ECE 634/CSE 646 InT: Practice Problems 4

Instructor: Manuj Mukherjee

1) Let X, Y be i.i.d. random variables taking values in $\mathbb{Z}$ with pmf $\mathrm{P}(n) = \frac{1}{2}(1-p)^{|n|-1}p$, for all $n \neq 0$, and $\mathrm{P}(0) = 0$. Let Z be a random variable whose conditional distribution given $X = x, Y = y$ is $N(\mu(x,y), K(x,y))$, where $\mu(x,y) = [x,y]^T$, and $K = \begin{bmatrix} 2^x & 0 \\ 0 & 2^y \end{bmatrix}$. Find $h(Z|X)$.

2) [*Le Cam's method*] Consider a family of distributions $\{\mathrm{P}_\theta : \theta \in \Theta\}$ and suppose $Y \sim \mathrm{P}_\theta$. An estimator of θ is defined to be any function $f : \mathcal{Y} \to \Theta$, where $\mathcal{Y}$ is the range of Y. Define the minmax risk of an estimator by $R_\Theta \triangleq \inf_{f:\mathcal{Y}\to\Theta} \sup_{\theta\in\Theta} \mathrm{P}_\theta(f(Y) \neq \theta)$.
   a) Argue that for any $\theta_1 \neq \theta_2$, $R_\Theta \geq \inf_f \max_{\theta\in\{\theta_1,\theta_2\}} \mathrm{P}_\theta(f(Y) \neq \theta)$.
   b) Use the fact that data processing inequality for relative entopies hold for arbitrary distributions.[1] Show that

$$\inf_f \max_{\theta\in\{\theta_1,\theta_2\}} \mathrm{P}_\theta(f(Y) \neq \theta) \geq \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\ln 2}{2} D(\mathrm{P}_{\theta_1}||\mathrm{P}_{\theta_2})}.$$

   [Hint: Use facts like $\max\{\mathrm{P}_{\theta_1}(f(Y) = \theta_2), 1 - \mathrm{P}_{\theta_2}(f(Y) = \theta_2)\} = \max\{2\mathrm{P}_{\theta_1}(f(Y) = \theta_2), 1 - \mathrm{P}_{\theta_2}(f(Y) = \theta_2) + \mathrm{P}_{\theta_1}(f(Y) = \theta_2)\} - \mathrm{P}_{\theta_1}(f(Y) = \theta_2)$ and Pinsker's inequality.]
   c) Now, consider the channel sensing problem where we have the following two hypothesis H_0 or there is no signal in the channel and H_1 or there is some signal in the channel. Under H_0 the channel outputs $Y \sim N(0, \sigma^2)$, while under H_1 the channel outputs $Y \sim N(\theta, \sigma^2)$ for some $\theta \geq \sqrt{P}$.[2] A channel sensing scheme is a hypothesis test $f : \mathbb{R} \to \{0,1\}$ and its minmax risk can be defined as $R(P, \sigma^2) \triangleq \inf_f \sup_{\theta\geq\sqrt{P}} \max\{\mathrm{P}_\theta(f(Y) = 0), \mathrm{P}_0(f(Y) = 1)\}$. Using the Le Cam's method, show that $R(P, \sigma^2) \geq \frac{P}{2\sigma^2}\log e$.

3) Consider the cascading of two channels with capacities C_1 and C_2. Show that capacity C of the cascaded channel satisfies $C \leq \min\{C_1, C_2\}$.

Answers

1. $\log(2\pi e)$.

[1]We didn't show this in class. We showed only the discrete case. However, this can be proved in exactly the same way by using the most general definition of relative entropy using Radon-Nikodym derivatives instead of pmfs.
[2]Basically H_0 implies receiver sees pure noise, whereas H_1 means the receiver sees noise added to some signal of power at least P.