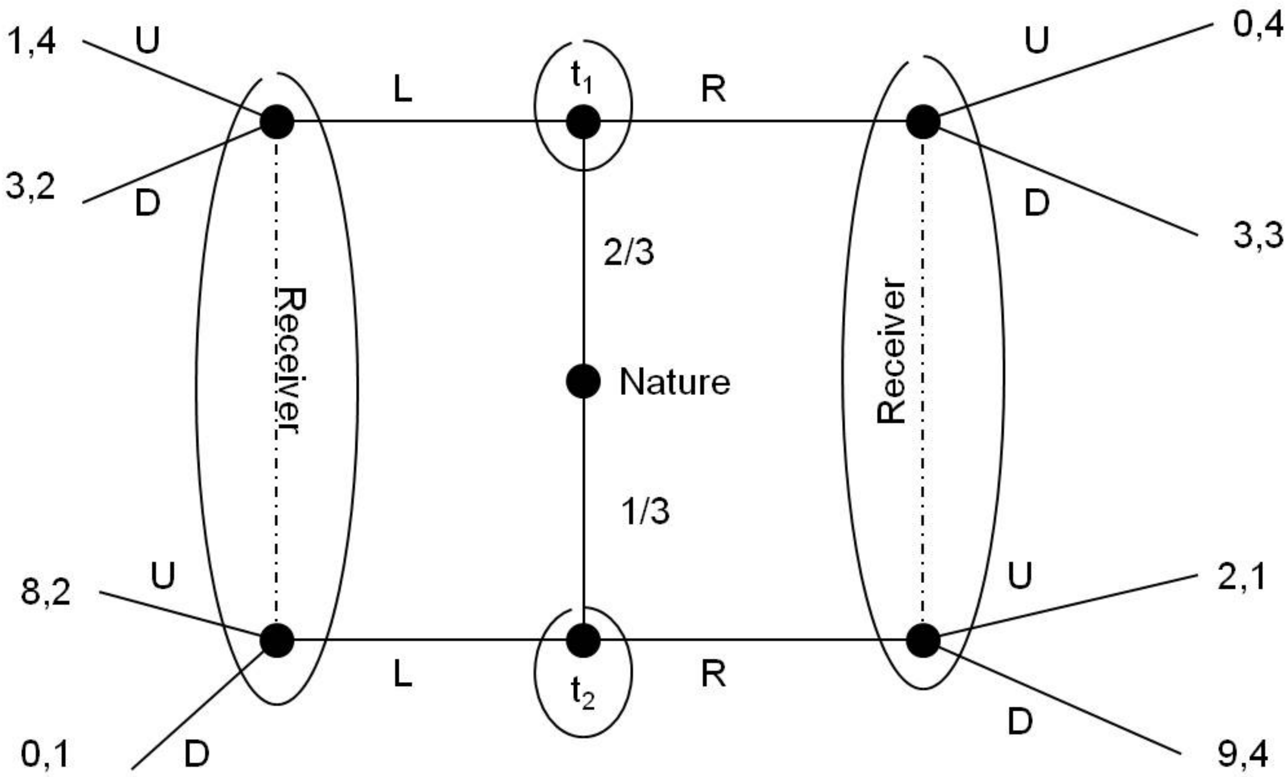
1. Consider the following game. Note that the probability of being a sender type t_1 is $\frac{2}{3}$ and the probability of being a sender type t_2 is $\frac{1}{3}$.



a Find all pure strategy separating perfect Bayesian equilibria. If there are none explain why not.

Answer:

Consider first the potential separating equilibrium where type t_1 chooses L and type t_2 chooses R. In this case the Receiver will choose U if L and D if R. Type t_1 receives 1 from choosing L and 3 if he switches to R so this will not be an equilibrium.

Consider now the potential separating equilibrium where type t_1 chooses R and type t_2 chooses L. In this case the Receiver will choose U if R and U if L. Type t_1 receives 0 from choosing R and 1 if he switches to L so this will not be an equilibrium.

Thus, there are no separating equilibria to this game.

b Find all pure strategy pooling perfect Bayesian equilibria. If there are none explain why not.

