ECE 351 DSP: Mid Semester Examination

Instructor: Manuj Mukherjee

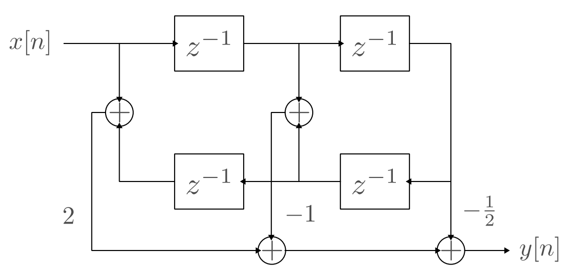
Total: 30 points

Instruction: You need to answer all the questions from 1-5. The bonus question 6 in the end carries no points, but you are strongly encouraged to attempt it after finishing all the regular questions.

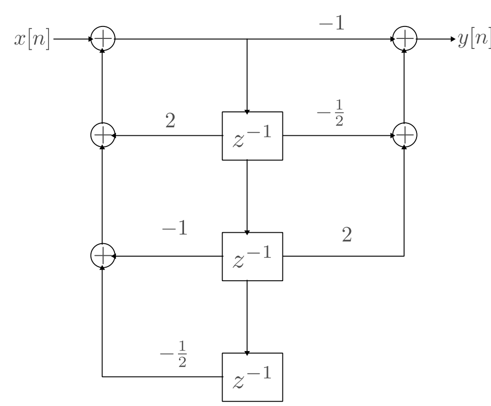
You can directly use any of the formulas attached with the question paper, or any other formulas covered in class. If you use anything else, you need to derive it.

All z-transforms are to be assumed to correspond to causal signals, unless stated otherwise.

1) Find the number of poles and zeros of the following systems. Recall that poles and zeros are respectively the roots of the denominator and the numerator polynomials of $H(z)$.
   a) The system shown in Figure 1a.
   b) The system shown in Figure 1b.



(a) Figure for Q.1a



(b) Figure for Q.1b

[1.5+1.5=3 points]

2) Consider the lattice-ladder representation shown in Figure 2.
   a) Find $H(z)$.
   b) Now, consider an input $x[n] = 3\cos(\pi n + \frac{\pi}{3}) - 2\sin(2\pi n)$ being fed to this system. What will be the output?

[8+4=12 points]

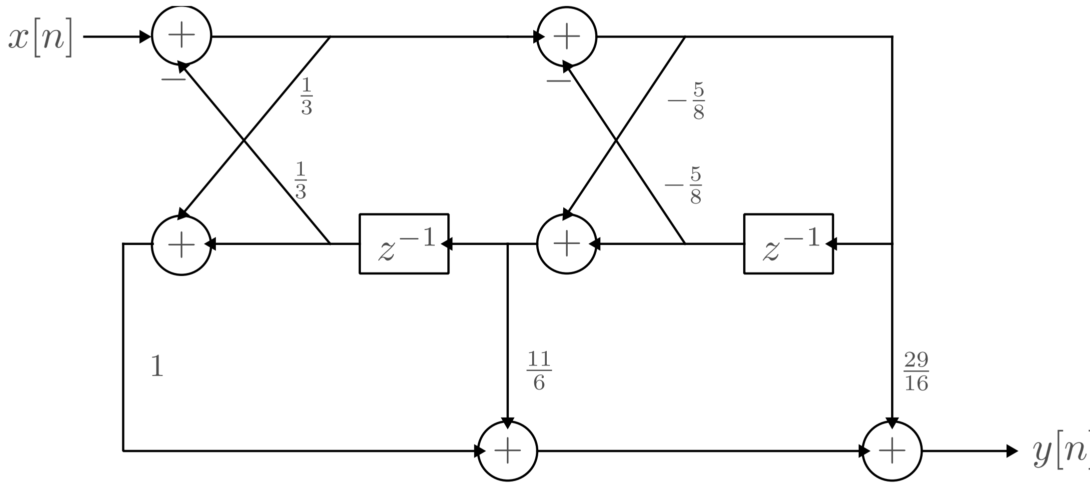


Fig. 2: Figure for Q.2

3) Consider the system given in Figure 3.

   a) Find $h[n]$.

   b) Let $\sum_{k=-\infty}^{\infty} \delta[n-3k]$ be the input to this system. Let $y[n]$ be the output. Calculate $y[n]$ for $n = 0, 1, 2, 3$.

   c) Let $Y(0), Y(1), Y(2), Y(3)$ be the 4-point DFT of $y[0], y[1], y[2], y[3]$. Compute $Y(2)$.

