ECE 351 DSP: Final Examination

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Total: 40 points

Instruction: Answer all the questions 1–6. You can use the formulas from the list of formulas provided directly.

1) Consider a type I linear phase system with group delay 2. Let one of its zeros be $0.5j$, and let $H(1) = \frac{25}{4}$.
   a) What is $H(z)$?
   b) Draw the cascade form realization of this system.

[4+3=7 points]

2) Let $x(t)$ be a continuous time signal that is bandlimited to B. Suppose a receiver receives $x(t)$ along with two-delayed echos, i.e., the receiver receives $w(t) = x(t) + ax(t - \tau_1) + bx(t - \tau_2)$.
   a) What is the continuous-time Fourier transform $W(F)$ in terms of $X(F)$?
   b) The receiver wants to recover $x(t)$ from $w(t)$ using a discrete-time filter $H_D(f)$, and assume that the receiver has access to an ideal sampler, and ideal reconstruction is possible. Give a sampling frequency F_s and the corresponding discrete-time filter $H_D(f)$ that does the job.

[2+5=7 points]

3) Let $x[n]$ be an N-point sequence, and let $y[n]$ be a $3N$-point sequence defined by $y[n] = x[\frac{n}{3}]$ whenever n is divisible by 3, and $y[n] = 0$ otherwise. Express the $3N$-point DFT $Y(k)$ in terms of the N-point DFTs $X(0), X(1), \ldots, X(N-1)$.

[5 points]

4) Consider the sequence $x[n] = [3, -1, 2, 5, 1, -2]$. Let $X(\omega)$ denote the discrete-time Fourier transform of $x[n]$. Now, define $Y(i) = X(\frac{i\pi}{2}), 0 \leq i \leq 3$, and let the sequence $y[n]$ be obtained by taking the 4-point IDFT of $Y(0), Y(1), Y(2), Y(3)$. Find $y[n]$.

[5 points]

5) Consider an 2-bit uniform quantizer with quantization levels $\pm\frac{(2i+1)\Delta}{2}, 0 \leq i \leq 1$, and the quantization intervals being $I_i = [i\Delta, (i+1)\Delta], -2 \leq i \leq 1$. Consider an incoming signal $x[n]$ whose magnitude is distributed as follows. The

probability that $x[n]$ lies in any of the quantization intervals I_i is uniformly distributed. Next, conditioned on the fact that $x[n]$ lies in I_i, the distribution of $x[n]$ follows the pdf

$$f(x|x \in I_i) = \begin{cases} \frac{3}{2\Delta}, & \text{if } x \in [i\Delta, i\Delta + \frac{\Delta}{3}] \\ 0, & \text{if } x \in [i\Delta + \frac{\Delta}{3}, i\Delta + \frac{2\Delta}{3}] \\ \frac{3}{2\Delta}, & \text{if } x \in [i\Delta + \frac{2\Delta}{3}, (i+1)\Delta]. \end{cases}$$

a) Let e be the quantization error. Argue that $e = |x[n] - \frac{(2i+1)\Delta}{2}|$ when $x[n] \in I_i$.

b) Show that $\mathbb{E}[x[n]] = 0$.

c) Find $\mathbb{E}[x[n]^2]$.

d) Find $\mathbb{E}[e^2]$ and hence obtain the SQNR.

[1+4+4+3=12 points]

6) Consider an analog low pass filter $H_{LP}(s)$ with passband edge $\Omega_p = 1$ rad/s and stopband edge $\Omega_s = \sqrt{3}$ rad/s. Let $G(z) = H_{LP}(\text{Bi}_2(z))$, and let $H(z) = G(z^3)$. Find the various passband edges and stopband edges located in $[0, \pi]$ for the filter $H(z)$.

[4 points]

List of formulas

Discrete time Fourier series

1) If $x[n]$ is periodic with period N then,

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn},$$

where

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}.$$

Discrete time Fourier transform

1) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$.
2) Time delay: $x[n-k] \longleftrightarrow e^{-j\omega k}X(\omega)$.
3) Symmetry: $X(\omega) = X^*(-\omega)$ if $x[n]$ is real.
4) Frequency shift: $e^{j\omega_0 n}x[n] \longleftrightarrow X(\omega - \omega_0)$.
5) Modulation: $x[n]\cos(\omega_0 n) \longleftrightarrow \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$.
6) Differentiation: $nx[n] \longleftrightarrow j\frac{dX(\omega)}{d\omega}$.
7) DTFT of Sinc: $\frac{\omega_0}{\pi}\text{sinc}(n\omega_0) \longleftrightarrow \text{rect}(\frac{\omega}{2\omega_0})$.
8) System $H(\omega)$ with exponential input $e^{j\omega_0 n}$: Output $H(\omega_0)e^{j\omega_0 n}$.
9) System $H(\omega)$ with cosine input $\cos(\omega_0 n + \theta)$: Output $|H(\omega_0)|\cos(\omega_0 n + \theta + \angle H(\omega_0))$.

