SOLUTION

11/15/2022

MTH210 – SUBMISSION_20221117

TIME: 15 minutes MARKS: 5

No consultation – open notes – **books and internet not allowed.** Marks will depend on the correctness and completeness of your answer. Any previous result used should be clearly referenced.

For n, m, k $\in \mathbb{N}$, n, m $\geq$ k $\geq$ 0, prove the following identity:

$$B(n,0)B(m,k) + B(n,1)B(m, k-1) + \ldots + B(n,k)B(m,0)$$
$$= B(n+m,k)$$

ID:

NAME:

GROUP:

~~Method~~ Consider:

$$B(n,0)B(m,k) + \cdots + B(n,k)B(m,0) = B(n+m,k) \qquad (1)$$

Method 1: Combinatorial Proof:

~~Let X~~ Let X and Y be disjoint sets with $|X| = n$, $|Y| = m$. Put $Z = X \cup Y$ so that $|Z| = n+m$.

Then, RHS of (1) = number of k-subsets of Z.

For the LHS, note that any k-subset of Z can be obtained by taking a j-subset of X for $j = k, k-1, \ldots, 0$ and taking its union with any $(k-j)$-subset of

Method 1 – continued

Y. The number of ways this can be done is $B(n,j)B(m,k-j)$.

Clearly, all these ways lead to distinct k-subsets of Z.

Hence, LHS of 1 also counts the number k-subsets of Z.

Hence, LHS = RHS.

Method 2: An analytical proof using the Binomial Theorem.

Consider: $(1+x)^n (1+y)^m$

$$= \left(\sum_{i=0}^{n} B(n,i)\, x^i\right)\left(\sum_{j=0}^{m} B(m,j)\, y^j\right) \qquad (2)$$

On the RHS, the co-efficient of $x^i y^j$ where $i+j=k$ is $B(n,i)B(m,j)$ where $j=k-i$. (3)

Now, put $y = x$ in (2).

Then, the LHS becomes $(1+x)^{n+m}$ and the coefficient of x^k becomes $B(n+m,k)$. (4)

The coefficient of x^k on RHS of (2) is the same as the LHS of (1), as explained by (3).

Hence, LHS of (1) = RHS of (2).

RUBRIC: This is a "one idea" proof.

Binary Marking: 5 CORRECT PROOF

0 NOT CORRECT