Solutions:

① a) $f(mn) = kmn$ & $f(m)f(n) = k^2 mn$. If $k=0$ or $k=1$, f is completely multiplicative & hence multiplicative.

Otherwise, f is not multiplicative & hence not completely multiplicative.

b) $f(mn) = (mn)^k = m^k n^k = f(m)f(n)$

$\Rightarrow f$ is completely multiplicative

$\Rightarrow f$ is multiplicative.

② Let m & n be positive integers.

a) If $m=1$ or $n=1$ (or both), the proof of complete multiplicativity is easy.

Assume that $m>1$ & $n>1$.

Let a & b be the # of not necessarily distinct prime nos. in the prime factorization of m & n, respectively.

Then the number of not necessarily distinct prime nos. in mn is $a+b$ &

$$\lambda(mn) = (-1)^{a+b} = (-1)^a(-1)^b$$
$$= \lambda(m)\lambda(n),$$

as claimed.

(b) Since λ is completely multiplicative by part (a) above, then λ is u &

F is multiplicative.

$\Rightarrow$ F is completely determined by its values at powers of prime nos.

Accordingly, let p be a prime & let a be a positive integer. Then

$$F(p^a) = \sum_{d/n} \lambda(p^a)$$
$$= \lambda(1) + \lambda(p) + \cdots + \lambda(p^a)$$
$$= 1 + (-1) + (-1)^2 + \cdots + (-1)^a$$
$$= \begin{cases} 1 & \text{if } a \text{ odd} \\ 0 & \text{'' '' even} \end{cases}$$

$$F(n) = \prod F(p_i^{a_i})$$

$$F(n) = \begin{cases} 1 & \text{iff each } a_i \text{ is even} \\ 0 & \text{o/w.} \end{cases}$$

(3) (i) $d(n) = 1$ iff $n = 1$

If $n = 1$, clear $d(n) = 1$

If $d(n) = 1 \Rightarrow d(n) = \prod_{i=1}^{r} (1 + a_i)$

where $n = p_1^{a_1} \cdots p_r^{a_r}$.

$a_i + 1 = 1 \quad \forall i$

$\Rightarrow a_i = 0 \quad \forall i$

(ii) $d(n) = 2$ iff n is a prime

Let $n = p_1^{a_1} \cdots p_r^{a_r}$

$\prod_{i=1}^{r} (1 + a_i) = 2 \Rightarrow a_j + 1 = 2$

$a_i + 1 = 1$ for all other i

$a_1 = 1$ & $a_i = 0$ for all other i

$\Rightarrow n = p$

(iii) $d(n) = 3$ iff $n = p^2$; p prime

$\prod_{i=1}^{r}(1+a_i) = 3 \Rightarrow$ $a_1+1 = 3 \Rightarrow a_1 = 2$

$a_i + 1 = 1$ for all other i

$\Rightarrow n = p^2$

(iv) $d(n) = 5$ iff $n = p^4$

(4) $d(n)$ is odd iff $n = \square$

let $n = p_1^{a_1} \cdots p_r^{a_r}$

$d(n)$ is odd iff $a_i + 1$ is odd for all i

$\Rightarrow a_i$ is even for all i

$\Rightarrow n$ is a perfect square.

(5) (a) Pf. similar to (2) (a).

$m = 2, n = 4$

$f(mn) = 2$ & $f(m)f(n) = 4$

not completely multiplicative

(b) let $p \in \mathbb{P}$ & $a \in \mathbb{N}$

$$f(p^a) = \sum_{d|n} f(p^a)$$

$$= f(1) + f(p) + \cdots + f(p^a)$$

$$= 1 + 2 + 2 + \cdots + 2 \quad (a \text{ times})$$

$$= 1 + 2a$$

$$f(n) = (1+2a_1)(1+2a_2) \cdots (1+2a_m)$$

(6) $\sigma_3(12) = 2044$

$\sigma_4(8) = 4369.$

Since $f(n) = n^k$ is completely multiplicative

$\Rightarrow \sigma_k(n)$ is multiplicative

$$\sigma_k(p^a) = 1^k + p^k + p^{2k} + \cdots + p^{ak}$$

$$= \frac{p^{(a+1)k} - 1}{p^k - 1}.$$

$$\sigma_k(n) = \prod_{i=1}^{r} \frac{p_i^{k(a_i+1)}}{p_i^k - 1}$$