Jacob Bird
4/29/24

PHYS 340 FINAL

① Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, thus $z_2^* = x_2 - iy_2$

if $z_1 z_2$ is purely real and non zero,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1 x_2 + iy_2 x_1 + iy_1 x_2 + i^2 y_1 y_2$$
$$= x_1 x_2 + i(x_1 y_2 + x_2 y_1) - y_1 y_2$$

then $x_1 y_2 + x_2 y_1 = 0.$

$$x_1 y_2 + x_2 y_1 = 0$$

$$-x_1 y_2 = x_2 y_1$$

$$-\frac{y_2}{x_2} = \frac{y_1}{x_1}$$

$$\left(\frac{r}{r}\right) - \frac{y_2}{x_2} = \frac{y_1}{x_1}$$

$$\frac{-ry_2}{rx_2} = \frac{y_1}{x_1}$$

$$y_1 = -ry_2$$
$$x_1 = rx_2$$

$$z_1 = x_1 + iy_1$$
$$= (rx_2) + i(-ry_2)$$
$$= r(x_2 - iy_2)$$

$$\boxed{z_1 = r z_2^*}$$

② See Jupyter Notebook

③ $\frac{dx}{dt} = 2x + y$ $\Rightarrow$ $\dot{x} = 2x + y$

$\frac{dy}{dt} = 16x + 2y$ $\quad$ $\dot{y} = 16x + 2y$

a) $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 16 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ where $\tilde{A} = \begin{bmatrix} 2 & 1 \\ 16 & 2 \end{bmatrix}$

b) Find eigenvalues of $\hat{A}$

$\det \begin{bmatrix} 2-\lambda & 1 \\ 16 & 2-\lambda \end{bmatrix} = 0$

$(2-\lambda)(2-\lambda) - 16 = 0$

$4 - 2\lambda - 2\lambda + \lambda^2 - 16 = 0$

$\lambda^2 - 4\lambda - 12 = 0$

$(\lambda + 2)(\lambda - 6) = 0$

$\boxed{\lambda = -2, 6}$

Find eigenvectors of $\tilde{A}$:

$\begin{bmatrix} 2-\lambda & 1 \\ 16 & 2-\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

for $\lambda = -2$

$\begin{bmatrix} 4 & 1 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4c_1 + c_2 = 0$

$16 + 4c_2 = 0$

$\boxed{-4c_1 = c_2}$

$\boxed{\vec{v}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}}$

for $\lambda = 6$

$\begin{bmatrix} -4 & 1 \\ 16 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-4c_1 + c_2 = 0$

$16c_1 - 4c_2 = 0$

$\boxed{4c_1 = c_2}$

$\boxed{\vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}}$

c) Form $\tilde{S}$ and $\tilde{S}^{-1}$ and $\lambda\mathcal{I}$:

$\tilde{S} = \begin{bmatrix} -4 & 4 \\ 1 & 1 \end{bmatrix}$ $\quad$ $\tilde{S}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & -4 \end{bmatrix}$ $\quad$ $\lambda\mathcal{I} = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \tilde{A}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\tilde{A} = \tilde{S}\lambda I \tilde{S}^{-1}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \tilde{S}\lambda I \tilde{S}^{-1}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\tilde{S}^{-1}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}r\\ s\end{pmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \tilde{S}\lambda I\begin{bmatrix}r\\ s\end{bmatrix}$$

$$\tilde{S}^{-1}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \tilde{S}^{-1}\tilde{S}\lambda I\begin{bmatrix}r\\ s\end{bmatrix}$$

$$\begin{bmatrix}\dot{r}\\ \dot{s}\end{bmatrix} = \lambda I\begin{bmatrix}r\\ s\end{bmatrix}$$

$$\begin{bmatrix}\dot{r}\\ \dot{s}\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}r\\ s\end{bmatrix}$$

