University Carlos III of Madrid **Departament of Economics**

Game Theory: Problem set 4: Repeated Games. Solutions.

Problem 1: Consider the following normal form game.

Player 2

		A	B	C
	A	$4,4$	$1,5$	$0,3$
Player 1	B	$7,1$	$3,4$	$0,1$
	C	$3,0$	$2,0$	$1,1$

The stage game G

1. Find the Nash equilibria in pure strategies.

   Solution: *There are two NE in pure strategies: (B,B) with payoffs $u_1 = 3$, $u_2 = 4$ and (C,C) with payoffs $u_1 = 1$, $u_2 = 1$.*

2. Assume that the above stage game is played two times. After the first round, players observed the moves done by the other player. The total payoffs of the repeated game are the sum of the payoffs obtained in each round. Is there a subgame perfect Nash equilibrium in pure strategies in which (A,A) is played in the first round?

   Solution: *Let us consider the following strategy profile S: player $i = 1,2$ at*

   - *$t = 1$ plays A;*
   - *$t = 2$ plays B if (A,A) was played at $t = 1$ and C otherwise.*

   Is S a SPNE of the repeated game? At $t = 2$ the strategy profile S suggests the players to play a NE. Hence, no player has an incentive to deviate at $t = 2$. Anticipating the payoffs at $t = 2$ the players anticipate the following payoffs at $t = 1$.

   Player 2

		A	B	C
	A	$7,8$	$2,6$	$1,4$
Player 1	B	$8,2$	$4,5$	$1,2$
	C	$4,1$	$3,1$	$2,2$

   and we see that playing A is not a best reply for player 1, when player 2 chooses A. Hence playing (A,A) at $t = 1$ is not part of a SPNE of the repeated game.

3. Assume that the above stage game is played three times. After the first and second round, players observed the moves done by the other player. The total payoffs of the repeated game are the sum of the payoffs obtained in each round. Is there a subgame perfect Nash equilibrium in pure strategies in which (A,A) is played in the first round?

   Solution: *Let us consider the following strategy profile S: player $i = 1,2$ at*

   - *$t = 1$ play (A,A);*
   - *$t = 2$ play (B,B) if (A,A) was played at $t = 1$. Otherwise, play C;*
   - *$t = 3$, play (B,B) if (A,A) was played at $t = 1$ and (B,B) was played at $t = 2$. Otherwise, play (C,C).*

   Let us check that the above strategy profile S is a SPNE. In period $t = 3$, it dictates players to play a NE of the stage game. Thus S induces a NE at all subgames that start at $t = 3$. Let us check that it is also a NE of the subgame that starts at $t = 2$.

   Let us first consider a subgame that starts at $t = 2$ after (A,A) was played in period $t = 1$. Ignoring the payoffs at $t = 1$ and anticipating the strategy S at $t = 3$, yields the following payoffs at $t = 2$.

		Player 2		
		A	B	C
Player 1	A	$5,5$	$2,6$	$1,4$
	B	$8,2$	$6,8$	$1,2$
	C	$4,1$	$3,1$	$2,2$

and (B,B) is a NE of this subgame.

Let us consider now a subgame that starts at $t=2$ after (A,A) was not played in period $t=1$. Ignoring the payoffs at $t=1$ and anticipating the strategy S at $t=3$, yields the following payoffs at $t=2$.

		Player 2		
		A	B	C
Player 1	A	$5,5$	$2,6$	$1,4$
	B	$8,2$	$4,5$	$1,2$
	C	$4,1$	$3,1$	$2,2$

The stage game G

and (C,C) is a NE of this subgame.

Let us check that it is also a NE of the subgame that starts at $t=1$. Anticipating the strategy S at $t=2,3$ yields the following payoffs at $t=1$.

		Player 2		
		A	B	C
Player 1	A	$10,12$	$3,7$	$2,5$
	B	$9,3$	$5,6$	$2,3$
	C	$5,2$	$4,2$	$3,3$

The stage game G

And (A,A) is a NE of this subgame. Hence, no player has incentives to deviate and S is a SPNE.

Problem 2: Consider the following normal form game.

