Homework 2

1. For an integer $k \geq 1$, if $c_k = \dfrac{p_k}{q_k}$ is the kth convergent of the simple continued fraction $< a_0, a_1, a_2, \cdots, a_n >$ and $a_0 > 0$, then show that

   (a)
   $$\frac{p_k}{p_{k-1}} =< a_k, a_{k-1}, \cdots, a_1, a_0 >,$$

   (b)
   $$\frac{q_k}{q_{k-1}} =< a_k, a_{k-1}, \cdots, a_2, a_1 > .$$

2. Write the infinite continued fraction expansion of

   (a) $\sqrt{2}$,

   (b) $\frac{1}{\sqrt{3}}$.

3. Prove that the positive integer n has as many representations as the sum of two squares as does the integer 2n.

   (Hint: Staring with a representation of n as a sum of two squares obtain a similar representation for 2n, and conversely.)

4. Prove that of any four consecutive integers, at least one is not representable as a sum of two squares.

5. (a) Find the least positive solution of $x^2 - 18y^2 = -1$ (if any) and $x^2 - 18y^2 = 1$. Given $\sqrt{18} =< 4, \dot{4}, \dot{8} >$

   (b) Find the least positive solution of $x^2 - 73y^2 = -1$ (if any) and $x^2 - 73y^2 = 1$. Given $\sqrt{73} =< 8, \dot{1}, \dot{1}, \dot{5}, \dot{5}, \dot{1}, \dot{1}, 1\dot{6} >$.

   (Hint: Assume that $x^2 - dy^2 = -1$ is solvable and (x_1, y_1) is its least positive solution, if the least positive solution (x_2, y_2) of $x^2 - dy^2 = 1$, then $x_2 + y_2\sqrt{d} = (x_1 + y_1\sqrt{d})^2$.)

6. Prove that of any two consecutive convergent of a irrational number ξ, at least one, a/b, satisfies the inequality
   $$\left| x - \frac{a}{b} \right| < \frac{1}{2b^2}.$$

7. The Pell numbers p_n and q_n are defined by

   $p_0 = 0$, $p_1 = 1$, $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$, and

   $q_0 = 1$, $q_1 = 1$, $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 2$.

   This gives two sequences

   $$0, 1, 2, 5, 12, 29, 70, \cdots$$

   $$1, 1, 3, 7, 17, 41, 99, \cdots$$

   If $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Show that the Pell numbers can be expressed as

   $$p_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \text{and } q_n = \frac{\alpha^n + \beta^n}{2}$$

   for $n \geq 0$.

8. For the Pell numbers, derive the relation below, where $n \geq 1$:

   (a) $p_{2n} = 2p_n q_n$.

   (b) $p_n + p_{n-1} = q_n$.

   (c) $2q_n^2 - q_{2n} = (-1)^n$.

   (d) $p_n + p_{n+1} + p_{n+3} = 3p_{n+2}$.

   (e) $q_n^2 - 2p_n^2 = (-1)^n$; hence q_n/p_n are the convergents of $\sqrt{2}$.