Answer:

One thing that you hopefully noticed from part a is that the Receiver will always choose U if L is observed. We can also see this by letting p be the probability that type t_1 chooses L and $1-p$ being the probability that type t_2 chooses L. Then:

$$\begin{aligned} E[U|L] &= 4p + 2(1-p) \\ E[D|L] &= 2p + 1(1-p) \end{aligned}$$

and so:

$$\begin{aligned} E[U|L] &> E[D|L] \\ 4p + 2 - 2p &> 2p + 1 - p \\ 2p + 2 &> p + 1 \\ p &> -1 \end{aligned}$$

Consider the pooling equilibrium where both types choose R. The Receiver's beliefs are $\Pr(t_1|R) = \frac{2}{3}$ and $\Pr(t_2|R) = \frac{1}{3}$. The Receiver's expected value of choosing U and D are:

$$\begin{aligned} E[U] &= 4 * \frac{2}{3} + 1 * \frac{1}{3} = \frac{9}{3} \\ E[D] &= 3 * \frac{2}{3} + 4 * \frac{1}{3} = \frac{10}{3} \end{aligned}$$

so that the Receiver would choose D if R is observed. We already know that the Receiver will choose U if L is observed (regardless of the probabilities) so we need to check that both types choosing R is a best response. Type t_1 receives 3 from choosing R and would receive 1 if he switched to L, and type t_2 receives 9 from choosing R and would receive 8 if he switched. So one pooling equilibrium is:

$$\begin{aligned} & \text{Type } t_1 \text{ choose } R \\ & \text{Type } t_2 \text{ choose } R \\ \Pr(t_1|R) &= \frac{2}{3} \\ \Pr(t_2|R) &= \frac{1}{3} \\ \Pr(t_1|L) &= p \in [0,1] \\ \Pr(t_2|L) &= 1-p \\ & \text{Receiver chooses } U \text{ if } L \\ & \text{Receiver chooses } D \text{ if } R \end{aligned}$$

Now consider the potential pooling equilibrium where both types choose L. We already know that the Receiver will choose U if L is observed so we need to determine when the Receiver will choose U and D for the off-the-equilibrium path choice of R. Let p be the probability that t_1 chooses R and $1-p$ be the probability that t_2 chooses R. Then the Receiver's expected values for choosing U and D respectively are:

$$\begin{aligned} E[U|R] &= 4 * p + 1 * (1-p) = 4p + 1 - p = 3p + 1 \\ E[D|R] &= 3 * p + 4 * (1-p) = 3p + 4 - 4p = 4 - p \end{aligned}$$

Then:

$$\begin{aligned} E[U|R] &\geq E[D|R] \\ 3p + 1 &\geq 4 - p \\ 4p &\geq 3 \\ p &\geq \frac{3}{4} \end{aligned}$$

So if $p \geq \frac{3}{4}$ then the Receiver will choose U and if $p \leq \frac{3}{4}$ the Receiver will choose D. When the Receiver chooses D both types would switch but when the Receiver chooses U neither type would switch. So there is a second pooling equilibrium where both types choose L:

$$\begin{aligned}
& \text{Type } t_1 \text{ chooses } L \\
& \text{Type } t_2 \text{ chooses } L \\
\Pr(t_1|L) &= \frac{2}{3} \\
\Pr(t_2|L) &= \frac{1}{3} \\
\Pr(t_1|R) &= p \in \left[\frac{3}{4}, 1\right] \\
\Pr(t_2|R) &= 1 - p \\
& \text{Receiver chooses } U \text{ if } L \\
& \text{Receiver chooses } U \text{ if } R
\end{aligned}$$

2. A buyer and seller have valuations v_b and v_s. It is common knowledge that there are gains from trade (i.e. $v_b > v_s$), but the size of the gains is private information as follows: the seller's valuation is uniformly distributed on $[0, 1]$; the buyer's valuation $v_b = k * v_s$, where $k > 1$ is common knowledge; the seller knows v_s (and hence v_b) but the buyer does not know v_s or v_b. Suppose the buyer makes a single offer, p, which the seller either accepts or rejects. Based on Samuelson (1984).

a What is the perfect Bayesian equilibrium when $k < 2$?

