_{max} reaches 1.57 x 10^6 kPa. In this way, the ultimate bearing capacity of the arch can be obtained.

1.3 Experimental parameters

Four rise-to-span variations and four section-to-rise variations are required for this report. For each case, the maximum capacity of the arch under a distributed load throughout the span and a point at ¼ of the span are required to be tested. The parameters of the test are shown below, where the span, height and side length information are listed for each case.

Case Number	Span (m)	Ratio	Height (m)	Side Length (m)	Value of 'a'
1	3	1	3	0.1	-1.65
2	3	1.5	4.5	0.1	-2.02
3	3	2	6	0.1	-2.2
4	3	4	12	0.1	-2.84
5	3	1	3	0.2	-1.65
6	3	1.5	4.5	0.2	-2.02
7	3	2	6	0.2	-2.2
8	3	4	12	0.2	-2.84
9	3	1	3	0.3	-1.65
10	3	1.5	4.5	0.3	-2.02
11	3	2	6	0.3	-2.2
12	3	4	12	0.3	-2.84
13	3	1	3	0.4	-1.65
14	3	1.5	4.5	0.4	-2.02
15	3	2	6	0.4	-2.2
16	3	4	12	0.4	-2.84

Table 1. Cases and parameters for this experiment

A trial run of segment length for Case 1, uniformly distributed load is carried out, and the maximum bearing load is shown in Table 2, Figure 12 and Figure 13. It can be observed that within the range of 0.1m-0.5m, segment length does not play a significant role in the result, whereas when segment length is exceeds 0.5m the result will have a larger discrepancy. Therefore, a segment length of 0.33m is determined for the code, taking into account of result accuracy and computational speed.

segment length (m)	Max W
0.1	4.58
0.3	4.58
0.5	4.45
1	3.03

Table 2. Testing the influence of segment length on maximum bearing capacity for Case 1

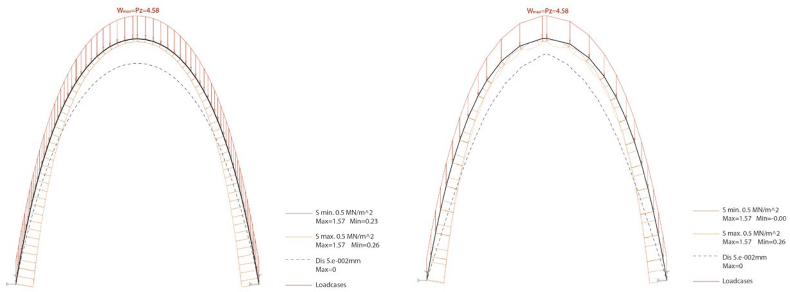


Figure 12. Stress and deformation diagrams for segment length 0.1m and 0.3m (left to right)

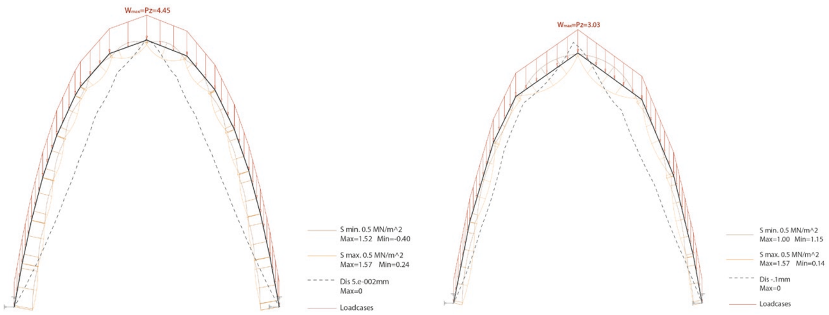
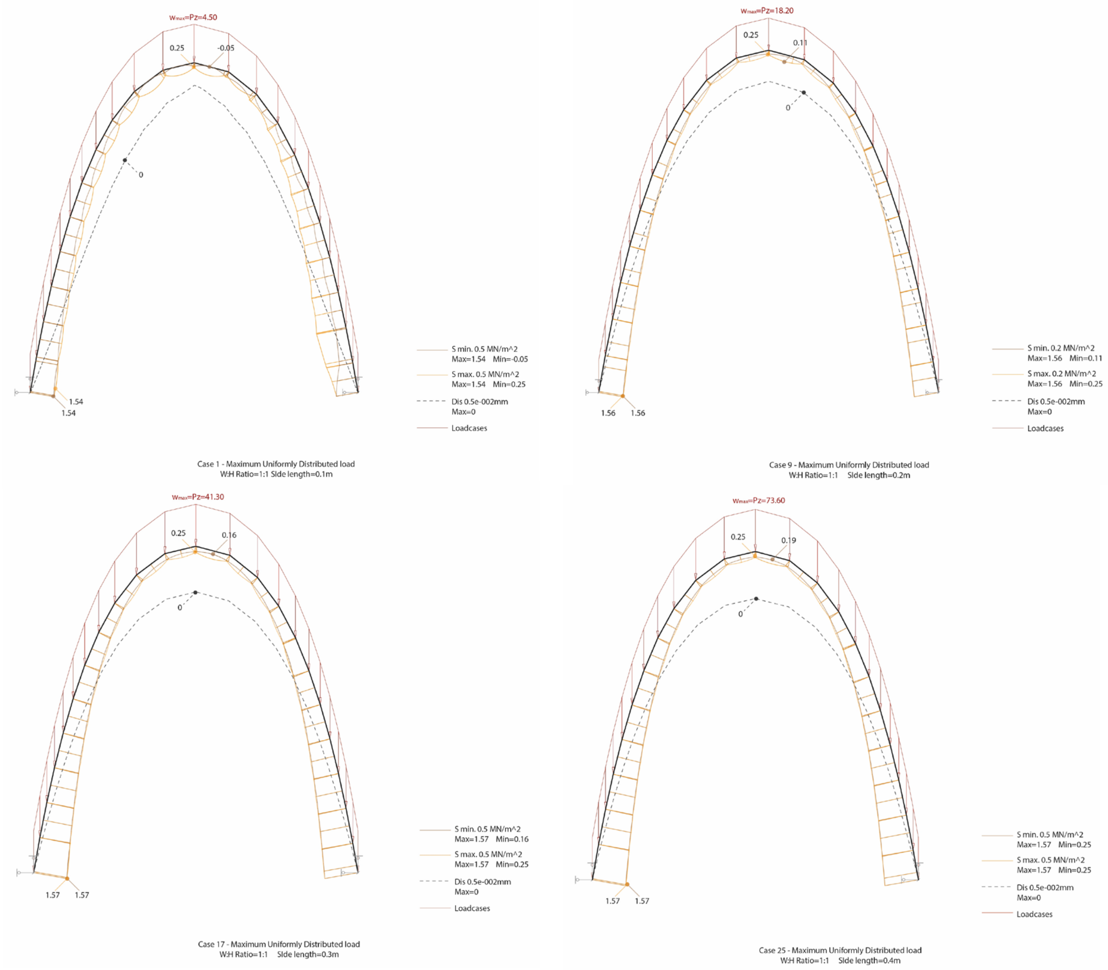


Figure 13. Stress and deformation diagrams for segment length 0.5m and 1m (left to right)

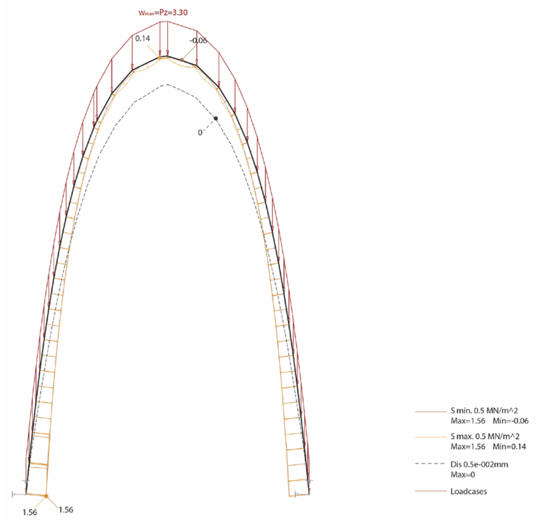
2. *RISE-TO-SPAN TEST: DISTRIBUTED LOAD*

2.1. Ratio 1:1 (Rise: Span) remain constant, change in thickness

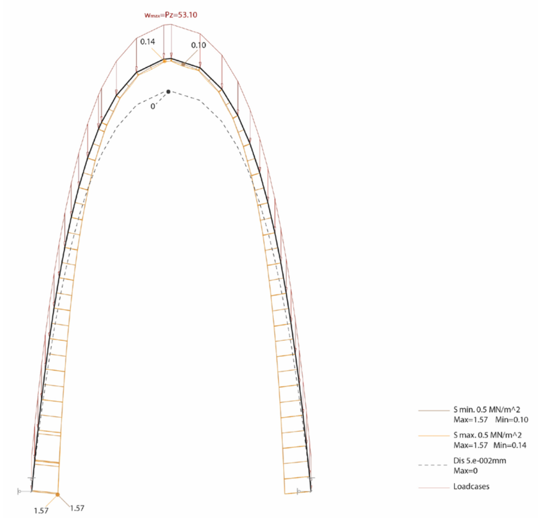


For all these 4 cases, the deformation shape is generally the same, however arch with thickness 100mm has a more deformed shape (slimmer) compared to others. The increase in stress also causes the deformation displacement to be larger. The stress on the tip of the arch is the smallest, and it gradually increases as it moves from the tip to the supports. As the distributed load increases the axial stress on the supports also increases, which is indicated in the graph as a wider stress diagram.

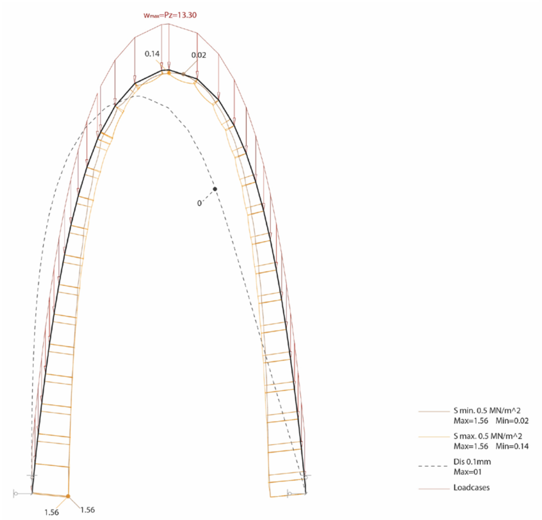
2.2. Ratio 1.5:1 remain constant, change in thickness



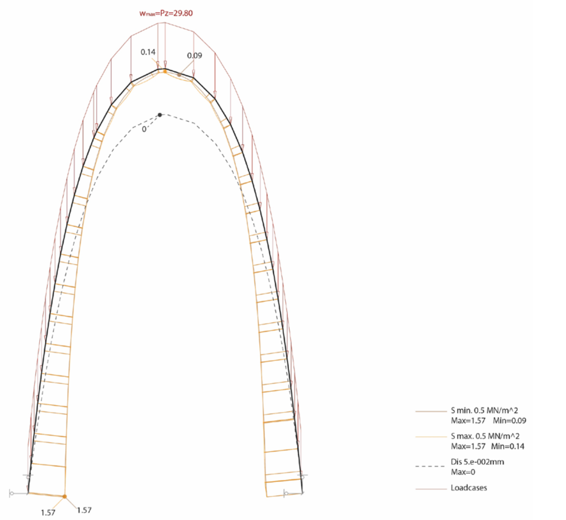
Case 3 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.1m



Case 27 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.4m



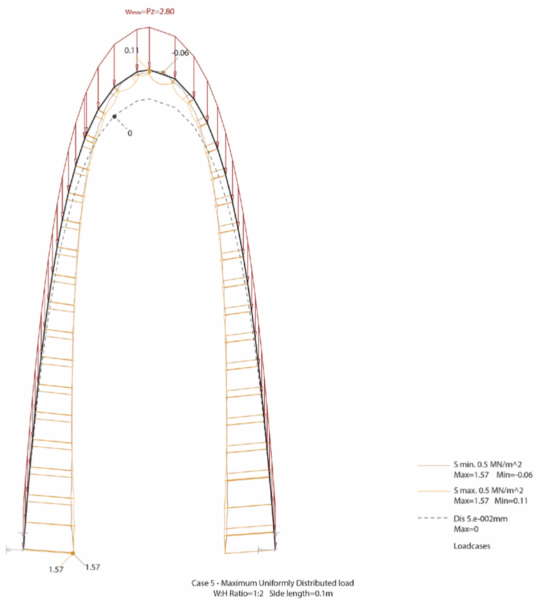
Case 11 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.2m

