Systems of Equations Using Matrices

Consider a system of N equations in N variables ($x_1, x_2, x_3, \ldots, x_N$)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1N}x_N = C_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2N}x_N = C_2$$

$$\vdots$$

$$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots a_{NN}x_N = C_N$$

We can write this in matrix form as:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} \vdots \\ x_N \end{pmatrix} \quad \begin{pmatrix} \vdots \\ c_N \end{pmatrix}$$

$$\boxed{\tilde{A}\tilde{X} = \tilde{C}}$$

We can solve this, using:

$$\tilde{X} = (\tilde{A})^{-1}\tilde{C}$$

where $(\tilde{A})^{-1}$ is the <u>inverse</u> of the coefficient matrix, $\tilde{A}$.

How does one find the <u>inverse</u> of a matrix? It's complicated ...

We can use the fact that

$$(A^{-1})(A) = \mathbb{I}$$

identity matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Easiest case: 1D: $\tilde{A} = (a_{11})$ $\tilde{C} = (c_1$

Equation :

$$(a_{11})(x_1) = (c_1)$$

$$\therefore (x_1) = \left(\frac{1}{a_{11}}\right)(c_1)$$

$$\therefore (\tilde{A})^{-1} = \frac{1}{a_{11}}$$

Next Easiest Case : 2D

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Let $(\tilde{A})^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} b_{11}a_{11} + b_{12}a_{21} &= 1 \\ b_{11}a_{12} + b_{12}a_{22} &= 0 \\ b_{21}a_{11} + b_{22}a_{21} &= 0 \end{aligned}\right\}$$

4 equations in 4 unknowns.

$$b_{21} a_{12} + b_{22} a_{22} = 1$$

a miracle occurs.

$$(\tilde{A})^{-1} = \frac{1}{a_{11} a_{22} - a_{21} a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

1. switch diagonals
2. negate off diagonals
3. prefactor $= \dfrac{1}{\text{determinant}(A)}$

determinant of a 2D matrix:

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| \equiv ad - bc$$

Example: $2x + y = 4$

$$x - 3y = 7$$

$$\tilde{A} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \qquad \tilde{c} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\det(\tilde{A}) = -6 - 1 = -7$$

$$(\tilde{A})^{-1} = \frac{1}{-7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\tilde{x} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 19 \\ -10 \end{pmatrix} = \begin{pmatrix} 19/7 \\ -10/7 \end{pmatrix}$$

<u>Check</u>:

$$2x + y = 2\left(\frac{19}{7}\right) + \left(\frac{-10}{7}\right)$$

$$= \frac{38-10}{7} = \frac{28}{7} = 4 \quad \checkmark$$

$$x - 3y = \frac{19}{7} - 3\left(\frac{-10}{7}\right)$$

$$= \frac{19 + 30}{7} = \frac{49}{7} = 7 \checkmark$$

Inverse of a 3D Matrix :

$$(\tilde{A})^{-1} = \frac{1}{\det(\tilde{A})} \times \text{Adjoint}(\hat{A})$$

So, we now have two questions...
how do we calculate the determinant of a 3D matrix?
how do we calculate the Adjoint $(\tilde{A})$?

(i) $\det(\tilde{A})$

1) Calculate the "co-factor" matrix.

3D ↘

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

2) Pick a row / column

3) determinant =

$$\sum_{\substack{r/c \\ \text{elements}}} (\text{value})(\text{cofactor}) \det(2\times 2)$$

Example:

$$\tilde{A} = \begin{pmatrix} 2^{+} & 1^{-} & -1^{+} \\ 0^{-} & 2^{+} & 1^{-} \\ 3^{+} & -3^{-} & 2^{+} \end{pmatrix}$$

$$\det(\tilde{A}) = +(2)\times \det\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \to 7$$

$$-(1)\times \det\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \to -3$$

$$+(-1)\times \det\begin{pmatrix} 0 & 2 \\ 3 & -3 \end{pmatrix} \to -6$$

$$= (2)(7) - (1)(-3) + (-1)(-6)$$

$$= 14 + 3 + 6$$

$$\boxed{\det(\tilde{A}) = 23}$$

(ii) Adjoint of a Matrix.

a) Start by taking the <u>transpose</u> of $\tilde{A}$

↖ rows ⟷ columns

$$\tilde{A}^T = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

b) Calculate all possible minor 2 x 2 determinants,

$$\overset{(+)}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = 7 \qquad \overset{(-)}{\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix}} = -1 \qquad \overset{(+)}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} = 3$$

$$\overset{(-)}{\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}} = -3 \qquad \overset{(+)}{\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}} = 7 \qquad \overset{(-)}{\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix}} = 2$$

(+) (-) (+)

$\begin{vmatrix} \cdot & \cdot \\ 2 & -3 \end{vmatrix} = -6 \qquad \begin{vmatrix} \cdot & \cdot \\ 1 & -3 \end{vmatrix} = -9 \qquad \begin{vmatrix} \cdot & \cdot \\ 1 & 2 \end{vmatrix} = 4$

c) calculate (cofactor)(det)

$$= \begin{pmatrix} 7 & 1 & 3 \\ 3 & 7 & -2 \\ -6 & 9 & 4 \end{pmatrix}$$

d) $\operatorname{adj}(\tilde{A}) = (\quad)^T$

$$= \begin{pmatrix} 7 & 3 & -6 \\ 1 & 7 & 9 \\ 3 & -2 & 4 \end{pmatrix}$$

$$\therefore (\tilde{A})^{-1} = \frac{1}{23} \begin{pmatrix} 7 & 3 & -6 \\ 1 & 7 & 9 \\ 3 & -2 & 4 \end{pmatrix}$$

<u>Check:</u>

$$\frac{1}{23} \begin{pmatrix} 7 & 3 & -6 \\ 1 & 7 & 9 \\ 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\frac{1}{\cdot} \begin{pmatrix} 23 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{23}\begin{pmatrix} 0 & 23 & 0 \\ 0 & 0 & 23 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Realistically, calculating the inverse of a matrix by hand is a nightmare, and we should just use programmed functions that have been tested and are not going to make mistakes!!

But, what we can take away from this is:

$$\det(\tilde{A}) \equiv \text{really important!}$$

If $\det(\tilde{A}) = 0$, then there are not any solutions!!!