Discrete Math, CSE 121 : End-Sem Exam, Monsoon 2023

General Instructions:

(a) Maximum marks = 50; Duration: 2 hours.

(b) This is a closed-book exam. The exam paper is self-contained. Any attempt to use any source during the exam will be dealt with according to the Academic Dishonesty Policy of the institute.

(c) In every proof/derivation clearly state your assumptions and give details of each step.

(d) A proof of an "if and only if" statement is incomplete unless implications on both ways proved.

(e) You will be evaluated for your attempt and approach. Therefore, you are encouraged to attempt the questions even if you can not complete an answer. The Academic Dishonesty Policy of the institute is equally applicable even for partial answers.

Questions:

1. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English texts. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.
   (a) All clear explanations are satisfactory. [1.5 marks]
   (b) Some excuses are unsatisfactory. [1.5 marks]
   (c) Some excuses are not clear explanations. [1.5 marks]

2. Prove or disprove that if A is an uncountable set and B is a countable set then $A - B$ is a countable set. [3.5 marks]

3. Let a and b be two nonzero integers. Then the set S of their common positive divisors is nonempty and finite. Thus, S contains its greatest element. This element is called the greatest common divisor of a and b and is denoted by $gcd(a, b)$. Prove that $gcd(a, bc) = 1$ if and only if $gcd(a, b) = 1$ and $gcd(a, c) = 1$, for any three nonzero integers a, b and c. [4 marks]

4. Let $f_1(x) : \mathbb{R} \to \mathbb{R}^+$ and $f_2(x) : \mathbb{R} \to \mathbb{R}^+$ be functions from the set of real numbers to the set of positive real numbers. Show that if $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, where $g(x) : \mathbb{R} \to \mathbb{R}^+$ is a function from the set of real numbers to the set of positive real numbers, then $f_1(x) + f_2(x)$ is $\Theta(g(x))$. Is this still true if $f_1(x)$ and $f_2(x)$ can take negative values? [(3+1) = 4 marks]

5. Consider an undirected connected acyclic graph $G = (V, E)$, where V is the set of vertices and E is the set of edges. Prove the following properties for G:
   (a) If $|V| = n$ then $|E| = n - 1$. [2.5 marks]
   (b) If $|V| \geq 2$ then there are at least two vertices in V with degree 1, where degree of v denotes the number of edges emanating from v. [2.5 marks]

6. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers. [5 marks]

7. Prove or disprove that the set of real numbers that are solutions of cubic equations $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are integers, is uncountable. [6 marks]

8. (a) Prove that a vertex v in a connected graph G is a cut vertex if and only if there exists 2 vertices v_1 and v_2 in G, such that every path between v_1 and v_2 passes through v. [3 marks]
   (b) Now, suppose that v is an endpoint of a cut edge. Using (a) prove that v is a cut vertex if and only if it has more than one neighbour. [3 marks]

9. A binary tree is called full if each node has either no child or two children. If T is a full binary tree, then prove that $n(T) \leq 2^{h(T)+1} - 1$, where n(T) and h(T) denote the number of nodes and the height of the tree T, respectively. [6 marks]

10. Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime integers such that $m_i \geq 2$ $\forall i \in \{1, \ldots, n\}$. Show that if $a \equiv b(mod\ m_i)$ $\forall i \in \{1, \ldots, n\}$, then $a \equiv b(mod\ m)$, where $m = m_1 m_2 \ldots m_n$. [6 marks]

Solutions:

1. (a) $\forall x(P(x) \to Q(x))$, (b) either $\exists x(R(x) \wedge \neg Q(x))$ or $\exists x \neg(R(x) \to Q(x))$ (c) either $\exists x(R(x) \wedge \neg P(x))$ or $\exists x \neg(R(x) \to P(x))$.

2. If $A - B$ is countable, then $(A - B) \cup B$ is also countable by using the fact that union of countable sets is countable. But $A \subseteq (A - B) \cup B$ is uncountable, which is a contradiction. Therefore, $A - B$ is uncountable.

3. If q divides c, then q divides bc. So, any common factor of a and c is also a common factor of a and bc. Same with b in place of c. So, we can apply contradiction to prove the forward direction: If a and bc have no common factors greater than 1, then certainly neither do a and b nor a and c.

   For the other part, applying Bezout's identity, fix integers x, y, z, w such that $ax + by = 1$ and $az + cw = 1$. Multiply these two equations, to obtain $a(axz + xcw + byz) + bc(yw) = 1$. Thus, $gcd(a, bc)$ must divide 1 because it divides a and bc. Therefore, $gcd(a, bc) = 1$.

   Notice that the last sentence or argument is required; only mentioning Bezout's identity is not enough which only says that if $gcd(a, b) = c$ the c can be written as $ax + by = c$ for some integers x and y.

4.

By definition there are positive constants C_1, C_1', C_2, C_2', k_1, k_1', k_2, and k_2' such that $f_1(x) \geq C_1|g(x)|$ for all $x > k_1$, $f_1(x) \leq C_1'|g(x)|$ for all $x > k_1'$, $f_2(x) \geq C_2|g(x)|$ for all $x > k_2$, and $f_2(x) \leq C_2'|g(x)|$ for all $x > k_2'$. We are able to omit the absolute value signs on the $f(x)$'s since we are told that they are positive; we are also told here that the $g(x)$'s are positive, but we do not need that. Adding the first and third inequalities we obtain $f_1(x) + f_2(x) \geq (C_1 + C_2)|g(x)|$ for all $x > \max(k_1, k_2)$; and similarly with the second and fourth inequalities we know $f_1(x) + f_2(x) \leq (C_1' + C_2')|g(x)|$ for all $x > \max(k_1', k_2')$. Thus $f_1(x) + f_2(x)$ meets the definition of being $\Theta(g(x))$.

