MTH 201: Probability and Statistics

Quiz 1

28/03/2023

Sanjit K. Kaul

No books, notes, or devices are allowed. Just a pen and eraser. Any exchange of information related to the quiz with a human or machine will be deemed as cheating. Institute rules will apply. Explain your answers. Show your steps. Approximate calculations are fine as long as the approximations are reasonable. Don't bother simplifying products and sums that are too time consuming. Leave your answer as products and sums, if you don't have the time. You have 50 minutes.

Question 1. 40 **marks** Data suggests that 10% of Indians have a B. Tech degree. We will call those with the degree engineers. Amongst the engineers, about 10% get a high-tech job. The kind of job an engineer gets is independent of jobs obtained by others. Answer the following questions.

(a) (8 marks) Derive the probability that there are at least two engineers in a room of 50 people. Explain your steps.

(b) Derive the probability that there are exactly two engineers with a high-tech job in a room of 50 people. Explain your steps.

(c) Derive the probability that in a randomly chosen pair of engineers, both the engineers have high-tech jobs.

(d) (20 marks) Suppose a room has 6 engineers of which 3 have high-tech jobs. We will create two pairs in the following manner. Randomly and without replacement choose an engineer as the first member of the first pair. If the chosen engineer has a high-tech job, choose without replacement the second member of the first pair to be one with a high-tech job with probability 0.8. If the first member doesn't have a high-tech job, the second member is chosen randomly and without replacement from the unpaired engineers.

To create the second pair, choose the first member of the pair randomly (and without replacement) from those that remain. Then, in case both types of engineers still remain unpaired, choose without replacement the second member of the second pair to be one with a high-tech job with probability 0.6 and one without a high-tech job otherwise. If one type of engineer had remained unpaired, randomly choose one of the remaining engineers as the second member of the second pair.

Calculate the probability that the two pairs together have three high-tech engineers. Draw a tree diagram to help you do so. Your tree diagram wants to record the type of engineer chosen at different stages of pairing.

Question 2. 60 **marks** A random heart patient has a *serious* heart ailment with probability 0.2 and a *moderate* heart ailment with probability 0.8. We have two kinds of surgeons, those that are risk taking (RT) and those that are risk averse (RA). A serious patient approaches a RT with probability 0.8 and otherwise approaches a RA. A patient with a moderate ailment approaches with equal probability a RA or a RT.

A serious patient who approaches a RT is accepted by the RT with probability 0.8 and with probability 0.2 the patient remains without a surgeon. Once accepted by a RT, a serious patient has a probability 0.6 of a successful surgery and 0.4 of a failed surgery.

A serious patient who approaches a RA is accepted by the RA with probability 0.3 and with probability 0.7 the patient remains without a surgeon. Once accepted by a RA, a serious patient has a probability 0.6 of a successful surgery and 0.4 of a failed surgery.

A patient with moderate ailment is accepted by whichever doctor the patient approaches with probability 1. Also, once accepted by a RT, a moderately ill patient has a probability 0.9 of a successful surgery and 0.1 of a failed surgery. These probabilities remain the same in case the patient had approached a RA instead. Answer the following questions.

(a) Draw the tree diagram.

(b) Derive the conditional probability that a patient is accepted by a RT, given that the patient is a serious heart patient and had a successful surgery.

(c) Derive the probability that a patient accepted by a RT has a successful surgery.

(d) Derive the probability that a patient accepted by a RA has a successful surgery.

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P[Any Indian is an Engineer] = 0.1.

P[Indian has high-tech job | Indian is an Engineer] = 0.1

(a) Any person in the room of 50 people is an engineer with probability 0.1.

P[At least two are engineers]

= 1 − P[0 Engineers in the room ∪ 1 Engineer in the room]

Expressing in terms of correct events. 4

= 1 − (P[0 Engineers in the room] + P[1 Engineer in the room])

$= 1 - (0.9)^{50} - {}^{50}C_1 \, (0.1)(0.9)^{49}$

Associating the correct probabilities with the identified events. 4

(b) P[Exactly 2 engineers in a room of 50]

$= \underbrace{{}^{50}C_2}_{\text{Number of ways in which 2 can be chosen out of 50.}} \; \underbrace{(0.1)^2 (0.9)^{48}}_{\text{Any selection of 2 engineers \& 48 non-engineers occurs with this probability.}}$

6 marks.

(c) We have been told that a pair of engineers was chosen. Denote the pair as (Eng 1, Eng 2).

