# Winter 2023 - CSE/ECE 344/544 Computer Vision
# Mid-Sem - Feb. 25, 2023

**Maximum score: 30** **Time: 60 minutes** 

**Instructions:**
- Try to attempt all questions.
- Keep your derivations clean and to the point. Do not skip any non-trivial steps.
- You may refer to your own 1-page cheat sheet, but you cannot share the cheat-sheet with others.
- In the unusual case that a question is not clear, please state your assumptions *clearly* and solve the question. Do this even if you have clarified your doubt with an invigilator. Reasonable assumptions will be accounted for while grading.
- Q4 is Extra Credit. You may choose not to attempt it, but we encourage attempting it.

**1.** (15 points) A permutation matrix is a binary matrix (with elements either 0 or 1) that has exactly one 1 in each row and column. It is called a permutation matrix as it permutes / reorders the elements of a vector. For example, a vector $[1, 2, 3]$ is permuted to $[3, 2, 1]$ by applying a permutation that maps the XYZ axes as $X \to Z$, $Y \to Y$ and $Z \to X$.

a) (2 points) Write the $3 \times 3$ matrix $\mathbf{P}$ that maps $X \to Y$, $Y \to Z$ and $Z \to X$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} z \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

[If partially correct: 1 point; Otherwise: 2 points]

b) (1+2=3 points) Show that the matrix $\mathbf{P}$ is a rotation matrix. What are the criteria for a matrix to qualify as a rotation matrix?

Criterion to qualify as a rotation matrix:

(a) Rows and columns of the matrix should be orthogonal and unit normal

(b) Determinant of the matrix should be 1

$$\mathbf{P}^{\mathrm{T}}\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$\det(\mathrm{P}) = 0 - 0 + 1(1 - 0) = 1$

$\therefore$ P is a rotation matrix

[If partially correct: $(0.5 + 1 + 1)$ points; Otherwise: $(1 + 1 + 1)$ points]

c) (2 points) What is the axis of rotation of $\mathbf{P}$? Identify the axis of rotation by inspecting $\mathbf{P}$ and the points that are invariant to the transformation $\mathbf{P}$.

The points $\alpha(1, 1, 1)$, for $\alpha \in \mathbb{R}$ are invariant to the permutation, which therefore make up the axis of rotation.

[If partially correct: 1.5 points; Otherwise: 2 points]

d) (3 points) Use the Euler-Rodriguez's formula to find out the axis and angle of rotation if $\mathbf{P}$ is a rotation matrix. Check if they are the same as in the previous part.
$\theta = \cos^{-1}\left(\frac{trace(P)-1}{2}\right) = \cos^{-1}\left(\frac{0-1}{2}\right) = \cos^{-1}\left(\frac{-1}{2}\right) = 2\pi/3 = 120°$

$$n = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (since } \theta = 2\pi/3)$$

[If partially correct: (1 + 1) points; Otherwise: (1.5 + 1.5) points]

e) (2+1=3 points) Find a $3 \times 3$ permutation matrix $\mathbf{P}'$, that is not a rotation. Comment on what kind of transformation is $\mathbf{P}'$. If your initial XYZ frame was a right-handed coordinate frame, does the resulting coordinate frame after applying $\mathbf{P}'$ still remain a right-handed coordinate frame?

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$det(P') = -1 \neq 1$

$X$ is in the same direction. $Z \rightarrow Y$ and $Y \rightarrow Z$ lead to a left-handed coordinate frame.

**OR**

$$P'' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$det(P'') = -1 \neq 1$

$Y$ is in the same direction. $Z \rightarrow X$ and $X \rightarrow Z$ lead to a left-handed coordinate frame.

[2 points (for non-rotation permutation matrix) + 1 point (for resulting coordinate frame)]

f) (2 points) Does this permutation matrix $\mathbf{P}'$ also have an axis (set of points) that is invariant to the transformation? If yes, what is that direction?

Yes, axis for $P'$ is $X$.

**OR**

Yes, axis for $P''$ is $Y$.

