HW5 Miscellaneous Techniques

HW5_1 Traj Generation

1.1 Modeling

对于经过 $N+1$ 个给定点 $x_0, x_1, \ldots, x_N$ 的 N 段轨迹 $p_i(s), i=0,\cdots,N$ ，每段轨迹都是一个三次样条曲线

$$p_i(s) = a_i + b_i s + c_i s^2 + d_i s^3, s \in [0,1], i = 0, \cdots, N-1$$

根据轨迹之间的一阶和二阶连续以及边界条件可得

$$\begin{aligned} a_i &= x_i \\ b_i &= D_i \\ c_i &= 3\left(x_{i+1} - x_i\right) - 2D_i - D_{i+1} \\ d_i &= 2\left(x_i - x_{i+1}\right) + D_i + D_{i+1} \end{aligned}, i = 0, \cdots, N-1$$

let $\mathbf{x} = [x_0, x_1, \cdots, x_{N-1}, x_N]^T$, then

$$\mathbf{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & 0 \end{bmatrix}_{N \times N+1} \mathbf{x}$$

$$\mathbf{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & 0 \end{bmatrix}_{N \times N+1} \mathbf{D}$$

$$\mathbf{D} = \begin{bmatrix} D_0 \\ \vdots \\ D_N \end{bmatrix} = \mathbf{A}_D \mathbf{x}$$

where,

$$\mathbf{A}_D = 3 \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_D \\ \mathbf{0} \end{bmatrix}_{N+1 \times N-1} \begin{bmatrix} -1 & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & \\ & & & & -1 & 0 & 1 \end{bmatrix}_{N-1 \times N+1}$$

$$
\mathbf{D}_D = \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix}^{-1}_{N-1\times N-1}
$$

With respect to $\mathbf{c}$,

$$
\mathbf{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \mathbf{A}_c \mathbf{x}
$$

Where,

$$
\mathbf{A}_c = 3\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}_{N\times N+1} + \begin{bmatrix} -2 & -1 & & & & \\ & -2 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & -2 & -1 & \\ & & & & -2 & -1 \end{bmatrix}_{N\times N+1} \mathbf{A}_D
$$

With respect to $\mathbf{d}$,

$$
\mathbf{d} = \begin{bmatrix} d_0 \\ \vdots \\ d_{N-1} \end{bmatrix} = \mathbf{A}_d \mathbf{x}
$$

Where,

$$
\mathbf{A}_d = 2\begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \end{bmatrix}_{N\times N+1} + \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix}_{N\times N+1} \mathbf{A}_D
$$

1.1.1 Stretch Energy

$$
\mathrm{Energy}(x_1, x_2, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} \int_0^1 \left\| p_i^{(2)}(s) \right\|^2 \mathrm{d}s
$$

Where,

$p_i^{(2)}(s) = 2c_i + 6d_i s$

$\left\| p_i^{(2)}(s) \right\| = 4c_i^2 + 24c_i d_i s + 36d_i^2 s^2$

$E_i = \int_0^1 \left\| p_i^{(2)}(s) \right\|^2 \mathrm{d}s = 4c_i^2 + 12c_i d_i + 12d_i^2$

then

$$
\mathbf{E} = 4\mathbf{c}^T\mathbf{c} + 12(\mathbf{c}^T\mathbf{d} + \mathbf{d}^T\mathbf{d}) = \mathbf{x}^T(4\mathbf{A}_c^T\mathbf{A}_c + 12\mathbf{A}_c^T\mathbf{A}_d + 12\mathbf{A}_d^T\mathbf{A}_d)\mathbf{x} = \mathbf{x}^T\mathbf{A}_E\mathbf{x}
$$

whose gradient is

$$\frac{\partial \mathbf{E}}{\partial \mathbf{x}} = (\mathbf{A}_E + \mathbf{A}_E^T)\mathbf{x}$$

When it comes to two-dimensional case, let $\mathbf{x} = [x_0, \cdots, x_N, y_0, \cdots, y_N]^T_{2(N+1)\times 1}$, then

$$\mathbf{E} = \mathbf{x}^T \mathbf{A}_E' \mathbf{x} = \mathbf{x}^T \begin{bmatrix} \mathbf{A}_E & \\ & \mathbf{A}_E \end{bmatrix}_{2(N+1)\times 2(N+1)} \mathbf{x}$$

$$\frac{\partial \mathbf{E}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{A}_E + \mathbf{A}_E^T & \\ & \mathbf{A}_E + \mathbf{A}_E^T \end{bmatrix}_{2(N+1)\times 2(N+1)} \mathbf{x}$$

1.1.2 Potential

$$\text{Potential}(x_1, x_2 \ldots, x_{N-1}) = 1000 \sum_{i=1}^{N-1} \sum_{j=1}^{M} d_{i,j}$$

Where,

$$d_{i,j} = \begin{cases} min(d_{s,i,1}, \cdots, d_{s,i,s_j}) & \text{, if } A_{\mathcal{P},j} x_i \le b_{\mathcal{P},j} \\ 0 & \text{, otherwise} \end{cases}$$

is the distance between x_i and the k-th segment of obstacle j.

