Notes on "Fracton Phases of Matter"

- Fracton models are often classified as "type I" if they possess stable mobile bound states, and as "type-II" if all mobile bound states can decay directly into the vacuum.
- It's important to notice that fracton physics has a very concrete realization as the topological lattice defects of ordinary crystals.

* Tensor gauge theory and higher moment conservation laws

- The gauge sector must be invariant under

$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha \quad (\text{scalar charge theory})$$

One can define a symmetric tensor electric field E_{ij} as the canonical conjugate to A_{ij}

$$[A_{ij}(x), E_{kl}(y)] = i(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})\,\delta(x-y)$$

- The magnetic field operator: $B_{ij} = \varepsilon_{ikl}\partial^k A^l{}_j$ (Traceless, $B^i{}_i = 0$)

One can also find $\partial_i B^{ij} = 0$. If A_{ij} is compactified, this condition relaxes to $\partial_i B^{ij} = \tilde{\rho}_j$, in which $\tilde{\rho}_i$ represents the density of vector-flavored magnetic monopole. (This is in contrast to two-dimensional compact theories, which are destablilized by instantons)

- We can then write down the Hamiltonian of such theory

$$H = \int d^3x \, \frac{1}{2}(E^{ij}E_{ij} + B^{ij}B_{ij})$$

Notes: $(E^i{}_i)^2$ term turns out to be an irrelavent perturbation to this fixed point. There are five gauge massless modes, may be regarded as "graviton".

- The generalised Gauss's Law : $\partial_i \partial_j E^{ij} = \rho$. (charge and dipole moment are conserved)
- Fracton Field Theory

One can first write down the theory with global symmetry: $\Phi \to e^{i\alpha}\Phi$, $\Phi \to e^{i\vec{\lambda}\cdot\vec{x}}\Phi$

The Lagrangian is: $\mathcal{L} = |\partial_t \Phi|^2 - m^2|\Phi|^2 - g|\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi|^2 - g'\Phi^{*2}(\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi)$

We can then gauge it $\Phi \to e^{i\alpha(x,t)}\Phi$, the gauge-covariant derivative is

$$\Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi - iA_{ij}\Phi^2$$

- Tensor Chern-Simons Theories

(Such field theory must both similar to, and qualitatively different from, TQFTs, in which the details of the underlying lattice are unimportant, and universal topology physics emerges.)

Fractonic flux attachment procedure introduces a non-commutative gauge structure and thus creates a deconfined $U(1)$ fracton theory.

This model shares some important features with chiral 2+1 D CS theory.

1) It creates self-statistical interactions between charged excitations.

2) It's gauge invariant only up to a boundary term, which implies that their boundaries host gapless surface states that connot be realised in 2 dimensions with subsystem symmetries.

- Generalized Witten Term

(Attaching electric charge to magnetic monopoles (Witten Effect) and leading to the Chern-Simons theory on the boundary)

$$\theta \vec{E} \cdot \vec{B}$$

* Fractons in Sovable Spin Models

- Type-I Fracton Model: X-Cube Model

Hamiltonian: $H = -\sum_v (A_v^x + A_v^y + A_v^z) - \sum_c B_c$

where $B_c = \prod_{\partial c} X$, $A_v^z = \prod_{+xy} Z$

The ground state degeneracy is $\log_2(GSD) = 2L_x + 2L_y + 2L_z - 3$ with periodic boundary condition.

Excitation: Fractons (Z on membrane)

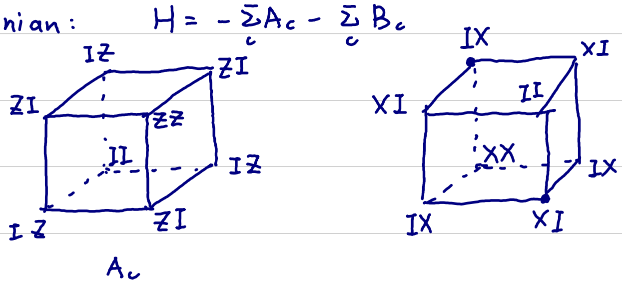
Lineon (X on a line)

Subregion Entanglement Entropy: Area Law + linear correction.

