ECE 351 DSP: Midterm

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Total: 30 points

1) Consider the LTI system given by the impulse response

$$h[n] = \left(\frac{3n+2}{2n+1}\right)^{3n} u[n].$$

a) Show that the system is unstable.

b) Give a bounded input signal (can be non-causal) which produces an unbounded output.

[3+2=5 points]

Solution: a) We need to show that $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$. To do so, we employ the root test. Observe that

$$\lim_{n\to\infty} (|h[n]|)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{3n+2}{2n+1}\right)^3 = \frac{27}{8} > 1.$$

Hence, by the root test, we have $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$, and so the system is unstable.

b) As shown in class, we consider the input signal

$$x[n] = \begin{cases} \frac{h[-n]}{|h[-n]|}, & n < 0 \\ 0, & n \geq 0. \end{cases}$$

It is easy to see that $|x[n]| \leq 1$ and hence it is bounded. Now, the output obtained at $n = 0$ is

$$\begin{aligned} y[0] &= \sum_{k=-\infty}^{\infty} x[k]h[-k] \\ &\overset{(a)}{=} \sum_{k=-\infty}^{0} x[k]h[-k] \\ &\overset{(b)}{=} \sum_{k=-\infty}^{0} \frac{|h[-k]|^2}{|h[-k]|} \\ &= \sum_{k=-\infty}^{0} |h[-k]| \\ &= \sum_{k=0}^{\infty} |h[k]| \\ &\overset{(c)}{=} \infty, \end{aligned}$$

where (a) follows by noting $h[n]$ is causal, (b) uses the definition of $x[n]$, and (c) uses the previous part. Hence, the corresponding output $y[n]$ is unbounded. ■

2) Consider an input $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$ to the LTI system with impulse response $h[n] = [1, -1, 2, -1, 1]$. The output $y[n]$ is then truncated to 4 samples at $n = 0, 1, 2, 3$, and then a 4-point DFT is taken. Find $Y(0)$.

[4 points]

Solution: Observe that the output is given by $h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[n-2k]$. Hence, $y[0] = 1 + 2 + 1 = 4$, $y[1] = -1 + (-1) = -2$, $y[2] = 2 + 1 + 1 = 4$, and $y[3] = -1 + (-1) = -2$. Hence, $Y(0) = y[0] + y[1] + y[2] + y[3] = 4$. ■

3) The system shown in Figure 1 is the transpose form representation of some system. Find its direct form representation.

[8 points]

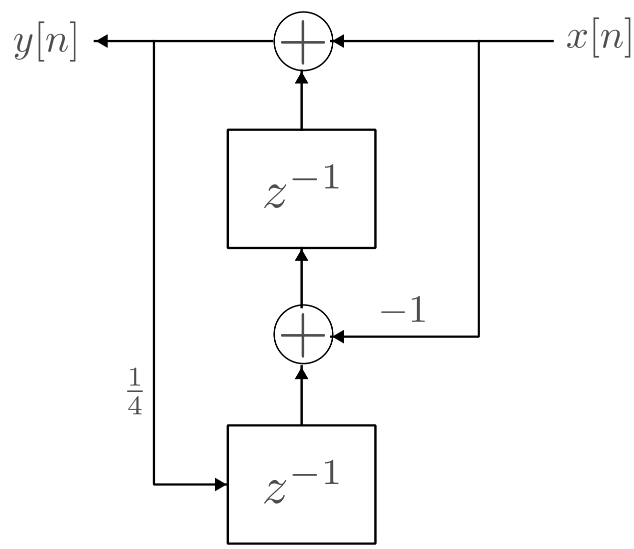

Fig. 1: Figure for Q.3

Solution: First we obtain the signal flow graph representation of the transpose form, which is given in Figure 2.

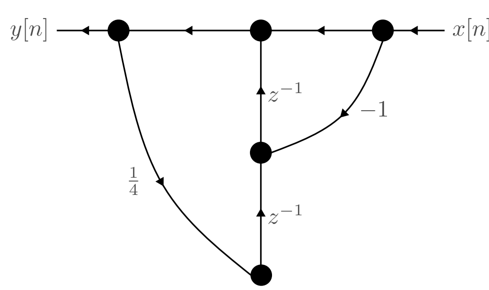

Fig. 2: Signal flow graph for the transpose form

Then we take a transpose of this signal flow graph to obtain the signal flow graph of the direct form, which is shown in Figure 3.

From here, we recover the direct form which is given in Figure 4. ■

4) Consider the 16-point radix-2 decimation-in-time FFT. Recall that in decimation-in-time algorithms the order of the input gets shuffled. What is the order in which the input needs to be arranged in this case?

