MTH 377/577 CONVEX OPTIMIZATION
Winter Semester 2023
Indraprastha Institute of Information Technology Delhi
MIDSEM EXAM SOLUTIONS

Q1. (5 points). Consider a function $f : \mathbb{R}^2_+ \mapsto \mathbb{R}$ defined by

$$f(x, y) = e^{2x_1+3x_2} + x_1 \log(x_1) + x_2 \log(x_2)$$

Is f convex ? If yes, prove it. If not, argue why not.

A1. We prove the convexity of f in a series of steps:
Step 1. $(x_1, x_2) \mapsto 2x_1 + 3x_2$ is a linear function.
Step 2. $t \mapsto e^t$ is a convex function.
Step 3. $(x_1, x_2) \mapsto e^{2x_1+3x_2}$ is a convex function as the convex transformation of a linear function.
Step 4. $(x_1, x_2) \mapsto x_1 \log(x_1)$ is a convex function.
Step 5. $(x_1, x_2) \mapsto x_2 \log(x_2)$ is a convex function.
Step 6. f is a convex function as the sum of three convex functions.

Grading. Allocate 2 points for Steps 1-3, 1 point for Steps 4-5, and 2 points for Step 6.

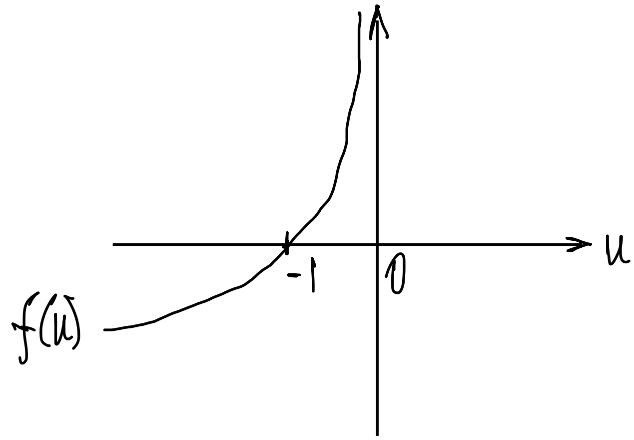
Q2. (4 points). Consider a function $f : (-\infty, 0) \mapsto \mathbb{R}$ defined by

$$f(u) = -\frac{1}{t} \log(-u)$$

where $t > 0$ is a parameter. Is f convex? Is f an increasing function? What is the limiting behavior of f as u approaches 0 ?
A2. Yes f is convex. Since $u :\mapsto -u$ is linear, $u :\mapsto \log(-u)$ is concave as a concave transformation of a linear function. Therefore, f is convex as a negatively scaling of a concave function.

Yes, f is increasing because $u_2 > u_1$ implies $-u_2 < -u_1$ which implies $\log(-u_2) < \log(-u_1)$ as log is a monotonically increasing function. This further implies $f(u_2) = -\frac{1}{t} \log(-u_2) > -\frac{1}{t} \log(-u_1) = f(u_1)$.

$f(u)$ tends to ∞ as u tends to 0.



Grading. Allocate 2 points for convexity, 1 point for increasing, and 1 point for limiting behavior. Student may show convexity graphically by depicting the graph of negative log kind of a function. This is fine too.

Q3. (8 points) Find the local minima, local maxima, or saddle points, if any, of the following functions. Show your full work and reasoning.

(a) $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $f(x, y) = (y - x^2)^2 - x^2$.
(b) $f : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by $f(x, y, z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9$.

Grading. Each part is worth 4 points. In each part, allocate 2 points to computing the Hessian correctly and 2 points to applying the Second-order Sufficiency Conditions for local minima/maxima/saddle point correctly.

