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MTH210 – MID-SEM EXAMINATION – 20221020

TIME: 60 MINUTES **MAXIMUM MARKS: 50**

NB: You may use any known result (i.e. theorems, propositions and lemmas, and tutorial problems) without proof; however, it should be identified clearly. This does not apply if you have been asked to prove a known result. Marks will depend on the correctness and completeness of your proofs. All questions have equal marks.

1. Consider the poset $P = <X, | >$, where $X = \{1,2,3,4,6,8,9,12,16,18,24\}$ and $|$ indicates the "is a divisor of" relation.
   a) Draw the Hasse diagram of P. *(3 marks)*
   b) Determine $\alpha(P)$ and $\omega(P)$. *(2 marks)*
   c) Decompose X into the (disjoint) union of $\omega(P)$ antichains. *(2 marks)*
   d) For any arbitrary finite poset $Q = < Y, \leq >$, is it possible to decompose Y into $< \omega(Q)$ antichains (YES/NO) ? Justify your answer. *(3 marks)*

2. Let $L = < X, \leq >$ be a distributive lattice with 0 and 1. Show that if $a, b \in X$ have complements, so do $a \vee b$ and $a \wedge b$.

3. Show that every positive integer can be expressed as the sum of **distinct** powers of 2, inclusive of $2^0 = 1$. For example, $5 = 2^2 + 2^0$.

4. Given $\sigma \in S_5$, where $\sigma = (45)(15)(13)(23)(43)$ as expressed as a product of transpositions. ***NB: For parts c) and d) below, you must show your steps.***
   a) Express σ in 2-array form. *(2 marks)*
   b) Express σ in disjoint cycle form. *(2 marks)*
   c) Determine the number of inversions in σ. *(3 marks)*
   d) Calculate the numerical value of the term corresponding to σ in det(A) in the determinant formula (Theorem 4) for the 5×5 matrix A below. *(3 marks)*

$$A = \begin{bmatrix} 1 & 0 & 4 & -1 & -2 \\ 6 & 2 & -1 & -2 & 5 \\ 2 & 4 & 6 & 8 & -3 \\ 0 & 3 & 0 & 8 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

5. For $n \geq 3$, how many permutations of S_n have cycle type $n-1,1$? Justify your answer.

SOLUTIONS FOLLOW

(NOT IN SAME ORDER)

Q5. For $n \geq 3$, how many permutations in S_n have cycle type $n-1, 1$? Justify your answer.

Ans: The number is $n\,(n-2)!$ (1)

For example, for $n = 3$, (1) gives

$3 \cdot 1! = 3$ (2)

The permutations can be listed as:-

$\sigma_1 = (23)(1)$, $\sigma_2 = (13)(2)$, $\sigma_3 = (12)(3)$.

Justify :- For $\sigma \in S_n$, with cycle type $n-1, 1$, let n be the symbol fixed by σ. Then, σ is a permutation in S_{n-1} with cycle type $n-1$, i.e. having a single cycle.

Using the known result (TUT 06, Q4), there are $((n-1)-1)! = (n-2)!$ such permutations,

Since the ~~permutation to be~~ symbol to be fixed can be chosen in n different ways, we get the answer.

Q3. Show that every positive integer can be expressed as the sum of distinct powers of 2, inclusive of $2^0 = 1$.

Answer:- Remark: Observe that the result holds vacuously for 0.

Method 1: By strong induction for $n \in \mathbb{Z}^+$.

Base Case: $n = 1$. $n = 2^0$ as stated in the question.

Inductive Step: Suppose the result holds for all positive integers $< n$ $(n \geq 2)$ — (IH).
~~[illegible]~~ Consider n.

Let 2^k be the highest power of 2 such that $2^k \leq n$.

Then $n = 2^k + m$, $0 \leq m < 2^k$.

If $m = 0$, $n = 2^k$ is the sum of distinct powers of 2.

If $m > 0$, $m < 2^k \leq n$, and so by (IH), m can be expressed as

Q3 (cont'd):-

the sum of distinct powers of 2. Clearly, none of these can be 2^k, since $m < 2^k$.

$\therefore n = 2^k + m = 2^k$ + distinct powers of 2 other than 2^k.

By SPMI, the result holds for all $n \in \mathbb{Z}^+$.

Method 2: Express $n = a_k a_{k-1} \cdots a_1 a_0$ (1) in place-value form but with binary digits rather than the usual decimal digits, i.e. in base two, not in base ten.

(*) Hence, $a_i = 1$ or 0 in the expression (1).

