ECE 634/CSE 646 InT: Practice Problems 3

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1) Show that for any discrete X, Y, Z, $H(X|Z) \leq H(X|Y) + H(Y|Z)$.

2) Let $K \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix. Show that $|\det(K)| \leq \Pi_{i=1}^{n} K_{i,i}$.

   [Of course you can show this using linear algebra. The goal here is to use information theory for its proof. Hint, you will need to use the subadditivity of differential entropy.]

3) [*A tweaked Fano's inequality:*] Let X be a discrete random variable taking values in $\mathcal{X}$, and let $x^* = \text{argmax}_{x \in \mathcal{X}} \mathrm{P}_X(x)$, and $p_{\max} = \mathrm{P}_X(x^*)$. Show that $H(X) \leq \log|\mathcal{X}|(1 - p_{\max}) + h(p_{\max})$.

4) Let $T = (V, E)$ be a tree (i.e., a connected graph with no cycles). Associate with each edge $e \in E$ i.i.d Be(1/2) random variables Y_e, and define $X_v = (Y_e : v \in e)$.[1]

   a) Show that $H(X_1, X_2, \ldots, X_V) = |V| - 1$.
   b) Now, let us order the set V by identifying it as $V = \{1, 2, \ldots, |V|\}$. Let $v \in V$ has degree $k > 1$, and let $v_1 < v_2 < \ldots v < k$ be its neighbours. Define $Z_v = (Y_{\{v,v_1\}} \oplus Y_{\{v,v_2\}}, Y_{\{v,v_2\}} \oplus Y_{\{v,v_3\}}, \ldots, Y_{\{v,v_{k-1}\}} \oplus Y_{\{v,v_k\}})$. If v has degree 1, then define Z_v to be a constant. One can show that since T is a tree, the constituent sums in $Z_v, v \in V$ are i.i.d. Use this fact to show that $H(Z^V) = |V| - 2$, where $Z^V \triangleq (Z_v : v \in V)$.
   c) Show that any of the neighbours v_l of v can recover X_v given (X_{v_l}, Z_v). Hence, argue that $(X_1, \ldots, X_V)$ can be recovered given any X_i and Z^V.

5) Consider the DMC whose input and output alphabets are given by $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$, and the inputs and outputs are related as $Y_i = X_i + Z_i \mod N$, such that Z_is are i.i.d. random variables independent of the X_is, whose distribution is given as,

$$\mathrm{P}_Z(i) = \begin{cases} (1-p)p^i, & 0 \leq i \leq N-2 \\ p^{N-1}, & i = N-1. \end{cases}$$

   Show that the capacity of this channel is given by $\log N - \frac{(1-p^{N-1})}{(1-p)} h(p)$.

[1] Note that we can view edges as two subsets of V.