Einstein Notation

It's really annoying to have to write

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

all the time. Plus, if we want to extend to four, five, ... N dimensions, it gets even more cumbersome.

Let's introduce a notation to make things more compact...

a) Insted of using x, y, z, let's use $1, 2, 3, \ldots, N$

b) Let's introduce a notation for the x, y, z directions:

$$x^1, x^2, x^3, \ldots$$

Why are we using superscripts????

That seems confusing ... more to come on this ...

c) We will get rid of the vector symbols

$$a \equiv \vec{a}$$

$$x' \equiv \vec{x}'$$

This is also confusing, but you've all smart, and understand context.

So,

$$\overset{(\rightarrow)}{a} = \sum_{i=1}^{3} c_i x^i$$

is the same as

$$\vec{a} = c_1 x^1 + c_2 x^2 + c_3 x^3$$

$$= c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$$

Now, more about the superscripts...

One of the reasons we use a superscript is that all of the terms have one subscript and one superscript.

When this happens, the Einstein notation convention is that there is an implied summation over dimensions. So, finally, we can write:

$$\boxed{a = c_i x^i}$$

This means the same as

$$a = \sum_{i=1}^{N} c_i x^i$$

More on Subsripts and Superscripts....

Vectors with superscripts are known as <u>contravavariant</u> vectors. Vectors with subscripts are known as <u>covariant</u> vectors, or covectors.

What does this mean? It is related to the transformation rules/properties of these objects. An easy way to think about this is to consider the <u>MATRIX</u> form of a vector or covector.

Contravariant $\rightarrow$ column matrix

$$a^i \rightarrow \begin{bmatrix} a^1 \end{bmatrix}$$

$$\begin{bmatrix} a^2 \\ a^3 \\ \vdots \\ a^N \end{bmatrix}$$

Covariant $\rightarrow$ row matrix

$$a_i \rightarrow [a_1 \; a_2 \; a_3 \ldots a_N]$$

MNEMONIC: Co - row - below

covariant

row matrix

subscript

With this representation in mind:

$$a = c_i x^i$$

$$= [c_1 \; c_2 \; c_3 \ldots c_N] \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x^3 \\ \vdots \\ x^N \end{bmatrix}$$

$$= c_1 x^1 + c_2 x^2 + c_3 x^3 \ldots + c_N x^N$$

as expected !!! ...

More about unit vectors: It is very common to use the notation

$$\hat{e}_i$$

For the unit vector in the i-th coordinate. Notice we are using a covariant representation. Then we would write:

$$\boxed{a = a^i e_i}$$

We can also write that

$$\hat{e}^i \cdot \hat{e}_j = \delta_{ij}$$

(no summation!)

Kronecker δ

$= 1$ if $i = j$

$= 0$ otherwise.

Scalar Product (a.k.a. Inner Product)

$$\vec{u} \cdot \vec{v} = u_j v^j$$

$$(= u_1 v^1 + u_2 v^2 + u_3 v^3)$$

Vector Product

This is a bit more complicated...

$$\boxed{\vec{u} \times \vec{v} = \varepsilon^{i}{}_{jk}\, u^{j} v^{k}\, \hat{e}_{i}}$$

Let's break this down...

$$\varepsilon^{i}{}_{jk} \equiv \underbrace{\delta^{il}}_{\text{Kronecker } \delta} \; \underbrace{\varepsilon_{ljk}}_{\text{Levi-Civita symbol.}}$$

$$\varepsilon_{ijk} = 0 \quad \text{if any two indices are the same}$$

$$= +1 \quad \text{if } ijk \text{ are in right-cyclical order}$$

$$= -1 \quad \text{if } ijk \text{ are in left-cyclical order}$$

<u>Example</u>: $\varepsilon_{121} \Rightarrow \varepsilon_{xyx} = 0$

3 $\hat{k}$ $\hat{j}$ 2

$$\varepsilon_{123} \Rightarrow \varepsilon_{xyz} = +1$$

$$\varepsilon_{312} \Rightarrow \varepsilon_{zxy} = +1$$

$$\varepsilon_{132} \Rightarrow \varepsilon_{xzy} = -1$$

$$\vec{u} \times \vec{v} = \underbrace{\varepsilon^{i}{}_{jk}\, u^{j} u^{k}\, \hat{e}_{i}}_{\text{sums over } i,\, j,\, \underline{\text{and}}\ k}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^{i}{}_{jk}\, u^{j} u^{k}\, \hat{e}_{i}$$

$$\varepsilon^{i}{}_{jk} = \sum_{l=1}^{3} \delta^{il} \varepsilon_{ljk}$$

$$3 \times 3 \times 3 \times 3$$

$= 81$ terms

All but <u>6</u> of these 81 terms are zero!!

$$= \hat{e}_1 (u^2 u^3 - u^3 u^2)$$
$$+ \hat{e}_2 (u^3 u^1 - u^1 u^3)$$
$$+ \hat{e}_3 (u^1 u^2 - u^2 u^1)$$

Look at the indices in those six terms

$1^{23} \rightarrow$ right-cyclic $\rightarrow +$

$1^{32} \rightarrow$ left-cyclic $\rightarrow -$

$2^{31} \rightarrow +$ $\qquad$ $3^{12} \rightarrow +$

$2^{13} \rightarrow -$ $\qquad$ $3^{21} \rightarrow -$