SOLUTION

9/21/2022

MTH210 – SUBMISSION_20220922

TIME: 17.5 minutes MARKS: 5

No consultation – open notes – books and internet not allowed.

Recall that if X is a non-empty set, then a partition of X is a family of non-empty subsets of X which are pairwise disjoint and whose union is X. If $\mathbb{P}$ and $\mathbb{Q}$ are two partitions of X, we say that $\mathbb{P}$ *is finer than* $\mathbb{Q}$ if every subset in $\mathbb{P}$ is contained in some subset in $\mathbb{Q}$, i.e. for any $A \in \mathbb{P}$, there is a $B \in \mathbb{Q}$ such that $A \subseteq B$. Let $\leq$ denote the *is finer than* relation and let Σ denote the family (set) of all partitions of X, i.e. $\Sigma = \{\mathbb{P} : \mathbb{P}$ is a partition of X$\}$.

a) Show that $\leq$ is a partial ordering on Σ. *(3 marks)*

b) Draw the Hasse diagram of $<\Sigma, \leq>$ if X = {1, 2, 3}. (2 *marks*)

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Let Σ be the family (set) of all partitions of X, and let $\leq$ be the "is finer than" relation on Σ.

a) To show that $<\Sigma, \leq>$ is a poset:-

Answer: (i) Reflexive Property:- Let $\mathbb{P}$ be a partition of X, and let $A \in \mathbb{P}$, i.e. $A \subseteq X$. Then, $A \subseteq A \in \mathbb{P}$. Hence, $\mathbb{P} \leq \mathbb{P}$, i.e. $\mathbb{P}$ is finer than $\mathbb{P}$.

(ii) Anti-Symmetric Property: Suppose $\mathbb{P} \leq \mathbb{Q}$ and $\mathbb{Q} \leq \mathbb{P}$. Let $A \in \mathbb{P}$. Then, $A \subseteq B$ for some $B \in \mathbb{Q}$, and

(PTO)

Continued

$B \subseteq A'$ for some $A' \in \mathbb{P}$,

i.e. $A \subseteq B \subseteq A'$. ①

But, subsets in a partition are mutually disjoint, so ① can hold only if $A = A'$, but then

$A = B$,

i.e. every element of $\mathbb{P}$ is an element of $\mathbb{Q}$. ②

Similarly, every element of $\mathbb{Q}$ is an element of $\mathbb{P}$. ③

From ② and ③, $\mathbb{P} = \mathbb{Q}$

[Note that a partition is a set whose elements are themselves sets.]

$\therefore \leq$ is anti-symmetric

(iii) Transitive Property:-

Suppose $\mathbb{P} \leq \mathbb{Q}$ and $\mathbb{Q} \leq \mathbb{W}$ are partitions of X, and let $A \in \mathbb{P}$.

Then, there exists $B \in \mathbb{Q}$ s.t. $A \subseteq B$ and there exists $C \in \mathbb{W}$ s.t. $B \subseteq C$.

$\therefore A \subseteq C$ and so

$\mathbb{P}$ is finer than $\mathbb{W}$, as required.

b) Let $X = \{1,2,3\}$, and let us consider $\langle X, \leq \rangle = \Sigma$.

Observe that $3 \in \mathbb{N}$ has exactly 3 partitions as follows:

$$\left.\begin{array}{l} 3 = 3 \\ 3 = 2+1 \\ 3 = 1+1+1 \end{array}\right\}.$$ These lead to the following partitions of X:

P_X consisting of: $\{1,2,3\}$

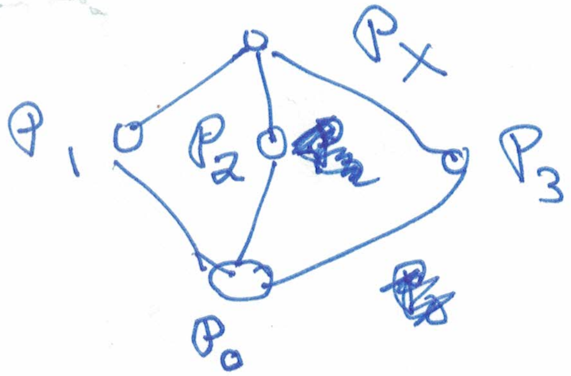
P_1 consisting of: $\{1\}, \{2,3\}$

P_2 consisting of: $\{2\}, \{1,3\}$

P_3 consisting of: $\{3\}, \{1,2\}$

P_0 consisting of: $\{1\}, \{2\}, \{3\}$.

The relation $\leq$ ~~here is~~ has the foll. Hasse Diagram:-



(PTO)

Remark: Clearly, Σ_3' is a lattice; in fact, it is the lattice which was given as an ~~latt~~ example in Wednesday's lecture (20220921). This suggests the following: Let $X_n = \{1, 2, \dots, n\}$, $n \geq 1$, and let $\langle \Sigma'_{X_n}, \leq \rangle$ be the corresponding poset with $\leq$ = "is finer than".

Is Σ'_n always a lattice?

Clearly, true for $n = 3$, as shown above, and trivially true for $n = 1, 2$.

Something for you to think about and answer !!!