Answer:

Suppose that the buyer offers price p to the seller. The seller will only accept if $p \geq v_s$. The buyer knows this, which is key. Thus, for any price p offered by the buyer, the expected value of the seller conditional on the offer price of p is:

$$E[v_s|p] = \frac{p}{2}$$

This is because the seller's value, if he accepts the offer, will now range between $[0, p]$, so the expected value of a seller who accepts the offer of p is just halfway between 0 and p. Thus, the expected value of the buyer, conditional on price p, is:

$$E[v_b|p] = \frac{kp}{2}$$

So the buyer's profit, for any p is:

$$\Pi_b = \frac{kp}{2} - p$$

Note that when $k < 2$ we have $\Pi_b < 0$, unless $p = 0$. To see this:

$$\begin{aligned}
\Pi_b &> 0 \\
\frac{kp}{2} - p &> 0 \\
\frac{kp}{2} &> p \\
k &> 2
\end{aligned}$$

Thus, the buyer should offer a price $p = 0$ when $k < 2$. The buyer believes that the seller's value, conditional on p, is $\frac{p}{2}$, and thus his own value is $\frac{kp}{2}$. The seller chooses to accept if $p \geq v_s$ and to reject if $p < v_s$.

b What is the perfect Bayesian equilibrium when $k > 2$?

Answer:

When $k > 2$ then the buyer can simply offer $p = 1$, the buyer believes the seller's value, conditional on p, is $\frac{p}{2}$, and the seller accepts the offer if $p \geq v_s$ and rejects the offer if $p < v_s$. Note that the outcome based on this equilibrium will always have the seller accepting the offer, since p is at least as large as the seller's highest possible value, but that we still need to specify what the seller would do if he saw a $p < 1$ to be complete. We can check to make sure that the buyer should not offer a different price, perhaps a $p > 1$ or a $p < 1$. Let's rule out the easy one first.

It is clear that the buyer should not offer some $p^* > 1$. Note that offering a $p^* > 1$ does not alter the buyer's expected value, does not change the seller's response, and simply lowers the profit which the buyer receives. In this case, it is like the Ultimatum Game – why give the other player any more than is necessary?

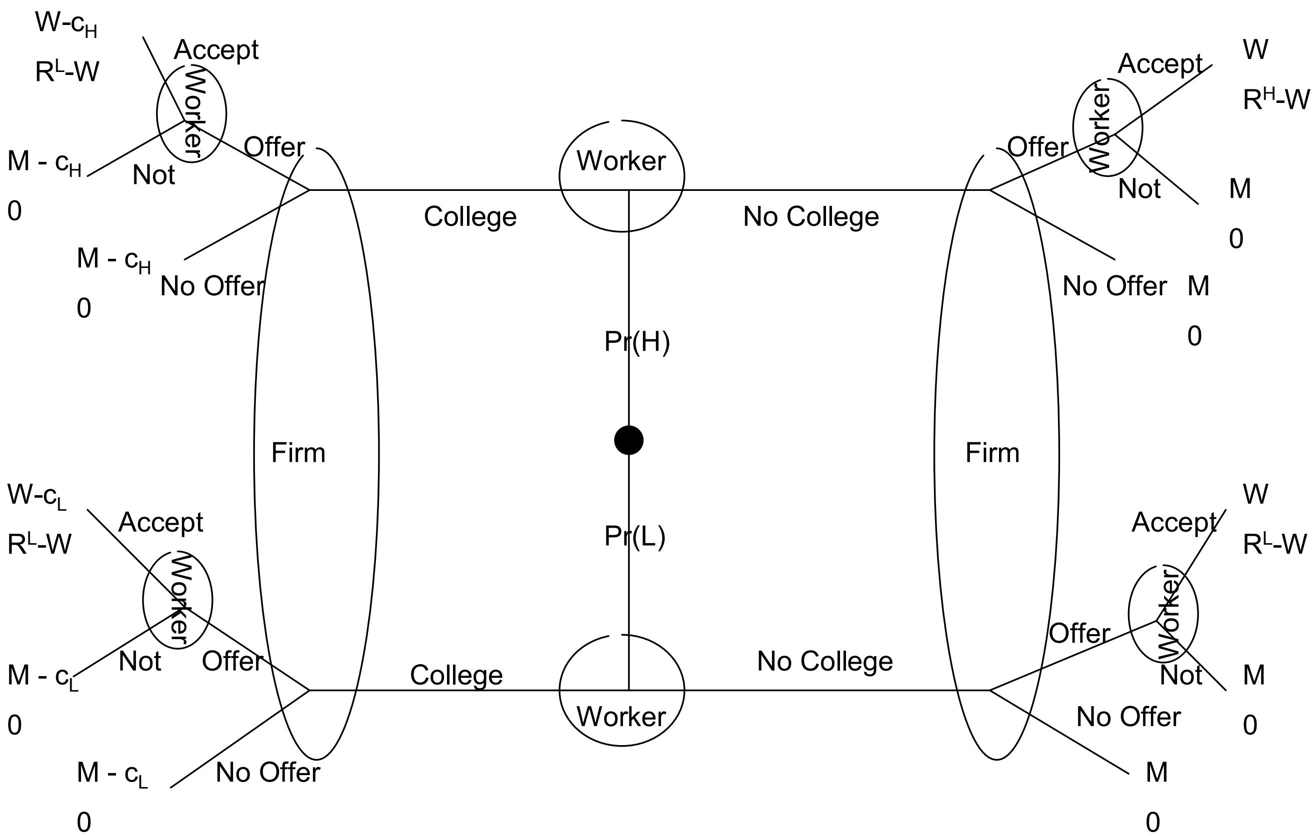
Now consider some $p^* < 1$. Note that this DOES change the buyer's expected value of the good when he receives the good – it is again $\frac{kp}{2}$. However, the buyer will make a profit now. Consider $k = 3$ and $p = 0.75$. The buyer now expects that the item will be worth 1.125 on average (if $p = 0.75$) and he is paying 0.75 for the item. Thus, he earns 0.375 on average. What if he offers \$1? Now he expects that the item is worth \$1.5 and only pays \$1 for it so earns 0.5 on average. Basically, the average utility (or profit) gain to the buyer by increasing his offer more than offsets the additional cost of the increased p.