Smoothing

对 $-max(-d_{s,i,1}, \cdots, -d_{s,i,s_j})$ 使用 LSE 光滑化，则

$$d_{i,j} = \begin{cases} -\epsilon \ln(\sum_{k=1}^{s_j} e^{-d_{s,i,k}/\epsilon}) & \text{, if } A_{\mathcal{P},j} x_i \le b_{\mathcal{P},j} \\ 0 & \text{, otherwise} \end{cases}$$

Potential 的梯度为

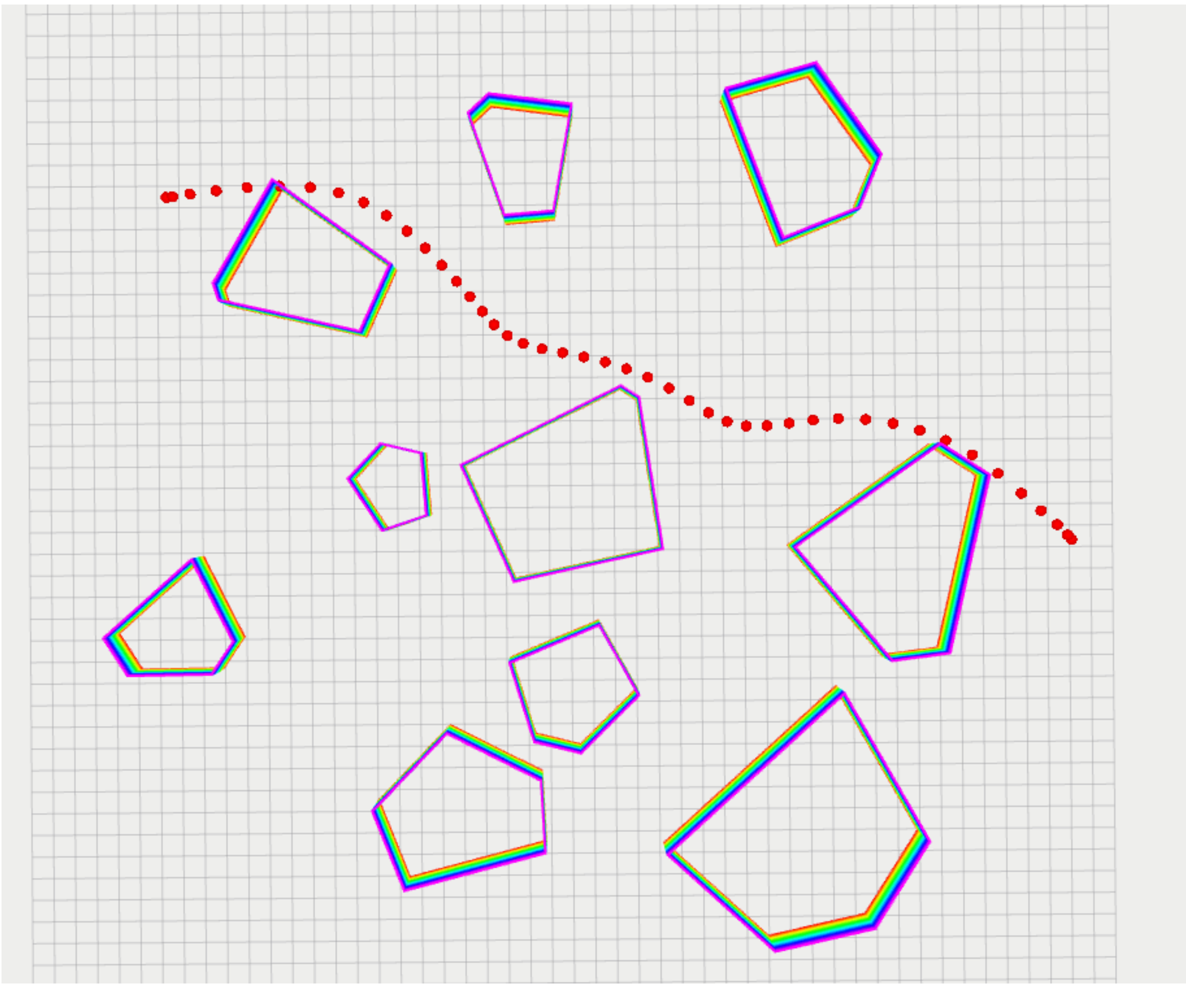
$$\frac{\partial P}{\partial x} = 1000 \begin{bmatrix} \sum_{j=1}^{M} g_{1,j} \\ \vdots \\ \sum_{j=1}^{M} g_{N-1,j} \end{bmatrix}$$

