HW4 Conic Programming

HW4_1

Theorem1：已知 $\xi(x) \in [0, +\infty)$ 为凸 w.r.t x，那么 $\xi^2(x)$ 也为凸。

Proof：因为 $\xi(x)$ 为凸，则 $\forall x_1 \neq x_2 \in \mathbb{R}^1$ 有

$$\xi(\frac{x_1+x_2}{2}) \leq \frac{\xi(x_1)+\xi(x_2)}{2} \tag{1}$$

又 $\xi(x) \in [0, +\infty)$，对不等式两边取平方可得

$$\xi^2(\frac{x_1+x_2}{2}) \leq \frac{\xi^2(x_1)+2\xi(x_1)\xi(x_2)+\xi^2(x_2)}{4} \tag{2}$$

又

$$\begin{aligned}
&\frac{\xi^2(x_1)+2\xi(x_1)\xi(x_2)+\xi^2(x_2)}{4} - \frac{\xi^2(x_1)+\xi^2(x_2)}{2} \\
&= -\frac{\xi^2(x_1)-2\xi(x_1)\xi(x_2)+\xi^2(x_2)}{4} \\
&= -\frac{(\xi(x_1)-\xi(x_2))^2}{4} \\
&\leq 0
\end{aligned} \tag{3}$$

由(2)和(3)可得

$$\xi^2(\frac{x_1+x_2}{2}) \leq \frac{\xi^2(x_1)+\xi^2(x_2)}{2}$$

所以 $\xi^2(x)$ 也为关于 x 的凸函数。证毕。

已知凸问题

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \leq 0 \\
& h(x) = 0
\end{aligned}$$

即 $f(x), g(x), h(x)$ 均为 x 的凸函数，又Positive weighted sum, point-wise max和对非负函数值的平方操作能够保持原函数的凸性，所以 PHR 增广拉格朗日函数

$$\mathcal{L}_\rho(x, \lambda, \mu) := f(x) + \frac{\rho}{2}\left\{\left\|h(x)+\frac{\lambda}{\rho}\right\|^2 + \left\|\max\left[g(x)+\frac{\mu}{\rho}, 0\right]\right\|^2\right\}$$

为关于 x 的凸函数。

HW4_2

对于 low-dimensional QP 问题

$$\begin{aligned}
&\min_{x\in\mathbb{R}^n} \frac{1}{2}x^{\mathrm{T}}M_{\mathcal{Q}}x + c_{\mathcal{Q}}^{\mathrm{T}}x, \\
&\text{s.t. } A_{\mathcal{Q}}x \leq b_{\mathcal{Q}}
\end{aligned}$$

$M_{\mathcal{Q}} \succeq 0$ 对称半正定。为了使问题能够使用 SDQP 进行求解，即使 $M_{\mathcal{Q}}$ 近似为一个对称正定矩阵。对于正定性，引入变量 x 的先验 $\overline{x}$ 使目标函数近似为

$$
\begin{aligned}
&\frac{1}{2}x^{\mathrm{T}}M_Q x+c_Q^{\mathrm{T}}x+\frac{1}{2\rho}\|x-\overline{x}\|^2\\
&=\frac{1}{2}x^{\mathrm{T}}M_Q x+c_Q^{\mathrm{T}}x+\frac{1}{2\rho}(x-\overline{x})^T(x-\overline{x})\\
&=\frac{1}{2}x^{\mathrm{T}}M_Q x+c_Q^{\mathrm{T}}x+\frac{1}{2\rho}(x^Tx-2\overline{x}^Tx+\overline{x}^T\overline{x})\\
&=\frac{1}{2}x^T(M_Q+\frac{1}{\rho}I)x+(c_Q-\frac{1}{\rho}\overline{x})^Tx+\frac{1}{2\rho}\overline{x}^T\overline{x}
\end{aligned}
$$

当 $\rho\to+\infty$ ，近似目标函数与原目标函数等价。可知 $M_Q+\frac{1}{\rho}I\succ 0$ 。则近似问题为

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^n}\frac{1}{2}x^T(M_Q+\frac{1}{\rho}I)x+(c_Q-\frac{1}{\rho}\overline{x})^Tx,\\
&\text{s.t. } A_Qx\le b_Q
\end{aligned}
\tag{4}
$$

通过反复求解问题(4)，并在每次求解前令 $\overline{x}=x^*$ ，其中 x^* 为上次求解得到的最优值，并逐渐增大 ρ ，使近似问题与原问题越来越接近。通过求解得到的最优值与给定的一致

```
# ck1201 @ ck1201-ubuntu20 in ~/workspace/Numerical_Optimization_in_Robotics/HW4/SD
$ ./sdqp_example
======================
iter times: 1
tol: 1.51422
Q_prox eigenvalues: 15.0828       1 2.91724
      optimal sol: -2.50965 0.163987    5.709
given optimal sol:  -1.06186 -0.958763  0.979381
      optimal obj: 15.4365
given optimal obj: -3.04124
cons precision:      -10.582     -16.4534     -9.83762     -5.52733 -4.44089e-15

======================
iter times: 2
tol: 2.84458
Q_prox eigenvalues: 14.1828 2.01724     0.1
      optimal sol:  -1.08431 -0.964583   1.01347
given optimal sol:  -1.06186 -0.958763  0.979381
      optimal obj: -3.02425
given optimal obj: -3.04124
cons precision:   -0.0623625      -4.9207     -6.15514 -8.88178e-16     -6.06236

======================
iter times: 3
tol: 0.0237818
Q_prox eigenvalues: 14.0928    0.01 1.92724
      optimal sol:  -1.06186 -0.958761   0.97938
given optimal sol:  -1.06186 -0.958763  0.979381
      optimal obj: -3.04124
given optimal obj: -3.04124
cons precision: -5.68434e-14     -4.83504     -6.14434  3.55271e-14           -6

======================
iter times: 4
tol: 2.40306e-06
Q_prox eigenvalues: 14.0838   0.001 1.91824
      optimal sol:  -1.06186 -0.958763  0.979381
given optimal sol:  -1.06186 -0.958763  0.979381
      optimal obj: -3.04124
given optimal obj: -3.04124
cons precision: 1.13687e-13    -4.83505    -6.14433 1.13687e-13           -6
```