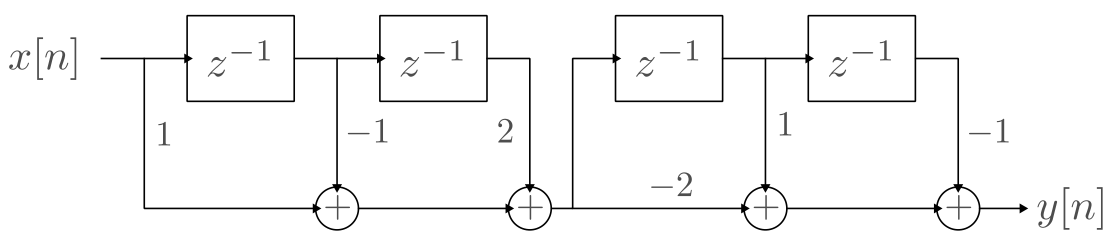


Fig. 3: Figure for Q.3

[2+2+2=6 points]

4) Consider the signal flow graph in Figure 4.

   a) Obtain its transpose-form representation.

   b) What is $h[n]$?

   c) Is the system stable?

[3+2+1=6 points]

5) Consider an FIR system with $h[n] \neq 0$ only if $n = 0, 1, 2, \ldots, M-1$.

   a) Let $x[n]$ be an input which is bounded. More precisely, let $B \geq |x[n]|$ for all n. Let $\max_{0 \leq n \leq M-1} |h[n]| = \eta$. Then, show that $|y[n]| \leq \eta M B$ for all n.

   b) Can FIR systems be unstable?

[2+1=3 points]

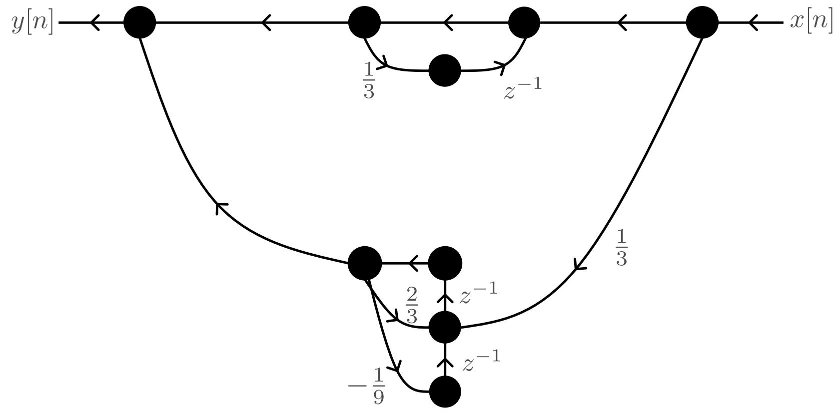


Fig. 4: Figure for Q.4

6) [**Bonus question**] Suppose $N = 9$ and you are using a radix-3 decimation in time algorithm to compute the DFT of a 9-point sequence $x[n]$. What is the order of the inputs in the butterfly diagram from top to bottom?

List of formulas

Convergence tests

1) *Ratio Test* for the series $\sum_{n=0}^{\infty} a[n]$. Let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If $L < 1$ the series converges, and if $L > 1$ the series diverges.
2) *Root Test* for the series $\sum_{n=0}^{\infty} a[n]$. Let $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. If $L < 1$ the series converges, and if $L > 1$ the series diverges.

Discrete time Fourier transform

1) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$.
2) Time delay: $x[n-k] \longleftrightarrow e^{-j\omega k}X(\omega)$.
3) Symmetry: $X(\omega) = X^*(-\omega)$ if $x[n]$ is real.
4) Frequency shift: $e^{j\omega_0 n}x[n] \longleftrightarrow X(\omega - \omega_0)$.
5) Modulation: $x[n]\cos(\omega_0 n) \longleftrightarrow \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$.
6) Differentiation: $nx[n] \longleftrightarrow j\frac{dX(\omega)}{d\omega}$.
7) DTFT of Sinc: $\frac{\omega_0}{\pi}\text{sinc}(n\omega_0) \longleftrightarrow \text{rect}(\frac{\omega}{2\omega_0})$.
8) System $H(\omega)$ with exponential input $e^{j\omega_0 n}$: Output $H(\omega_0)e^{j\omega_0 n}$.
9) System $H(\omega)$ with cosine input $\cos(\omega_0 n + \theta)$: Output $|H(\omega_0)|\cos(\omega_0 n + \theta + \angle H(\omega_0))$.