Z-transform properties

1) Time shift: $x[n-k] \longleftrightarrow z^{-k}X(z)$.
2) Z-scaling: $a^n x[n] \longleftrightarrow X(a^{-1}z)$.
3) Z-differentiation: $nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}$.
4) Initial value theorem: For causal $x[n]$, $x[0] = \lim_{z\to\infty} X(z)$.

Z-transform of common signals

1) $\delta[n] \longleftrightarrow 1$.
2) $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}$.
3) $na^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$.
4) $a^n e^{j\omega_0 n}u[n] \longleftrightarrow \frac{1}{1-ae^{j\omega_0}z^{-1}}$.
5) $a^n\cos(\omega_0 n)u[n] \longleftrightarrow \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$.
6) $a^n\sin(\omega_0 n)u[n] \longleftrightarrow \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$.

Formulas for partial fraction expansion coefficient for rational Z-transform

1) Coefficient for single-pole at p is $(z-p)\frac{X(z)}{z}|_{z=p}$.
2) Coefficient for the term $\frac{1}{(z-p)^i}$, $i<k$, for a pole of multiplicity k at p is $\frac{d^{k-i}}{dz^{k-i}}\left(\frac{(z-p)^k}{(k-i)!}\frac{X(z)}{z}\right)|_{z=p}$.
3) Coefficient for the term $\frac{1}{(z-p)^k}$ for a pole of multiplicity k at p is $(z-p)^k\frac{X(z)}{z}|_{z=p}$.
4) $\frac{n(n-1)\dots(n-i+2)}{(i-1)!}p^{n-i+1}u[n-i+2] \longleftrightarrow \frac{z}{(z-p)^i}$.

Formulas for filters

1) **FIR Low Pass Filter**: $H(z)=\frac{1}{2^M}(1+z^{-1})^M$.

   Cutoff frequency $2\cos^{-1}(2^{-\frac{1}{2M}})$.
2) **FIR High Pass Filter**: $H(z)=\frac{1}{2^M}(1-z^{-1})^M$.

   Cutoff frequency $2\sin^{-1}(2^{-\frac{1}{2M}})$.
3) **IIR Low Pass Filter**: $H(z)=\frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha|< 1$.

   Cutoff frequency $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.
4) **IIR High Pass Filter**: $H(z)=\frac{1-\alpha}{2}\frac{1-z^{-1}}{1+\alpha z^{-1}}, |\alpha|< 1$.

   Cutoff frequency $\pi-\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.
5) **IIR Bandpass Filter**: $H(z)=\frac{1-\alpha}{2}\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, |\alpha|< 1, |\beta|< 1$.

   3-dB bandwidth $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

   Centre frequency $\cos^{-1}\beta$.
6) **IIR Notch Filter**: $H(z)=\frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, |\alpha|< 1, |\beta|< 1$.

   3-dB bandwidth $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

   Notch frequency $\cos^{-1}\beta$.
7) **Linear phase systems** with $h[n]=0$, if $n<0$, and $n\geq N$.

   Phase response: $-\omega(\frac{N-1}{2})+\pi\mathbf{1}\{H(\omega)<0\}$.

   Group delay: $\frac{N-1}{2}$.

Formulas for continuous-time Fourier transform

1) $X(F)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi Ft}dt$, $x(t)=\int_{-\infty}^{\infty}X(F)e^{j2\pi Ft}dF$.
2) Time shift: $x(t-\tau)\longleftrightarrow X(F)e^{-j2\pi F\tau}$.
3) Duality: $x(t)\longleftrightarrow X(F) \iff X(t)\longleftrightarrow x(-F)$.
4) Frequency shift: $x(t)e^{j2\pi F_0 t}\longleftrightarrow X(F-F_0)$.
5) Time scaling: $x(at)\longleftrightarrow \frac{1}{|a|}X(\frac{F}{a})$.

Sampling

1) Relation between DTFT frequency and CTFT frequency: $fF_s=F$, where F_s is the sampling frequency.
2) Relation between DTFT of sampled signal $x[n]$ and CTFT of the original signal $x_a(t)$: $X(f)=F_s\sum_{k=-\infty}^{\infty}X_a(F-kF_s)$, where F_s is the sampling frequency.

3) Perfect reconstruction (with $F_s \geq 2B$): $x_a(t) = \sum_{n=-\infty}^{\infty} x[n]\text{Sinc}(\pi F_s(t - nT_s))$, where $T_s = \frac{1}{F_s}$.
4) If $x_a(t)$ is bandlimited to B and $H(F)$ is an analog filter, then the output of $H(F)$ with input $x_a(t)$ can be achieved by first sampling $x_a(t)$ with $F_s = 2B$, then applying the discrete-time filter $H_D(f) = \frac{1}{F_s}H(fF_s), f \in [-\frac{1}{2}, \frac{1}{2}]$, and then applying perfect reconstruction.
5) Bandpass Sampling: Bandpass signal from F_L to F_H needs sampling frequency F_s satisfying $\frac{2F_H}{l+1} \leq F_s \leq \frac{2F_L}{l}, l = 1, 2, \ldots, l_{\max}$, where $l_{\max} = \lfloor \frac{F_L}{B} \rfloor$, where $B = F_H - F_L$.