[2 points]

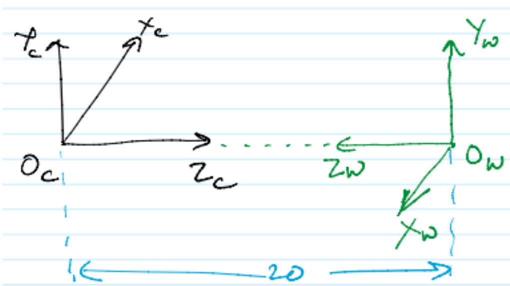
**Note (for e and f): Either example is enough. Doesn't have to be both. There are other examples of non-rotation permutation as well.**

**2.** (5 points) Let's say we are given a camera that has a VGA resolution, i.e., $640 \times 480$ pixels. Assume that $f_x = f_y = 250$ is the focal length, and $(p_x, p_y) = (320, 240)$ are the principal point coordinates (projection of the center of projection on the imaging grid / chip), and there is no skewness (i.e., skewness=0). Let's say the world origin is 20 units away along the positive $Z$ axis of the camera frame of reference. The angle between Z-axis in the camera frame, $Z_C$ and the Z-axis in the world frame $Z_W$ is $\pi$ radians. Similarly the angle between $X_C$ and $X_W$ is also $\pi$ radians. Given two points in the world: $\mathbf{P}_W^1 = [40, 40, 0]^\top$ and $\mathbf{P}_W^2 = [20, 10, 0]$, determine which point is within the field of view of the camera. A point is said to be in the field of view of the camera, if when projected on to the imaging grid, it will lie within the bounds of the imaging grid of pixels.

$$K = \begin{bmatrix} 250 & 0 & 320 \\ 0 & 250 & 240 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \textbf{0.5 marks}$$

$$O_\mathrm{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \& \quad [O_\mathrm{w}]_\mathrm{c} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 1 \end{bmatrix} \rightarrow \textbf{0.5 marks}$$

Since $\angle \mathrm{Z_c Z_w} = \pi$ and $\angle \mathrm{X_c X_w} = \pi$, the rotation is about the Y-axis. $\rightarrow$ **0.25 marks**




$$R = \begin{bmatrix} \cos\pi & 0 & \sin\pi \\ 0 & 1 & 0 \\ -\sin\pi & 0 & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \textbf{0.25 marks}$$

$$P^1{}_\mathrm{c} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 20 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -40 \\ 40 \\ 20 \end{bmatrix} \rightarrow \textbf{0.25 marks}$$

$$P^2{}_\mathrm{c} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 20 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \\ 20 \end{bmatrix} \rightarrow \textbf{0.25 marks}$$

$$KP^1{}_\mathrm{c} = \begin{bmatrix} 250 & 0 & 320 \\ 0 & 250 & 240 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -40 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} (250 \times (-40))/20 + 320 \\ (250 \times 40)/20 + 240 \end{bmatrix} = \begin{bmatrix} -180 \\ 740 \end{bmatrix} \rightarrow \textbf{0.5 marks}$$

No, $\mathrm{P}^1$ is not in the field of view. $\rightarrow$ **1 marks**

$$KP^2{}_\mathrm{c} = \begin{bmatrix} 250 & 0 & 320 \\ 0 & 250 & 240 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} (250 \times (-20))/20 + 320 \\ (250 \times 10)/20 + 240 \end{bmatrix} = \begin{bmatrix} 70 \\ 365 \end{bmatrix} \rightarrow \textbf{0.5 marks}$$

Yes, $\mathrm{P}^2$ is in the field of view. $\rightarrow$ **1 marks**

**3.** (5+5=10 points) Show that distances between a pair of points are invariant under Euclidean transformations. Show that angles between vectors are invariant under similarity transformations. {*Hint*: Showing that the cosine of an angle does not change before and after the transformation establishes that the angles themselves do not change. }

**Solutions:**

**NOTE: If the invariance is shown for 2D then it should be argued why it should generalize to higher or at least 3 dimensions to get full credit.**

