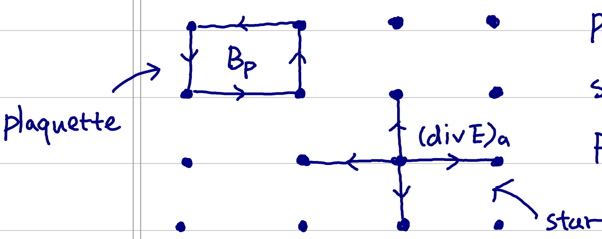
* Presentation at group meeting

1. Lattice QED

Consider a square lattice, we can define electric field and magnetic vector potential as E_{ab} and A_{ab}, and they are anti-symetry. $E_{ab} = -E_{ba}$, $A_{ab} = -A_{ba}$. And $[A_{ab}, E_{ab}] = i$



From static EM, we know that

$$\vec{B} = \vec{\nabla} \times \vec{A} , \quad \vec{\nabla} \cdot \vec{E} = \rho$$

Then on lattice, there are

$$B_p = (\vec{\nabla} \times \vec{A})_p = \sum_{l \in \partial(p)} A_l , \quad (\mathrm{div}\, E)_a = \sum_{b \in nn(a)} E_{ab} = q_a \quad (\text{Gauss's Law})$$

We can write the Hamiltonian as

$$H = \underbrace{-K \sum_p \cos(\vec{\nabla} \times \vec{A})}_{\text{magnetic term}} + \underbrace{K' \sum_a (\mathrm{div}\, E)_a^2}_{\text{charge term}} + \underbrace{\frac{V}{2} \sum_{\langle ab \rangle} E_{ab}^2}_{\text{electric term}}$$

When B_p is a small fluctuation, we can Taylor expand the first term

$$H \approx K' \sum_a q_a^2 + \frac{V}{2} \sum_{\langle ab \rangle} E_{ab}^2 + \frac{K}{2} \sum_p B_p^2$$

- In vacuum, there is no charge, this Hamiltonian becomes (in continuum limit)

$$H \approx \int d^3x \left[\frac{\varepsilon}{2} |\vec{E}|^2 + \frac{1}{2\mu} |\vec{B}|^2 \right]$$

which is familiar to us.

- In the limit $V \to 0$, both $(\mathrm{div}\, E)_a$ and B_p commute with H. So the ground state should be found just by choosing $q_a = (\mathrm{div}\, E)_a = 0$ in all sites and $B_p = 0$ in all plaquettes, which is analogous to toric code.

2. Toric code (2D surface code)

Lattice QED is a $U(1)$ gauge theory, which means locally one can change A_{ab} by a scalar and the theory leaves unchanged. Kitaev's toric code is a kind of $\mathbb{Z}_2$ gauge theory, we can have a map from one to each other.

$$\sigma_i^z \sim e^{iA_{ab}}, \text{ with } A_{ab} = 0, \pi , \quad P \sim \cos(\vec{\nabla} \times \vec{A})$$

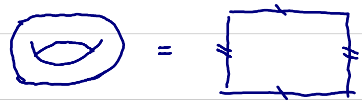
$$\sigma_i^x \sim e^{i\pi E_{ab}}, \text{ with } E_{ab} = 0, \pi , \quad S \sim e^{i\pi (\mathrm{div}\, E)_a}$$

Then, kitaev's toric code can be written as

Hamiltonian

$$H_{tc} = -K\sum_{f}\prod_{\ell\in\partial f}\sigma^z_\ell - K'\sum_{v}\prod_{\ell:v\in\partial\ell}\sigma^x_\ell = -K\sum_f P_f - K'\sum_v S_v$$

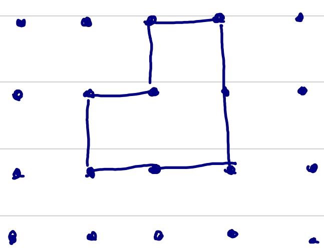
plaquette　　　　star



periodic boundary condition.

g.s.

