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Q-1. $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$; both are unknown.

a) $H_0: \mu \leq \mu_0$ v/s $H_1: \mu > \mu_0$] right tailed test

(Here, σ^2 is unknown but a nuisance parameter. We need to estimate it but we are not making direct inference of σ^2)

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$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$; $\underset{\sim}{\theta} = (\underset{\theta_1}{\underset{\downarrow}{\mu}}, \underset{\theta_2}{\underset{\downarrow}{\sigma^2}})$

MLE of $\underset{\sim}{\theta}$

$$L(\theta) = \frac{1}{(2\pi\theta_2)^{n/2}} \exp\left[\frac{-1}{2} \frac{\Sigma(x_i - \theta_1)^2}{\theta_2}\right]$$

$$l(\theta) = \frac{-n}{2} \log(2\pi\theta_2) - \frac{1}{2} \frac{\Sigma(x_i - \theta_1)^2}{\theta_2}$$

$$= c - \frac{n}{2} \log \theta_2 - \frac{1}{2} \frac{\Sigma(x_i - \theta_1)^2}{\theta_2}$$

$$\frac{dl}{d\theta_1} = 0 \quad ; \quad \frac{dl}{d\theta_2} = 0$$

$$\frac{dl}{d\theta_1} = \frac{\Sigma(x_i - \theta_1)}{\theta_2} = 0$$

$$\frac{dl}{d\theta_2} = \frac{-n}{2} \frac{1}{\theta_2} + \frac{1}{2} \frac{\Sigma(x_i - \theta_1)^2}{\theta_2^2} = 0$$

$$\hat{\theta}_1 = \bar{x} \quad ; \quad \hat{\theta}_2 = \frac{\Sigma(x_i - \hat{\theta}_1)^2}{n}$$

$$\Rightarrow \hat{\theta}_2 = \frac{1}{n} \Sigma(x_i - \bar{x})^2$$

a) $\left.\dfrac{d^2 l}{d\theta_1^2}\right|_{\hat\theta_1, \hat\theta_2} < 0$ or $\left.\dfrac{d^2 l}{d\theta_2^2}\right|_{\hat\theta_1, \hat\theta_2} < 0$

and (b) $\left|\begin{matrix} \dfrac{d^2 l}{d\theta_1^2} & \dfrac{d^2 l}{d\theta_2 d\theta_1} \\ \dfrac{d^2 l}{d\theta_1 d\theta_2} & \dfrac{d^2 l}{d\theta_2^2} \end{matrix}\right|_{\hat\theta_1, \hat\theta_2} > 0$

$$\left.\frac{d^2 l}{d\theta_1^2}\right|_{\hat\theta_1, \hat\theta_2} = \frac{-n}{\hat\theta_2} = \frac{-n^2}{\Sigma(x_i-\bar{x})^2} < 0$$

$$\left.\frac{d^2 l}{d\theta_1 d\theta_2}\right|_{\hat\theta_1 \hat\theta_2} = 0$$

$\rightarrow |J| > 0$

Thus, $\hat\theta_1 = \hat\mu_{MLE} = \bar{X}$ and $\hat\theta_2 = \hat\sigma^2_{MLE} = \frac{1}{n}\Sigma(x_i-\bar{x})^2$

* Calculating MLE, $\hat{\mu}_{MLE} = \bar{X}$ and $\hat{\sigma}^2_{MLE} = \frac{1}{n} \Sigma (X_i - \bar{X})^2$

(under unrestricted parameter space)

For unrestricted parameter space,

$\Theta = \{-\infty < \mu < \infty, \sigma^2 > 0\}$

for restricted parameter space

$H_0 = \{-\infty < \mu \leq \mu_0 ; \sigma^2 > 0\}$

$H_1 = \{\mu > \mu_0 ; \sigma^2 > 0\}$

$\Theta_0 \cup \Theta_1 = \Theta$

$$\lambda(x) = \frac{N^r}{D^r} = \frac{\text{restricted MLE}}{\text{unrestricted MLE}}$$

$$D^r = \sup_{\theta \in \Theta} L(\theta) = \max_{(\mu, \sigma^2)} L(\theta)$$

$$= L(\hat{\theta}_{MLE}) = L(\hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE})$$

