MTH-204: ODEs/PDEs
Semester: Winter 2024
MidSem Solutions
Date: 29 Feburary 2024

1. (e) x

2. (b) $\frac{\pi}{2} < t < 3$

3. (d) $y^3 + (x+1)e^{-x} = 2$

4. (a) $\lim_{t\to-\infty} y(t) = 2$, $\lim_{t\to\infty} y(t) = 0$, inflection point at $y = 1$

5. (e) $\frac{t^2+3}{2te^t}$

6. (e) $y\log(x) + 3x^2 - 2y = C$

7. (c) It has an integrating factor that is a function of x alone.

Problem 8 :

let's denote

- $A(t)$ as the amount of salt in the tank at time t (in grams).
- $C(t)$ as the concentration of salt in the tank at time t (in grams per litre).

Given :

- Initial amount of salt in the tank :
  $A(0) = 100 \text{ litres} \times 0.05 \text{ gm/l} = 5 \text{ grams}$
- Rate at which brine is pumped into the tank :
  5 l/s
- Concentration of salt in the incoming brine : 0.02 g/l
- Rate at which the mixture is pumped out :
  4 l/s.

The differential equation governing the amount of salt in the tank :

$$\frac{dA}{dt} = (\text{rate in}) - (\text{rate out})$$

rate in = concentration of salt in the incoming brine multiplied by the rate at which brine is pumped into the tank.

rate in $= 0.02 \times 5 = 0.1 \; g/s$ — 1/2

rate out = concentration of salt in the tank multiplied by the rate at which the mixture is pumped out.

rate out $= 4\,c(t)$ — 1/2

so, the differential equation becomes:

$$\frac{dA}{dt} = 0.1 - 4\,c(t)$$

$c(t) = \dfrac{A(t)}{V(t)}$, where $V(t)$ is the volume of solution in the tank at time t. Since the volume is changing due to inflow and outflow, we need to express $V(t)$ as well.

$$V(t) = 100 + (5-4)t = 100 + t$$

so, differential equation is

$$\frac{dA}{dt} = 0.1 - \frac{4A}{100+t} \quad \text{—— (1)}$$

$$\frac{dA}{dt} + \frac{4A}{100+t} = 0.1$$

Integrating factor, $e^{\int \frac{4}{100+t}dt} = e^{4\ln(100+t)}$

$$= (100+t)^4$$

Solution,

$$(A(t))(100+t)^4 = \int (0.1)(100+t)^4 dt + c$$

$$(A(t))(100+t)^4 = \frac{0.1}{5}(100+t)^5 + c$$

$$A(t) = (0.02)(100+t) + \frac{c}{(100+t)^4} \quad \text{—— (1)}$$

Initial condition : $A(0) = 5$ grams — $\frac{1}{2}$

$$5 = 2 + \frac{C}{(105)^4} \; ; \quad C = 3(105)^4$$

$$C(t) = \frac{A(t)}{100+t} = 0.02 + \frac{C}{(100+t)^5}$$

$$C(t) = 0.02 + \frac{3(105)^4}{(100+t)^5}$$ — $\frac{1}{2}$

as $t \to \infty$

$C(t) \to 0.02$ — 1

Problem 9: [5 Marks] Consider the differential equation

$$2t^2y'' + 3ty' - y = 0 \qquad , t > 0$$

and assume that $y_1(t) = t^{-1}$ is a solution. Use the reduction of order method to find a second linearly independent solution.

Solution: We seek a solution of the form $y_2(t) = v(t)\, t^{-1}$.
This gives us :

$$y_2'(t) = -v(t)\, t^{-2} + v'(t)\, t^{-1}$$

$$y_2''(t) = 2v(t)\, t^{-3} - 2v'(t)\, t^{-2} + v''(t)\, t^{-1}$$

(1)

Substituting in the original equation gives;

$$\Rightarrow 2t^2\left(2v(t)\, t^{-3} - 2v'(t)\, t^{-2} + v''(t)\, t^{-1}\right) + 3t\left(-v(t)t^{-2} + v(t)\, t^{-1}\right) - v'(t)t^{-1} = 0$$

$$\Rightarrow \; 2t\, v''(t) - v'(t) = 0 \qquad \text{—— (1)}$$

Seperating the variable gives $\dfrac{v''(t)}{v'(t)} = \dfrac{1}{2t}$ which has a solution $v'(t) = t^{1/2}$ and taking an anti-derivative, we get; (1)

$$\Rightarrow v(t) = \frac{2}{3}t^{3/2} \text{ or } C\, t^{3/2} \text{ or } t^{3/2} \qquad \text{—— (1)}$$

$$\Rightarrow \therefore\; y_2(t) = v(t)\cdot t^{-1} = \frac{2}{3}t^{1/2} \text{ or } t^{1/2} \qquad \text{—— (1)}$$

