ECE/CSE 636: Communications Networking

Mid Semester Exam

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No books, notes, or devices are allowed. Just a pen/ pencil and eraser. Institute rules will apply with regards to cheating. Show your steps. You have about 120 minutes.

Question 1. 5 **marks** Consider a renewal process $\{N(t); t > 0\}$. How are the probabilities $P[N(t) = m]$ and $P[S_m \leq t]$ ordered? Explain your answer.

Question 2. 10 **marks** Consider a renewal process with inter-arrival times that are distributed uniformly over $(0, 1)$. You observe arrivals at $0.5, 0.65, 1.2, 2.0, 2.8$ seconds.

1) What is the probability that the next arrival does not occur on or before 3.8 seconds?
2) Suppose the next arrival does not occur on or before 3.2 seconds. Derive the distribution of the time to the next arrival, conditioned on the above fact?

Question 3. 15 **marks** For a Poisson process, derive the conditional PDF $f_{X_n|S_n}(x|s)$. Are X_n and S_n independent random variables?

Question 4. 15 **marks** Consider an interval $(0, t)$ that sees n arrivals from a Poisson process of rate λ. Let the n^{th} arrival occur at time τ, where $0 < \tau < t$. Derive the distribution of the interval $(t - \tau)$. Assume the definitions of the Poisson process. The rest you must show as a part of the derivation.

Question 5. 15 **marks** You toss a coin N times, where N is a Poisson random variable of rate μ. A toss gives heads with probability p, tails with probability q, and returns undefined with probability $1 - (p + q)$. All tosses are independent of each other. Derive the following.

1) PMF of the number of heads,
2) PMF of the number of tails,
3) Joint PMF of the numbers of heads and tails.

Question 6. 15 **marks** Consider a system with three servers 1, 2, and 3. All servers provide service to a job that is exponentially distributed with rate μ and independent of service provided to other jobs. At time 0 all servers are assigned a job. Further, a new job is assigned to a server as soon as it finishes service of the current job. What is the probability that the job assigned to server 1 at time 0 is the tenth job to leave the system? [Hint: How would you describe the processes that count the departures from the servers.]

Question 7. 30 **marks** Let A_t and R_t be the age and residual time random variables at time t. We will derive the joint distribution of A_t and R_t. To do so, consider the two cases $t < S_1$ and $t > S_1$.

1) What value or range of values does A_t take when $t < S_1$?
2) What value or range of values does A_t take when $t > S_1$?
3) Derive the joint distribution $f_{A_t,R_t}(a, r)$ of A_t and R_t for the case when $t < S_1$.
4) Derive the joint distribution $f_{A_t,R_t}(a, r)$ of A_t and R_t for the case when $t > S_1$. Note that $A_t = a, R_t = r$ implies that either $S_1 = t - a, S_2 = t + r$ or $S_2 = t - a, S_3 = t + r$ or (and so on). Use this to write down

$f_{A_t,R_t}(a,r)$ as a sum of appropriately chosen joint PDFs. Simplify the sum using a well known property of the Poisson PMF.

5) What happens to the joint distribution in the limit as $t \to \infty$?

I. Useful Formula

The PDF of the Erlang RV S_n with rate λ is $f_{S_n}(s) = \lambda^n s^{n-1} e^{-\lambda s}/(n-1)!$.
The PMF of the Poisson RV is $P[N = k] = (\lambda t)^k e^{-\lambda t}/k!$, $k = 0, 1, \ldots$.