To find $\hat{\mu}$, we need to find max $L(\theta)$ under restricted parameter space

$$\max L(\underset{\sim}{\theta}) \left\{ \begin{matrix} \mu \leq \mu_0 \\ \sigma^2 > 0 \end{matrix} \right\}$$

$$L(\underset{\sim}{\theta}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{-1}{2} \frac{\Sigma(x_i - \mu)^2}{\sigma^2}\right]$$

$$l(\underset{\sim}{\theta}) = \log(L(\underset{\sim}{\theta})) = \frac{-n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i - \mu)^2$$

$$\frac{d(l(\underset{\sim}{\theta})}{d\sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i - \mu)^2$$

We confirm that $\left.\frac{d^2 l}{d\sigma^2}\right|_{\hat{\sigma}^2} < 0$

Putting $\hat{\sigma}^2$ in $l(\underset{\sim}{\theta})$ to get $\hat{\mu}$

$$l(\theta) = c - \frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2}n\Sigma(x_i - \mu)^2$$

$$l(\theta) = c - \frac{n}{2}\log\left[\frac{1}{n}\Sigma(x_i - \mu)^2\right] - \frac{1}{2}\Sigma(x_i - \mu)^2 \quad \text{(i)}$$

$$\frac{d(l(\theta)}{d\mu} = 0$$

$$\Rightarrow \quad + \frac{n}{2}\frac{\Sigma(x_i - \mu)}{\Sigma(x_i - \mu)^2} + \frac{2}{2} n\Sigma(x_i - \mu) = 0$$

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$$\Rightarrow \frac{n}{\Sigma x_i - n\mu} = 0$$

$\Rightarrow$ $\hat{\mu}_{MLE}$ can't be obtained

We find $\hat{\mu}_{MLE}$ using algebric approach,

In (i) max $l(\theta)$ w.r.t μ implies min $\Sigma(x_i - \mu)^2$

$$\min_{\{\mu\}} \Sigma(x_i - \mu)^2 = \min_{\mu} \Sigma(\underbrace{x_i - \bar{x}}_{a} + \underbrace{\bar{x} - \mu}_{b})^2$$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

$$\hat{\mu} = \bar{X} \quad ; \quad \hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i - \hat{\mu})^2$$

$H_0 : \mu \le \mu_0$ v/s $H_1 : \mu > \mu_0$

From $\max\limits_{\substack{\{\mu \le \mu_0\} \\ \sigma^2 > 0}} L(\theta)$

$\hat{\mu} = \bar{X}$ when $\bar{X} \le \mu_0$

if $\bar{X} > \mu_0$ $\hat{\mu} = \mu_0$

$$\hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2 \quad \text{if } \bar{X} \le \mu_0$$

$$\hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i - \mu_0)^2 \quad \text{if } \bar{X} > \mu_0$$

Case I when $\bar{X} \leq \mu_0$

$\hat{\hat{\mu}} = \bar{X}$; $\hat{\hat{\sigma}}^2 = \frac{1}{n} \Sigma(X_i - \bar{X})^2$

$L(\hat{\hat{\mu}}, \hat{\hat{\sigma}}^2)$

$$\lambda(x) = \frac{\frac{1}{(2\pi \hat{\hat{\sigma}}^2)^{n/2}} \exp\left[-\frac{1}{2} \frac{\Sigma(x_i - \hat{\hat{\mu}})^2}{\hat{\hat{\sigma}}^2} \right]}{\frac{1}{(2\pi \hat{\sigma}^2)^{n/2}} \exp\left[-\frac{1}{2} \frac{\Sigma(x_i - \hat{\mu})^2}{\hat{\sigma}^2} \right]}$$

$$= 1 \qquad \left[\because \hat{\hat{\sigma}}^2 = \hat{\sigma}^2 \text{ and } \hat{\mu} = \hat{\hat{\mu}} \right]$$

