MTH-204: Worksheet 3 (Solutions)

10 points

(1 pt.)

Q-1 Consider $(2x+3y)\,dx + (3x+4y)\,dy = 0$

$$M = 2x+3y \qquad N = 3x+4y$$

$$\Rightarrow \frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$$

$\therefore$ ODE is exact

Now, $u(x,y) = \int M\,dx + p(y) = x^2+3xy+p(y)$

$$\Rightarrow \frac{\partial u}{\partial y} = 3x + p'(y) = N \Rightarrow 3x+p'(y) = 3x+4y$$

$$\Rightarrow p'(y) = 4y$$

$$\Rightarrow p(y) = 2y^2 + k \quad \text{f.s. constant } k$$

$$\therefore u(x,y) = x^2+3xy+2y^2+k = C_1$$

$$\Rightarrow \boxed{x^2+3xy+2y^2 = C}$$

(2 pts.)

Q-2 Let $y(t)$ be number of infected persons at time t. Then

$$\frac{dy}{dt} \propto y(1-y) \Rightarrow \frac{dy}{dt} = ky(1-y), \; k>0 \; \left(\because \text{No. of infected persons are increasing}\right)$$

$$\Rightarrow \frac{dy}{dt} = ky - ky^2$$

Equilibrium Solutions: $ky(1-y) = 0$

$\Rightarrow y=0$ & $y=1$ are equilibrium sol's

Now, $f(y) \begin{cases} >0 & , 0<y<1 \\ <0 & , y<0 \\ <0 & , y>1 \end{cases}$, where $f(y) = ky(1-y) = ky - ky^2$

$$\Rightarrow f'(y) = k - 2ky$$

$$\Rightarrow f'(0) = k > 0 \Rightarrow 0 \text{ is unstable}$$

$$\& \; f'(1) = -k < 0 \Rightarrow 1 \text{ is stable.}$$

Now, $\frac{dy}{dt} = ky - ky^2$

Comparing it with Logistic equations $y' = Ay - By^2$, we have

$$\boxed{y(t) = \frac{1}{1+Ce^{-kt}}} \Rightarrow y(t) \to 1 \text{ as } t \to \infty$$

Hence, eventually everybody in the population will be infected.

(2 pts.)

Q-3 Let $y(t)$ be the population of fishes at time t, Then

$$y'(t) = Ay - By^2 - Hy \quad \text{[Schaefer Model]}$$
$$= (A-H)y - By^2$$

So, again by comparing with Logistic equation, we have

$$\boxed{y(t) = \frac{1}{Ce^{-(A-H)t} + \frac{B}{A-H}}}$$

Equilibrium solutions are: $(A-H)y - By^2 = 0$

$$\Rightarrow y[A-H-By] = 0$$

$$\Rightarrow y_1 = 0 \text{ and } y_2 = \frac{A-H}{B} \ (>0) \ (\because H < A)$$

$$f(y) = (A-H)y - By^2 \Rightarrow f'(y) = A-H-2By$$

$$\Rightarrow f'\left(\frac{A-H}{B}\right) = A-H - 2\cancel{B} \times \frac{A-H}{\cancel{B}} = -A+H < 0$$

$\Rightarrow y_2$ is stable

$\therefore$ Population y_2 remains unchanged under harvesting.

$\Rightarrow$ The fraction Hy_2 of y_2 can be harvested indefinitely, hence Hy_2 is the equilibrium harvest.

(1 pt.)

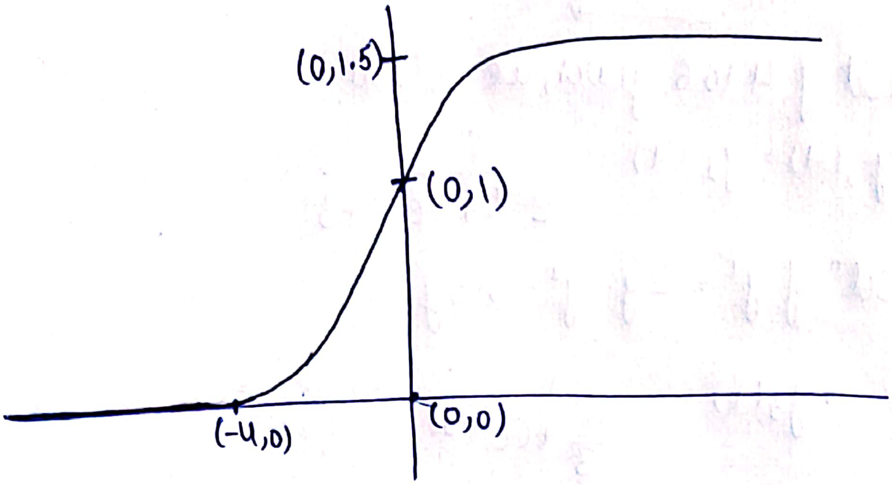
Q-4 By problem 3, $y(t) = \frac{1}{Ce^{-(A-H)t} + \frac{B}{A-H}}$