**For Euclidean Transformation:**
Lets consider 2 points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are coordinates in $\mathbb{R}^n$. After applying the euclidean transformation on them, we get $\boldsymbol{P'}_1$ and $\boldsymbol{P'}_2$. Therefore,
$\boldsymbol{P'}_1 = \boldsymbol{R}\boldsymbol{P}_1 + \boldsymbol{t}$
$\boldsymbol{P'}_2 = \boldsymbol{R}\boldsymbol{P}_2 + \boldsymbol{t}$,
where $\boldsymbol{R}$ is the rotation matrix and $\boldsymbol{t}$ is the translation vector. $\rightarrow$ **1 MARK**

Now, $\boldsymbol{P'}_1 - \boldsymbol{P'}_2 = \boldsymbol{R}\boldsymbol{P}_1 + \boldsymbol{t} - \boldsymbol{R}\boldsymbol{P}_2 - \boldsymbol{t} = \boldsymbol{R}(\boldsymbol{P}_1 - \boldsymbol{P}_2)$ $\rightarrow$ **1 MARK**

We know that $||\boldsymbol{P'}_1 - \boldsymbol{P'}_2||_{\mathbf{2}}^{\mathbf{2}} = (\boldsymbol{P'}_1 - \boldsymbol{P'}_2)^{\mathrm{T}}(\boldsymbol{P'}_1 - \boldsymbol{P'}_2)$ $\rightarrow$ **1 MARK**

Therefore, $(\boldsymbol{P'}_1 - \boldsymbol{P'}_2)^{\mathrm{T}}(\boldsymbol{P'}_1 - \boldsymbol{P'}_1) = [\boldsymbol{R}(\boldsymbol{P}_1 - \boldsymbol{P}_2)]^{\mathrm{T}}[\boldsymbol{R}(\boldsymbol{P}_1 - \boldsymbol{P}_2)]$
$= (\boldsymbol{P}_1 - \boldsymbol{P}_2)^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{P}_1 - \boldsymbol{P}_2)$
$= (\boldsymbol{P}_1 - \boldsymbol{P}_2)^{\mathrm{T}}(\boldsymbol{P}_1 - \boldsymbol{P}_2)$ $\rightarrow$ **2 MARK**
Hence proved that the distance between 2 points is invariant to Euclidean Transformation.

**Similarity Transformation**
In homogeneous form, we have a similarity transformation as:

$$\tilde{P''}_1 = \begin{bmatrix} \mathbf{sR} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tilde{P}_1$$

$$\implies \mathbf{P}_1^{''} = \mathbf{sRP}_1 + \mathbf{t} \quad \text{and} \quad \mathbf{P}_2^{''} = \mathbf{sRP}_2 + \mathbf{t} \quad \rightarrow \mathbf{1}\ [\textbf{point}]$$

To get the new vectors, take difference from transformed origin

$$\begin{aligned} \mathbf{O}'' &= \mathbf{sRO} + \mathbf{t} \\ &= \mathbf{sR} \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \mathbf{t} \\ &= \mathbf{t} \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{O}^{''}\mathbf{P}_1^{''} &= \mathbf{P}_1'' - \mathbf{O}'' \\ &= \mathbf{sRP}_1;\ \text{and} \\ \mathbf{O}^{''}\mathbf{P}_2^{''} &= \mathbf{P}_2'' - \mathbf{O}'' \\ &= \mathbf{sRP}_2 \end{aligned}$$

The original vectors are $\mathbf{OP_1}$ and $\mathbf{OP_2}$.
Then, $\mathbf{OP_1} = \mathbf{P_1}$ and $\mathbf{OP_2} = \mathbf{P_2}$, as $\mathbf{O}$ is $[\mathbf{0,...0}]^{\mathbf{T}}$.

