# ECE 351 DSP: Mid Semester Examination

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**Total:** 30 points

**Instruction:** You need to answer all the questions from 1-6. The bonus question in the end carries no points, but you are strongly encouraged to attempt it after finishing all the regular questions. You can directly use any of the formulas attached with the question paper, or any other formulas covered in class. If you use anything else, you need to derive it.

1) Which of the regions shown in Figure 1 can be the region of convergence (ROC) of some z-transform?

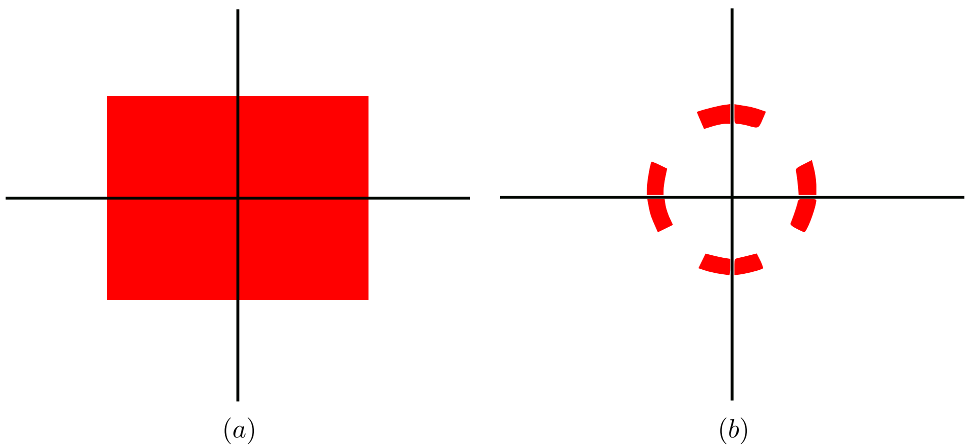

Fig. 1: Figure for question 1

[2 points]

2) Consider the signal $x[n] = 2\cos(\frac{\pi}{4}n) + \cos(\frac{\pi}{2}n) + \frac{1}{2}\cos(\frac{3\pi}{4}n)$. What is its fundamental period?

[3 points]

3) Consider the IIR low pass filter with cutoff frequency $\frac{\pi}{3}$. What is the output of the filter when the input is $x[n] = 2\cos(\frac{\pi}{2}n - \frac{\pi}{4})$?

[5 points]

4) Consider a notch filter with notch frequency $\frac{\pi}{3}$ and 3-dB bandwidth $\frac{\pi}{6}$.

   a) What is its transfer function?
   b) Show its implementations with adders, delays, and multipliers.

c) Now, suppose we design a comb filter with $L = 3$ using this notch filter. Within $[-\pi, \pi]$, where are the notches of the comb filter located?

[3+4+3=10]

5) Consider the system with the transfer function $H(z) = \frac{(1-2z^{-1})(1-3z^{-1})(1-\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}e^{-j\frac{\pi}{4}}z^{-1})(1-\frac{1}{4}e^{j\frac{\pi}{4}}z^{-1})}$.

a) What kind (min,max, or mixed) phase system is this?
b) Construct a min-phase system having the same amplitude response as the above system, but a lower group delay.

[1+4=5 points]

6) Find the group delay of the system shown in Figure 2.

[5 points]

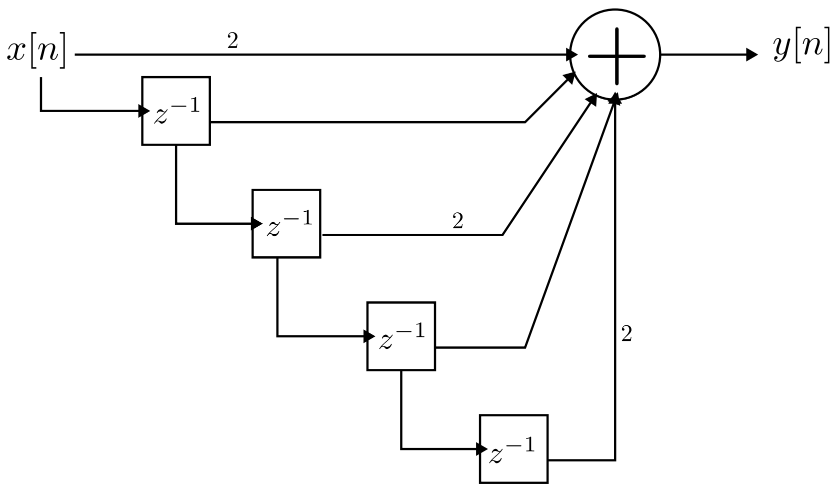


Fig. 2: Figure for question 5

7) For what positive values of $p$ is the system with impulse response $h[n] = \binom{2n}{n} p^n u[n]$ stable? Here $\binom{m}{r}$ denotes $m$ choose $r$.