And we have $\boxed{P_f|\psi\rangle = |\psi\rangle}$, $S_v|\psi\rangle = |\psi\rangle$, so the plaquette and star operators are also called stabilizers. If we consider the state $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ as light link and the state $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ as dark link, then the ground state of toric code is all the dark loops configuration. (trivial loops, quantum loop gas)



Wilson loop is defined as

$$W_c = Tr(P\exp(i\oint_c A_\mu dx^\mu))$$

topologial.

In field theory, the behaviour vacuum expectation value of Wilson loop is always used see the phase where the theory is (e.g. in confined phase, $\langle W\rangle \sim e^{-A}$, in deconfined phase, $\langle W\rangle \sim e^{-L}$).

For a topological theory, the VEV of Wilson loop should be constant. Because a topologial theory is not depend on metric, then two topologically equivalent wilson loop should give the same result.

As we have known, Wilson loop in toric code is defined as

$$W_c = \prod_c \sigma^z_c$$

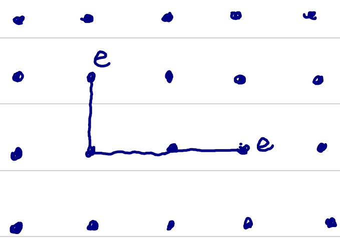
By the property of stabilizer, we get

$$\langle W_c\rangle_{trivial} = 1.$$

p.s. one can also see the topological property by its effective field theory, which is called Chern-Simon theory. But I will not talk about that in this presentation.

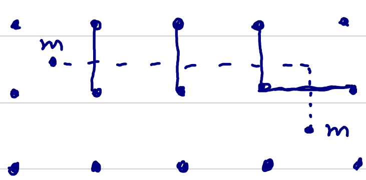
excitation: There are two kinds of excitations in toric code.

① electric particle, which can be detected by star operator on the end.



They should appear in pairs and each e particle costs energy $2k'$.

② magnetic particle. which can be detected by plaquette operator on the end.



Each costs energy $2k$.

People usually set $2k > 2k'$. Then excitations will always be e-type and the ground state becomes all the 1-7 loop configuration. (These excitations are also called anyon)

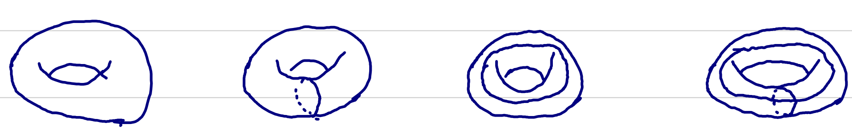
Degeneracy: What about the non-trivial loops?

① They are closed loops, which means there are no excitations (g.s)

② Such loops are not equivalent to trivial loops.

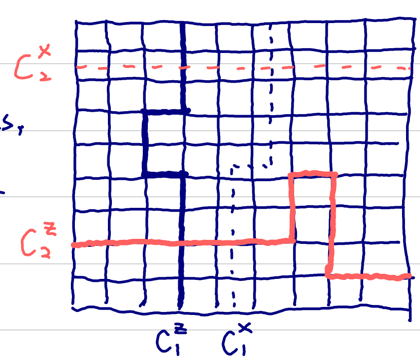
So there is ground states degeneracy!

There is two kinds of non-trivial loops on torus, which means the degree of degeneracy is 4. Then it means toric code can encode 2 logical qubits.



$|\bar{0}\bar{0}\rangle$ $|\bar{0}\bar{1}\rangle$ $|\bar{1}\bar{0}\rangle$ $|\bar{1}\bar{1}\rangle$

On planar code. it drawn as



$\bar{X}_1 = \prod_{C_1^x} \sigma^x$ $\quad \bar{Z}_1 = \prod_{C_1^z} \sigma^z$

$\bar{X}_2 = \prod_{C_2^x} \sigma^x$ $\quad \bar{Z}_2 = \prod_{C_2^z} \sigma^z$

More generally speaking, this is the first homology group on torus, which is $Z_2 \oplus Z_2$. We will not talk about this in the presentation.

