Eigenvalues and Eigenvectors

It is often very interesting to consider the answer to the question:

Are there any vectors whose direction is left unchanged after applying a transformation? Can we determine this by looking only at the transformation matrix, T. $\Rightarrow$

Answer: yes, and yes.

Suppose there is some vector, $\vec{v}$, whose direction is unchanged by a transformation:

$$\underset{\substack{\uparrow \\ N \times N \\ \text{matrix}}}{T} \underset{\substack{\uparrow \\ (N \times 1) \\ \text{column}}}{\vec{v}} = \underbrace{\lambda \vec{v}}_{\substack{\text{just some scalar} \\ \times \vec{v}, \text{ so same} \\ \text{direction.}}} \leftarrow \begin{array}{c}(N \times 1) \\ \text{column}\end{array}$$

$$\therefore (T - \lambda \underset{\substack{\uparrow \\ \text{identity} \\ \text{matrix.}}}{I}) \vec{v} = 0$$

$$\begin{pmatrix} T_{xx} - \lambda & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} - \lambda & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} - \lambda \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = 0$$

$\boxed{\text{Key Idea}}$: Solve :

$$\det (T - \lambda I) = 0$$

<u>Example:</u> Let $T = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

$$T - \lambda \mathbb{I} = \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix}$$

$$\det(T - \lambda \mathbb{I}) = \lambda(3+\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\therefore \boxed{\lambda = -2, -1}$$

These are the two eigenvalues of this transformation matrix.

this means that:

$$T \vec{v}_1 = -2 \vec{v}_1$$

$$T \vec{v}_2 = -1 \vec{v}_2$$

$$T\vec{v}_2 = -1\vec{v}_2$$

The $\vec{v}_1$ and $\vec{v}_2$ are the two eigenvectors of this transform matrix, and we still have the job of finding these. In fact, these two vectors are more important than the eigenvalues!

$$T\vec{v}_1 = -2v_1$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = -2\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix}$$

$$\begin{pmatrix} v_{1y} \\ -2v_{1x} - 3v_{1y} \end{pmatrix} = \begin{pmatrix} -2v_{1x} \\ -2v_{1y} \end{pmatrix}$$

$$v_{1y} = -2v_{1x}$$

$$-2v_{1x} - 3v_{1y} = -2v_{1y}$$

$$2v_{1x} + v_{1y} = 0$$

$$-2v_{1x} - v_{1y} = 0$$

Oops !!

det = 0 !!

So, we do not have a unique solution.

All we know is that :

$$2v_{1x} + v_{1y} = 0$$

$$\boxed{v_{1y} = -2v_{1x}}$$

But : We can also make the choice to have $|\vec{v}_1| = 1$ (normalized)

$$v_{1x}^2 + v_{1y}^2 = 1$$

$$v_{1x}^2 + 4v_{1x}^2 = 1$$

$$v_{1x}^2 = \frac{1}{5}$$

$$v_{1x} = \frac{}{\sqrt{5}}$$

$$\therefore v_{1y} = -\frac{2}{\sqrt{5}}$$

$$\vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} .4472\ldots \\ -.8944\ldots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = -\begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix}$$

$$\begin{pmatrix} v_{2y} \\ -2v_{2x} - 3v_{2y} \end{pmatrix} = \begin{pmatrix} -v_{2x} \\ -v_{2y} \end{pmatrix}$$

$$v_{2x} + v_{2y} = 0 \quad \leftarrow \quad v_{2y} = -v_{2x}$$

$$-2v_{2x} - 2v_{2y} = 0$$

$$v_{2x}^2 + v_{2y}^2 = 1 \qquad v_{2x} = \frac{1}{\sqrt{2}}$$

$$v_{2x}^2 + v_{2x}^2 = 1 \qquad v_{2y} = -\frac{1}{\sqrt{2}}$$

$$\vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \end{pmatrix}$$

$$v_2 \qquad \begin{pmatrix} -\frac{1}{\sqrt{2}} \end{pmatrix}$$