Problem 1. Critical points: $y_1' = 0 \Rightarrow y_2 = 0$

$y_2' = 0 \Rightarrow -y_1 + \frac{1}{2}y_1^2 = 0 \Rightarrow y_1\left(-1+\frac{1}{2}y_1\right) = 0$

$\Rightarrow y_1 = 0 \;\&\; y_1 = 2$

Two critical points: $(0,0)$ & $(2,0)$

Let $f(y_1,y_2) = y_2$ & $g(y_1,y_2) = -y_1 + \frac{1}{2}y_1^2$

$\frac{\partial f}{\partial y_1} = 0$, $\frac{\partial f}{\partial y_2} = 1$ & $\frac{\partial g}{\partial y_1} = -1+y_1$, $\frac{\partial g}{\partial y_2} = 0$

$\underline{\underline{(0,0)}}$ $\quad y_1' = y_2$, $\; y_2' = -y_1$ $\Rightarrow Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y \Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$

$\Rightarrow$ center (stable)

$\underline{\underline{(2,0)}}$ $\quad y_1' = y_2$, $\; y_2' = y_1 - 2$ $\Rightarrow Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$

$\Rightarrow$ saddle (unstable)

Problem 2. $Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t}$

$\underline{\underline{H.}}$ $\quad \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\underline{\lambda = 1}$. $\quad y = x \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = -1}$ $\quad y = -x \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Y_H = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$Y_0(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \Rightarrow Y_0^{-1} = \frac{1}{-2} \begin{pmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

So, $U'(t) = Y_0^{-1}\begin{pmatrix}1\\-3\end{pmatrix}e^{3t}$

$$= \frac{1}{2}\begin{pmatrix} e^{-t} & e^{-t}\\ e^{t} & -e^{t}\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix}e^{3t}$$

$$= \frac{1}{2}\begin{pmatrix}-2e^{-t}\\ 4e^{t}\end{pmatrix}e^{3t} = \begin{pmatrix}e^{2t}\\ 2e^{4t}\end{pmatrix}$$

$$U(t) = \begin{pmatrix}e^{2t}/2\\ e^{4t}/2\end{pmatrix}$$

$$Y_p(t) = \frac{1}{2}\begin{pmatrix} e^{t} & e^{-t}\\ e^{t} & -e^{-t}\end{pmatrix}\begin{pmatrix}e^{2t}\\ e^{4t}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2e^{3t}\\ 0\end{pmatrix} = \begin{pmatrix}e^{3t}\\ 0\end{pmatrix}$$

$$\boxed{Y(t) = Y_H + Y_p = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + \begin{pmatrix}e^{3t}\\ 0\end{pmatrix}}$$

Problem 3.

$$Y' = \begin{pmatrix}4 & 1\\ 2 & 3\end{pmatrix}Y + \begin{pmatrix}0.6t\\ -2.5t\end{pmatrix}$$

H. $\lambda = 2 \Rightarrow \begin{pmatrix}-1\\2\end{pmatrix}$

$\lambda = 5 \Rightarrow \begin{pmatrix}1\\1\end{pmatrix}$

$$Y_H = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + c_2\begin{pmatrix}-1\\2\end{pmatrix}e^{2t}$$

Let $Y_p(t) = U + Vt \Rightarrow Y_p' = V$

$$V = \begin{pmatrix}4 & 1\\ 2 & 3\end{pmatrix}(U + Vt) + \begin{pmatrix}0.6\\ -2.5\end{pmatrix}t$$

$$\Rightarrow \begin{pmatrix}4 & 1\\ 2 & 3\end{pmatrix}V + \begin{pmatrix}0.6\\ -2.5\end{pmatrix} = 0 \Rightarrow V = \begin{pmatrix}-0.43\\ 1.12\end{pmatrix}$$

$$\& \quad V = \begin{pmatrix}4 & 1\\ 2 & 3\end{pmatrix}U \Rightarrow U = \begin{pmatrix}-0.241\\ 0.534\end{pmatrix}$$

So, $\boxed{Y(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{5t} + c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{2t} + \begin{pmatrix}-0.241\\ 0.534\end{pmatrix} + \begin{pmatrix}-0.43\\ 1.12\end{pmatrix}t}$

Problem 4. Let us define

$$y_1 = y$$
$$y_2 = y_1'$$

then $y_2' - 9y_1 + y_1^3 = 0$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = 9y_1 - y_1^3 = y_1(3-y_1)(3+y_1) \end{cases}$$

critical points are $(0,0)$, $(3,0)$, $(-3,0)$

$f(y_1, y_2) = y_2$ $\qquad g(y_1, y_2) = 9y_1 - y_1^3$

$\frac{\partial f}{\partial y_1} = 0$, $\frac{\partial f}{\partial y_2} = 1$ $\qquad \frac{\partial g}{\partial y_1} = 9 - 3y_1^2$, $\frac{\partial g}{\partial y_2} = 0$

$(0,0)$ $\quad Y' = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix} Y \Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$

$\hookrightarrow$ saddle (unstable)

$(3,0)$ $\quad \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix} \Rightarrow \lambda^2 + 18 = 0 \Rightarrow \lambda = \pm i\sqrt{18}$

$\hookrightarrow$ center (stable)

$(-3,0)$ $\quad \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix} \rightarrow$

$\hookrightarrow$ center (stable)