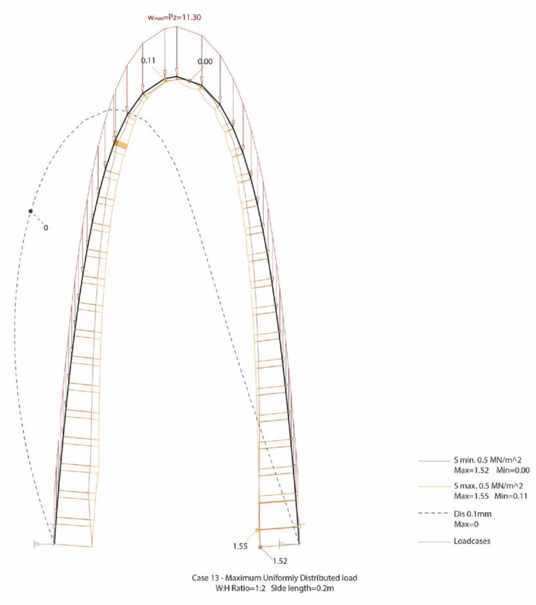


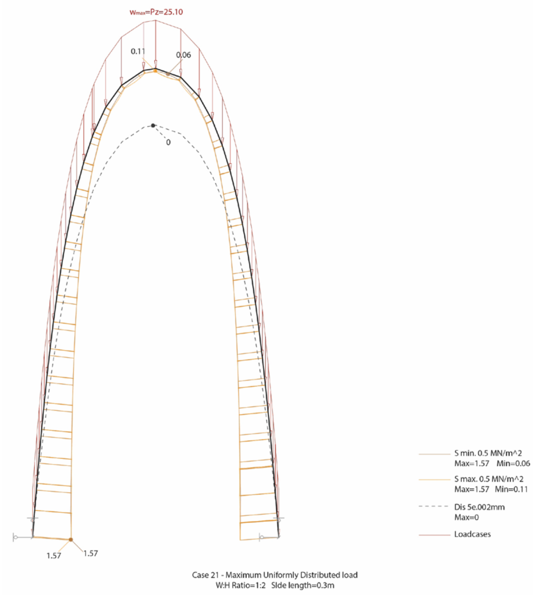
Case 19 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.3m

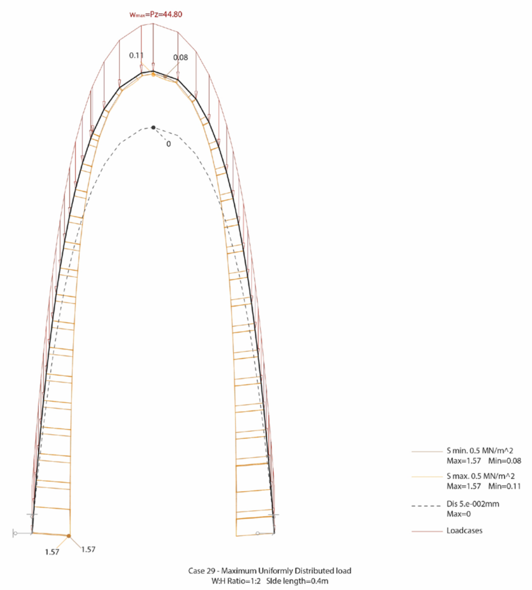
Similar to previous cases, for all these 4 cases, the deformation shape is generally the same, however arch with thickness 100mm has a more deformed shape (slimmer) compared to others. Also with the thickness at 200mm, the deformation shape is tilted slight to the left, indicating the arch failing at around 1/4 of the span. The increase in stress also causes the deformation displacement to be larger.The stress on the tip of the arch is the smallest, and it gradually increases as it moves from the tip to the supports. As the distributed load increases the axial stress on the supports also increases, which is indicated in the graph as a wider stress diagram.

2.3. Ratio 2:1 remain constant, change in thickness



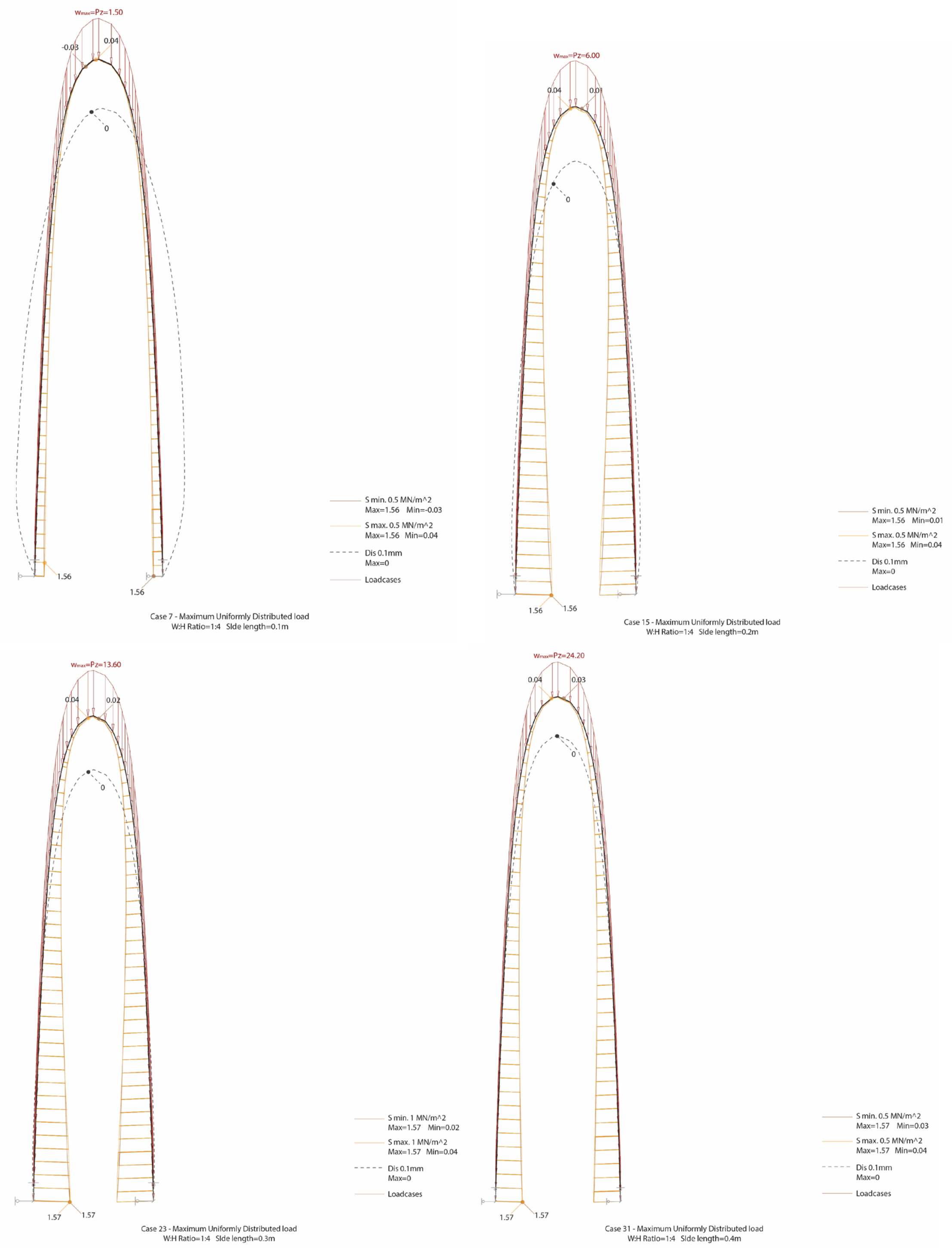






For all these 4 cases, the deformation shape is generally the same, however with the thickness at 200mm, the deformation shape is tilted slight to the left, indicating the arch failing at around 1/4 of the span. The increase in stress also causes the deformation displacement to be slightly larger. The stress on the tip of the arch is the smallest, and it gradually increases as it moves from the tip to the supports. As the distributed load increases the axial stress on the supports also increases, which is indicated in the graph as a wider stress diagram.

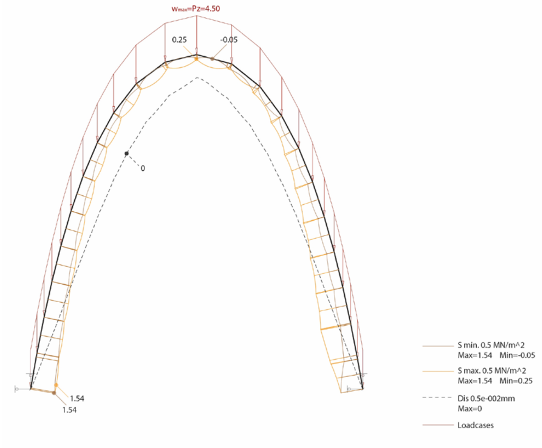
2.4. Ratio 4:1 remain constant, change in thickness



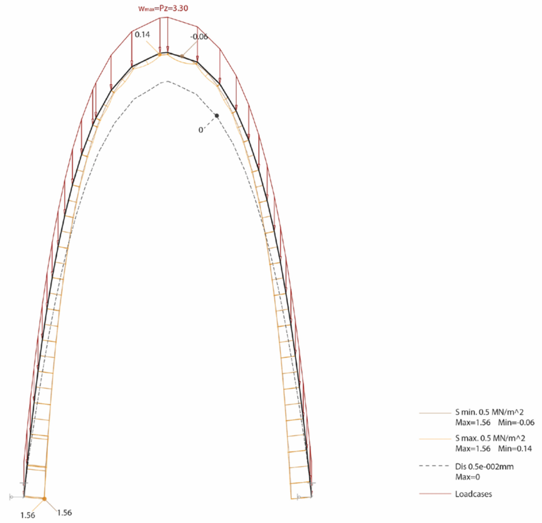
For all these 4 cases, the deformation shape is generally the same, however unlike the previous cases, the deformation shape decreases in height as well as deflecting outwards at the support, forming a shorter and wider shape. The deformation is the most significant in arch with thickness 100mm. The increase in stress also causes the deformation displacement to be slightly larger. The stress on the tip of the arch is the smallest, and it gradually increases as it moves from the tip to the supports. As the distributed load increases the axial stress on the supports also increases, which is indicated in the graph as a wider stress diagram.

3. THICKNESS-TO-RISE TEST: DISTRIBUTED LOAD

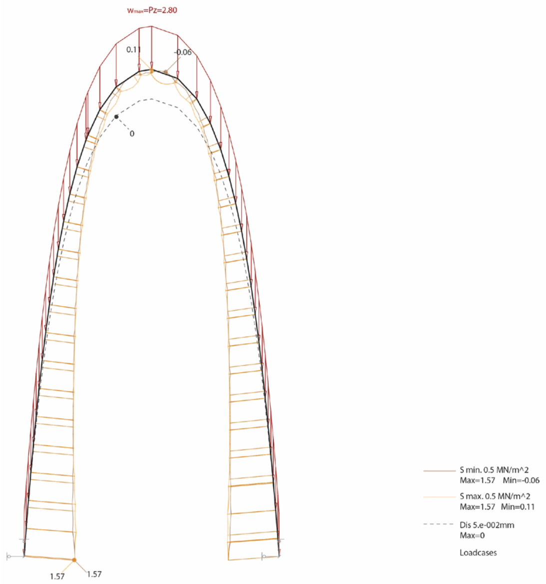
3.1. Thickness (100mm x 100mm) remains constant, change in rise to span ratio



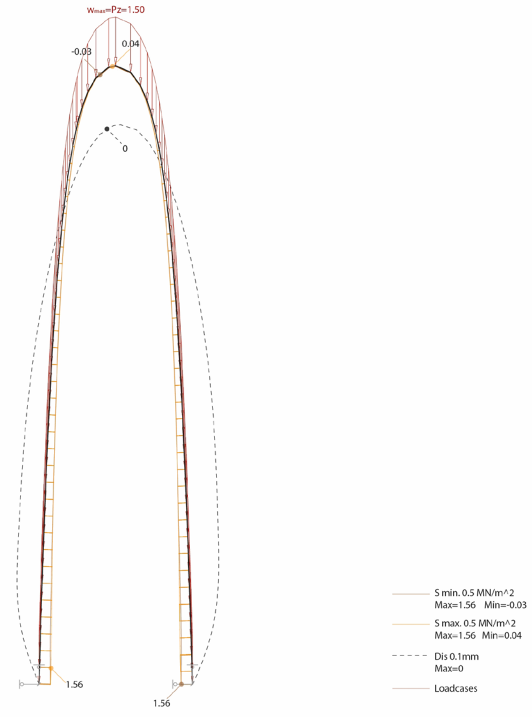
Case 1 - Maximum Uniformly Distributed load
W:H Ratio=1:1 Side length=0.1m



Case 3 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.1m



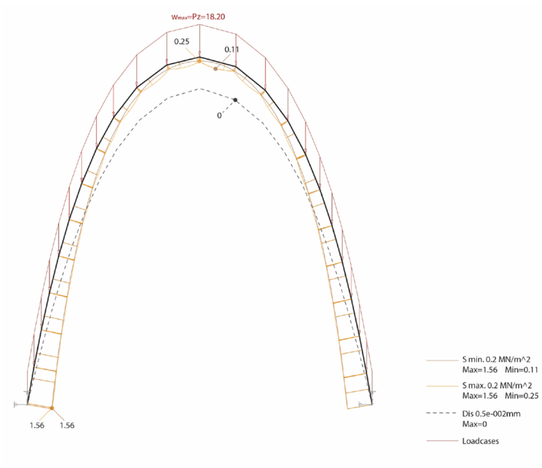
Case 5 - Maximum Uniformly Distributed load
W:H Ratio=1:2 Side length=0.1m



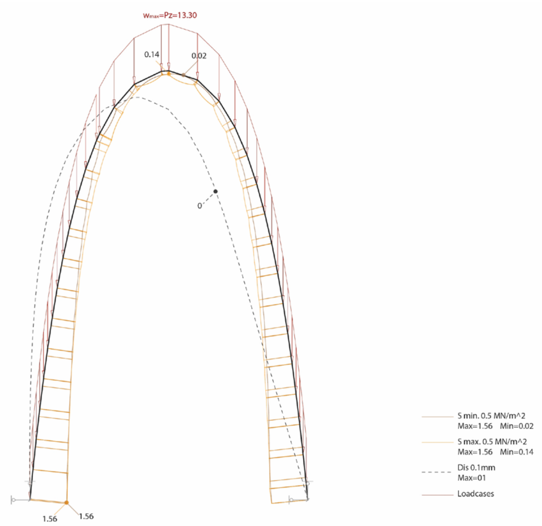
Case 7 - Maximum Uniformly Distributed load
W:H Ratio=1:4 Side length=0.1m

As the rise to span ratio increases, the deformation shape of the arch gradually changing from deforming inwards to outwards. In addition, ratio 4:1 shows a significant change in shape nearer to supports, which deflects outwards in a curved manner.

