Worksheet - 10

Problem 1:

We first note that

$$\cos(t) = -\cos(t-\pi)$$

Then we can write the differential equation as

$$y'' + 5y' + 6y = \delta\left(t-\frac{\pi}{2}\right) - u(t-\pi)\cos(t-\pi)$$

Take the laplace transform of the differential equation

$$s^2Y(s) + 5sY(s) + 6Y(s) = e^{-\frac{\pi}{2}s} + e^{-\pi s}\frac{s}{s^2+1}$$

$$Y(s) = e^{-\frac{\pi}{2}s}\frac{1}{s^2+5s+6} + e^{-\pi s}\frac{s}{s^2+1}\frac{1}{s^2+5s+6}$$

$$Y(s) = e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\pi s}\left(\frac{1}{10}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} - \frac{2}{5}\frac{1}{s+2} + \frac{3}{10}\frac{1}{s+3}\right)$$

Take the inverse laplace transform

$$y(t) = \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right) u(t-\pi/2) +$$

$$\left(\frac{\cos(t-\pi) + \sin(t-\pi)}{10} - \frac{2}{5} e^{-2(t-\pi)} + \frac{3}{10} e^{-3(t-\pi)}\right) u(t-\pi)$$

<u>Problem 2</u> : let us define $f(t) = e^{at}$ and $g(t) = e^{bt}$.

$$\mathcal{L}\{f(t) * g(t)\} = F(s)\,G(s)$$

$$= \frac{1}{s-a}\,\frac{1}{s-b}$$

$$= \frac{1}{a-b}\,\frac{1}{s-a} + \frac{1}{b-a}\,\frac{1}{s-b}$$

$$= \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)$$

The inverse Laplace transform of this is

$$f(t) * g(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)$$

Problem 3: let $F(s) = \log \frac{s+a}{s+b} = \log(s+a) - \log(s+b)$

let's calculate its derivative

$$G(s) = F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

It's inverse laplace transform is

$$g(t) = e^{-at} - e^{-bt}$$

But we know that

$$g(t) = \mathcal{L}^{-1}\{F'(s)\} = t f(t)$$

$$f(t) = \frac{g(t)}{t} = \frac{e^{-at} - e^{-bt}}{t}$$

Problem 4: If we take the laplace transform of both equations, we get

$$(s Y_1 - y_1(0)) + Y_2 = 0$$

$$Y_1 + (sY_2 - y_2(0)) = \frac{2s}{s^2+1}$$

$$\Rightarrow \quad sY_1 - 1 + Y_2 = 0$$

$$Y_1 + sY_2 = \frac{2s}{s^2+1}$$

which can be rewritten as

$$\begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2s}{s^2+1} \end{pmatrix}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{2s}{s^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s}{s^2+1} \\ \frac{1}{s^2+1} \end{pmatrix}$$

Taking the inverse laplace transform,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Problem 5 :

$$\mathcal{L}\{3 \sinh(4t)\} = 3 \frac{4}{s^2-4^2} = F(s)$$

$$\mathcal{L}(3t \sinh(4t)\} = -F'(s) = (3) \frac{8s}{(s^2-4^2)^2}$$

Problem 6 : Take the laplace transform of both equations, we get

$$sY_1 - y_1(0) = -Y_1 + 4Y_2$$
$$sY_2 - y_2(0) = 3Y_1 - 4Y_2$$

$$\Rightarrow \quad sY_1 - 3 = -Y_1 + 4Y_2$$
$$sY_2 - 4 = 3Y_1 - 4Y_2$$

which can be rewritten as

$$\begin{pmatrix} s+1 & -4 \\ -3 & s+4 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$Y_1 = \frac{\begin{vmatrix} 3 & -4 \\ 4 & s+4 \end{vmatrix}}{\begin{vmatrix} s+1 & -4 \\ 3 & s+4 \end{vmatrix}} = \frac{3s+28}{s^2+5s+16} = \frac{3s+28}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}$$

$$Y_2 = \frac{\begin{vmatrix} s+1 & 3 \\ -3 & 4 \end{vmatrix}}{\begin{vmatrix} s+1 & -4 \\ 3 & s+4 \end{vmatrix}} = \frac{4s+13}{s^2+5s+16} = \frac{4s+13}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}$$

Their inverse laplace transforms are

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{3s+28}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} + 28\,\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\}$$

$$= 3\cos\left(\frac{\sqrt{39}}{2}t\right) + \frac{56}{\sqrt{39}}\sin\left(\frac{\sqrt{39}}{2}t\right)$$

Same way,

$$y_2(t) = 4\cos\left(\frac{\sqrt{39}}{2}t\right) + \sqrt{\frac{26}{3}}\sin\left(\frac{\sqrt{39}}{2}t\right)$$