MTH 371: Mid Semester Exam

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Instructions

- Show all your work to score full marks. Detailed answers are a must.
- You can use a calculator. No phones or other electronic devices may be used.
- In all the questions where process or distribution is to be identified write all the corresponding parameters. Incomplete information will lead to deduction of marks.
- Wherever needed, define an appropriate random variable and then solve the question.

Questions

1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. The probability density function of X_i is given by

$$f_{X_i}(x_i) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2-1)} e^{-x/2} \quad x_i \geq 0$$

   (a) (1.5 points) Find the moment generating function of X_i.

   (b) (1.5 points) Derive the distribution of $S_n = X_1 + X_2 + \ldots + X_n$.

2. When a two way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again. The paging message is sent again and again until a success occurs. Answer the following

   (a) (1 point) How will you model the process, explain.

   (b) (1.5 points) Find the PMF of X, the number of times the system sends the same message.

   (c) (1.5 points) Find the probability that out of 10 attempts only one was successful.
   (Hint: Define a corresponding random variable and then solve it.)

   (d) (1.5 points) Find the probability that the first success was obtained after 3 attempts.
   (Hint: Define a corresponding random variable and then solve it.)

3. Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute, respectively. Suppose N, the number of vehicles (cars, trucks, or buses) that arrive follows Poisson with $\lambda = 1.2 + 0.9 + 0.7 = 2.8$ vehicles/minute. Answer the following questions

   (a) (1 point) Is it a renewal process, explain.

   (b) (1 point) Find the probability that 2 vehicles arrived in $(0, 2]$ minutes.

   (c) (1.5 points) Find the probability that at least 2 vehicles arrived in $(3, 5]$ minutes and utmost 1 vehicle arrived in $(5, 8]$.

   (d) (1.5 points) Find the probability that the time taken to arrival of two vehicles is more than 5 minutes.

4. The input to a digital filter is an independent and identically distributed random sequence $\ldots, X_{-1}, X_0, X_1, \ldots$ following Gaussian with $E[X_i] = 0$ and $Var[X_i] = 1$. The output $\ldots, Y_{-1}, Y_0, Y_1, \ldots$ is related to the input by the formula $Y_n = X_n + X_{n-1}$ for all integers in n. Answer the following questions

   (a) (1.5 points) Find the distribution of Y_n.

   (b) (1 point) Find the expected value, $E[Y_n]$.

   (c) (1.5 points) Find the autocovariance function $C_Y[m, k]$. The autocovariance is given by $C_Y[m, k] = Cov[Y_m, Y_k]$.

   (d) (1.5 points) If $\mathbf{X} = (X_1, \ldots, X_n)$ are jointly Gaussian. Can you find the distribution of $\mathbf{Y} = (Y_1, \ldots, Y_n)$. If yes, find it. If not, comment in detail why not.

5. (1.5 points) The time X required to repair a machine is an exponential distributed random variable with mean 1/2. What is the probability that a repair time takes at least 12 and a half-hour hours given that its duration exceeds 12 hours.

6. (1.5 points) The cars at a bridge arrive according to a Poisson process at rate $\lambda = 2$ per hour. Evaluate $Cov(N(5) - 2N(6), 3N(10))$, here $N(t)$ is number of arrivals in time t.