		Player 2	
		A	B
Player 1	A	$2,2$	$-2,6$
	B	$6,-2$	$0,0$

The stage game G

1. Find the all the Nash equilibria of the game G.

   Solution: *There is a unique NE (B,B) with payofss $u_1 = u_2 = 0$. It is in dominant strategies.*

2. Assume that the above stage game is played infinitely many times. After each round, players observe the moves done by the other player. The total payoffs of the repeated game are the discounted (with discount factor δ) sums of the payoffs obtained in each round. For what values of the discount factor δ is there a subgame perfect Nash equilibrium in pure strategies in which (A,A) is played in every round?

   Solution: *Let us consider trigger strategies: Player $i=1,2$ at*

   - $t=1$ *plays A;*
   - $t>1$ *plays A if (A,A) was played at $t=1,\ldots,t-1$. Otherwise, play B.*

Let us check that it is a SPNE of the repeated game. We first check that it is a NE of the repeated game. The payoffs obtained by both players with the trigger strategy are,

$$u^c = 2 + 2\delta + \cdots + 2\delta^t + 2\delta^{t+1} + 2\delta^{t+2} + 2\delta^{t+3} + \cdots$$

If one player deviates at stage t and the other player follows the trigger strategy, the payoff of the player which deviates is

$$u^d = 2 + 2\delta + \cdots + 2\delta^t + 6\delta^{t+1} + 0\delta^{t+2} + 0\delta^{t+3} + \cdots$$

The trigger strategy is NE iff $u^c \geq u^d$. That is,

$$2\delta^{t+1} + 2\delta^{t+2} + 2\delta^{t+3} + \cdots \geq 6\delta^{t+1}$$

That is

$$2(1 + 2\delta + 2\delta^2 + \cdots \geq 6$$

which is the same as

$$\frac{2}{1-\delta} \geq 6$$

Thus, the trigger strategy is a NE of the repeated game iff

$$\delta \geq \frac{2}{3}$$

Now, the standard argument shows that it is also a NE in every subgame: There are two types of subgames starting at a stage t.

- *Subgames in which at every stage $1, 2, \ldots, t-1$ it was played (A, A). Then, the situation in the subgame that starts at this node is exactly as above, except that the payoffs are multiplied by δ^{t-1}. The above argument shows that the trigger strategy is also a NE of that subgame.*
- *Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile (A, A) was not played. In these subgames the trigger strategy prescribes that (C, C), a NE of the stage game G, is played. But, this is a SPNE of this subgame.*

Therefore, if $\delta \geq \frac{2}{3}$, the trigger strategy is a SPNE of the repeated game. Note that in this strategy profile, the players cooperate at every stage of the repeated game.

Problem 3: Consider the following normal form game.

		Player 2	
		C	D
Player 1	C	$6, 6$	$0, 8$
	D	$0, 2$	$4, 0$

The stage game G

1. Find the all the Nash equilibria of the game G.

   Solution: *There is no NE in pure strategies. We look for a NE in mixed strategies of the form*

$$\begin{aligned} \sigma_1 &= pC + (1-p)D \\ \sigma_2 &= qC + (1-q)D \end{aligned}$$

We have that

$$\begin{aligned} u_1(C, \sigma_2) &= 6q \\ u_1(D, \sigma_2) &= 4 - 4q \end{aligned}$$

and

$$\begin{aligned} u_2(\sigma_1, C) &= 6p + 2(1-p) = 4p + 2 \\ u_2(\sigma_1, D) &= 8p \end{aligned}$$

We must have that

$$\begin{aligned} 6q &= 4 - 4q \\ 4p + 2 &= 8p \end{aligned}$$

That is,

$$p = \frac{1}{2}, \quad q = \frac{2}{5}$$

Thus,

$$\begin{aligned} \sigma_1 &= \frac{1}{2}C + \frac{1}{2}D \\ \sigma_2 &= \frac{2}{5}C + \frac{3}{5}D \end{aligned}$$

is a NE of G with payoffs

$$u_1 = \frac{12}{5}, \quad u_2 = 4$$

2. Assume that the above stage game is played 27 times. After each round, players observe the moves done by the other player. The total payoffs of the repeated game are the sum of the payoffs obtained in each round. Find all the subgame perfect Nash equilibrium of the repeated game.