Case II: when $\bar{X} > \mu_0$

$$\lambda(x) = \frac{\frac{1}{(\hat{\hat{\sigma}}^2)^{n/2}} \exp\left[\frac{-1}{2\hat{\hat{\sigma}}^2} \Sigma(x_i - \hat{\hat{\mu}})^2 \right]}{\frac{1}{(\hat{\sigma}^2)^{n/2}} \exp\left[\frac{-1}{2\hat{\sigma}^2} \Sigma(x_i - \hat{\mu})^2 \right]}$$

$$= \frac{\left(\Sigma(x_i - \bar{x})^2 \right)^{n/2}}{\left(\Sigma(x_i - \mu_0)^2 \right)^{n/2}}$$

$$= \left[\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right]^{n/2}$$

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Using LRT we reject H_0 if

$$\lambda(x) \leq c$$

$$\left[\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2}\right]^{n/2} \leq c$$

$$\left[\frac{\hat{\hat{\sigma}}^2}{\hat{\sigma}^2}\right]^{n/2} \geq c$$

$$\frac{\Sigma(x_i - \mu_0)^2}{\Sigma(x_i - \bar{x})^2} \geq c_1$$

$$\frac{\Sigma(x_i - \bar{x} + \bar{x} - \mu_0)^2}{\Sigma(x_i - \bar{x})^2} \geq c$$

$$\frac{\Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}{\Sigma(x_i - \bar{x})^2} \geq c_1$$

$$\frac{n(\bar{x} - \mu_0)^2}{\Sigma(x_i - \bar{x})^2} \geq c_2$$

$$\frac{n(\bar{x} - \mu_0)^2}{(n-1)\, s^2} \geq c_2 \qquad \because S^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2$$

$$\Rightarrow \frac{n(x - \mu_0)^2}{s^2} \geq c_3$$

$T =$

$\alpha \rightarrow$ probability of rejecting a null hypothesis when it is true.

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$$T = \frac{\sqrt{n}(\bar{x} - \mu)}{s} \sim t_{(n-1)}$$

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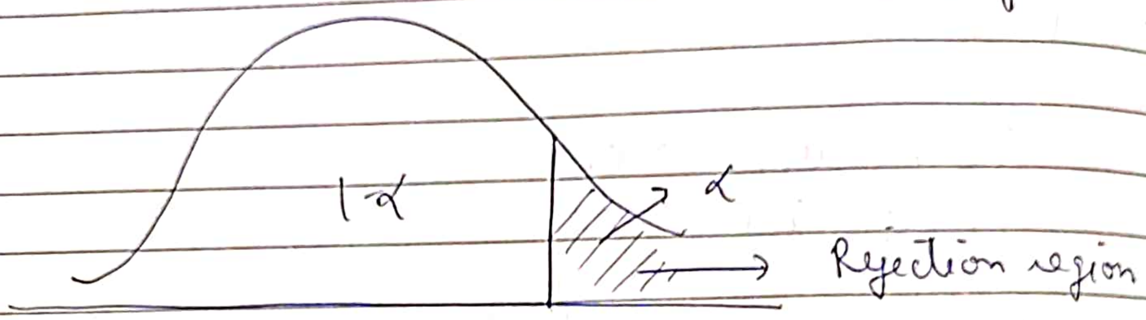
The test rejects H_0 if $T > t_\alpha$ where t_α is the $(1-\alpha)$th percentile of the t-distribution with $(n-1)$ degrees of freedom. (where α is the level of significance)



Right tailed test

We reject the null hypothesis at level of significance when calculated t-value is larger than critical value from t-distribution

Probability of rejecting null hypothesis is equal to level of significance (α)

$$P(T \geq c) = \alpha$$

2\. The likelihood ratio test statistic for testing $H_0 : \theta \in \Theta_0$ v/s $H_1 : \theta \in \Theta_0^C$ is defined as

$$\lambda(x) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} = \frac{\text{restricted MLE}}{\text{unrestricted MLE}}$$

For unrestricted parameter space,

$$\Theta = \{ -\infty < \mu < \infty , \; \sigma^2 > 0 \}$$

For restricted parameter space,

$$\Theta_0 = \{ \mu = \mu_0 , \; \sigma^2 > 0 \}$$

$$\Theta_1 = \{ \mu \in \mathbb{R} \setminus \{\mu_0\} , \; \sigma^2 > 0 \}$$

Now, unrestricted MLE has already calculated in part (1).

