Worksheet 6

1. Determine (with a proof or counter example) whether each of the arithmetic functions below is completely multiplicative, multiplicative or both. Here k is fixed number.
   (a) $f(n) = kn$,
   (b) $f(n) = n^k$.
2. Let $n \in \mathbb{N}$. The Liouville λ-function, denoted by $\lambda(n)$, is

$$\lambda(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1 p_2 \cdots p_k \text{ where } p_1, p_2, \cdots, p_k \text{ are not necessarily distinct primes.} \end{cases}$$

$$\lambda(12) = (-1)^3 = -1, \ 12 = 2.2.3$$

   (a) Prove that λ is a completely multiplicative function.
   (b) Let $F(n)$ be
$$F(n) = \sum_{d|n} \lambda(d).$$
   Prove that $F(n) = \begin{cases} 1, & \text{if n is a perfect square,} \\ 0, & \text{otherwise.} \end{cases}$
3. Characterize those positive integers n for which each of the following property holds.
   (a) $d(n) = 1$,
   (b) $d(n) = 2$,
   (c) $d(n) = 3$,
   (d) $d(n) = 5$.
4. Characterize those positive integers n for which $d(n)$ is odd.
5. Let $n \in \mathbb{N}$. Define an arithmetic function ρ by $\rho(1) = 1$ and $\rho(n) = 2^r$, where r is the number of distinct prime numbers in the prime factorization of n.

   Example: $\rho(12) = 2^2 = 4$.
   (a) Prove that ρ is multiplicative but not completely multiplicative.

(b) Let $f(n) = \sum_{d|n} \rho(d)$. If $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the prime factorization of n , then find a formula for $f(n)$ in terms of prime factorization.

Hint: note that f is multiplicative and it is determined by the prime powers of n, so use (a) above.

6. Let $n \in \mathbb{N}$. If $k \in \mathbb{N}^*$, then define

$$\sigma_k(n) = \sum_{d|n} d^k.$$

Note that this is a generalization of $d(n)$ and $\sigma(n)$, as when $k = 0$ we have $\sigma_0(n) = d(n)$ and $k = 1$ we have $\sigma_1(n) = \sigma(n)$.

(a) Find $\sigma_3(12)$ and $\sigma_4(8)$.

(b) Prove that $\sigma_k(n)$ is multiplicative.

(c) Let $p \in \mathbb{P}$ and $a \in \mathbb{N}$. Find a formula for $\sigma_k(p^a)$.

(d) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where p_i's are distinct primes. Use (b) and (c) to find the formula for $\sigma_k(n)$.