3. There are two basic schools of thought on what exactly a college education is. The first school of thought is that students go to college to learn skills that will help them in their future career. The second school of thought is that a college degree is useful in the sense that it sends a signal to your prospective employers that you are able to be trained and willing to work hard enough for 4 (or 5 or 6 or 7 or perhaps only 3) years to get a degree, but that it does not really prepare you for your future career. Consider the following: There are two types of workers, high ability and low ability. The type of worker is randomly determined by chance with $\Pr(high)$ being the probability of a high type and $\Pr(low)$ being the probability of a low type, where $\Pr(low) = 1 - \Pr(high)$. The workers must make a decision regarding school. They can either go to college or not go to college. If a high ability worker attends college it costs him C^H. If a low ability worker attends college it costs him C^L. Assume that $C^L > C^H$ because the low ability worker will have to work harder in college than the high ability worker.

   There is an employer who must make a hiring decision. The employer is unable to observe type but is able to observe whether or not the worker has a college education. The employer must choose to either offer the worker the job or not offer the worker the job, and this must be made for both types of people – those who went to college and those who did not. If the employer offers a job, the wage is W regardless of worker type and whether or not they went to college. If the employer does not offer a job, then the worker receives his opportunity wage of M. If a job is offered the worker must decide whether or not to accept the job – if the job is accepted then the worker receives the wage of W (minus any costs incurred if he went to college) and if the job is not accepted the worker receives M (minus any costs incurred if he went to college). A high ability worker is worth R^H to the employer if he accepts the offer while a low ability worker is worth R^L to the employer, regardless of whether or not the worker went to college. Assume that $R^H > R^L$. If the worker does not accept the job then the employer receives 0.

   a Draw the extensive form of this game. Leave the payoffs as variables for now.

   Answer:



b Propose a Perfect Bayesian Equilibrium that is a pooling equilibrium where all workers attend college. Verify that this is in fact a Perfect Bayesian Equilibrium by stating the relevant constraints.

Answer:

A Perfect Bayesian equilibrium that is a pooling equilibrium is as follows:

Worker's strategy: Attend college if high type, attend college if low type, accept offer regardless of type and decision to attend college.

Firm's beliefs: Since all workers attend college in this equilibrium there is no additional information conveyed by the action taken by the workers. Thus, the probability that a worker who attends college is high ability, $\Pr(High|College) = \Pr(high)$ and the probability that a worker who attends college is low ability, $\Pr(Low|College) = \Pr(low)$. Since no workers attend college, let $\Pr(High|No\ College) = q$ and $\Pr(Low|No\ College) = 1 - q$

Firm's strategy: Offer a job if the worker attends college, do not offer a job if he does not attend

The key here is to note that the firm needs to not offer a job if the worker does not attend college. If the firm offered a job to those who did not attend college then no one would go to college as they could receive W without incurring the costs of college.

