1) 1) $\left(\frac{a}{r}\right)^3 \sin 2V \cancel{} = x$

$$\bar{x} = \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)}^3 \sin V \, dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \sin V \frac{r^2}{a^2\sqrt{1-e^2}}\, dV$$

$$= \frac{1}{2\pi}\cdot\frac{a}{\sqrt{1-e^2}}\int_0^{2\pi}\frac{\sin 2V}{r}\, dV$$

$$= \underbrace{\frac{a}{2\pi(1-e^2)\sqrt{1-e^2}}}\int_0^{2\pi} \sin 2V\cdot(1+e\cos V)\, dV$$

$$= \quad '' \quad \int_0^{2\pi} \overbrace{\sin 2V\, dV}^{0} + e\int_0^{2\pi}\overbrace{\cos V\cdot \sin 2V\, dV}^{0}$$

$$= \boxed{0 = \left(\frac{a}{r}\right)^3 \sin 2V}$$

1) 2) $\left(\frac{a}{r}\right)^4 \cos V$

$$\overline{\left(\frac{a}{r}\right)^4 \cos V} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^4 \cos V \, dM$$

$$= \frac{a^2}{2\pi\sqrt{1-e^2}}\int_0^{2\pi} r^{-2}\cos V \, dV$$

$$= \underbrace{\frac{a^2}{2\pi \cancel{\pi}\sqrt{1-e^2}\cdot a^2(1-e^2)^2}}\int_0^{2\pi}\cos V(1+e\cos V)^2 dV$$

$$= \text{"} \int_0^{2\pi}\cos V\,(1+2e\cos V+e^2\cos^2 V)\,dV$$

$$= \text{"} \int_0^{2\pi}\cancel{\cos V}^{\,0}\,dV + \int_0^{2\pi}\cos V\cdot 2e\cos V + \int_0^{2\pi} e^2\cancel{\cos^3 V}\,dV$$

$$\pi\cdot 2e \qquad\qquad 0$$

$$= \frac{1}{\cancel{2}\cancel{\pi}\cdot\sqrt{1-e^2}\cdot(1-e^2)^2}\cdot\cancel{2}e\cdot\cancel{\pi} = \boxed{\frac{e}{\sqrt{1-e^2}\cdot(1-e^2)^2} = \overline{\left(\frac{a}{r}\right)^4\cos V}}$$

1) 3) $\left(\frac{a}{r}\right)^5 \cos 2v$

$$\overline{\left(\frac{a}{r}\right)^5 \cos 2v} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^5 \cos 2v \, dv$$

$$= \frac{1}{2\pi}\cdot\frac{a^2}{\sqrt{1-e^2}}\int_0^{2\pi} r^{-3}\cos 2v \, dv$$

$$= \frac{1}{2\pi}\cdot\frac{\cancel{a^2}}{\sqrt{1-e^2}}\cdot\frac{1}{\cancel{a^2}(1-e^2)^3}\int_0^{2\pi}\cos 2v\,(1+e\cos v)^3 \, dv$$

$$= \frac{1}{2\pi\cdot\sqrt{1-e^2}\cdot(1-e^2)^3}\int_0^{2\pi}\cos 2v\cdot\left(1+3e\cos v+3e^2\cos^2 v + e^3\cos^3 v\right) dv$$

$$= \frac{1}{2\pi\sqrt{1-e^2}\cdot(1-e^2)^3}\int_0^{2\pi}\underbrace{\cos 2v}_{0} + \underbrace{3e\cos v\cos 2v}_{0} + \underbrace{3e^2\cos^2 v\cos 2v}_{\frac{3\pi\cdot e^2}{2}} + \underbrace{e^3\cos^3 v\cos 2v}_{0} \, dv$$

$$= \frac{1}{2\cancel{\pi}\sqrt{1-e^2}(1-e^2)^3}\cdot\frac{3\cancel{\pi}e^2}{2} = \boxed{\frac{3e^2}{4\sqrt{1-e^2}(1-e^2)^3} = \overline{\left(\frac{a}{r}\right)^5\cos 2v}}$$

2) $R \equiv \overline{R}_{sec} = -\frac{3}{2} \frac{n^2 R_\oplus^2 J_2}{(1-e^2)^{3/2}} \left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)$

1) $\dot{a}_{sec} = \frac{2}{na} \underbrace{\frac{\partial \overline{R}_{sec}}{\partial M_0}}_{\to 0}$ $\therefore \boxed{\dot{a}_{sec} = 0}$

