Quiz 2 - Coordinate Transformations - CSE/ECE 344/544 Solutions

1. Show that the Euclidean distance ($||\mathbf{P}_1 - \mathbf{P}_2||_2$) is invariant to a rotation transformation applied via a 2×2 orthogonal matrix $\mathbf{R}$. Here, $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{R}^2$ are any two points in the 2D plane. {*Hint*: Show that the Euclidean distance between the two points before and after the transformation is the same.}

   Solution:

   We want to show that $||\mathbf{P}_1 - \mathbf{P}_2||_2$ is invariant to a rotation transform via the 2×2 rotation matrix $\mathbf{R}$.

   We have: $\mathbf{P}_1' = \mathbf{R}\mathbf{P}_1$ and $\mathbf{P}_2' = \mathbf{R}\mathbf{P}_2$. Then,

$$\begin{aligned}
\mathbf{P}_1' - \mathbf{P}_2' &= \mathbf{R}\mathbf{P}_1 - \mathbf{R}\mathbf{P}_2 \\
\Rightarrow (\mathbf{P}_1' - \mathbf{P}_2') &= \mathbf{R}(\mathbf{P}_1 - \mathbf{P}_2) \\
\Rightarrow (\mathbf{P}_1' - \mathbf{P}_2')^\top(\mathbf{P}_1' - \mathbf{P}_2') &= (\mathbf{R}(\mathbf{P}_1 - \mathbf{P}_2))^\top (\mathbf{R}(\mathbf{P}_1 - \mathbf{P}_2)) \\
\Rightarrow (\mathbf{P}_1' - \mathbf{P}_2')^\top(\mathbf{P}_1' - \mathbf{P}_2') &= \left((\mathbf{P}_1 - \mathbf{P}_2)^\top \mathbf{R}^\top\right) (\mathbf{R}(\mathbf{P}_1 - \mathbf{P}_2)) \\
\Rightarrow ||\mathbf{P}_1' - \mathbf{P}_2')||_2^2 &= \left((\mathbf{P}_1 - \mathbf{P}_2)^\top \left(\mathbf{R}^\top\mathbf{R}\right) (\mathbf{P}_1 - \mathbf{P}_2)\right) \\
\Rightarrow ||\mathbf{P}_1' - \mathbf{P}_2')||_2^2 &= ((\mathbf{P}_1 - \mathbf{P}_2)^\top (\mathbf{P}_1 - \mathbf{P}_2)) \\
\Rightarrow ||\mathbf{P}_1' - \mathbf{P}_2')||_2^2 &= ||\mathbf{P}_1 - \mathbf{P}_2||_2^2
\end{aligned}$$

   Since $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, we see that the square of the length (and hence its positive square root, the length) of the vector $\mathbf{P}_1\mathbf{P}_2$ does not change under rotation. Thus, we conclude that the Euclidean distance between $\mathbf{P}_1$ and $\mathbf{P}_2$ is preserved under a rotation transformation.

   Alternatively, you may prove it as below by expanding the vectors and the multiplication with a rotation matrix. The above proof can be extended to any dimensions compactly.

   $\|P_1 - P_2\|_2$ is invariant to rotation transformation.

   $P_1 = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{y}_1 \end{bmatrix}, P_2 = \begin{bmatrix} \mathrm{x}_2 \\ \mathrm{y}_2 \end{bmatrix}$

   $\|P_1 - P_2\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

   Rotation transformation matrix $= R = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

   $P_1' = R.P_1 = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{y}_1 \end{bmatrix} = \begin{bmatrix} \mathrm{x}_1\cos(\alpha) - \mathrm{y}_1\sin(\alpha) \\ \mathrm{x}_1\sin(\alpha) + \mathrm{y}_1\cos(\alpha) \end{bmatrix}$

   $P_2' = R.P_2 = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \mathrm{x}_2 \\ \mathrm{y}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{x}_2\cos(\alpha) - \mathrm{y}_2\sin(\alpha) \\ \mathrm{x}_2\sin(\alpha) + \mathrm{y}_2\cos(\alpha) \end{bmatrix}$

   $\|P_1' - P_2'\|$

   $= \sqrt{(x_1\cos(\alpha) - y_1\sin(\alpha) - x_2\cos(\alpha) + y_2\sin(\alpha))^2 + (x_1\sin(\alpha) + y_1\cos(\alpha) - x_2\sin(\alpha) - y_2\cos(\alpha))^2}$

