Time: 45 minutes Max Marks: 15

Instructions:
- Do not plagiarize. Do not assist your classmates in plagiarism.
- Show your full solution for the questions to get full credit.
- Attempt all questions that you can. Q3 may look familiar and is for extra credit.
- In the unlikely case a question is not clear, discuss it with an invigilator. Please ensure that you clearly include any assumptions you make, even after clarification from the invigilator.

1. (3+2+1+2+2=10 points) In homogeneous representations, a point $\mathbf{x}$ in 2D lies on a line ℓ if $\ell^\top \mathbf{x} = 0$. (a) Derive the condition that the two lines given by $\ell = [a, b, c]^\top$ and $\ell' = [a', b', c']^\top$ are parallel (by relating a, b, a' and b'). (b) Show that the intersection of parallel lines ℓ and ℓ' results in a point at infinity. (c) Show that other lines parallel to ℓ and ℓ' will also intersect at the same point at infinity. (d) Show that the slope of the lines ℓ, ℓ' can be recovered from the point at infinity corresponding to those parallel lines. (e) Show that any two points at infinity $\mathbf{x}_\infty$ and $\mathbf{x}'_\infty$ lie on a line at infinity ℓ_∞. Write the vector representing the line. {*Hint*: The slope of a line is given by m when we write the equation as $y = mx + k$.}

   Solution:

   See the handwritten solution below.

2. (2+2+1=5 points) Let $\mathbf{I}_1$ be an image with a resolution of 1024×1024. (a) If you wish to scale an image down by a factor of 2 to get an image $\mathbf{I}_2$ with a resolution of 512×512, which transformations would you apply to the pixels in $\mathbf{I}_1$? Find the parameters of that transformation. (b) Since pixel locations have to be integers in an image (which is a grid of pixels), an additional discretization step is also required. This would map multiple pixels from $\mathbf{I}_1$ to a single pixel in $\mathbf{I}_2$. Give an example of the pixels from $\mathbf{I}_1$ which would get mapped to a particular pixel in $\mathbf{I}_2$. (c) What would be the transformation for obtaining an image $\mathbf{I}_4$ of resolution 256×256? {*Hint*: You may assume any origin in the pixel space. It need not even be a valid pixel location, e.g., it can be (0,0) when the first pixel is at (1,1). Although, a corner location of the origin *may not* be the most convenient for the transformations being considered here.}

   Solution:

   You may show this as a similarity transformation, which together includes scaling and translation. One solution without any translation is given below.

3. (5 points, *Extra Credit*) Given a 3×4 camera matrix $\mathbf{P}$, such that a world 3D point $\widetilde{\mathbf{X}}$ (homogeneous coordinates) gets mapped to the 2D point $\widetilde{\mathbf{x}}$ (again in homogeneous coordinates) via the relation $\widetilde{\mathbf{x}} = \mathbf{P}\widetilde{\mathbf{X}}$, show that the camera centre (center of projection or the optical centre of the camera; also the origin on the camera frame of reference) in the world coordinate frame is the null-space of $\mathbf{P}$. Is this the only vector in the null space of $\mathbf{P}$? Explain why or why not.

   Solution:

   Please see Mid-sem solution to Q4 b). This vector would be the only vector (apart from its scaled variants) in the null-space of $\mathbf{P}$. The reason is that $\mathbf{P}$ is a 3×4 matrix with rank 3, which will only have a one dimensional null-space.

Q1

(a) Given lines $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$

We know above lines are of format

line $l \Rightarrow \quad ax + by + c = 0$

Finding its' slope by converting it to form
$y = mx + c$ where m is the slope $\rightarrow$

line $l \Rightarrow$
$$ax + by + c = 0$$
$$x + \frac{b}{a}y + \frac{c}{a} = 0$$
$$\frac{b}{a}y = -x - \frac{c}{a}$$
$$\boxed{y = -\frac{a}{b}x - \frac{c}{b}}$$

Thus slope is $\left(\frac{-a}{b}\right)$, as $\left\{\begin{matrix}\text{in accordance with} \\ y = mx + c \text{ with } m \\ \text{as slope}\end{matrix}\right\}$ $\boxed{m = \frac{-a}{b}}$

for line $l' \Rightarrow$
$$a'x + b'y + c' = 0$$
$$x + \frac{b'}{a'}y + \frac{c'}{a'} = 0$$
$$\frac{b'}{a'}y = -x - \frac{c'}{a'}$$
$$\boxed{y = -\frac{a'}{b'}x - \frac{c'}{b'}}$$

Thus slope is $\left(\frac{-a'}{b'}\right)$ $\boxed{m' = \frac{-a'}{b'}}$

We know that,
two lines are parallel when their slopes are equal. Equating slopes of lines l and l'.

$$m = m'$$
$$\frac{-a}{b} = \frac{-a'}{b'}$$

$$\boxed{\frac{a}{b} = \frac{a'}{b'}}$$

Thus, above is the condition for lines to be parallel to each other.

