## Submission Solution (3 April, 2022)

Q(1) $T(x, y, z) = (2x + 2y + z,\ 2y + z,\ 2x + 3y + z)$

a) $T(e_1) = T(1,0,0) = (2, 0, 2)$ , $T(e_2) = T(0,1,0) = (2,2,3)$

$T(e_3) = T(0,0,1) = (1,1,1)$

$$[T]_\alpha = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

b) $[e_1]_\beta = \left(\frac{1}{2}, \frac{-1}{2}, 0\right)$ $\quad [e_2]_\beta = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ $\quad [e_3]_\beta = (-1, 0, 1)$

$$P_{\alpha \to \beta} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $$[T]_\beta = PAP^{-1} = \begin{bmatrix} -2 & 0 & -2 \\ -1 & 1 & -1 \\ 5 & 1 & 6 \end{bmatrix}, \quad AP^{-1} = \begin{bmatrix} 4 & 0 & 5 \\ 2 & 2 & 3 \\ 5 & 1 & 6 \end{bmatrix}$$

d) $$[Tv]_\beta = [T]_\beta [v]_\beta = \begin{bmatrix} -3 \\ -1 \\ 12 \end{bmatrix}, \quad [v]_\beta = \begin{bmatrix} 5/2 \\ 1/2 \\ 4 \end{bmatrix}$$

Rubric: Marks can be given just for getting correct answer But deduct 50%, if steps are not shown.

Note: Any correct method can be used for calculations

## Additional Calculations:

$\rightarrow [e_1]_\beta = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = (\alpha_1 - \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3)$

$\Rightarrow \alpha_1 - \alpha_2 + \alpha_3 = 1, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_3 = 0$

$\Rightarrow \alpha_1 - \alpha_2 = 1 \quad \alpha_1 + \alpha_2 = 0$

$$\Rightarrow \begin{array}{r} \alpha_1 + \alpha_2 = 0 \\ \hline 2\alpha_1 = 1 \end{array} \quad \Rightarrow \alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2}$$

$[e_1]_\beta = (\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{2}, -\frac{1}{2}, 0)$

$e_2 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \Rightarrow \alpha_1 - \alpha_2 + \alpha_3 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \alpha_3 = 0$

$\Rightarrow \alpha_1 - \alpha_2 = 0 \quad \alpha_1 + \alpha_2 = 1$

$\Rightarrow \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2}$

$[e_2]_\beta = (\frac{1}{2}, \frac{1}{2}, 0)$

$e_3 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \Rightarrow \alpha_1 - \alpha_2 + \alpha_3 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_3 = 1$

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 = -1 \\ \alpha_1 + \alpha_2 = -1 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_1 = -1 \\ \alpha_2 = 0 \end{array}$$

$[e_3]_\beta = (-1, 0, 1)$

$\rightarrow v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \Rightarrow \alpha_1 - \alpha_2 + \alpha_3 = 1, \quad \alpha_1 + \alpha_2 + \alpha_3 = 2, \quad \alpha_3 = 4$

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 = -3 \\ \alpha_1 + \alpha_2 = -2 \end{array}\right\} \Rightarrow \alpha_1 = -\frac{5}{2}, \quad \alpha_2 = \frac{1}{2}$$

$[v]_\beta = [-\frac{5}{2}, \frac{1}{2}, 4)$

MTH100 Monsoon Semester – 2021-22 Section B

**Submission for Sunday 3rd April 2022 – 30 minutes. Max Marks: 5 + 5**

**Q2.** Given the matrix A below.

a. Determine the characteristic polynomial of A and verify that A satisfies its characteristic polynomial. (*2.5 marks*)

b. Is A invertible (YES/NO) ? (*One word answer in capitals.*) (*0.5 marks*)

c. If your answer to b. is YES, determine the inverse of A. If your answer is NO, justify your answer with reference to suitable calculations and results. (*2 marks*)

**NB: Your answer to c. must use your answer to a. DO NOT USE ANY OTHER METHOD.**

$$A = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 5 & 8 \\ 2 & -2 & -2 \end{bmatrix}$$

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SOLUTION

a. The characteristic polynomial of A is $\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 - 2\lambda$ (1)

[See last page for calculations]

For the remainder of the question, we will work with the polynomial

$q(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda^2 - 3\lambda + 2)$ (2)

This does not affect the answers.

Note that $A^2 = \begin{bmatrix} 4 & 2 & 8 \\ 6 & 5 & 16 \\ 0 & -2 & -4 \end{bmatrix}$ (3)

and $A^3 = \begin{bmatrix} 12 & 2 & 16 \\ 22 & 5 & 32 \\ -4 & -2 & -8 \end{bmatrix}$ (4)

Hence, $q(A) = A^3 - 3A^2 + 2A$

(PTO)

(Cont'd)

$$\begin{bmatrix} 12 & 2 & 16 \\ 22 & 5 & 32 \\ -4 & -2 & -8 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 & 8 \\ 6 & 5 & 16 \\ 0 & -2 & -4 \end{bmatrix} + 2\begin{bmatrix} 0 & 2 & 4 \\ -2 & 5 & 8 \\ 2 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [0], \text{ as required} \quad (5)$$

b) NO

c) Now, $A^3 - 3A^2 + 2A = [0]$

or $A[A^2 - 3A + 2] = [0]$

or $AB = [0]$, where $B = [A^2 - 3A + 2]$ (6)

Suppose $A$ is invertible. (BWOC)

By (6), $A$ is a zero-divisor since $B \neq [0]$.

However, by a known result

( See Q4(a), TUT 04, week commencing 20220131)

an Invertible matrix <u>cannot be a zero divisor.</u>

This is a contradiction. Result follows.

# RUBRIC



a. Calculation of characteristic polynomial
→ 1 mark

Verification → 1.5 marks
For an incomplete or partial verification, 0.5 marks may be given if $A^2$ is correct & 0.5 marks more if $A^3$ is correct.

However: Steps must be shown. If correct answer, but no steps, cut 50% marks.

b. NO → 0.5 marks

c. Justify (only if NO is answer for b.) → 2 marks.
The method must be the one given (details of argument may be slightly different).
Any other method → RREF, ∇ IT, etc. → 0 marks.

Additional calculations: (4)

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 2 & 4 \\ -2 & 5-\lambda & 8 \\ 2 & -2 & -2-\lambda \end{bmatrix}$$

$$= -\lambda\left[(5-\lambda)(-2-\lambda)+16\right] - 2\left[(-2)(-2-\lambda)-16\right]$$

$$+4\left[4-2(5-\lambda)\right]$$

$$= -\lambda\left[-10-3\lambda+x^2+6\right] - 2\left[4+2\lambda-16\right] + 4\left[-6+2\lambda\right]$$

$$= -\lambda\left[\lambda^2-3\lambda+6\right] - 2\left[2\lambda-12\right] + 4\left[2\lambda-6\right]$$

$$= -\lambda^3+3\lambda^2-6\lambda-4\lambda+24+8\lambda-24$$

$$= -\lambda^3+3\lambda^2-2\lambda \quad \checkmark$$