A3. (a)

$$
\begin{aligned}
Df(x, y) &= \begin{bmatrix} 4x^3 - 4xy - 2x & 2(y - x^2) \end{bmatrix} \\
D^2 f(x, y) &= \begin{bmatrix} 12x^2 - 4y - 2 & -4x \\ -4x & 2 \end{bmatrix} \\
D^2 f(x, y) - \lambda I &= \begin{bmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{bmatrix}
\end{aligned}
$$

The stationary points (x, y) of f are points where the gradient is zero. Such points are solutions to the system of equations

$$
\begin{aligned}
4x^3 - 4xy - 2x &= 0 \\
2(y - x^2) &= 0
\end{aligned}
$$

The only solution to this system is $x = 0, y = 0$. At this stationary point, the Hessian is

$$
\begin{aligned}
D^2 f(0, 0) &= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \\
D^2 f(0, 0) - \lambda I &= \begin{bmatrix} -2 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix}
\end{aligned}
$$

The eigenvalues of the Hessian $D^2 f(0,0)$ are simply the diagonal entries -2 and 2. So by the eigenvalue test, $D^2 f(0,0)$ is indefinite and so $(0,0)$ is a saddle point of f.

(b)

$$Df(x,y) = \begin{bmatrix} 4x+y-6 & x+2y+z-7 & y+2z-8 \end{bmatrix}$$
$$D^2 f(x,y) = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The stationary points (x,y,z) of f are points where the gradient is zero. Such points are solutions to the system of equations

$$\begin{aligned} 4x+y-6 &= 0 \\ x+2y+z-7 &= 0 \\ y+2z-8 &= 0 \end{aligned}$$

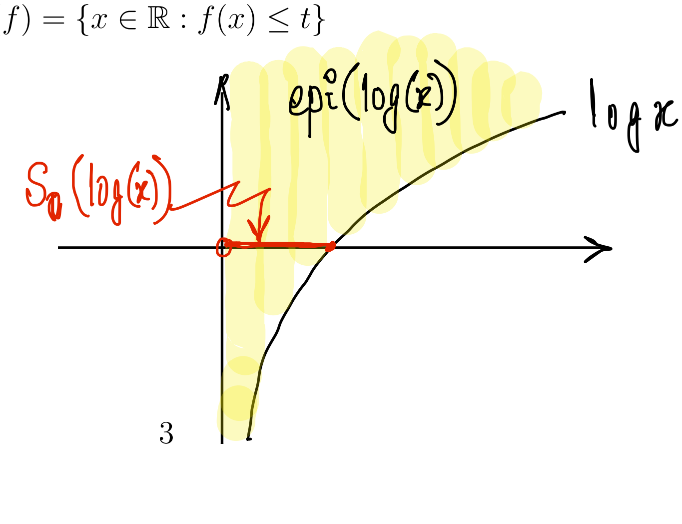
The only solution to this system is $x = 1.2, y = 1.2, z = 3.4$. The Hessian is the same throughout the domain and its eigenvalues are all positive. (Use the eigenvalue calculator!). So by the eigenvalue test, the Hessian is positive definite and so f is a convex function. Hence the stationary point $x = 1.2, y = 1.2, z = 3.4$ is a global minimum (and so a local minimum) of f.

Q4. (5 points) Define set theoretically (in formal mathematics) the notions of epigraph $\mathbf{epi}(f)$ and t-sublevel set $\mathbf{S}_t(f)$ for a univariate function f. Answer the following questions:

(a) What is $\mathbf{epi}(\log(x))$? Draw it as a shaded region.
(b) What is $\mathbf{epi}(\log(x)) \cap \{(x,y) \in \mathbb{R}^2 : x = 0\}$?
(c) What is the 0-sublevel set of $\log(x)$? Draw it and write it precisely as a set.
(d) Is the 0-sublevel set of $\log(x)$ a convex set ?

A4. The definitions are

$$\begin{aligned} \mathbf{epi}(f) &= \{(x,t) \in \mathbb{R}^2 : t \geq f(x)\} \\ \mathbf{S}_t(f) &= \{x \in \mathbb{R} : f(x) \leq t\} \end{aligned}$$



(a) all points on or above the graph of $\log(x)$.
(b) emptyset. the epigraph never intersects the y-axis.
(c) the left open right closed interval $(0, 1]$
(d) Yes

Grading: 1 point for definitions. 1 point for every part (a) - (d).

Q5. (5 points). Let $S \subset \mathbb{R}^n$ be a convex set and $v \in \mathbb{R}^n$. Show that the set C defined as

$$C = \{\theta v + (1-\theta)s : \theta \in [0,1] \text{ and } s \in S\}$$

is a convex set.

