MTH-204: Worksheet 1 (Solutions)

10 points

(2pts.)

Q-1 Let $y = cx - c^2 \Rightarrow y' = c$

Putting these values in LHS of DE, we get

$$y'^2 - xy' + y = \cancel{c^2} - \cancel{cx} + \cancel{cx} - \cancel{c^2} = 0$$

Therefore, $y = cx - c^2$ is a general solution of DE since it satisfies it and has an arbitrary constant as well.

Next, let $y = \frac{x^2}{4} \Rightarrow y' = \frac{x}{2}$

Then again putting values in LHS of DE, we get

$$y'^2 - xy' + y = \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0$$

Therefore, $y = \frac{x^2}{4}$ is a particular solution of DE.

Now, if $cx - c^2 = \frac{x^2}{4} \Rightarrow c = c^2 = 0 \Rightarrow c = 0$

Therefore, $\frac{x^2}{4}$ can't be obtained from $cx - c^2$ for any value of c.

Hence, $\frac{x^2}{4}$ is a singular solution of DE.

($1 \times 2 = 2$pts.)

Q-2 Let $y(t)$ be amount of Radium left after time t. Then, we know that

$$y'(t) = ky \quad \text{for some constant } k.$$

Solving this, we get

$$y(t) = Ce^{kt} \quad \text{——} \quad (1)$$

Putting $t = 0$ in (1), we get $C = y(0)$

$$\Rightarrow y(t) = y(0)\, e^{kt} \quad \text{——} \quad (2)$$

Now, half-life is given as 3.6 days, so, for $t = 3.6$, we have

$$y(3.6) = \frac{y(0)}{2}$$

Putting in ②, we get

$$\frac{y(0)}{2} = y(0)\, e^{k \times 3.6}$$

$$\Rightarrow e^{3.6k} = \frac{1}{2} \quad \Rightarrow 3.6k = -\ln 2$$

$$\Rightarrow k = -\frac{\ln 2}{3.6}$$

So, $\boxed{y(t) = y(0)\, e^{-\frac{\ln 2}{3.6} t}}$

(a) $y(0) = 1g$

Then, $y(1) = y(0)\, e^{-\frac{\ln 2}{3.6} \cdot 1} = e^{-\frac{\ln 2}{3.6}}$

$$\Rightarrow y(1) = e^{\ln 2^{-1/3.6}} = 2^{-\frac{1}{3.6}} \approx 0.825$$

$$\therefore \boxed{y(1) \approx 0.825g}$$

(b) $y(365) = 1 \cdot e^{-\frac{\ln 2}{3.6} \cdot 365} = 2^{-\frac{365}{3.6}} \approx 3.012 \times 10^{-31}$

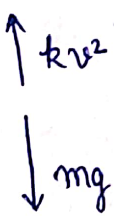
$$\therefore \boxed{y(365) \approx 3.012 \times 10^{-31} g}$$

(2 pts.)