For now, we can see for a topological phase, usually it will have those properties, which are ground state degeneracy, gapped excitations (usually have fractional statistics). (It may also have gapless edge mode, but not for toric code.)

3. 3D topological code.

- In 3D topological codes, logical operators become strings and membranes. Usually people pick logical Z operator as string and logical X operators as membrane.
- Transversal Gate: Unitary operators that transform encoded states by acting seperately on suitable subsystems, in the simplest case independently on each qubit.
- To achieve universal quantum computation, one need Clifford gates as well as non-Clifford gates. The simplest way to have non-Clifford gate (In this presentation CCZ) is to build a 3D topological codes.

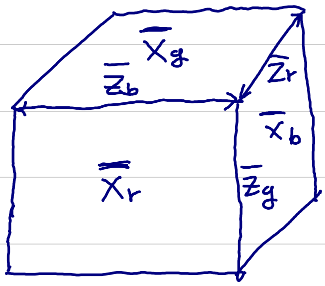
The generators of Clifford group are $Cl_n = \{U : UPU^\dagger \in P_n, \forall P \in P_n\} = \langle H_i, P_i, CNOT_{ij} \rangle / U(1)$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad CNOT = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

- Then is the model construction.

* Proof of transversal CCZ

Definition • Let C^c be an $m \times n$ binary matrix with m equals to the number of X stabilizers of SC_c.



$$m \overbrace{\begin{pmatrix} 1 & \cdots & 1 \\ & \cdots & \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}}^{n}$$

- Each row of C^c has a 1 at column j if the stabilizer generator correspond to that row act non-trivially on qubit q_j.
- Let G_0^c be the linear span of all rows of C^c
- For each code, we choose a canonical $\bar{X}_c$ operator which act on one of the c-boundaries of the lattice. Let X^c be an n-bit binary vector describing the support of $\bar{X}_c$. X^c has 1 at position j if $\bar{X}_c$ acts non-trivially on q_j.
- Let G_1^c be the coset $\{X^c + g : g \in G_0^c\}$ (In homology theory, g is cocycles.)
- Then the states can be written as

$$|\bar{\alpha}\rangle_c = \frac{1}{\sqrt{|G_\alpha^c|}} \sum_{g \in G_\alpha^c} |g\rangle_c \quad , \alpha \in \{0,1\}$$

Proof • The initial state is

$$|\overline{\alpha\beta\gamma}\rangle_{rgb} = \sum_{t \in G_\alpha^r, u \in G_\beta^g, v \in G_\gamma^b} |t\rangle_r |u\rangle_g |v\rangle_b$$

If we act $\overline{CCZ}$ gate on it, we get

$$\overline{CCZ}\,|\overline{\alpha\beta\gamma}\rangle_{rgb} = \sum_{t \in G_\alpha^r, u \in G_\beta^g, v \in G_\gamma^b} CCZ^{\otimes n} |t\rangle_r |u\rangle_g |v\rangle_b$$

$$= \sum_{t \in G_\alpha^r, u \in G_\beta^g, v \in G_\gamma^b} (-1)^{|t \cdot u \cdot v|} |t\rangle_r |u\rangle_g |v\rangle_b$$

only $|111\rangle$ can get a (-1) factor, which means the parity of $|t\cdot u\cdot v|$ can decide the signature of the state.

$$t\cdot u\cdot v = (\alpha X^r + t')\cdot(\beta X^g + u')\cdot(\gamma X^b + v')$$
$$= \alpha\beta\gamma X^r X^g X^b + \cdots$$

only $X^r X^g X^b$ term survive.

- Transversality of $\overline{CZ}$ gate can be proved by setting one of the logical qubit to be fixed at $|\bar{0}\rangle$ and $|\bar{1}\rangle$.