Worksheet 8 Solutions

Q1. (a) The undamped motion of a mass on an spring is given by

$$my'' + ky = 0$$

where $y = y(t)$ is the displacement of the mass.

Hence, we obtain,

$$m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$

$$m_2 y_2'' = -k_2(y_2 - y_1)$$

for the unknown displacements $y_1 = y_1(t)$ of the first mass m_1 and $y_2 = y_2(t)$ of the second mass m_2.

Now, $m_1 = m_2 = 1$; $k_1 = 3$, $k_2 = 2$, So

$$y_1'' = -3y_1 + 2(y_2 - y_1) = -5y_1 + 2y_2$$

$$y_2'' = -2y_2 + 2y_1$$

$$y'' = \begin{bmatrix} -5 & 2 \\ +2 & -2 \end{bmatrix} y$$

(b) As for a single equation, we try exponential function of t,

$$y = xe^{wt}$$

$$\Rightarrow \quad y'' = w^2 x e^{wt} = Axe^{wt} \qquad [\text{Since } y'' = Ay]$$

Let $w^2 = \lambda$

$$\Rightarrow \quad Ax = \lambda x$$

For eigen values: $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda = -1 \ \& \ -6$$

Eigen vectors : For $\lambda = -1$

$Ax = -x \Rightarrow -5x_1 + 2x_2 = -x_1 \Rightarrow 2x_1 = x_2$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = -6$

$AX = -6 \Rightarrow -5x_1 + 2x_2 = -6x_1 \Rightarrow x_1 = -2x_2$

$\Rightarrow v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Since, $\omega = \sqrt{\lambda}$, So $\sqrt{-1} = \pm i$ & $\sqrt{-6} = \pm i\sqrt{6}$

we obtain,

$y(t) = v_1 (c_1 \cos t + c_2 \sin t) + v_2 (c_3 \cos\sqrt{6}t + c_4 \sin\sqrt{6}t)$

$\Rightarrow y_1 = c_1 \cos t + c_2 \sin t + 2c_3 \cos\sqrt{6}t + 2c_4 \sin\sqrt{6}t$

$y_2 = 2c_1 \cos t + 2c_2 \sin t - c_3 \cos\sqrt{6}t - c_4 \sin\sqrt{6}t$

Q2. Let $y_1(t)$ and $y_2(t)$ be amount of fertilizer in tank T_1 and T_2 respectively at time t.

$y_1(0) = 400$ lb & $y_2(0) = 200$ lb

We can model the system with the following diff. eqn -

$y_1' = \frac{4}{200} y_2 - \frac{16}{200} y_1 = -0.08y_1 + 0.02y_2$

$y_2' = \frac{16}{200} y_1 - \frac{16}{200} y_2 = 0.08y_1 - 0.08y_2$

$$y' = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.02 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Eigen values -

$$\begin{vmatrix} -0.08-\lambda & 0.02 \\ 0.08 & -0.08-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 0.16\lambda + 0.0048 = 0$$

$$\Rightarrow \lambda = -0.12 \;\&\; -0.04$$

Eigen vectors - $AX = \lambda X$, where $A = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.08 \end{bmatrix}$

For $\lambda = -0.12$

$$\Rightarrow -0.08X_1 + 0.02X_2 = -0.12X_1$$

$$\Rightarrow -0.04X_1 = 0.02X_2 \Rightarrow X_2 = -2X_1$$

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For $\lambda = -0.04$

$$\Rightarrow -0.08X_1 + 0.02X_2 = -0.04X_1$$

$$\Rightarrow 0.04X_1 = 0.02X_2 \Rightarrow X_2 = 2X_1$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General sol^n - $y(t) = c_1 v_1 e^{-0.12t} + c_2 v_2 e^{-0.04t}$

Given $y(0) = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$

$$\Rightarrow c_1 + c_2 = 40$$

$$-2c_1 + 2c_2 = 20$$

$$\Rightarrow 4c_2 = \cancel{60}\,100 \Rightarrow c_2 = 25$$

$$\Rightarrow c_1 = 15$$

$$y(t) = 15\cancel{0}\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-0.12t} + 25\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.04t}$$

Q3. Model for current $I_1(t)$:

Voltage drop across the circuit $= 0$

Voltage drop across C + Voltage drop across $R = 0$

$$\frac{Q}{C} + RI_1 - RI_2 = 0 \quad ; \quad \frac{dQ}{dt} = I_1$$

$$\Rightarrow Q = \int I_1 dt$$

$$\Rightarrow \frac{1}{C}\int I_1 dt + R(I_1 - I_2) = 0 \quad — ①$$

Model for current $I_2(t)$:

Voltage drop across the circuit $= 0$

Voltage drop across L + Voltage drop across $R = 0$

$$LI_2' + R(I_2 - I_1) = 0 \quad — ②$$

By ①, we have

$$\frac{1}{C} I_1 + R(I_1' - I_2') = 0$$

$$\Rightarrow I_1' - I_2' + \frac{1}{RC} I_1 = 0$$

$$\Rightarrow I_1' + \frac{R}{L}(I_2 - I_1) + \frac{1}{RC} I_1 = 0$$

Now, $R = 5$, $L = 5$ & $C = \frac{1}{25}$

$$I_1' + I_2 - I_1 + \frac{25}{5} I_1 = 0$$

$$\Rightarrow I_1' = -4I_1 - I_2$$

$$I_2' = I_1 - I_2$$

$$I'(t) = \begin{bmatrix} -4 & -1 \\ 1 & -1 \end{bmatrix} I(t)$$

Eigen values :

$$\begin{vmatrix} -4-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4+\lambda)(1+\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 5 = 0$$

$$\Rightarrow \lambda = -1.38 \text{ \& } -3.62$$

Eigen vectors -

$$AX = -1.38X \quad , \text{ where } A = \begin{bmatrix} -4 & -1 \\ 1 & -1 \end{bmatrix}$$

$$-4X_1 - X_2 = -1.38X_1$$

$$\Rightarrow -2.62X_1 = X_2 \qquad \Rightarrow v_1 = \begin{bmatrix} 1 \\ -2.62 \end{bmatrix}$$

$$AX = -3.62X$$

$$\Rightarrow -4X_1 - X_2 = -3.62X_1$$

$$\Rightarrow -0.38X_1 = X_2 \qquad \Rightarrow v_2 = \begin{bmatrix} 1 \\ -0.38 \end{bmatrix}$$

General solution -

$$I(t) = c_1 v_1 e^{-1.38t} + c_2 v_2 e^{-3.62t}$$

$$= c_1 \begin{bmatrix} 1 \\ -2.62 \end{bmatrix} e^{-1.38t} + c_2 \begin{bmatrix} 1 \\ -0.38 \end{bmatrix} e^{-3.62t}$$