HW2_1 LBFGS-example

Lewis & Overton line search

- weak Wolfe conditions
- no interpolation used

$l \leftarrow 0$

$u \leftarrow +\infty$

$\alpha \leftarrow 1$

repeat

    if $S(\alpha)$ fails

        $u \leftarrow \alpha$

    else if $C(\alpha)$ fails

        $l \leftarrow \alpha$

    else

        return α

    if $u < +\infty$

        $\alpha \leftarrow (l + u)/2$

    else

        $\alpha \leftarrow 2l$

end (repeat)

结果

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Iteration: 39
Function Value: 9.935e-13
Gradient Inf Norm: 1.717e-06
Variables:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
================================
Iteration: 40
Function Value: 2.429e-18
Gradient Inf Norm: 6.202e-09
Variables:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
================================
L-BFGS Optimization Returned: 0
Minimized Cost: 2.429e-18
Optimal Variables:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
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HW2_2 Smooth Navigation Path Generation

建模

对于经过 $N+1$ 个给定点 $x_0, x_1, \ldots, x_N$ 的 N 段轨迹 $p_i(s), i=0,\cdots,N$ ，每段轨迹都是一个三次样条曲线

$$p_i(s) = a_i + b_i s + c_i s^2 + d_i s^3, s \in [0,1], i = 0,\cdots,N-1$$

根据轨迹之间的一阶和二阶连续以及边界条件可得

$$\begin{aligned} a_i &= x_i \\ b_i &= D_i \\ c_i &= 3\left(x_{i+1} - x_i\right) - 2D_i - D_{i+1} \\ d_i &= 2\left(x_i - x_{i+1}\right) + D_i + D_{i+1} \end{aligned}, i = 0,\cdots,N-1$$

令 $\mathbf{x} = [x_0, x_1, \cdots, x_{N-1}, x_N]^T$ ，则

$$\mathbf{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & 1 & 0 \end{bmatrix}_{N \times N+1} \mathbf{x}$$

$$\mathbf{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & 1 & 0 \end{bmatrix}_{N \times N+1} \mathbf{D}$$

$$\mathbf{D} = \begin{bmatrix} D_0 \\ \vdots \\ D_N \end{bmatrix} = \mathbf{A}_D \mathbf{x}$$

其中,

$$\mathbf{A}_D = 3 \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_D \\ \mathbf{0} \end{bmatrix}_{N+1 \times N-1} \begin{bmatrix} -1 & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & \\ & & & & -1 & 0 & 1 \end{bmatrix}_{N-1 \times N+1}$$

$$\mathbf{D}_D = \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix}^{-1}_{N-1 \times N-1}$$

对于系数 $\mathbf{c}$,

$$\mathbf{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \mathbf{A}_c \mathbf{x}$$

其中,

$$\mathbf{A}_c = 3 \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}_{N \times N+1} + \begin{bmatrix} -2 & -1 & & & & \\ & -2 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & -2 & -1 & \\ & & & & -2 & -1 \end{bmatrix}_{N \times N+1} \mathbf{A}_D$$

对于系数 $\mathbf{d}$,

$$\mathbf{d} = \begin{bmatrix} d_0 \\ \vdots \\ d_{N-1} \end{bmatrix} = \mathbf{A}_d \mathbf{x}$$

其中,

$$\mathbf{A}_d = 2 \begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \end{bmatrix}_{N \times N+1} + \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix}_{N \times N+1} \mathbf{A}_D$$

Stretch Energy

$$\mathrm{Energy}(x_1, x_2, \ldots, x_{N-1}) = \sum_{i=0}^{N-1} \int_0^1 \left\| p_i^{(2)}(s) \right\|^2 \mathrm{d}s$$

其中

$p_i^{(2)}(s) = 2c_i + 6d_i s$

$\left\| p_i^{(2)}(s) \right\| = 4c_i^2 + 24c_i d_i s + 36d_i^2 s^2$

$E_i = \int_0^1 \left\| p_i^{(2)}(s) \right\|^2 \mathrm{d}s = 4c_i^2 + 12c_i d_i + 12d_i^2$