Find the general Solution

Q 10 $\quad y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

This is a linear inhomogeneous Second order differential eqⁿ of form →

$$y'' + Py' + qy = S$$

where

$$P = -6$$
$$q = 9$$
$$S = +6x^2 + 2 - 12e^{3x}$$

First of all, we need to solve Corresponding Homogeneous eqⁿ →

$$y'' + Py' + qy = 0$$

In our Case $\quad y'' - 6y' + 9y = 0$

characterstick eqⁿ will be →

$$K^2 - 6K + 9 = 0$$
$$K^2 - 3K - 3K + 9 = 0$$
$$K(K-3) - 3(K-3) = 0$$
$$(K-3)^2 = 0$$

— 1

$K = 3$ is the only root of characterstick eqⁿ.

So the Solⁿ of Homogenous eqⁿ will be of form →

$$y_c(x) = C_1 e^{K_1 x} + C_2 x e^{K_1 x}$$
$$y_c(x) = C_1 e^{3x} + C_2 x e^{3x}$$

0.5

Now we will solve inhomogeneous eqⁿ →

Particular Solⁿ will be of form →

$$y_P = A + Bx + Cx^2 + De^{3x} + Exe^{3x} + Fx^2e^{3x}$$

$$y_P' = B + 2Cx + 3De^{3x} + Ee^{3x} + 3Exe^{3x} + 2Fxe^{3x} + 3Fx^2e^{3x}$$

1

$$y_P'' = 2C + 9De^{3x} + 3Ee^{3x} + 9Exe^{3x} + 3Ee^{3x} + 2Fe^{3x} + 6Fxe^{3x} + 6Fxe^{3x} + 9Fx^2e^{3x}$$

Now Put it in eqⁿ → $y_P'' - 6y_P' + 9y_P = 6x^2 + 2 - 12e^{3x}$

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$$2C + 9De^{3x} + 3Ee^{3x} + 9Exe^{3x} + 3Ee^{3x} + 2Fe^{3x}$$
$$+ 6Fxe^{3x} + 6Fxe^{3x} + 9Fx^2e^{3x} - 6B - 12Cx - 18De^{3x}$$
$$- 6Ee^{3x} - 18Exe^{3x} - 12Fxe^{3x} - 18Fx^2e^{3x} + 9A + 9Bx$$
$$+ 9Cx^2 + 9De^{3x} + 9Exe^{3x} + 9Fx^2e^{3x} = 6x^2 + 2 - 12e^{3x} \quad (1)$$

$$2C + 2Fe^{3x} - 6B - 12Cx + 9A + 9Bx + 9Cx^2 = 6x^2 + 2 - 12e^{3x}$$

$$2C - 6B + 9A + (9B - 12C)x + 9Cx^2 + 2Fe^{3x} = 6x^2 + 2 - 12e^{3x}$$

Comparing Coefficient of x, x^2, e^{3x} and Constants on both sides →

$$2F = -12 \Rightarrow F = -6$$

$$9C = 6 \Rightarrow C = 2/3$$

$$9B - 12C = 0 \Rightarrow B = 4C/3 = 8/9$$

$$2C - 6B + 9A = 2 \Rightarrow A = \frac{2 - 2C + 6B}{9} = \frac{2}{3}$$

0.5

Substituting these values in y_p gives →

$$y_p = A + Bx + Cx^2 + De^{3x} + Exe^{3x} + Fx^2e^{3x}$$

$$y_p = \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2e^{3x}$$

General Solⁿ → $y = y_c + y_p$ (1)

$$y = C_1e^{3x} + C_2xe^{3x} + \frac{2}{3} + \frac{8}{9}x + \frac{2}{3}x^2 - 6x^2e^{3x}$$

Problem 11: 7 Marks

First find the solution of the homogeneous problem.
We will find the roots of the characteristic equation.

