Lagrange Points

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Setup

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate
import astropy.units as u
import astropy.constants as const
plt.rc('axes', labelsize=14, titlesize=20)
plt.rc('figure', titlesize=20)
```

In [2]:
```
# global variable
G = 4 * np.pi**2
```

Three Body Problem

The three key differential equations are

$$\begin{aligned}
\frac{d^2\vec{r}_1}{dt^2} &= \frac{GM_2}{r_{1,2}^3}\vec{r}_{1,2} + \frac{GM_3}{r_{1,3}^3}\vec{r}_{1,3} &&= \frac{GM_2}{r_{1,2}^3}(\vec{r}_2 - \vec{r}_1) + \frac{GM_3}{r_{1,3}^3}(\vec{r}_3 - \vec{r}_1) \\
\frac{d^2\vec{r}_2}{dt^2} &= \frac{GM_1}{r_{2,1}^3}\vec{r}_{2,1} + \frac{GM_3}{r_{2,3}^3}\vec{r}_{2,3} &&= \frac{GM_1}{r_{2,1}^3}(\vec{r}_1 - \vec{r}_2) + \frac{GM_3}{r_{2,3}^3}(\vec{r}_3 - \vec{r}_2) \\
\frac{d^2\vec{r}_3}{dt^2} &= \frac{GM_1}{r_{3,1}^3}\vec{r}_{3,1} + \frac{GM_2}{r_{3,2}^3}\vec{r}_{3,2} &&= \frac{GM_1}{r_{3,1}^3}(\vec{r}_1 - \vec{r}_3) + \frac{GM_2}{r_{3,2}^3}(\vec{r}_2 - \vec{r}_3)
\end{aligned}$$

Note: vector $\vec{r}_{i,j} = \vec{r}_j - \vec{r}_i$ goes from body i to body j.

We can separate the three key equations into a system of eighteen first-order ODEs:

$$
\begin{aligned}
\frac{dx_1}{dt} &= v_{x1} & \frac{dv_{x1}}{dt} &= \frac{GM_2}{r_{1,2}^3}x_{1,2} + \frac{GM_3}{r_{1,3}^3}x_{1,3} \\
\frac{dy_1}{dt} &= v_{y1} & \frac{dv_{y1}}{dt} &= \frac{GM_2}{r_{1,2}^3}y_{1,2} + \frac{GM_3}{r_{1,3}^3}y_{1,3} \\
\frac{dz_1}{dt} &= v_{z1} & \frac{dv_{z1}}{dt} &= \frac{GM_2}{r_{1,2}^3}z_{1,2} + \frac{GM_3}{r_{1,3}^3}z_{1,3} \\
\frac{dx_2}{dt} &= v_{x2} & \frac{dv_{x2}}{dt} &= \frac{GM_1}{r_{2,1}^3}x_{2,1} + \frac{GM_3}{r_{2,3}^3}x_{2,3} \\
\frac{dy_2}{dt} &= v_{y2} & \frac{dv_{y2}}{dt} &= \frac{GM_1}{r_{2,1}^3}y_{2,1} + \frac{GM_3}{r_{2,3}^3}y_{2,3} \\
\frac{dz_2}{dt} &= v_{z2} & \frac{dv_{z2}}{dt} &= \frac{GM_1}{r_{2,1}^3}z_{2,1} + \frac{GM_3}{r_{2,3}^3}z_{2,3} \\
\frac{dx_3}{dt} &= v_{x3} & \frac{dv_{x3}}{dt} &= \frac{GM_1}{r_{3,1}^3}x_{3,1} + \frac{GM_2}{r_{3,2}^3}x_{3,2} \\
\frac{dy_3}{dt} &= v_{y3} & \frac{dv_{y3}}{dt} &= \frac{GM_1}{r_{3,1}^3}y_{3,1} + \frac{GM_2}{r_{3,2}^3}y_{3,2} \\
\frac{dz_3}{dt} &= v_{z3} & \frac{dv_{z3}}{dt} &= \frac{GM_1}{r_{3,1}^3}z_{3,1} + \frac{GM_2}{r_{3,2}^3}z_{3,2}
\end{aligned}
$$

We choose to work in astronomical units, where mass is in $\mathrm{M}_\odot$, length is in AU, and time is in yr. We choose these units for two reasons: (i) the value of the gravitational constant is simplified greatly in these units, $G = 4\pi^2\ \mathrm{AU}^3\ \mathrm{M}_\odot^{-1}\ \mathrm{yr}^{-2}$; and (ii) the input values of a system resembling the Earth-Sun system remain simple, e.g. $M_1 = 1\ \mathrm{M}_\odot,\ a = 1\ \mathrm{AU}$.