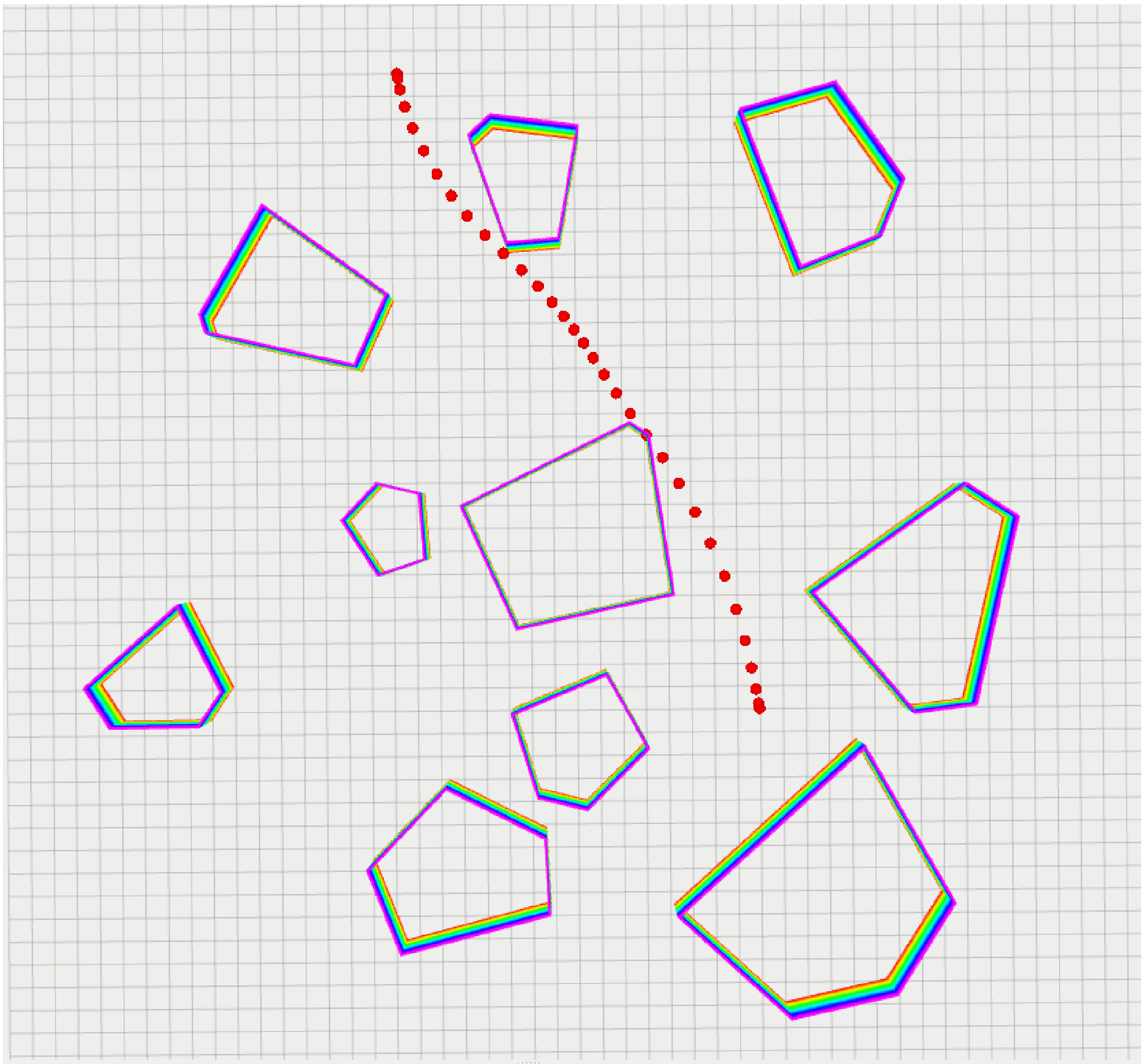
Where,

$$g_{i,j} = \begin{cases} \frac{1}{\sum_{k=1}^{s_j} e^{-d_{s,i,k}/\epsilon}} \sum_{k=1}^{s_j} e^{-d_{s,i,k}/\epsilon} \frac{x_i - o_{i,j,k}}{\|x_i - o_{i,j,k}\|} & \text{, if } A_{\mathcal{P},j} x_i \le b_{\mathcal{P},j} \\ 0 & \text{, otherwise} \end{cases}$$

where $o_{i,j}$ is the closest point to x_i on $A_{\mathcal{P},j} x_i = b_{\mathcal{P},j}$ 。 use LBFGS to solve this problem。