[4 points]

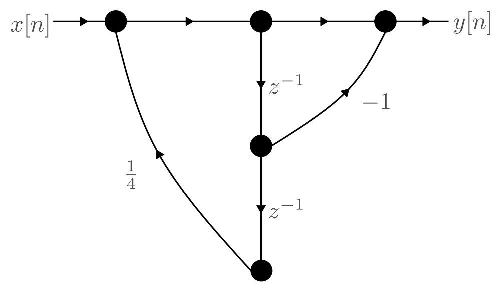

Fig. 3: Signal flow graph for the direct form

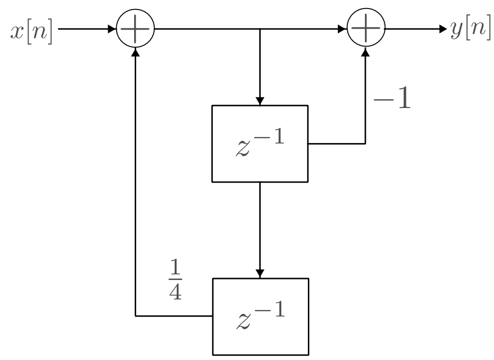

Fig. 4: Direct form

Solution: Recall that in decimation-in-time, the input should be arranged in the bit reversed order. Table I gives the binary representation of the numbers 0-15. Thus the order in which the input should be arranged is $x[0]$, $x[8]$, $x[4]$, $x[12]$, $x[2]$, $x[10]$, $x[6]$, $x[14]$, $x[1]$, $x[9]$, $x[5]$, $x[13]$, $x[3]$, $x[11]$, $x[7]$, $x[15]$. ■

TABLE I: 4-bit representation

Number	4-bit equivalent
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

5) Consider the system shown in Figure 5.

a) Consider an input of length 7 to the system shown. If we have to calculate the output using N-point DFT, what is the minimum possible value of N?

b) Now consider the input signal

$$x[n] = \begin{cases} \binom{n}{2} 3^{2n}, & 0 \le n \le 44 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose the output is to be computed using the overlap-and-save method using 16-point DFTs. What is the number of zeros that need to be padded to the right of $x[n]$?

c) Show that $H(\omega) = 4\cos^2 \frac{\omega}{2} e^{-j\omega}$.

d) Show that the group delay of the system at any $\omega \in [-\pi, \pi]$ is 1.

[2.5+2.5+2+2=9 points]

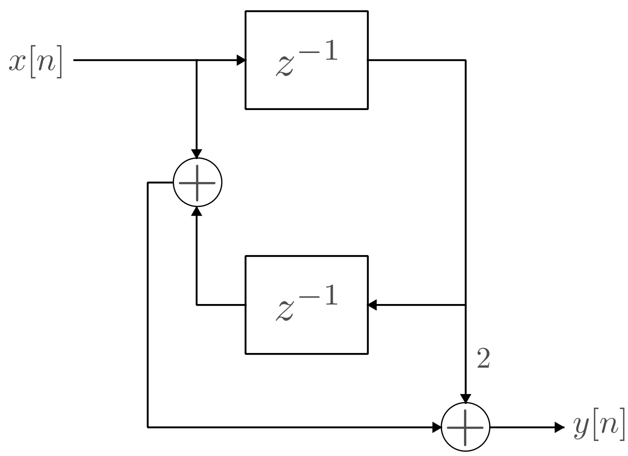

Fig. 5: Figure for Q.5

Solution: a) By inspection we see that $h[n] = [1, 2, 1]$, and hence of length 3. Thus the output will be of length $7+3-1 = 9$. Hence, the minimum value required for N is 9.

b) With 16-point DFTs, and $h[n]$ being of length 3, the size of each input chunk in the overlap-and-save method is given by $16 - 3 + 1 = 14$. Since the input is of length 45, we need to first pad 3-1=2 zeros to it. This makes it of length 47. However, the input chunk size 14 must divide the length of the input. Thus, the input needs to be made of length at least 56. Thus, in total 56-45=11 zeros needs to be padded to the right of $x[n]$.

c) Observe that $H(z) = 1 + 2z^{-1} + z^{-2} = (1 + z^{-1})^2$. Therefore,

$$\begin{aligned} H(\omega) &= (1 + e^{-j\omega})^2 \\ &= (1 + \cos\omega - j\sin\omega)^2 \\ &= 4\cos^2\frac{\omega}{2}(\cos\frac{\omega}{2} - j\sin\frac{\omega}{2})^2 \\ &= 4\cos^2\frac{\omega}{2}e^{-j\omega}. \end{aligned}$$

d) Hence, using the formula for group delay, we see that it is equal to $\frac{d}{d\omega}\omega = 1$. ■