ECE 634/CSE 646 InT: Midterm Examination

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Total: 20 points

Instruction: Answer question 1 and ANY TWO of the following questions.

1) Consider a source with alphabet $\mathcal{X} = \{a, b, c, d, e\}$ and a pmf P_X given by $\mathrm{P}_X(a) = \mathrm{P}_X(c) = 0.1$, $\mathrm{P}_X(b) = 0.3, \mathrm{P}_X(d) = 0.45$. Give a Huffman code for this source.

[6 points]

2) Consider a source with alphabet $\mathcal{X}$ and pmf P_X, and fix $\epsilon > 0$. Define the sets

$$A_{\epsilon,a}^{(n)} \triangleq \left\{x^n \in \mathcal{X}^n : \left|\frac{1}{n}N(a, x^n) - \mathrm{P}_X(a)\right| \leq \epsilon \mathrm{P}_X(a)\right\}, \forall a \in \mathcal{X},$$

where $N(a, x^n) = \sum_{i=1}^{n} \mathbf{1}\{x_i = a\}$.

   a) Show that for any sequence x^n, $\mathrm{P}_X(x^n) = \prod_{a\in\mathcal{X}} \mathrm{P}_X(a)^{N(a,x^n)}$.
   b) Show that if $x^n \in \cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}$, then $\log \frac{1}{\mathrm{P}_X(x^n)} \leq n(1 + \epsilon)H(\mathrm{P}_X)$. [Hint: If $x^n \in \cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}$, then $N(a, x^n)$ is bounded on both sides for all $a \in \mathcal{X}$ by definition of $A_{\epsilon,a}^{(n)}$. Use that bound.]
   c) Show that $|\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}| \leq 2^{n(1+\epsilon)H(\mathrm{P}_X)}$.[1] [Hint: This is very similar to the AEP proof. Start with $\mathrm{P}_X(\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}) \leq 1$, and then use the result from part (b).]

[1+3+3=7 points]

3) Consider a sequence $a_n, n \geq 1$, and define $\liminf_{n\to\infty} a_n = \sup_{k\geq 1} \inf_{n\geq k} a_n$. Show that $\liminf_{n\to\infty} a_n \geq a$ if and only if for every $\epsilon > 0$, the sequence a_n satisfies $a_n \geq_n a - \epsilon$.

[7 points]

4) Consider a complete binary tree of depth d. Now, consider a random path from root to a leaf of this tree as follows. At every node, choose to go to its left child (marked with 0) with probability p, and go to its right child (marked with 1) with probability $1 - p$.

   a) What is the probability of reaching a leaf, path to which contains exactly i zeros?
   b) What is the number of leaves, the path to whom contains exactly i zeros? [Hint: Argue that the set of leaves is 'same' as the set of binary strings of length d.]

[1] For those who are curious, not related to exam. This method gives an alternative proof for the achievability part of the lossless source coding theorem. In this proof, you will be encoding only the sequences from $\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}$, and by part (c), this requires $k(n) = n(1 + \epsilon)H(\mathrm{P}_X)$ bits. Hence, we get a rate of $(1 + \epsilon)H(\mathrm{P}_X)$. Also, this is a good encoding since the probability of error goes down to zero. This follows from a practice problem, where we showed that $\mathrm{P}_X(\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}) \to 1$ as $n \to \infty$. This is a better proof in the following sense. Here, we can use concentration inequalities to further show that the probability of error $\mathrm{P}_X((\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)})^c) \leq 2^{-nC_n}$, for some $C_n > 0$, thereby showing that the probability of error decays exponentially fast. Due to this stronger result, the set $\cap_{a\in\mathcal{X}} A_{\epsilon,a}^{(n)}$ is referred to as a strong typical set.

c) Let Y be a random variable denoting the leaf that was reached by this random path. Find $H(Y)$.

[2+2.5+2.5=7 points]

5) Consider a *symmetric* binary matrix $L \in \{0,1\}^{n\times n}$, whose diagonal elements are all 1.

a) Show that with binary additions and multiplications, for any $x \in \{0,1\}^n$, we have $x^T L x = \sum_{i:x_i=1} L_{i,i} + \sum_{\substack{i\neq j:\\ x_i=x_j=1}} L_{i,j}$.

b) Argue using the previous part that because L is symmetric and has ones in the diagonal, $x^T L x = 0$ if x has an even number of ones, and $x^T L x = 1$ if x has an odd number of ones.

c) Now, assume $Y = X^T L X$, where X is a vector of n i.i.d. uniform bits. What is $H(Y)$? [Hint: P$(Y = 0) =$ P$(X$ has even number of zeros$)$ by the previous part.]

[2.5+2.5+2=7 points]

6) [DIFFICULT. Question meant for A, A+ separation.] Prove Shearer's lemma. Let $[n] \triangleq \{0, 1, \ldots, n\}$, and $S_1, S_2, \ldots, S_k \subseteq [n]$ such that each $i \in [n]$ appears in at least t of these k subsets $S_1, \ldots, S_k$. Show that $H(X_{[n]}) \leq \frac{1}{t}\sum_{i=1}^k H(X_{S_i})$, where $X_A \triangleq (X_i : i \in A)$ for every $i \in A$. [Hint: You start by noting that $\sum_{i=1}^k H(S_i) = \sum_{i=1}^k \sum_{j\in S_i} H(X_j|X_{\{l<j:l\in S_i\}}) \geq \sum_{i=1}^k \sum_{j\in S_i} H(X_j|X_{\{l<j\}})$.]

[7 points]