Basic Arithmetic and Algebra of Complex Numbers.

As imaginary numbers and real numbers are different things, we cannot combine them ... they have to be kept separate.

We write $\underline{\underline{c}} = \underline{\underline{a}} + \underline{\underline{b\,i}}$

real part

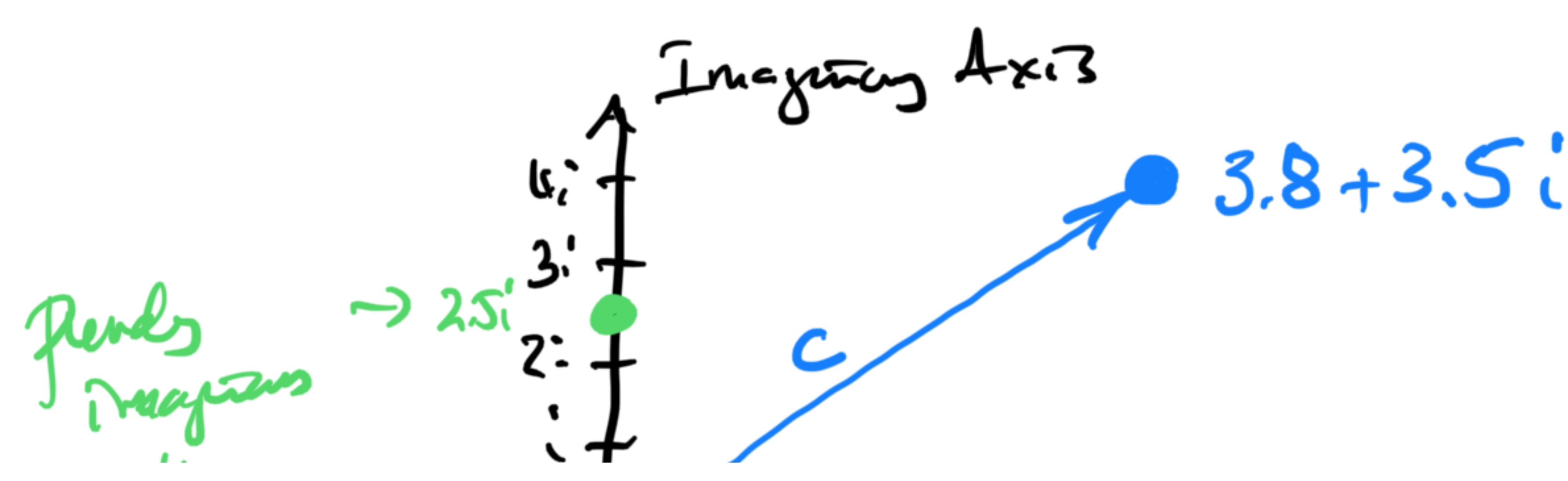
purely imaginary part (bi)

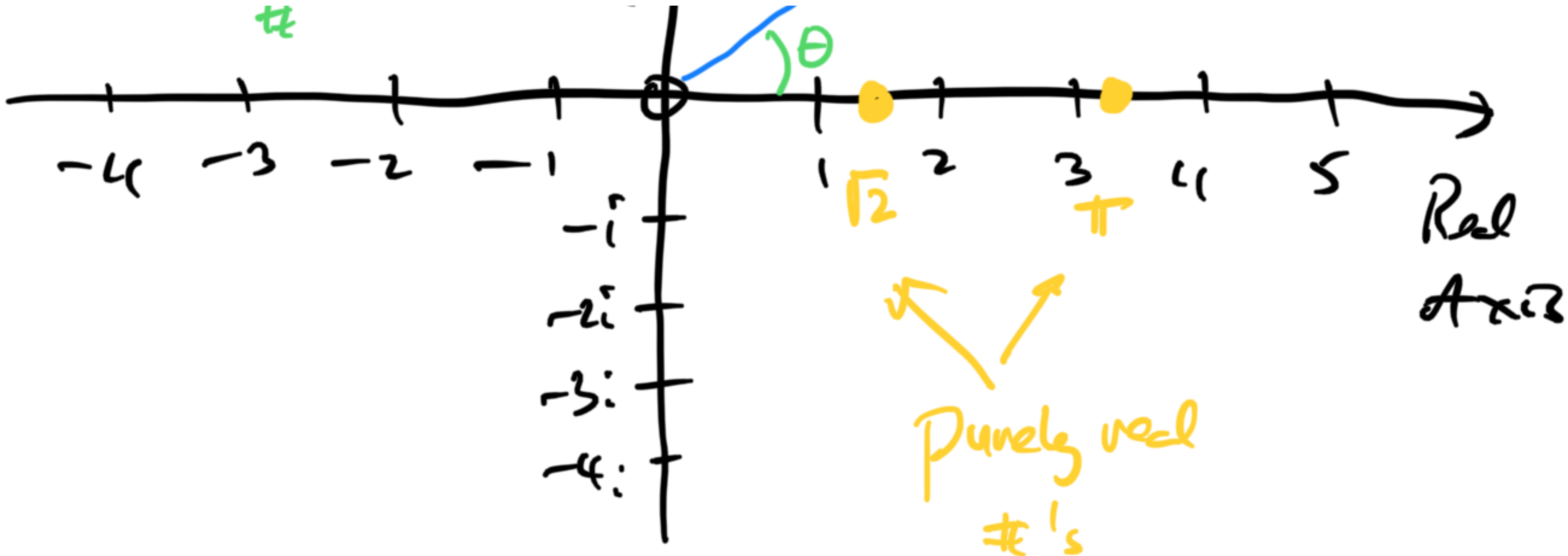
→ b is a real # !!

