Quiz 5-Solutions

Q1. If (x, y, z) is a Pythagorean triple, then prove that gcd(x, y) = gcd(x, z) = gcd(y, z).

Answer:
Given: (x, y, z) is a Pythagorean triple

$$\Rightarrow x^2 + y^2 = z^2$$

To prove: gcd(x, y) = gcd(x, z) = gcd(y, z)

Proof:

Let, gcd(x, y) = d

$\Rightarrow d \mid x \quad and \quad d \mid y$

$\Rightarrow d^2 \mid x^2 \quad and \quad d^2 \mid y^2$

$\Rightarrow d^2 \mid (x^2 + y^2)$ $\{ if\ a|b\ and\ a|c,\ then\ a|(b + c) \}$

$\Rightarrow d^2 \mid z^2$ $\{ Given: x^2 + y^2 = z^2 \}$

$\Rightarrow d \mid z$

Now, $d \mid x\ and\ d \mid z \Rightarrow gcd(x, z) = d.m$ $where\ m \in \{1, 2, ... \}$ ————————————— (i)

Let, gcd(x, z) = k

$\Rightarrow k \mid x \quad and \quad k \mid z$

$\Rightarrow k^2 \mid x^2 \quad and \quad k^2 \mid z^2$

$\Rightarrow k^2 \mid (z^2 - x^2)$ $\{ if\ a|b\ and\ a|c,\ then\ a|(b - c) \}$

$\Rightarrow k^2 \mid y^2$ $\{ Given: x^2 + y^2 = z^2 \}$

$\Rightarrow k \mid y$

Now, $k \mid x\ and\ k \mid y \Rightarrow gcd(x, y) = k.n$ $where\ n \in \{1, 2, ... \}$

$\Rightarrow d = k.n$ $\{ gcd(x, y) = d \}$ ————————————— (ii)

Also, $gcd(x, z) = d.m$ { Using (i)}

$\Rightarrow k = d.m$ $\{ gcd(x, z) = k \}$

By, putting the value of k in eq (ii), we get

$$d = d.m.n$$

$$\Rightarrow m.n = 1$$

Here, m,n ∈ {1, 2, ... }, therefore m = 1 and n = 1
Putting n value in eq (ii), we get, $d = k$

Therefore, gcd(x, y) = gcd(x, z) { gcd(x, y) = d and gcd(x, z) = k }
Similarly, gcd(x, y) = gcd(y, z)

Therefore, gcd(x, y) = gcd(x, z) = gcd(y, z)

Q2. Prove that there are no solutions in positive integers to the equation.

$x^n + y^n = z^n$ **for n ∈ N and n is a multiple of 4.**

Answer:

Given: $n \in N$ and n is a multiple of 4.

To prove: $x^n + y^n = z^n$ has no solutions in positive integers.

Proof:

Theorem: The equation $x^4 + y^4 = z^2$ has no solution in non-zero integers.

Here, $n \in N$ and n is a multiple of 4, therefore, n can be written as:

$$n = 4k \;\; s.t. \; k \in N$$

Now,

$$x^n + y^n = z^n$$
$$\Rightarrow x^{4k} + y^{4k} = z^{4k}$$
$$\Rightarrow (x^k)^4 + (y^k)^4 = (z^{2k})^2$$

$Let,\; x' = x^k,\; y' = y^k \; and \; z' = z^{2k}$

$$\Rightarrow (x')^4 + (y')^4 = (z')^2$$

Now, by the above-mentioned theorem, this equation has no solution in non-zero integers.
Therefore, $x^{4k} + y^{4k} = z^{4k}$ has no solution in non-zero integers.

Hence, $x^n + y^n = z^n$ has no solutions in positive integers for $n \in N$ a multiple of 4.