Here R is a rot$^n$ matrix, $\mathbf{t}$ is a transl$^n$ vector and $\mathbf{s}$ is a scalar from the previous part, we have

$$\begin{aligned}
(\mathbf{O''P_1''}\text{- }\mathbf{O''P_2''})^{\mathbf{T}}(\mathbf{O''P_1''}\text{- }\mathbf{O''P_2''}) &= [\mathbf{sR(P_1-P_2)}]^{\mathbf{T}}[\mathbf{sR(P_1-P_2)}] \quad \rightarrow \mathbf{1\ MARK}\\
&= \mathbf{s}^2(\mathbf{P_1}\text{-}\mathbf{P_2})^{\mathrm{T}}\mathbf{R^TR}(\mathbf{P_1}\text{-}\mathbf{P_2})\\
&= \mathbf{s}^2(\mathbf{P_1}\text{-}\mathbf{P_2})^{\mathrm{T}}(\mathbf{P_1}\text{-}\mathbf{P_2})\\
&= \mathbf{s}^2(\mathbf{P_1}^{\mathbf{T}}\mathbf{P_1} + \mathbf{P_2}^{\mathbf{T}}\mathbf{P_2}\ \text{-}\ \mathbf{2P_1}^{\mathbf{T}}\mathbf{P_2})\\
&= \mathbf{s}^2(||\mathbf{P_1}||_2{}^2 + ||\mathbf{P_2}||_2{}^2 - 2||\mathbf{P_1}||||\mathbf{P_2}||cos(\theta)) \quad \rightarrow \mathbf{1\ MARK}
\end{aligned}$$

$$||\mathbf{O''P_1}||_2^2 + ||\mathbf{O''P_2''}||_2^2 - 2||\mathbf{O''P_1''}|| \cdot ||\mathbf{O''P_2''}||cos(\phi) = \mathbf{s}^2||\mathbf{P}_1||_2^2 + \mathbf{s}^2||\mathbf{P}_2||_2^2 - 2\mathbf{s}||\mathbf{P}_1|| \cdot \mathbf{s}||\mathbf{P}_2||cos(\theta) \quad (1)$$

Here $\phi$ is angle between $\mathbf{P}_1''$ and $\mathbf{P}_2''$, and $\theta$ is angle between $\mathbf{P}_1$ and $\mathbf{P}_2$.
Now, $||\mathbf{O''P_1''}||_2^2 = \mathbf{s}^2||\mathbf{P}_1||_2^2$.
This can be proved as follows:-

$$\begin{aligned}
\mathbf{O''P_1''} &= \mathbf{P_1''} - \mathbf{O''}\\
&= \mathbf{sRP_1} + \mathbf{t} - \mathbf{t}\\
&= \mathbf{sRP_1}\\
=> ||\mathbf{P_1''} - \mathbf{O''}||_2{}^2 &= (\mathbf{P_1''} - \mathbf{O''})^{\mathrm{T}}(\mathbf{P_1''} - \mathbf{O''})\\
&= (\mathbf{sRP_1})^{\mathrm{T}}(\mathbf{sRP_1})\\
&= \mathbf{s}^2\mathbf{P_1}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{RP_1}\\
&= \mathbf{s}^2||\mathbf{P_1}||_2{}^2
\end{aligned}$$

Similarly, $||\mathbf{P_2''} - \mathbf{O''}||_2{}^2 = \mathbf{s}^2||\mathbf{P_2}||_2{}^2$
Taking square root (positive), we have $||\mathbf{O''P_1''}||_2 = \mathbf{s}^2||\mathbf{P_1}||_2$,

$$||\mathbf{O''P_2''}||_2 = \mathbf{s}^2||\mathbf{P_2}||_2 \rightarrow \mathbf{2\ MARK}$$

Then, from (1), $cos(\phi) = cos(\theta) => \phi = \theta =>$ angle is preserved.