   $= \sqrt{((x_1 - x_2)\cos(\alpha) - (y_1 - y_2)\sin(\alpha))^2 + ((y_1 - y_2)\cos(\alpha) - (-x_1 + x_2)\sin(\alpha))^2}$

   $= \sqrt{\begin{array}{l} \cos^2(\alpha)(x_1 - x_2)^2 + \sin^2(\alpha)(y_1 - y_2)^2 - 2\sin(\alpha)\cos(\alpha)(x_1 - x_2)(y_1 - y_2) \\ + \cos^2(\alpha)(y_1 - y_2)^2 + \sin^2(\alpha)(x_2 - x_1)^2 - 2\sin(\alpha)\cos(\alpha)(y_1 - y_2)(x_2 - x_1) \end{array}}$

$= \sqrt{\cos^2(\alpha)(x_1 - x_2)^2 + \sin^2(\alpha)(y_1 - y_2)^2 + \cos^2(\alpha)(y_1 - y_2)^2 + \sin^2(\alpha)(x_2 - x_1)^2}$

$= \sqrt{(x_1 - x_2)^2(\cos^2(\alpha) + \sin^2(\alpha)) + (y_1 - y_2)^2(\sin^2(\alpha) + \cos^2(\alpha))}$

$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \|P_1 - P_2\|$

2. Show that the Euclidean distance between the two points $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{R}^2$ is invariant to a translation transformation, applied using a 2×1 vector $\mathbf{t}$. {*Hint*: Show that the Euclidean distance between the two points before and after the transformation is the same.}

   Solution:

   $P_1 = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{y}_1 \end{bmatrix}, P_2 = \begin{bmatrix} \mathrm{x}_2 \\ \mathrm{y}_2 \end{bmatrix}$

   $\|P_1 - P_2\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

   Translation matrix $= T = \begin{bmatrix} \mathrm{t}_x \\ \mathrm{t}_y \end{bmatrix}$

   $P_1' = P_1 + T$

   $P_2' = P_2 + T$

   $\|P_1' - P_2'\| = \sqrt{((x_1 + t_x) - (x_2 + t_x))^2 + ((y_1 + t_y) - (y_2 + t_y))^2}$

   $= \sqrt{(x_1 + t_x - x_2 - t_x)^2 + (y_1 + t_y - y_2 - t_y)^2}$

   $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

   $= \|P_1 - P_2\|$

   This proof could also be furnished compactly for any dimensions by writing it as in Soln. 1 above.

3. (a) Show that a 2D translation $\mathbf{t}$ applied to a 2D point $\mathbf{P}_1$ represented in Cartesian coordinates is not a linear function. (b) Write the point $\mathbf{P}_1$ in its homogeneous coordinate representation. Now design a 3×3 transformation matrix $\mathbf{T}_t$ that implements the same translation $\mathbf{t}$. (c) Show that this transformation applied to the point $\mathbf{P}_1$ in its homogeneous coordinates is linear.

   Solution: Condition for a transformation to be linear:

   $$\text{Superposition: } f(x + y) = f(x) + f(y)$$

   $$\text{Homogeneity: } f(cx) = cf(x)$$

   OR

   $$f(ax + by) = af(x) + bf(y)$$

   Part (a): Proof by contradiction.

   Check for Homogeneity Condition:

   $\mathbf{P}_1 = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{y}_1 \end{bmatrix}, \mathbf{t} = \begin{bmatrix} \mathrm{t}_x \\ \mathrm{t}_y \end{bmatrix},$

   Translation in Cartesian coordinates: $f(\mathbf{P}_1) = \mathbf{P}_1 + \mathbf{t}$

   $f(c\mathbf{P}_1) = \begin{bmatrix} \mathrm{cx}_1 + t_x \\ \mathrm{cy}_1 + t_y \end{bmatrix}$

$cf(\mathbf{P}_1) = \begin{bmatrix} \mathrm{cx}_1 + ct_x \\ \mathrm{cy}_1 + ct_y \end{bmatrix}$

f(c$\mathbf{P}_1$) != cf($\mathbf{P}_1$), Hence translation in cartesian coordinates is not Linear.