1.2 Result





HW5_2 TOPP

The trajectory produced from HW5_1 is

$$\begin{aligned} x_i(p) &= a_{x,i} + b_{x,i}p + c_{x,i}p^2 + d_{x,i}p^2 \\ y_i(p) &= a_{y,i} + b_{y,i}p + c_{y,i}p^2 + d_{y,i}p^2 \end{aligned} \quad i = 1, \cdots, N, \; p \in [0,1]$$

set the step as d_p , assume that the trajectory between two points is a straight line. Then the arc length of the trajectory is

$$s_k = \sum_{m=0}^{n_k-1} \sqrt{(x((m+1)d_p) - x(md_p))^2 + (y(x((m+1)d_p) - y(md_p))^2}, \;\; 0 \le k \le K$$

The derivative of arc length w.r.t arc length is

$$q'_m(s^k) = \begin{cases} \frac{q_m(s^{k+1}) - q_m(s^k)}{s_{k+1} - s_k} & , k = 0 \\ \frac{q_m(s^{k+1}) - q_m(s^{k-1})}{s_{k+1} - s_{k-1}} & , 1 \le k \le K-1 \;, \;\; m = x, y \\ \frac{q_m(s^k) - q_m(s^{k-1})}{s_k - s_{k-1}} & , k = K \end{cases}$$

second order derivative:

$$
q_m''(s^k) = \begin{cases} \frac{q_m'(s^{k+1}) - q_m'(s^k)}{s_{k+1} - s_k} & ,k = 0 \\ \frac{q_m'(s^{k+1}) - q_m'(s^{k-1})}{s_{k+1} - s_{k-1}} & ,1 \le k \le K-1 \quad m = x, y \\ \frac{q_m'(s^k) - q_m'(s^{k-1})}{s_k - s_{k-1}} & ,k = K \end{cases}
$$

2.1 Formulate

$$
\begin{aligned}
\min_{a^k, b^k, c^k, d^k} \quad & \sum_{k=0}^{K-1} 2\left(s^{k+1} - s^k\right) d^k \\
\text{s.t.} \quad & \left\| \begin{matrix} 2 \\ c^{k+1} + c^k - d^k \end{matrix} \right\| \le c^{k+1} + c^k + d^k, && 0 \le k \le K-1 \\
& \left\| \begin{matrix} 2c^k \\ b^k - 1 \end{matrix} \right\| \le b^k + 1, && 0 \le k \le K \\
& b^k \ge 0, && 0 \le k \le K \\
& b^{k+1} - b^k = 2\left(s^{k+1} - s^k\right) a^k, && 0 \le k \le K-1 \\
& \left\| q'(s^k)\sqrt{b^k} \right\|_\infty \le v_{\max}, && 0 \le k \le K \\
& \left\| q''\left(s^k\right) b^k + q'\left(s^k\right) a^k \right\|_\infty \le a_{\max}, && 0 \le k \le K-1 \\
& b^0 = b_0, b^K = b_K.
\end{aligned}
$$

Let

$$
\begin{aligned}
\boldsymbol{x} &= [a_0, \cdots, a_{K-1}, b_0, \cdots, b_K, c_0, \cdots, c_K, d_0, \cdots, d_{K-1}]^T \\
\boldsymbol{s} &= [s_0, \cdots, s_K]^T
\end{aligned}
$$

Then,

$$
\begin{bmatrix} s_1 - s_0 \\ \vdots \\ s_K - s_{K-1} \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}_{K \times K+1} \qquad \boldsymbol{s} = \boldsymbol{A}_d \boldsymbol{s}
$$

2.1.1 Object function

$$
\boldsymbol{f}^T \boldsymbol{x}
$$

$$
\boldsymbol{f} = 2\left[\boldsymbol{0}_{K \times K} \;\; \boldsymbol{0}_{K \times K+1} \;\; \boldsymbol{0}_{K \times K+1} \;\; \boldsymbol{I}_{K \times K}\right]^T \boldsymbol{A}_d \boldsymbol{s}
$$

2.1.2 Inequality constraint

for the first inequality constraint

$$
\left\| \begin{matrix} 2 \\ c^{k+1} + c^k - d^k \end{matrix} \right\|_2 \le c^{k+1} + c^k + d^k, \;\; 0 \le k \le K-1
$$

