Worksheet-4
Course Name: Math-III (Section-A)
Total marks = 20
Date: 28/09/2022

1. Find the directions in which the function $f(x, y) = x^2y + e^{xy} \sin y$ increase and decrease most rapidly at $P_0 = (1, 0)$. Then find the derivative of the function in these directions. (2+2)
2. Sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point $(-1, 2)$. Then write an equation for the tangent line.
   $x^2 - xy + y^2 = 7$ (2+2)
3. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of i + j is $2\sqrt{2}$ and in the direction of -2j is -3 . What is the derivative of f in the direction of - i - 2j ? Give reasons for your answer. (Don't compute from the scratch; use the algebra rules for gradient to arrive at the solution) (4)
4. Find equations for the (a) tangent plane and (b) normal line at the point $P_0(3, 5, -4)$ on the given surface
   $x^2 + y^2 - z^2 = 18$ (2+2)
5. Find the linearization L(x, y) of the function at each point.
   $f(x, y) = x^2 + y^2 + 1$ at (a) (0,0) and (b) (1,1) (2+2)

Q. 1.

Given $f(x,y) = x^2y + e^{xy}\sin y$

$P_0 = (1,0)$

$f_x = 2xy + ye^{xy}\sin y$. (1)

$f_y = x^2 + xe^{xy}\sin y + e^{xy}\cos y$.

$\therefore \nabla f\Big|_{(1,0)} = f_x\Big|_{(1,0)}\hat{i} + f_y\Big|_{(1,0)}\hat{j}$

$= 0\cdot\hat{i} + 2\cdot\hat{j} = 2\hat{j}$

$\therefore \vec{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2\hat{j}}{\sqrt{2^2}} = \hat{j}$ (1)

Hence f increases most rapidly in the direction $\vec{u} = \hat{j}$

& $(D_u f)\Big|_{(1,0)} = \nabla f\cdot\vec{u} = 2\hat{j}\cdot\hat{j} = 2$. (1)

and f decreases most rapidly in the direction $-\vec{u} = -\hat{j}$ and

$(D_{-u}f)\Big|_{(1,0)} = \nabla f\cdot(-\vec{u})$

$= 2\hat{j}\cdot(-\hat{j})$

$= -2$. (1)

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Q.2. Here,

$f(x,y) = C$ is $x^2 - xy + y^2 = 7$.

$\therefore \nabla f = f_x \hat{i} + f_y \hat{j}$

$= (2x - y)\hat{i} + (2y - x)\hat{j}$

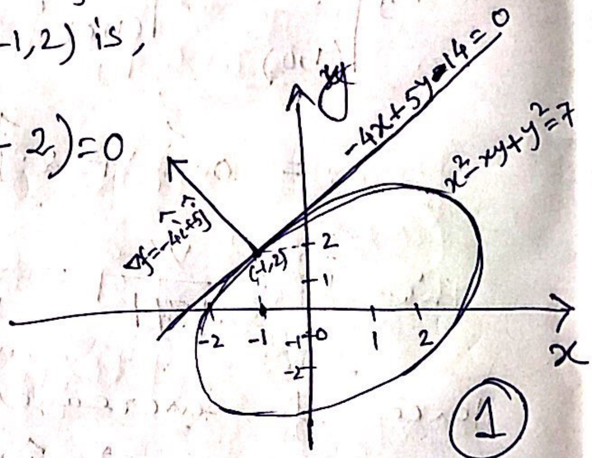
$\therefore \nabla f \big|_{(-1,2)} = -4\hat{i} + 5\hat{j}$ (1)

& Hence the equation of tangent line at $(-1,2)$ is,

$-4(x-(-1)) + 5(y-2) = 0$

$\Rightarrow -4x + 5y - 14 = 0$

(2)



(1)

Q.3. Given, $\vec{u_1} = i + j \Rightarrow \vec{u_1} = \frac{u}{|u|} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$(D_{u_1} f)_{P_0} = f_x(1,2) \cdot \frac{1}{\sqrt{2}} + f_y(1,2)\frac{1}{\sqrt{2}} = 2\sqrt{2}$ (given)

(1) $\Rightarrow f_x(1,2) + f_y(1,2) = 4 \longrightarrow$ (I)

Also, for $\vec{v} = -2\hat{j} \Rightarrow \vec{v_1} = \frac{v}{|v|} = \frac{-2\hat{j}}{\sqrt{(-2)^2}} = -\hat{j}$

$\therefore (D_{v_1} f)_{P_0} = -3$ (given)

$\Rightarrow f_x(1,2) \times 0 + f_y(1,2)(-1) = -3 \longrightarrow$ (II)

$\therefore$ (II) $\Rightarrow f_y(1,2) = 3$. (1)

Substituting (II) in (I), we have,

$$f_x(1,2) = 1.$$

(1)

$$\therefore \nabla f(1,2) = \hat{i} + 3\hat{j}$$

Now, in the direction $-\hat{i}-2\hat{j}$, we have,

(1)

$$\left(D_{(-\hat{i}-2\hat{j})} f\right) = \nabla f \cdot \left(\frac{-\hat{i}-2\hat{j}}{\sqrt{(-1)^2+(-2)^2}}\right)$$

$$= \left(\hat{i}+3\hat{j}\right) \cdot \left(-\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right)$$

$$= -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

Q.4.

$$f(x,y) = x^2+y^2-z^2-18.$$

$$\therefore \nabla f = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

(1)

$$\nabla f|_{(3,5,-4)} = 6\hat{i} + 10\hat{j} + 8\hat{k}$$

(a) Tangent line at $(3,5,-4)$:

$$6(x-3) + 10(y-5) + 8(z+4) = 0$$

$$\Rightarrow 6x + 10y + 8z = 36$$

(1.5)

$$\Rightarrow 3x + 5y + 4z = 18.$$

(b) Normal line:

$$\frac{x-3}{6} = \frac{y-5}{10} = \frac{z+4}{8} = t \text{ (say)}$$

$$\therefore x = 3+6t;\ y = 5+10t;\ z = -4+8t.$$

(1.5)

Q.5.

$f(x,y) = x^2+y^2+1$.

$f(0,0) = 1$ and $f(1,1) = 3$.

$f_x = 2x$ and $f_y = 2y$. (1)

$f_x|_{(0,0)} = 0$, $f_y|_{(0,0)} = 0$

$f_x|_{(1,1)} = 2$, $f_y|_{(1,1)} = 2$. (1)

(a) at $(0,0)$.

The linearization $L(x,y)$ of the function at $(0,0)$:

$$L(x,y) = f(0,0) + f_x|_{(0,0)}(x-0) + f_y|_{(0,0)}(y-0)$$

$$= 1 + 0\times(x-0) + 0(y-0)$$

$$= 1.$$ (1)

(b) The linearization $L(x,y)$ of the function at $(1,1)$:

$$L(x,y) = f(1,1) + f_x|_{(1,1)}(x-1) + f_y|_{(1,1)}(y-1)$$

$$= 3 + 2(x-1) + 2(y-1)$$

$$= 3 + 2x - 2 + 2y - 2$$

$$= 2x + 2y - 1.$$ (1)

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