2) $\dot{e}_{sec} = \frac{1-e^2}{na^2 e} \underbrace{\frac{\partial \overline{R}_{sec}}{\partial M_0}}_{\to 0} - \frac{\sqrt{1-e^2}}{na^2 e} \underbrace{\frac{\partial \overline{R}_{sec}}{\partial \omega}}_{\to 0}$

$\therefore \boxed{\dot{e}_{sec} = 0}$

3) $\dot{i}_{sec} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i \cdot \underbrace{\frac{\partial \overline{R}_{sec}}{\partial \omega}}_{\to 0} - \underbrace{\frac{\partial \overline{R}_{sec}}{\partial \Omega}}_{\to 0}\right)$

$\therefore \boxed{\dot{i}_{sec} = 0}$

4) $\dot{\omega}_{sec} = \frac{\sqrt{1-e^2}}{na^2 e} \underbrace{\frac{\partial \overline{R}_{sec}}{\partial e}}_{\to -\frac{9}{2}\frac{n^2 R_\oplus^2 J_2 e}{(1-e^2)^{5/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)} - \frac{\cot i}{na^2\sqrt{1-e^2}} \underbrace{\frac{\partial \overline{R}_{sec}}{\partial i}}_{\to -\frac{3}{2}\frac{n^2 R_\oplus^2 J_2}{(1-e^2)^{3/2}} \sin i \cos i}$

$\frac{\cancel{\sqrt{1-e^2}}}{na^2 \cancel{e}} \frac{n^2 R^2 J_2 \cancel{e}}{(1-e^2)^{5/2}}$

$= \frac{\sqrt{1-e^2}}{na^2 e} \cdot \frac{-9}{2} \frac{n^2 R_\oplus^2 J_2 e}{(1-e^2)^{5/2}} \left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) - \frac{\overset{\cos i / \cancel{\sin i}}{\cot i}}{na^2\sqrt{1-e^2}} \cdot \frac{-3}{2} \frac{n^2 R_\oplus^2 J_2}{(1-e^2)^{3/2}} \cancel{\sin i} \cos i$

$= \frac{3}{4} \frac{n R_\oplus^2 J_2}{(1-e^2)^2} \left(-3\sin^2 i + 2 + 2\cos^2 i\right)$, $2\cos^2 i \to 2 - 2\sin^2 i$

$= \boxed{\frac{3n R_\oplus^2 J_2}{4(1-e^2)^2}\left(-5\sin^2 i + 4\right) = \dot{\omega}_{sec}}$

2) 5) $\dot{M}_{0,sec} = \frac{1-e^2}{na^2e}\frac{\partial \bar{R}_{sec}}{\partial e} - \frac{2}{na}\frac{\partial \bar{R}_{sec}}{\partial a}$

$\hookrightarrow -\frac{9}{2}\frac{n^2 R_\oplus^2 J_2 e}{(1-e^2)^{5/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)$

$\frac{-6n R_\oplus^2 J_2}{(1-e^2)^{3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)\left(\frac{-3}{2}\frac{n}{a}\right) \leftarrow \frac{\partial \bar{R}_{sec}}{\partial n}\cdot\frac{\partial n}{\partial a}$

$= \frac{9n^2 R_\oplus^2 J_2}{a(1-e^2)^{3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)$

$\dot{M}_{0,sec} = \frac{\cancel{1-e^2}}{\cancel{n}a^2\cancel{e}}\left(-\frac{9}{2}\frac{n^{\cancel{2}} R_\oplus^2 J_2 \cancel{e}}{(1-e^2)^{\cancel{5/2}\, 3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)\right)$

$- \frac{2}{\cancel{n}a}\left(\frac{9 n^{\cancel{2}} R_\oplus^2 J_2}{a(1-e^2)^{3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)\right)$

$= -\frac{9}{2}\frac{n R_\oplus^2 J_2}{a^2(1-e^2)^{3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) - \frac{18 n R_\oplus^2 J_2}{a^2(1-e^2)^{3/2}}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right)$

$= \frac{n R_\oplus^2 J_2}{a^2(1-e)^{3/2}}\left(-\frac{54}{2}(3\sin^2 i - 2) - 108(3\sin^2 i - 2)\right)$

$= -\frac{135 n R_\oplus^2 J_2}{a^2(1-e)^{3/2}}(3\sin^2 i - 2) = \dot{M}_{0,sec}$