Let

$$
\boldsymbol{A}_{1,k}\boldsymbol{x} + \boldsymbol{b}_{1,k} = \begin{bmatrix} \boldsymbol{0}_{2K+1+k} & 1 & 1 & \boldsymbol{0}_{K-1} & 1 & \boldsymbol{0}_{K-k-1} \\ & & \boldsymbol{0} & \boldsymbol{0} & & \\ \boldsymbol{0}_{2K+1+k} & 1 & 1 & \boldsymbol{0}_{K-1} & -1 & \boldsymbol{0}_{K-k-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \in \mathcal{Q}^3, \;\; 0 \le k \le K-1
$$

Similarly, for the second inequality constraint

$$
\boldsymbol{A}_{2,k}\boldsymbol{x} + \boldsymbol{b}_{2,k} = \begin{bmatrix} \boldsymbol{0}_{K+k} & 1 & \boldsymbol{0}_{3K-k+1} \\ \boldsymbol{0}_{2K+k+1} & 2 & \boldsymbol{0}_{2K-k} \\ \boldsymbol{0}_{K+k} & 1 & \boldsymbol{0}_{3K-k+1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \mathcal{Q}^3, \;\; 0 \le k \le K
$$

for the third inequality constraint

$$\boldsymbol{A}_{3,k}\boldsymbol{x}+\boldsymbol{b}_{3,k}=\left[\mathbf{0}_{K+k}\quad 1\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+[0]\in\mathcal{Q}^1,\ \ 0\le k\le K$$

for the fourth inequality constraint

$$\boldsymbol{A}_{4,1,k}\boldsymbol{x}+\boldsymbol{b}_{4,1,k}=\left[\mathbf{0}_{K+k}\quad -q_x'^2\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[v_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K$$

$$\boldsymbol{A}_{4,2,k}\boldsymbol{x}+\boldsymbol{b}_{4,2,k}=\left[\mathbf{0}_{K+k}\quad -q_y'^2\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[v_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K$$

$$\boldsymbol{A}_{4,3,k}\boldsymbol{x}+\boldsymbol{b}_{4,3,k}=\left[\mathbf{0}_{K+k}\quad q_x'^2\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[v_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K$$

$$\boldsymbol{A}_{4,4,k}\boldsymbol{x}+\boldsymbol{b}_{4,4,k}=\left[\mathbf{0}_{K+k}\quad q_y'^2\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[v_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K$$

for the fifth inequality constraint

$$\boldsymbol{A}_{5,1,k}\boldsymbol{x}+\boldsymbol{b}_{5,1,k}=\left[\mathbf{0}_{k}\quad q_x'\left(s^k\right)\quad \mathbf{0}_{K-1}\quad q_x''\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[a_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K-1$$

$$\boldsymbol{A}_{5,2,k}\boldsymbol{x}+\boldsymbol{b}_{5,2,k}=\left[\mathbf{0}_{k}\quad q_y'\left(s^k\right)\quad \mathbf{0}_{K-1}\quad q_y''\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[a_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K-1$$

$$\boldsymbol{A}_{5,3,k}\boldsymbol{x}+\boldsymbol{b}_{5,3,k}=\left[\mathbf{0}_{k}\quad -q_x'\left(s^k\right)\quad \mathbf{0}_{K-1}\quad -q_x''\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[a_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K-1$$

$$\boldsymbol{A}_{5,4,k}\boldsymbol{x}+\boldsymbol{b}_{5,4,k}=\left[\mathbf{0}_{k}\quad -q_y'\left(s^k\right)\quad \mathbf{0}_{K-1}\quad -q_y''\left(s^k\right)\quad \mathbf{0}_{3K-k+1}\right]\boldsymbol{x}+\left[a_{max}\right]\in\mathcal{Q}^1,\ \ 0\le k\le K-1$$

2.1.3 Equality constraint

rewrite $b^{k+1}-b^k=2\left(s^{k+1}-s^k\right)a^k,\ 0\le k\le K-1$

$$\boldsymbol{A}_d[\mathbf{0}\ \ \boldsymbol{I}\ \ \mathbf{0}\ \ \mathbf{0}]_{K+1\times 4K+2}\boldsymbol{x}=2diag(\boldsymbol{A}_d\boldsymbol{s})[\boldsymbol{I}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}]_{K\times 4K+2}\boldsymbol{x}$$

all Equality constraints can be written as follow

$$\begin{bmatrix}\boldsymbol{A}_d[\mathbf{0}\ \ \boldsymbol{I}\ \ \mathbf{0}\ \ \mathbf{0}]_{K+1\times 4K+2}-2diag(\boldsymbol{A}_d\boldsymbol{s})[\boldsymbol{I}\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}]_{K\times 4K+2}\\ [\mathbf{0}_K\ \ 1\ \ \mathbf{0}_{3K+1}]\\ [\mathbf{0}_{2K}\ \ 1\ \ \mathbf{0}_{2K+1}]\end{bmatrix}\boldsymbol{x}=\boldsymbol{G}\boldsymbol{x}=\begin{bmatrix}\mathbf{0}\\ b_0\\ b_K\end{bmatrix}=\boldsymbol{h}$$

where $\mathbf{0}$ and $\boldsymbol{I}$ are zero matrix and identity matrix with appropriate size.

