Eigenvalues of Covariance matrix and Lagrange Multipliers

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Recall we wish to find the eigenvalues of $A = X^T X$. My claim is that the largest eigenvalue is the solution to

$$\max_u u^T A u \tag{0.1}$$

$$\sum_i u_i^2 = 1. \tag{0.2}$$

Let's work in $\mathbb{R}^2$ to simplify things. Letting $u = \begin{bmatrix} x \\ y \end{bmatrix}$ (ie. a vector in $\mathbb{R}^2$) and

$$A := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then the above can be written as

$$u^T A u = u^T X^T X u = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiplying the matrix by the vector u on the right, the above becomes

$$u^T A u = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}.$$

Now multiplying these matrices (note the first vector is just a 1×2 matrix). We obtain

$$u^T A u = ax^2 + 2bxy + cy^2.$$

Now let's define

$$f(x, y) = ax^2 + bxy + cy^2.$$

Then we seek to solve

$$\max_{(x,y)} f(x, y,) \tag{0.3}$$

$$x^2 + y^2 = 1. \tag{0.4}$$

This is a Lagrange Multiplier Problem. Let $g(x) = x^2 + y^2$, then

$$\nabla g = (2x, 2y).$$

And

$$\nabla f = (2ax + 2by, 2cy + 2bx).$$

Using the method of Lagrange Multipliers we have $\nabla f(x) = \lambda \nabla g(x)$ for some λ. Or with the above,

$$(2ax + 2by, 2cy + 2bx) = \lambda(2x, 2y).$$

BUT! What is this? WE can rewrite the left side as

$$2 \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

Or in matrix notation, $Au = \lambda u$.