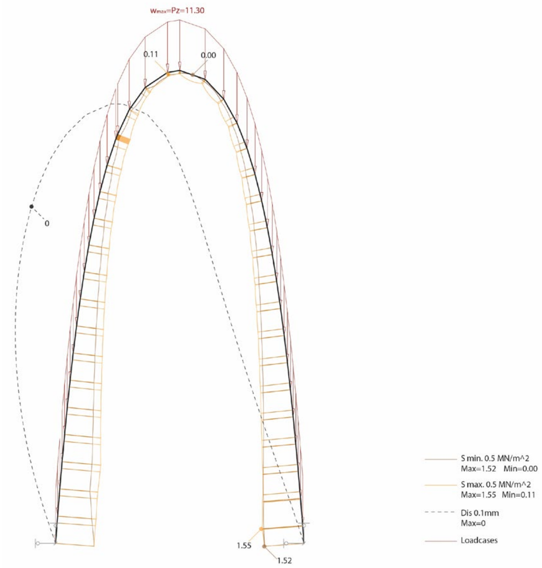
3.2. Thickness (200mm x 200mm) remains constant, change in rise to span ratio



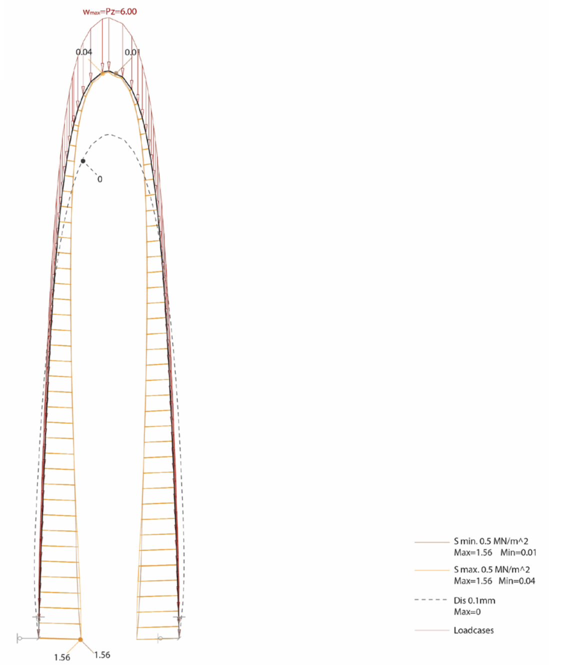
Case 9 - Maximum Uniformly Distributed load
W:H Ratio=1:1 Side length=0.2m



Case 11 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.2m



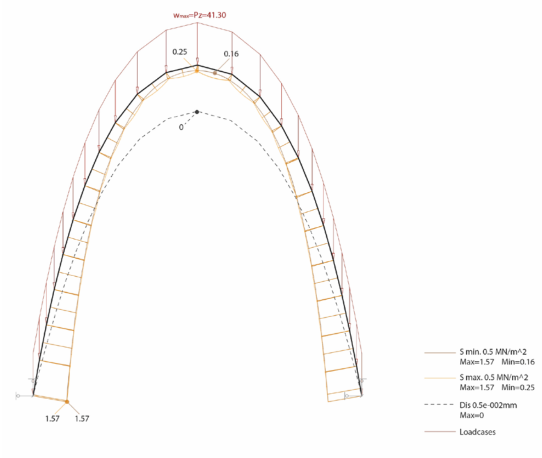
Case 13 - Maximum Uniformly Distributed load
W:H Ratio=1:2 Side length=0.2m



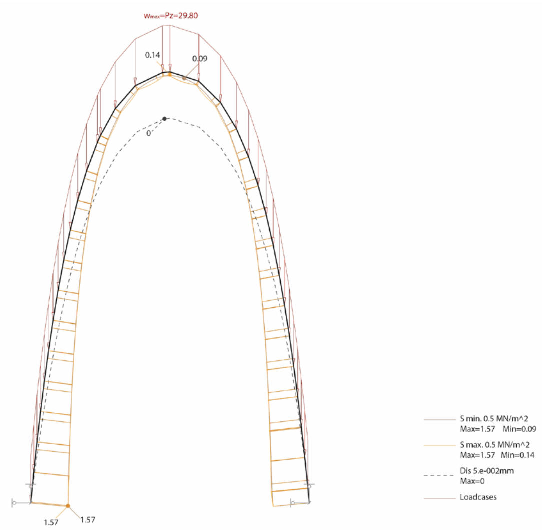
Case 15 - Maximum Uniformly Distributed load
W:H Ratio=1:4 Side length=0.2m

The deformation tendency is the similar for these four cases, except ratio 1.5:1 and 2:1 shows the deformation tends to tilt to the left, and the deflection is larger as the ratio increases. 4:1 ratio here shows that increase in thickness decreaes deformation compare to previous four cases, especially comparing to the 4:1 ratio with thickness 100mm.

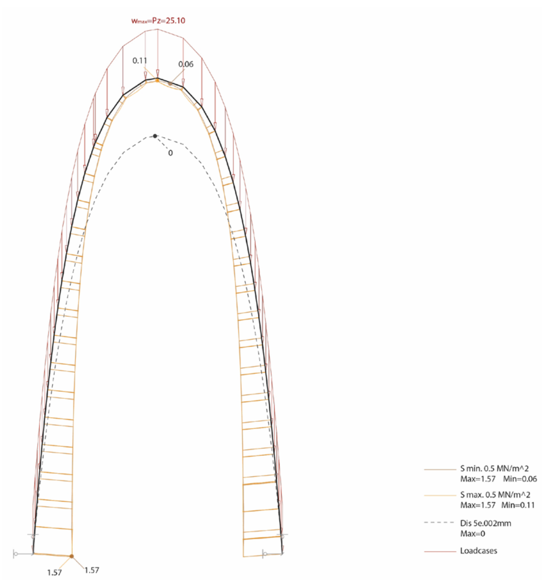
3.3. Thickness (300mm x 300mm) remains constant, change in rise to span ratio



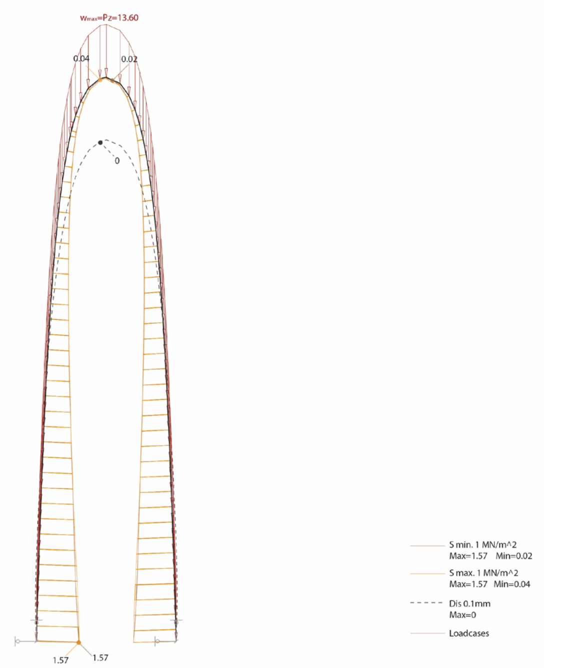
Case 17 - Maximum Uniformly Distributed load
W:H Ratio=1:1 Side length=0.3m



Case 19 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.3m



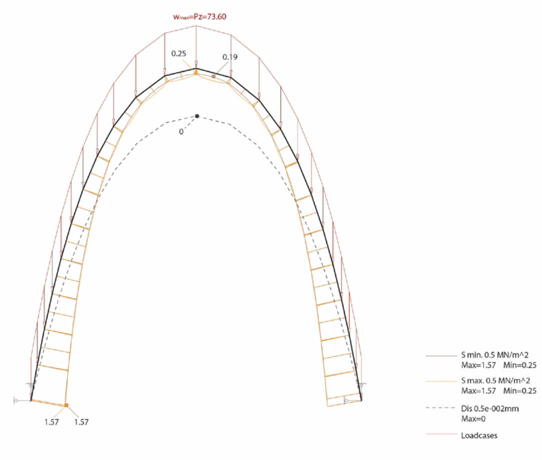
Case 21 - Maximum Uniformly Distributed load
W:H Ratio=1:2 Side length=0.3m



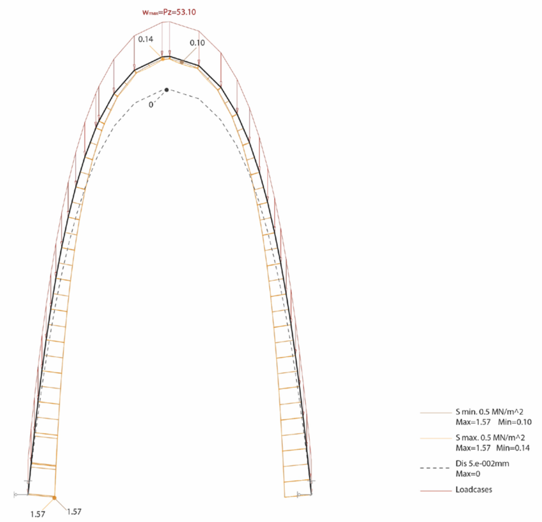
Case 23 - Maximum Uniformly Distributed load
W:H Ratio=1:4 Side length=0.3m

As the rise to span ratio increases, the deformation shape of the arch gradually changing from deforming inwards to outwards. However, after the thickness increased, the defomation shapes of each ratio gradually looking similar to each other, in terms of decreasing in height and deforming inwards.

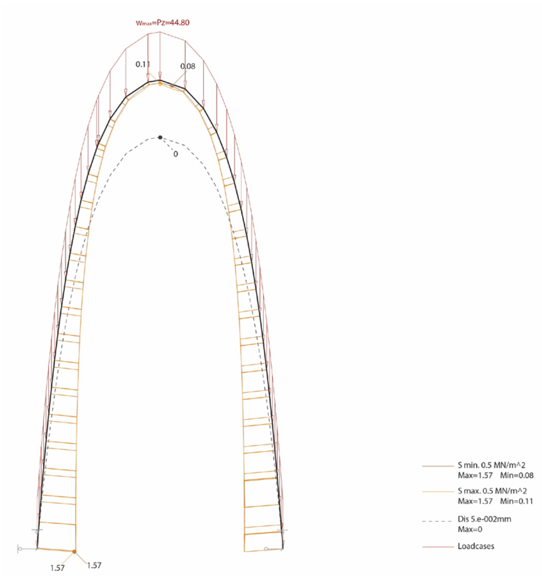
3.4. Thickness (400mm x 400mm) remains constant, change in rise to span ratio



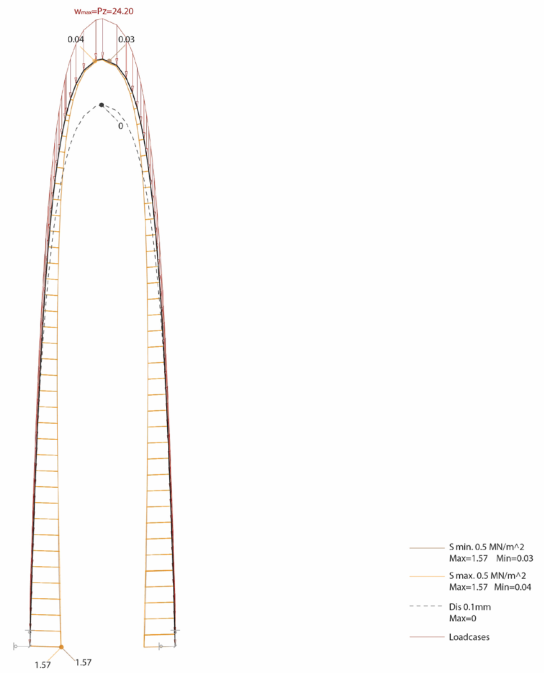
Case 25 - Maximum Uniformly Distributed load
W:H Ratio=1:1 Side length=0.4m



