Worksheet-9
Course Name: Math-III (Section-A)
Total marks = 20
Date: 23/11/2022

1. Find the circulation and flux of the field $\mathbf{F} = -y^2\hat{i} + x^2\hat{j}$ around and across the closed semicircular path that consists of the semicircular arch $r_1(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}, 0 \leq t \leq \pi$, followed by the line segment $r_2(t) = t\hat{i}, -2 \leq t \leq 2$. (5 marks)

2. Find the flux of the field $\mathbf{F} = (x+y)\hat{i} - (x^2+y^2)\hat{j}$ outward across the triangle with vertices (1,0), (0,1) and (-1,0). (5 marks)

3. Verify whether the field $F = (e^x \cos y)\hat{i} - (e^x \sin y)\hat{j} + z\hat{k}$ is conservative or not. (5 marks)

4. Find a potential function $\mathbf{f}$ for the field $\mathbf{F} = (\ln x + \sec^2(x+y))\hat{i} + (\sec^2(x+y) + \frac{y}{y^2+z^2})\hat{j} + \frac{z}{y^2+z^2}\hat{k}$ (5 marks)

Rubric & solution for Worksheet - 9

Q.1. Given $F = -y^2\hat{i} + x^2\hat{j}$

$C1: \vec{r}_1(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}$; $0 \le t \le \pi$

$\therefore F\big|_{C_1} = -4\sin^2 t\,\hat{i} + 4\cos^2 t\,\hat{j}$

$\frac{d\vec{r}_1}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$

$\therefore F\big|_{C_1} \cdot \frac{d\vec{r}_1}{dt} = 8\sin^3 t + 8\cos^3 t$

$\therefore$ circulation $\Big|_{C_1} = \int_{C_1}\left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt = 8\int_0^\pi(\cos^3 t + \sin^3 t)\,dt$

$= 2\int_0^\pi(4\cos^3 t + 4\sin^3 t)\,dt$

1 $\quad = 2\int_0^\pi\left[\cos 3t + 3\cos t + 3\sin t - \sin 3t\right]dt$

$= 2 \times \frac{16}{3} = \frac{32}{3}$.

Flux $\Big|_{C_1} = \int_{C_1}(M\,dy - N\,dx) = \int_0^\pi\left[-4\sin^2 t(2\cos t) - 4\cos^2 t(-2\sin t)\right]dt$

$= 8\int_0^\pi(\cos^2 t\sin t - \sin^2 t\cos t)\,dt$.

1 $\quad = 8\left[-\cos^3 t - \sin^3 t\right]_0^\pi$

$= -8\left[-\frac{1}{3} - \frac{1}{3}\right] = 16/3$

$C_2: \vec{r}_2(t) = t\hat{i}$ $\quad -2 \le t \le 2$

$F\big|_{C_2} = t^2\hat{j}$ $\quad$ & $\frac{d\vec{r}_2(t)}{dt} = \hat{i}$

circulation $|_{C_2}$

$$= \int_{C_2} \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int_{-2}^{2} (t^2 \hat{j}) \cdot \hat{i} \, dt = 0.$$ 1

$$\text{Flux} |_{C_2} = \int_{C_2} M dy - N dx = \int_{-2}^{2} -t^2 dt = -\frac{t^3}{3} \Big]_{-2}^{2}$$
$$= -\frac{16}{3}.$$ 1

$$\therefore \text{Total circulation} = \int_{C_1} + \int_{C_2} = \frac{32}{3} + 0 = \frac{32}{3}$$ 0.5

$$\text{Total Flux} = \int_{C_1} + \int_{C_2} = \frac{16}{3} - \frac{16}{3} = 0$$ 0.5

Q.2. Flux from $(1,0)$ to $(0,1)$:

$$r_1 = (1-t)\hat{i} + t\hat{j} \ ; \ 0 \le t \le 1$$
$$F = (x+y)\hat{i} - (x^2+y^2)\hat{j} = \hat{i} - (1-2t+2t^2)\hat{j}$$
$$n1 = 1 \cdot \hat{i} - (-1)\hat{j} = \hat{i} + \hat{j}$$ 0.5

$$\text{Flux 1} = \int_0^1 f \cdot n1 \, ds = \int_0^1 (1 - 1 + 2t - 2t^2) dt$$ 1
$$= \int_0^1 (2t - 2t^2) dt.$$
$$= \left[t^2 - \frac{2}{3} t^3 \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Flux from $(0,1)$ to $(-1,0)$:

$$r_2 = -t\hat{i} + (1-t)\hat{j} \ ; \ 0 \le t \le 1$$ 0.5
$$F = (x+y)\hat{i} - (x^2+y^2)\hat{j} = (1-2t)\hat{i} - (1-2t+2t^2)\hat{j}$$
$$n2 = -\hat{i} + \hat{j}$$
$$\therefore \text{Flux 2} = \int_0^1 F \cdot n2 \, ds = \int_0^1 (2t - 1 - 1 + 2t - 2t^2) dt.$$

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$$= \int_0^1 (-2+4t-2t^2)\,dt$$ 1

$$= -2+2-\frac{2}{3} = -2/3$$

Flux From $(-1,0)$ to $(1,0)$:→

$r_3 = (-1+2t)\hat{i} \qquad 0 \le t \le 1$ 0.5

$F = (x+y)\hat{i} - (x^2+y^2)\hat{j} = (-1+2t)\hat{i} - (-1+2t)^2\hat{j}$

$n3 = 0\cdot\hat{i} - 2\hat{j} = -2\hat{j}$

$$\text{flux}3 = \int_0^1 2(2t-1)^2\,dt$$

$$= \int_0^1 2(4t^2-4t+1)\,dt.$$

$$= \frac{8}{3} - 4 + 2 = \frac{8}{3} - 2 = \frac{2}{3}.$$ 1

∴ Total Flux = Flux 1 + Flux 2 + Flux 3
$= 1/3 - 2/3 + 2/3 = 1/3.$ 0.5

Q.3. $F = e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j} + z\hat{k}$

$M = e^x \cos y$; $N = -e^x \sin y$; $P = z$ 1

$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$; $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$;

$\frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y}$ 1 + 1 + 1

Hence F is conservative 1

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Q.4. $F = \left(\ln x + \sec^2(x+y)\right)\hat{i} + \left(\sec^2(x+y) + \dfrac{y}{y^2+z^2}\right)\hat{j} + \dfrac{z}{y^2+z^2}\hat{k}$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$\cancel{\therefore \frac{\partial f}{\partial x} = \ln x + \sec^2(x+y)}$

~~Integrating, we get,~~

$\cancel{f(x,y,z) = x\ln x - x + \tan(x+y)}$

1 $\dfrac{\partial f}{\partial z} = \dfrac{z}{y^2+z^2} \Rightarrow f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + g(x,y)$

1 $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial x} = \ln x + \sec^2(x+y)$

$\Rightarrow g(x,y) = x\ln x - x + \tan(x+y) + h(y)$

1 $\therefore f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + x\ln x - x + \tan(x+y) + h(y)$

1 Now, $\dfrac{\partial f}{\partial y} = \dfrac{y}{y^2+z^2} + \sec^2(x+y) + h'(y) = \sec^2(x+y) + \dfrac{y}{y^2+z^2}$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = C$ (constant)

1 $\therefore f(x,y,z) = \frac{1}{2}\ln(y^2+z^2) + x\ln x - x + \tan(x+y) + C$

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