[**Bonus Question**]

# List of formulas

## Discrete time Fourier series

1) If $x[n]$ is periodic with period $N$ then,

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn},$$

where

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}.$$

## Discrete time Fourier transform

1) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$.
2) Time delay: $x[n-k] \longleftrightarrow e^{-j\omega k}X(\omega)$.
3) Symmetry: $X(\omega) = X^*(-\omega)$ if $x[n]$ is real.
4) Frequency shift: $e^{j\omega_0 n}x[n] \longleftrightarrow X(\omega - \omega_0)$.
5) Modulation: $x[n]\cos(\omega_0 n) \longleftrightarrow \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$.
6) Differentiation: $nx[n] \longleftrightarrow j\frac{dX(\omega)}{d\omega}$.
7) DTFT of Sinc: $\frac{\omega_0}{\pi}\mathrm{sinc}(n\omega_0) \longleftrightarrow \mathrm{rect}(\frac{\omega}{2\omega_0})$.
8) System $H(\omega)$ with exponential input $e^{j\omega_0 n}$: Output $H(\omega_0)e^{j\omega_0 n}$.
9) System $H(\omega)$ with cosine input $\cos(\omega_0 n + \theta)$: Output $|H(\omega_0)|\cos(\omega_0 n + \theta + \angle H(\omega_0))$.

## Z-transform properties

1) Time shift: $x[n-k] \longleftrightarrow z^{-k}X(z)$.
2) Z-scaling: $a^n x[n] \longleftrightarrow X(a^{-1}z)$.
3) Z-differentiation: $nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}$.
4) Initial value theorem: For causal $x[n]$, $x[0] = \lim_{z\to\infty} X(z)$.

## Z-transform of common signals

1) $\delta[n] \longleftrightarrow 1$.
2) $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}$.
3) $na^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$.
4) $a^n e^{j\omega_0 n}u[n] \longleftrightarrow \frac{1}{1-ae^{j\omega_0}z^{-1}}$.
5) $a^n\cos(\omega_0 n)u[n] \longleftrightarrow \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$.
6) $a^n\sin(\omega_0 n)u[n] \longleftrightarrow \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$.

## FORMULAS FOR PARTIAL FRACTION EXPANSION COEFFICIENT FOR RATIONAL Z-TRANSFORM

1) Coefficient for single-pole at $p$ is $(z-p)\frac{X(z)}{z}|_{z=p}$.
2) Coefficient for the term $\frac{1}{(z-p)^i}$, $i<k$, for a pole of multiplicity $k$ at $p$ is $\frac{d^{k-i}}{dz^{k-i}}\left(\frac{(z-p)^k}{(k-i)!}\frac{X(z)}{z}\right)|_{z=p}$.
3) Coefficient for the term $\frac{1}{(z-p)^k}$ for a pole of multiplicity $k$ at $p$ is $(z-p)^k\frac{X(z)}{z}|_{z=p}$.
4) $\frac{n(n-1)\ldots(n-i+2)}{(i-1)!}p^{n-i+1}u[n-i+2] \longleftrightarrow \frac{z}{(z-p)^i}$.

## FORMULAS FOR FILTERS

1) **FIR Low Pass Filter**: $H(z) = \frac{1}{2^M}(1+z^{-1})^M$.

   Cutoff frequency $2\cos^{-1}(2^{-\frac{1}{2M}})$.

2) **FIR High Pass Filter**: $H(z) = \frac{1}{2^M}(1-z^{-1})^M$.

   Cutoff frequency $2\sin^{-1}(2^{-\frac{1}{2M}})$.

3) **IIR Low Pass Filter**: $H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1$.

   Cutoff frequency $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

4) **IIR High Pass Filter**: $H(z) = \frac{1-\alpha}{2}\frac{1-z^{-1}}{1+\alpha z^{-1}}, |\alpha| < 1$.

   Cutoff frequency $\pi - \cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

5) **IIR Bandpass Filter**: $H(z) = \frac{1-\alpha}{2}\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, |\alpha| < 1, |\beta| < 1$.

   3-dB bandwidth $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

   Centre frequency $\cos^{-1}\beta$.

6) **IIR Notch Filter**: $H(z) = \frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, |\alpha| < 1, |\beta| < 1$.

   3-dB bandwidth $\cos^{-1}(\frac{2\alpha}{1+\alpha^2})$.

   Notch frequency $\cos^{-1}\beta$.

7) **Linear phase systems** with $h[n] = 0$, if $n < 0$, and $n \geq N$.

   Phase response: $-\omega(\frac{N-1}{2}) + \pi\mathbf{1}\{H(\omega) < 0\}$.

   Group delay: $\frac{N-1}{2}$.