MTH204: Worksheet 9 (Solutions)

(10 points)

(1 pt.)

Q-1 $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ — ①

Substituting $y = \sum_{m=0}^{\infty} a_m x^m$ in ①, we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0 \quad \text{②}$$

In ②, set $m-2=s$ in the first series and $m=s$ in rest of three series, which gives:

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0$$

(1 pt.)

Q-2 x^0: $2a_2 + ka_0 = 0$

x^1: $3\cdot 2a_3 - 2a_1 + ka_1 = 0$

For $s \geq 2$, x^s: $(s+2)(s+1)a_{s+2} - s(s-1)a_s - 2sa_s + ka_s = 0$

$$\Rightarrow a_2 = -\frac{k}{2}a_0, \quad a_3 = \frac{(2-k)}{3\cdot 2}a_1 \quad \& \quad a_{s+2} = \frac{s(s-1)+2s-k}{(s+1)(s+2)}a_s, \quad s \geq 2$$

Note that a_{s+2} form holds for all $s \geq 0$ as a_2 and a_3 can also be driven from them.

(1 pt.)

Q-3 From recurrence relation,

$$a_{s+2} = \frac{s(s-1)+2s-k}{(s+1)(s+2)}a_s, \quad s \geq 0$$

it is immediate that all even coefficients can be expressed in terms of a_0 & every odd coefficient can be expressed in terms of a_1.

Therefore, by taking even & odd powers it gives us the division

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

(2 pts)

Q-4 $S=0: \; a_2 = -\frac{k}{2} a_0$

$S=1: \; a_3 = \frac{2-k}{6} a_1$

$S=2: \; a_4 = \frac{6-k}{12} a_2 = \frac{k(k-6)}{24} a_0$

$S=3: \; a_5 = \frac{12-k}{20} a_3 = \frac{(k-2)(k-12)}{120} a_1$

$S=4: \; a_6 = \frac{20-k}{30} a_4 = \frac{k(k-6)(20-k)}{720} a_0$

$S=5: \; a_7 = \frac{30-k}{42} a_5 = \frac{(k-2)(k-12)(30-k)}{5040} a_1$

Then,

$$y_1(x) = \frac{1}{a_0}(a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots)$$

$$= 1 - \frac{k}{2} x^2 + \frac{k(k-6)}{24} x^4 - \frac{k(k-6)(k-20)}{720} x^6 + \dots$$

$$y_2(x) = \frac{1}{a_1}(a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \dots)$$

$$= x - \frac{(k-2)}{6} x^3 + \frac{(k-2)(k-12)}{120} x^5 - \frac{(k-2)(k-12)(k-30)}{5040} x^7 + \dots$$

(2 pts.)

Q-5 Note that $a_{s+2} = \frac{s(s-1) + 2s - k}{(s+1)(s+2)} a_s$

$$= \frac{s^2 + s - n(n+1)}{(s+1)(s+2)} a_s$$

$$= \frac{(s-n)(n+s) + (s-n)}{(s+1)(s+2)} a_s$$

$$= -\frac{(n-s)(n+s+1)}{(s+1)(s+2)} a_s$$

If $n \geq 0$ & n is even, then $a_{n+2} = 0$ and by recurrence

$$a_{n+4} = a_{n+6} = \ldots = 0$$

$\Rightarrow y_1(x)$ is a polynomial of degree n.

If $n \geq 0$ & n is odd, then $a_{n+2} = 0$ and by recurrence

$$a_{n+4} = a_{n+6} = \ldots = 0$$

$\Rightarrow y_2(x)$ is a polynomial of degree n.

(3 pts.)

Q-6 $p_0(x) = a_0$

Since a_0 is leading term with $n=0$, so $a_0 = \dfrac{(0!)^2}{2^0 \times 1!} = 1$

$$\Rightarrow \boxed{p_0(x) = 1}$$

$p_1(x) = a_1 x$

Since a_1 is leading term, $a_1 = \dfrac{(1!)^2}{2^1 \times 2!} = \dfrac{1}{4}$

$$\Rightarrow \boxed{p_1(x) = \frac{x}{4}}$$

$p_2(x) = a_0 + a_2 x^2$

Since a_2 is leading term, $a_2 = \dfrac{(2!)^2}{2^2 \times 3!} = \dfrac{4}{4 \times 6} = \dfrac{1}{6}$

Now, using $a_{s+2} = \dfrac{-(n-s)(n+s+1)}{(s+2)(s+1)} a_s$, we have $a_s = \dfrac{-(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2}$

For $s=0, n=2$, $a_0 = \dfrac{-2}{2 \cdot 3} a_2 = -\dfrac{2}{6} \times \dfrac{1}{6} = -\dfrac{1}{18}$

$$\Rightarrow \boxed{p_2(x) = \frac{1}{6}x^2 - \frac{1}{18}}$$

$p_3(x) = a_1 x + a_3 x^3$

Since a_3 is leading term, $a_3 = \dfrac{(3!)^2}{2^3 \times 4!} = \dfrac{3}{16}$

For $s=1, n=3$, $a_1 = \dfrac{6}{-2 \times 5} a_3 = -\dfrac{6}{10} \times \dfrac{3}{16} = -\dfrac{9}{80}$

$$\Rightarrow \boxed{p_3(x) = \frac{3}{16}x^3 - \frac{9}{80}x}$$

$$p_4(x) = a_0 + a_2 x^2 + a_4 x^4$$

Since a_4 is leading term, $a_4 = \frac{(4!)^2}{2^4 \times 5!} = \frac{3}{10}$

For $s=2, n=4$, $a_2 = \frac{-4\times3}{2\times7} a_4 = -\frac{9}{35}$

For $s=0, n=4$, $a_0 = -\frac{2\times1}{4\times5} a_2 = \frac{9}{350}$

$$\Rightarrow \boxed{p_4(x) = \frac{3}{10}x^4 - \frac{9}{35}x^2 + \frac{9}{350}}$$

$$p_5(x) = a_1 x + a_3 x^3 + a_5 x^5$$

Since a_5 is leading term, $a_5 = \frac{(5!)^2}{2^5 \times 6!} = \frac{5}{8}$

For $s=3, n=5$, $a_3 = -\frac{5\times4}{2\times9} a_5 = -\frac{25}{36}$

For $s=1, n=5$, $a_1 = -\frac{3\times2}{4\times7} a_3 = \frac{25}{168}$

$$\Rightarrow \boxed{p_5(x) = \frac{5}{8}x^5 - \frac{25}{36}x^3 + \frac{25}{168}x}$$