We need the following:

- In order for workers to accept we need:

$W - C^H \geq M - C^H$ for the high ability type, if they go to college

$W - C^L \geq M - C^L$ for the low ability type, if they go to college

$W \geq M$ for those who do not go to college.

All of these lead to needing just $W \geq M$.

- In order for workers to go to college we need:

$W - C^H \geq M$ for high ability workers who attend college (they would get M if they chose no college)

$W - C^L \geq M$ for low ability workers who attend college (they would get M if they chose no college)

Since $C^L > C^H$, all we really need for the workers is $W - C^L \geq M$. Both $W \geq M$ and $W - C^H \geq M$ follow from that assumption.

- The firm must also be maximizing its expected utility using its strategy, so:

$\Pr(high) * \left(R^H - W\right) + \Pr(low) * \left(R^L - W\right) \geq 0$ for offering to those that attend college

$q * \left(R^H - W\right) + (1-q) * \left(R^L - W\right) \leq 0$ for not offering to those that do not attend college

For q we would need:

$$\begin{aligned} qR^H + R^L - qR^L - W &\leq 0 \\ q\left(R^H - R^L\right) &\leq W - R^L \\ q &\leq \frac{W - R^L}{R^H - R^L} \end{aligned}$$

Essentially, if the employer believes the probability that a high ability worker is choosing no college is too high, then the employer will offer the job if no college is observed. But if they put very low probability on observing a high ability worker given no college, then they will not offer a job. So a perfect Bayesian equilibrium in which all workers attend college is:

Worker's strategy:

Attend if high type, attend if low type; accept offer regardless of type and decision to attend college

Employer's beliefs:

$\Pr(High|College) = \Pr(High)$; $\Pr(Low|College) = \Pr(Low)$

$\Pr(High|NoCollege) \leq \frac{W-R^L}{R^H-R^L}$; $\Pr(Low|NoCollege) = 1 - \Pr(High|NoCollege)$

Employer's strategy:

Offer if worker attends college, not offer if the worker does not attend college

This is an equilbrium if: $W - C^L \geq M$; $\Pr(high) * \left(R^H - W\right) + \Pr(low) * \left(R^L - W\right) \geq 0$; and the restrictions on q hold.

c Propose a Perfect Bayesian Equilibrium that is a separating equilibrium where only high ability workers attend college. Verify that this is in fact a Perfect Bayesian Equilibrium by stating the relevant constraints.

Answer:

A Perfect Bayesian equilibrium that is a separating equilibrium is as follows:

Worker's strategy: Attend college if high ability, do not attend college if low ability; accept offer regardless of type and decision to attend college

Firm's beliefs: The probability that a worker who attends college is high ability is now 1, or $\Pr(High|College) = 1$. The probability that a worker who attends college is low ability is now 0 or $\Pr(Low|College) = 0$. The probability that a worker who does not attend college is low ability is 1. The probability that a worker who does not attend college is high ability is 0. In a separating equilibrium agents are able to update their beliefs based on some action taken – in this game, the action that allows the firm to update its belief is the worker's decision to attend college.

Firm's strategy: Offer if college, do not offer if no college

The conditions that must now hold are:

$W - C^H \geq M$ for high ability workers who go to college

$W - C^L \leq M$ for low ability workers who go to college

$W \geq M$ for all workers

For the firm, the conditions are now:

$R^H - W \geq 0$ for offering to those that attend college

$R^L - W \leq 0$ for not offering to those that do not attend college

d Suppose that $R^H = 20$, $R^L = 15$, $C^L = 12$, $C^H = 8$, $M = 12$, $W = 19$, and $\Pr(high) = 0.5$.

- Will there be a pooling equilibrium or a separating equilibrium?

- Which workers will attend college and which will not?

Answer:

There will be a pooling equilibrium in which NO workers attend college. To see this note that it is a dominant strategy for both types of worker to not attend college. If the high ability worker attends college, the most he will be able to receive is 11, if the firm offers a job. However, he receives 12 just by staying at his job paying M, so there is no incentive to go to college. Also, note that the firm will choose not to offer any jobs. The expected revenue to the firm if it offers a job is $.5 * 1 + .5 * (-4) = -1.5$. Since this is worse than the payoff of 0 to the firm if it did not offer a job, it will choose not to offer a job. Thus we have a pooling equilibrium where no workers attend college. Remember, a pooling equilibrium only requires that all the players act in the same way, not that they act in the "right" way (if you assume going to college is the right way).