   Solution: *Since, the stage game has a unique NE, the unique SPNE of the game repeated finitely many times consists in playing the NE of part 1 at every stage of the repeated game.*

3. Assume that the above stage game is played infinitely many times. After each round, players observed the moves done by the other player. The total payoffs of the repeated game are the discounted (with discount factor $\delta = 0.9$) sums of the payoffs obtained in each round. Is there a subgame perfect Nash equilibrium in pure strategies in which (C, C) is played in every round?

   Solution: *Let us consider trigger strategies: Player $i = 1, 2$ at*

   - $t = 1$ *plays* C;
   - $t > 1$ *plays* C *if* (C, C) *was played at* $t = 1, \ldots, t-1$. *Otherwise, play* (σ_1, σ_2).

   Let us check that it is a SPNE of the repeated game. We first check that it is a NE of the repeated game. The payoffs obtained by both players with the trigger strategy are,

$$u_1^c = u_2^c = 6 + 6\delta + \cdots + 6\delta^t + 6\delta^{t+1} + 6\delta^{t+2} + 62\delta^{t+3} + \cdots$$

   Note that $\mathrm{BR}_1(C) = C$. *Hence, player 1 has no incentives to deviate. If player 2 deviates at stage t and player 1 follows the trigger strategy, the payoff of player 2 is*

$$u_2^d = 6 + 6\delta + \cdots + 6\delta^t + 8\delta^{t+1} + 4\delta^{t+2} + 4\delta^{t+3} + \cdots$$

   The trigger strategy is NE iff $u^c - u^d \geq 0$. That is,

$$\delta^{t+1}\left(-2 + 2\delta + 2\delta^2 + \cdots\right) \geq 0$$

   That is

$$\delta + \delta^2 + \cdots \geq 1$$

   which is the same as

$$\frac{\delta}{1-\delta} \geq 1$$

Thus, the trigger strategy is a NE of the repeated game iff

$$\delta \geq \frac{1}{2}$$

Now, the standard argument shows that it is also a NE in every subgame: There are two types of subgames starting at a stage t.

- *Subgames in which at every stage $1, 2, \ldots, t-1$ it was played (C, C). Then, the situation in the subgame that starts at this node is exactly as above, except that the payoffs are multiplied by δ^{t-1}. The above argument shows that the trigger strategy is also a NE of that subgame.*
- *Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile (C, C) was not played. In these subgames the trigger strategy prescribes to play (σ_1, σ_2), a NE of the stage game G. But, this is a SPNE of this subgame.*

Therefore, if $\delta \geq \frac{1}{2}$, the trigger strategy is a SPNE of the repeated game. Note that in this strategy profile, the players cooperate at every stage of the repeated game.

Problem 4: Two firms compete in a Cournot market with a homogeneous good. The market demand is $p = 33 - q_1 - q_2$ where q_1 and q_2 are the quantities produced buy the firms. Both firms have constant marginal cost $c = 3$.

(a) Suppose that the market opens only in one period. If there were only one firm in the market, what would be the monopoly profit? what would be the amount produced?

Solution: *If there is a monopolist in the market, it maximizes the profit function*

$$\max_q (30 - q)q$$

The solution is $q^m = 15$, with profit $\pi^m = 225$.

(b) Suppose that the market opens only in one period and the firms know it. Find the Cournot equilibrium and the profits in equilibrium.

Solution: *The profit of firm $i = 1, 2$ is*

$$\pi_i(q_1, q_2) = (30 - q_1 - q_2)q_i$$

The best reply of firm $i = 1, 2$ is

$$\mathrm{BR}_i(q_j) = \frac{30 - q_j}{2}$$

and the NE is the solution to the system of equations

$$q_1 = \frac{30 - q_2}{2}, \quad q_2 = \frac{30 - q_1}{2}$$

We obtain the Cournot-Nash equilibrium $q_1^* = q_2^* = 10$, with profits $\pi_1^* = \pi_2^* = 100$.

(c) Assuming that the market opens only in one period and the firms know it, show that the profits if each firm produces half the monopoly quantity found in part (a) are bigger than the profits in the Cournot equilibrium computed in (b).

Solution: *We have that,*

$$\frac{\pi^m}{2} = \frac{225}{2} > \pi_i^* = 100$$

(d) Argue why if the market opens only in one period and firms know it, each firm producing half the monopoly quantity found in part (a) is not a reasonable outcome.