If the f's can take on negative values, then this is no longer true. For example, let $f_1(x) = x^2 + x$, let $f_2(x) = -x^2 + x$, and let $g(x) = x^2$. Then each $f_i(x)$ is $\Theta(g(x))$, but the sum is $2x$, which is not $\Theta(g(x))$.

5. (a) Notice that weak induction may not work here.

   Proof. We apply strong induction on n. Take a tree on $n \geq 2$ vertices and delete an edge e. Then, we get two subtrees T_1, T_2 of order n_1, n_2, respectively, where $n_1 + n_2 = n$. So, $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$. By induction hypothesis $\|T\| = \|T_1\| + \|T_2\| + 1 = n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$. ∎

   (b) One can do it using induction.

   Proof. Let T be any tree on $n \geq 2$ vertices. Then $\sum_{v \in V(T)} d(v) = 2\|E(T)\| = 2(n-1) = 2n - 2$. By PHP, T has at least two vertices of degree 1. ∎

6. See Slide 27 of Class 22 lecture notes:
   Solution: Write each of the $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n+1$, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i. ◀

7. There is a bijection between set of cubic polynomials $ax^3 + bx^2 + cx + d$ to $\mathbb{Z}^4$ as $f(ax^3 + bx^2 + cx + d) = (a, b, c, d)$. Now, $\mathbb{Z}^4$ is countable by its enumeration by finding a sequence. So, the set of cubic polynomials is countable. Now a cubic polynomial can have at most 3 real roots.

Taking the sets of 1st, 2nd and 3rd roots as three sets A, B and C, these are countable. Thus, the set of real numbers that are solutions to the cubic equations $ax^3 + bx^2 + cx + d = 0$ will be union of these countable sets A, B and C. Thus it will be countable.

8. (a) Let G be a connected graph. If v is a cut vertex, then $G-v$ has at least two components, say, G_1 and G_2. Choose any $x \in V(G_1)$ and any $y \in V(G_2)$. Since G is connected, there must exists at least one x to y path in G. If any of the x to y paths in G does not contain v, then the same path will connect x and y in $G - v$, thereby implying that x and y lie in the same component of $G - v$. This contradiction shows that every x to y path in G must contain v.

   Conversely, suppose $x, y \in V(G)$ such that every x to y path contains v. Then the removal of v clearly disconnects x and y, so that $G - v$ is disconnected. Thus v is a cut vertex of G.

   (b) Let (u, v) be a cut edge with $u \in V(G_1)$ and $v \in V(G_2)$ where G_1 and G_2 are two connected components of graph G by removing (u, v).

   Now, if v is a cut vertex then $G_2 - v$ can not be empty. Thus, v has at least one more neighbour other than u.

   Conversely, suppose v has two or more neighbours including u as mentioned above. Then removing disconnects u from $G_2 - v$, which is nonempty. Thus, v is a cut vertex.

9. See Slides 23, 24, 25, and 26 of Class 8 Lecture notes:

   Proof: We prove this inequality using structural induction.

   BASIS STEP: For the full binary tree consisting of just the root r the result is true because $n(T) = 1$ and $h(T) = 0$, so that $n(T) = 1 \le 2^{0+1} - 1 = 1$.

   RECURSIVE STEP: For the inductive hypothesis we assume that $n(T_1) \le 2^{h(T_1)+1} - 1$ and $n(T_2) \le 2^{h(T_2)+1} - 1$ whenever T_1 and T_2 are full binary trees. By the recursive formulae for $n(T)$ and $h(T)$ we have $n(T) = 1 + n(T_1) + n(T_2)$ and $h(T) = 1 + \max(h(T_1), h(T_2))$.
   We find that

$$
\begin{aligned}
n(T) &= 1 + n(T_1) + n(T_2) && \text{by the recursive formula for } n(T) \\
&\le 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) && \text{by the inductive hypothesis} \\
&\le 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 && \text{because the sum of two terms is at most 2 times the larger} \\
&= 2 \cdot 2^{\max(h(T_1),h(T_2))+1} - 1 && \text{because } \max(2^x, 2^y) = 2^{\max(x,y)} \\
&= 2 \cdot 2^{h(T)} - 1 && \text{by the recursive definition of } h(T) \\
&= 2^{h(T)+1} - 1.
\end{aligned}
$$

   This completes the recursive step. ◁

10. We will argue for the truth of this statement using the Fundamental Theorem of Arithmetic. What we must show is that $m_1 m_2 \cdots m_n \mid a - b$. Look at the prime factorization of both sides of this proposition. Suppose that p is a prime appearing in the prime factorization of the left-hand side. Then $p \mid m_j$ for some j. Since the m_i's are relatively prime, p does not appear as a factor in any of the other m_i's. Now we know from the hypothesis that $m_j \mid a - b$. Therefore $a - b$ contains the factor p in its prime factorization, and p must appear to a power at least as large as the power to which it appears in m_j. But what we have just shown is that each prime power p^r in the prime factorization of the left-hand side also appears in the prime factorization of the right-hand side. Therefore the left-hand side does, indeed, divide the right-hand side.