MTH 377/577 CONVEX OPTIMIZATION
Winter Semester 2022
Indraprastha Institute of Information Technology Delhi
Problem Set 2: Convex Functions

Q1. (Exercise 3.1 in [BV]). (a) Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is convex and $a, b \in \mathrm{dom}\ f$ with $a < b$. Show that for all $x \in [a, b]$

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(a)$$

(b) Show that for all $x \in (a, b)$

$$\frac{f(x) - f(a)}{x-a} \leq \frac{f(b) - f(a)}{b-a} \leq \frac{f(b) - f(x)}{b-x}$$

Q2. (Exercise 3.3 in [BV]). Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity and concavity of g ?

Q2. (Exercise 3.6 in [BV]). When is the epigraph of a function a halfspace? When is the epigraph of a function a polyhedron?

Q3. (Exercise 3.15 in [BV]). For $\alpha \in (0, 1]$, let $u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$ with dom $u_\alpha = \mathbb{R}_+$. We also define $u_0(x) = \log(x)$ with dom $u_0 = \mathbb{R}_{++}$. This family of functions is used to model risk-averse preferences in Economics.

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$
(b) Show that u_α are concave, monotone increasing and all satisfy $u_\alpha(1) = 0$.

Q4. (Exercise 3.16 in [BV]). For each of the following functions determine whether it is convex or concave
(a) $f(x) = e^x - 1$ on $\mathbb{R}$
(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$
(c) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$
(d) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ on $\mathbb{R}^2_{++}$ where $\alpha \in [0, 1]$. This is often called the Cobb-Douglas

function in Economics used to model both consumer preferences and production technology.

Q5. (Exercise 3.17 in [BV]). Suppose $p < 1, p \neq 0$. Show that the function $f(x_1, x_2) = (x_1^p + x_2^p)^{1/p}$ with dom $f = \mathbb{R}^2_{++}$ is concave. This is called the constant elasticity of substitution (CES) utility function in Economics used to model consumer preferences and production technology.