$\Rightarrow$ $\dot{r} = -2r$

$\dot{s} = 6s$

These DE hold solutions of: $r = r_0 e^{-2t}$, $s = s_0 e^{6t}$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}r_0 e^{-2t}\\ s_0 e^{6t}\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-4r_0 e^{-2t} + 4s_0 e^{6t}\\ r_0 e^{-2t} + s_0 e^{6t}\end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= -4r_0 e^{-2t} + 4s_0 e^{6t}\\ y(t) &= r_0 e^{-2t} + s_0 e^{6t}\end{aligned}}$$

d) initial conditions: $x(0) = 1$, $y(0) = 0$

$1 = -4r_0 e^0 + 4s_0 e^0$

$0 = r_0 e^0 + s_0 e^0$

$1 = -4r_0 + 4s_0$

$0 = r_0 + s_0$

$s_0 = -r_0$

$1 = -4(-s_0) + 4s_0$

$1 = 4s_0 + 4s_0$

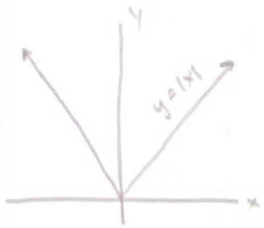
$1 = 8s_0$

$\frac{1}{8} = s_0$

$-\frac{1}{8} = r_0$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2}e^{-2t} + \frac{1}{2}e^{6t}\\ y(t) &= -\frac{1}{8}e^{-2t} + \frac{1}{8}e^{6t}\end{aligned}}$$

④ For the function $|x|$ on $-\pi \le x \le \pi$, begin with the general form of the Fourier series over a symmetric interval:



$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{\pi n t}{T}\right) + b_n \sin\left(\frac{\pi n t}{T}\right)\right]$$

Calculate a_0, b_n, a_n:

$$a_0 = \frac{1}{2T}\int_{-T}^{T} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx$$

by symmetry:

$$= \frac{1}{2\pi}\cdot 2\int_0^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi}$$

$$\boxed{= \frac{1}{2}\pi}$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{\pi n t}{T}\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos\left(\frac{\pi n x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

Integration by parts:
$\int v\,du = vu - \int u\,dv$
$v = x \qquad dv = dx$
$u = \frac{\sin(nx)}{n} \qquad du = \cos(nx)$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi\sin(n\pi)}{n} - \frac{\sin(n\cdot 0)}{n} - \frac{1}{n}\left[\frac{1}{n}(-\cos nx)\right]_0^{\pi}\right]$$

$\sin(n\pi) = 0$

$$= \frac{2}{\pi}\left[0 - 0 - \frac{1}{n^2}\left(-\cos(n\pi) + \cos(n\cdot 0)\right)\right]$$

$\cos(n\pi) = (-1)^n$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right]$$

$$\boxed{= \frac{2}{\pi n^2}\left((-1)^n - 1\right)}$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(t)\sin\left(\frac{\pi n t}{T}\right)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin\left(\frac{\pi n x}{\pi}\right)dx$$

Notice, this is an even function times an odd function, so overall it is the integral of an odd function over a symmetric interval

$$\boxed{=0}$$

Thus,

$$\boxed{|x| = \frac{1}{2}\pi + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos(nx)}$$

c) $|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos(nx)$

at $x=0$:

$$0 = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)(1)$$

$$\sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1-(-1)^n\right) = \frac{\pi}{2}$$

← notice, this will be 0 at even values of n.

So let $n = 2k-1$

$$\sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)^2} = \frac{\pi}{2}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}}$$

See Jupyter notebook for expansion of first 5 non-zero terms

⑤ $\iint_S G(x,y,z)\,dS$ for $z = 2 - x^2$ bounded by $x=0,\ y=0,\ y=4,\ z=0$

$z = 2 - x^2$

$\frac{\partial z}{\partial x} = -2x$

$\frac{\partial z}{\partial y} = 0$

$$\iint_S G(x,y,z)\,dS = \iint_D G(x,y,z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$= \int_0^4 \int_0^{\sqrt{2}} x\sqrt{1+4x^2}\,dx\,dy$$

let $u = 4x^2+1$

$du = 8x\,dx$

$u = 1+4(0)^2 = 1$

$u = 1+4(\sqrt{2})^2 = 9$

$$= \int_0^4 \int_1^9 \sqrt{u}\,du\,dy$$

$$= \int_0^4 \frac{2}{3}u^{3/2}\Big|_1^9\,dy$$

$$= \int_0^4 \frac{13}{6}\,dy$$

$$= \frac{13}{6}\cdot 4 = \boxed{\frac{26}{3}}$$