Part 1 : Number Systems

Binary to Decimal

$b_n b_{n-1} \dots b_2 b_1 b_0$

$=$

$(b_n \times 2^n) + (b_{n-1} \times 2^{n-1}) + \dots + (b_1 \times 2) + b_0$

Decimal to Binary

repeatedly $\div 2$ and get remainders $r_0, r_1, r_2 \dots$

Binary $= r_n r_{n-1} \dots r_2 r_1 r_0$

Conversion Algorithm

- let $b > 1 \in \mathbb{Z}$ be the new base
- divide repeatedly by b, converting remainders to digits in base b. $(r_1, r_1, r_3, \dots, r_n)$
- base ~~ten~~ b representation is:

$r_n r_{n-1} \dots r_2 r_1 r_0$

Given any integers $a, b \in \mathbb{Z}$ s.t. $b \neq 0$, there are unique integers $q, r \in \mathbb{Z}$ s.t. $a = qb + r$ and $0 \leq r < |b|$

The GCD of 2 Ints m and n is denoted by $GCD(m, n)$.

Euclidean Algorithm

Input: $m, n \in \mathbb{Z}$ s.t. $0 \leq m < n$

Output: $GCD(m, n)$

Algorithm:

let $r_1 = m$ and $r_0 = n$

Repeat for each $i = 1, 2, 3, \dots$

If $r_i = 0$, output r_{i-1} + Halt

Otherwise, divide r_{i-1} by r_i and let r_{i+1} be the remainder

Congruence

We say 2 ints (a and b) are congruent modulo $n > 1$

$a \equiv b \mod n$ or $a \overset{n}{\equiv} b$

if $a - b$ is an integer multiple of n

i.e. $a = b + kn$.

Part 1: Number Systems Continued

Computer representation of ints

An Int x is represented as $x \bmod 2^N$ in binary

$N=3$

x	$x \bmod 2^3$	bin	x	$x \bmod 2^3$	bin
0	0	000	-4	4	100
1	1	001	-3	5	101
2	2	010	-2	6	110
3	3	011	-1	7	111

Real Numbers

Can be thought of as corresponding to points on an infinite straight line.

$\mathbb{R}$

Rational numbers

- Subset of $\mathbb{R}$
- A rational number has the form m/n where $m, n \in \mathbb{Z} \wedge n \neq 0$

$\mathbb{Q}$

Algebraic Axioms

1. Commutativity: $x+y=y+x$ & $x*y=y*x$
2. Associativity: $x+(y+z)=(x+y)+z$ & $x*(y*z)=(x*y)*z$
3. Distributivity: $x*(y+z)= x*y + x*z$
4. additive identity: 0
5. multipliative identity: 1
6. 4 & 5 are distinct: $1 \neq 0$
7. Every element has additive inverse: $-x$
8. Every non-zero element has a multiplicative inverse $\frac{1}{x}$ s.t. $x \neq 0$.

Ordering Axioms

9. Transitivity of ordering
   if $x<y$ and $y<z$ then $x<z$
10. trichotomy law
    Exactly 1 is true: $x<y$, $x>y$, $x=y$
11. Preservation of ordering under addition
    If $x<y$ then $x+z<y+z$
12. preservation of ordering under multiplication
    If $0<z$ and $x<y$ then $zx<zy$

Part 1: Number Systems Contd.

Completeness axiom.

13. Every non-empty subset that is bounded above has a least upper bound

Modulus

$|x|$ of a real number is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Complex numbers

$a+bi \quad a,b \in \mathbb{R}$

Complex Conjugate

$\operatorname{conj}(a+bi) = a-bi$

Re(z)

$$= (z+\bar{z})/2$$

Im(z)

$$= (z-\bar{z})/2i$$

Complex modulus

$$|z| = \sqrt{a^2+b^2}$$

Polar Representation

let $z = x+yi$, we now represent z in polar coordinates

$$z = x+yi = r(\cos\theta + i\sin\theta)$$

Polar to Complex translation

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}$$

De moivre's theorem

For any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Fundamental theorem of algebra

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

has n roots: $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$

$$\Downarrow$$

$$f(x) = a_n(x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n)$$

Vectors

Part 2 – Linear Algebra and Matrices – CS13①1

Vectors:

