MTH 372: Mid Semester Exam

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Instructions

- Show all your work to score full marks. Incomplete explanations will lead to deduction of marks.
- This is a closed book exam.
- No phones or other electronic devices may be used.
- If required, you can use the following information

  1. The density of $T = X_{(1)}$ is given by

     $$f_T(t) = nf_X(t)[1 - F_X(t)]^{n-1}.$$

  2. The density of $T = X_{(n)}$ is given by

     $$f_T(t) = nf_X(t)[F_X(t)]^{n-1}.$$

  3. The density of $X \sim$Beta (α, β) distribution is given by

     $$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}.$$

Questions

1. (2 points) Let $X_1 \ldots, X_n$ be i.i.d. from the geometric distribution given by

$$f_\theta(x) = \theta(1-\theta)^{x-1},\ x = 1, 2, \ldots$$

where $0 < \theta < 1$. Let $T = \sum_{i=1}^{n} X_i$ be the sample total.

Using definition of sufficiency, verify if T is a sufficient statistic for θ.

2. Let $X_1, \ldots, X_n$ be i.i.d. with the following pdf

$$f_\theta(x_i) = \frac{2\ x_i}{\theta^2},\ \ 0 < x_i < \theta.$$

Answer the following questions

(a) (2 points) Apart from the data itself, find sufficient statistic(s) for θ.

(b) (2 points) Find minimal sufficient statistic(s) for θ.

(c) (3 points) Is $T(X) = X_{(n)}$ a complete statistic?

3. Let $Y_1, \ldots, Y_n$ be independent random variables from Normal $(\beta x_i, \sigma^2)$. Here, $\boldsymbol{\theta} = (\beta, \sigma^2)$ are unknown and x_i's are known. Answer the following questions

(a) (2 points) Find method of moments estimator (MME) for β and σ^2.

(b) (4 points) Find maximum likelihood estimator (MLE) for β and σ^2.

(c) (2 points) Find an unbiased estimator of β.

(d) (2 points) Find an unbiased estimator of σ^2.

(e) (1 point) Does it belong to an exponential family. Explain.

(f) (1 point) Does it belong to a location-scale family. Explain.

4. Let $X_1, \ldots, X_n$ be i.i.d. from Uniform $(0, \theta)$, $\theta > 0$.

(a) (1 point) Let $R = X_{(n)} - X_{(1)}$ and if $R \sim$ Beta $(n-1, 2)$. Is R an ancillary statistic. Explain.

(b) (2 points) Find an unbiased estimator of θ.

(c) (2 points) Find MSE of $T = X_{(n)}$.

(d) (4 points) Find a uniformly minimum variance unbiased estimator (UMVUE) of θ.