Mid-semester exam	**Name:**	______________________	
Date : 29/02/2024	**Section:**	______________________	
MTH204: ODEs/PDEs	**Maximum Time:**	60 Minutes	
Semester: Winter 2024	**Maximum Marks:**	60	

DO NOT SHOW ANY WORK FOR PROBLEMS 1 to 7. JUST INDICATE THE RIGHT OPTION. ONLY ONE OPTION IS CORRECT. FOR ALL OTHER PROBLEMS, YOU NEED TO SHOW YOUR WORK.

Problem 1. [2] Find the integrating factor that would make the following equation exact:

$$y^2 + \sin x + xy\frac{dy}{dx} = 0.$$

(a) $e^{xy^2/2}$

(b) e^{y^2}

(c) $\frac{x^2y^2}{2}$

(d) $y \sin x$

(e) x

Problem 2. [2] Consider the IVP

$$\log(t)\frac{dy}{dt} - \frac{2y}{\cos(t)} = \frac{t^2}{2^t - 8}, \quad y(2) = \pi.$$

What is the largest interval on which a solution $y(t)$ is guaranteed to exist?

(a) $t > 0$

(b) $\frac{\pi}{2} < t < 3$

(c) $t < 1$

(d) $1 < t < \frac{3\pi}{2}$

(e) $3 < t < \frac{3\pi}{2}$

Problem 3. [2] Which formula describes implicitly the solution of the IVP

$$3e^x\frac{dy}{dx} - \frac{x}{y^2} = 0, \quad y(0) = 1.$$

(a) $3ye^x = x^2 + 3$

(b) $3e^x = \frac{x}{y} + 3$

(c) $e^x(x + y) = y^2$

(d) $y^3 + (x + 1)e^{-x} = 2$

(e) $y^3 + 2y = 3e^x + x$

Problem 4. [3] Consider the IVP

$$\frac{dy}{dt} = 2y^2 - 4y, \quad y(5) = 1.$$

Which of the following describes the nature of the solution?

(a) $\lim_{t\to-\infty} y(t) = 2$, $\lim_{t\to\infty} y(t) = 0$, inflection point at $y = 1$

(b) $\lim_{t\to-\infty} y(t) = 2$, $\lim_{t\to\infty} y(t) = \infty$, concave up

(c) $\lim_{t\to-\infty} y(t) = 0$, $\lim_{t\to\infty} y(t) = 4$, inflection point at $y = 2$

(d) $\lim_{t\to-\infty} y(t) = -\infty$, $\lim_{t\to\infty} y(t) = 0$, concave down

(e) $\lim_{t\to-\infty} y(t) = 0$, $\lim_{t\to\infty} y(t) = -\infty$, inflection point at $y = 1/2$

Problem 5. [3] The solution of the IVP

$$ty' + (t+1)y = te^{-t}, \quad t > 0, \quad y(1) = 2/e.$$

is

(a) $2e^{-t}$

(b) $te^{-t} + 1$

(c) $(t^2+1)e^{-t}$

(d) $\frac{1+t}{e^t}$

(e) $\frac{t^2+3}{2te^t}$

Problem 6. [2] Consider the exact first-order equation

$$\frac{y}{x} + 6x + (\log(x) - 2)y' = 0.$$

Which of the following is the general implicit solution to this equation?

(a) $y\log(x) + 3x^2 = C$

(b) $\frac{y^2}{2x} + 6xy = C$

(c) $(\log(x) - 1)x - 2x = C$

(d) $y\log(x) - 2y = C$

(e) $y\log(x) + 3x^2 - 2y = C$

Problem 7. [3] Which of the following is true about the differential equation

$$(3y^2 - 4x(y^3+1))\,dx + xy(2-3xy)\,dy = 0$$

(a) It is exact.

(b) It is homogeneous.

(c) It has an integrating factor that is a function of x alone.

(d) It has an integrating factor that is a function of y alone.

(e) None of the above.

Problem 8. [5] A tank originally has 100 liters of a brine with a concentration of 0.05 grams of salt per liter. Brine with concentration of 0.02 grams of salt per liter is pumped into the tank at a rate of 5 liters per second. The mixture is kept stirred and is pumped out at a rate of 4 liters per second. Find the amount of salt in the tank as a function of time. What will be the concentration (grams/litre) of salt in the tank as time tends to infinity.

Problem 9. [5] Consider the differential equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

and assume that $y_1(t) = t^{-1}$ is a solution. Use the reduction of order method to find a second linearly independent solution.

Problem 10. [5] Find the general solution of

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}.$$

Problem 11. [7] Find the general solution of

$$y''' - 3y'' + 2y' = \frac{e^{2x}}{1+e^x}.$$

Problem 12. [10] Find the general solution of

$$\begin{aligned}\frac{dx}{dt} &= x - 2y + 2z\\ \frac{dy}{dt} &= -2x + y - 2z\\ \frac{dz}{dt} &= 2x - 2y + z\end{aligned}$$

Problem 13. [4] For the system of equations

$$\frac{dx}{dt} = 3x - 18y$$
$$\frac{dy}{dt} = 2x - 9y$$

determine the classification and stability/instability of the critical point $(0, 0)$. If it is stable, is it also asymptotically stable?

Problem 14. [7] A 64 lb weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant being 18 lb/ft. The weight comes to rest in its equilibrium position. It is then pulled down 6 inches below its equilibrium position and released at $t = 0$. At this instant an external force given by $F(t) = 3\cos(\omega t)$ is applied to the system. (Use gravitational constant, $g = 32$ ft/s^2)

(a) Assuming there is no damping, determine the value of ω which gives rise to undamped resonance.

(b) Assuming that there is a damping force present and is numerically equal to $4(dy/dt)$, where dy/dt is the instantaneous velocity in feet per second, determine the resonance frequency of resulting motion. Remember that resonance frequency is the frequency which gives the maximum amplitude when external force $F(t)$ is absent.