

End-Semester Exam
DM, Monsoon 2021

Duration : 120 mins
Max marks : 25+1

1. (3 marks) Determine the number of subgraphs of the complete bipartite graph $K_{3,3}$ which are isomorphic to the cycle C_6 .
2. (3 marks) For each positive integer n , define the integers a_n and b_n by

$$(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}.$$

For instance, since $(1 + \sqrt{2})^3 = 7 + 5\sqrt{2}$, we have $a_3 = 7, b_3 = 5$. Prove that $\gcd(a_n, b_n) = 1$ for each positive integer n .

3. (3 marks) Determine the number of equivalence relations on the set $\{1, 2, \dots, 8\}$ which contain precisely 2 distinct equivalence classes.
4. (3 marks) What is the remainder when

$$12^{34^{56^{78}}}$$

is divided by 80?

5. (3 marks) Prove that

$$\frac{(3 + \sqrt{13})^n - (3 - \sqrt{13})^n}{\sqrt{13}}$$

is an integer divisible by 2^n for each nonnegative integer n .

6. (3 marks) Let a_n denote the number of surjective (onto) functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, 3\}$ such that $f(1) < f(2)$. Give a Θ estimate for a_n .
7. (3 marks) Give a combinatorial proof that

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{\ell} 2^k = \binom{n}{\ell} 2^\ell 3^{n-\ell}.$$

8. (4 marks) An engineer designs at least one robot a day for 30 days. If a total of 45 robots have been designed, then show that there must have been a series of consecutive days when exactly 14 robots were designed.
9. (1 mark (bonus)) Show that $\sqrt[5]{5}$ is irrational by using the well ordering principle.