$$\text{Real}(c) = a$$

$$\text{Imag}(c) = b$$

Visual Representation of Complex Numbers.





We imagine c as a vector in the complex plane .... Why? Well, then the algebra of complex #'s becomes extremely similar to the algebra of 2D vectors. ☺

Example: $|c| = \sqrt{a^2 + b^2}$

$$\tan\theta = \frac{b}{a}$$

[Aside: a purely real number is just a complex number with zero imaginary part!

$$\pi \Rightarrow \pi + 0i \qquad (a=\pi,\ b=0)$$

is just

a purely imaginary

$3i = 0 + 3i$ a complex number with zero real part.]

$(a=0, b=3)$

Addition :

$$c_1 = a_1 + b_1 i$$
$$c_2 = a_2 + b_2 i$$

$$c_1 + c_2 = (a_1 + a_2) + (b_1 + b_2) i$$

(just like 2D vectors!)

Multiplication :

$$c_1 c_2 = (a_1 + b_1 i)(a_2 + b_2 i)$$

(FOIL)

$$= a_1 a_2 + b_1 a_2 i + a_1 b_2 i + b_1 b_2 \boxed{i^2}$$

$i = \sqrt{-1}$

$\therefore i^2 = -1$

$$c_1 c_2 = (a_1 a_2 - b_1 b_2) + (b_1 a_2 + a_1 b_2) i$$

Example :

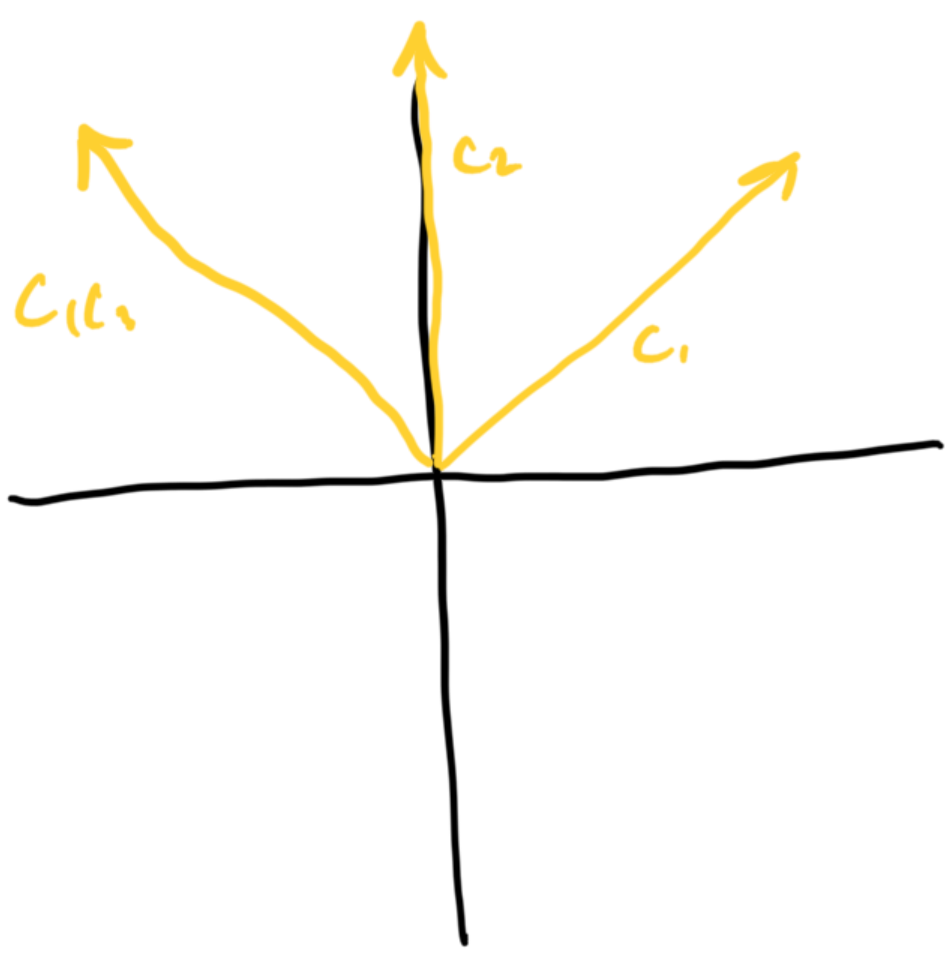
$$c_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