Case 27 - Maximum Uniformly Distributed load
W:H Ratio=1:1.5 Side length=0.4m



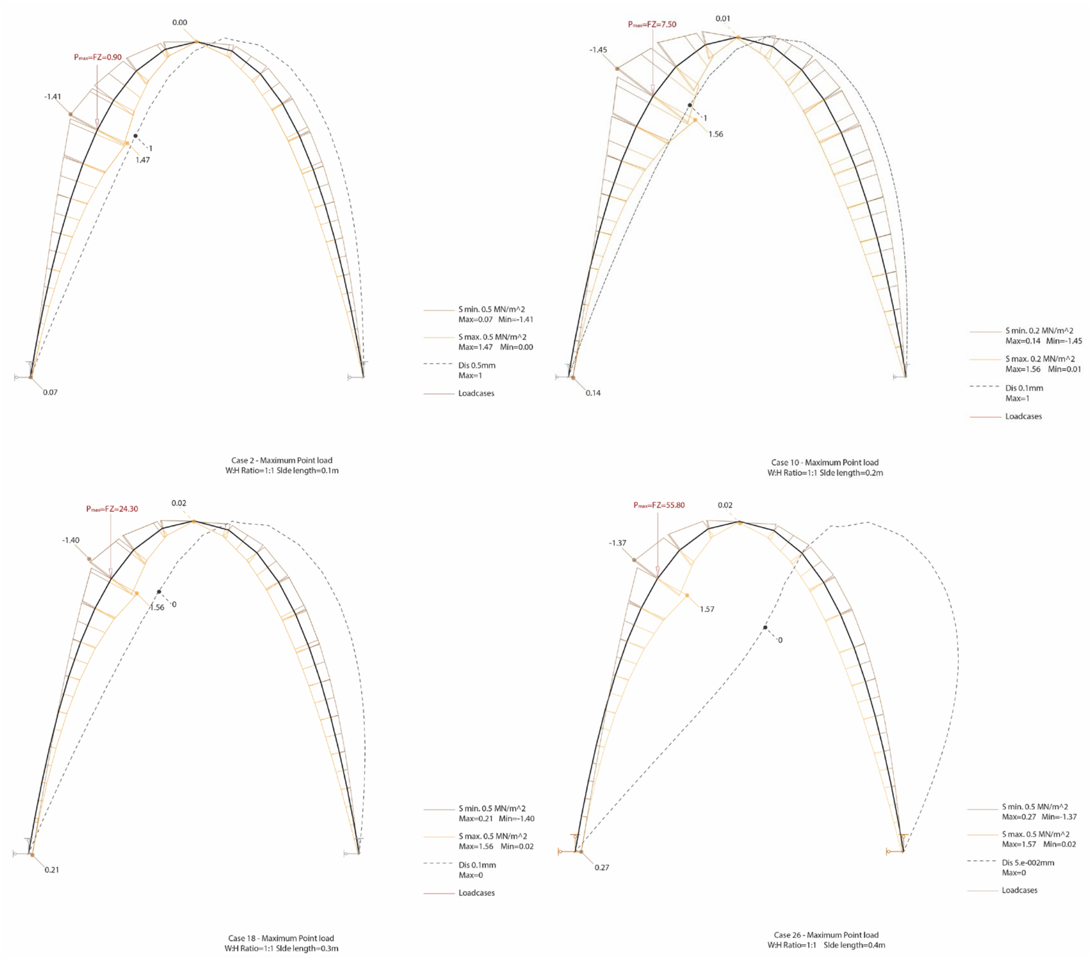
Case 29 - Maximum Uniformly Distributed load
W:H Ratio=1:2 Side length=0.4m



Case 31 - Maximum Uniformly Distributed load
W:H Ratio=1:4 Side length=0.4m

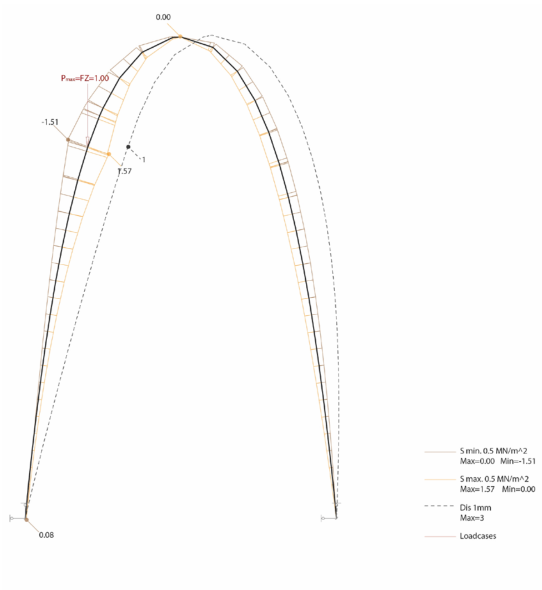
4. *RISE-TO-SPAN TEST: POINT LOAD*

4.1. Ratio 1:1 (Rise: Span) remain constant, change in thickness

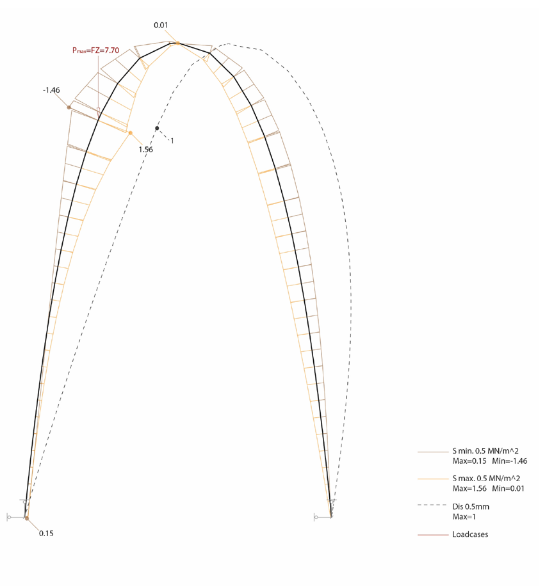


The 1:1 ratio case scenario focuses on changing its thickness, varying from a side length of 0.1m to 0.4m, therefore having 4 cases. The loading is quite different for each increasing side length, leading to increasing values of the maximum and minimum stresses. The deflection is also worth mentioning, this ending up in a dramatic shape for the point load of F = 55.80, with the side length of 0.4m.

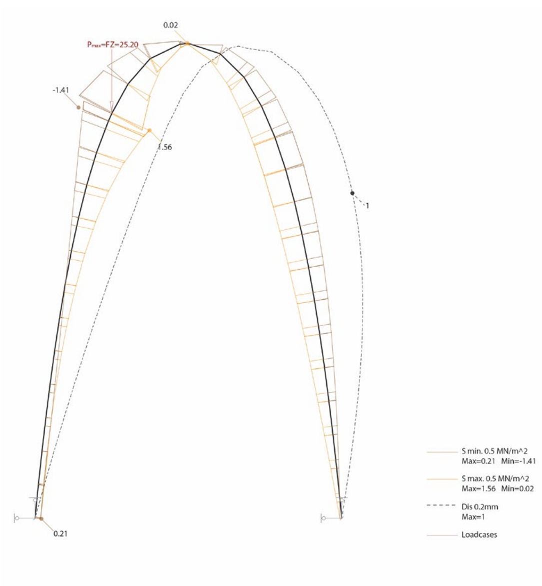
4.2. Ratio 1.5:1 (Rise: Span) remain constant, change in thickness



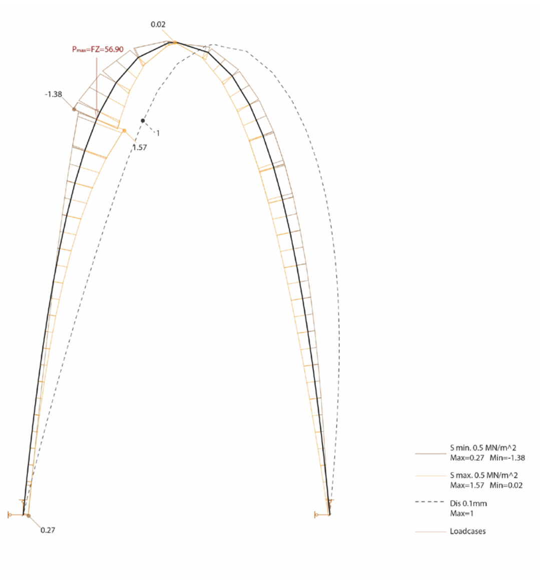
Case 4 - Maximum Point load
W:H Ratio=1:1.5 Side length=0.1m



Case 12 - Maximum Point load
W:H Ratio=1:1.5 SIde length=0.2m



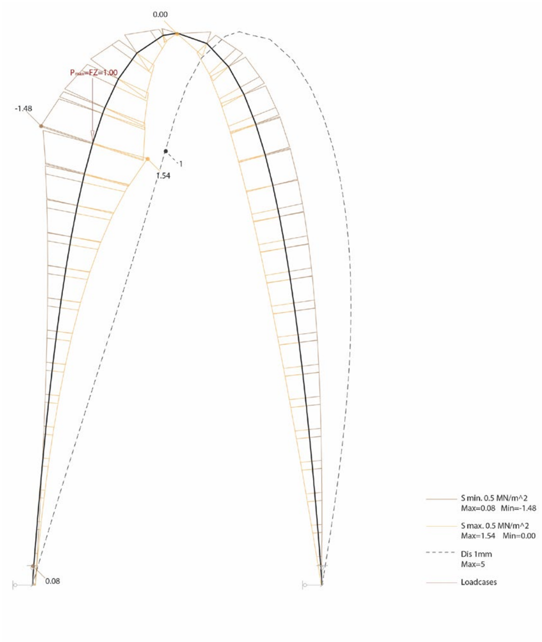
Case 20 - Maximum Point load
W:H Ratio=1:1.5 Side length=0.3m



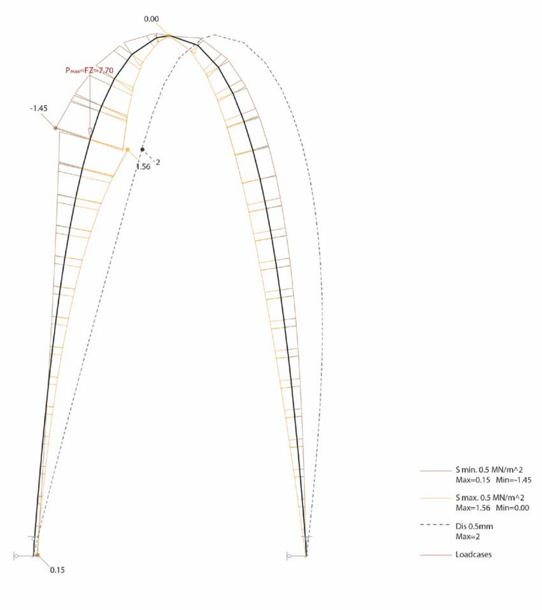
Case 28 - Maximum Point load
W:H Ratio=1:1.5 SIde length=0.4m

For the 1.5:1 ratio case scenario, the side lengths also go from 0.1 to 0.4m, with increasing point loads, the maximum one equaling 56.90. The same process of the stresses' increase is happening as for the previous case scenario; therefore the maximum and minimum stresses increase directly proportional with the load applied. This ratio does act better when it comes to deflection than most of the other cases, which can be seen in the example with the side length of 0.4m.

