Mid-Semester Exam
DM, Monsoon 2021

Duration : 90 mins
Max marks : 20

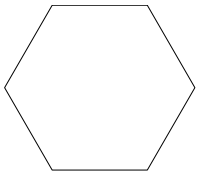
1. (2 marks) Show that $n^3 - 3n^2 + 2n$ is divisible by 6 for each positive integer n.

2. (3 marks) Use the Well Ordering Principle to prove that any integer greater than or equal to 23 can be represented as the sum of nonnegative integer multiples of 6, 7 and 17.

3. (3 marks) Let α, β be roots of the equation $x^2 - 3x - 1 = 0$. For each nonnegative integer n, let $y_n = \alpha^n + \beta^n$. Show that $\gcd(y_n, y_{n+1}) = 1$ for each nonnegative integer n.

4. (3 marks) Give a combinatorial proof of the following identity for nonnegative integers n and k.
$$\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1}.$$

5. (3 marks) Given 13 points inside a convex regular hexagon of unit side length, show that we can find 3 of them such that the area of the triangle formed by them has area at most $\frac{\sqrt{3}}{4}$.



6. (3 marks) Guess the value of the following sum and give a combinatorial proof.
$$\sum_{k=0}^{n}\sum_{\ell=0}^{k} \binom{n}{k}\binom{k}{\ell}\binom{\ell}{m}.$$

7. (3 marks) Determine a_n, the number of words of length n on the alphabet $\{a, b, c\}$ which do not contain the substring ab. For instance, $a_3 = 21$ since there are 21 such words with 3 letters, namely:

$$\begin{array}{ccccccc} aaa & aac & aca & acb & acc & baa & bac \\ bba & bbb & bbc & bca & bcb & bcc & caa \\ cac & cba & cbb & cbc & cca & ccb & ccc. \end{array}$$