Transformation of Vectors.

Suppose we have some vector, $\vec{a}$, and we apply to this vector some "transformation" function, T, which will turn $\vec{a}$ into a new vector, $\vec{b}$:

$$\vec{b} = T\vec{a}$$

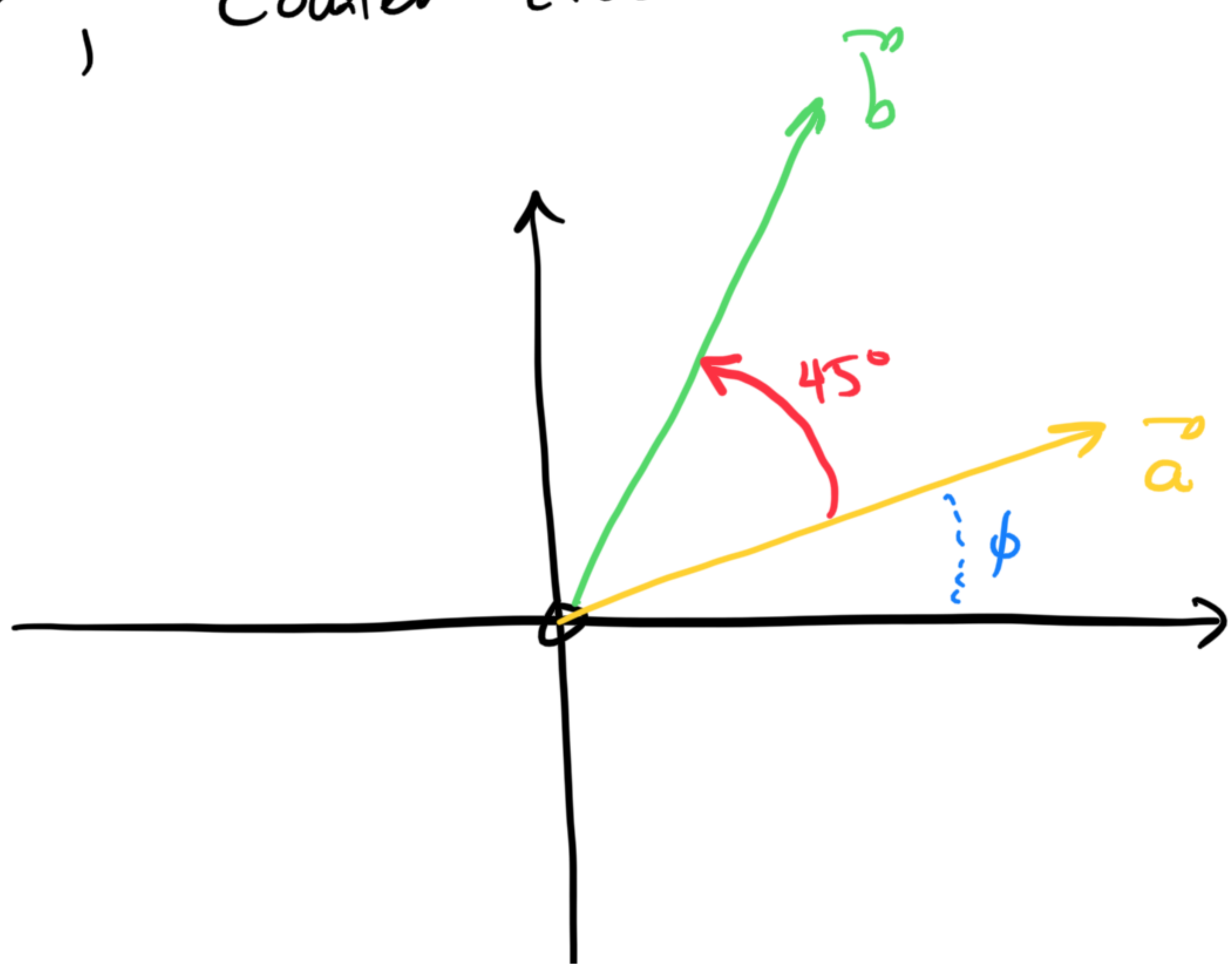
What does "T" look like, mathematically?? Think of matrices....

$$\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$\vec{b} = T \cdot \vec{a}$$

So, a transformation matrix is a $N \times N$ matrix which acts on a vector of dimension N, and produces a new vector of dimension N.

Example in 2D: Consider a transformation matrix which takes a vector, and <u>rotates</u> it by $45°$, counter-clockwise.



What is the transformation matrix, T, for this situation?

$a_x = a\cos\phi$ $\qquad |\vec{a}| = |\vec{b}| = a$

$a_y = a\sin\phi$

$b_x = b\cos(\phi + 45^\circ) = a\cos(\phi + 45^\circ)$

$\frac{1}{\sqrt{2}}$

$= a[\cos\phi\cos 45^\circ - \sin\phi\sin 45^\circ]$

$\frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}a\cos\phi - \frac{1}{\sqrt{2}}\sin\phi$

$$\boxed{b_x = \frac{1}{\sqrt{2}}a_x - \frac{1}{\sqrt{2}}a_y}$$

$b_y = b\sin(\phi + 45^\circ) = a\sin(\phi + 45^\circ)$

$1/\sqrt{2}$

$= a[\sin\phi\cos 45^\circ + \cos\phi\sin 45^\circ]$

$\frac{1}{\sqrt{2}}$

$= \frac{1}{} a\sin\phi$

$\sqrt{2}$

$+ \frac{1}{\sqrt{2}} a \cos\phi$

$$\boxed{b_y = \frac{1}{\sqrt{2}} a_x + \frac{1}{\sqrt{2}} a_y}$$

$$\therefore \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{T} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$\boxed{\therefore T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

CCW Rotation!

→ The transformation properties of vectors is at the heart of physics!

→ Why? → Symmetries!! Conservation Laws!!!

→ similarly, we learn a lot from understanding the symmetries / properties of the transformation matrix itself!!

→ this is what we want to spend some time studying.....

Linear Transformations:

A linear transformation is one that preserves the operations of <u>ADDITION</u> and <u>scalar multiplication</u>

i.e

$$T(k\vec{a}) = kT(\vec{a})$$

$$T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$$

Is the rotation matrix above a linear transformation?

$$T\vec{a} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\, a_x - \sin\theta\, a_y \\ \sin\theta\, a_y + \cos\theta\, a_x \end{pmatrix}$$

$$T(k\vec{a}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} ka_x \\ ka_y \end{pmatrix}$$

$$= \begin{pmatrix} k\cos\theta\, a_x - k\sin\theta\, a_y \\ k\sin\theta\, a_x + k\cos\theta\, a_y \end{pmatrix}$$

$$= k \begin{pmatrix} \cos\theta\, a_x - \sin\theta\, a_y \\ \sin\theta\, a_x + \cos\theta\, a_y \end{pmatrix}$$

$\left(\sin\theta\, a_x + \cos\theta\, a_y \right)$

$$\boxed{T(k\vec{a}) = k\,T\vec{a}}$$

→ It is easy to similarly show that

$$T[\vec{a} + \vec{b}] = T\vec{a} + T\vec{b}$$

$\therefore$ $T(\theta)$ is a linear transformation.