Worksheet-8
Course Name: Math-III (Section-A)
Total marks = 20
Date: 15/11/2022

1. Evaluate the triple integrals: (3+3= 6 marks)

   (a) $\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{6-3x-2y} (1-x)\, dzdydx$

   (b) $\int\int\int_S 2y\ dxdydz$ Where, S is the region in the first octane that lies below the plane $x + y + \frac{z}{2} = 2$ and above the region in the xy-plane bounded by the lines $x = 0$,$y = 0$ and $x + y = 2$.

2. Find the volume of an ice-cream cone (Fig-1) bounded by the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $x^2 + y^2 + z^2 = 18$. (5 marks)

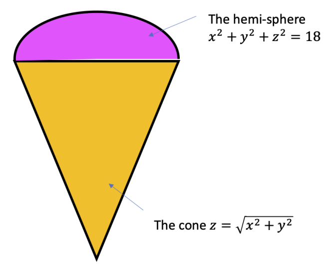

Figure 1: Ice-cream Cone

3. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from (0,0,0) to (1,1,1) (Fig-2) given by

   $C1 : \overline{r(t)} = t\hat{i} + t^2\hat{j}$, $0 \leq t \leq 1$

   $C2 : \overline{r(t)} = \hat{i} + \hat{j} + t\hat{k}$, $0 \leq t \leq 1$ (5 marks)

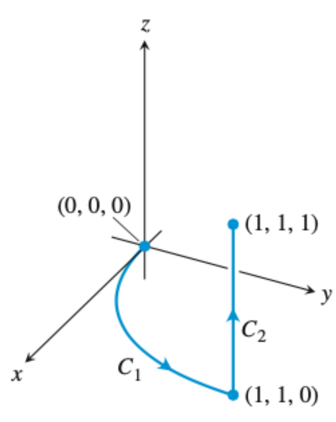

Figure 2: The paths of integration

4. Find the line integral of the function $f(x, y) = x^2 - y$ over the curve C: $x^2 + y^2 = 4$ in the first quadrant from (0,2) to $\sqrt{2}, \sqrt{2}$ (4 marks)

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Rubric and ~~ans~~ solution of WorkSheet - 8

Q.1. (a) $$\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{6-3x-2y} (1-x)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} \Big[(1-x)z\Big]_0^{6-3x-2y} dy\, dx$$

1 $$= \int_0^2 \int_0^{\frac{6-3x}{2}} (1-x)(6-3x-2y)\, dy\, dx$$

$$= \int_0^2 \Big[(1-x)(6y-3xy-y^2)\Big]_0^{\frac{6-3x}{2}} dx$$

$$= \int_0^2 (1-x)\left(18-9x-3x\left(3-\frac{3}{2}x\right)-\left(\frac{6-3x}{2}\right)^2\right)dx$$

$$= \int_0^2 (1-x)\left(18-9x-\cancel{9x}+\frac{9}{2}x^2-\frac{9x^2}{4}+\cancel{9x}-9\right)dx$$

1 $$= \int_0^2 (1-x)\left(9-9x+\frac{9x^2}{4}\right)dx$$

$$= \frac{9}{4}\int_0^2 (1-x)(x^2-4x+4)\, dx$$

$$= \frac{9}{4}\int_0^2 (x^2-4x+4-x^3+4x^2-4x)\, dx$$

$$= \frac{9}{4}\int_0^2 (-x^3+5x^2-8x+4)\, dx$$

1 $$= \frac{9}{4}\left[-\frac{x^4}{4}+\frac{5}{3}x^3-4x^2+4x\right]_0^2 = \frac{9}{4}\times\frac{4}{3} = 3.$$

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Q.1.(b) $\iiint_S 2y \, dx\,dy\,dz$.

(1) $$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2y \, dz\, dy\, dx.$$

$$= 2\int_0^2 \int_0^{2-x} y(4-2x-2y)\, dy\, dx.$$

$$= 4\int_0^2 \int_0^{2-x} (2y - xy - y^2)\, dy\, dx.$$

(1) $$= 4\int_0^2 \left[y^2 - \frac{xy^2}{2} - \frac{y^3}{3}\right]_0^{2-x} dx.$$

$$= 4\int_0^2 \left[(2-x)^2 - \frac{x}{2}(2-x)^2 - \frac{(2-x)^3}{3}\right] dx$$

$$= 4\int_0^2 (2-x)^2 \left(1 - \frac{x}{2} - \frac{2-x}{3}\right) dx.$$

$$= \frac{4}{6}\int_0^2 (2-x)^3\, dx.$$

$$= \frac{2}{3}\left[-\frac{(2-x)^4}{4}\right]_0^2$$

(1) $$= \frac{2}{3} \times \frac{2^4}{4} = \frac{8}{3}.$$

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Q.2. In cylindrical coordinates, we have,

$$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r. \quad \text{(The half cone)}$$