```
In [3]: # vec = [x1, y1, z1, vx1, vy1, vz1, x2, y2, z2, vx2, vy2, vz2, x3, y3, z1, vx
        3, vy3, vz3]
        def df_3body(vec, t, M1, M2, M3):
            x1 = vec[0]
            y1 = vec[1]
            z1 = vec[2]
            vx1 = vec[3]
            vy1 = vec[4]
            vz1 = vec[5]
            x2 = vec[6]
            y2 = vec[7]
            z2 = vec[8]
            vx2 = vec[9]
            vy2 = vec[10]
            vz2 = vec[11]
            x3 = vec[12]
            y3 = vec[13]
            z3 = vec[14]
            vx3 = vec[15]
            vy3 = vec[16]
            vz3 = vec[17]

            x12 = x2 - x1
            y12 = y2 - y1
            z12 = z2 - z1
            r12 = np.sqrt(x12**2 + y12**2 + z12**2)

            x21, y21, z21, r21 = (-x12, -y12, -z12, r12)

            x13 = x3 - x1
            y13 = y3 - y1
            z13 = z3 - z1
            r13 = np.sqrt(x13**2 + y13**2 + z13**2)

            x31, y31, z31, r31 = (-x13, -y13, -z13, r13)

            x23 = x3 - x2
            y23 = y3 - y2
            z23 = z3 - z2
            r23 = np.sqrt(x23**2 + y23**2 + z23**2)

            x32, y32, z32, r32 = (-x23, -y23, -z23, r23)


            dvec = np.zeros(len(vec))

            dvec[0 ] = vx1
            dvec[1 ] = vy1
            dvec[2 ] = vz1
            dvec[3 ] = G * M2 / r12**3 * x12 + G * M3 / r13**3 * x13
            dvec[4 ] = G * M2 / r12**3 * y12 + G * M3 / r13**3 * y13
            dvec[5 ] = G * M2 / r12**3 * z12 + G * M3 / r13**3 * z13
            dvec[6 ] = vx2
            dvec[7 ] = vy2
            dvec[8 ] = vz2
            dvec[9 ] = G * M1 / r21**3 * x21 + G * M3 / r23**3 * x23
```

```
    dvec[10] = G * M1 / r21**3 * y21 + G * M3 / r23**3 * y23
    dvec[11] = G * M1 / r21**3 * z21 + G * M3 / r23**3 * z23
    dvec[12] = vx3
    dvec[13] = vy3
    dvec[14] = vz3
    dvec[15] = G * M1 / r31**3 * x31 + G * M2 / r32**3 * x32
    dvec[16] = G * M1 / r31**3 * y31 + G * M2 / r32**3 * y32
    dvec[17] = G * M1 / r31**3 * z31 + G * M2 / r32**3 * z32

    return dvec
```

```
In [4]: def calc_energy(vec, M1, M2, M3):
            x1 = vec[:,0]
            y1 = vec[:,1]
            z1 = vec[:,2]
            vx1 = vec[:,3]
            vy1 = vec[:,4]
            vz1 = vec[:,5]
            x2 = vec[:,6]
            y2 = vec[:,7]
            z2 = vec[:,8]
            vx2 = vec[:,9]
            vy2 = vec[:,10]
            vz2 = vec[:,11]
            x3 = vec[:,12]
            y3 = vec[:,13]
            z3 = vec[:,14]
            vx3 = vec[:,15]
            vy3 = vec[:,16]
            vz3 = vec[:,17]