Solution: *Because if, for example, firm 1 knows that firm 2 is going to produce*

$$q_2 = \frac{q^m}{2} = \frac{15}{2}$$

its best reply is to produce

$$q_1 = \mathrm{BR}_1\left(\frac{15}{2}\right) = \frac{30 - \frac{15}{2}}{2} = \frac{45}{4}$$

and obtain the profit

$$\pi_1\left(\frac{45}{4}, \frac{15}{2}\right) = \frac{2025}{16} \approx 126.563 > \frac{225}{2} = \frac{\pi^m}{2}$$

(e) Assume now that the market opens periodically in the following way: In every period, the probability that the market will continue tomorrow is $\frac{3}{4}$. With probability $\frac{1}{4}$ the market finishes at that period and never reopens. The discount factor is $\delta = 4/5$. Show that the trigger strategy sustains the monopoly outcome found in part (a) with each firm producing half of the monopoly quantity.

Solution:

Note that this is equivalent to a game repeated infinitely many times with discount factor

$$\delta_0 = \frac{3}{4}\frac{4}{5} = \frac{3}{5}$$

Let us consider the following (trigger strategies) strategy profile T. Firm $i = 1, 2$ choose the following q_i,

- *At $t = 1$, choose half the monopoly output*

$$q_i = \frac{q^m}{2} = \frac{15}{2}$$

- *At $t > 1$, if*

$$q_1 = q_2 = \frac{15}{2}$$

  was chosen at $t = 1, \ldots, t-1$ then chose

$$q_i = \frac{q^m}{2} = \frac{15}{2}$$

  Otherwise, choose the Cournot-Nash output

$$q_i = q_i^* = 10$$

For what values of δ_0 is the strategy profile T is a NE of the repeated game? The utility obtained by any player under the strategy profile T is

$$\begin{aligned}
u_T &= \frac{225}{2} + \frac{225}{2}\delta_0 + \frac{225}{2}\delta_0^2 + \cdots \frac{225}{2}\delta_0^t + \frac{225}{2}\delta_0^{t+1} + \frac{225}{2}\delta_0^{t+2} + \frac{225}{2}\delta_0^{t+3} + \cdots \\
&= \frac{225}{2}\left[\left(1 + \delta_0 + \delta_0^2 + \cdots + \delta_0^t\right) + \delta_0^{t+1} + \left(\delta_0^{t+2} + \delta_0^{t+3} + \cdots\right)\right] \\
&= \frac{225}{2}\left[\left(1 + \delta_0 + \delta_0^2 + \cdots + \delta_0^t\right) + \delta_0^{t+1} + \frac{\delta_0^{t+2}}{1 - \delta_0}\right]
\end{aligned}$$

If a firm $i = 1, 2$ deviates in stage $t + 1$, then its best option is to deviate to

$$\mathrm{BR}_i(q_2^c) = \frac{45}{4}$$

with a profit of

$$\frac{2025}{16}$$

After that, in period $t+2$ the firm j that is following the T strategy profile, will switch to the Cournot-Nash equilibrium. Hence at period $t + 2$ the best action for firm i is to swith to the Cournot-Nash *equilibrium with profit* 100*. Thus, if firm* $i = 1, 2$ *deviates in stage* $t+1$*, the maximum payoff it can obtain is (note that we are using that* $\frac{2025}{16} = \frac{225}{2} \times \frac{9}{8}$*)*

$$\begin{aligned}
u_d &= \frac{225}{2} + \frac{225}{2}\delta_0 + \frac{225}{2}\delta_0^2 + \cdots \frac{225}{2}\delta_0^t + \frac{2025}{16}\delta_0^{t+1} + 100\delta_0^{t+2} + 100\delta_0^{t+3} + \cdots \\
&= \frac{225}{2}\left[\left(1 + \delta_0 + \delta_0^2 + \cdots \delta_0^t\right) + \frac{9\delta_0^{t+1}}{8}\right] + 100\left(\delta_0^{t+2} + \delta_0^{t+3} + \cdots\right) \\
&= \frac{225}{2}\left[\left(1 + \delta_0 + \delta_0^2 + \cdots \delta_0^t\right) + \frac{9\delta_0^{t+1}}{8}\right] + \frac{100\delta_0^{t+2}}{1-\delta_0}
\end{aligned}$$

