ECE 634/CSE 646 InT: Practice Problems 1

Instructor: Manuj Mukherjee

1) Consider a sequence $a_n, n \geq 1$, $a_n \in \mathbb{R}$. Define

$$\limsup_{n\to\infty} a_n \triangleq \inf_{k\geq 1} \sup_{n\geq k} a_n.$$

Show that:

a) $\limsup_{n\to\infty} a_n = a$ implies that for any $\epsilon > 0$, we have $a_n \leq_n a + \epsilon$. [Hint: Define $b_k = \sup_{n\geq k} a_n$, and note that b_k is thus a decreasing sequence.]

b) Show that for any $\epsilon > 0$, if a sequence a_n satisfies $a_n \leq_n a + \epsilon$, then $\limsup_{n\to\infty} a_n \leq a$.

c) Find $\limsup_{n\to\infty} (-1)^n$.

2) We say that a property P happens infinitely often for some sequence a_n, if for every $n \geq 1$, there exists a $k \geq n$ such that a_k satisfies the property P.

a) Show that if $\limsup_{n\to\infty} a_n = a$, then for any $\epsilon > 0$, $a_n \geq a - \epsilon$ occurs infinitely often.

b) Show that if for any $\epsilon > 0$ a sequence a_n satisfies $a_n \geq a - \epsilon$ infinitely often, then $\limsup_{n\to\infty} a_n \geq a$.

c) Find $\limsup_{n\to\infty}\ \sin(\frac{n\pi}{2})(1 - e^{-n})$.

3) Consider a discrete memoryless source with alphabet $\mathcal{X}$ and pmf P_X and fix $\delta > 0$. Let $f^{(n)*}, g^{(n)*}, n \geq 1$ be the sequence of optimal encoder decoders, i.e.,

$$R^{(n)*} = \min_{\substack{f^{(n)}, g^{(n)}, n\geq 1: \\ \mathrm{P}_X(X^n \neq g^{(n)}(f^{(n)}(X^n))) \leq_n \delta}} R^{(n)}.$$

Show that $R^{(n)^*} \to H(\mathrm{P}_X)$.

[Hint: Tweak the proof of the lossless source coding theorem slightly, by noting that the optimal encoder will always have lower rate than AEP based encoder.]

4) Consider a discrete memoryless source with an alphabet $\mathcal{X}$ and pmf P_X. Fix an $\epsilon > 0$, and for every $a \in \mathcal{X}$, define

$$A_{\epsilon,a}^{(n)} \triangleq \{x^n : |\frac{1}{n}N(a) - \mathrm{P}_X(a)| \leq \mathrm{P}_X(a)\epsilon\},$$

where $N(a)$ is number of times a appears in x^n, i.e., $N(a) = \sum_{i=1}^n \mathbf{1}\{x_i = a\}$. Show that $\mathrm{P}_X(\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}) \to 1$. [Hint: Define random variables carefully, and you can reduce this problem to a WLLN and a union bound.]

I. Answers

1) c) 1.
2) c) 1.