Q-3 Attraction by earth $= mg$

Air resistance $= kv^2$, k is proportionality constant

Resultant of the forces $= mg - kv^2$



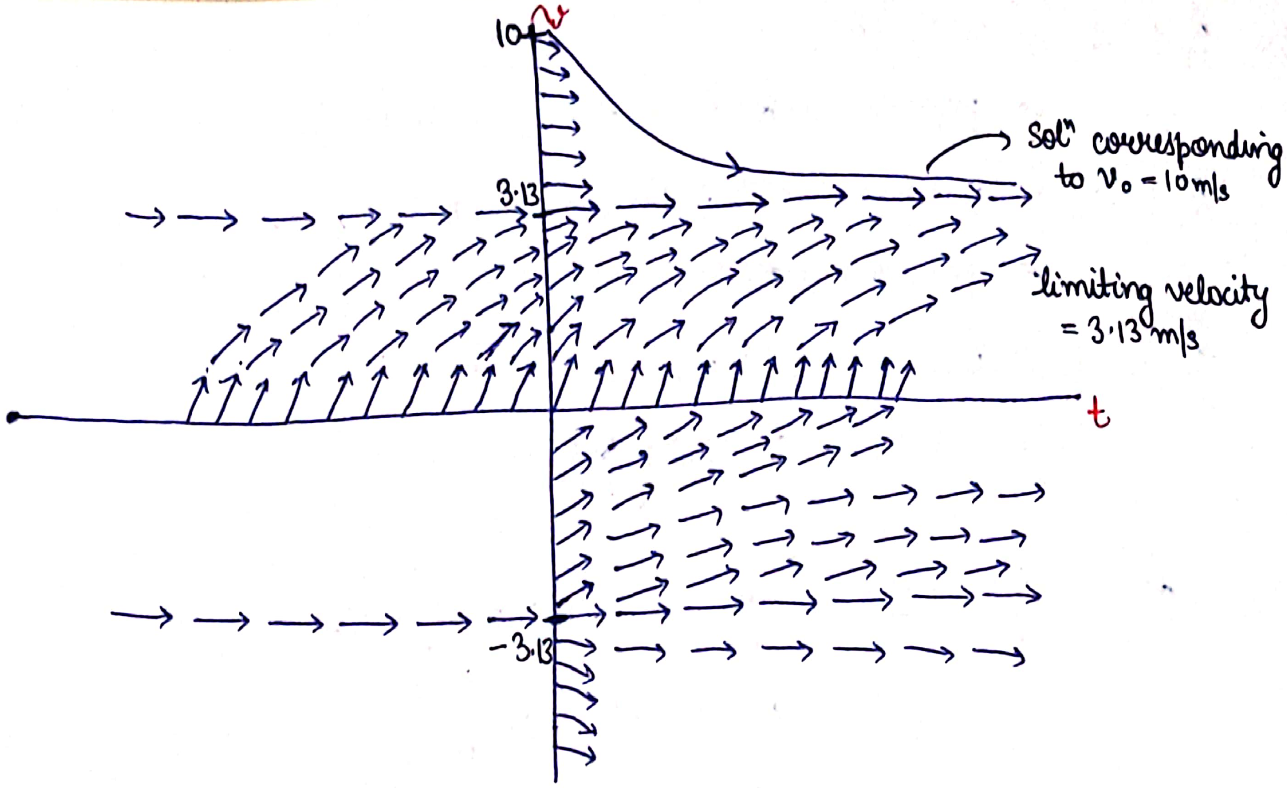
Using Newton's 2nd law,

$$mv' = mg - kv^2$$

Let $m = k = 1$, then

$$v' = 9.8 - v^2 = f(t, v)$$

For limiting velocity, $v^2 - 9.8 = 0 \Rightarrow v = \pm 3.13$



Yes, parachute will still be sufficient if air resistance $= kv$.

(1×2=2pts.)

Q-4 Let $y(t)$ be the number of bacteria at time t.
Let $B(t)$ be the number of bacteria born at time t.
Let $D(t)$ be the number of bacteria dead at time t.

(a) Now, $\frac{dy}{dt} = \frac{dB}{dt} - \frac{dD}{dt}$

$$\Rightarrow \frac{dy}{dt} = k_1 y - k_2 y \Rightarrow \frac{dy}{dt} = (k_1 - k_2)\, y$$

$$\Rightarrow \frac{dy}{y} = (k_1 - k_2)\,dt$$

$$\Rightarrow \ln y = (k_1 - k_2)t + \ln C$$

$$\Rightarrow y = C e^{(k_1-k_2)t}$$

At $t=0$, $y(0) = C e^{(k_1-k_2)\cdot 0} \Rightarrow C = y(0)$

$$\therefore \boxed{y(t) = y(0)\, e^{(k_1-k_2)t}}$$

(b) If $k_1 = k_2 \Rightarrow y(t) = y(0)$

$\Rightarrow$ population doesn't change with time.

If $k_1 > k_2 \Rightarrow y(t) = y(0)e^{kt}$ where $k = k_1 - k_2 > 0$

$\Rightarrow$ population grows exponentially fast.

If $k_1 < k_2 \Rightarrow y(t) = y(0)e^{-kt}$ where $k = k_2 - k_1 > 0$

$\Rightarrow$ population decreases exponentially.

(1 pt.)

Q-5 $\frac{dV}{dp} = -\frac{V}{p} \Rightarrow \frac{dV}{V} = -\frac{dp}{p}$

$\Rightarrow \ln V = -\ln p + \ln C = \ln \frac{C}{p}$

$\Rightarrow \boxed{V = \frac{C}{p}}$, for some constant C.

(1 pt.)

Q-6 $y' = -Ay\ln y$, $A > 0$

$$y\ln y = \begin{cases} >0 & \text{for } y>1 \\ =0 & \text{for } y=1 \\ <0 & \text{for } y<1 \;\&\; y>0 \; (0<y<1) \end{cases}$$

So, if tumor mass at any time t, $y(t) > 1$, then growth of tumor cells will decrease since $y'(t) < 0$.

Conversely, if $y(t) < 1$, then tumor cells will grow since $y'(t) > 0$.

$y \equiv 1$ is a constant solution of ODE.

$$\int \frac{dy}{y\ln y} = -\int A\,dt$$

$\Rightarrow \ln(\ln y) = -At + \ln C$

$\Rightarrow \ln y = Ce^{-At}$

$\Rightarrow \ln y(t) = Ce^{-At}$

$\Rightarrow \boxed{y(t) = e^{Ce^{-At}}}$