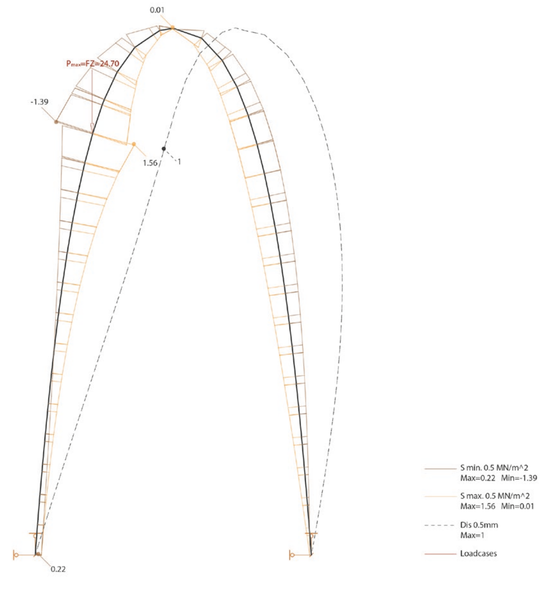
4.3. Ratio 2:1 (Rise: Span) *REMAIN CONSTANT, CHANGE IN THICKNESS*



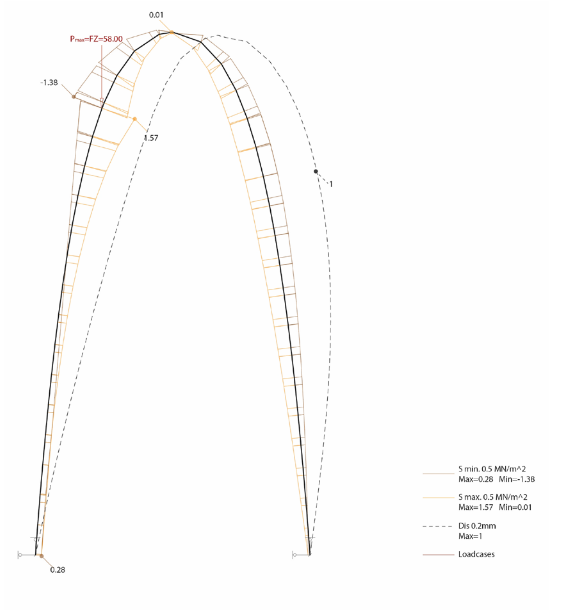
Case 6 - Maximum Point load
W:H Ratio=1:2 Side length=0.1m



Case 14 - Maximum Point load
W:H Ratio=1:2 Side length=0.2m



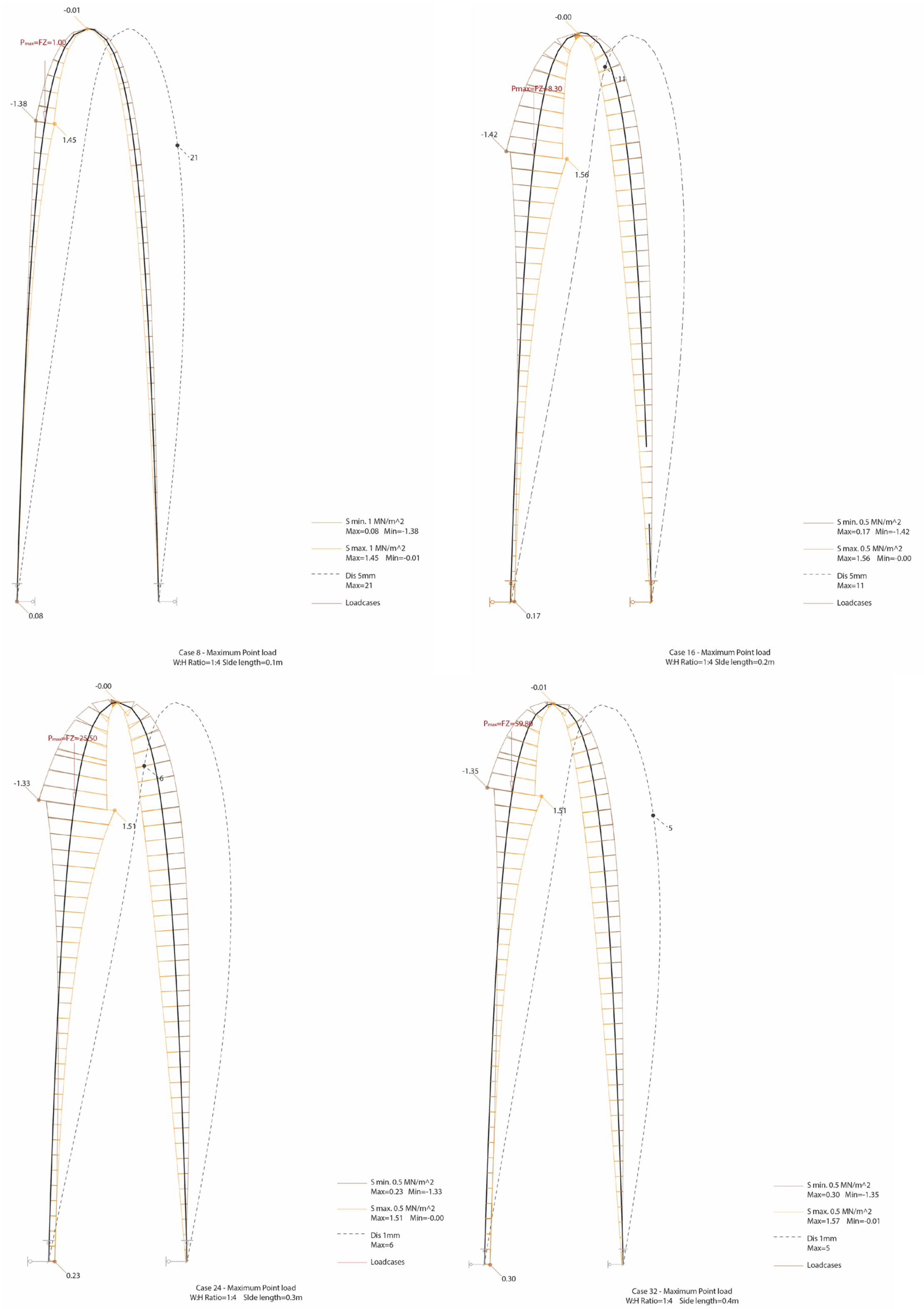
Case 22 - Maximum Point load
W:H Ratio=1:2 Side length=0.3m



Case 30 - Maximum Point load
W:H Ratio=1:2 Side length=0.4m

When it comes to the 2:1 ratio case scenario, the side lengths are the same as before, with increasing point loads. The maximum stress varies from 1.54 to 1.57, while the minimum is between 0.08 to 0.28. The arch deflects to the right as usual, due to the point load applied in the left side, which has a maximum value of 58 for the side length of 0.4.

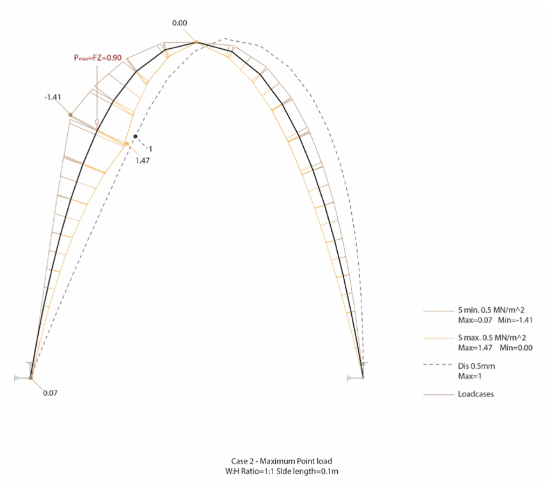
4.4. Ratio 4:1 (Rise: Span) remain constant, change in thickness

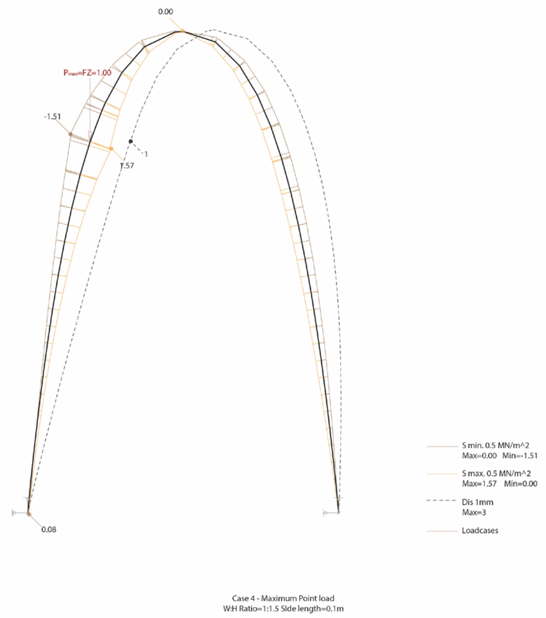


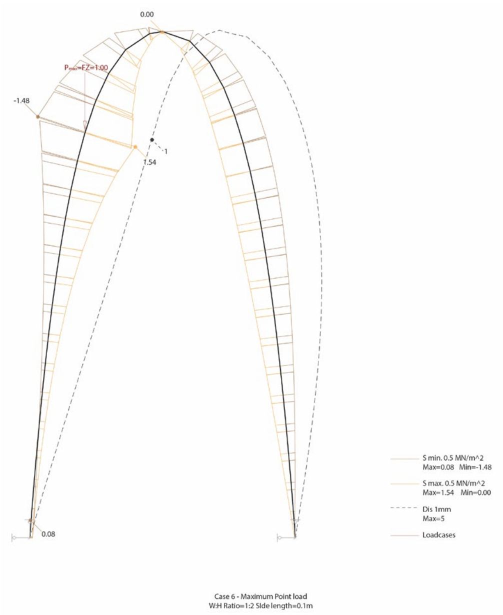
The side lengths are the same for the 4:1 ratio case scenario, with increasing point loads, from 1.00 to 59.80. Although this time, the maximum stress equating 1.56 is recorded for the second case, with a side length of 0.2m. This decreases when moving on to larger lengths, being a constant of 1.51. The same happens with the minimum stress as well, this being -1.42. This can also be seen in the arch's deflection.

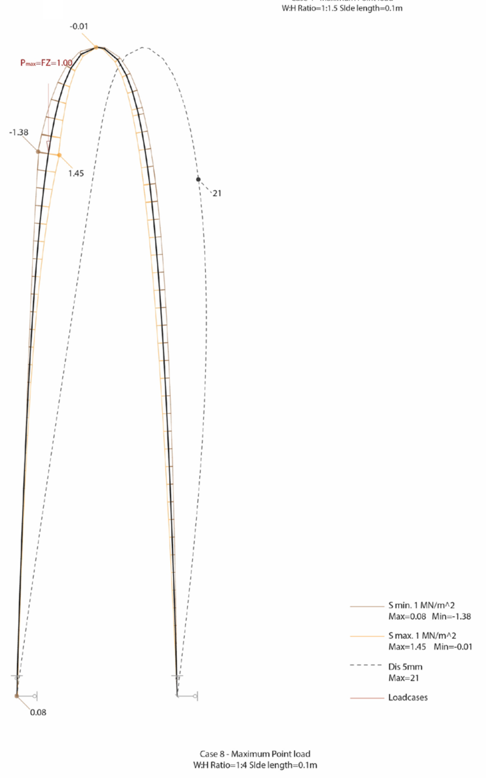
5. THICKNESS-TO-RISE TEST: POINT LOAD

5.1. Thickness (100mm x 100mm) *remains constant, change in rise to span ratio*





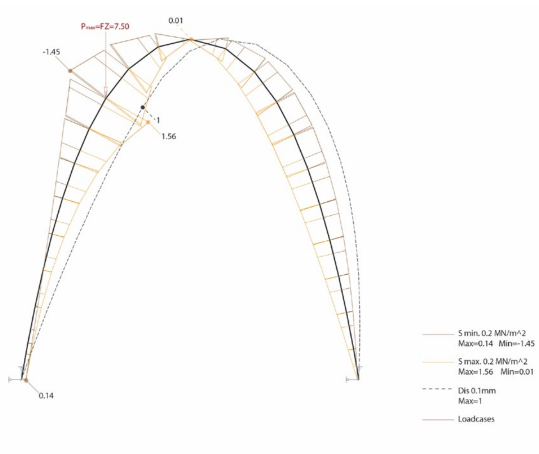




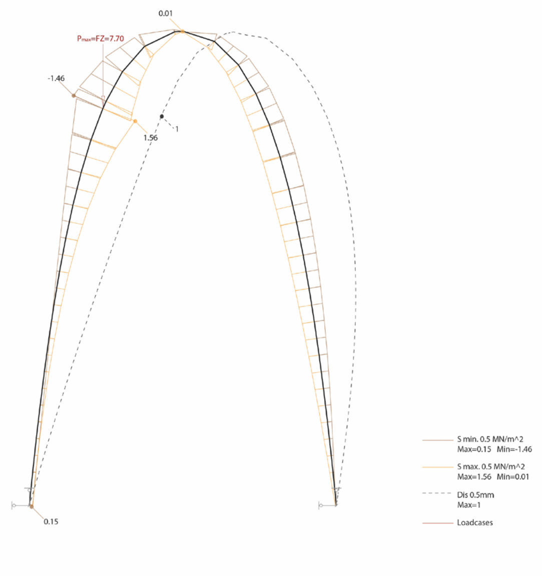
For all these 4 cases, the deformation shapes are similar, which bend towards the opposite direction from the position of the point load.

The maximum stress and minimum stress are at the location of the point load, and the stresses diminish towards the bottom support and the highest point.
With the increase in the rise and span ratio, the stress curve and deformation shape seem to be stretched in proportion. The displacement value also increases with height. The maximum displacement point is on the same side of the point load for the first three cases except for the case for 4:1 rise and fall, the maximum displacement is on the opposite side but similar location.

