ECE 634/CSE 646 InT: Assignment 2

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Total: 10 points

1) Consider the discrete memoryless channel where input and output alphabets are binary, and the channel transition probability is given as $\mathrm{P}_{Y|X}(0|0) = 1, \mathrm{P}_{Y|X}(0|1) = \epsilon$. Show that the capacity of this channel is given by $h(\frac{1}{1+2^{f(\epsilon)}}) - \frac{f(\epsilon)}{1+2^{f(\epsilon)}}$, where $h(\cdot)$ is the binary entropy function and $f(\epsilon) = \frac{h(\epsilon)}{1-\epsilon}$.

   [**Hint:** First, assume $X \sim \mathrm{Be}(p)$, and expand $I(X;Y) = H(Y) - H(Y|X)$. Next, compute them and use basic calculus.]

   [5 points]

2) Prove that for any jointly distributed random variables X^n, Y^n the following holds:

$$\sum_{i=1}^{n} I(X_{i+1}^n; Y_i | Y^{i-1}) = \sum_{i=1}^{n} I(Y^{i-1}; X_i | X_{i+1}^n),$$

   where $X_i^j \triangleq (X_i, X_{i+1}, \dots, X_j)$, and $Y^0 = X_{n+1}^n = \text{constant}$.

   [**Hint:** One way of solving this will involve you proving $H(X^n) = \sum_{i=1}^n [H(X_i^n | Y^{i-1}) - H(X_{i+1}^n | Y^i)]$.]

   [3 points]

3) Let $(X_i, Y_i) \sim$ i.i.d. P_{XY}, and let $N \sim \text{unif}\{[n]\}$ such that $N \perp (X^n, Y^n)$. Show that $I(X_N; Y_N | N) = I(X; Y)$, where $I(X;Y)$ is the mutual information with respect to the joint distribution P_{XY}.

   [2 points]