Quantization

1) Uniform quantizer with quantization intervals of size Δ, and L levels, have FSR $= L\Delta$.
2) Signal to quantization noise ratio SQNR$= \frac{2\sqrt{3}\sigma_x 2^b}{\text{FSR}}$, for a b-bit quantizer where $\sigma_x = \mathbb{E}[x[n]^2]$, UNDER THE ASSUMPTION that quantization error is distributed as unif$[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

Discrete Fourier Transform

1) Let $Y(k) = X(\frac{2\pi k}{N}), k = 0, 1, \ldots, N-1$, where $X(\omega)$ is the DTFT of $x[n]$. Then, $Y(0), Y(1), \ldots, Y(N)$, are the N-point DFT of $x_p[0], x_p[1], \ldots, x_p[N-1]$, where $x_p[n] = \sum_{l=-\infty}^{\infty} x[n]$.
2) Circular convolution: $\sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N] \longleftrightarrow X_1(k)X_2(k)$.
3) Time reversal: $x[(-n)_N] \longleftrightarrow X(-(k)_N)$.
4) Circular shift: $x[(n-l)_N] \longleftrightarrow X(k)e^{-j\frac{2\pi}{N}kl}$.
5) Circular frequency shift: $x[n]e^{j\frac{2\pi}{N}ln} \longleftrightarrow X((k-l)_N)$.
6) Complex conjugate: $x^*[n] \longleftrightarrow X^*((-k)_N)$.
7) Time multiplication: $x_1[n]x_2[n] \longleftrightarrow \frac{1}{N}\sum_{l=0}^{N-1} X_1(l)X_2((k-l)_N)$.

Time Truncation

1) If signal is truncated to L samples, the main lobe of an angular frequency component at ω_0 has a width of $\frac{4\pi}{L}$, i.e., the main lobe is spread $\frac{2\pi}{L}$ on both sides of ω_0.

IIR Filter Design

1) Butterworth filter: $|H_B(\Omega)|^2 = \frac{1}{1+\epsilon^2(\frac{\Omega}{\Omega_p})^{2N}} = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}}$.
   - $\epsilon = \sqrt{\frac{1}{(1-\delta_1)^2} - 1}$.
   - $N = \frac{\log(\frac{\sqrt{1-\delta_2^2}}{\delta_2 \epsilon})}{\log(\frac{\Omega_s}{\Omega_p})}$.
   - $\Omega_c = \Omega_p \epsilon^{-\frac{1}{N}}$.
2) Chebyshev Polynomial:

$$T_N(x) = \begin{cases} \cos(N\cos^{-1} x), & |x| \leq 1 \\ \cosh(N\cosh^{-1} x), & |x| > 1. \end{cases}$$

3) Chebyshev Filters: Type I - $|H_{C1}(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\frac{\Omega}{\Omega_s})}$.

   Type II - $|H_{C2}(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[\frac{T_N^2(\frac{\Omega_s}{\Omega_p})}{T_N^2(\frac{\Omega_s}{\Omega})}\right]}$.

   - $\epsilon = \sqrt{\frac{1}{(1-\delta_1)^2} - 1}$.
   - $N = \frac{\log\left(\frac{\sqrt{1-\delta_2^2}+\sqrt{1-\delta_2^2(1+\epsilon^2)}}{\delta_2 \epsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}+\sqrt{\frac{\Omega_s^2}{\Omega_p^2}-1}\right)}$.

4) Frequency Transformations:

   a) Low Pass (Ω_p) to Low Pass ($\Omega_{p'}$): $s \to \frac{\Omega_p s}{\Omega_{p'}}$.

   b) Low Pass (Ω_p) to High Pass ($\Omega_{p'}$): $s \to \frac{\Omega_p \Omega_{p'}}{s}$.

   c) Low Pass (Ω_p) to Band Pass ($\Omega_{p_u}, \Omega_{p_l}$): $s \to \frac{\Omega_p(s^2+\Omega_{p_u}\Omega_{p_l})}{s(\Omega_{p_u}-\Omega_{p_l})}$.

   d) Low Pass (Ω_p) to Band Stop ($\Omega_{p_u}, \Omega_{p_l}$): $s \to \frac{\Omega_p s(\Omega_{p_u}-\Omega_{p_l})}{s^2+\Omega_{p_u}\Omega_{p_l}}$.

5) Bilinear Transform: $\mathrm{Bi}_T(z) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$.

   Relation between ω and Ω under Bilinear transform: $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.