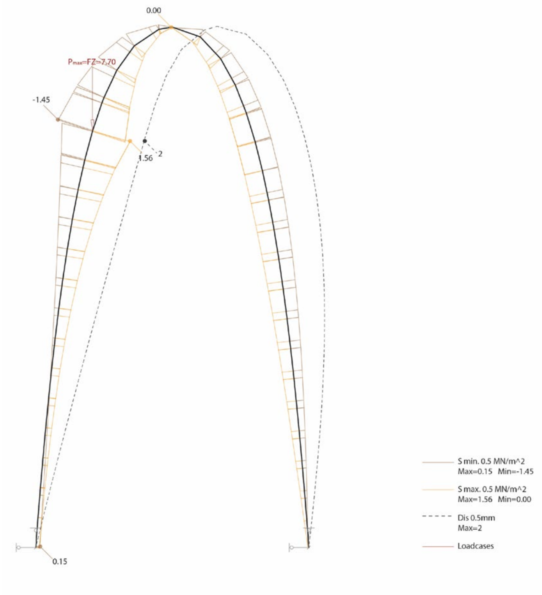
5.2. Thickness (200mm x 200mm) remains constant, change in rise to span ratio



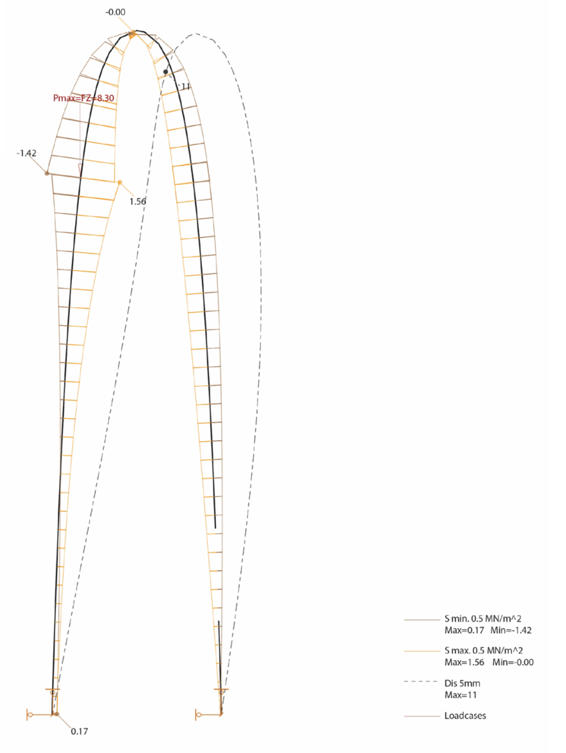
Case 10 - Maximum Point load
W:H Ratio=1:1 Side length=0.2m



Case 12 - Maximum Point load
W:H Ratio=1:1.5 Side length=0.2m



Case 14 - Maximum Point load
W:H Ratio=1:2 Side length=0.2m



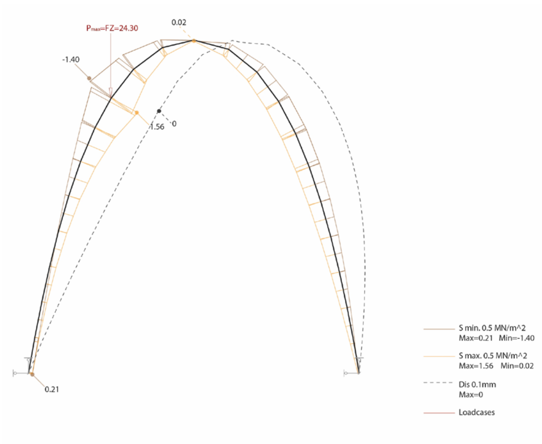
Case 16 - Maximum Point load
W:H Ratio=1:4 Side length=0.2m

For all these 4 cases, the deformation shapes are similar, which bend towards the opposite direction from the position of the point load.

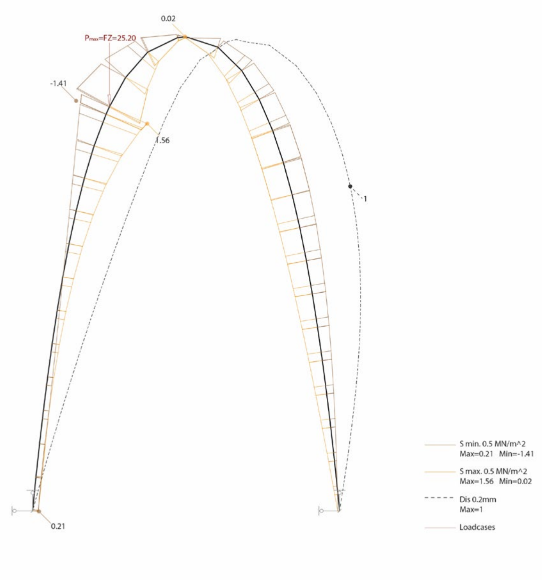
The maximum stress and minimum stress are at the location of the point load, and the stresses diminish towards the bottom support and the highest point.

With the increase in the rise and span ratio, the stress curve and deformation shape are seeming to be stretched in proportion. The displacement value also increases with the height. The maximum displacement points are all on the same side of the point load at a similar length to the point load location, except for the case for 4:1 rise and fall, the maximum displacement is higher close to the top of the shape

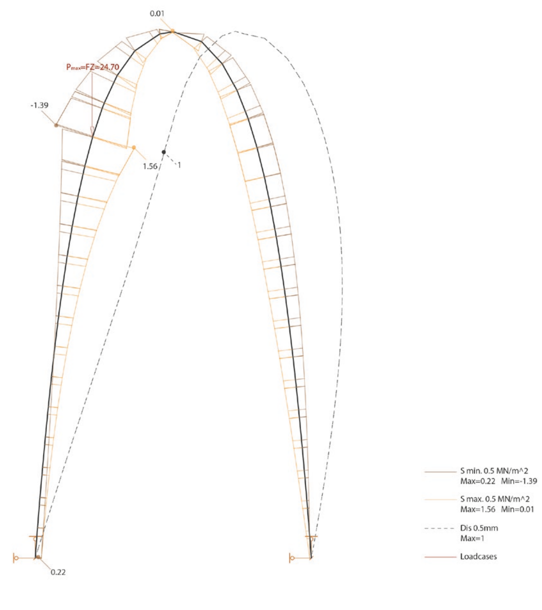
5.3. Thickness (300mm x 300mm) remains constant, change in rise to span ratio



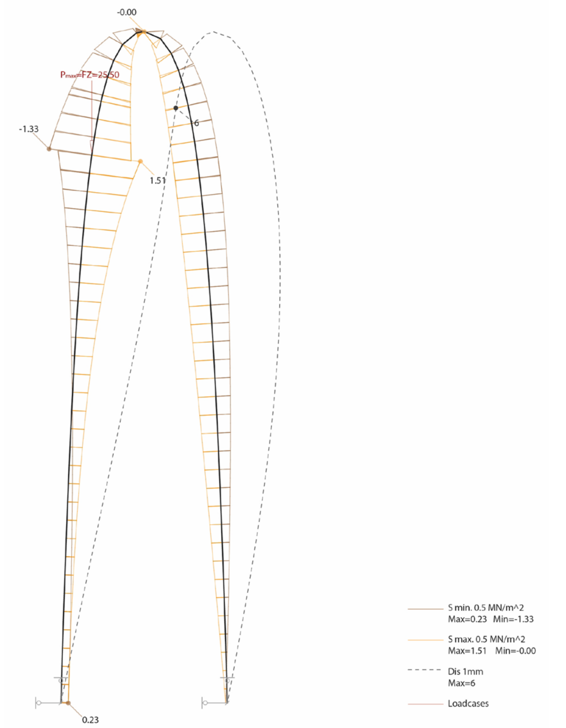
Case 18 - Maximum Point load
W:H Ratio=1:1 SIde length=0.3m



Case 20 - Maximum Point load
W:H Ratio=1:1.5 SIde length=0.3m



Case 22 - Maximum Point load
W:H Ratio=1:2 SIde length=0.3m



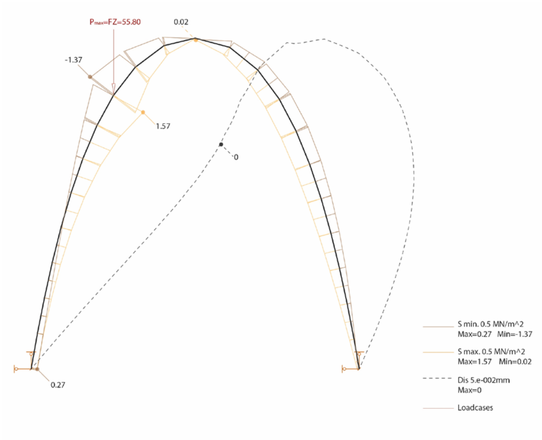
Case 24 - Maximum Point load
W:H Ratio=1:4 SIde length=0.3m

For all these 4 cases, the deformation shapes are similar, which bend towards the opposite direction from the position of the point load.

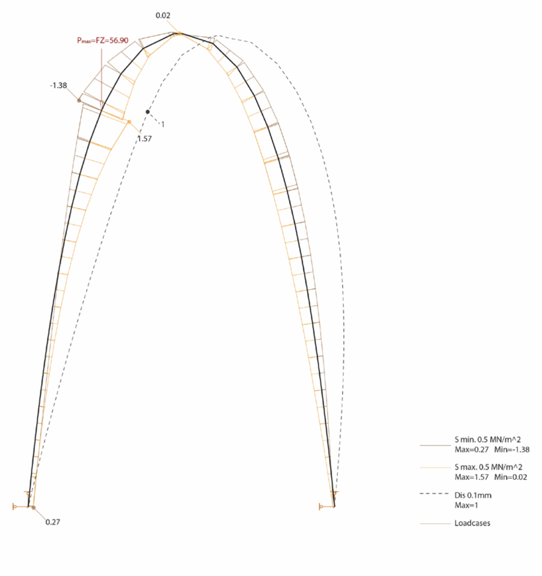
The maximum stress and minimum stress are at the location of the point load, and the stresses diminish towards the bottom support and the highest point.

With the increase in the rise and span ratio, the stress curve and deformation shape are seeming to be stretched in proportion. The displacement value also increases with the height. The maximum displacement locations are on the same side and next to the point load for the 1:1, 2:1 and 4:1 rise and fall cases, the for 1.5:1 rise and fall case, the maximum displacement point is at the opposite side geometry from the point load.

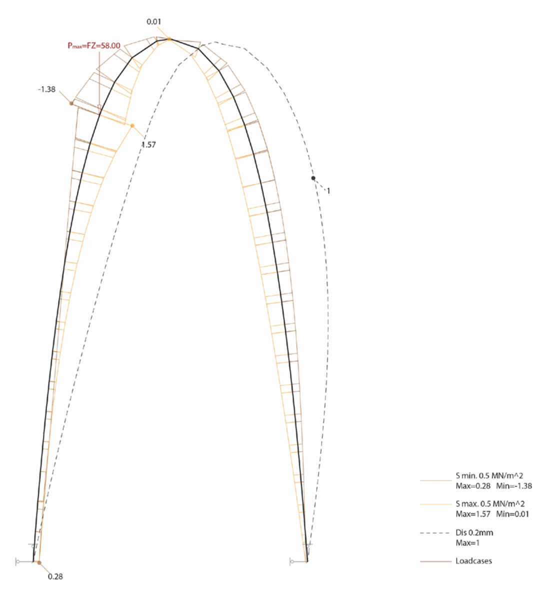
5.4. Thickness (400mm x 400mm) remains constant, change in rise to span ratio



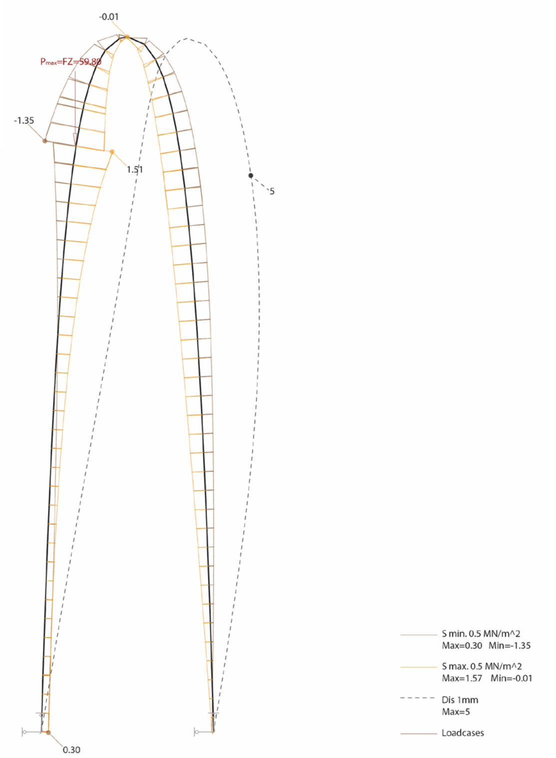
Case 26 - Maximum Point load
W:H Ratio=1:1 Side length=0.4m



Case 28 - Maximum Point load
W:H Ratio=1:1.5 Side length=0.4m



Case 30 - Maximum Point load
W:H Ratio=1:2 Side length=0.4m



Case 32 - Maximum Point load
W:H Ratio=1:4 Side length=0.4m

For all these 4 cases, the deformation shapes are similar, which bend towards the opposite direction from the position of the point load.

