MTH100 **Monsoon Semester – 2021-22** **Section B**

Submission for Tuesday 6th April 2022 – 15 minutes. **Max Marks: 5**

Instructions: *Open notes and textbook; consultation and use of calculators, computers and internet not allowed. You may use any* ***known*** *result. This includes all propositions and observations in the lecture slides, and results from tutorials. If you use any other result from any other source, including the textbook, you have to give a full proof of that result. All your steps must be shown.*

a) Give an example of a finite-dimensional vector space V over the field $\mathbb{R}$, and a linear operator T: V → V such that the matrix of T relative to some ordered basis of V has all non-zero entries on the diagonal, but T is not invertible. (2 *marks*)

b) Give an example of a finite-dimensional vector space V over the field $\mathbb{R}$, and a linear operator T: V → V such that the matrix of T relative to some ordered basis of V has all zero entries on the diagonal, but T is invertible. (*3 marks*)

NB: ***In your examples, you may either take dim V = n, where n is a fixed but arbitrary positive integer, or you may take any particular n, but n should be at least 3. You must make this clear in your answer. You can take different dimensions for a) and b).***

Answer: <u>Method 1</u>: The technically best method is to use Prop. 26(b) to construct suitable linear operators.

(b) ~~For (b)~~ We give a general answer: Let $\beta = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be any basis of V and define $T: V \to V$ by $T\bar{v}_i = v_{i+1}$ for $i = 1, \ldots, n-1$; $T\bar{v}_n = \bar{v}_1$.

Then $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$ which has

all 0's on the diagonal. However, since T takes a basis of V to a basis of V, T is an isomorphism, hence invertible, (Q1, TheT 10, week of 20220321)

Continued:-

(a) We take $n=3$ and use the standard ordered basis $\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

Define T by: $T\bar{e}_1 = \bar{e}_1 + \bar{e}_2$

$$T\bar{e}_2 = \bar{e}_2 + \bar{e}_3$$

$$T\bar{e}_3 = \bar{e}_2 + \bar{e}_3$$

Then, $[T]_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Clearly, not invertible but all non-zero entries on diagonal.

Method 2: For both (a) and (b), ~~def~~ take suitable matrices meeting the conditions, and define the linear operator as T_A = left multiplication by A (*)

Rubric: For method 1 – for each part:

definition of T → 1 mark (if correct)

Construction of matrix → 0.5 mark

Justification → 1 mark

For Method 2: Selection of Matrix → 1 mark (if correct)

~~Construction~~ Definition of T → 0.5 mark (step *)

Justification → 0.5 marks