SOLUTION

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MTH210 – SUBMISSION_20220929

TIME: 17.5 minutes MARKS: 5

No consultation – open notes – books and internet not allowed.

For $n \in \mathbb{Z}^+$, put $X_n = \{1,2,\ldots,n\}$. Put $V = \{0,1\}$, and finally, put V_n = Cartesian product of n copies of V, i.e. V_n is the set of all ordered n-tuples with 0-1 entries.

Construct a partial order R on V_n such that $<V_n, R>$ is isomorphic to $<\mathbb{P}(X_n), \subseteq>$.

Note: *You must firstly define the relation R, secondly show that it is a partial order, and thirdly prove the isomorphism.*

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Firstly, for $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n) \in V_n$, define $\bar{a} \, R \, \bar{b}$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \ldots, n$. For V, we use the standard partial ordering $\leq$, i.e. $0 \leq 1$.

Clearly, R is a well-defined relation on V_n.

Secondly, to show R is a partial ordering:-

- Reflexive Property:- For $\bar{a} \in V_n$, $a_i \leq a_i$ for all $i = 1, 2, \ldots, n$, so $\bar{a} R \bar{a}$ holds.
- Anti-symmetric Property: Suppose $\bar{a} R \bar{b}$ and $\bar{b} R \bar{a}$. Then:

$a_i \leq b_i$ for $i = 1, \ldots, n$ ①

and $b_i \leq a_i$ for $i = 1, \ldots, n$ ②

But then $a_i = b_i$ for all i, and so $\bar{a} = \bar{b}$.

Cont'd

- Transitive Property: suppose $\bar{a} R \bar{b}$ and $\bar{b} R \bar{c}$, where $\bar{c} = (c_1, \dots, c_n) \in V_n$.

Then $\left.\begin{array}{l} a_i \leq b_i \\ \text{and } b_i \leq c_i \end{array}\right\}$ for all $i = 1, \dots, n$

$\therefore a_i \leq c_i$ for all i, by the transitive property for $\langle V_{~~n~~}, \leq \rangle$.

$\therefore \bar{a} R \bar{c}$ as required.

Thirdly, to show the isomorphism of $\langle P(X_n), \subseteq \rangle$ and $\langle V_n, R \rangle$. Note that $|P(X_n)| = 2^n = |V_n|$ so the first requirement of an isomorphism is fulfilled.
Define a mapping (function) $\Psi : P(X_n) \to V_n$ by $\Psi(A) = \bar{a}$ ~~s~~such that

$a_i = 1 \;\{$ if $i \in A \}$
and $a_i = 0 \;\{$ if $i \notin A \}$, for all $A \subseteq X_n$.

We show that Ψ is injective:- WOLOG, if ~~?~~ $A \neq B$, there exists some i s.t. $i \in A$ and $i \notin B$. Then $a_i = 1$ and $b_i = 0$, where $\bar{a} = \Psi(A)$ and $\bar{b} = ~~?~~ \Psi(B)$. $\therefore \bar{a} \neq \bar{b}$, i.e. $\Psi(A) \neq \Psi(B)$.
Since Ψ is an injective mapping between two finite sets with an equal number of elements, Ψ is also surjective, i.e. Ψ is bijective.
Finally, to show that Ψ is order-preserving, i.e. an isomorphism, suppose $A, B \in P(X)$ with $A \subseteq B$.
Then, if $i \in A$, $i \in B$ also, i.e. $a_i = 1 = b_i$.
If $i \notin A$, $a_i = 0 \leq b_i$. $\therefore$ ~~???~~ $\bar{a} R \bar{b}$, as required.