Part (b): $\mathbf{P}_1$ in its homogeneous coordinate representation

$\mathbf{P}_1 = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{y}_1 \\ 1 \end{bmatrix}$, $\mathbf{T}_t = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

Translation in homogeneous coordinate representation: $f(\mathbf{P}_1) = \mathbf{T}_t\mathbf{P}_1$

Part (c): Prove translation in homogeneous coordinate representation is linear.

$f(a\mathbf{P}_1 + b\mathbf{P}_2) = \mathbf{T}_t(a\mathbf{P}_1 + b\mathbf{P}_2)$

$f(a\mathbf{P}_1 + b\mathbf{P}_2) = \mathbf{T}_t(a\mathbf{P}_1) + \mathbf{T}_t(b\mathbf{P}_2)$

$f(a\mathbf{P}_1 + b\mathbf{P}_2) = a\mathbf{T}_t(\mathbf{P}_1) + b\mathbf{T}_t(\mathbf{P}_2)$

$f(a\mathbf{P}_1 + b\mathbf{P}_2) = af(\mathbf{P}_1) + bf(\mathbf{P}_2)$

Since both homogeneity and superposition are satisfied, the translation transformation in homogeneous coordinates is linear.

4. Find the 3×3 Euclidean transformation (rotation and translation) matrix ${}^{AB}\mathbf{T}_{XY}$ that maps points from the XY co-ordinate frame to the AB co-ordinate frame. The angle $\alpha = \pi/4$ and the point $\mathbf{P}_{XY} = [-1, 2]^\top$. The transformation is shown in the figure below

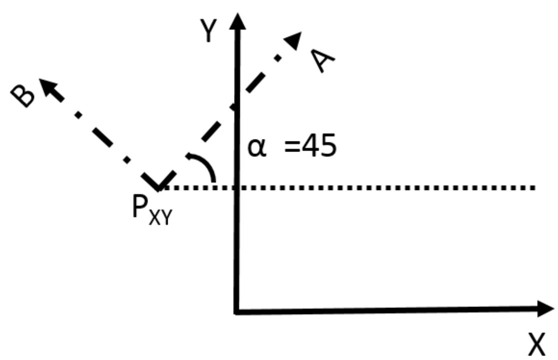

Figure 1: Geometry of the coordinate frames XY and AB.

Solution:

In this 2D coordinate transformation, there are three variables that need to be determined, i.e., a total of three degrees of freedom: two for the 2D translation $\mathbf{t} = [t_x, t_y]^\top$ and θ, the angle of rotation (angle by which the points are rotated).

There are two ways to approach this problem. Either approach is acceptable, however, the important thing to remember is that *transformations are applied to points*, which give us the equations (or constraints) that are used to solve for the variables.

We want to find the transformation matrix ${}^{AB}\mathbf{T}_{XY}$, which maps points in the XY frame of reference to the AB frame of reference (or eq. coordinate frame). In other words, it will map every point $\mathbf{Q}$'s coordinates in the XY frame, i.e., $\mathbf{Q}_{XY}$ to its coordinates $\mathbf{Q}_{AB}$ in the AB frame $\left(\mathbf{Q}_{AB} = {}^{AB}\mathbf{T}_{XY}\mathbf{Q}_{XY}\right)$.

Since both translation and rotation are involved, there could be two approaches to solve this problem.

Approach 1) Translation followed by Rotation.
In this approach, we'll first solve for translation and then for the rotation. Therefore our transformation can be written as the following composition:
${}^{AB}\mathbf{T}_{XY} = \mathbf{T}_{rot}\mathbf{T}_{trans}$
where $\mathbf{T}_{rot}$ and $\mathbf{T}_{trans}$ denote the pure rotation and pure translation transformations respectively.

Part (i). Estimating the translation.

Let $\mathbf{O}_{XY} = [0,0]^\top$ be the origin in the XY coordinate frame. We observe that the point $\mathbf{P}_{XY}$ in the XY coordinate frame is translated to the origin $\mathbf{O}_{AB}$ in the AB coordinate frame. Therefore, we need to estimate the parameters (t_x, t_y) of the translation transformation (in homogeneous coordinates[1]) as follows:

$$\mathbf{O}_{AB} = \mathbf{T}_{trans}\mathbf{P}_{XY}$$
$$\mathbf{O}_{AB} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{AB} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}_{XY}$$

The subscripts XY and AB denote that the points are represented in the XY and AB coordinate frames respectively.

Solving the above equations, we compute $t_x = 1$ and $t_y = -2$ making the pure translation transformation as

$$\mathbf{T}_{trans} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (ii). Estimating the rotation.