The maximum stress and minimum stress are at the location of the point load, and the stresses diminish towards the bottom support and the highest point.

With the increase in the rise and span ratio, the stress curve and deformation shape are seeming to be stretched in proportion. The maximum displacement locations are on either side of the geometry close to the point load height.

6. TEST RESULTS AND ANALYSIS

The test results of distributed load and point load for all cases are shown in Table 3. The impact of thickness and rise-to-span ratio on the bearing capacity of the arches are plotted and further analyzed.

Case Number	side length	Ratio	Max W (kN/m)	Max P (kN)
1	0.1	1	4.5	0.9
2		1.5	3.3	1.0
3		2	2.8	1.0
4		4	1.5	1.1
5	0.2	1	18.2	7.5
6		1.5	13.3	7.7
7		2	11.3	7.7
8		4	6	8.3
9	0.3	1	41.3	24.3
10		1.5	29.8	25.3
11		2	25.1	25
12		4	13.6	26.5
13	0.4	1	73.6	55.8
14		1.5	53.1	56.9
15		2	44.8	58
16		4	24.2	59.8

Table 3. Test results of the maximum capacity of all cases

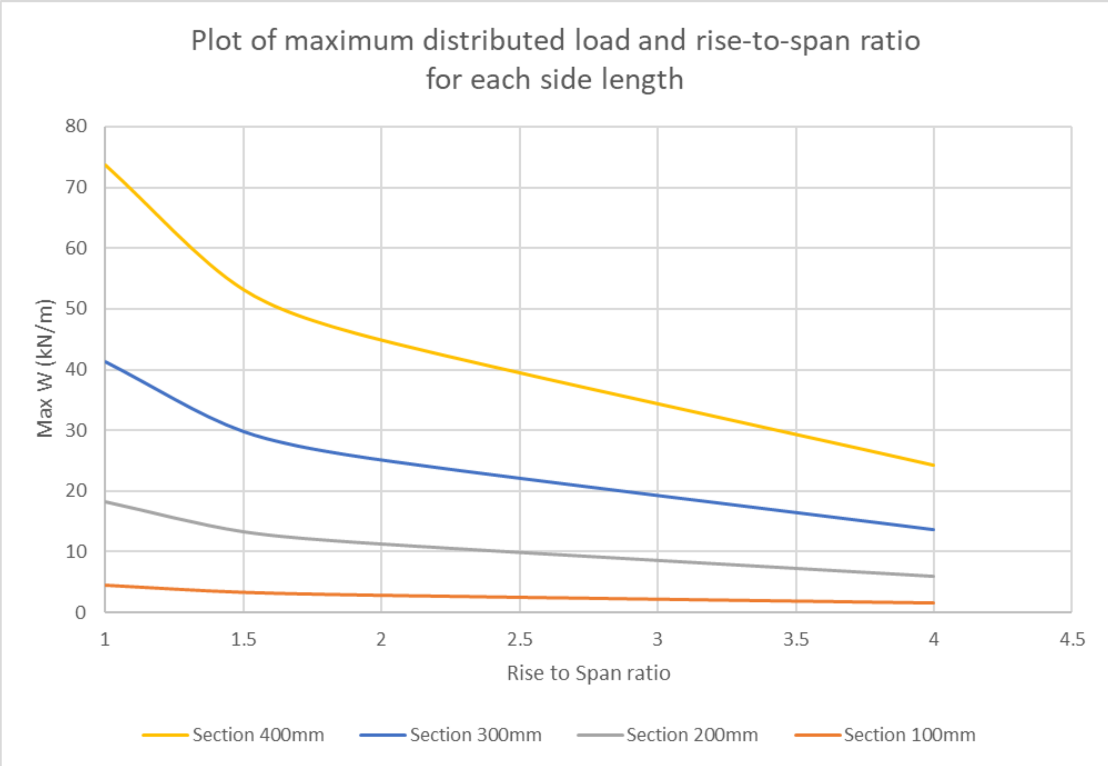


Figure 12. Impact of rise-to-span ratio on distributed load capacity for four sections

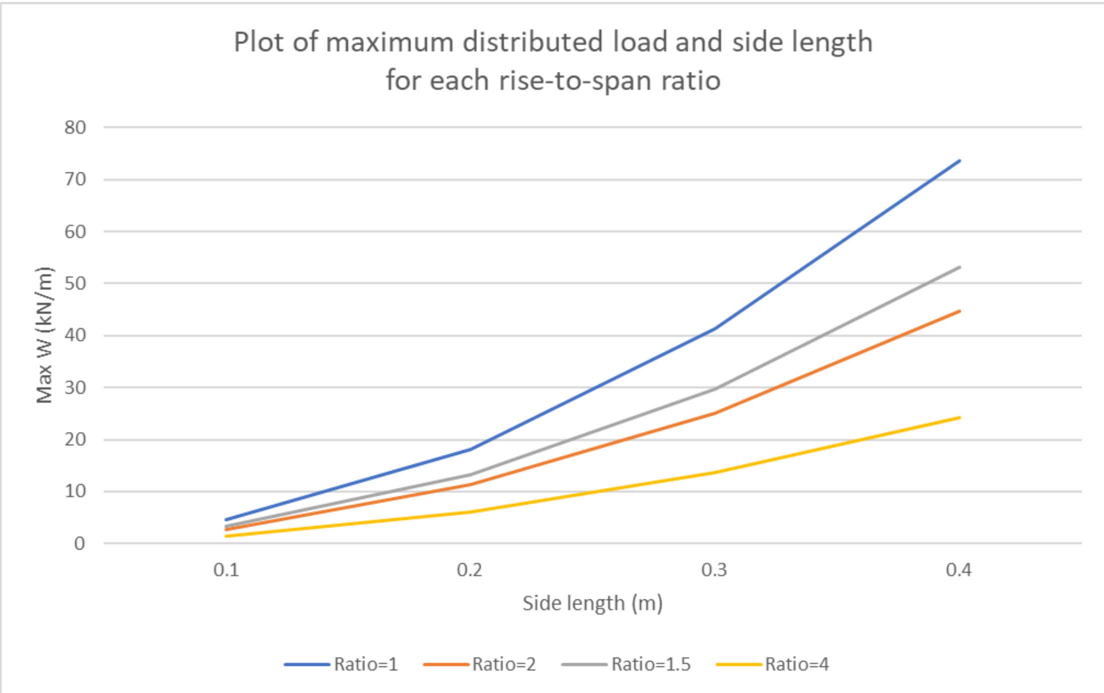


Figure 13. Impact of section dimensions on distributed load capacity for four rise-to-span ratios

When testing with the distributed loads acting upon the arches, the results show that the thicker the section is, the larger the ultimate distributed load can act on each rise-to-span ratio arches. Each time the thickness increase 100mm, the ultimate load increases in an exponential manner, almost double the ultimate value compare to the previous case. As the section remains constant, the graph indicates that the 1:1 ratio is the strongest, and as the rise-to-span ratio increases the ultimate distributed load decreases. The drop in ultimate load changing from ratio 1:1 to 1.5:1

is significantly larger in the case of section 400mm, and as the section decreases, the drop decreases. For example, arch with section 100mm has the least fluctuation in values of ultimate distributed load compared to the other cases. In this comparison, arch with 400mm thickness and ratio 1:1 is the strongest, and the one with 100mm thickness and ratio 4:1 is the weakest.

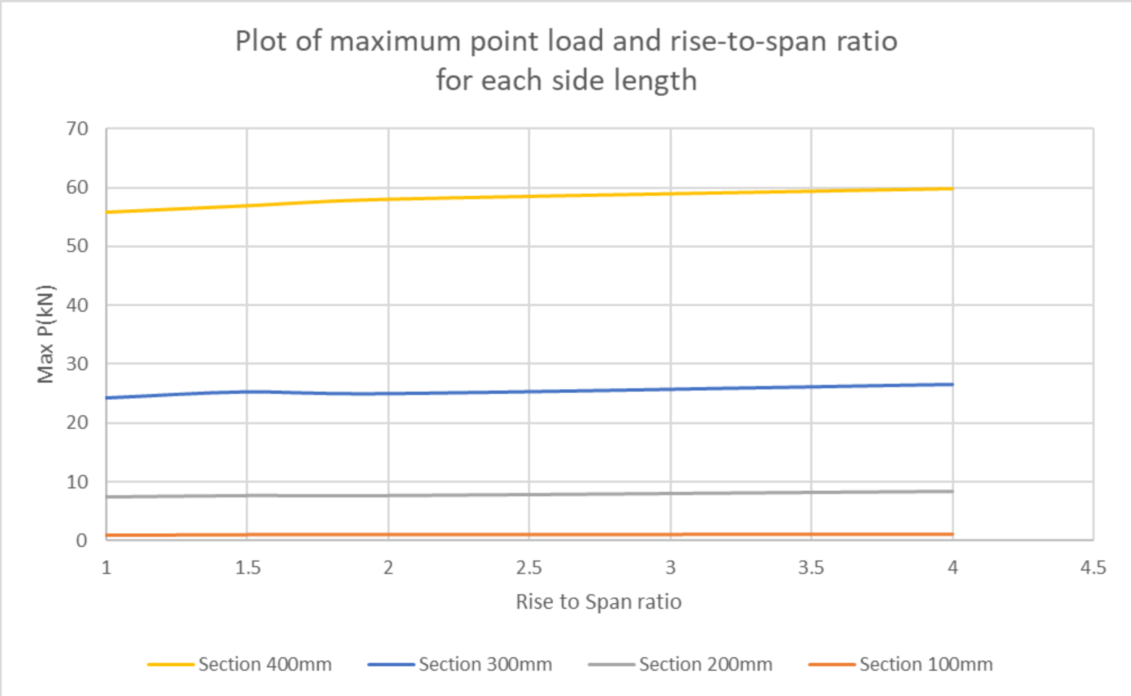


Figure 14. Impact of rise-to-span ratios on point load capacity for four section dimensions

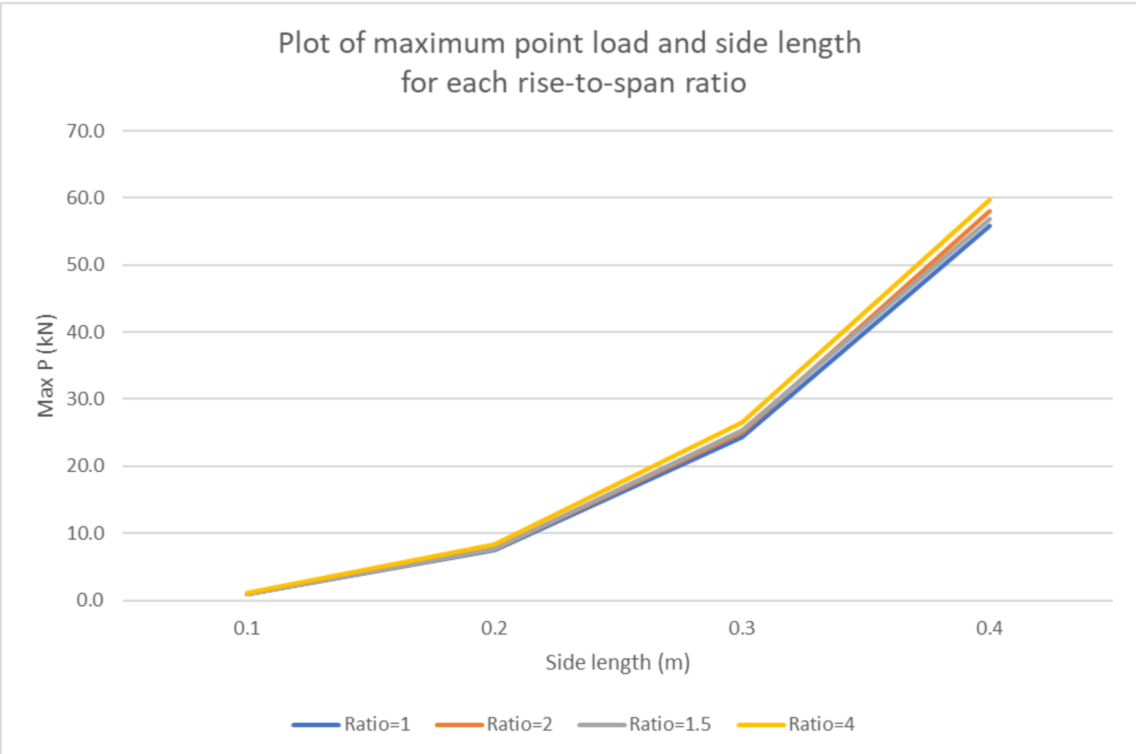


Figure 14. Impact of section dimensions on point load capacity for four rise-to-span ratios

When testing with the point load acting upon the arches, the results show that the thicker the section is, the larger the ultimate point load can act on each rise-to-span ratio arches. The difference in ultimate load also increases as the thickness increases, also in an exponential manner. As the section remains constant, the four graphs show that the change in rise to span ratio does not affect much of the ultimate point load, the values are almost constant, but the load does increase slightly as the rise to span ratio rises. 400mm section arch shows a clearer increase in ultimate load as the ratio increases. In this comparison, arch with 400mm thickness and ratio 4:1 is the strongest, and the one with 100mm thickness and ratio 1:1 is the weakest.