Given $A=2$, $B=1$ and $H=0.5$, $y(0)=1$

$$\Rightarrow y(t) = \frac{1}{Ce^{-(2-0.5)t} + \frac{1}{2-0.5}} = \frac{1}{\frac{2}{3} + Ce^{-1.5t}}$$

$$y(0)=1 \Rightarrow 1 = \frac{1}{C+\frac{2}{3}} \Rightarrow C+\frac{2}{3}=1 \Rightarrow C=\frac{1}{3}$$

$$\therefore\ y(t) = \frac{1}{\frac{2}{3}+\frac{e^{-1.5t}}{3}} = \frac{3}{2+e^{-1.5t}} \longrightarrow \frac{3}{2} \text{ as } t\to\infty$$



If there were no fishing, then $H=0 \Rightarrow y'(t)=2y-y^2$

$$\Rightarrow y(t) = \frac{1}{Ce^{-2t}+\frac{1}{2}} = \frac{2}{2Ce^{-2t}+1} \longrightarrow 2 \text{ as } t\to\infty$$

(2 pts.)

Q-5 By problem 4, for first 2 years we have the solution

$$y_1(t) = \frac{3}{2+e^{-1.5t}}$$

$$\Rightarrow y_1(2) = \frac{3}{2+e^{-3}}$$

Now, by continuity, $y_1(3)$ at the end of the first period is the initial value for the solution y_2 during the next period, i.e.

$$y_2(2) = y_1(2) = \frac{3}{2+e^{-3}}$$

Now, y_2 is the solution of $y'=2y-y^2$ (no fishing during this period)

$$\Rightarrow y_2(t) = \frac{2}{2Ce^{-2t}+1} \Rightarrow y_2(2) = \frac{2}{2Ce^{-4}+1} = \frac{3}{2+e^{-3}}$$

$$\Rightarrow 4+2e^{-3} = 6Ce^{-4}+3 \Rightarrow 6Ce^{-4} = 1+2e^{-3}$$

$$\Rightarrow C = \frac{e^4}{6}+\frac{e}{3}$$

$$\Rightarrow y_2(t) = \frac{2}{2\left(\frac{e^4}{6}+\frac{e}{3}\right)e^{-2t}+1} = \frac{6}{e^{4-2t}+2e^{1-2t}+3}$$

$$\Rightarrow y_2(2) = \frac{6}{1+2e^{-3}+3} = \frac{6}{4+2e^{-3}} = \frac{3}{2+e^{-3}}$$

