Second Order Differential Equations Procedure

Given:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} & \tilde{A} & \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x(0) = x_0$$
$$y(0) = y_0$$
$$\dot{x}(0) = V_{0x}$$
$$\dot{y}(0) = V_{0y}$$

Step 1: Solve for the eigenvalues and eigenvectors of $\tilde{A}$

$\rightarrow \lambda_1, \lambda_2$

$\rightarrow \vec{S}_1, \vec{S}_2$

Step 2: Construct the matrix, $\tilde{S}$, with the eigenvectors of $\tilde{A}$ as columns.

$$\tilde{S} = \begin{vmatrix} S_{1x} & S_{2x} \end{vmatrix}$$

$$S = \begin{pmatrix} \cdots & \cdots \\ S_{1y} & S_{2y} \end{pmatrix}$$

<u>Step 3</u>: Construct the matrix, Λ, with the eigenvalues of $\tilde{A}$ as the <u>diagonal</u> elements.

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

<u>Step 4</u>: Find the <u>inverse</u> of $\tilde{S}$,

$$\tilde{S}_{inv} = \frac{1}{\det(S)} \begin{pmatrix} S_{2y} & -S_{2x} \\ -S_{1y} & S_{1x} \end{pmatrix}$$

<u>Step 5</u>: Let $\ddot{r}_1 = \lambda_1 r_1$

Let $\ddot{r}_2 = \lambda_2 r_2$

Case <u>1</u>: $\lambda_i < 0$

$$r_i = C_i \sin \sqrt{\lambda_i}\, t + K_i \cos \sqrt{\lambda_i}\, t$$

Case 2: $\lambda > 0$

$$r_i = c_i e^{\sqrt{\lambda_i} t} + k_i e^{-\sqrt{\lambda_i} t}$$

Now, need to determine c_1, c_2, k_1, k_2 from initial conditions.

Step 6 : Using that $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = S_{inv} \begin{pmatrix} x \\ y \end{pmatrix}$

$$r_1(t) = \frac{1}{\det S} \left(S_{2y} x(t) - S_{2x} y(t) \right)$$

$$r_2(t) = \frac{1}{\det S} \left(-S_{1y} x(t) + S_{1x} y(t) \right)$$

1 $\Rightarrow$ $$r_{10} = \frac{1}{\det S} \left(S_{2y} x_0 - S_{2x} y_0 \right)$$

2 $\Rightarrow$ $$r_{20} = \frac{1}{\det S} \left(-S_{1y} x_0 + S_{1x} y_0 \right)$$

Evaluate based on step 5 solutions !!

Using that $\begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \end{pmatrix} = S_{inv} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

$$\dot{r}_1(t) = \frac{1}{\det S} \left(S_{2y} \dot{x}(t) - S_{2x} \dot{y}(t) \right)$$

$$\dot{r}_2(t) = \frac{1}{\det S}\left(-S_{1y}\dot{x}(t) + S_{1y}\dot{y}(t)\right)$$

③ $\Rightarrow$ $$\dot{r}_{10} = \frac{1}{\det S}\left(S_{2y}V_{0x} - S_{2x}V_{0y}\right)$$

④ $\Rightarrow$ $$\dot{r}_{20} = \frac{1}{\det S}\left(-S_{1y}V_{0x} + S_{1y}V_{0y}\right)$$

Use ①, ②, ③, ④ to determine

C_1, K_1, C_2, K_2 !! $\rightarrow s(t), r(t)$

Step 6: Using $\begin{pmatrix} x \\ y \end{pmatrix} = \tilde{S}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

$$\boxed{\begin{aligned} x(t) &= S_{1x}\, r_1(t) + S_{2x}\, r_2(t) \\ y(t) &= S_{1y}\, r_1(t) + S_{2y}\, r_2(t) \end{aligned}}$$