            #kinteic energy
            KE1 = 0.5 * M1 * (vx1**2 + vy1**2 + vz1**2)
            KE2 = 0.5 * M2 * (vx2**2 + vy2**2 + vz2**2)
            KE3 = 0.5 * M3 * (vx3**2 + vy3**2 + vz3**2)
            KE = KE1 + KE2 + KE3

            #potential energy
            x12 = x2 - x1
            y12 = y2 - y1
            z12 = z2 - z1
            r12 = np.sqrt(x12**2 + y12**2 + z12**2)

            x21, y21, z21, r21 = (-x12, -y12, -z12, r12)

            x13 = x3 - x1
            y13 = y3 - y1
            z13 = z3 - z1
            r13 = np.sqrt(x13**2 + y13**2 + z13**2)

            x31, y31, z31, r31 = (-x13, -y13, -z13, r13)

            x23 = x3 - x2
            y23 = y3 - y2
            z23 = z3 - z2
            r23 = np.sqrt(x23**2 + y23**2 + z23**2)

            PE12 = -G * M1 * M2 / r12
            PE13 = -G * M1 * M3 / r13
            PE23 = -G * M2 * M3 / r23
            PE = PE12 + PE13 + PE23

            E = KE + PE

            return E
```

```
In [5]: def correct_CoM(vec0, M1, M2, M3):
            marr = np.array([M1, M2, M3])
            rarr = np.array([vec0[6*i:6*i+3] for i in range(3)])
            varr = np.array([vec0[6*i+3:6*i+6] for i in range(3)])

            rcom = np.dot(marr, rarr) / np.sum(marr)
            vcom = np.dot(marr, varr) / np.sum(marr)

            rarr = rarr - rcom
            varr = varr - vcom

            vec0 = np.array([rarr[0], varr[0], rarr[1], varr[1], rarr[2], varr[2]]).fl
        atten()
            return vec0
```

```
In [6]: def plot_orbits(vec, legend=True):

            x1 = vec[:,0]
            y1 = vec[:,1]
            x2 = vec[:,6]
            y2 = vec[:,7]
            x3 = vec[:,12]
            y3 = vec[:,13]

            f = plt.figure(figsize=(8,8))
            plt.plot(x1, y1, label='$M_1$')
            plt.plot(x2, y2, label='$M_2$')
            plt.plot(x3, y3, label='$M_3$')
            plt.xlabel('x (AU)')
            plt.ylabel('y (AU)')
            plt.title('Orbits')
            plt.gca().set_aspect('equal')
            if plt.ylim()[0]>-0.5: plt.ylim(-0.5,plt.ylim()[1])
            if plt.ylim()[1]<0.5: plt.ylim(plt.ylim()[0], 0.5)
            if legend: plt.legend()

            f.tight_layout()