A 2D rotation matrix has the following form:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle by which *the points are rotated*, with counter-clockwise being the positive direction. The corresponding 3×3 rotation transformation (in homogeneous coordinates) is given by

$$\mathbf{T}_{rot} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similar to the translation case, we need to identify points and their corresponding transformed points that would provide the constraints to solve for the variable θ. Since the origin is invariant to the rotation operation, we cannot use the origin as one of the points to solve for θ. Let's consider the point represented by the unit vector $\widehat{\imath}_{AB}$, which is the unit vector along the first axis in the AB coordinate frame (i.e., the vector $\overrightarrow{PA}$). This point will have the following coordinates in the XY frame of reference

$$\left[\widehat{\imath}_{AB}\right]_{XY} = \overrightarrow{OP}_{XY} + \frac{\overrightarrow{PA}_{XY}}{||\overrightarrow{PA}_{XY}||}$$

[1]We don't necessarily have to follow homogeneous coordinates here though, but we are doing it for convenience, as it allows composition of transformations simply via matrix multiplication.

As before, the subscript XY indicates the coordinate frame in which the above vectors are represented. Therefore, we have

$$\left[\widehat{i}_{AB}\right]_{XY} = \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}\cos\alpha\\\sin\alpha\end{bmatrix} \tag{1}$$

We can write the above sum from the geometry as shown in Fig. 1. Drawing a unit length vector parallel to $\overrightarrow{PA}$ that has its base at the origin ($\mathbf{O}_{XY}$), we see its coordinates will be $[\cos\alpha, \sin\alpha]^\top$. We also know that $\widehat{i}_{AB} = [1,0]^\top$, as it is the first axis of the AB coordinate frame[2]. The transformation ${}^{AB}\mathbf{T}_{XY}$ should map $\left[\widehat{i}_{AB}\right]_{XY}$ to $\widehat{i}_{AB}$. Therefore, in homogeneous coordinates, we have:

$$\begin{aligned}
\widehat{i}_{AB} &= {}^{AB}\mathbf{T}_{XY}\left[\widehat{i}_{AB}\right]_{XY}\\
\widehat{i}_{AB} = \begin{bmatrix}1\\0\\1\end{bmatrix} &= \mathbf{T}_{rot}\mathbf{T}_{trans}\begin{bmatrix}\cos\alpha-1\\\sin\alpha+2\\1\end{bmatrix} = {}^{AB}\mathbf{T}_{XY}\left[\widehat{i}_{AB}\right]_{XY}\\
\begin{bmatrix}1\\0\\1\end{bmatrix} &= \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha-1\\\sin\alpha+2\\1\end{bmatrix}\\
\begin{bmatrix}1\\0\\1\end{bmatrix} &= \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha-1+1\\\sin\alpha+2-2\\1\end{bmatrix}\\
\begin{bmatrix}1\\0\\1\end{bmatrix} &= \begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha\\\sin\alpha\\1\end{bmatrix}\\
\begin{bmatrix}1\\0\\1\end{bmatrix} &= \begin{bmatrix}\cos(\theta+\alpha)\\\sin(\theta+\alpha)\\1\end{bmatrix}
\end{aligned}$$

From the above, we have the following constraints: $\cos(\theta+\pi/4) = 1$ and $\sin(\theta+\pi/4) = 0$. By taking arccos and arcsin, we can see that $\theta+\pi/4 = 0$, i.e., $\theta = -\pi/4$ radians. Therefore, our 2D rotation matrix is
$\mathbf{R} = \begin{bmatrix}\cos(-\pi/4) & -\sin(-\pi/4)\\\sin(-\pi/4) & \cos(\pi/4)\end{bmatrix}$. Notice the negative sign. This is because the *axes* have rotated counter-clockwise, and therefore the transformation, when applied to points would rotate them *clockwise*, thus yielding the negative sign.

The translation vector was $\mathbf{t} = [1,-2]^\top$ from part (i). Therefore, the transformation ${}^{AB}\mathbf{T}_{XY}$ is then given as the product:

$$\begin{aligned}
{}^{AB}\mathbf{T}_{XY} &= \begin{bmatrix}\cos(-\pi/4) & -\sin(-\pi/4) & 0\\\sin(-\pi/4) & \cos(\pi/4) & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}\mathbf{R} & \mathbf{Rt}\\0\ 0 & 1\end{bmatrix}
\end{aligned}$$

Here, $\mathbf{R}$ and $\mathbf{t}$ are as derived above.