Z-transform properties

1) Time shift: $x[n-k] \longleftrightarrow z^{-k}X(z)$.
2) Z-scaling: $a^n x[n] \longleftrightarrow X(a^{-1}z)$.
3) Z-differentiation: $nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}$.
4) Initial value theorem: For causal $x[n]$, $x[0] = \lim_{z\to\infty} X(z)$.

Z-transform of common signals

1) $\delta[n] \longleftrightarrow 1$, ROC: $\mathbb{C}$.
2) $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}$, ROC: $|z| > |a|$.
3) $na^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$, ROC: $|z| > |a|$.
4) $a^n e^{j\omega_0 n}u[n] \longleftrightarrow \frac{1}{1-ae^{j\omega_0}z^{-1}}$, ROC: $|z| > |a|$.
5) $a^n\cos(\omega_0 n)u[n] \longleftrightarrow \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$, ROC: $|z| > |a|$.
6) $a^n\sin(\omega_0 n)u[n] \longleftrightarrow \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$, ROC: $|z| > |a|$.

Formulas for partial fraction expansion coefficient for rational Z-transform

1) Coefficient for single-pole at p is $(z-p)\frac{X(z)}{z}|_{z=p}$.
2) Coefficient for the term $\frac{1}{(z-p)^i}$, $i < k$, for a pole of multiplicity k at p is $\frac{d^{k-i}}{dz^{k-i}}\left(\frac{(z-p)^k}{(k-i)!}\frac{X(z)}{z}\right)|_{z=p}$.

3) Coefficient for the term $\frac{1}{(z-p)^k}$ for a pole of multiplicity k at p is $(z-p)^k \frac{X(z)}{z}|_{z=p}$.

4) $\frac{n(n-1)\ldots(n-i+2)}{(i-1)!} p^{n-i+1} u[n-i+2] \longleftrightarrow \frac{z}{(z-p)^i}$.

DISCRETE FOURIER TRANSFORM

1) Let $Y(k) = X(\frac{2\pi k}{N}), k = 0, 1, \ldots, N-1$, where $X(\omega)$ is the DTFT of $x[n]$. Then, $Y(0), Y(1), \ldots, Y(N-1)$, are the N-point DFT of $x_p[0], x_p[1], \ldots, x_p[N-1]$, where $x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$.
2) Circular convolution: $\sum_{m=0}^{N-1} x_1[m] x_2[(n-m)_N] \longleftrightarrow X_1(k) X_2(k)$.
3) Time reversal: $x[(-n)_N] \longleftrightarrow X(-(k)_N)$.
4) Circular shift: $x[(n-l)_N] \longleftrightarrow X(k) e^{-j\frac{2\pi}{N}kl}$.
5) Circular frequency shift: $x[n] e^{j\frac{2\pi}{N}ln} \longleftrightarrow X((k-l)_N)$.
6) Complex conjugate: $x^*[n] \longleftrightarrow X^*((-k)_N)$.
7) Time multiplication: $x_1[n] x_2[n] \longleftrightarrow \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) X_2((k-l)_N)$.

LATTICE-LADDER FORM

1) Lattice recursions:
   a) $A_i(z) = A_{i-1}(z) + K_i z^{-1} B_{i-1}(z)$.
   b) $B_i(z) = z^{-i} A_i(z^{-1})$.
   c) $A_0(z) = B_0(z) = 1$.
   d) $A_{i-1}(z) = \frac{A_i(z) - K_i B_i(z)}{1 - K_i^2}$.
   e) $\alpha_i[0] = \beta_i[i] = 1$.
   f) $\alpha_i[i] = \beta_i[0] = K_i$.
   g) $\alpha_i[k] = \beta_i[i-k]$ for all $0 \leq k \leq i$.
2) Ladder recursions:
   a) $C_i(z) = C_{i-1}(z) + v_i B_i(z)$.
   b) $v_i = c_i[i]$.