1

and $z = \sqrt{18-x^2-y^2} = \sqrt{18-r^2}$ (The hemi-sphere)

∴ their intersection point is :

$$r = \sqrt{18-r^2}$$
$$\Rightarrow r^2 = 18-r^2$$
$$\therefore r^2 = 9.$$
$$\Rightarrow r = 3.$$

1

∴ The volume of the ice-cream cone is.

$$= \iiint r\,dr\,d\theta\,dz.$$

$$= \int_0^{2\pi}\int_0^3\int_r^{\sqrt{18-r^2}} r\,dz\,dr\,d\theta.$$

1

$$= \int_0^{2\pi}\int_0^3 r\left(\sqrt{18-r^2}-r\right)dr\,d\theta.$$

$$= -\int_0^{2\pi}\left(9-18\sqrt{2}+9\right)d\theta. = +\left(18\sqrt{2}-18\right)\times 2\pi.$$

$$= 36\pi\left(\sqrt{2}-1\right)$$

2

$$\boxed{\begin{aligned} &= 9\left(2\sqrt{2}\ \cancel{-1}\right)\times 2\pi \\ &= 18\pi\left(2\sqrt{2}\ \cancel{-1}\right) \\ &= 36\sqrt{2}\,\pi \end{aligned}}$$

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Q.3. In C_1: $r(t) = t\hat{i} + t^2\hat{j}$

$0 \le t \le 1$.

Now, $\frac{dr}{dt} = \hat{i} + 2t\hat{j}$

1 $\left|\frac{dr}{dt}\right| = \sqrt{1+4t^2}$

Now, $f(x,y,z) = x + \sqrt{y} - z^2$
$= t + \sqrt{t^2} - 0^2$
$= 2t$ $\quad (\because t \ge 0 \Rightarrow \sqrt{t^2} = |t| = t)$

$\Rightarrow \int_{C_1} f(x,y,z)\,ds$

$= \int_0^1 2t\sqrt{1+4t^2}\,dt$.

1

$= \frac{1}{6}(5\sqrt{5} - 1)$

In C_2: $r(t) = \hat{i} + \hat{j} + t\hat{k}$
$0 \le t \le 1$.

1

$\therefore \frac{dr}{dt} = \hat{k} \Rightarrow \left|\frac{dr}{dt}\right| = 1$.

Now, $f(x,y,z) = x + \sqrt{y} - z^2 = 2 - t^2$.

1 $\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 (2-t^2)\,1 \cdot dt$.

$= 5/3$.

1 $\therefore \int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \frac{1}{6}(5\sqrt{5}-1) + \frac{5}{3}$

$= \frac{5\sqrt{5}}{6} + \frac{3}{2}$.

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Q.4. $C: x^2+y^2=4.$

$\therefore r(t) = (2\sin t)\hat{i} + (2\cos t)\hat{j},$

(1) $0 \le t \le \pi/4$

$t=0.$
$r(t) = 0\cdot\hat{i} + 2\cdot\hat{j}$
$= (0,2)$
$t=\pi/4$
$r(t) = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$
$= (\sqrt{2}, \sqrt{2})$

$\therefore \frac{dr}{dt} = 2\cos t\,\hat{i} - 2\sin t\,\hat{j}$

$\Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2.$

(1)

& $f(x,y) = x^2 - y$
$= (2\sin t)^2 - 2\cos t$
$= 4\sin^2 t - 2\cos t.$

$\therefore \int_C f\,ds = \int_0^{\pi/4} (4\sin^2 t - 2\cos t)\,2\,dt.$

$= 4.\int_0^{\pi/4} (1-\cos 2t - \cos t)\,dt.$

(2)

$= \cancel{4.\left(\frac{\pi}{4} - \frac{1}{2} - \frac{1}{\sqrt{2}}\right) = \pi - 4\sqrt{2} - 4}$

$= 4.\left[t - \frac{\sin 2t}{2} - \sin t\right]_0^{\pi/4}$

$= 4\left[\frac{\pi}{4} - \frac{1}{2} - \frac{1}{\sqrt{2}}\right]$

$= \pi - 2(1+\sqrt{2})$

— x —