# MTH 204 Quiz 5

(Time : 15 mins, Maximum Marks : 10)

April 19, 2023

**Question 1.**

[4 points] Consider the 2nd order nonlinear ODE

$$① \quad \frac{d^2y}{dt^2} = t^2 + y\frac{dy}{dt} + y^2, \quad y(0) = 0, \ y'(0) = 1.$$

Assuming that a series solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

exists, find values of $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$.

Soln: The Taylor series of $y(t)$ around $t=0$ is given by:

$$② \quad y(t) = y(0) + y'(0)t + y''(0)\frac{t^2}{2!} + y'''(0)\frac{t^3}{3!} + y^{iv}(0)\frac{t^4}{4!} + y^{v}(0)\frac{t^5}{5!} + \dots$$

$$y(0)=0, y'(0)=1 \Rightarrow y(t) = t + y''(0)\frac{t^2}{2!} + y'''(0)\frac{t^3}{3!} + y^{iv}(0)\frac{t^4}{4!} + y^{v}(0)\frac{t^5}{5!} + \dots \quad ①$$

Now, using ① we have $y''(0) = y(0)y'(0) + (y(0))^2 = 0$

$$\& \ \frac{d^3y}{dt^3} = 2t + \left(\frac{dy}{dt}\right)^2 + y\frac{d^2y}{dt^2} + 2y\frac{dy}{dt} \Rightarrow y'''(0) = (y'(0))^2 + y(0)y''(0) + 2y(0)y'(0) = 1$$

$$\Downarrow$$

$$\frac{d^{iv}y}{dt^{iv}} = 2 + 2\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + y\frac{d^3y}{dt^3} + 2\left(\frac{dy}{dt}\right)^2 + 2y\frac{d^2y}{dt^2} \Rightarrow y^{iv}(0) = 4$$

$$\Downarrow$$

$$\frac{d^{v}y}{dt^{v}} = 3\left(\frac{d^2y}{dt^2}\right)^2 + 3\frac{dy}{dt}\frac{d^3y}{dt^3} + \frac{dy}{dt}\frac{d^3y}{dt^3} + y\frac{d^{iv}y}{dt^{iv}} + 4\frac{dy}{dt}\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} + 2y\frac{d^3y}{dt^3}$$

$$\Rightarrow y^{v}(0) = 4 \quad ②$$

$$\therefore y(t) = t + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{30}t^5 + \dots \quad \text{(Using ②)}$$

Hence, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{1}{6}$, $a_4 = \frac{1}{6}$, $a_5 = \frac{1}{30}$. ①

**Question 2.**

[6 points] Consider a mass-spring system with values of constants as $m = 1$, $c = 3$, and $k = 2$. An external force of 2 units is acting on the system till 6 units of time, after which the external force increased to 4 units. Set up an IVP to describe this mass-spring system. Write the function representing external force in terms of unit-step function. Take Laplace Transform of the IVP and find Laplace transform of the solution. (Note. All units are SI units. You dont have to find inverse Laplace transform.)

Solⁿ:

2 The equation of motion for a mass-spring system with damping and external forcing can be written as:

$$my'' + cy' + ky = F_{ext}(t) \quad \text{—(1)}$$

where $y$ is the displacement of the mass from its equilibrium point,
$m$ is the mass of the object.
$c$ is the damping coefficient
$k$ is the spring constant
$F_{ext}(t)$ is the external force at time $t$ acting on the mass

1 Here, $m=1$, $C=3$, $k=2$ and

$$F_{ext} = \begin{cases} 2u(t) & , 0 \le t < 6 \\ 4u(t-6) & , t \geq 6 \end{cases}$$ for unit-step funcⁿ $u(t)$

1 ∴ (1) gives $y'' + 3y' + 2y = 2u(t) + 4u(t-6)$

To write an IVP for this eqⁿ, we need initial conditions.

Let $y(0)=0$, $y'(0)=0$ ⇒ mass is at rest at the equilibrium position at $t=0$

Thus, IVP is
$$\left.\begin{array}{l} y'' + 3y' + 2y = 2u(t) + 4u(t-6) \\ y(0)=0,\ y'(0)=0 \end{array}\right\} \text{(2)}$$

1 Taking Laplace Transform of the IVP (2) and using initial value theorem, we get

$$s^2 Y(s) - s y(0) - y'(0) + 3s Y(s) - 3y(0) + 2Y(s) = \frac{2}{s} + \frac{4e^{-6s}}{s} \quad \text{—(3)}$$

On applying initial conditions, we get from ③

$$s^2 Y(s) + 3s Y(s) + 2Y(s) = \frac{2+4e^{-6s}}{s}$$

$$\Rightarrow Y(s) = \frac{2+4e^{-6s}}{s(s^2+3s+2)}$$

$$\Rightarrow \boxed{Y(s) = \frac{2+4e^{-6s}}{s(s+1)(s+2)}}$$

① 

Alternate solution of Q1:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$\Rightarrow y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} \;\; \& \;\; y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \quad Ⓐ$$

∴ ① gives

① $$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = t^2 + \left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right)^2$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} n a_n a_m t^{m+n-1} - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n a_m t^{m+n} = t^2 \quad — ②$$

Using initial conditions, we have

$$y(0) = \boxed{a_0 = 0} \quad \& \quad y'(0) = \boxed{a_1 = 1} \quad (\text{By Ⓐ})$$

Comparing coefficients of $t^k$, we have

For $k=0$, ② gives

$$2a_2 - a_1 a_0 - a_0^2 = 0 \Rightarrow \boxed{a_2 = 0}$$

6×0.5 = ③

For $k=1$, ② gives

$$6a_3 - a_1^2 - 2a_2 a_0 - 2a_0 a_1 = 0 \Rightarrow 6a_3 - 1 = 0 \Rightarrow \boxed{a_3 = \frac{1}{6}}$$

For $k=2$, ② gives

$$12a_4 - 3a_1 a_2 - a_1^2 - 2a_0 a_2 = 1 \Rightarrow 12a_4 - 1 = 1 \Rightarrow \boxed{a_4 = \frac{1}{6}}$$

For $k=3$, ② gives

$$20a_5 - 4a_1 a_3 - 2a_2^2 - 4a_4 a_0 - 2a_0 a_3 - 2a_2 a_1 = 0 \Rightarrow 20a_5 - \frac{4}{6} = 0$$

$$\Rightarrow \boxed{a_5 = \frac{1}{30}}$$