Worksheet-3
Course Name: Math-III (Section-A)
Total marks = 20
Date: 21/09/2022

1. If $f(x, y) = \frac{(x+y)}{(2+\cos x)}$ and $\epsilon = 0.02$, then show that there exists a $\delta > 0$ such that for all (x, y), $\sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0, 0)| < \epsilon$. (5)

2. Show that the function $h(x, y) = \ln(\sqrt{x^2 + y^2})$ satisfies a Laplace equation (The two-dimensional Laplace equation is given by $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$). (5)

3. $w = \ln(x^2 + y^2 + z^2), x = \cos t, y = \sin t, z = 4\sqrt{t}$
   Express (a) $\frac{dw}{dt}$ as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then (b) evaluate $\frac{dw}{dt}$ at the value $t = 3$. (2.5+2.5)

4. Find the derivative of the function $g(x, y) = x - \frac{y^2}{x} + \sqrt{3}\sec^{-1}(2xy)$ at $P_0 = (1, 1)$ in the direction of $A = 12i + 5j$. (5)

Q. 1. Page-I | Worksheet - 3 | Solution + Rubric |.

~~Take Suppose there exists~~ Suppose there exists a $\delta > 0$.

$\because \quad -1 \le \cos x \le 1$

$\Rightarrow -1+2 \le 2+\cos x \le 2+1$

$\Rightarrow \quad 1 \le 2+\cos x \le 3.$ (1)

$\Rightarrow \quad \frac{1}{3} \le \frac{1}{2+\cos x} \le 1$

$\Rightarrow \quad \frac{|x+y|}{3} \le \left| \frac{x+y}{2+\cos x} \right| \le |x+y|$

$\therefore \quad \left| \frac{x+y}{2+\cos x} \right| \le |x+y| \le |x|+|y|$ (1)

[Now, $f(x,y) = \frac{x+y}{2+\cos x}$ & $f(0,0) = 0$.],

Then for $|x| < \delta$ and $|y| < \delta$

$\Rightarrow |f(x,y) - f(0,0)| < 2\delta = \varepsilon.$

So, $2\delta = \varepsilon$ & $\varepsilon = 0.02$ (2)

$\therefore \quad \delta = 0.01 > 0.$

Note: $|x| < \sqrt{x^2+y^2}$ & $|y| < \sqrt{x^2+y^2}$

Hence, we have proved that there exists a $\delta = 0.01 > 0$ s.t. for all (x,y)

$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \varepsilon.$ (1)

Q② Given $h(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

So, $\frac{\partial h}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$

& $\frac{\partial h}{\partial y} = \frac{y}{x^2+y^2}$ ①

$$\frac{\partial^2 h}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \to \text{I}$$

$$\& \quad \frac{\partial^2 h}{\partial y^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \to \text{II}$$

①

~~Hence the~~ Adding I + II, we have,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

$$\therefore \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$ ③

Hence, h satisfies the 2D-Laplace equation

Q.3. Given $w = \ln(x^2+y^2+z^2)$

$x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$.

ⓐ

Now, $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}$; $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$; $\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

0.5

Now, $\frac{dx}{dt} = -\sin t$; $\frac{dy}{dt} = \cos t$; $\frac{dz}{dt} = \frac{2}{\sqrt{t}}$

0.5

Now, $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$

$$= \frac{-2x\sin t + 2y\cos t + 4t^{-1/2}z}{x^2+y^2+z^2}$$

$$= \frac{-2\cos t\sin t + 2\sin t\cos t + 4t^{-1/2}4t^{1/2}}{\cos^2 t + \sin^2 t + 16t}$$

$$= \frac{16}{1+16t}$$ 0.5

~~OR~~

2nd part:

$$W = \ln(x^2+y^2+z^2)$$

$$= \ln(\sin^2 t + \cos^2 t + 16t) = \ln(1+16t)$$

$$\therefore \frac{dw}{dt} = \frac{16}{1+16t}$$ 0.5

(b) $\left.\frac{dw}{dt}\right|_{(t=3)} = \frac{16}{1+48} = \frac{16}{49}$ 2.5

Q.4. Here $A = 12\hat{i} + 5\hat{j}$

$$\therefore \vec{u} = \frac{12\hat{i}+5\hat{j}}{\sqrt{12^2+5^2}} = \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$$ 0.5