Thus, the strategy profile T *is a NE of* G_R *if and only if*

$$0 \leq u_T - u_d = \frac{225\delta_0^{t+1}}{2}\left(1 - \frac{9}{8}\right) + \frac{25\delta_0^{t+2}}{2(1-\delta_0)} = \frac{25\delta_0^{t+2}}{2(1-\delta_0)} - \frac{225\delta_0^{t+1}}{2}\frac{1}{8} = \frac{25\delta_0^{t+1}}{2}\left(\frac{\delta_0}{1-\delta_0} - \frac{9}{8}\right)$$

That is, the strategy profile T *is a NE of* G_R *if and only if*

$$\frac{9}{8} \leq \frac{\delta_0}{1-\delta_0}$$

i.e,. if $\delta_0 \geq \frac{9}{17}$*. Since* $\frac{3}{5} > \frac{9}{17}$*, we conclude that the strategy profile* T *is a NE of the repeated game.*

(f) In the context of the previous part show that the trigger strategy of part (e) is a subgame perfect Nash equilibrium.

Solution: *There are two types of subgames starting at a stage* t*.*

- *Subgames in which at every stage* $1, 2, \ldots, t-1$ *it was played* (q^m, q^m)*. Then, the situation in the subgame that starts at this node is exactly like in the original repeated game, except that the payoffs are multiplied by* δ_0^{t-1}*. So, the above argument shows that the strategy profile* T *is also a NE of that subgame.*
- *Subgames in which at some stage* $1, 2, \ldots, t-1$ *the strategy profile*

  $$\left(\frac{q^m}{2}, \frac{q^m}{2}\right)$$

  was not played. In these subgames the strategy profile T *prescribes that* (q^*, q^*)*, a NE of the stage game* G*, is played. But, this is a SPNE of this subgame.*

Therefore, if $\delta_0 \geq \frac{9}{17}$*, the strategy profile* T *is a SPNE of the game repeated infinitely many times. In this strategy profile, the players cooperate at every stage of the repeated game.*

Problem 5: Consider the following normal form game.

		Player 2	
		C	D
Player 1	C	$4, 4$	$0, 6$
	D	$6, 0$	$1, 1$

The stage game G

(a) Find the all the Nash equilibria of the game G.

Solution: *The NE is* (D, D) *with payoffs* $u_1 = u_2 = 1$*. It is in dominant strategies.*

(b) Assume that the above stage game is played infinitely many times. After each round, players observe the moves done by the other player. The total payoffs of the repeated game are the discounted (with discount factor $\delta = \frac{1}{2}$) sums of the payoffs obtained in each round. Consider the following strategy profile:

i. both players play (C, C) until nobody deviates.

ii. If somebody deviates, then, in the following period both players play (D, D) for n periods.

iii. After this n periods of punishment, both players go back to the strategy in (i).

For what values of n does the above strategy profile constitute a subgame perfect Nash equilibrium?

Solution: *The payoff with the proposed strategy profile is*

$$u_c = 4 + 4\delta + 4\delta^2 + \cdots + 4\delta^{t-1} + 4\delta^t + 4\delta^{t+1} + \cdots + 4\delta^{t+n} + 4\delta^{t+n+1} + 4\delta^{t+n+2} + \cdots$$

If one player deviates in stage t he gets the payoff

$$u_d = 4 + 4\delta + 4\delta^2 + \cdots + 4\delta^{t-1} + 6\delta^t + \delta^{t+1} + \cdots + \delta^{t+n} + 4\delta^{t+n+1} + 4\delta^{t+n+2} + \cdots$$

The proposed strategy profile is a NE iff

$$0 \leq u_c - u_d = -2\delta^t + 3\delta^{t+1} + \cdots + 3\delta^{t+n} = \delta^t\left(3\delta + \cdots + 3\delta^n - 2\right) = \delta^t\left(\frac{3\delta - 3\delta^{n+1}}{1-\delta} - 2\right)$$

Thus, the strategy profile is a NE iff

$$3\delta - 3\delta^{n+1} \geq 2 - 2\delta$$

i.e.

$$5\delta \geq 2 + 3\delta^{n+1}$$

With $\delta = \frac{1}{2}$ this means

$$\frac{5}{2} \geq \frac{3}{2^{n+1}}$$

or $2^n \geq 3$. So, it is enough to take $n = 2$.

