Complex Numbers

Consider an equation like:

$$x^2 + 3x + 2 = 0$$

What are the solutions?

Method 1:

$$(x+2)(x+1) = 0$$

$$\therefore x = -2 \text{ or } x = -1$$

Method 2: Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2}$$

$$= \frac{-3 \pm \sqrt{1}}{2} = \frac{-3 \pm 1}{2}$$

$$\therefore x = -\frac{3}{2} + \frac{1}{2} = -1$$ ✓

$$x = -\frac{3}{2} - \frac{1}{2} = -2$$ ✓

Where does the Quadratic Formula come from?

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (a \neq 0)$$

Recall: $(x+q)^2 = x^2 + 2qx + q^2$

$2q = \frac{b}{a} \qquad q^2 = \frac{b^2}{4a^2}$

$\therefore q = \frac{b}{2a}$

$$\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right] - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Consider the following Equation:

$$x^2 + 1 = 0$$

$$x^2 = -1$$

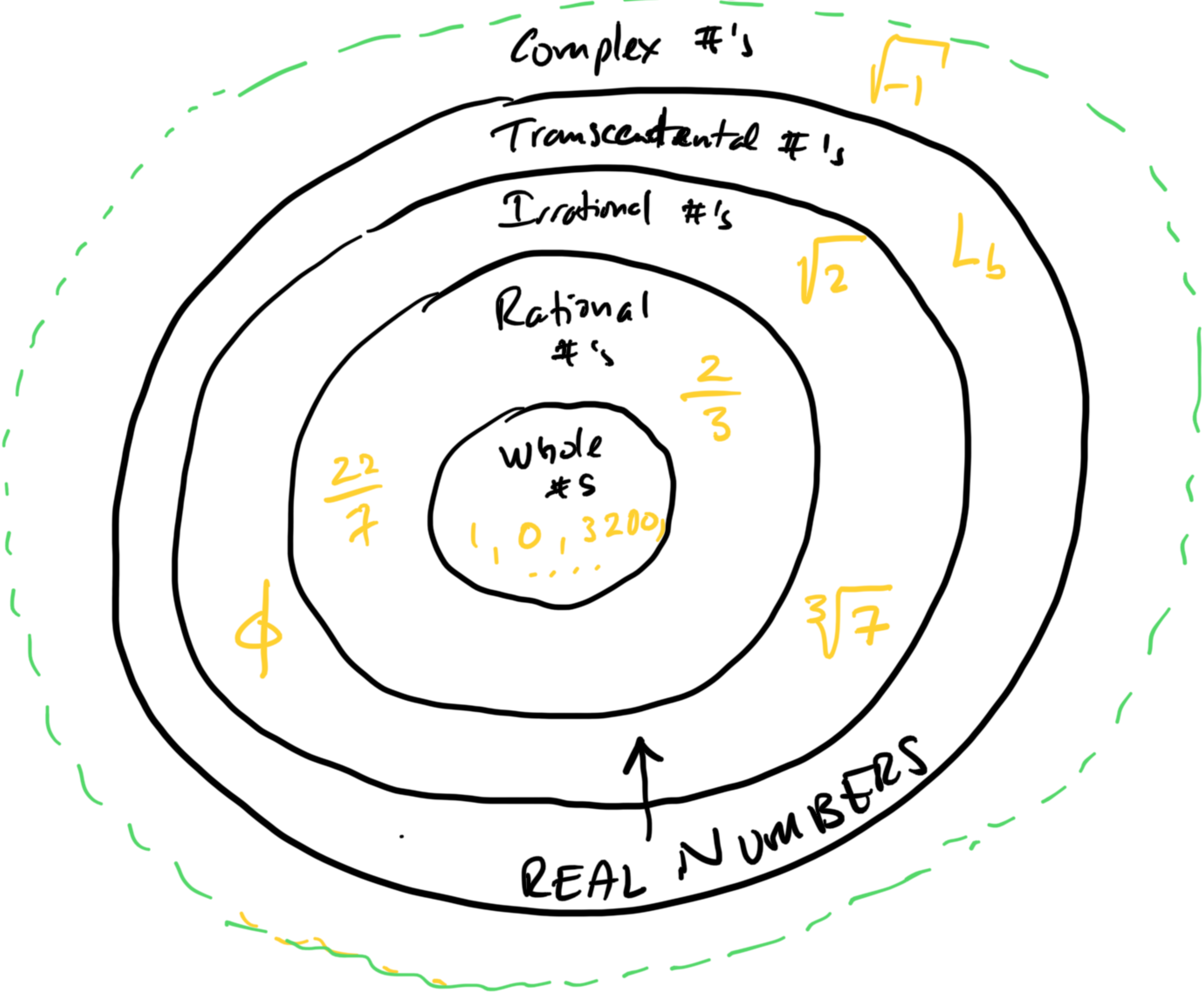
$$x = \pm\sqrt{-1}$$ ... uh oh

This equation has no <u>real</u> solutions.

But, as it turns out, we can make a <u>lot</u> of progress in mathematics if we don't give up.

We choose to define $\sqrt{-1} \equiv i$

This is not a real number. We are not sure what meaning it has yet.



The set of (whole #'s, Rational #s, and irrational #s) are known as the <u>algebraic</u> numbers. They are the numbers you get from solving

equation.

Some algebraic equation...

e.g. $3x = 7 \quad \therefore x = \frac{7}{3}$

$x^2 = 41 \quad \therefore x = \pm\sqrt{41}$

Transcendental Numbers are <u>not</u> algebraic numbers.

e.g. $L_b = \sum_{n=1}^{\infty} 10^{-n!}$

$$= 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + 10^{-120} + 10^{-720} + \dots$$

$$= 0.\underset{1}{1}\underset{2}{1}000\underset{6}{1}00000000000000000\underset{24}{1}00\dots$$