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$
$$\Rightarrow \lambda(\lambda^2 - 3\lambda + 2) = 0$$
$$\Rightarrow \lambda = 0, 1, 2.$$

So, the homogeneous solution is

$$y_h = c_1 + c_2 e^x + c_3 e^{2x} \qquad - ①$$

Wronskian of the functions y_1, y_2 and y_3 is

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x} = 2e^{3x}$$

Particular solution is given by

$$y_p = y_1 \int \frac{W_1}{W} r(x)\,dx + y_2 \int \frac{W_2}{W} r(x)\,dx + y_3 \int \frac{W_3}{W} r(x)\,dx \qquad - ①$$

$$W_1 = \begin{vmatrix} 0 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 1 & e^x & 4e^{2x} \end{vmatrix} = e^{3x}$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & 1 & 4e^{2x} \end{vmatrix} = -2e^{2x} \qquad - ①$$

$$W_3 = \begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & 1 \end{vmatrix} = e^x$$

$$\int \frac{W_1}{W} r(x)\,dx = \int \frac{e^{3x}}{2e^{3x}} \cdot \frac{e^{2x}}{1+e^x}\,dx = \frac{1}{2}\int \frac{e^{2x}}{1+e^x}\,dx$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$= \frac{1}{2}\int \frac{t\,dt}{1+t} = \frac{1}{2}\int \frac{t+1-1}{1+t}\,dt = \frac{1}{2}\int dt - \frac{1}{2}\int \frac{1}{1+t}\,dt = \frac{1}{2}t - \frac{1}{2}\log(1+t)$$

$$= \frac{1}{2}e^x - \frac{1}{2}\log(1+e^x) \qquad - ①$$

$$\int \frac{W_2}{W} r(x)\,dx = \int -\frac{2e^{2x}}{2e^{3x}} \cdot \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{e^{x}}{1+e^x}\,dx$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$= -\int \frac{dt}{1+t} = -\log(1+t) = -\log(1+e^x) \qquad -①$$

$$\int \frac{W_3}{W} r(x)\,dx = \int \frac{e^x}{2e^{3x}} \cdot \frac{e^{2x}}{1+e^x}\,dx = \frac{1}{2}\int \frac{1}{1+e^x}\,dx$$

Put $1+e^x = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{t-1}$

$$= \frac{1}{2}\int \frac{1}{t(t-1)}dt = \frac{1}{2}\int\left[\frac{1}{t-1} - \frac{1}{t}\right]dt = \frac{1}{2}\left[\log(t-1) - \log(t)\right]$$

$$= \frac{1}{2}\left[\log(e^x) - \log(1+e^x)\right] = \frac{1}{2}\log\frac{e^x}{1+e^x} \qquad -①$$

So, $y_p = \frac{1}{2}e^x - \frac{1}{2}\log(1+e^x) + e^x\left[-\log(1+e^x)\right] + \frac{e^{2x}}{2}\cdot\log\left(\frac{e^x}{1+e^x}\right)$

The general solution is

$y = y_h + y_p$

$$= c_1 + c_2e^x + c_3e^{2x} + \frac{1}{2}e^x - \frac{1}{2}\log(1+e^x) - e^x\log(1+e^x) + \frac{e^{2x}}{2}\log\left(\frac{e^x}{1+e^x}\right).$$

$$= c_1 - \frac{1}{2}\log(1+e^x) + \left[c_2 + \frac{1}{2} - \log(1+e^x)\right]e^x$$

$$+ \left[c_3 + \frac{1}{2}\log\left(\frac{e^x}{1+e^x}\right)\right]e^{2x} \qquad -①$$

12. Given the System of ODE's

$$\frac{dx}{dt} = x - 2y + 2z$$

$$\frac{dy}{dt} = -2x + y - 2z$$

$$\frac{dz}{dt} = 2x - 2y + z$$

In matrix representation, it can be written as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad - (\#)$$

Let $U = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ — ($\frac{1}{2}$ marks)

So (#) $\Rightarrow$ $\frac{dU}{dt} = AU$ — ($\frac{1}{2}$ marks)

Let λ be the Eigenvalue of A

then $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(\lambda-1)^2 - 4\right] + 2\left[2(\lambda-1) + 4\right] + 2\left[4 + 2(\lambda - 1)\right] = 0$$