2.1.4 Conic ALM

AL:

$$\mathcal{L}_\rho(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu})=\boldsymbol{f}^T\boldsymbol{x}+\frac{\rho}{2}\left(\left\|\boldsymbol{G}\boldsymbol{x}-\boldsymbol{h}+\frac{\boldsymbol{\lambda}}{\rho}\right\|+\sum_{i=1}^{12K+6}\left\|\boldsymbol{P}_{\mathcal{K}_i}\left(\frac{\boldsymbol{\mu}_i}{\rho}-\boldsymbol{A}_i x-\boldsymbol{b}_i\right)\right\|^2\right)$$

whose gradient is

$$\nabla_x\mathcal{L}_\rho(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu})=\boldsymbol{f}+\boldsymbol{G}^T(\boldsymbol{\lambda}+\rho(\boldsymbol{G}\boldsymbol{x}-\boldsymbol{h}))-\sum_{i=1}^{12K+6}\boldsymbol{A}_i^T\boldsymbol{P}_{\mathcal{K}_i}\left(\boldsymbol{\mu}-\rho(\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{b}_i)\right)$$

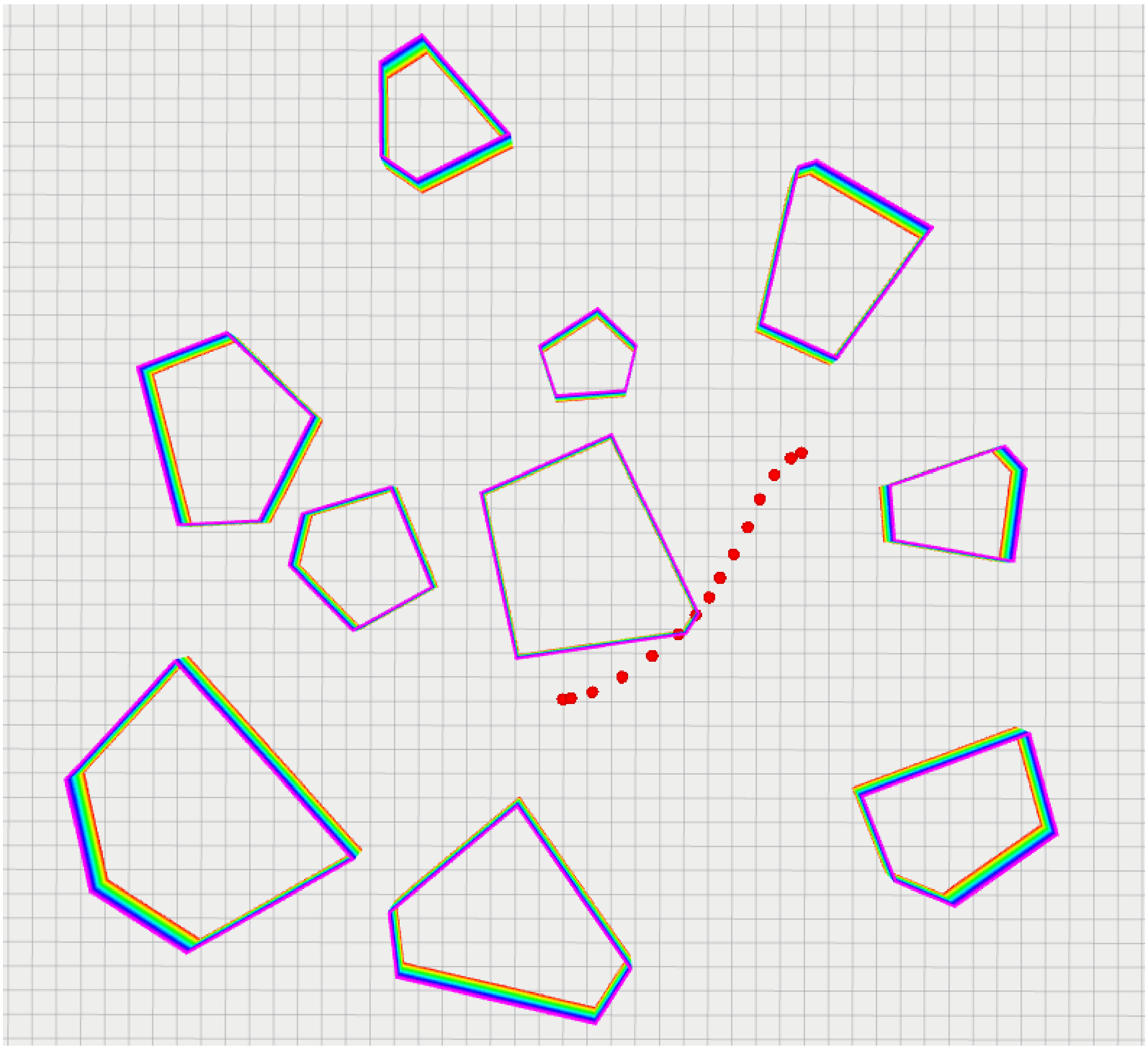
Where,

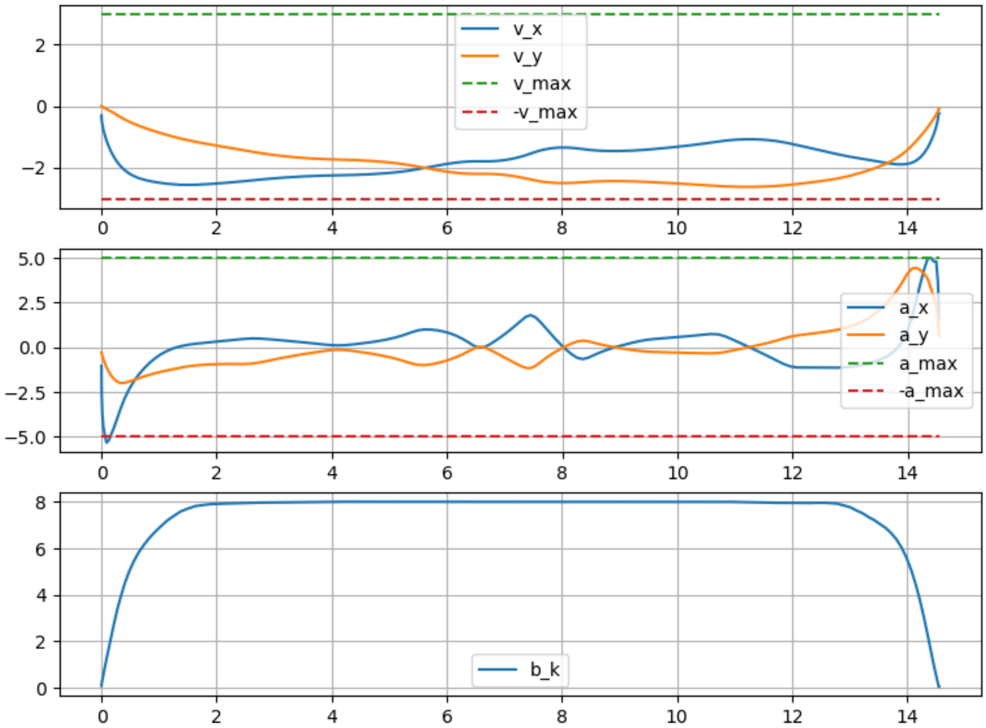
$$P_{\mathcal{K}=\mathcal{Q}^n}(v)=\begin{cases}0, & v_0\le-\|v_1\|_2\\ \frac{v_0+\|v_1\|_2}{2\|v_1\|_2}\left(\|v_1\|_2,v_1\right)^{\mathrm{T}}, & |v_0|<\|v_1\|_2\\ v, & v_0\ge\|v_1\|_2\end{cases},\ \ n\ge 2$$

$$P_{\mathcal{K}=\mathcal{Q}^n}(v)=\begin{cases}0, & v\le 0\\ v, & v\ge 0\end{cases},\ \ n=1$$