Problem 6: Consider the following stage game

		Player 2: A	Player 2: B
Player 1	A	$0,5$	$1,1$
	B	$1,1$	$5,0$

The stage game G

Assume that G is played infinitely many times. For what values of the discount factor δ is there a subgame perfect Nash equilibrium in which in the equilibrium path players play (A, A) in odd periods and play (B, B) in even periods.

Solution: *Let us consider the following strategy profile: Player 1 at*

- $t = 1$ *play A;*
- $t = 2s + 1,\ s = 1, 2, \ldots$ *If (A, A) was played in all previous odd periods and (B, B) was played in all previous even periods then play A;*
- $t = 2s,\ s = 1, 2, \ldots$ *If (A, A) was played in all previous odd periods and (B, B) was played in all previous even periods then play B;*
- *Otherwise, play B.*

Player 2 at

- $t = 1$ *play A;*
- $t = 2s + 1,\ s = 1, 2, \ldots$ *If (A, A) was played in all previous odd periods and (B, B) was played in all previous even periods then play A;*
- $t = 2s,\ s = 1, 2, \ldots$ *If (A, A) was played in all previous odd periods and (B, B) was played in all previous even periods then play B;*
- *Otherwise, play A.*

The utilities of the players in the above strategy are

$$\begin{aligned}
u_1^c &= 5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1} + 5\delta^{2t+1} + 5\delta^{2t+3} + \cdots \\
&= 5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1} + 5\delta^{2t+1}\left(1 + \delta^2 + \delta^4 + \cdots\right) \\
&= 5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1} + 5\frac{\delta^{2t+1}}{1-\delta^2} \\
u_2^c &= 5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t} + 5\delta^{2t+2} + 5\delta^{2t+4} + \cdots \\
&= 5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t} + 5\delta^{2t+2}\left(1 + \delta^2 + \delta^4 + \cdots\right) \\
&= 5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t} + 5\frac{\delta^{2t+2}}{1-\delta^2}
\end{aligned}$$

If player 1 deviates in period $2t+1$ his payoff will be

$$\begin{aligned}
u_1^d &= \underbrace{5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1}}_{cooperation} + \underbrace{\delta^{2t}}_{deviation} + \underbrace{\delta^{2t+1} + \delta^{2t+2} + \cdots}_{punishment} \\
&= 5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1} + \delta^{2t}\left(1 + \delta + \delta^2 + \cdots\right) \\
&= 5\delta + 5\delta^3 + 5\delta^5 + \cdots + 5\delta^{2t-1} + \frac{\delta^{2t}}{1-\delta}
\end{aligned}$$

Thus, $u_1^c \geq u_1^d$ if and only if

$$5\frac{\delta^{2t+1}}{1-\delta^2} \geq \frac{\delta^{2t}}{1-\delta}$$

which is equivalent to

$$\frac{5\delta}{(1-\delta)(1+\delta)} \geq \frac{1}{1-\delta}$$

Thus, we need that $5\delta \geq 1+\delta$. That is, $\delta \geq \frac{1}{4}$. If player 2 deviates in period $2t+2$ his payoff will be

$$\begin{aligned}
u_2^d &= \underbrace{5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t}}_{cooperation} + \underbrace{\delta^{2t+1}}_{deviation} + \underbrace{\delta^{2t+2} + \delta^{2t+3} + \cdots}_{punishment} \\
&= 5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t} + \delta^{2t+1}\left(1 + \delta + \delta^2 + \cdots\right) \\
&= 5 + 5\delta^2 + 5\delta^4 + \cdots + 5\delta^{2t} + \frac{\delta^{2t+1}}{1-\delta}
\end{aligned}$$

Thus, $u_2^c \geq u_2^d$ if and only if

$$5\frac{\delta^{2t+2}}{1-\delta^2} \geq \frac{\delta^{2t+1}}{1-\delta}$$

That is, $\delta \geq \frac{1}{4}$.