**4.** (5+5=10 points;*Extra Credit*) The Camera Center (center of projection) is [0,0,0] in the camera coordinate frame.

a) Given the intrinsic parameter matrix $\mathbf{K}$ and the extrinsic parameters, the rotation and translation forming the $3 \times 4$ extrinsic matrix $[\mathbf{R}\ \mathbf{t}]$, how would you compute the camera center in the world coordinate frame?

$$O_{\mathrm{c}} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}$$

$$\tilde{P}_{\mathrm{c}} = \begin{bmatrix}\mathbf{P_c}\\ \mathbf{1}\end{bmatrix}$$

$$\tilde{P}_{\mathrm{w}} = \begin{bmatrix}\mathbf{P_w}\\ \mathbf{1}\end{bmatrix}$$

$$\tilde{O_c} = \begin{bmatrix} \mathbf{O_c} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \rightarrow \mathbf{0.5\ marks}$$

Given [**R, t**], which is the world to camera coordinate tranformation, we have

$\tilde{\mathbf{P_c}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \tilde{\mathbf{Pw}} \rightarrow \mathbf{1\ marks}$

$\tilde{\mathbf{P_w}} = [{}^{\mathbf{c}}\mathbf{T_w}]^{\mathbf{-1}}\ \tilde{\mathbf{P_c}} \rightarrow \mathbf{1\ marks}$

This applies when $\tilde{\mathbf{P_c}} = \tilde{\mathbf{O_c}}$ too. $\rightarrow \mathbf{0.5\ marks}$

$[\tilde{\mathbf{O_c}}]_{\mathrm{w}} = [{}^{\mathbf{c}}\mathbf{T_w}]^{\mathbf{-1}}\ \tilde{\mathbf{O_c}}$

$$[\tilde{\mathbf{O_c}}]_{\mathrm{w}} = \begin{bmatrix} \mathbf{R^T} & \mathbf{-R^Tt} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \rightarrow \mathbf{2\ marks}$$

$[\mathbf{O_c}]_{\mathbf{w}} = \mathbf{-R^Tt}$

b) If the intrinsic and extrinsic matrices are not explicitly given, but only the $3 \times 4$ camera projection matrix $\mathbf{M}$ is provided. Can you estimate the camera center in the world frame of reference using only $\mathbf{M}$ (without decomposing it into $\mathbf{K}, \mathbf{R}, \mathbf{t}$)?

Solutions:

Note that $\tilde{\boldsymbol{\beta}}_{in} = \boldsymbol{M}\tilde{\boldsymbol{P}}_{\boldsymbol{w}}$, where, $\tilde{\boldsymbol{P}}_{\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{P_w} \\ \mathbf{1} \end{bmatrix}$

Pluggin in $[\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}}$ instead of $\tilde{\boldsymbol{P}}_{\boldsymbol{w}}$, we get

$\tilde{\boldsymbol{\theta}}_{in} = \boldsymbol{M}[\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}}$,

An important observation to make is that

$$\tilde{\boldsymbol{\theta}}_{in} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We show that by expanding $\boldsymbol{M}$ into the image formation pipeline as follows:

$$\tilde{\boldsymbol{\theta}}_{in} = \boldsymbol{K} \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{R} & t \\ 0 & 1 \end{bmatrix} [\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}},$$

$$\begin{bmatrix} \mathrm{R} & t \\ 0 & 1 \end{bmatrix} [\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}} = [\tilde{\boldsymbol{O}}_{\boldsymbol{c}}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{\boldsymbol{\theta}}_{in} = \boldsymbol{K} \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \boldsymbol{K} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tilde{\boldsymbol{\theta}}_{in} == \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

When the 3D point is the camera origin in the world coordinate frame.

$=> \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = M[\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}}$

$=> [\tilde{\boldsymbol{O}}_{\boldsymbol{c}}]_{\boldsymbol{w}}$ will lie in the null space of $\boldsymbol{M}$.

Null space of $\boldsymbol{M}$ will be 1D because $\boldsymbol{M}$ has rank 3 as it is a product of K (full rank $3 \times 3$) and $[Rt]$ which also has rank 3 as $\boldsymbol{R}$ is orthogonal (and therefore full rank).

Full credit only if all steps are shown especially showing why $\tilde{\boldsymbol{\theta}}_{in}$ will be $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

**6 MARKS**