MTH 204 Quiz 4

(Time : 15 mins, Maximum Marks : 10)

April 5, 2023

Question 1.

[5 points] Linearize the following system of nonlinear ODEs around the critical point $(3, 0)$.

$$\frac{dA}{dt} = 3A - A^2 - AB,$$
$$\frac{dB}{dt} = 6B - AB - 2B^2.$$

(1 mark)

Critical point $\rightarrow (3,0)$

Change of variables $\tilde{A} = A - 3$ and $\tilde{B} = B - 0$

$$\frac{d\tilde{A}}{dt} = 3(\tilde{A}+3) - (\tilde{A}+3)^2 - (\tilde{A}+3)\tilde{B} = f_1$$

$$\frac{d\tilde{B}}{dt} = 6\tilde{B} - (\tilde{A}+3)\tilde{B} - 2\tilde{B}^2 = f_2$$

(2 marks)

Jacobian at $(0,0)$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial \tilde{A}} & \frac{\partial f_1}{\partial \tilde{B}} \\ \frac{\partial f_2}{\partial \tilde{A}} & \frac{\partial f_2}{\partial \tilde{B}} \end{pmatrix}_{(0,0)} = \begin{bmatrix} 3 - 2(\tilde{A}+3) - \tilde{B} & -(\tilde{A}+3) \\ -\tilde{B} & 6 - (\tilde{A}+3) - 4\tilde{B} \end{bmatrix}_{(0,0)} = \begin{bmatrix} -3 & -3 \\ 0 & 3 \end{bmatrix}$$

(2 marks)

linearization of the system is

$$y' = \begin{bmatrix} \tilde{A}' \\ \tilde{B}' \end{bmatrix} = J\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$$

$$\frac{d\tilde{A}}{dt} = -3\tilde{A} - 3\tilde{B}$$

$$\frac{d\tilde{B}}{dt} = 3\tilde{B}$$

Question 2.

[5 points] Solve the following nonhomogeneous system of ODEs

$$\frac{dx}{dt} = x + 2y + 3t,$$
$$\frac{dy}{dt} = 2x + y + 2.$$

You can choose any method you like.

First consider the homogeneous system

$$\frac{dx}{dt} = x + 2y \quad \& \quad \frac{dy}{dt} = 2x + y$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(1 mark)

The characteristic eq^n is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

eigen values are 3 and -1

eigen vector corresponding to $\lambda = 3$

$$(A - 3I)u = 0 \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = u_2 \Rightarrow u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2 marks)

For $\lambda = -1$

$$(A + I)v = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Y_h(t) = c_1 u e^{3t} + c_2 v e^{-t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

For particular Solution

$$Y'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = AY + gt + h \quad \text{where } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \; g = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \;\&\; h = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$Y_p(t) = at + b \quad \text{where } a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$Y_p'(t) = a$$

$$Y_p'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = AY + gt + h$$

$$\Rightarrow \quad a = A(at+b) + gt + h$$

$$\Rightarrow \quad Aa + g = 0 \quad \text{and} \quad Ab + h = a$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a_1 + 2a_2 = -3$$
$$2a_1 + a_2 = 0$$

$$\Rightarrow \quad a_1 = 1 \;\&\; a_2 = -2$$

(2 marks) $$a = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Ab = a - h$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad b_1 + 2b_2 = 1$$
$$2b_1 + b_2 = -4$$

$$\Rightarrow \quad b_1 = -3 \;\&\; b_2 = 2$$

$$b = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad Y_p(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Hence, Solution for non homogeneous system of ODE's is

$$Y(t) = Y_h(t) + Y_p(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$