则

$$\mathbf{E} = 4\mathbf{c}^T\mathbf{c} + 12(\mathbf{c}^T\mathbf{d} + \mathbf{d}^T\mathbf{d}) = \mathbf{x}^T(4\mathbf{A}_c^T\mathbf{A}_c + 12\mathbf{A}_c^T\mathbf{A}_d + 12\mathbf{A}_d^T\mathbf{A}_d)\mathbf{x} = \mathbf{x}^T\mathbf{A}_E\mathbf{x}$$

梯度

$$\frac{\partial \mathbf{E}}{\partial \mathbf{x}} = (\mathbf{A}_E + \mathbf{A}_E^T)\mathbf{x}$$

对于二维的情况，令 $\mathbf{x} = [x_0, \cdots, x_N, y_0, \cdots, y_N]^T_{2(N+1)\times 1}$ ，则

$$\mathbf{E} = \mathbf{x}^T\mathbf{A}_E^{'}\mathbf{x} = \mathbf{x}^T \begin{bmatrix} \mathbf{A}_E & \\ & \mathbf{A}_E \end{bmatrix}_{2(N+1)\times 2(N+1)} \mathbf{x}$$

$$\frac{\partial \mathbf{E}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{A}_E + \mathbf{A}_E^T & \\ & \mathbf{A}_E + \mathbf{A}_E^T \end{bmatrix}_{2(N+1)\times 2(N+1)} \mathbf{x}$$

Potential

$$\mathrm{Potential}(x_1, x_2 \ldots, x_{N-1}) = 1000 \sum_{i=1}^{N-1} \sum_{j=1}^{M} \max\left(r_j - \|x_i - o_j\|, 0\right)$$

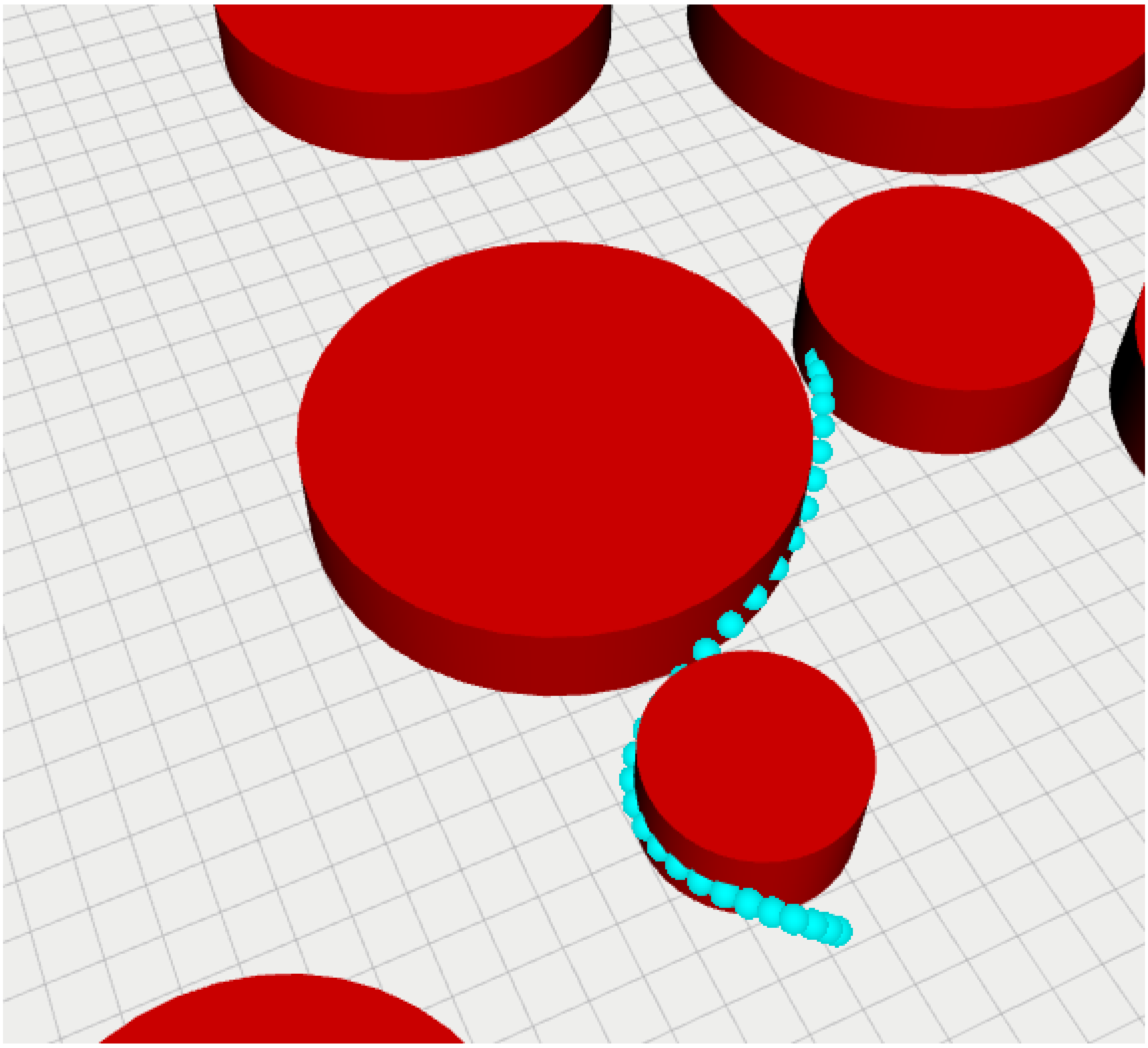
$$\frac{\partial P}{\partial x} = 1000 \begin{bmatrix} \sum_{j=1}^M g_{1,j} \\ \vdots \\ \sum_{j=1}^M g_{N-1,j} \end{bmatrix}$$