Approach 2) Rotation followed by Translation.

[2]This is equivalent to the X-axis in the local coordinate frame.

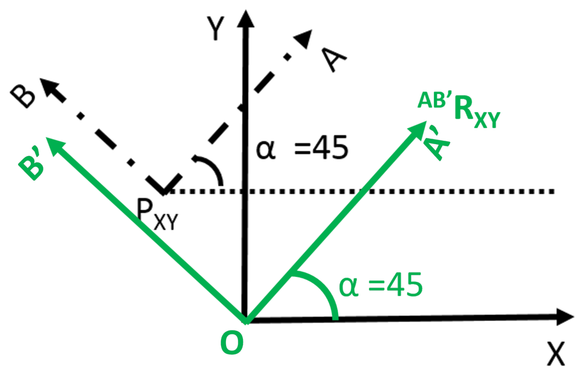

Figure 2: Intermediate coordinate frame AB' in green, obtained by applying pure rotation first.

In this approach, we will first identify the rotation transformation. This will be a pure rotation transformation to an intermediate coordinate frame AB', which has axes parallel to AB, but still has the origin at the same point as the origin of XY, i.e., **O** as shown in Fig. 2. By following the same process as in part (ii) of Approach 1), we observe that the unit vector along $\overrightarrow{OA}$, i.e., the point $\left[\widehat{i}_{AB'}\right]_{XY} = [\cos\alpha, \sin\alpha]^\top$ gets mapped to $\widehat{i}_{AB'} = [1, 0]^\top$. Following the steps in part (ii) in Approach 1), we obtain the same rotation matrix $\mathbf{R}$ with $\theta = -\pi/4$, and the corresponding pure rotation transformation matrix (in homogeneous coordinates) as $\mathbf{T}_{rot} = \begin{bmatrix} \mathbf{R} & 0 \\ 0\ 0 & 1 \end{bmatrix}$.

In part (i) of Approach 1), the translation vector $\mathbf{t}$ turned out to be $-\overrightarrow{OP}_{XY}$, the translation in this case would be via the same vector $\overrightarrow{OP}$, however, expressed in the AB' coordinate frame, i.e., $\overrightarrow{OP}_{AB'}$[3]. Thus, following the same steps as before, the translation vector in this case will be $-\overrightarrow{OP}_{AB'}$, which is the same as the rotation applied to the vector $-\overrightarrow{OP}_{XY}$, i.e., $\mathbf{R} * \left(-\overrightarrow{OP}_{XY}\right) = \mathbf{Rt}$. The resulting pure translation transformation matrix is given by

$$\mathbf{T}_{trans} = \begin{bmatrix} \mathbf{I} & \mathbf{Rt} \\ 0\ 0 & 1 \end{bmatrix}$$

The final transformation ${}^{AB}\mathbf{T}_{XY}$ is then given as the product

$${}^{AB}\mathbf{T}_{XY} = \begin{bmatrix} \mathbf{I} & \mathbf{Rt} \\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & 0 \\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{Rt} \\ 0\ 0 & 1 \end{bmatrix}$$

which is the same as the transformation matrix we obtained by following Approach 1).

[3]Why? Because, we have first applied the rotation transformation to map each point to the AB' coordinate frame.

Things to remember while solving for Euclidean coordinate transformations:

- Euclidean coordinate transformations are invertible, so regardless of how you solve for them, you would *always* get the same solution.
- These transformations operate on points and with the homogeneous representations turn out to be matrix-vector multiplication. Often, it makes sense to solve them step-by-step, e.g., first translation, then rotation.
- Canonical unit vectors $(\widehat{i}, \widehat{j}, \widehat{k}$ are very useful for determining the rotation / translation parameters. They are great for sanity checks as well. Unit vectors in 2D lie on the unit circle with their Cartesian coordinates having the form $[\cos\theta, \sin\theta]^\top$, where θ can be obtained from the pola coordinate / rotation angle.
- Remember when *the point* is rotated counter-clockwise, the angle of rotation is positive.
- The origin is invariant to rotation, and hence is not a good choice of a point to be used while solving / validating your rotation solution. It is good for translation though.
- If you solve for rotation and translation separately, the order in which you apply the transformation matters.