Now, $g(x,y) = x - \frac{y^2}{x} + \sqrt{3}\sec^{-1}(2xy)$.

$$\therefore g_x = 1 + \frac{y^2}{x^2} + \sqrt{3}\cdot\frac{\cancel{2}y}{\cancel{2}xy\sqrt{4x^2y^2-1}}$$ 0.5

$$\therefore g_x(1,1) = 1+1+\cancel{\sqrt{3}}\frac{1}{\cancel{\sqrt{3}}} = 3.$$

$$g_y = -\frac{2y}{x} + \sqrt{3}\frac{\cancel{2}x}{\cancel{2}xy\sqrt{4x^2y^2-1}}$$

(page-IV)

$\therefore g_y(1,1) = -2 + \frac{\sqrt{3}}{\sqrt{3}} = -1$. 0.5

Now, $\nabla g = g_x \hat{i} + g_y \hat{j} = 3\hat{i} - \hat{j}$

Hence Derivative of the function g in the direction of $A = 12\hat{i} + 5\hat{j}$ at $P_0(1,1)$ is

$$(D_{\vec{u}} g)_{P_0} = \nabla g \cdot \vec{u}$$ 0.5

$$= (3\hat{i} - \hat{j}) \cdot \left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right)$$

$$= \frac{36}{13} - \frac{5}{13} = \frac{31}{13}$$ 3

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Math-III

Page No.
Date : / /

"WorkSheet-3 Solution"

Q.1. Solution: Suppose there exists a $\delta > 0$ s.t. for all (x,y)

$$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \varepsilon.$$

where $\varepsilon = 0.02 > 0$ (given)

Let, $\delta = 0.01 \ (>0)$.

Now, $f(x,y) = \dfrac{x+y}{2+\cos x}$.

Notice that,

$$-1 \le \cos x \le 1 \quad (1)$$
$$\Rightarrow 1 \le 2+\cos x \le 3.$$
$$\Rightarrow \frac{1}{3} \le \frac{1}{2+\cos x} \le 1$$
$$\Rightarrow \frac{x+y}{3} \le \frac{x+y}{2+\cos x} \le x+y \longrightarrow (*)$$

And $f(0,0) = 0$.

Now, $$|f(x,y) - f(0,0)| = \left| \frac{x+y}{2+\cos x} - 0 \right|$$
$$\le |x+y| \quad \text{(from the inequality (*))}$$
$$\le 2\sqrt{x^2+y^2}$$
$$< 2\cdot\delta \quad (2)$$
$$= 2 \times 0.01.$$
$$= 0.02$$
$$= \varepsilon.$$

Hence, $\exists$ a $\delta = 0.01 > 0$ s.t. $\forall (x,y)$

$$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \varepsilon. \quad (2)$$

where $\varepsilon = 0.02$.

Page No.
Date : / /

OR Alternative Solution of Q. 1 :→

Let, $\delta = 0.01$.

$\because \quad -1 \le \cos x \le 1$

$\Rightarrow 1 \le 2+\cos x \le 3$.

$\Rightarrow \dfrac{x+y}{3} \le \dfrac{x+y}{2+\cos x} \le x+y$. 2

$\Rightarrow \dfrac{|x+y|}{3} \le \left|\dfrac{x+y}{2+\cos x}\right| \le |x+y| \le |x|+|y|$.

Then $|x| < \delta$ & $|y| < \delta \Rightarrow |f(x,y) - f(0,0)|$ 1

$(f(0,0) = 0)$

$$\begin{aligned} &= \left|\frac{x+y}{2+\cos x}\right| \\ &\le |x+y| \\ &\le |x|+|y| \\ &< 2\delta \\ &= 2 \times 0.01 \\ &= 0.02 \\ &= \varepsilon. \end{aligned}$$ 2

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