Quiz #3		**Name:**	____________
Date : 27/03/2024		**Section:**	____________
MTH204: ODEs/PDEs		**Maximum Time:**	25 Minutes
Semester: Winter 2024		**Maximum Marks:**	15

DO NOT SHOW ANY WORK HERE. JUST WRITE WHAT IS BEING ASKED. THERE IS NO STEP MARKING.

Problem 1. [4] For the ODE $y' = -2xy$, let $\sum_{m=0}^{\infty} a_m x^m$ be a series solution. If $a_1 = c_1 a_0$, $a_2 = c_2 a_0$, $a_3 = c_3 a_0$, and $a_4 = c_4 a_0$, then what are values of c_1, c_2, c_3, c_4?

$$y = \sum_{m=0}^{\infty} a_m x^m \Rightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} 2 a_m x^{m+1} = 0$$

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m = 0$$

$$a_1 + \sum_{m=1}^{\infty} \left((m+1) a_{m+1} + 2 a_{m-1}\right) x^m = 0$$

$$a_1 = 0, \quad (m+1) a_{m+1} + 2 a_{m-1} = 0, \quad m \geq 1$$

$\underline{m=1}$ $2a_2 + 2a_0 = 0$

$\Rightarrow a_2 = -a_0$

$\underline{m=2}$ $3a_3 + 2a_1 = 0$

$\Rightarrow a_3 = 0$

$\underline{m=3}$ $4a_4 + 2a_2 = 0$

$\Rightarrow a_4 = \frac{1}{2} a_0$

So, $\boxed{c_1 = 0,\ c_2 = -1,\ c_3 = 0,\ c_4 = \frac{1}{2}}$

+1 +1 +1 +1

Problem 2. [2] For the ODE $xy'' + y = 0$, a series solution using Frobenius method takes the form $y = x^r \sum_{m=0}^{\infty} a_m x^m$, then what are possible values of r?

$$y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots$$

$$y' = r a_0 x^{r-1} + (r+1) a_1 x^r + (r+2) a_2 x^{r+1} + \cdots$$

$$y'' = r(r-1) a_0 x^{r-2} + r(r+1) a_1 x^{r-1} + (r+1)(r+2) a_2 x^r + \cdots$$

$$xy'' + y = 0$$

Least power is x^{r-1} whose coeff. is

$$r(r-1) a_0 = 0$$

Assuming $a_0 \neq 0$ $\boxed{r = 0, 1}$

+1 +1

Problem 3. [6] For the ODE, $y'' - y' - x^2 y = 0$, if the series solution $y = \sum_{m=0}^{\infty} a_m x^m$ takes the form

$$y = a_0(b_0 + b_1 x^4 + b_2 x^5 + \cdots) + a_1(c_1 x + c_2 x^2 + c_3 x^3 + \cdots),$$

what are values of $b_0, b_1, b_2, c_1, c_2, c_3$?

$$y = \sum_{m=0}^{\infty} a_m x^m, \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=2}^{\infty} a_{m-2} x^m = 0$$

$$\boxed{\begin{matrix} \underset{+1}{b_0 = 1}, & \underset{+1}{b_1 = \frac{1}{12}}, & \underset{+1}{b_2 = \frac{1}{16}} \\ \underset{+1}{c_1 = 1}, & \underset{+1}{c_2 = \frac{1}{2}}, & \underset{+1}{c_3 = \frac{1}{6}} \end{matrix}}$$

$m=0$: $2a_2 - a_1 = 0 \Rightarrow a_2 = \frac{1}{2} a_1$

$m=1$: $6a_3 - 2a_2 = 0 \Rightarrow a_3 = \frac{1}{3} a_2 = \frac{1}{6} a_1$

$m \geq 2$: $(m+2)(m+1) a_{m+2} - (m+1) a_{m+1} - a_{m-2} = 0$

$m=2$: $12 a_4 - 3 a_3 - a_0 = 0 \Rightarrow a_4 = \frac{1}{4} a_3 + \frac{1}{12} a_0 = \frac{1}{24} a_1 + \frac{1}{12} a_0$

$m=3$: $20 a_5 - 4 a_4 - a_1 = 0 \Rightarrow a_5 = \frac{1}{5} a_4 + \frac{1}{20} a_1 = \frac{7}{120} a_1 + \frac{1}{16} a_0$

$$y = a_0 + a_1 x + \frac{1}{2} a_1 x^2 + \frac{1}{6} a_1 x^3 + \left(\frac{1}{24} a_1 + \frac{1}{12} a_0\right) x^4 + \left(\frac{7}{120} a_1 + \frac{1}{16} a_0\right) x^5$$

Problem 4. [3] If the inverse Laplace transform of $\frac{5s+1}{s^2-25}$ is $A\sinh(at) + B\cosh(at)$, then what are values of A, a, B? $\left(\text{Hint. } \frac{s}{s^2-a^2} = \mathcal{L}(\cosh(at)), \frac{a}{s^2-a^2} = \mathcal{L}(\sinh(at)).\right)$

$$\mathcal{L}^{-1}\left(\frac{5s+1}{s^2-25}\right) = 5\mathcal{L}^{-1}\left(\frac{s}{s^2-5^2}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{5}{s^2-5^2}\right)$$

$$= 5\cosh(5t) + \frac{1}{5}\sinh(5t)$$

$$\boxed{\underset{+1}{A = \frac{1}{5}}, \quad \underset{+1}{a = 5}, \quad \underset{+1}{B = 5}}$$