            return f
```

Calculating orbits and Conservation Laws

Two-body

When working with Lagrange points, we assume that the third body, the body in a lagrange point, is much less massive than the other two bodies \ ($M_1, M_2 \gg M_3$). Therefore, we initially treat the system as a two body problem and insert the third body at the end. First, we take in the eccentricity, the semimajor axis, and the masses of the two main bodies. From these, we must calculate the initial positions and velocities. Since the two masses are always on opposite sides of the center of mass, the origin, we take the initial positions to be on the x-axis with initial velocities along the y-axis such that the orbits are counterclockwise. Without loss of generality, we assume that M_1 start on the positive x-axis and M_2 starts on the negative x-axis. For a given semimajor axis and an eccentricity, the radius r as a function of the semimajor axis and the angle ϕ, measured from the x-axis, is given by

$$r = \frac{a(1-e^2)}{1+e\cos\phi}.$$

We are assuming that $\phi = 0$ initially, though, and so we use

$$r = \frac{a(1-e^2)}{1+e} = a(1-e).$$

The radius of each mass is then given by

$$r_1 = \frac{M_2}{M}r \qquad r_2 = \frac{M_1}{M}r$$

where $M = M_1 + M_2$. In the astronomical units we have chosen, the period P and the semimajor axis a are related by

$$P^2 = \frac{1}{M}a^3.$$

We use the conservation of angular momentum to calculate the initial velocity.

$$\ell = |\vec{r} \times \vec{v}| = \left[GMa(1-e^2)\right]^{1/2}$$

Since the radius and velocity are perpendicular at the pericenter and apocenter, and we assume that the masses start at the pericenter on the x-axis, we have that $\vec{r} = r\hat{x}$ and $\vec{v} = v\hat{y}$.

$$|\vec{r} \times \vec{v}| = |(r\hat{x}) \times (v\hat{y})| = rv \qquad \Longrightarrow \qquad v = \frac{\sqrt{GMa(1-e^2)}}{r}$$

As with the radii, we then find the velocity for each mass.

$$v_1 = \frac{M_2}{M}v \qquad v_2 = \frac{M_1}{M}v$$

Adding the third body

Now that we have a a stable two body orbit, we simply place the much less massive third body at a lagrange point, with the correct initial velocity.

Once we have the three bodies set up, we make a small correction to the initial positions and velocities of all three bodies so that we remain in the center of mass frame. We do this by calculating the center of mass and the center of mass's velocity, then subtracting off these values from the initial positions and velocities.

$$M = \sum_{i=1}^{3} M_i, \qquad \vec{R}_{\text{CoM}} = \frac{1}{M}\sum_{i=1}^{3} M_i\vec{r}_i, \qquad \vec{V}_{\text{CoM}} = \frac{1}{M}\sum_{i=1}^{3} M_i\vec{v}_i$$

$$\vec{r}_{i,\text{new}} = \vec{r}_{i,\text{old}} - \vec{R}_{\text{CoM}}, \qquad \vec{v}_{i,\text{new}} = \vec{v}_{i,\text{old}} - \vec{V}_{\text{CoM}}$$

L_1, L_2, L_3, and L_4

Since the third body always has the same period as the two-body system and this is a circular orbit, the velocity is always given by the equation

$$v = \frac{2\pi r}{P}$$

where r is the radius of the orbit and P is the period of the orbit. Therefore, we only need to find the initial position of each Lagrange point. The following equations are from Cornish (1998), where r is the same as above and $\alpha = M_2/(M_1 + M_2)$.

$$\begin{aligned} x_{3,\mathrm{L}_1} &= r\left[1 - \left(\frac{\alpha}{3}\right)^{1/3}\right] \\ x_{3,\mathrm{L}_2} &= r\left[1 + \left(\frac{\alpha}{3}\right)^{1/3}\right] \\ x_{3,\mathrm{L}_3} &= r\left[1 + \frac{5}{12}\alpha\right] \end{aligned}$$

For L_4, we take the initial position and velocity of M_2 and rotate them by 60°.

$$\begin{aligned} x_{3,\mathrm{L}_4} &= x_2\cos(60^\circ), \qquad & y_{3,\mathrm{L}_4} &= x_2\sin(60^\circ) \\ v_{x3,\mathrm{L}_4} &= v_{y2}\sin(60^\circ), \qquad & v_{y3,\mathrm{L}_4} &= v_{y2}\cos(60^\circ) \end{aligned}$$

In [7]:
```python
r = 1
M1 = 1
M2 = 1e-3
M3 = 1e-10

# initial_2body
M = M1 + M2
r1 = M2 / M * r
r2 = M1 / M * r
P = np.sqrt(r**3 / M)
v = np.sqrt(G * M / r)
v1 = M2 / M * v
v2 = M1 / M * v

