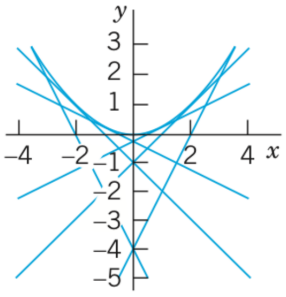
MTH-204: Worksheet 1

25 January, 2023

1. **Singular Solution.** An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution*. The ODE $y'^2 - xy' + y = 0$ is of this kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = \frac{x^2}{4}$. Explain the figure below (2)



2. **Half-life.** Radium ${}^{224}_{88}Ra$ has a half-life of about 3.6 days.
   (a) Given 1 gram, how much will still be present after 1 day? (1)
   (b) After 1 year? (1)

3. **Parachutist.** Two forces act on a parachutist, the attraction by the earth mg (m = mass of person plus equipment, $g = 9.8$ m/sec^2 the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using **Newton's second law** of motion (mass × acceleration = resultant of the forces), set up a model (an ODE for $v(t)$). Graph a direction field (choosing m and the constant of proportionality equal to 1). Assume that the parachute opens when $v = 10$ m/sec. Graph the corresponding solution in the field. What is the limiting velocity? Would the parachute still be sufficient if the air resistance were only proportional to $v(t)$? (2)

4. **Another population model.**
   (a) If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time. (1)
   (b) What is the limiting situation for increasing time? Interpret it. (1)

5. **Boyle-Mariotte's law for ideal gases.** Experiments show for a gas at low pressure p (and constant temperature) the rate of change of the volume $V(p)$ equals $-V/p$. Solve the model. (1)

6. **Gompertz growth in tumors.** The Gompertz model is (1)

$$y' = -Ay \, ln \, y \; (A > 0),$$

where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.