Systems of Differential Equations.

There are just tons of physical systems that are governed by not just one, but <u>many</u> differential equations: circuits, coupled oscillators, biological systems, ...

Imagine a system like this:

$$\frac{dx}{dt} = -2x$$

$$\frac{dy}{dt} = -2y$$

x and y could be any two parameters of the system (current and voltage, KE and PE, ... whatever)

We write this as:

We will write ...

$$\dot{x} = -2x$$
$$\dot{y} = -2y$$

We already know the solutions here:

$$x(t) = x_0 e^{-2t}$$
$$y(t) = y_0 e^{-3t}$$

Where $x_0 = x(0)$, $y_0 = y(0)$ $\equiv$ initial conditions.

Suppose we are told $x(0) = 3$, $y(0) = 2$

Then
$$\boxed{\begin{aligned} x(t) &= 3e^{-2t} \\ y(t) &= 2e^{-3t} \end{aligned}}$$

Done!

Let's write this now as a matrix equation !!

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}}_{\text{think of this as a transformation matrix.}} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now, this matrix is <u>already</u> diagonal, and so we don't have to do anything $\Rightarrow$ $\lambda = -2, -3$ are the eigenvalues; and the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Check:</u>

$$\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \checkmark$$

Now, suppose some other system is described by:

$$\frac{dx}{dt} = 4x + 2y$$

$$\frac{dy}{dt} = -x + y$$

COUPLED Diff. Equations!!! Much harder to solve!!!!

Write as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \overbrace{\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

→ easy to show that

$\lambda_1, \lambda_2 = 3, 2$

$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So, $S = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

$S^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$

$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

$$\boxed{S^{-1} A S = \Lambda}$$

$A = S \Lambda S^{-1}$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = S \Lambda S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} \vdots \\ \dot{y} \end{pmatrix}$$

$$\equiv \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = S\Lambda \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\underbrace{S^{-1}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}_{= \begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix}} = \Lambda \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\therefore \begin{pmatrix} \dot{r} \\ \dot{s} \end{pmatrix} = \Lambda \begin{pmatrix} r \\ s \end{pmatrix}$$

Super simple!!

$$r = r_0 e^{3t}$$

$$s = s_0 e^{2t}$$

<u>Now</u> $\begin{pmatrix} r \\ s \end{pmatrix} = S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$

So $\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} r \\ s \end{pmatrix}$

$$= \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} r_0 e^{3t} \\ s_0 e^{2t} \end{pmatrix}$$

$$\therefore \quad x = -2 r_0 e^{3t} + s_0 e^{2t}$$

$$y = r_0 e^{3t} - s_0 e^{2t}$$

Suppose $x(0) = 1$, $y(0) = 0$

$$1 = -2r_0 + s_0$$

$$0 = r_0 - s_0$$

$$\therefore r_0 = -1, \quad s_0 = -1$$

$$\boxed{\begin{aligned} x &= 2e^{3t} - e^{2t} \\ y &= -e^{3t} + e^{2t} \end{aligned}}$$

Done!