其中

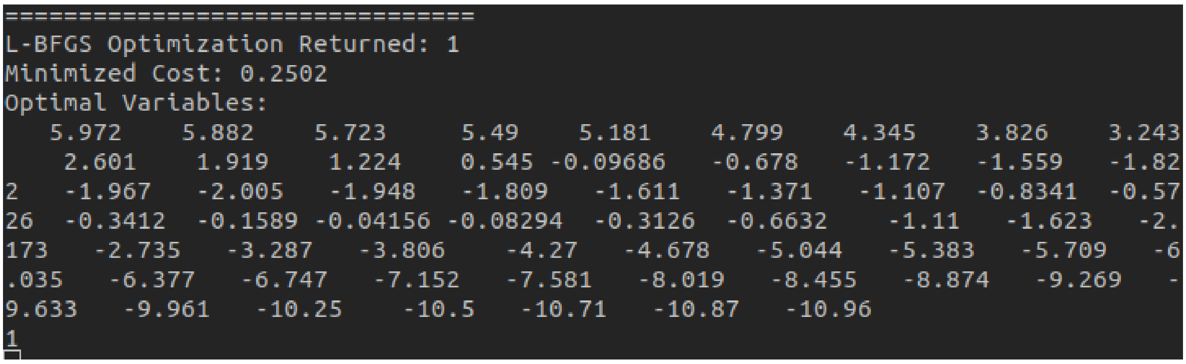
$$g_{i,j} = \begin{cases} -\frac{x_i - o_j}{\|x_i - o_j\|} & \text{, if } r_j - \|x_i - o_j\| > 0 \\ 0 & \text{, otherwise} \end{cases}$$

结果

演示视频见attachments文件夹中的 HW2_example.mp4 。



成功规划出一条无碰撞的最小拉伸能量轨迹。



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================================
L-BFGS Optimization Returned: 1
Minimized Cost: 0.2502
Optimal Variables:
   5.972     5.882     5.723     5.49     5.181     4.799     4.345     3.826     3.243
    2.601     1.919     1.224     0.545 -0.09686    -0.678    -1.172    -1.559    -1.82
2   -1.967    -2.005    -1.948    -1.809    -1.611    -1.371    -1.107   -0.8341   -0.57
26  -0.3412   -0.1589 -0.04156 -0.08294   -0.3126   -0.6632     -1.11    -1.623    -2.
173    -2.735    -3.287    -3.806     -4.27    -4.678    -5.044    -5.383    -5.709    -6
.035    -6.377    -6.747    -7.152    -7.581    -8.019    -8.455    -8.874    -9.269    -
9.633    -9.961    -10.25     -10.5    -10.71    -10.87    -10.96
1
```