MTH 204: Worksheet 7 (Solutions)

(10 points)

(2×2.5=5 pts.)

Q-1 (a) $e^{3x}, \sin x, \cos x$, $y''' - 3y'' + y' - 3y = 0$ — ①

$y_1(x) = e^{3x}$, $y_2(x) = \sin x$, $y_3(x) = \cos x$

For $y_1(x)$: $27e^{3x} - 3\times 9e^{3x} + 3e^{3x} - 3e^{3x} = 0$

$\therefore y_1$ is a solution of ①.

For $y_2(x)$: $-\cos x + 3\sin x + \cos x - 3\sin x = 0$

$\therefore y_2$ is a solution of ①.

For $y_3(x)$: $\sin x + 3\cos x - \sin x - 3\cos x = 0$

$\therefore y_3$ is a solution of ①.

Wronskian: $W(y_1, y_2, y_3) = \begin{vmatrix} e^{3x} & \sin x & \cos x \\ 3e^{3x} & \cos x & -\sin x \\ 9e^{3x} & -\sin x & -\cos x \end{vmatrix}$

$= e^{3x}[-\cos^2 x - \sin^2 x] - \sin x[-3e^{3x}\cos x + 9e^{3x}\sin x] + \cos x[-3e^{3x}\sin x - 9e^{3x}\cos x]$

$= -e^{3x} + 3e^{3x}\cancel{\sin x}\cos x - 9e^{3x}\sin^2 x - 3e^{3x}\cancel{\cos x}\sin x - 9e^{3x}\cos^2 x$

$= -10e^{3x} \neq 0$

$\Rightarrow \{y_1, y_2, y_3\}$ is linearly independent on any interval, hence forms basis.

(b) $1, x^2, x^5$, $x^3 y''' - 4x^2 y'' + 4x y' = 0$ — ②, $x > 0$

$y_1(x) = 1$, $y_2(x) = x^2$, $y_3(x) = x^5$

For $y_1(x)$: $x^3 \times 0 - 4x^2 \times 0 + 4x \times 0 = 0$

$\therefore y_1$ is a solution of ②.

For $y_2(x)$: $x^3 \times 0 - 8x^2 + 8x^2 = 0$

$\therefore$ y_2 is a solution of ②.

For $y_3(x)$: $x^3 \times 60x^2 - 4x^2 \times 20x^3 + 4x \times 5x^4 = 0$

$(60x^5 - 80x^5 + 20x^5 = 0)$

$\therefore$ y_3 is a solution of ②

Wronskian: $W(y_1, y_2, y_3) = \begin{vmatrix} 1 & x^2 & x^5 \\ 0 & 2x & 5x^4 \\ 0 & 2 & 20x^3 \end{vmatrix} = 40x^4 - 10x^4 = 30x^4 \neq 0$ for $x \neq 0$

$\Rightarrow$ $\{y_1, y_2, y_3\}$ is linearly independent on any interval not containing 0, and hence forms a basis

Q-2 (a) Let $y_2(x) = u(x)\, y_1(x)$

$$\Rightarrow y_2'(x) = u' y_1 + y_1' u$$

$$\Rightarrow y_2'' = u'' y_1 + u' y_1' + y_1'' u + y_1' u'$$

$$\Rightarrow y_2''' = u''' y_1 + u'' y_1' + u'' y_1' + u' y_1'' + y_1''' u + y_1'' u' + y_1'' u' + y_1' u''$$

Put in the ODE,

$$u''' y_1 + 2u'' y_1' + 3u y_1'' + y_1''' u + y_1'' u + y_1' u'' + p_2(x)[u'' y_1 + u' y_1' + y_1'' u + y_1' u'] + p_1(x)[u' y_1 + y_1' u] + p_0(x) u(x) y_1(x) = 0$$

$$\Rightarrow u'''(y_1) + u''(3y_1' + p_2(x) y_1) + u'(3y_1'' + 2p_2(x) y_1' + p_1(x) y_1] + u\underbrace{[y_1''' + y_1'' p_2(x) + p_1(x) y_1' + p_0(x) y]}_{0} = 0$$

$$\Rightarrow y_1 u''' + (3y_1' + p_2(x) y_1) u'' + (3y_1'' + 2p_2(x) y_1' + p_1(x) y_1] = 0 \quad —①$$

Let $u'(x) = z(x)$, that is, $u(x) = \int z(x)\, dx$. Then ① becomes

$$\boxed{y_1 z'' + (3y_1' + p_2(x) y_1) z' + (3y_1'' + 2p_2(x) y_1' + p_1(x) y_1) z = 0}$$

(6) $(2-x)y''' + (2x-3)y'' - xy' + y = 0$, $y_1(x) = x$

$\Rightarrow p_0(x) = \dfrac{2x-3}{2-x}$, $p_1(x) = \dfrac{-x}{2-x}$, $p_2(x) = \dfrac{1}{2-x}$

Reduced ODE is:

$$x \times z'' + \left(3 \times 1 + \frac{1}{2-x} \times x\right) z' + \left(3 \times 0 + \frac{2}{2-x} \times 1 + \left(\frac{-x}{2-x}\right) x\right) z = 0$$

$$\Rightarrow x z'' + \frac{6-2x}{2-x} z' + \left(\frac{2-x^2}{2-x}\right) z = 0 \Rightarrow \boxed{x z'' + \left(\frac{6-2x}{2-x}\right) z' + (2+x) z = 0}$$

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