# initial values for two bodies
vec0 = np.zeros(18)
vec0[0] = -r1
vec0[4] = -v1
vec0[6] = r2
vec0[10] = v2
```

In [8]:
```python
# L_1 initial values
x3 = r * (1 - (M2 / M / 3)**(1/3))
y3=0
vx3 = 0
vy3 = 2 * np.pi * x3 / P

vec1 = np.array([x3,y3,0,vx3,vy3,0])
```

In [9]:
```python
# L_2 initial values
x3 = r * (1 + (M2 / M / 3)**(1/3))
y3=0
vx3 = 0
vy3 = 2 * np.pi * x3 / P

vec2 = np.array([x3,y3,0,vx3,vy3,0])
```

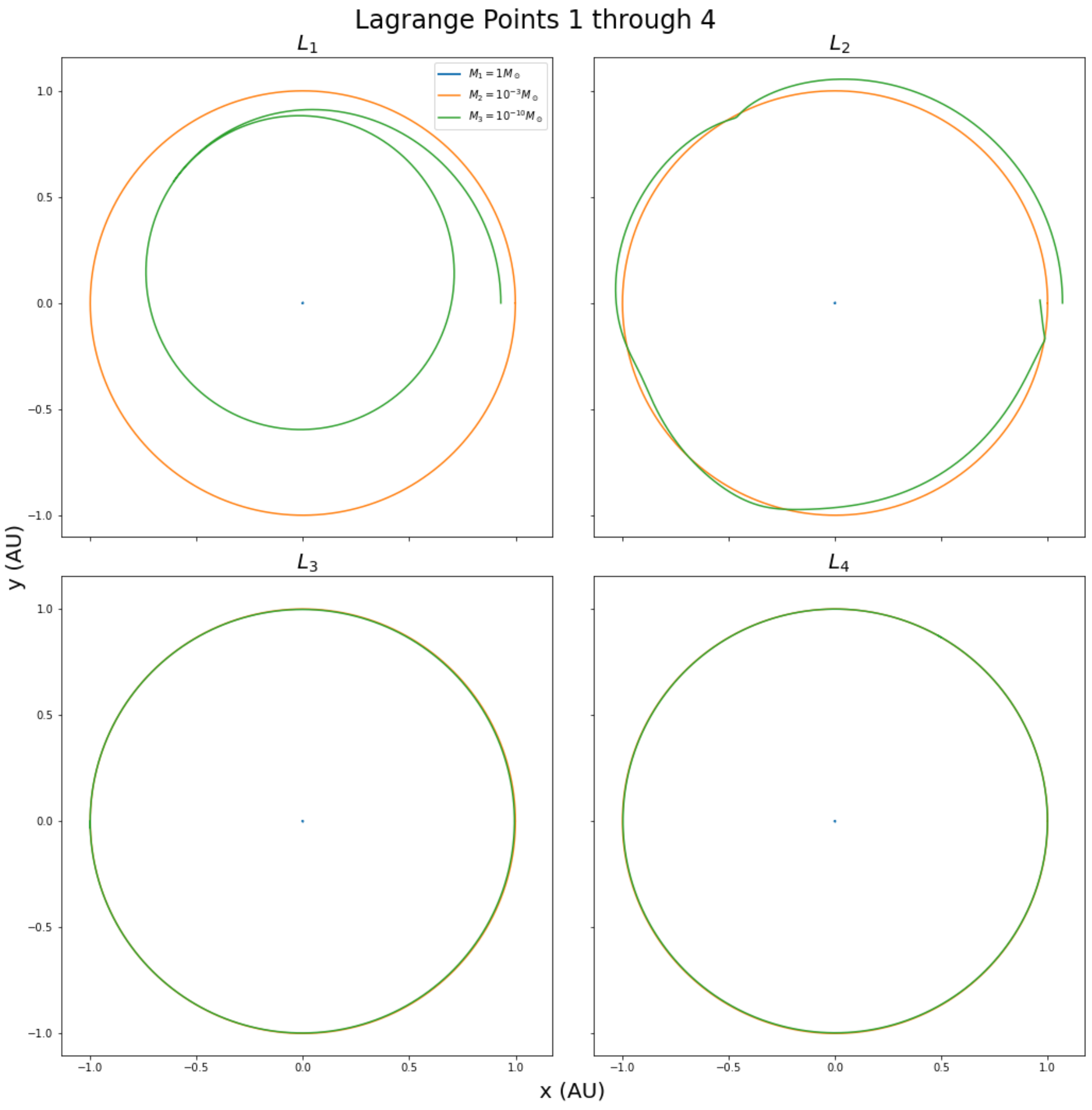
In [10]:
```python
# L_3 initial values
x3 = -r * (1 - 5 * M2 / M / 12)
y3=0
vx3 = 0
vy3 = 2 * np.pi * x3 / P