HW4_3

$$
\begin{aligned}
\min_{x\in\mathbb{R}^7}\quad & f^Tx\\
\text{s.t.}\quad & \|Ax+b\|\le c^Tx+d.
\end{aligned}
$$

其中， $f=[1,2,3,4,5,6,7]^T$ ， $b=[1,3,5,7,9,11,13]^T$ ， $c=[1,0,0,0,0,0,0]^T$ ， $d=1$ ，

$$
A=\begin{bmatrix}
7 &&&&&& \\
& 6 &&&&& \\
&& 5 &&&& \\
&&& 4 &&& \\
&&&& 3 && \\
&&&&& 2 & \\
&&&&&& 1
\end{bmatrix}
$$

易得

$$
\bar{A}x+\bar{b}=\begin{pmatrix} c^{\mathrm{T}} \\ A \end{pmatrix} x+\begin{pmatrix} d \\ b \end{pmatrix}=\begin{pmatrix} c^{\mathrm{T}}x+d \\ Ax+b \end{pmatrix}\in\mathcal{Q}^{8}
$$

增广拉格朗日函数为

$$
\mathcal{L}_{\rho}(x,\mu)=f^{\mathrm{T}}x+\frac{\rho}{2}\left\| P_{\mathcal{K}=\mathcal{Q}^{n}}\left(\frac{\mu}{\rho}-\bar{A}x-\bar{b}\right)\right\|^{2}
$$

其中,

$$
P_{\mathcal{K}=\mathcal{Q}^{n}}(v)=\begin{cases} 0, & v_0\le -\|v_1\|_2 \\ \frac{v_0+\|v_1\|_2}{2\|v_1\|_2}\left(\|v_1\|_2, v_1\right)^{\mathrm{T}}, & |v_0|<\|v_1\|_2 \\ v, & v_0\ge \|v_1\|_2 \end{cases}
$$

梯度为

$$
\nabla_x\mathcal{L}_{\rho}(x,\mu)=f-\rho\bar{A}^{T}\partial_B P_{\mathcal{K}=\mathcal{Q}^{n}}(v)\,P_{\mathcal{K}=\mathcal{Q}^{n}}\left(\frac{\mu}{\rho}-\bar{A}x-\bar{b}\right)
$$

其中,

$$
\partial_B P_{\mathcal{K}=\mathcal{Q}^{n}}(v)=\begin{cases} 0, & v_0\le -\|v_1\|_2 \\ \begin{bmatrix} \frac{1}{2} & \frac{v_1^T}{2\|v_1\|} \\ \frac{v_1}{2\|v_1\|} & \frac{v_0+\|v_1\|}{2\|v_1\|}I-\frac{v_0 v_1 v_1^T}{2\|v_1\|^3} \end{bmatrix}, & |v_0|<\|v_1\|_2 \\ I, & v_0\ge \|v_1\|_2 \end{cases}
$$

更新策略

$$
\begin{cases} x\leftarrow \operatorname{argmin}_x \mathcal{L}_{\rho}(x,\mu) \\ \mu\leftarrow P_{\mathcal{Q}^{8}}\left(\mu-\rho\left(\bar{A}x+\bar{b}\right)\right) \\ \rho\leftarrow \min[(1+\gamma)\rho,\beta] \end{cases}
$$

通过求解得到的最优值与给定的一致

```
# ck1201 @ ck1201-ubuntu20 in ~/workspace/Numerical_Optimization_in_Robotics/HW4/SOCP/build [16
$ ./socp_ALM
================================
iter times: 1
L-BFGS Optimization Returned: 0
      Minimized Cost: -283.429
given Minimized Cost: -156.959
      Optimal Variables:  0.161626 -0.619227  -1.25753  -2.28652  -4.19227  -8.71912  -28.0226
given Optimal Variables: -0.127286 -0.506097  -1.01317  -1.77744  -3.06097  -5.66462  -13.7682

================================
iter times: 2
L-BFGS Optimization Returned: 0
      Minimized Cost: -156.959
given Minimized Cost: -156.959
      Optimal Variables: -0.127286 -0.506097  -1.01317  -1.77744  -3.06097  -5.66462  -13.7682
given Optimal Variables: -0.127286 -0.506097  -1.01317  -1.77744  -3.06097  -5.66462  -13.7682
```