3)

$$U_{4,0} = -J_4 \frac{\mu R_\oplus^4}{8r^5}\left(35\sin^4\phi - 30\sin^2\phi + 3\right)$$

$$\overline{\left(\frac{a}{r}\right)^5} = (1-e^2)^{-\frac{7}{2}}\left(1+\frac{3}{2}e^2\right)$$

$$\sin^4\phi = \sin^4 i \sin^4(\omega + \nu)$$

$$= \sin^4 i\left(\frac{3}{8} - \frac{1}{2}\cancel{\cos}\, 2(\omega+\nu) + \frac{1}{8}\cancel{\cos 4(\omega+\nu)}\right)$$

$$= \frac{3}{8}\sin^4 i$$

$$\sin^2\phi = \sin^2 i \sin^2(\omega+\nu)$$

$$= \sin^2 i\left(\frac{1}{2} - \frac{1}{2}\cancel{\cos}\, 2(\omega+\nu)\right)$$

$$= \frac{1}{2}\sin^2 i$$

$$U_{40} = \frac{-J_4 \mu R_\oplus^4}{8r^5}\left(\frac{105}{8}\sin^4 i - 15\sin^2 i + 3\right)$$

$$\overline{U}_{4,0} = -\frac{J_4 \mu R_\oplus^4}{8}\cdot\overline{\left(\frac{a^5}{r^5}\right)}\cdot\frac{1}{a^5}\left(\frac{105\sin^4 i}{8} - 15\sin^2 i + 3\right)$$

$$= \boxed{\frac{-J_4 \mu R_\oplus^4}{8a^5}\left(\frac{105\sin^4 i}{8} - 15\sin i + 3\right)\cdot(1-e^2)^{-7/2}\cdot\left(1+\frac{3}{2}e^2\right) = \overline{U}_{0,4}}$$

4) $\dot{\Omega}_{sec}$?

$$\overline{U_{4,0}} = \frac{-J_4 \mu R_\oplus^4}{8a^5}\left(\frac{105 \sin^4 i}{8} - 15\sin^2 i + 3\right)(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)$$

$$\dot{\Omega}_{sec,J_4} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial \overline{R}_{sec J_4}}{\partial i} \rightarrow \overline{U_{4,0}}$$

∂i

$$\frac{\partial \overline{R}_{sec J_4}}{\partial i} = \frac{-J_4 \mu R_\oplus^4}{8a^5}(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\cdot\left(\frac{105}{2}\sin^3 i\cos i - 30\sin i\cos i\right)$$

$\mu = n^2a^3$

$$\dot{\Omega}_{sec,J_4} = \frac{1}{\cancel{n}a^2(1-e^2)^{1/2}\cancel{\sin i}}\left(-\frac{J_4 \mu R_\oplus^4}{8a^5}(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\left(\frac{105}{2}\sin^{\cancel{3}\,2} i\cos i - 30\cancel{\sin i}\cos i\right)\right)$$

$$= \frac{1}{a^2(1-e^2)^4}\cdot\frac{-J_4 n R_\oplus^4}{a^2}\left(1+\frac{3}{2}e^2\right)\left(\frac{105}{2}\sin^2 i\cos i - 30\cos i\right)$$

$$\downarrow$$

$$= \frac{-J_4 n R_\oplus^4}{a^4(1-e^2)^4}\left(1+\frac{3}{2}e^2\right) \qquad 15\cos i\left(\frac{7}{2}\sin^2 i - 2\right)$$

$$= \boxed{\frac{-15 J_4 n R_\oplus^4 \cos i}{a^4(1-e^2)^4}\left(1+\frac{3}{2}e^2\right)\left(\frac{7}{2}\sin^2 i - 2\right) = \dot{\Omega}_{sec}}$$

5) $r_p = 7200$ km $\quad r_a = 7250$ km $\quad i = 98°$

1) $$\dot{\Omega}_{sec, J_2} = -\frac{3}{2} \cdot \frac{\sqrt{\mu}\, J_2\, R_\oplus^2}{(1-e^2)^2\, a^{7/2}} \cos i$$

2) $$\dot{\Omega}_{sec, J_4} = \frac{-15\, J_4\, n\, R_\oplus^4 \cos i}{a^4 (1-e^2)^4} \left(1 + \frac{3}{4} e^2\right)\left(\frac{7}{2} \sin^2 i - 2\right)$$

3) J_2 is orders of magnitude larger & thus could lead to greater error.