$$c_2 = i$$

$$c_1 c_2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)(i)$$

$$= \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} i^2$$

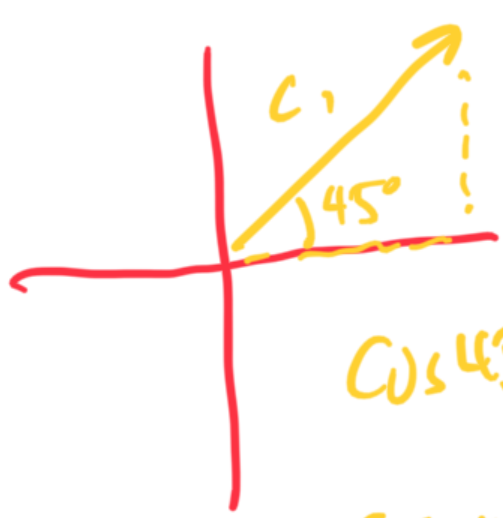
$$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$



Is there a way we can use this visualization more effectively?

Leonhard Euler →

write $c_1 = \cos\theta + i \sin\theta$



$\cos 45° = \frac{1}{\sqrt{2}}$

$\sin 45° = \frac{1}{\sqrt{2}}$

$$c_1 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

Similarly,

$|c_1| = 1$

$c_2 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

$\therefore\ a = |c_1| \cos 45° = \frac{1}{\sqrt{2}}$

$b = |c_1| \sin 45° = \frac{1}{\sqrt{2}}$

$c_2 = \cos\frac{\pi}{2} \ldots$

$= 0 + i$

$c_2 = i$

Definition:

Let $\boxed{e^{i\theta} = \sin\theta + i\cos\theta}$

$$\therefore\quad c_1 = e^{i\pi/4}$$

$$c_2 = e^{i\pi/2}$$

$$\begin{aligned}
\therefore\quad c_1 c_2 &= e^{i\pi/4}\, e^{i\pi/2} \\
&= e^{i(\pi/4 + \pi/2)} \\
&= e^{i\frac{3\pi}{4}} \\
&= \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \\
&= -\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}
\end{aligned}$$

Division :

What is $\left(\frac{4+3i}{2-i}\right)$?

Method 1 : Define the complex conjugate, c^*

If $c = a+bi$

$c^* = a-bi$

Interesting Fact : $c^*c = (a-bi)(a+bi)$

$= \underbrace{a^2+b^2}_{\text{purely real !}}$

$$\frac{c_1}{c_2} = \frac{4+3i}{2-i} \times \frac{2+i}{2+i}$$

$$= \frac{(4+3i)(2+i)}{(2^2+1^2)}$$

$$= \frac{8 + 6i + 4i - 3}{5}$$

$$= \frac{5 + 10i}{5}$$

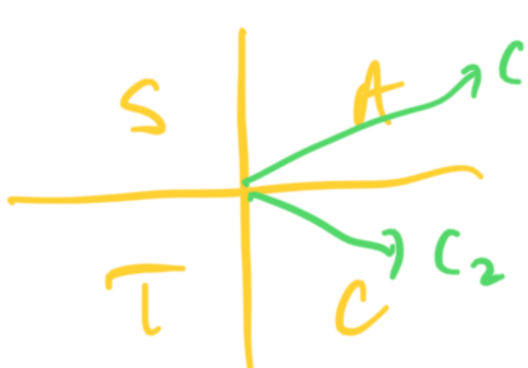
$$\boxed{\frac{c_1}{c_2} = 1 + 2i}$$

Method 2:

Convert numerator and denominator to Euler form.

