MTH100 **Monsoon Semester – 2021-22** **Section B**

Submission for Wednesday 30th March 2022 – 15 minutes. Max Marks: 5

Instructions: *Open notes and textbook; consultation and use of calculators, computers and internet not allowed. You may use any* ***known*** *result. This includes all propositions and observations in the lecture slides, and results from tutorials. If you use any other result from any other source, including the textbook, you have to give a full proof of that result.*

Q. PROVE or DISPROVE: If T is a linear operator on a **finite-dimensional** vector space V over the field F, such that rank (T^2) = rank (T), then Range (T) $\cap$ Kernel (T) = $\{\mathbf{0}\}$.

Remark: You must clearly write PROVE or DISPROVE at the top of your answer. 1 mark is reserved for this. If not written, you will directly get 0 marks. For PROVE, you must give a general proof using known results; use of examples is not acceptable. For DISPROVE, you must give a concrete (numerical) counter-example.

ANSWER: PROVE — 1 MARK

Proof: For convenience, put $W = \text{Range}(T)$ and $U = \text{Range}(T) \cap \text{Kernel}(T)$

Let $n = \dim V$, and $r = \text{rank}(T) = \dim W$. (1)

Clearly, $\text{Range}(T^2) \subseteq \text{Range}(T)$, but since $\text{rank}(T^2) = \text{rank}(T)$, $\text{Range}(T^2) = \text{Range}(T) = W$ (2)

Let $T_1 : W \to W$ be the linear operator defined by $T_1(\bar{w}) = T(\bar{w})$

(Cont'd)

for all $\bar{w} \in W$.

T_1 is simply the restriction of T to the domain W. But

$$\text{Range}(T_1) = \text{Range}(T^{2}) = W \quad ③$$

Applying the Rank Theorem to T_1,

$$\text{rank}(T_1) + \text{nullity}(T_1) = \dim W,$$

and applying ①, ②, ③, this becomes: $r + \text{nullity}(T_1) = r$ ④

Hence, $\text{Nullity}(T_1) = 0 \Rightarrow$

$$\text{Kernel}(T_1) = \{\bar{0}\} \quad ⑤$$

Finally, suppose $\bar{u} \in U \subseteq W$

Then, $T_1(\bar{u}) = T(\bar{u}) = \bar{0}$

$\Rightarrow \bar{u} \in \text{Kernel}(T_1)$

$\Rightarrow \bar{u} = \bar{0}$, by ⑤.

$\therefore U = \{\bar{0}\}$, as required.

Rubric:- 4 marks for a correct proof. A proof which is incomplete, but reaches as far as ② may be given 1.5 marks. Applying Rank Theorem may be given another 0.5 marks.

Remark: Result can also be proved from basics - similar to proof of Rank Theorem for linear Transformations.