(b) $l = [a \ \ b \ \ c]^T \qquad l' = [a' \ \ b' \ \ c']^T$

~~We~~ Let intersection point of 2 lines be x.
We know that intersection point of 2 lines is ~~to~~ their cross product →.

$$x = l \times l'$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ a' & b' & c' \end{vmatrix}$$

$$= \begin{bmatrix} bc' - cb' \\ -ac' + a'c \\ ab' - a'b \end{bmatrix} \longrightarrow = 0.$$

Since $\frac{a}{b} = \frac{a'}{b'}$

$$\Rightarrow \boxed{ab' - a'b = 0}$$

$$x = \begin{bmatrix} bc' - cb' \\ -ac' + a'c \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} b\left(c' - \frac{b'}{b}c\right) \\ -a\left(c' - \frac{a'}{a}c\right) \\ 0 \end{bmatrix}$$

$$\frac{a}{b} = \frac{a'}{b'} \Rightarrow \text{Let } \frac{a'}{a} = \frac{b'}{b} = t$$

$$x = \begin{bmatrix} b(c' - tc) \\ -a(c' - tc) \\ 0 \end{bmatrix}$$

Let $c' - tc = \tau$ be some constant

$$x = \begin{bmatrix} b\tau \\ -a\tau \\ 0 \end{bmatrix}$$

$$x = \tau \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

Thus x is proportional to $[b \ -a \ 0]^T$ which is a point at ∞.

$$l \times l' \propto \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} = x_\infty$$

Thus l & l' intersect at a point at ∞.

(c) $$x_\infty = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

Let us assume a new line $L = [A \ B \ C]^T$ parallel to l and l'.
Since the line is parallel their slopes will be equal. & thus following holds —

$$\boxed{\frac{a}{b} = \frac{a'}{b'} = \frac{A}{B}}$$

We know that a point in x 2D lies ~~on~~ on a line l if $\boxed{l^T x = 0}$

Thus if L intersects with l & l' at the same point then x_∞ lies on L.

Computing $L^T \cdot x_\infty$

$$= \begin{bmatrix} A & B & C \end{bmatrix} \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

$$= Ab - aB$$

Since $\frac{a}{b} = \frac{A}{B}$

$$\Rightarrow \cancel{AB} = aB = bA.$$

$$L^T x_\infty = Ab - aB$$
$$= Ab - Ab$$
$$= 0.$$

$\boxed{L^T x_\infty = 0}$ Thus x_∞ lies on line L.

In other words any line parallel to l and l' will also intersect at the same point at x_∞.

~~word~~

(d) $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ $x_\infty = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$

We know that for a line L if the point at ∞ is $x_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, then its slope is y/x.

Similarly for above lines with intersection point

$$X_\infty = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

The slope is $-a/b$.

Slope for $l = \frac{-a}{b}$.

Slope for $l' = \frac{-a}{b} = \frac{-a'}{b'}$ { Since they are parallel }.

(e) $x_\infty = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$ and $x_\infty' = \begin{bmatrix} b' \\ -a' \\ 0 \end{bmatrix}$

The line passing through these 2 points can be represented by the cross product of their homogenous coordinate vectors. Let the line be l then →.

$$l = x_\infty \times x'_\infty$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b & -a & 0 \\ b' & -a' & 0 \end{vmatrix}.$$

$$= \begin{bmatrix} 0 \\ 0 \\ -a'b + ab' \end{bmatrix}$$

Let $-a'b + ab' = \tau$. then.

$$l = \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix}$$

$$l = \tau \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

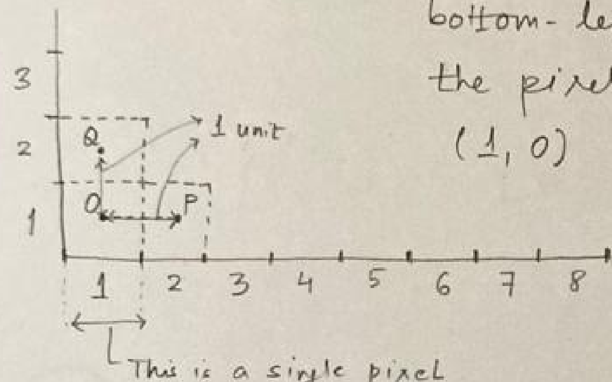
$$x_\infty \times x_\infty' \propto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = l_\infty$$

Therefore these two points intersect at a line at ∞ i.e. $l_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

Hence $l_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the vector representation of the line at infinity.

Quiz 3, Solution to Q2

Pixel Space : the figure shows our pixel space. Each pixel is 1 unit square. The origin O (0,0) of the space is at the center of the bottom-left corner pixel. Accordingly, the pixel coordinates of P are (1, 0) and Q are (0, 1).



(a) Let (x_2, y_2) represent a pixel in I_2 and (x_1, y_1) a pixel in I_1. Then, one possible transformation from I_1 to I_2 is :

$$x_2 = \lfloor 0.5\, x_1 \rfloor \qquad y_2 = \lfloor 0.5\, y_1 \rfloor$$

where $\lfloor\ \rfloor$ represents the "floor" function.

The parameter of this transformation is the Scale factor 0.5.

(b) The pixels (255, 255) and (254, 254) both map to (127, 127)

since

$\lfloor 0.5 \times 255 \rfloor = \lfloor 127.5 \rfloor = 127$

and

$\lfloor 0.5 \times 254 \rfloor = \lfloor 127 \rfloor = 127.$

(c)

$$x_2 = \lfloor 0.25\, x_1 \rfloor$$

$$y_2 = \lfloor 0.25\, y_1 \rfloor$$

In matrix form,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \left\lfloor \begin{bmatrix} r_x & 0 \\ 0 & r_y \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right\rfloor$$

where the large $\lfloor \; \rfloor$ is a floor operator applied to the entire matrix (this is an abuse of notation), r_x is the resize (or scale) factor for x and r_y for y coordinates.