vec3 = np.array([x3,y3,0,vx3,vy3,0])
```

In [11]:
```python
# L_4 initial values
r3 = r2
v3 = v2

x3 = r3 * np.cos(np.pi/3)
y3 = r3 * np.sin(np.pi/3)
vx3 = -v3 * np.sin(np.pi/3)
vy3 = v3 * np.cos(np.pi/3)

vec4 = np.array([x3,y3,0,vx3,vy3,0])
```

In [12]:
```python
veclist = []
for vecn in (vec1, vec2, vec3, vec4):
    vec0[12:] = vecn
    vec0 = correct_CoM(vec0, M1, M2, M3)

    tarr = np.linspace(0, P, 1000)
    vec = integrate.odeint(df_3body, vec0, tarr, args=(M1, M2, M3))

    veclist.append(vec)
```

Plotting

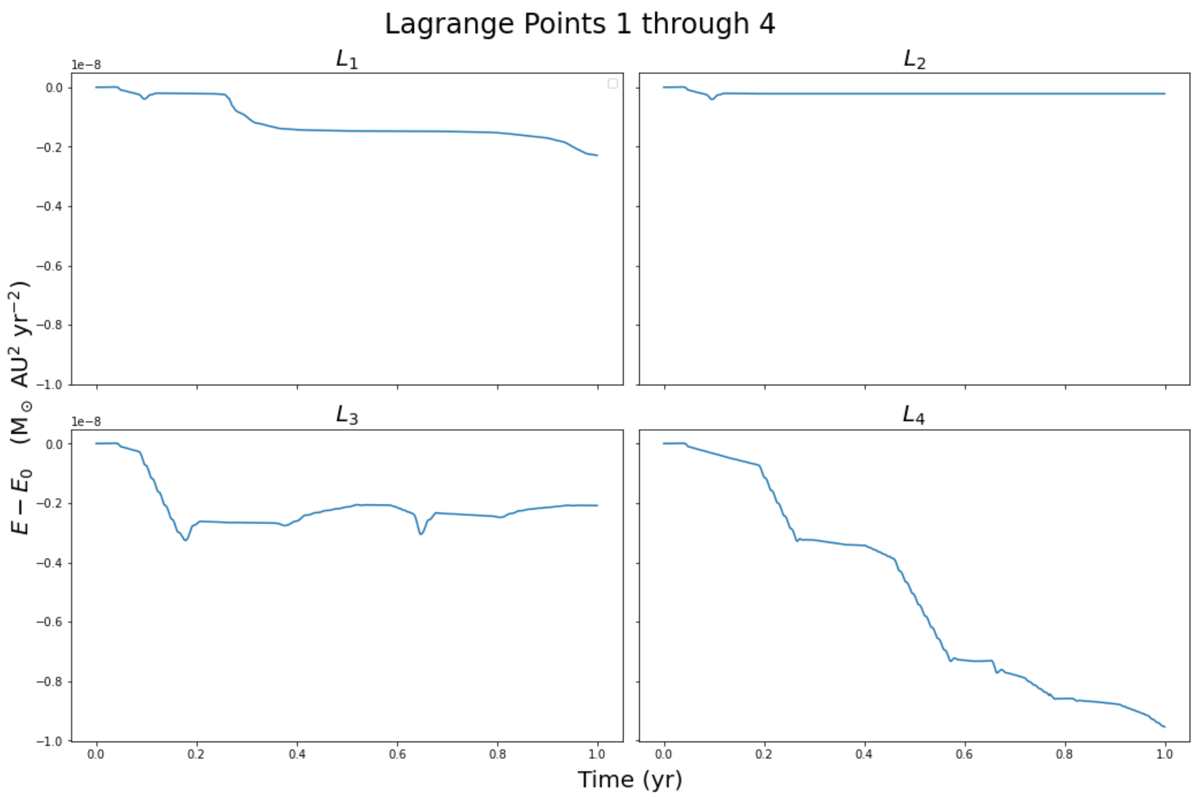
In [13]:
```
fig, axs = plt.subplots(2,2, figsize=(15,15), sharex=True, sharey=True)