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$

$\underline{a} + \underline{b} = (a_1 + b_1, a_2 + b_2, \ldots)$

$\lambda \underline{a} = (\lambda a_1, \lambda a_2, \ldots)$

$|\underline{a}| = \sqrt{a_1^2 + a_2^2 + \ldots}$

$\underline{a} . \underline{b} = a_1 b_1 + a_2 b_2 + \ldots$

Scalar product

$\underline{a} . \underline{b} = a_1 b_1 + a_2 b_2 \ldots = |\underline{a}||\underline{b}| \cos\theta$

Span

If $U = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m\}$ is a finite set of vectors in $\mathbb{R}^n$ then the span of U is the set of all linear combinations of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m$

$\text{Span } U = \{\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \ldots + \alpha_m \underline{u}_m \mid \alpha_1, \alpha_2, \ldots \alpha_m \in \mathbb{R}\}$

Linear dependence

A set $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if there are numbers $\alpha_1, \alpha_2, \ldots \alpha_m$, not all zero s.t.

$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \ldots + \alpha_m \underline{u}_m = \underline{0}$

Basis

Given subspace S of $\mathbb{R}^n$, a set of vectors is called a basis of S if it is a linearly independent set that spans S.

Theorem: If the set $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m\}$ spans S, a subspace $\mathbb{R}^n$, then any linearly independent subset of S contains at most m vectors

distance between 2 vectors

$|\underline{a} - \underline{b}|$

Unit vector and Zero vector

$|\underline{a}| = 1 \qquad \underline{0} = (0, 0, 0, \ldots)$

orthogonal / perpendicular

iff $\underline{a} . \underline{b} = 0$

Linear combination

Given vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3, \ldots \underline{u}_m \in \mathbb{R}^n$ and reals $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ any vector of form

$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \ldots + \alpha_m \underline{u}_m$

is a linear combination of the given vectors

Subspaces

A subspace of $\mathbb{R}^n$ is a nonempty subset S of $\mathbb{R}^n$ with the following properties.

1. $\underline{u}, \underline{v} \in S \Rightarrow \underline{u} + \underline{v} \in S$
2. $\underline{u} \in S \; \lambda \in \mathbb{R} \Rightarrow \lambda \underline{u} \in S$

Linear independence

A set $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m\}$ of vectors in $\mathbb{R}^n$ is linearly independent if $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \ldots + \alpha_m \underline{u}_m = 0 \Rightarrow$

$\Rightarrow \alpha_i = 0 \;\; \forall i \in \{1..m\}$

Dimension

dimension of a subspace of $\mathbb{R}^n$ is the number of vectors in the basis for the subspace.

Constructing basis

remove vectors from subspace if they are a linear combination of other subspaces. This creates basis.

Matrices

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Matrix addition

$A+B$, simply sum each element to the other corresponding element.

Both matrices must have the same order

Multiplying matrices

order: $n \times m \cdot m \times k$

- must match
- order of resulting matrix

e.g. $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \times \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \begin{pmatrix} ag+bh+ci \\ dg+eh+fi \end{pmatrix}$

Properties of matrix mult.

1. $(AB)C = A(BC)$

2a. $A(B+C) = AB+BC$

2b. $(A+B)C = AC+BC$

3. $IA = A = AI$
4. $0A = 0 = A0$

5a. $A^pA^q = A^{p+q} = A^qA^p$

5b. $(A^p)^q = A^{pq}$

Matrices & linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$\equiv$ $A = [a_{ij}]\ m \times n$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$A\underline{x} = \underline{b}$

Properties of Addition and Scalar multiplication

1. $A+(B+C) = (A+B)+C$
2. $A+0 = A = 0+A$
3. $A+(-A) = 0 = (-A)+A$
4. $A+B = B+A$
5. $(\lambda+\mu)A = \lambda A + \mu A$
6. $\lambda(A+B) = \lambda A + \lambda B$
7. $\lambda(\mu A) = (\lambda\mu)A$

Diagonal matrices

Must be square matrix.

only \ diagonal has nonzero values

e.g. identity matrix

Transpose

If $A = [a_{ij}]_{m \times n}$ then $A^T = [a'_{ij}]_{n \times m}$

where $a'_{ij} = a_{ji}$

properties

1. $(A^T)^T = A$ 2. $(A+B)^T = A^T + B^T$

3. $(\lambda A)^T = \lambda A^T$ 4. $(AB)^T = B^TA^T$

Matrix inverse

B is inverse of A iff.

$$AB = I = BA$$

Not all matrices have an inverse

Elementary row operations & inverse

If a sequence of elementary row operation turns A to I then the same operations on I gives you A^{-1}