- Type - II Fracton Model: Haah's Code

Hamiltonian: $H = -\sum_c A_c - \sum_c B_c$

where



Excitations: Application of a single spin operator creates four quasiparticles at the corners of a tetrahedron.

Ground state degeneracy: $\log_2(GSD) < 4L$

Subregion Entanglement Entropy: Area Law + linear correction.

- Higgsing

The scalar charge theory becomes non-fractonic once Higgsed while a modified 'hollow' tensor gauge theory remains fractonic even upon Higgsing

?

some references about it:

Phys. Rev. B 98, 035111 (2018), Phys. Rev. B 97, 235112 (2018)

- Geometric Aspects

Spatial curvature can induce a stable ground state degeneracy for the X-cube model.

1) Coupled Layer Construction: Phys. Rev. B 95, 245126 (2017)

arXiv: 1701.00762

arXiv: 1910.04765 (Haah's code construction)

2) Cage-net Model: It generalises the coupled layer construction from stacks of Toric code to stacks of other string-net states.

Phys. Rev. X 9, 021010 (2019)

(an interesting example: stacking 2D doubled Ising models)

3) String - Membrane - Net Construction

This model is shown equal to most of the known foliated fracton models up to trivial degrees of freedom and local unitary transformations.

* Foliation (type -I)

- Basic Idea: Starting from a model with a larger system size, we can apply a finite depth local unitary transformation and map the model to a smaller system size together with decoupled layers of 2D gapped states.

  (Shirley, Slage and Chen)

  We define two foliated fracton models to have the same foliated fracton order (FFO) if they can be related through a finite depth local unitary transformation upon the addition of decoupled stacks of 2D layers of gapped states.

- In particular, we construct a lattice by embedding a large number of transversely intersecting surfaces, referred to as leaves, into the 3-manifold M.

- Universal properties

- entanglement entropy: S_{FFO} (wire frame)
- fractional excitations: Quotient super-selection sectors

  (mod out non-fractional excitations and dimension-2 fractional excitations which come from foliation layers)

- Twisted Phases (twisted $\mathbb{Z}_2$ gauge theory)

* Realization in Elasticity Theory

- Fracton - Elasticity Duality

  - The connection between fractons and lattice defects can be seen by studying the conventional elasticity theory of two-dimensional cystals, which turns out to have an exact duality mapping with the scalar charge fracton tensor gauge theory (enriched by an extra global symmetry)

  - To lowest order in derivatives, the most general low-energy effective action for a crystal can be written as:

$$S = \int d^2x\, dt\, \frac{1}{2}\left((\partial_t u_i)^2 - C^{ijkl} u_{ij} u_{kl} \right)$$

    where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

    The anti-symmetric strain $\varepsilon^{ij}\partial_i u_j$ doesn't appear to lowest order, which is a consequence of the underlying spontaneously broken rotational symmetry.

    This is the action for two gapless modes.

    Disclination defects can be represented as $\varepsilon^{il}\varepsilon^{jk}\partial_l \partial_k u_{ij} = \rho$

    where ρ is disclination density.

    A crystal also hosts stable dipolar bound states of disclinations, which correspond to dislocation defects.

    In elasticity theory, there is a enriched global $U(1)$ symmetry, which restrict the mobility of dipoles. (Phys. Rev. Lett. 120, 195301 (2018))

- Extensions

  - In 3D, there are "fractonic lines". (Rank four tensor gauge theory)

    The symmetry of indices is $(ij) \leftrightarrow (kl)$ symmetric

    $i \leftrightarrow j$, $k \leftrightarrow l$ anti-symmetric

    So it's a combination of symmetric tensor gauge theory (point like fracton) and higher form theory (extended objects).

    The Gauss's Law becomes: $\partial_i \partial_k E^{ijkl} = \rho^{jl}$

- Supersolid. (Phys. Rev. B 100, 094105 (2019))

superfluid : Spontaneous symmetry broken $U(1)$

solid : Enriched global $U(1)$ (symmetric tensor gauge theory)

combining fracton - elasticity duality and particle - vortex duality.