A5. Let $\theta_1 v + (1-\theta_1)s_1$ and $\theta_2 v + (1-\theta_2)s_2$ be two points in C. Take $\lambda \in [0,1]$. Then a λ-convex combination of these points is

$$\begin{aligned}
&\lambda[\theta_1 v + (1-\theta_1)s_1] + (1-\lambda)[\theta_2 v + (1-\theta_2)s_2] \\
&= \Big(\lambda\theta_1 + (1-\lambda)\theta_2\Big)v + \lambda(1-\theta_1)s_1 + (1-\lambda)(1-\theta_2)s_2
\end{aligned}$$

We will have shown that C is a convex set if we can find a point $s \in C$ such that

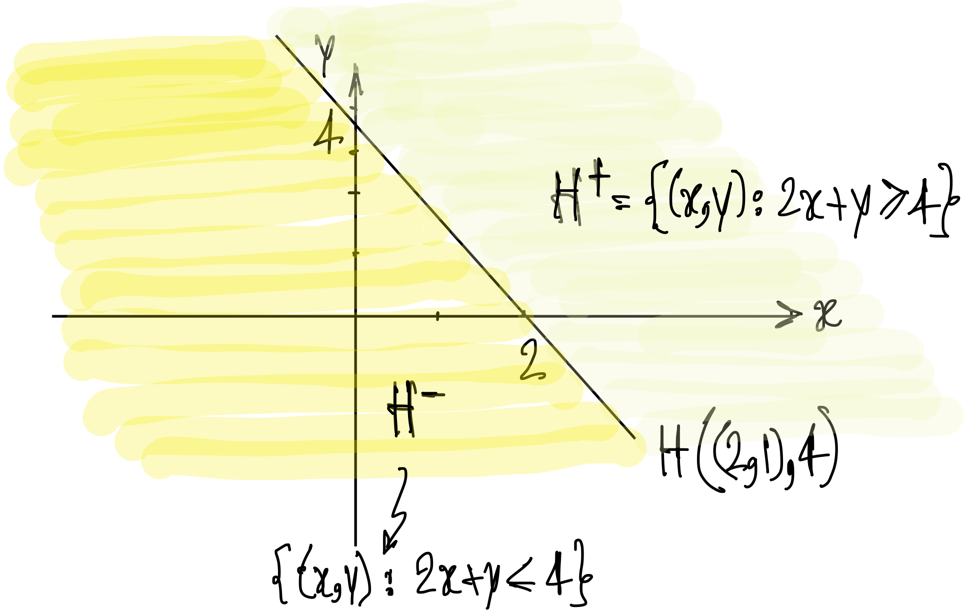
$$\begin{aligned}
\lambda(1-\theta_1)s_1 + (1-\lambda)(1-\theta_2)s_2 &= \Big(1 - \lambda\theta_1 - (1-\lambda)\theta_2\Big)s \\
\text{or, equivalently } s &= \frac{\lambda(1-\theta_1)}{1-\lambda\theta_1-(1-\lambda)\theta_2}s_1 + \frac{(1-\lambda)(1-\theta_2)}{1-\lambda\theta_1-(1-\lambda)\theta_2}s_2 \\
\text{or, equivalently } s &= \theta_3 s_1 + (1-\theta_3)s_2 \text{ where } \theta_3 = \frac{\lambda(1-\theta_1)}{1-\lambda\theta_1-(1-\lambda)\theta_2}
\end{aligned}$$

But such a point s which is a convex combination of s_1 and s_2 must belong to C. Hence C is convex.

Grading: 1 point for taking the right two arbitrary points in C. In particular, these two points may have different θ's and different s's. 1 point if the solution displays that one needs to show the convex combination of two arbitrary points should be in C. 3 points for demonstrating the latter part of the answer.

Q6. (3 points). Is the line passing through the point $(3/2, 1)$ and slope -2 a hyperplane ? If yes, identify this hyperplane with its (normal vector, scalar) and indicate the positive and negative halfspaces associated with it. If not, argue why not.

A6. Yes, a line in the plane is a hyperplane in $\mathbb{R}^2$. Its equation is $2x + y = 4$. So its normal vector is $(2, 1)$ and its scalar is 4.



Grading: 2 points for affirmation of hyperplane and identifying (normal vector, scalar). 1 point for indicating the halfspaces correctly.