for i, (ax,vec) in enumerate(zip(axs.flatten(), veclist),1):
    x1 = vec[:,0]
    y1 = vec[:,1]
    x2 = vec[:,6]
    y2 = vec[:,7]
    x3 = vec[:,12]
    y3 = vec[:,13]

    ax.plot(x1, y1, label='$M_1 = 1 M_\odot$', alpha=1, lw=2)
    ax.plot(x2, y2, label='$M_2 = 10^{-3} M_\odot$')
    ax.plot(x3, y3, label='$M_3 = 10^{-10} M_\odot$')
    ax.set_aspect('equal')
    ax.set_title('$L_{{{:.0f}}}$'.format(i), fontsize=20)


fig.add_subplot(111, frameon=False, xticks=(), yticks=())
plt.xlabel('x (AU)', fontsize=20, labelpad=25)
plt.ylabel('y (AU)', fontsize=20, labelpad=20)

handles, labels = ax.get_legend_handles_labels()
axs[0,0].legend(handles, labels, loc='upper right')

fig.suptitle('Lagrange Points 1 through 4', fontsize=24)

fig.tight_layout(rect=(0,0,1,1))
```



We can see in the figure above that the instability of the first two points. L_3 is also unstable, but we posit that since it is much farther from M_2, it takes longer for the instability to be apparent. L_4 is truly stable for this mass ratio ans we can see the stability in the corresponding plot. Animations of these plots were also submitted.

In [14]:
```python
fig, axs = plt.subplots(2,2, figsize=(15,10), sharex=True, sharey=True)

for i, (ax,vec) in enumerate(zip(axs.flatten(),veclist),1):
    E = calc_energy(vec, M1, M2, M3)
    print('E{0:.0f} = {1:.8e}'.format(i,E[0]))
    dE = E - E[0]
    ax.plot(tarr, dE)
    ax.set_title('$L_{{{:.0f}}}$'.format(i), fontsize=20)

fig.add_subplot(111, frameon=False, xticks=(), yticks=())
plt.xlabel('Time (yr)', fontsize=20, labelpad=25)
plt.ylabel('$E - E_0 \quad (\mathrm{M_\odot \ AU^2 \ yr^{-2}})$', fontsize=20,
labelpad=30)

handles, labels = ax.get_legend_handles_labels()
axs[0,0].legend(handles, labels, loc='upper right')

fig.suptitle('Lagrange Points 1 through 4', fontsize=24)

fig.tight_layout(rect=(0,0,1,1))
print()
```

```
E1 = -1.97392114e-02
E2 = -1.97392103e-02
E3 = -1.97392108e-02
E4 = -1.97392108e-02
```



The figure above shows the change in the total energy of the system as a function of time. We can see that the energy losses are on the order of 10^{-8} while the initial energies are on the order of 10^{-2}, meaning the energy losses are irrelevant for simulations of this length (one period).