Hence, verified

For the period of 4 to 6 years, we obtain

$$y_3(4) = y_2(4) = \frac{6}{e^{-4}+2e^{-7}+3}$$

Now, y_3 is solⁿ of $y' = 2y - y^2 - 0.5y$

$$\Rightarrow y_3(t) = \frac{1}{\frac{2}{3}+Ce^{-1.5t}}$$

$$\Rightarrow y_3(4) = \frac{3}{2+3Ce^{-6}} = \frac{6}{e^{-4}+2e^{-7}+3}$$

$$\Rightarrow e^{-4}+2e^{-7}+3 = 4+6Ce^{-6}$$

$$\Rightarrow 6Ce^{-6} = e^{-4}+2e^{-7}-1$$

$$\Rightarrow C = \frac{e^2}{6}+\frac{e^{-1}}{3}-\frac{e^6}{6}$$

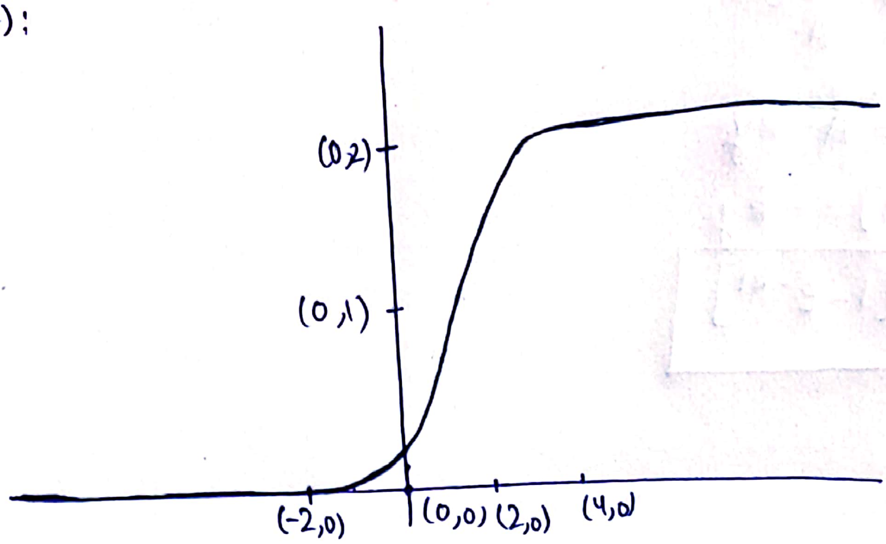
$$\Rightarrow y_3(t) = \frac{1}{\frac{2}{3}+\frac{1}{6}(e^2+2e^{-1}-e^6)e^{-1.5t}} = \frac{6}{4+e^{2-1.5t}+2e^{-1-1.5t}-e^{6-1.5t}}$$

$$\Rightarrow y_3(4) = \frac{6}{4+e^{-4}+2e^{-7}-1} = \frac{6}{e^{-4}+2e^{-7}+3}$$

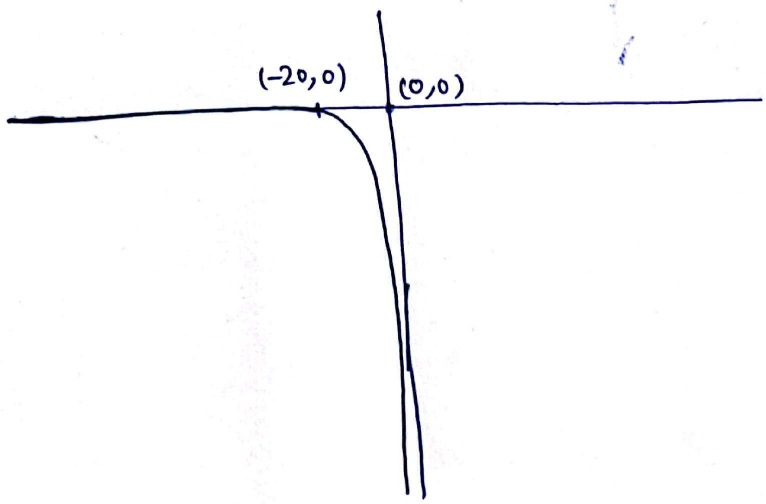
Hence, verified

Graphs: $y_1(t)$: Same as $y(t)$ of problem 4

$y_2(t)$:



$y_3(t)$:



(2pts.)

Q-6 Let $y(t)$ be the amount of the drug present at time t.

Then, $y'(t) = A - ky$ ($\because$ Drug removed at time $t = ky(t)$)

$$\Rightarrow \frac{dy}{dt} = A - ky$$

$$\Rightarrow \frac{dy}{A-ky} = dt$$

$w = A - ky$
$\Rightarrow dw = -k\,dy$
$\Rightarrow dy = -\frac{dw}{k}$

$$\Rightarrow \frac{-dw}{kw} = dt$$

$$\Rightarrow -\log w = kt + \log c$$

$$\Rightarrow -\log(A-ky) = kt + \log c$$

$$\Rightarrow \frac{1}{A-ky} = Ce^{kt}$$

Given $y(0) = 0 \Rightarrow \frac{1}{A} = C$

$$\therefore \quad \frac{1}{A-ky} = \frac{1}{A} e^{kt}$$

$$\Rightarrow A e^{-kt} = A - ky$$

$$\Rightarrow A[e^{-kt} - 1] = -ky$$

$$\Rightarrow \boxed{y(t) = \frac{A}{k}[1 - e^{-kt}]}$$