Inverse to solve linear equations

$A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow$

$\Rightarrow I\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$

Matrices 2

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Cofactors

Given a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

To get cofactor A_{11} remove row a_{12} and a_{21}, find determinant of resulting matrix

Do this for all A_{ij} then Apply ~~matrix~~ this matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & & \cdots \\ + & - & + & & \vdots \\ - & + & - \\ \vdots & & \vdots & \cdots \end{bmatrix}$$

Linear transformation

A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if

$T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$

$T(\lambda \underline{u}) = \lambda T(\underline{u})$

$\forall\ \underline{u}, \underline{v} \in \mathbb{R}^m$ and $\forall\ \lambda \in \mathbb{R}$

Characteristic equation

$$|A - \lambda I| = 0$$

Eigenvalues and Eigenvectors

Use characteristic equation to find eigenvalues

then use equation:

$$A\underline{r} = \lambda \underline{r}$$

to find eigenvector $\underline{r}$ that corresponds to eigenvalue λ

Determinant

determinant of $n \times n$ matrix $A = [a_{ij}]$ is

$|A| = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$

where A_{ij} is the ij^{th} cofactor of A.

determinant and inverse

A square matrix is invertible iff its determinant is non-zero

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$$

Adjoint $(\mathrm{adj}(A))$

transpose of its matrix of cofactors

Every matrix defines a linear transformation.

Suppose T is a matrix ~~transformation~~ transformation s.t. $T(\underline{x}) = A\underline{x}$ then:

$$\begin{aligned} T(c\underline{u} + d\underline{v}) &= A(c\underline{u} + d\underline{v}) \\ &= A(c\underline{u}) + A(d\underline{v}) \\ &= cA\underline{u} + dA\underline{v} \\ &= cT(\underline{u}) + dT(\underline{v}) \end{aligned}$$

$\therefore$ linear transformation

Diagonalising a matrix

Let a have eigenvectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and have a matrix V have column vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ then

$V^{-1}DV = A$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

Sequences and Series - CS131

Sequence

Infinite list of numbers

Combination Rules for Convergent Sequences

Sum rule:

$a_n + b_n \to \alpha + \beta$

Scalar multiple rule:

$\lambda a_n \to \lambda \alpha$

product rule

$a_n b_n \to \alpha \beta$

reciprocal rule

$1/a_n \to 1/\alpha$

quotient rule

$b_n/a_n \to \beta/\alpha$

'hybrid' rule

$b_n c_n/a_n \to \beta\gamma/\alpha$

Basic Convergent Sequences

$\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $p>0$

$\lim_{n\to\infty} c^n = 0$ for any c where $|c|<1$

$\lim_{n\to\infty} c^{1/n} = 1$ for any $c>0$

$\lim_{n\to\infty} n^p c^n = 0$ for $p>0$ and $|c|<1$

$\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for any $c \in \mathbb{R}$

$\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n = e^c$ for any $c \in \mathbb{R}$

non-homogenous

1. Find general solution h_n
2. Find particular solution p_n
3. $x_n = h_n + p_n$

Limit of a Sequence

A sequence (a_n) of real numbers is said to converge to a limit $l \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N with $|a_n - l| < \epsilon$ $\forall n > N$.

Basic properties of Convergent Sequences

1. A convergent sequence has a unique limit
2. If $a_n \to l$, then every subsequence of (a_n) also converges to l.
3. If $a_n \to l$ then $|a_n| \to |l|$
4. SQUEEZE RULE
   If $a_n \to l$ and $b_n \to l$ and $a_n \leq c_n \leq b_n$ then $c_n \to l$
5. A convergent sequence (a_n) is bounded, i.e., there exists a $B>0$ with $-B \leq a_n \leq B$ $\forall n$.
6. Any increasing sequences bounded above or decreasing sequence bounded below converges.

