MTH100 Monsoon Semester – 2021-22 Section B (I)

Submission for Tuesday 29th March 2022 – 15 minutes. Max Marks: 5

Instructions: *Open notes and textbook; consultation and use of calculators, computers and internet not allowed. You may use any* **known** *result. This includes all propositions and observations in the lecture slides, and results from tutorials. Obviously, this does not apply if you are asked to prove a known result itself, as below; you may use all prior results. If you use any other result from any other source, including the textbook, you have to give a full proof of that result.*

Q. Let V be a vector space over the field F, and consider the space L(V,V) of linear operators on V. Show that multiplication of linear operators (i.e. composition) satisfies the following property:

$U(T_1 + T_2) = UT_1 + UT_2$ for all $U, T_1, T_2 \in L(V,V)$.

ANSWER: Recall that functions $f, g : X \to Y$ are equal if and only if

$f(x) = g(x)$ for all $x \in X$ (1)

We apply this idea here.

So let $\bar{v} \in V$.

Then: $U(T_1 + T_2)(\bar{v})$

$= U[(T_1 + T_2)\bar{v}]$ – definition of composition

$= U[T_1(\bar{v}) + T_2(\bar{v})]$ → definition of addition

$= U(T_1(\bar{v})) + U(T_2(\bar{v}))$ – since U is linear (*)

(PTO)

Cont'd

2

$= (UT_1)\bar{v} + (UT_2)\bar{v}$ – defn. of composition

$= (UT_1 + UT_2)\bar{v}$ – definition of addition

2

Since 2 holds for all $\bar{v} \in V$,

$U(T_1 + T_2) = UT_1 + UT_2$

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5 marks for a correct proof – the steps need to come in the right order. Additional Remarks :–

a. The idea of 1 is essential to the proof. Algebraic manipulations not including an element of V are not enough. 0 marks for any such answer.

b. The step shown with (*) is key, because it is the only place where linearity is used. If this is not explicitly mentioned, DEDUCT 1 mark

c. Other steps are simpler, reason need not be stated.