Solutions #W 8

Problem 1: let $y = \sum_{m=0}^{\infty} a_m x^m$

$$y' = \sum_{m=1}^{\infty} a_m \, m \, x^{m-1} \quad , \quad y'' = \sum_{m=2}^{\infty} a_m \, m(m-1) \, x^{m-2}$$

Substituting the series in the ODE, we get

$$2a_2 + 6a_3 x - (a_1 + 2a_2 x) + \sum_{m=4}^{\infty} \big((m-1) m a_m - (m-1) a_{m-1} + a_{m-4} \big) x^{m-2} = 0$$

$$\Rightarrow \quad 2a_2 - a_1 = 0 \Rightarrow a_2 = \frac{a_1}{2}$$

$$6a_3 - 2a_2 = 0 \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3!}$$

when $m = 4$

$$12a_4 - 3a_3 + a_0 = 0 \Rightarrow a_4 = \frac{3a_3 - a_0}{12} = \frac{a_1}{4!} - \frac{a_0}{12}$$

when $m = 5$

$$20a_5 - 4a_4 + a_1 = 0 \Rightarrow a_5 = -a_0 \quad - a_1$$

60 4!

The general solution is then

$$y = a_0\left(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5 \ldots\ldots\right) + a_1\left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 \ldots\ldots\right)$$

Problem 2 : let $u = \frac{x}{a}$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{1}{a}$$

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{du}\frac{1}{a}\right)\frac{du}{dx} = \frac{d^2y}{du^2}\frac{1}{a^2}$$

Substituting into the differential Equation, we get

$$(a^2 - a^2u^2)\frac{d^2y}{du^2}\frac{1}{a^2} - 2(au)\frac{dy}{du}\frac{1}{a} + n(n+1)y = 0$$

$$(1-u^2)\frac{d^2y}{du^2} - 2u\frac{dy}{du} + n(n+1)y = 0$$

whose general solution is

$$y(u) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x) = c_1 y_1\left(\frac{x}{a}\right) + c_2 y_2\left(\frac{x}{a}\right)$$

Problem 3:

If $z = \sqrt{x}$, then

$$z' = \frac{1}{2\sqrt{x}} = \frac{1}{2} z^{-1}$$

$$y' = \frac{dy}{dz} z' = \frac{dy}{dz} \frac{1}{2} z^{-1}$$

$$y'' = \frac{dy'}{dz}\frac{dz}{dx} = \frac{1}{2}\left(-z^{-2}\frac{dy}{dz} + z^{-1}\frac{d^2y}{dz^2}\right)\frac{1}{2}z^{-1}$$

$$= \frac{1}{4}\left(z^{-2}\frac{d^2y}{dz^2} - z^{-3}\frac{dy}{dz}\right)$$

With these, we can rewrite the ODE as

$$\frac{z^2}{4}\left(z^{-2}\frac{d^2y}{dz^2} - z^{-3}\frac{dy}{dz}\right) + \frac{1}{2}z^{-1}\frac{dy}{dz} + \frac{1}{4}y = 0$$

$$\frac{1}{4}\frac{d^2y}{dz^2} - \frac{1}{4}z^{-1}\frac{dy}{dz} + \frac{1}{2}z^{-1}\frac{dy}{dz} + \frac{1}{4}y = 0$$

$$\frac{1}{4}\frac{d^2y}{dz^2} + \frac{1}{4}z^{-1}\frac{dy}{dz} + \frac{1}{4}y = 0$$

Multiplying by $4z^2$, we get

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + z^2y = 0$$

which is a Bessel's equation. It's general solution needs Bessel's functions of second kind.

General solution is

$$y = c_1 J_0(z) + c_2 Y_0(z)$$

$$y = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})$$

Problem 4 : If $z = \sqrt{x}$ then $z' = \frac{1}{2\sqrt{x}} = \frac{1}{2}z^{-1}$

$$y = u\sqrt{x} = uz$$

$$y' = \frac{dy}{dz}\frac{dz}{dx} = \left(\frac{du}{dz} z + u\right)\left(\frac{1}{2} z^{-1}\right)$$

$$y'' = \frac{dy'}{dz}\frac{dz}{dx}$$

$$= \frac{1}{4} z^{-1} \frac{d^2u}{dz^2} + \frac{1}{4} z^{-2} \frac{du}{dz} - \frac{1}{4} z^{-3} u$$

So we can rewrite the ODE as

$$z^4\left(\frac{1}{4} z^{-1} \frac{d^2u}{dz^2} + \frac{1}{4} z^{-2} \frac{du}{dz} - \frac{1}{4} z^{-3} u\right) + \frac{1}{4}\left(z^2 + \frac{3}{4}\right)uz = 0$$

$$\Rightarrow \frac{1}{4} z^3 \frac{d^2u}{dz^2} + \frac{1}{4} z^2 \frac{du}{dz} + \frac{1}{4} z^3 u - \frac{1}{16} uz = 0$$

Multiplying the whole equation by $4z^{-1}$, we get

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + z^2 u - \frac{1}{4} u = 0$$

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - \frac{1}{4}\right)u = 0$$

That is Bessel's equation.

It's general solution is

$$u = c_1 J_{1/2}(z) + c_2 J_{-1/2}(z) = c_1 J_{1/2}(\sqrt{x}) + c_2 J_{-1/2}(\sqrt{x})$$

$$y = uz = \left(c_1 J_{1/2}(\sqrt{x}) + c_2 J_{-1/2}(\sqrt{x})\right)\sqrt{x}$$

<u>Problem 5 :</u>

This is Bessel's equation. It's general solution is $y = c_1 J_4(x) + c_2 Y_4(x)$.