MTH 204 : Worksheet 6 Solutions

Q1.

$$I = \frac{dQ}{dt}$$

$$\Rightarrow \quad Q(t) = \int I(t)\,dt$$

Voltage drop across circuit = External voltage

V.d. across R + V.d. across L + V.d. across C = V(t)

$$RI + LI' + \frac{Q}{C} = V(t)$$

$$\Rightarrow \quad LI' + RI + \frac{1}{C}\int I(t)\,dt = V(t) = V_0 \sin(\omega t)$$

Differentiate once to get rid of integral

$$\Rightarrow \quad \boxed{LI'' + RI' + \frac{I}{C} = V_0 \omega \cos(\omega t)} \quad \text{— ①}$$

Q2.

Homogeneous part is $\quad LI'' + RI' + \frac{1}{C} I = 0$

Char. eq^n is $\quad Lm^2 + Rm + \frac{1}{C} = 0$

$$\Rightarrow \quad m = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

$$\boxed{m = \frac{-R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}}$$

Q3.

$$I_p = a\cos(\omega t) + b\sin(\omega t)$$

$$I_p' = -a\omega \sin(\omega t) + b\omega \cos(\omega t)$$

$$I_p'' = -a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t)$$

Substitute it in ODE in eq^n ① and collect $\cos(\omega t)$ & $\sin(\omega t)$ terms, we get

$$\left(\left(-L\omega^2 + \tfrac{1}{C}\right)a + R\omega b\right)\cos(\omega t) + \left(-R\omega a + \left(-L\omega^2 + \tfrac{1}{C}\right)b\right)\sin\omega t$$
$$= V_0\,\omega\cos(\omega t)$$

$$\Rightarrow \quad \left(-L\omega^2 + \tfrac{1}{C}\right)a + R\omega b = V_0\,\omega$$

$$\& \quad -R\omega a + \left(-L\omega^2 + \tfrac{1}{C}\right)b = 0$$

Let $\quad S := L\omega - \frac{1}{C\omega}$ (it is known as Impedance)

$$\Rightarrow \quad -Sa + Rb = V_0 \quad \& \quad -Ra - Sb = 0$$

$$\Rightarrow \quad a = \frac{-V_0 S}{R^2+S^2} \quad \& \quad b = \frac{V_0 R}{R^2+S^2}$$

Q4.

If $\quad I_p = a\cos(\omega t) + b\sin(\omega t)$

We want $\quad I_p = C\sin(\omega t - \delta)$

$$\Rightarrow \quad I_p = C\left(\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta\right)$$

$$\Rightarrow \quad -C\sin\delta = a \quad \& \quad C\cos\delta = b$$

$$\Rightarrow \quad C = \sqrt{a^2+b^2} \quad \& \quad \delta = \tan^{-1}\frac{a}{b}$$

$$\Rightarrow \quad \boxed{C = \frac{V_0}{\sqrt{R^2+S^2}} \qquad \& \quad \delta = \tan^{-1}\frac{S}{R}}$$

Q5.

From Q2.

$$m = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

$$= -\frac{13}{0.6} \pm \frac{1}{0.6}\sqrt{169 - \frac{1.2}{10^{-2}}}$$

$$= \frac{-13 \pm \sqrt{49}}{0.6} = \frac{-13 \pm 7}{0.6}$$

$\Rightarrow \quad m_1 = -10 \quad \& \quad m_2 = \frac{-100}{3}$

So, $\quad I_h = c_1 e^{-10t} + c_2 e^{\frac{-100}{3}t}$

For I_p,

$$S = L\omega - \frac{1}{C\omega} = 0.3 \times 100\pi - \frac{1}{10^{-2} \times 100\pi} \approx 93.93$$

Then

$$I_p = \frac{110}{\sqrt{13^2 + (93.93)^2}} \operatorname{Sin}\left(100\pi t - \tan^{-1}\frac{93.93}{13}\right)$$

$$I_p \simeq 1.16 \operatorname{Sin}(100\pi t - 82.13^\circ)$$

So, General solution is

$$I(t) = c_1 e^{-10t} + c_2 e^{\frac{-100}{3}t} + 1.16 \operatorname{Sin}(100\pi t - 82.13^\circ) \quad \text{—} \circledast$$

We Know, $\quad I(0) = 0 \quad \& \quad Q(0) = 0$

Since, $\quad LI'(t) + RI(t) + \frac{Q(t)}{C} = V_0 \operatorname{Sin}(\omega t)$

$$I(0) = 0 \;\&\; Q(0) = 0 \quad \Rightarrow \quad I'(0) = 0$$

Substituting $I(0) = 0 \;\&\; I'(0) = 0$ in $\circledast$, we obtain

$$c_1 = 0.54 \quad \& \quad c_2 = -1.69$$

Hence,

$$\boxed{I(t) = 0.54 e^{-10t} - 1.69 e^{\frac{-100}{3}t} + 1.16 \operatorname{Sin}(100\pi t - 82.13^\circ)}$$