Discrete Mathematics, CSE 121 : Pre-End-Semester Exam, Monsoon 2022

General Instructions:

(a) Maximum marks = 10; Duration: Up to 5 hours.

(b) ***Bonus marks on scoring full marks***

  (a) Submission < 1 hour: 5 marks
  (b) Submission < 2 hours: 3 marks
  (c) Submission < 3 hours: 1 mark

(c) The exam paper is self-contained. Any use of any source without acknowledgement will be dealt with according to the Academic Dishonesty Policy of the institute.

(d) In every proof/derivation clearly state your assumptions and give details of each step.

(e) You will be evaluated for your attempt and approach. Therefore, you are encouraged to attempt the questions even if you can not complete an answer. The Academic Dishonesty Policy of the institute is equally applicable even for partial answers.

(f) Submission guidelines:

  (a) Please write your name and roll number clearly on the top of the first page of your submission.
  (b) Please write your answers clearly; a script which can not be clearly read will be returned unchecked.
  (c) Only the final uploaded version to GC within the submission timeline will be accepted as the submission. The written copy of this version has to be submitted during the end-sem exam along with the end-sem exam answer sheet.
  (d) **The submission timeline will be strictly adhered to for bonus marks. No submission after 5 hours will be accepted in any circumstance.**

Questions:

1. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$. You may find Theorems 1 and 2 given in Figures 1 and 2, respectively, useful here. [2 marks]

2. Consider the ICC T20 world cup championship of cricket. The rule of Super over applies in the event that the two teams score equally at the end of 20 overs of batting by each side. Consider the following "Super overs" rule. Each team selects 6 bowlers and 6 batsmen. The first game of the super overs starts and bowlers from the first

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots,$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Figure 1: Theorem 1

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Figure 2: Theorem 2

team bowl, whereas the batsmen from the second team try hitting the balls out of boundary for "sixers", one after another. *A hit not scoring a sixer is not counted to a score.* Also, suppose that a bowler will be allowed to re-bowl a wide or a no ball. In essence, every batsman from each team gets to hit exactly one valid bowl in a super over to score a six. If the score remains the same at the end of the first game of super overs, i.e., after both the teams would have played one over each, both the teams are allowed to play one more game of one over each. If even after four overs of play the score remains the same the winner is decided by a toss of a fair coin.

(a) Give the possible ways of scoring if the game is finished after one super over play by each team. As the teams score, the play can end if it is impossible for the other team to equalize. [2 marks]

(b) Give the possible ways of scoring if the game is finished after two super overs of play by each team. [2 marks]

3. Kruskal's algorithm for producing the minimum spanning tree in a graph is given in Figure 3. Deriving from this algorithm devise an algorithm for constructing a connected weighted undirected acyclic subgraph of maximum possible weight. [1 mark]

4. Prove that the devised algorithm in the last answer works as desired. [3 marks]

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for $i := 1$ **to** $n - 1$
  e := any edge in G with smallest weight that does not form a simple circuit
    when added to T
  $T := T$ with e added
return T {T is a minimum spanning tree of G}

Figure 3: Kruskal's Minimum Spanning Tree Algorithm