$c_1 = 4 + 3i \qquad \theta_1 = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$

$c_2 = 2 - i \qquad \theta_2 = \tan\left(\frac{-1}{2}\right) = -26.565^\circ$



$$\frac{c_1}{c_2} = \frac{|c_1|\, e^{i\theta_1}}{|c_2|\, e^{i\theta_2}} = \frac{5}{\sqrt{5}} e^{i(\theta_1 - \theta_2)}$$

$$= \sqrt{5}\, e^{i(63.435^\circ)}$$

$|c_1| = \sqrt{4^2 + 3^2}$

$= 5$

$$= \sqrt{5}\left(\cos(63.435^\circ) + i \sin(63.435^\circ)\right)$$

$|c_2| = \sqrt{2^2 + 1^2}$

$= \sqrt{5}$

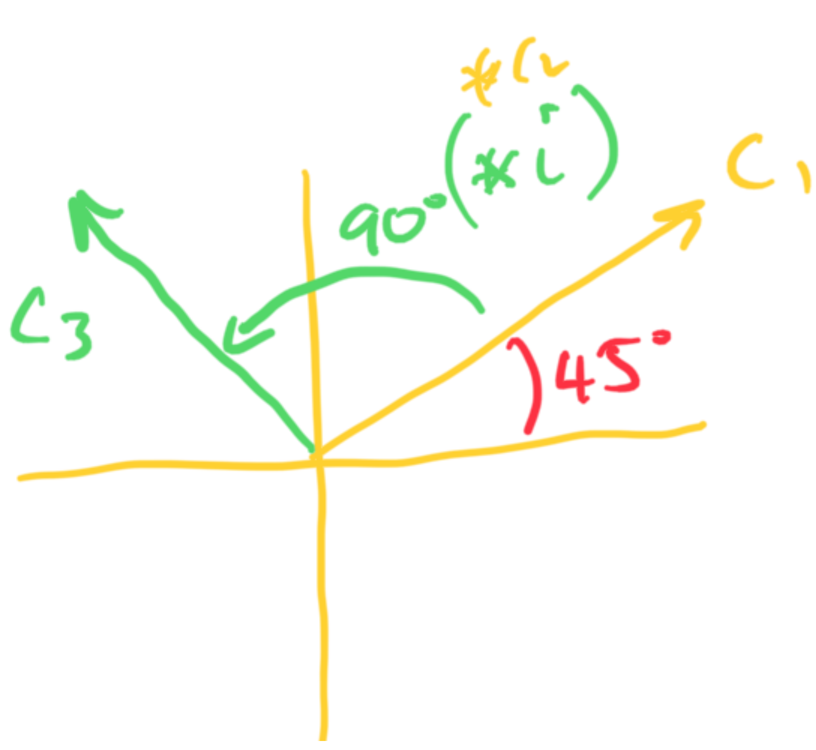
$$= 1.0000 + 2.0000\, i$$

Conclusion: If we want exact algebraic answers, use

$$\frac{c_1}{c_2} = \frac{c_1 c_2^*}{c_2 c_2^*} \quad \ldots$$

If we want numerical answers, and in Python code, ... convert to the Euler form ... or, maybe the code will handle complex division without issue?

[Aside: multiplication / Division of complex numbers is just adding / subtracting **Angles**:



$$c_1 c_2 = |c_1||c_2| e^{i(\theta_1+\theta_2)}$$

$$c_1 / c_2 = \frac{|c_1|}{|c_2|} e^{i(\theta_1-\theta_2)}$$

Exponentiation:

What is c_1^n ??

→ Euler: write $c_1 = |c_1| e^{i\theta_1}$

$$c_1^n = \left(|c_1| e^{i\theta_1}\right)^n$$

$$= |c_1|^n e^{i(n\theta_1)}$$

$$c_1^n = |c_1|^n \left[\cos(n\theta_1) + i\sin(n\theta_1)\right]$$

Ex. let $c_1 = 4 + 3i$

what is $(c_1)^7$?

$|c_1| = \sqrt{3^2 + 4^2}$

$= 5$

$\theta_1 = 36.87°$

$c_1^7 = (5)^7 \Big(\cos(258.0893)$

$+ i\sin(258.0893)\Big)$

$= 5^7 \big(-.2064$

$7\theta_1 = 258.089°$

$\sim 0.9785\,i\,)$

$= -.16124$

$-.76443\,i$