Solving homogenous recurrences

$x_n + a x_{n-1} + b x_{n-2} = 0$

Let λ_1, λ_2 be solutions to $\lambda^2 + a\lambda + b = 0$.

if $\lambda_1 \neq \lambda_2$, $A\lambda_1^n + B\lambda_2^n$

else $A\lambda_1^n + Bn\lambda_2^n$

Sequences and series 2 - CS131

Series

A Series is summing together a sequence

$S_n = \sum_{n=0}^{\infty} a_n$

Comparison Test

Suppose $0 \leq a_n \leq b_n$ for every n. Then:

1. If $\sum b_n$ converges then so does $\sum a_n$
2. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Ratio Test

If $|a_{n+1}/a_n| \to L$ then

1. If $0 \leq L < 1$ then $\sum a_n$ converges
2. If $L > 1$ then $\sum a_n$ diverges
3. If $L=0$ the test is inconclusive

Basic Divergent Series

$\sum \frac{1}{n^k}$ diverges for any $k \leq 1$

Power Series

$\sum a_n x^n$

General Binomial Theorem

For any rational q and $x \in (-1;1)$ then

$(1+x)^q = \sum_{n=0}^{\infty} \binom{q}{n} x^n$

Where

$\binom{q}{n} = \frac{q(q-1)\cdots(q-(n-1))}{n!}$

Basic properties of convergent Series

Sum Rule

If $\sum a_n$ converges to S and $\sum b_n$ converges to t then, $\sum(a_n+b_n)$ converges to $S+t$.

Multiple rule

If $\sum a_n$ converges to S and ~~$\sum b_n$ converges to t~~ $\lambda \in \mathbb{R}$ then $\sum \lambda a_n \to \lambda S$.

Sequence of a Convergent Series

If $\sum a_n \to S$ then $a_n \to 0$.

Modulus rule

If $\sum |a_n|$ converges then $\sum a_n$ converges

Basic convergent Series

$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ Converges for any r with $|r|<1$

$\sum \frac{1}{n^k}$ Converges for any $k>1$

$\sum n^k r^n$ Converges for $k>0$ and $|r|<1$

$\sum_{n=0}^{\infty} \frac{c^n}{n!} = e^c$ for any $c \in \mathbb{R}$

Radius of Convergence

The radius of convergence of a power series $\sum a_n x^n$ is the number $R \geq 0$ s.t. the series converges whenever $|x|<R$ and diverges whenever $|x|>R$

Calculus - CS131

Combination rules for limits

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$ then

Sum rule

$\lim_{x \to a}(f(x) + g(x)) = l + m$

multiple rule

$\lim_{x \to a} \lambda f(x) = \lambda l$ $(\lambda \in \mathbb{R})$

product rule

$\lim_{x \to a} f(x) g(x) = lm$

quotient rule

$\lim_{x \to a} f(x)/g(x) = l/m$

Squeeze rule

If $f(x) \leq g(x) \leq h(x)$ for $x \neq a$

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$

Differentiation

f' = derivative of f.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Combination rules for derivatives

If f and g are differentiable

Sum rule

$(f+g)' = f' + g'$

multiple rule

$(\lambda f)' = \lambda f'$

product rule

$(fg)' = fg' + f'g$

Continuous function

Let $f: D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$.

we say f is continuous at a point $a \in D$ if $\lim_{x \to a} f(x)$ exists and $= f(a)$.

we say f is continuous if it is continuous at a, $\forall a \in D$.

Intermediate value theorem

If $f: [a;b] \to \mathbb{R}$ is a continuous function, and $f(a)$ and $f(b)$ have opposite signs then $f(c) = 0$ for some $c \in (a;b)$.

minimum/maximum point

minimum of $f: [a,b] \to \mathbb{R}$ is a point $m \in [a,b]$ with $f(x) \geq f(m)$ $\forall x \in [a,b]$

maximum of $f: [a,b] \to \mathbb{R}$ is a point $M \in [a,b]$ with $f(x) \leq f(M)$ $\forall x \in [a,b]$

Theorem

f is differentiable at $a \Rightarrow$ f is continuous at a.

quotient rule

$(f/g)' = (f'g - fg')/g^2$

Chain rule

$(g \circ f)'(x) = g'(f(x)) f'(x)$

Calculus 2 - CS131

Stationary points

A point a where
$f'(a) = 0$

Rolle's theorem

If $f : [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b) and $f(a) = f(b)$, then there is a point $c \in (a,b)$ with $f'(c) = 0$

Second derivative test

If $f'(x) = 0$ and $f''(x) < 0$ then x is a maximum
If $f'(x) = 0$ and $f''(x) > 0$ then x is a minimum

Partial derivatives

partial derivative of $f(x,y)$ is obtained by differentiating $f(x,y)$ with respect to either x or y