(2)

$$\Rightarrow (1-\lambda)\left[(\lambda-1-2)(\lambda-1+2)\right] + 8\left[\lambda-1+2\right] = 0$$

$$\Rightarrow (1-\lambda)\left[(\lambda-3)(\lambda+1)\right] + 8(\lambda+1) = 0.$$

$$\Rightarrow (\lambda+1)\left[(1-\lambda)(\lambda-3) + 8\right] = 0$$

$$\Rightarrow (\lambda+1)(-3+4\lambda-\lambda^2+8) = 0$$

$$\Rightarrow (\lambda+1)(\lambda^2-4\lambda-5) = 0$$

$$\Rightarrow (\lambda+1)(\lambda-5)(\lambda+1) = 0$$

So, Eigenvalues of A are $-1, -1, 5$ — Calculation of Eigenvalues (2) marks

Eigenvectors Corresponding $\lambda_1 = -1$ be v_1 then

$$(A+I)v_1 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By, Gaussian Elimination

$$\Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \beta = \alpha + \gamma$$ — 1 mark for the relation

$$\Rightarrow v_1 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha+\gamma \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the eigenvectors corresponding to $\lambda = -1$

— (2) marks | 1 mark for each E.V

(3)

- Let v_2 be the Eigenvector corresponding to $\lambda = 5$

then

$$(A - 5I)\, v_2 = 0$$

$$\Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \Rightarrow \begin{bmatrix} 2 & -2 & -4 \\ -2 & -4 & -2 \\ -4 & -2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to \tfrac{1}{2} R_1 \Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ -2 & -4 & -2 \\ -4 & -2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1 \Rightarrow R_3 \to R_3 + 4R_1 \quad \begin{bmatrix} 1 & -1 & -2 \\ 0 & -6 & -6 \\ 0 & -6 & -6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \Rightarrow R_2 \to \tfrac{-1}{6} R_2 \quad \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to R_1 + R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \alpha = \gamma, \quad \beta = -\gamma \Rightarrow v_2 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \gamma \\ -\gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

1 mark for relation

(4)

$\Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is ~~the~~ an eigenvector for eigenvalue $\lambda = 5$

— (1) mark for the eigenvector

So General solution of given System is

$$U(t) = e^{-t}\left(C_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) + C_3 e^{5t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= C_1 e^{-t} + C_3 e^{5t} \\ y(t) &= C_1 e^{-t} + C_2 e^{-t} - C_3 e^{5t} \\ z(t) &= \cancel{e^{-t} C_2}\, C_2 e^{-t} + C_3 e^{5t} \end{aligned}}$$

— (2) marks for final solution

Problem - 13

Solution: $\frac{dx}{dt} = 3x - 18y$

$\frac{dy}{dt} = 2x - 9y$

$$A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$$

First find the eigen values of A :-

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(-9-\lambda) + 36 = 0$

$-27 - 3\lambda + 9\lambda + \lambda^2 + 36 = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0$

$\lambda = -3, -3$

Nature of eigen values :-

Both eigen values are real, equal and negative — (2)

i.e $\boxed{\lambda_1 = \lambda_2 = -3 < 0}$

Classification : Improper Node — (1)

~~Sta~~ The critical point (0,0) is stable. (1/2)

It is also asymptotically stable. (1/2)

Problem #14. There are two unit issues in this problem:

① Weight is given in pounds (lb) but it should be given in Newton (N) or some other Force unit.

② Spring was pulled down 6 inches. Other distance units are in feet (ft).

If students take following values then it could be considered correct.

① If they take weight $= mg = 64 \Rightarrow m = 2$
or directly $m = 64$

② Pulling distance $= 6$ or 0.5

Depending on this, possible answers are

(a) +1 $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{18}{m}}$

if $m = 64 \Rightarrow \omega = \sqrt{\frac{18}{64}} = \frac{3}{4\sqrt{2}} = 0.53$

if $m = 2 \Rightarrow \omega = \sqrt{\frac{18}{2}} = 3$

+1

(b) The eq. in this case will be

$$m\left(\frac{d^2y}{dt^2}\right) + a\left(\frac{dy}{dt}\right) + ky = F(t)$$

where $m = 64$ or 2, $a = 4$, $k = 18$, $F(t) = 3\cos(\omega t)$

Resonance frequency $= \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{a^2}{2m^2}}$ +3

if $m = 2 \Rightarrow \frac{1}{2\pi}\sqrt{\frac{18}{2} - \frac{16}{8}} = \frac{\sqrt{7}}{2\pi} = 0.42$

if $m = 64 \Rightarrow \frac{1}{2\pi}\sqrt{\frac{18}{64} - \frac{16}{2\times(64)^2}}$

+2