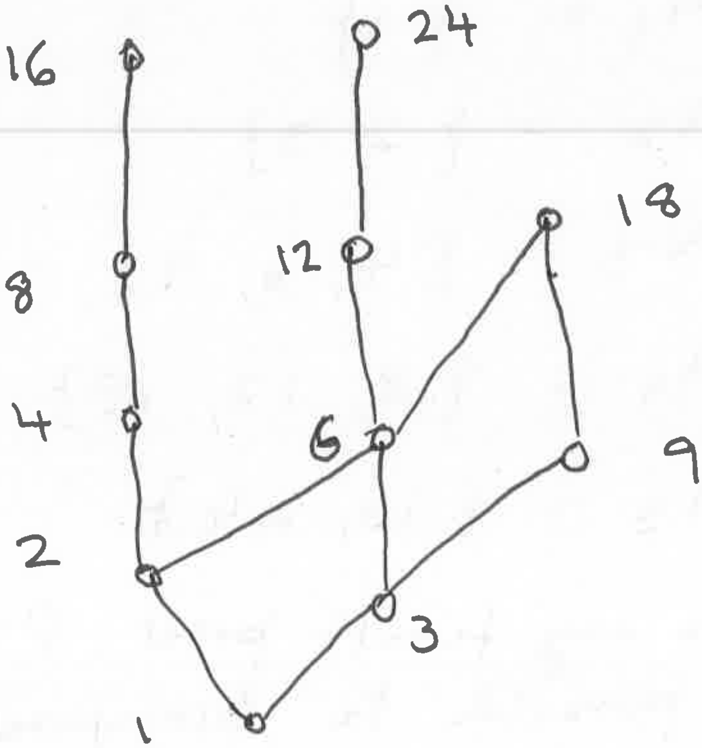
$\therefore n = \sum_{i=0}^{k} a_i 2^i$ is an expression of n as a sum of distinct powers of 2.

<u>Remark</u>: This only works for base two.

Q1. Given $P = \langle X, | \rangle$, where $X = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24\}$ and $|$ indicates the divisor of relation.

NB: $|X| = 11$

a) Hasse Diagram of P (1)



(b) $\alpha(P) = 3$

$\omega(P) = 5$

Q 1 (cont'd)

(c) Expression X as the disjoint union of $\omega(P)$ antichains.

Ans: The easiest way to do this is to mimic the proof of Prop. 8: ~~resso~~ successively remove minimal elements of each poset obtained.

So: $A_1 = \{1\}$

$A_2 = \{2, 3\}$

$A_3 = \{4, 6, 9\}$

$A_4 = \{8, 12, 18\}$

$A_5 = \{16, 24\}$

(d) For any finite poset $Q = \langle Y, R \rangle$, is it possible to decompose Y into $< \omega(Q)$ antichains (YES/NO)? Why?

Ans: <u>NO</u>.

Suppose BWOC that it is possible. Let C be a chain with $\omega(Q)$ elements. Since there are $< \omega(Q)$ antichains, two elements of C must lie in the same antichain $\Rightarrow\!\Leftarrow$

Q2: Let $L = \langle X, \leq \rangle$ be a distributive lattice with 0 and 1. Show that if $a, b \in X$ have complements, so do $a \vee b$ and $a \wedge b$.

Ans: Let $\bar{a}$ and $\bar{b}$ be the complements of a and b.

Put $c = \bar{a} \wedge \bar{b}$

Then: $(a \vee b) \vee c = (a \vee b) \vee (\bar{a} \wedge \bar{b})$

$= [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}]$, using distributive property

$= (1 \vee b) \cancel{\vee b} \wedge (1 \vee a) = 1 \wedge 1$

$= 1$ (1)

~~Similarly~~ Again:-

$(a \vee b) \wedge c = (a \vee b) \wedge (\bar{a} \wedge \bar{b})$

$= [a \wedge (\bar{a} \wedge \bar{b})] \vee [b \wedge (\bar{a} \wedge \bar{b})]$

$= (0 \wedge \bar{b}) \vee (0 \wedge \bar{a}) = 0 \vee 0$

$= 0$ (2)

From (1) and (2), c is the complement of

Q2 - cont'd

of $a \vee b$.

Similarly, $d = \bar{a} \vee \bar{b}$ is the complement of $a \wedge b$. (3)

(Duality)

Q 4. $\sigma \in S_5$ is given by $\sigma = (45)(15)(13)(23)(43)$.

(a) σ in 2-array form:-

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}$$

(b) σ in disjoint cycle form:-

$$\sigma = (1\ 3\ 5)(2\ 4)$$

(d) Since σ has been expressed as a product of 5 transpositions, σ is odd. $\therefore$ $\operatorname{sgn}(\sigma) = -1$

Hence, the term corresponding to σ in

$\det A = (-1)\, a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} a_{4\sigma(4)} a_{5\sigma(5)}$ (1)

$= (-1)\, a_{13} a_{24} a_{35} a_{42} a_{51}$ (2)

$= (-1)(4)(-2)(-3)(3)(1) = -72$ (3)

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Q 4 (c) :

i	j	σ(i)	σ(j)	Inversion
1	2	3	4	NO
1	3	3	5	NO
1	4	3	2	YES
1	5	3	1	YES
2	3	4	5	NO
2	4	4	2	YES
2	5	4	1	YES
3	4	5	2	YES
3	5	5	1	YES
4	5	2	1	YES

ANSWER: 7 INVERSIONS (ODD no.)