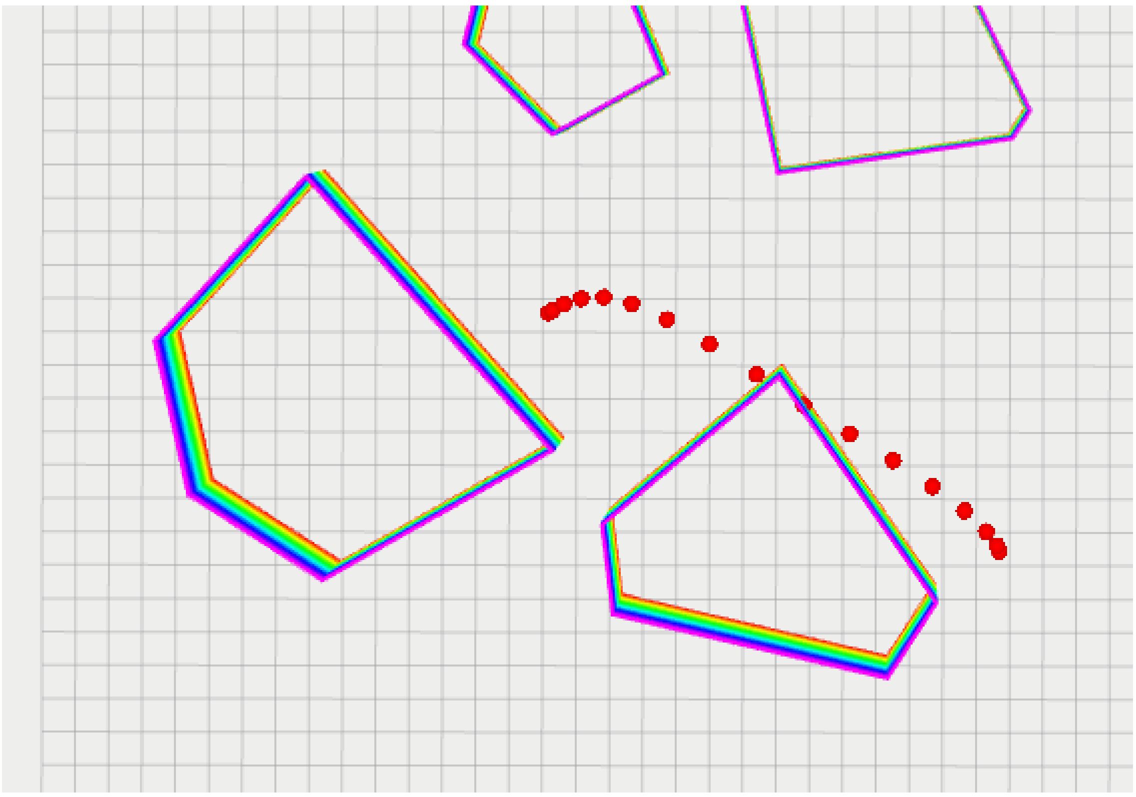
2.2 Result

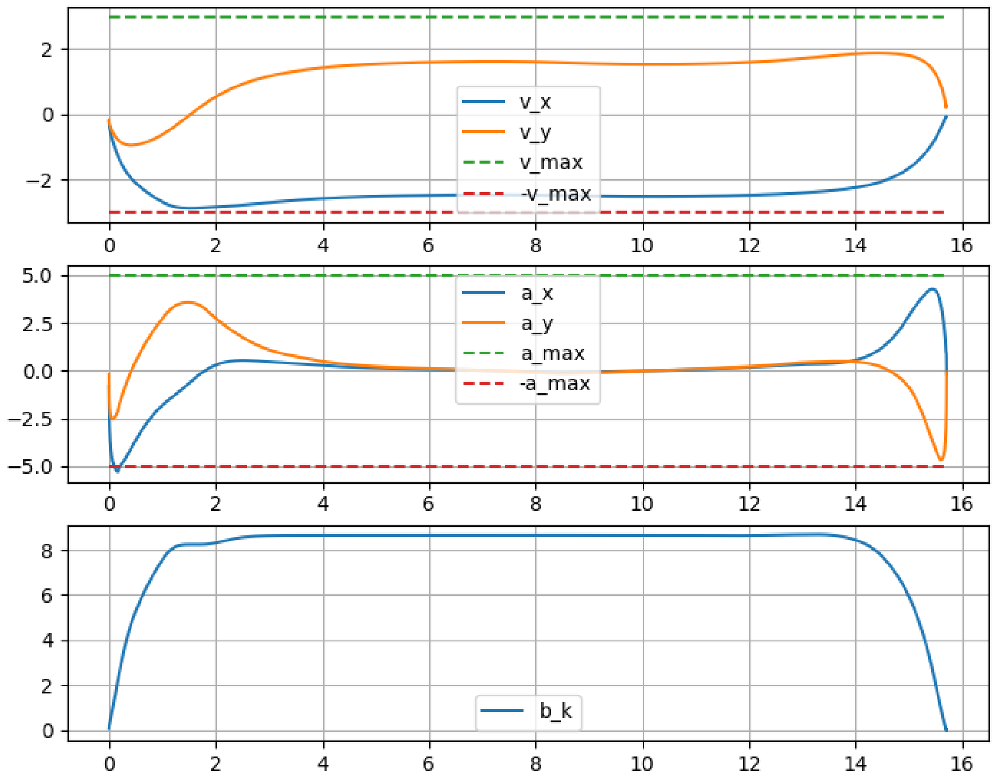
case 1





case 2





v_x
v_y
v_max
-v_max
a_x
a_y
a_max
-a_max
b_k