MTH 373/573:
Scientific Computing

# Solutions to Quiz 2

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## Problem 1: Block LU Decomposition    4 + 3.5 = 7.5 points

Suppose a square matrix $M \in \mathbb{R}^{n \times n}$ is written in block form as:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{R}^{k \times k}$, $k < n$ is square and invertible.

(a) Verify that we can decompose $M$ as the product:

$$M = \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & \mathbb{O} \\ \mathbb{O} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix}.$$

Here $I$ and $\mathbb{O}$ denote the identity and block zero matrices of appropriate sizes, respectively.

We can write many similar solutions but here I wish to highlight two. In the first, we can think of columns of the matrix product $AB$ as a combination of columns of the left matrix $A$ using the elements of each column of the right matrix $B$ as coefficients successively.

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} =$$

$$\left[ \begin{bmatrix} a_1 \end{bmatrix} b_{11} + \begin{bmatrix} a_2 \end{bmatrix} b_{12} + \begin{bmatrix} a_3 \end{bmatrix} b_{13} \quad \begin{bmatrix} a_1 \end{bmatrix} b_{21} + \begin{bmatrix} a_2 \end{bmatrix} b_{22} + \begin{bmatrix} a_3 \end{bmatrix} b_{23} \quad \begin{bmatrix} a_1 \end{bmatrix} b_{31} + \begin{bmatrix} a_2 \end{bmatrix} b_{32} + \begin{bmatrix} a_3 \end{bmatrix} b_{33} \right]$$

Alternatively, we can think of $AB$ as a combination of rows of $B$ using the elements of rows of $A$ as the linear combination's coefficients.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}\begin{bmatrix} b_1 \end{bmatrix} + a_{12}\begin{bmatrix} b_2 \end{bmatrix} + a_{12}\begin{bmatrix} b_3 \end{bmatrix} \\ a_{21}\begin{bmatrix} b_1 \end{bmatrix} + a_{22}\begin{bmatrix} b_2 \end{bmatrix} + a_{23}\begin{bmatrix} b_3 \end{bmatrix} \\ a_{31}\begin{bmatrix} b_1 \end{bmatrix} + a_{32}\begin{bmatrix} b_2 \end{bmatrix} + a_{33}\begin{bmatrix} b_3 \end{bmatrix} \end{bmatrix}$$

Thus, using that matrix matrix product is a combination of columns of the left matrix with the columns of the right one as coefficients, we have that:

$$
\begin{aligned}
M &= \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & \mathbb{O} \\ \mathbb{O} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A \\ \mathbb{O} \end{bmatrix} & \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} \mathbb{O} \\ D - CA^{-1}B \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} I \\ CA^{-1} \end{bmatrix} A + \begin{bmatrix} \mathbb{O} \\ I \end{bmatrix} \mathbb{O} & \begin{bmatrix} I \\ CA^{-1} \end{bmatrix} \mathbb{O} + \begin{bmatrix} \mathbb{O} \\ I \end{bmatrix} \left(D - CA^{-1}B\right) \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A \\ CA^{-1}A \end{bmatrix} + \begin{bmatrix} \mathbb{O}\mathbb{O} \\ I\mathbb{O} \end{bmatrix} & \begin{bmatrix} I\mathbb{O} \\ CA^{-1}\mathbb{O} \end{bmatrix} + \begin{bmatrix} \mathbb{O}\left(D - CA^{-1}B\right) \\ I\left(D - CA^{-1}B\right) \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \end{bmatrix} & \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ D - CA^{-1}B \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} A & \mathbb{O} \\ C & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A & \mathbb{O} \\ C & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I \\ \mathbb{O} \end{bmatrix} & \begin{bmatrix} A & \mathbb{O} \\ C & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} A^{-1}B \\ I \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} I + \begin{bmatrix} \mathbb{O} \\ D - CA^{-1}B \end{bmatrix} \mathbb{O} & \begin{bmatrix} A \\ C \end{bmatrix} \left(A^{-1}B\right) + \begin{bmatrix} \mathbb{O} \\ D - CA^{-1}B \end{bmatrix} I \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} AI \\ CI \end{bmatrix} + \begin{bmatrix} \mathbb{O}\mathbb{O} \\ \left(D - CA^{-1}B\right)\mathbb{O} \end{bmatrix} & \begin{bmatrix} A\left(A^{-1}B\right) \\ C\left(A^{-1}B\right) \end{bmatrix} + \begin{bmatrix} \mathbb{O}I \\ \left(D - CA^{-1}B\right)I \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \end{bmatrix} & \begin{bmatrix} B \\ CA^{-1}B \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ D - CA^{-1}B \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\end{aligned}
$$

We could also have verified this using the other interpretation of the matrix matrix product as the combination of rows of the right matrix using the rows of the left matrix as coefficients.

(b) Suppose that you are given the LU decompositions $A = L_1U_1$ and $D - CA^{-1}B = L_2U_2$, show how to construct an LU decomposition of $M$ given these additional matrices.