In conclusion, the larger the thickness of the arches' sections, the larger ultimate load it can take in general. In distributed load's cases, the arch is stronger when rise to span ratio is smaller, whereas in point load's cases (at 1/4 of the span), the arch is stronger as this ratio increases.

7. HAND CALCULATIONS FOR MODEL VALIDATION:

To validate the results obtained via parametric modelling, hand calculations were done to calculate the maximum load a catenary arch could withstand (related to the horizontal reaction at its bases), dependent upon the loading conditions and its height-to-span ratio. Bending moment profiles vary along the arch dependent upon its geometric properties.

Parameter of specific case being investigated:

Span: 3 m.
Height: 3 m. Thus, ratio is 1:1
Side Length of arch cross-section: 0.1 m

Moment of Inertia for a square cross-section given by the equation:

$$I = b^4/12$$

Where b is the side length of the cross-section in metres. Thus,

$$I = 0.1^4/12 = 8.33 \times 10^{-6}\ m^4$$

Distance from the neutral axis given by c:

$$c = b/2$$
$$c = 0.1/2 = 0.05\ m.$$

σ= mean tensile strength of C12/15 concrete

$$\sigma = 1.57 \times 10^6\ Pa$$

Flexural Formula:

$$\sigma = \frac{M_{max} c}{I}$$

To calculate M_{max}, the equation stated above was rearranged to give:

$$M_{max} = \frac{\sigma I}{c}$$

$$M_{max} = \frac{\sigma * \frac{b^4}{12}}{\frac{b}{2}}$$

$$M_{max} = b^3 * \frac{1.57*10^6}{6}$$

$$M_{max} = 0.1^3 * \frac{1.57*10^6}{6}$$

In this specific instance (Case 1), the M_{max} empirically obtained equals 261.667 Nm. If the arch was to be subjected to moments exceeding this at any point in its span, it would theoretically fail.

The Catenary Arch is given by the equation:

$$y = \frac{(e^{ax}+e^{-ax})}{2a} = \frac{\cosh(ax)}{a}$$

where a is a constant varying with arch geometry and the lowest point of the chain can be found at $(0, a)$.

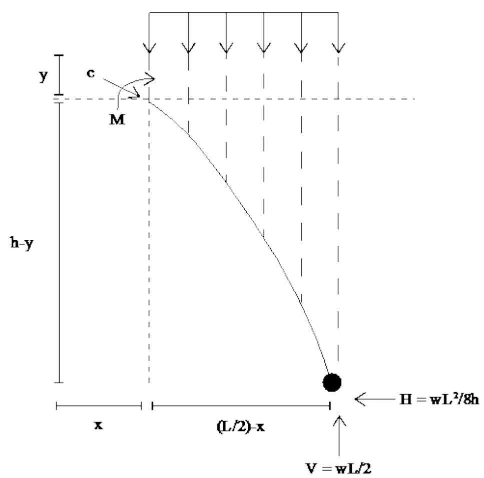
In this specific case, via observation of the shape of the curve, the value of a corresponding to the arch geometry being investigated is -3.8.

The two scenarios investigated maximum point load applied at quarter of arch span and maximum distributed load are then detailed below:

<u>Distributed Load Case (w_{max}):</u>

Moment inspected at 0.1 m. intervals from the arch's centre (0 – 1.5 m., symmetry here is assumed) at any given cross-section:

By inspection of the forces acting on the arch:



and assuming equilibrium, the following equation can be derived:

$$\sum M = 0$$

$$M + w\left[\frac{L}{2} - x\right]\left[\frac{\frac{L}{2} - x}{2}\right] - \left[\left(\frac{wL}{2}\right)\left(\frac{L}{2} - x\right)\right] + \left[\left(\frac{wL^2}{8h}\right)(h - y)\right] = 0$$

Therefore,

$$M = \left(\frac{\mathrm{wL}^2}{8h}\right)(y) - (\frac{w}{2})(x^2)$$

Thus, substituting the catenary arch equation in place of y gives:

$$M = \left(\frac{\mathrm{wL}^2}{8h}\right)\left(\frac{(e^{ax}+e^{-ax})}{2a}\right) - (\frac{w}{2})(x^2)$$

To find the max distributed load, M_{max} can be substituted in place of M.

$$M_{max} = \left(\frac{\mathrm{wL}^2}{8h}\right)\left(\frac{(e^{ax}+e^{-ax})}{2a}\right) - (\frac{w}{2})(x^2)$$

Via rearrangement of the equation again, an equation for the max distributed load can be obtained:

$$M_{max} = w(\left(\frac{\mathrm{L}^2}{8h}\right)\left(\frac{(e^{ax}+e^{-ax})}{2a}\right) - \left(\frac{1}{2}\right)(x^2))$$

$$\mathrm{w} = \mathrm{M}_{\mathrm{max}} /(\left(\frac{\mathrm{L}^2}{8h}\right)\left(\frac{(e^{ax}+e^{-ax})}{2a}\right) - \left(\frac{1}{2}\right)(x^2))$$

By substituting the appropriate values of L, h, a and the previously calculated M_{max} into the equation above, values of w were obtained for each 0.1 m. section of the arch. For more detailed process calculations and evidence of equation implementation, please refer to the spreadsheet attached.

Summary of Results:

Distance from Centre, x, (m.)	Max Distributed Load Sustainable at Point X (N)
0.0	1001.2
0.1	1462.3
0.2	2858.3
0.3	-93124.3
0.4	-2468.2
0.5	-1189.8
0.6	-755.3
0.7	-536.8
0.8	-405.6
0.9	-318.5

1.0	-256.6
1.1	-210.7
1.2	-175.5
1.3	-147.7
1.4	-125.4
1.5	-107.1

Table 4. Results of maximum distributed load at each distance

Via inspection of the change in moment across span, it is evident that the arch will fail at its supports first as the value sustainable is lowest here. In this instance, the maximum distributed load the arch can withstand is 107.1 Nm.

From our hand calculations, we have obtained a value which is far lower than what we found using Robot. This may be due to the simplification of the problem using linear elastic theory. The two processes, while solving the same problem, use very different methods, and it is possible that small deviations have lead to a large difference in the outcome. Despite this, the disparity between the two values is larger than we would usually expect.

<u>Maximum Point Load Applied at 1/4th of the Span (P_{max}):</u>

However, to find max point load, we must consider the left and right sides separately, either side of the hinge. Firstly, we can consider the equation for the moment.

Left Side:

$$M = \frac{PL}{8}\left[\frac{6x}{L} - \frac{y}{h}\right]$$

Right Side:

$$M = \frac{PL}{8}\left[\frac{2x}{L} - \frac{y}{h}\right]$$

Where P is the point load, L is the length of the arch, x is the horizontal distance (as measured from the left and right support, respectively), y is the vertical distance from the base, and h is the height of the arch.

Using these two equations, we can substitute in $y = \frac{e^{ax}+e^{-ax}}{2a}$, the equation for the catenary arch, and we obtain:

Left Side:

$$M = \frac{PL}{8}\left[\frac{6x}{L} - \frac{e^{ax}+e^{-ax}}{2ah}\right]$$

Right Side:

$$M = \frac{PL}{8}\left[\frac{2x}{L} - \frac{e^{ax}+e^{-ax}}{2ah}\right]$$

We can then rearrange this to find the value of P.

Left Side:

$$P = \frac{8M}{6x - \frac{L(e^{ax} + e^{-ax})}{2ah}}$$

Right Side:

$$P = \frac{8M}{2x - \frac{L(e^{ax} + e^{-ax})}{2ah}}$$

Using the previously determined values for M_{max} and by substituting in values for the other coefficients, we can calculate the moment at each point along the curve from 0 to 3m

Distance from Centre, x, (m.), Left Side	Max Point Load (acting at L/4) Sustainable at Point X (N)	Distance from Centre, x, (m.), Right Side	Max Point Load (acting at L/4) Sustainable at Point X (N)
0.0	-3003.5	0.0	-3003.5
0.1	19121.1	0.1	-7205.4
0.2	2319.4	0.2	20414.3
0.3	1240.5	0.3	4294.1
0.4	847.7	0.4	2407.9
0.5	643.5	0.5	1670.5
0.6	517.7	0.6	1273.5
0.7	431.9	0.7	1022.8
0.8	369.4	0.8	848.5
0.9	321.5	0.9	719.1
1.0	283.4	1.0	618.3
1.1	252.3	1.1	537.1
1.2	226.2	1.2	469.8
1.3	203.8	1.3	412.8
1.4	184.4	1.4	363.9
1.5	167.3	1.5	321.3

Table 5. Hand calculation results

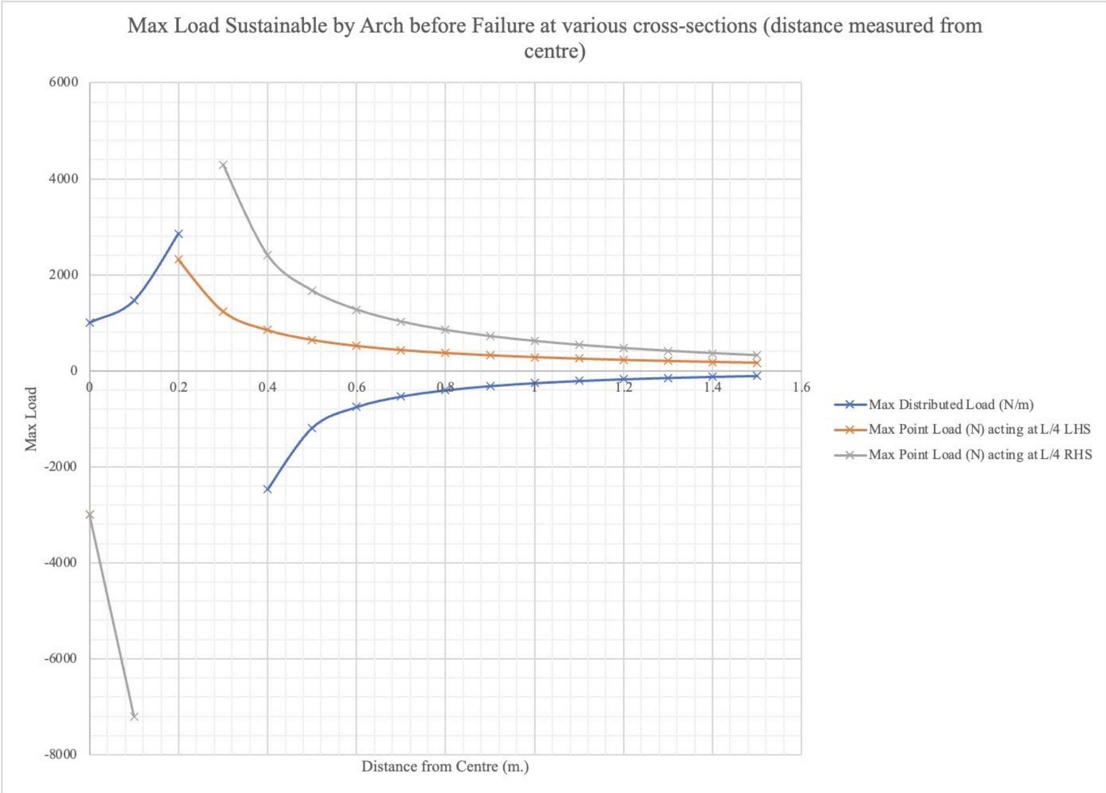


Figure 15. Plot of hand maximum load capacity

From this graph, we can see that the weakest points on the graph are at the supports, furthest from the centre. Therefore, in both the point load and distributed load case, we can expect the failure to occur at the supports, where the maximum stress is.

8. LIMITATIONS

As the model assumes that the arch is perfectly linear elastic, there are some limitations and aspects that lead to a gap between modelled and real-life scenarios.

The assumption suggests that the loads do not cause any permanent deformation and strain responses directly proportional to the applied stress. Important phenomena such as buckling, cracking and other non-linear effects are neglected for the hand calculation. In usual cases, as the stress increases, materials gradually behave from linear state to nonlinear state. Assuming the material being perfectly linear elastic can lead to an underestimation of values of ultimate stress[5].

The brief also requires that we don't consider self-weight, which can increase the simulation result of bearing capacity because axial stress is not considered. Robot takes into account of self-weight automatically, which can cause discrepancies between hand calculated results and software-generated results.

Other well-known limitations would also be human errors, in terms of estimating and approximating certain values wrong. The deformation of the arch might prove to be quite far from reality due to the structure's scale, example found in this simulation.

9. REFERENCES

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