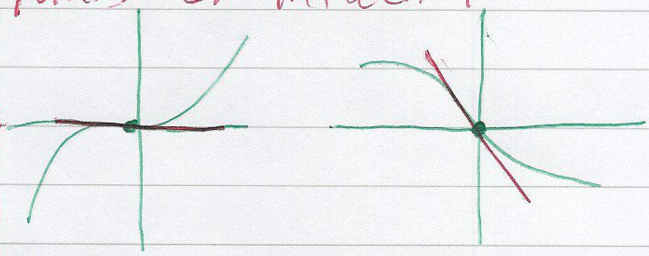
$$\frac{\partial f(x,y)}{\partial x} \qquad \frac{\partial f(x,y)}{\partial y}$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y}$$

Implicit differentiation

$$x^2 + y^2 = 1$$
$$\Downarrow$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\Downarrow$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Points of inflection



Stationary   non-Stationary

Mean value theorem

If f is continuous on $[a,b]$ and differentiable on (a,b) then there is a $c \in (a,b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Curve sketching

1. Find stationary points
2. Find nature of stationary points
3. Find x when $f(x) = 0$
4. determine behaviour of $f(x)$ as $x \to \pm\infty$.
5. Investigate nature as $f(x) \to \pm\infty$.

L'Hôpitals rule

Suppose $f(a) = g(a) = 0$ for some point a.

If $\lim_{x \to a} f'(x)/g'(x)$ exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Calculus 3 - CS131

Inverse function

$$f^{-1}(y) = x \iff f(x) = y$$

Integration

$$A = \int_a^b f(x)\,dx$$

And

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)(x_r - x_{r-1})$$

Calculation of definite Integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If $\frac{d}{dx}F(x) = f(x)\ \forall x$ and f is continuous, then $\int f(x)\,dx = F(x) + C$ for some constant C

Logarithmic functions

$$\int_1^x \frac{1}{t}\,dt = \log(x)$$

$\log(1) = 0$ $\quad \frac{d(\log x)}{dx} = \frac{1}{x}$

$\log(xy) = \log x + \log y$

$\log\left(\frac{x}{y}\right) = \log x - \log y$

differentiation of inverse function

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

or

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

properties of definite integrals

Sum rule:

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Multiple rule:

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

Split integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Theorem

If $f(x) \leq g(x)$, for every $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Exponential functions

$y = \exp(x) \iff x = \log y$

$\exp(0) = 1$ $\quad \frac{d(\exp(x))}{dx} = \exp(x)$

$\exp(x+y) = \exp(x)\exp(y)$

Taylor's theorem

f is an $n+1$-differentiable function on an open interval containing a and x. Then $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + R_n(x)$

Where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

Taylor's Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Calculus 4 - CS131

First order ODEs

Seperable equations

$\frac{dy}{dx} = f(x)g(y) \implies$

$\implies \int \frac{1}{g(y)} dy = \int f(x)\, dx$

Homogenous equations

$\frac{dy}{dx} = f(y/x)$

$\Downarrow$ $v = y/x$

$\log(x) = \int \frac{1}{f(v)-v} dv$

Linear equations

$\frac{dy}{dx} + P(x)y = Q(x)$

1. $Q(x) = 0$ then
   $y = e^{-\int P(x)dx}$
2. $Q(x) \neq 0$ then
   $I(x) = e^{\int P(x)dx}$
   $yI(x) = \int Q(x)I(x)\, dx$

Second order ODEs

$ay'' + by' + cy = f(x)$

Let $y = P(x)$ be a particular Solution

$y = H(x)$ be a general Solution.

$y = H(x) + P(x)$ is the general Solution of

$ay'' + by' + cy = f(x)$

Homogenous equations

$ay'' + by' + cy = 0$

$\Downarrow$

$a\lambda^2 + b\lambda + c = 0$

Cases:

1. 2 real roots
   $Y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2. 1 shared root
   $Y = (A+Bx)e^{\lambda x}$
3. Complex roots
   $Y = e^{\alpha x}(A\cos \beta x + B\sin(\beta x))$

particular Solutions

$f(x)$	form of particular Solution
$e^{\alpha x}$	$y = Ae^{\alpha x}$ if α is not root. $y = Axe^{\alpha x}$ if α non-repeated root $y = Ax^2e^{\alpha x}$ if α repeated root
n degree Polynomial	if 0 is not a root - n degree if 0 is non repeated root - $n+1$ degree if 0 is repeated root - $n+2$ degree
$A\cos\alpha x + B\sin\alpha x$	if $i\alpha$ not a root $y = C\cos\alpha x + D\sin\alpha x$ otherwise $y = x(C\cos\alpha x + D\sin\alpha x)$