For restricted MLE, $\hat{\hat{\mu}} = \mu_0$

and $\hat{\hat{\sigma}}^2 = \frac{1}{n} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 \right)$

$\{ \because$ calculated in part (1) $\}$

$\therefore$ restricted MLE $= \dfrac{1}{(2\pi \hat{\hat{\sigma}}^2)^{n/2}} \exp\left[\dfrac{-1}{2\hat{\hat{\sigma}}^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right]$

$$\therefore \quad \lambda(x) = \frac{\frac{1}{(2\pi \hat{\hat{\sigma}}^2)^{n/2}} \exp\left[\frac{-1}{2\hat{\hat{\sigma}}^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right]}{\frac{1}{(2\pi \hat{\sigma}^2)^{n/2}} \exp\left[\frac{-1}{2\hat{\sigma}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right]}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right)^{n/2} \exp\left[\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{2\hat{\sigma}^2} - \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{2\hat{\hat{\sigma}}^2} \right]$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right)^{n/2} \exp\left[\frac{\sum_{i=1}^{n} (x_i - \bar{x})}{2 \frac{1}{n}\left(\sum_{i=1}^{n} (x_i - \bar{x})^2 \right)} - \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{\frac{2}{n}\left(\sum_{i=1}^{n} (x_i - \mu_0)^2 \right)} \right]$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right)^{n/2} \exp\left[\frac{n}{2} - \frac{n}{2} \right]$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right)^{n/2}$$

Using LRT we reject H_0 if

$$\lambda(x) \leq c$$

$$\left(\frac{\hat{\sigma}^2}{\hat{\hat{\sigma}}^2} \right)^{n/2} \leq c$$

This has already solved in part (1).

& we got, $\dfrac{n(\bar{x} - \mu_0)^2}{s^2} \geq c_1$

where $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$\Rightarrow \left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \geq c_1$$

Test statistic, $T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$

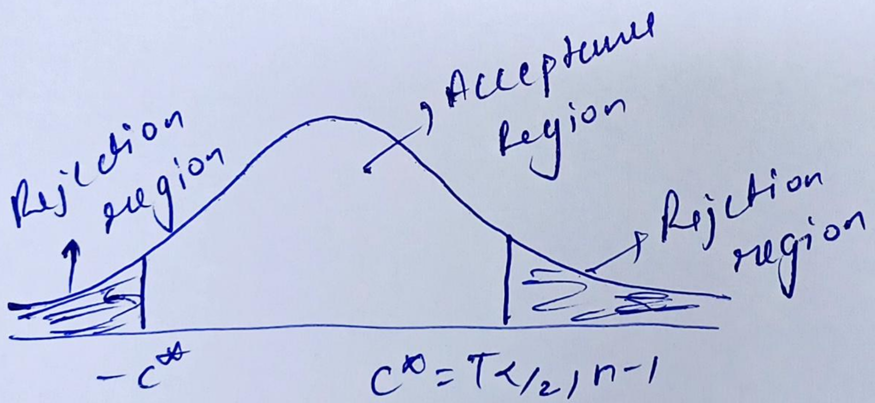
Now, $P\left(\left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \geq c_1^* \right) = \alpha$

$$\Rightarrow P(|T| \geq c^*) = \alpha$$

$$\Rightarrow P(T \geq c^* \text{ or } T \leq -c^*) = \alpha$$

$$\Rightarrow P(T \geq c^*) + P(T \leq -c^*) = \alpha$$

where, $c^* = T_{\alpha/2, n-1}$



Because of symmetry, we have

$$P(T \geq c^*) = P(T \leq -c^*)$$

$$\Rightarrow P(T \geq c^*) = \alpha/2$$

3)
To compare the results obtained from the two parts, we need to compare the rejection regions of the two LRTs.
For part (a), the rejection region is $T > c$, and c is the critical value that satisfies $P\left(T > c \mid H_0\right) = \alpha$.
For part (b), the rejection region is $T > c$ or $T <- c$, c is chosen such that
$P\left(T > c \ or \ T < c|H_0\right) = \alpha$.
We can see that the rejection regions of the two LRTs are different. The LRT in part (b) has a two-sided rejection region, while the LRT in part (a) has a one-sided rejection region. This is because the alternative hypothesis in part (b) is two-sided, while the alternative hypothesis in part (a) is one-sided.