Using the LU decompositions for $A$ and $D - CA^{-1}B$ in $M$, we obtain:

$$
\begin{aligned}
M &= \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & \mathbb{O} \\ \mathbb{O} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} L_1U_1 & \mathbb{O} \\ \mathbb{O} & L_2U_2 \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} I & \mathbb{O} \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} L_1 & \mathbb{O} \\ \mathbb{O} & L_2 \end{bmatrix} \begin{bmatrix} U_1 & \mathbb{O} \\ \mathbb{O} & U_2 \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ \mathbb{O} & I \end{bmatrix} \\
&= \begin{bmatrix} IL_1 & \mathbb{O}L_2 \\ CA^{-1}L_1 & IL_2 \end{bmatrix} \begin{bmatrix} U_1 I & U_1 A^{-1}B \\ U_2\mathbb{O} & U_2 I \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} L_1 & \mathbb{O} \\ CA^{-1}L_1 & L_2 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_1 & U_1 A^{-1}B \\ \mathbb{O} & U_2 \end{bmatrix}}_{U}
\end{aligned}
$$

## Problem 2: Gaussian Elimination with Partial Pivoting 7.5 points

Solve the following linear system using Gaussian elimination with partial pivoting and show all intermediate matrices at each step.

$$
\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}.
$$

*Step 1:* The first column and its pivot are highlighted below, and therefore the first permutation and elementary elimination matrices are as shown:

$$
A_1 = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ \boxed{4} & -1 & 2 \end{bmatrix} \implies P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} \implies M_1 P_1 A_1 = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1/2 & 0 \\ 0 & -3/4 & 3/2 \end{bmatrix}.
$$

*Step 2:* The part of second column to be processed and its pivot are highlighted below, and therefore the second permutation and elementary elimination matrices are as shown:

$$
A_2 = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1/2 & 0 \\ 0 & \boxed{-3/4} & 3/2 \end{bmatrix} \implies P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}, \implies M_2 P_2 A_2 = \begin{bmatrix} 4 & -1 & 2 \\ 0 & -3/4 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}.
$$

*Step 3:* The matrix is already in upper triangular form, and there's nothing left to be done. Next, we can solve the linear system by operating on $b$ the same matrices that we used to transform $A$ into an upper triangular matrix, and then solving the upper triangular linear system. Note that the permutation matrices do not permute the variables $x_1, x_2, x_3$ because it's equations that are been rearranged by them.

$$
Ux = M_2 P_2 M_1 P_1 Ab
$$

$$\begin{bmatrix} 4 & -1 & 2 \\ 0 & -3/4 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1/2 \\ -1/4 \end{bmatrix}(-1) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}(-2)\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 3/2 \\ -7/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -7/4 \\ 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}(-1) + \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}(-7/4) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}(3/2)$$

$$\implies \begin{bmatrix} 4 & -1 & 2 \\ 0 & -3/4 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -7/4 \\ 1/3 \end{bmatrix} \implies x_3 = 1/3, x_2 = 3, x_1 = 1/3 \text{ from back substitution.}$$

Thus, the solution via Gaussian elimination with partial pivoting is $x = \begin{bmatrix} 1/3 \\ 3 \\ 1/3 \end{bmatrix}$.

## Problem 3: True or False with Justification — 7.5 points

For each of the following statements, state whether it is *True* or *False* and provide a short justification, proof or counterexample as appropriate.

(1) The product of two symmetric matrices is symmetric. (1.5 points)

**False.**
Let $A = A^T$ and $B = B^T$. Then $(AB)^T = B^TA^T = BA$. But, matrix product does not necessarily commute, that is $AB \neq BA$ in general. For example, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ while $BA = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

(2) A system of linear equations $Ax = b$ has a solution if, and only if, the $m \times n$ matrix $A$ and the augmented $m \times (n+1)$ matrix $\begin{bmatrix} A & b \end{bmatrix}$ have the same rank. (2 points)

**True.**
Consider the forward direction first ( $\implies$ ): Since $Ax = b$ has a linear system solution, thus $b \in \text{im}(A)$ and therefore $b$ is expressible as a linear combination of the columns of $A$ using $x$. Thus, rank of $A$ and the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ will be the same.
For the converse direction ( $\impliedby$ ): Given that rank of $A$ and rank of the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ are the same, this necessarily means that $b$ is not independent of the columns of $A$. Hence, $b \in \text{im}(A)$ and thus $Ax = b$ admits a linear system solution.

(3) If a matrix is singular then it cannot have an LU factorization. (1.5 points)

**False.**
Consider the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which has a LU decomposition given by $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ which can be easily verified by computing the product $LU$. Thus, nonsingular matrices can admit an LU decomposition.

(4) The norm of a singular matrix is zero. (2.5 points)

**False.**
Consider the singular matrix in the previous part $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Its 1- and $\infty$-norms are both 2.

## Problem 4: Properties of Matrix Norm — 1.5 + 2 + 4 = 7.5 points

Show that for an $m \times n$ matrix $A$, the norm defined by $\|A\|_{\max} := \max_{i,j} |a_{ij}|$ satisfies the three properties for being a matrix norm. (For quick reference, the three properties are: (a) $\|A\| > 0$ if $A \neq \mathbb{O}$, (b) $\|\gamma A\| = |\gamma|\,\|A\|$ for any scalar $\gamma$, and (c) $\|A + B\| \leq \|A\| + \|B\|$.)

(a) The first property can be verified easily since if $A$ is the zero matrix, all it entries are $0$, and $\max_{i,j} |a_{ij}| = \max_{i,j} |0| = 0$. Conversely, if $\max_{i,j} |a_{ij}| = 0$, then the largest entry of the matrix is $0$, thus all entries have to be $0$. For any nonzero matrix, the largest magnitude entry of a matrix will be greater than $0$.
(b) For the second part, scaling a matrix $A$ by $\gamma$ scales all the entries of the matrix, and hence the largest magnitude entry is also scaled by $|\gamma|$.
(c) For triangle inequality, $\|A + B\|_{\max} = \max_{i,j} \left(|a_{ij} + b_{ij}|\right) \leq \max_{i,j} \left(|a_{ij}| + |b_{ij}|\right)$ where we use the triangle inequality for any two real numbers $a_{ij}$ and $b_{ij}$. Finally, $\max_{i,j} \left(|a_{ij}| + |b_{ij}|\right) \leq \max_{i,j} |a_{ij}| + \max_{i,j} |b_{ij}| = \|A\|_{\max} + \|B\|_{\max}$.