SOLUTION

MTH210 – SUBMISSION_20221110

TIME: 15 minutes MARKS: 5

No consultation – open notes – **books and internet not allowed.** Marks will depend on the correctness and completeness of your answer. Any previous result used should be clearly referenced.

For $n \in \mathbb{Z}^+$, $n \geq 1$, prove the Binomial Theorem:

$$(1 + x)^n = \sum_{k=0 \text{ to } n} B(n,k)\, x^k$$

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We will show that the formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad (1)$$

holds for all $n \in \mathbb{Z}^+$ by PMI, making use of Pascal's identity:-

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{for } n \geq k \geq 1 \qquad (2)$$

Let $X = \{n \in \mathbb{Z}^+ : (1) \text{ holds for } n\}$.

Base Case: $n=1$. Then, the RHS of (1)

$= \binom{1}{0}x^0 + \binom{1}{1}x^1 = 1 + x = (1+x)^1$

$=$ LHS of (1). So $1 \in X$.

Induction Step: Suppose that 1 holds for some +ve integer $n-1$, $n-1 \geq 1$. *

(IH)

Consider: $(1+x)^n = (1+x)(1+x)^{n-1}$

$= (1+x)\left(1 + \cancel{\binom{n}{1}x} + \binom{n-1}{1}x + \binom{n-1}{2}x^2 + \cdots + \binom{n-1}{n-1}x^{n-1}\right)$, applying the IH. 3

Collecting the coefficients in 3, we get

$$(1+x)^n = 1 + \left[1 + \binom{n-1}{1}\right]x + \left[\binom{n-1}{2} + \binom{n-1}{1}\right]x^2$$

$$+ \left[\ \cdots\ \right]x^3 + \cdots + \left[\binom{n-1}{n-1} + \binom{n-1}{n-2}\right]x^{n-1}$$

$$+ \binom{n-1}{n-1}x^n \quad 4$$

~~Note~~ We apply 2 to each of the coefficients in 4, noting that the coefficient of x is $\binom{n-1}{0} + \binom{n-1}{1} = \binom{n}{1}$ and the coefficient of x^n is $\binom{n-1}{n-1} = 1 = \binom{n}{n}$, to get

$$= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \quad 5,$$

i.e. RHS of 1, as required.

$\therefore X = \mathbb{Z}^+$ by PMI.

(*) Using $n-1$ is slightly more convenient; we could use n and show that 1 holds for $n+1$ in a similar way.