→ this number is not a solution to any algebraic equation... but it is a <u>real</u> number.

→ The first number to be proven to be transcendental (that was ... to prove transcendental not constructed ... (it's exist) was ... e.

(Charles Hermite - 1873)

Proof: We start with the fact that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Assume $e = \frac{a}{b}$ (i.e. that it is rational)

$$\text{Let } x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right)$$

$$= b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right)$$

$$= \underbrace{a(b-1)!}_{\text{integer}} - \sum_{n=0}^{b} \underbrace{\frac{b!}{n!}}_{\text{all integers, because } n \leq b}$$

$\therefore$ x is an integer.

Now $x = b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right)$

Now, $x = \left(\sum_{n=0}^{\infty} \ldots\right.$

$$= \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0$$

$\therefore$ x is **positive**.

Consider $\frac{b!}{n!} = \frac{b(b-1)\cdots(2)(1)}{n(n-1)\cdots(b+1)(b)\cdots(2)(1)}$

$(n \geq b+1)$ $$= \frac{1}{n(n-1)(n-2)\cdots(b+1)}$$

$$= \frac{1}{(b+1)(b+2)\cdots(b+(n-b))}$$

$$\leq \frac{1}{(b+1)^{n-b}}$$

$$\therefore x = \sum_{n=b+1}^{\infty} \frac{b!}{n!} \leq \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}}$$

geometric series $$= \sum_{k=1}^{\infty} \frac{1}{(b+1)^{k}}$$

$$= \frac{1}{\ldots}\left(\frac{1}{\ldots}\right.$$

$$\frac{}{b+1}\left(1-\frac{}{b+1}\right)$$

$$= \frac{1}{b} < 1$$

$$\therefore \quad \boxed{0 < x < 1}$$

But, we showed above that x is an <u>integer</u> if $e = \frac{a}{b}$.

Since there is not an integer between 0 and 1, $e \neq \frac{a}{b}$!!

$\therefore$ e is irrational

The proof that e is transcendental is more involved ... but follows a similar "proof by contradiction" approach.