MTH 204 Quiz 1

Maximum Points: 20 (Maximum Time: 20 mins)

March 6, 2021

Question 1.

(2 points) Mention the correct option in the answer sheet (Do not show your work).

Let $u(x, y)$ be a harmonic function, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Which of the following is an exact differential?

1. $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$
2. $\frac{\partial u}{\partial x} dx - \frac{\partial u}{\partial y} dy = 0$
3. $\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = 0$
4. $\frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0$

Solution: **(1) and (4)**

For (1).

$$M = \frac{\partial u}{\partial x} \qquad and \qquad N = \frac{\partial u}{\partial y} \qquad \Longrightarrow \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

For (4).

$$M = \frac{\partial u}{\partial y} \quad and \quad N = -\frac{\partial u}{\partial x} \quad \Longrightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} = \frac{\partial N}{\partial x} \qquad [\because u(x, y)\, is\, harmonic]$$

Question 2.

(2 points) Mention the correct option in the answer sheet (Do not show your work).

Consider the ODE

$$x^2 \frac{d^2 y}{dx^2} - 6y = 0.$$

Which of the following form a basis of solutions for this ODE?

1. $\{x^2, x^3\}$
2. $\{x^2, x^{-3}\}$
3. $\{x^{-2}, x^3\}$

4. $\{x^{-2}, x^{-3}\}$

Solution: **(3)**

The given ODE is Cauchy-Euler equation. Let $y = x^m$ is the solution. Hence, the auxiliary equation would be,

$$m(m-1) - 6 = 0 \qquad or \qquad m^2 - m - 6 = 0$$
$$(m-3)(m+2) = 0 \qquad or \qquad m = -2,\, 3$$

Hence, the set $\{x^{-2}, x^3\}$ would form the basis of solutions of the given ODE.

Question 3.

(2 points) Fill in the blanks to make the following sentence correct (Just write your answer, do not show work).

The ODE

$$(6x^5 - xy)dx + (-x^2 + xy^2)dy = 0$$

can be converted into an exact ODE by multiplying it with ______.

Solution: $\mathbf{\frac{1}{x}}$

$$M = (6x^5 - xy) \qquad and \qquad N = (-x^2 + xy^2)$$
$$\frac{\partial M}{\partial y} = -x \qquad and \qquad \frac{\partial N}{\partial x} = -2x + y^2$$
$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x} = p(x) \text{ [function of x only]}$$

Hence the ODE will be exact after multiplying with

$$\text{Integrating Factor} = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Question 4.

(2 points) Fill in the blanks to make the following sentence correct (Just write your answer, do not show work).

If $y(x) = e^{-x^2}$ is a solution of the ODE

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^3 y = 0$$

for some $\alpha, \beta \in \mathbb{R}$, then the value of $\alpha\beta$ is ______.

Solution: -4

Since $y(x) = e^{-x^2}$ is a solution of the ODE, we get

$$x \cdot (-2e^{-x^2} + 4x^2e^{-x^2}) + \alpha \cdot (-2xe^{-x^2}) + \beta x^3 e^{-x^2} = 0$$
$$-2(\alpha+1)xe^{-x^2} + (4+\beta)x^3e^{-x^2} = 0$$

implying

$$\alpha = -1 \qquad and \qquad \beta = -4, \qquad \therefore \quad \alpha\beta = 4$$

Question 5.

(2 points) Mention whether the following statement is TRUE (Do not show work).

One particular solution of ODE
$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = -e^x$$
is xe^x.

Solution: **False**

The auxiliary equation for the CHO (corresponding homogeneous ODE) is
$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$
$$(\lambda - 1)^2(+1) = 0 \qquad \therefore \quad \lambda = -1, 1, 1$$
Hence, the basis of CHO would be $\{e^{-x}, e^x, xe^x\}$. According to Modification Rule, the particular solution should be linearly independent w.r.t. basis of CHO. That's why xe^x can't be a particular solution of given ODE, rather it would be of the form Ax^2e^x, where A is a constant to be found.

Question 6.

(2 points) Mention whether the following statement is TRUE or FALSE (Do not show work).

Consider the following ODE
$$\frac{dy}{dt} = y\left(1 - \frac{y}{10}\right)$$
For the initial condition $y(0) = 20$, if the solution is $y(t)$ then
$$\lim_{t\to\infty} y(t) = 20.$$

Solution: **False**

The given ODE represents logistic-growth with carrying capacity 10. Hence for $t \to \infty$,
$$y\left(1 - \frac{y}{10}\right) = 0 \qquad or \qquad y = 0(\text{can't due to carrying capacity limitation}), \text{ hence} \quad y = 10.$$

Question 7.

(4 points) Show your full work for this problem.

Consider the ODE
$$\frac{dy}{dt} + 5y = 10 + 29\cos 2t.$$
If $y(0) = 0$ then find $y(\pi)$.

Solution: **7(approximately)**

The given ODE is a linear first order ODE,
$$p(t) = 5 \qquad \therefore \quad F(t) = e^{5t}$$

Hence, our solution would be

$$y(t) = e^{-5t}\int e^{5t}(10 + 29\cos 2t)dt + ce^{-5t}$$

$$y(t) = 2 + \frac{29}{5^2+2^2}(5\cos 2t + 2\sin 2t) + ce^{-5t}$$

$$y(t) = 2 + 5\cos 2t + 2\sin 2t + ce^{-5t}$$

Now, by applying the initial condition $y(0) = 0$, we have

$$2 + 5 + 0 - c = 0 \qquad \therefore \quad c = -7$$

Hence

$$y(t) = 2 + 5\cos 2t + 2\sin 2t - 7e^{-5t}$$

Therefore,

$$y(\pi) = 2 + 5 - 7e^{-5\pi} \approx 7$$

Question 8.

(4 points) Show your full work for this problem.

Find the general solution of the ODE

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 10\cos x + 5\sin x.$$

Solution: $y(x) = c_1 + c_2e^{2x} + c_3e^{-2x} + \cos x - 2\sin x$

CHO is

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0$$

Characteristic equation is $m^3 - 4m = 0 \implies m = 0, \pm 2$
C.F. is $c_1 + c_2e^{2x} + c_3e^{-2x}$
Let the particular solution is

$$y_p = A\cos x + B\sin x, \quad y_p' = -A\sin x + B\cos x, \quad y_p'' = -A\cos x - B\sin x, \quad y_p''' = A\sin x - B\cos x$$

Substitute the values in given ODE, we obtain

$$5A\sin x - 5B\cos x = 10\cos x + 5\sin x \implies A = 1, B = -2$$

Therefore,

$$y_p = \cos x - 2\sin x$$

Therefore, solution of given ODE is

$$y(x) = c_1 + c_2e^{2x} + c_3e^{-2x} + \cos x - 2\sin x$$