Solution Worksheet 9

Problem 1:

(a) $$\mathcal{L}\{\cos^2 \omega t\} = \int_0^\infty \cos^2(\omega t)\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{1+\cos(2\omega t)}{2}\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\, dt + \frac{1}{2}\int_0^\infty \cos(2\omega t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\left[\frac{-1}{s} e^{-st}\right]_0^\infty + \frac{1}{2}\int_0^\infty \cos(2\omega t) e^{-st}\, dt$$

$$= \frac{1}{2s} + \frac{1}{2}\int_0^\infty \cos(2\omega t)\, e^{-st} dt$$

As we know $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}\{\cos^2(\omega t)\} = \frac{1}{2s} + \frac{1}{2(s^2+4\omega^2)}$$

(b) $L\{t^2\} = \dfrac{2!}{s^3}$

$L\{e^{at} f(t)\} = F(s-a)$

so,

$L\{t^2 e^{-3t}\} = \dfrac{2!}{(s+3)^3}$

(c) let us define $f = t\cos at$.

$f' = \cos(at) - at\sin at$

$f'' = -a\sin(at) - a\sin(at) - a^2 t\cos(at)$

$= -2a\sin(at) - a^2 t\cos(at)$

$L\{f''\} = -2a\dfrac{a}{s^2+a^2} - a^2 L\{t\cos(at)\} = \dfrac{-2a^2}{s^2+a^2} - a^2 F$

where $F = L\{f\}$

As we know,

$$L\{f''\} = s^2 f - s f(0) - f'(0)$$

$f(0) = 0$, $f'(0) = 1$, we get

$$L\{f''\} = s^2 F - 1$$

Equating both expressions for $L\{f''\}$ we get

$$\frac{-2a^2}{s^2+a^2} - a^2 f = s^2 F - 1$$

solving for F,

$$F = \frac{s^2 - a^2}{(s^2+a^2)^2}$$

In particular, for $a = 4$, we get

$$F = \frac{s^2 - 16}{(s^2+16)^2}$$

Problem 2:

(a)

$$L^{-1}\left\{\frac{5s+1}{s^2-25}\right\} = 5L^{-1}\left\{\frac{s}{s^2-25}\right\} + L^{-1}\left\{\frac{1}{s^2-25}\right\}$$

$$= 5\cosh(5t) + \frac{1}{5}\sinh(5t)$$

(b) we know that

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{e^{at}f(t)\} = F(s-a)$$

we have the inverse laplace transform

$$L^{-1}\left\{\frac{21}{s^4}\right\} = \frac{21}{3!}L^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{7}{2}t^3$$

and

$$L^{-1}\left\{\frac{21}{(s+\sqrt{2})^4}\right\} = \frac{7}{2}t^3e^{-\sqrt{2}t}$$

(c) let $F = \frac{1}{s^2}\ \frac{20}{s-2\pi}$

The inverse Laplace transform of $\frac{20}{s-2\pi}$ is $20e^{2\pi t}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{20}{s-2\pi}\right\} = \int_0^t 20\, e^{2\pi\tau}\, d\tau = 20\,\frac{e^{2\pi t}-1}{2\pi}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{20}{s-2\pi}\right\} = \int_0^t \frac{20}{2\pi}\left(e^{2\pi\tau}-1\right)d\tau = \frac{20}{2\pi}\left(-t+\frac{e^{2\pi t}-1}{2\pi}\right)$$

<u>Problem 3</u> :

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = sY - y(0)$$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

we can write the ODE as

$$(s^2Y - 11s - 28) - (sY - 11) - 6Y = 0$$

$$(s^2 - s - 6)Y - 11s - 17 = 0$$

$$Y = \frac{11s+17}{s^2-s-6} = \frac{11s+17}{(s-3)(s+2)} = \frac{1}{s+2} + \frac{10}{s-3}$$

It's inverse laplace transform is

$$y = e^{-2t} + 10e^{3t}$$

<u>Problem 4 :</u>

We make a change of variable

$$\bar{t} = t - 1.5 \Rightarrow t = \bar{t} + 1.5$$

Then

$$y'(t) = \bar{y}'(\bar{t}) \quad , \quad y''(t) = \bar{y}''(\bar{t})$$

Then, we can rewrite the IVP as

$$\bar{y}''(\bar{t}) + 3\bar{y}'(\bar{t}) - 4\bar{y}(\bar{t}) = 6e^{2\bar{t}},$$

$$\bar{y}(0) = 4 \,, \; \bar{y}'(0) = 5$$

Making the laplace transform of both sides, we get

$$\left(s^2\bar{Y} - 4s - 5\right) + 3(s\bar{Y} - 4) - 4\bar{Y} = \frac{6}{s-2}$$

$$(s^2 + 3s - 4)\bar{Y} - 4s - 17 = \frac{6}{s-2}$$

$$(s+4)(s-1)\,\bar{Y} = \frac{6}{s-2} + 4s + 17$$

$$\bar{Y} = \frac{3}{s-1} + \frac{1}{s-2}$$

It's inverse laplace transform is

$$\bar{y}(\bar{t}) = 3e^{\bar{t}} + e^{2\bar{t}}$$

$$y(t) = \bar{y}(t - 1.5) = 3e^{t-1.5} + e^{2(t-1.5)}$$

Problem 5 : $L\{y'\} = sY$

$L\{y''\} = s^2 Y$

Take the laplace transform of the whole equation, then the ODE becomes

$$s^2Y + 6sY + 8Y = \frac{1}{s+3} - \frac{1}{s+5}$$

$$(s+4)(s+2)Y = \frac{2}{(s+3)(s+5)}$$

$$Y = \frac{2}{(s+2)(s+3)(s+4)(s+5)}$$

$$Y = \frac{1}{3(s+2)} - \frac{1}{s+3} + \frac{1}{s+4} - \frac{1}{3(s+5)}$$

It's inverse transform is

$$y(t) = \frac{1}{3}e^{-2t} - e^{-3t} + e^{-4t} - \frac{1}{3}e^{-5t}$$