We want

P[Eng 1 & Eng 2 are both in high-tech jobs]

= P[Eng 1 is in high-tech job ∩ Eng 2 is in high-tech job]

The event 3

Since it is given that the kind of job an engineer gets is independent of the job that any other engineer gets, we can simplify the above probability as

(P[Eng 1 is in high-tech job]) (P[Eng 2 is in high-tech job])

Use of independence 3

$= (0.1)^2 = 0.01.$

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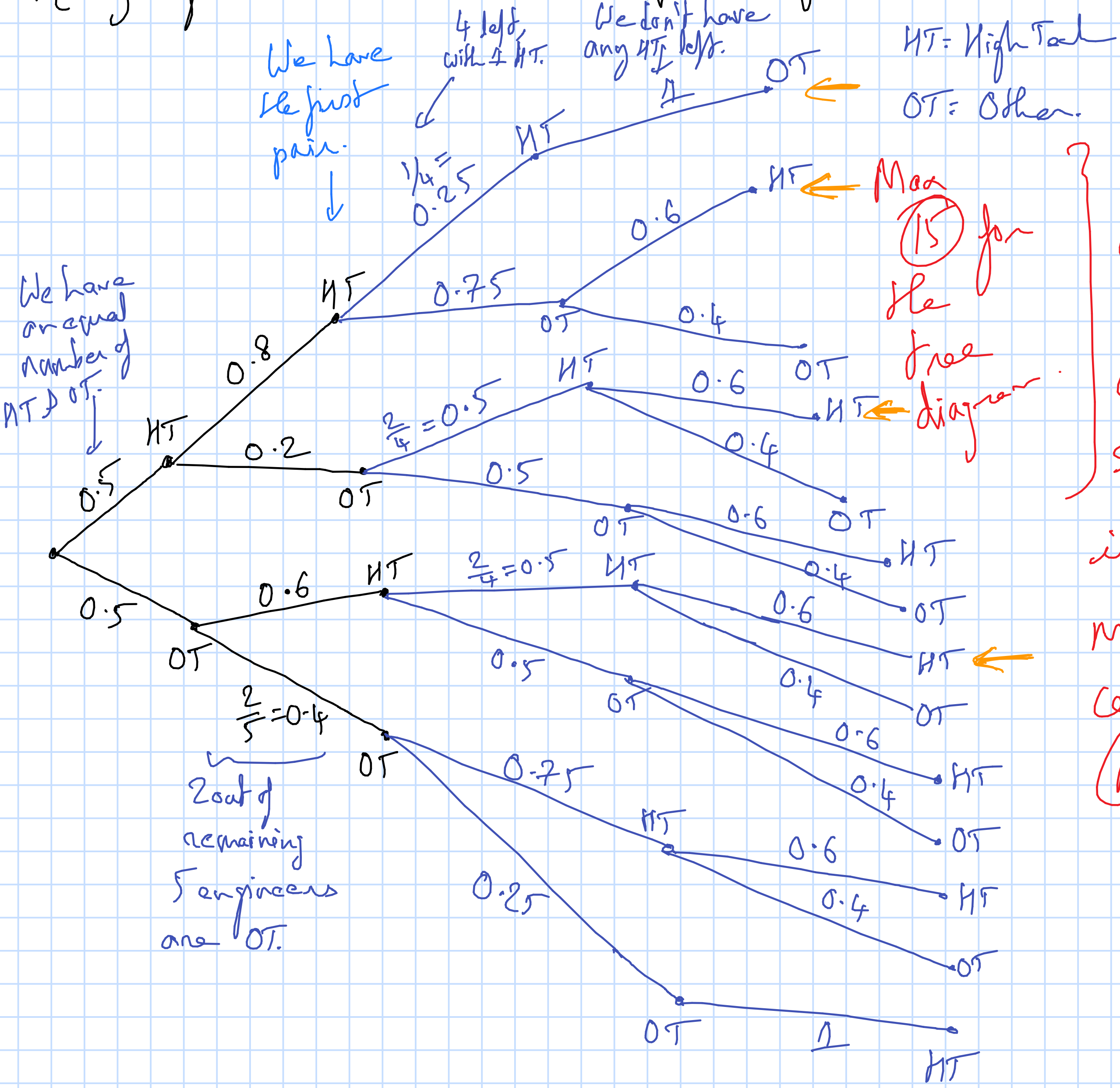
Calculate the probability that the two pairs together have three high-tech engineers. Draw a tree diagram to help you do so. Your tree diagram wants to record the type of engineer chosen at different stages of pairing.

P[Any engineer in the room has a high-tech job] $= \frac{3}{6} = 0.5$.



TA(s) may use their descretion. One way of splitting the 15 is giving 1 mark for every correct path. (Assuming there are correct paths)

P[Two pairs together have three HT]

= P[Paths marked ← in the tree diagram]

$= (0.5)(0.8)(0.25)(1)$ ← 1

$+ (0.5)(0.8)(0.75)(0.6)$ ← 1

$+ (0.5)(0.2)(0.5)(0.6)$ ← 1

$+ (0.5)(0.6)(0.5)(0.6)$ ← 1

+1 for getting all the four correct.

$= 0.4\,(0.25 + (0.6)(0.75))$

$+ 0.15\,(0.8)$

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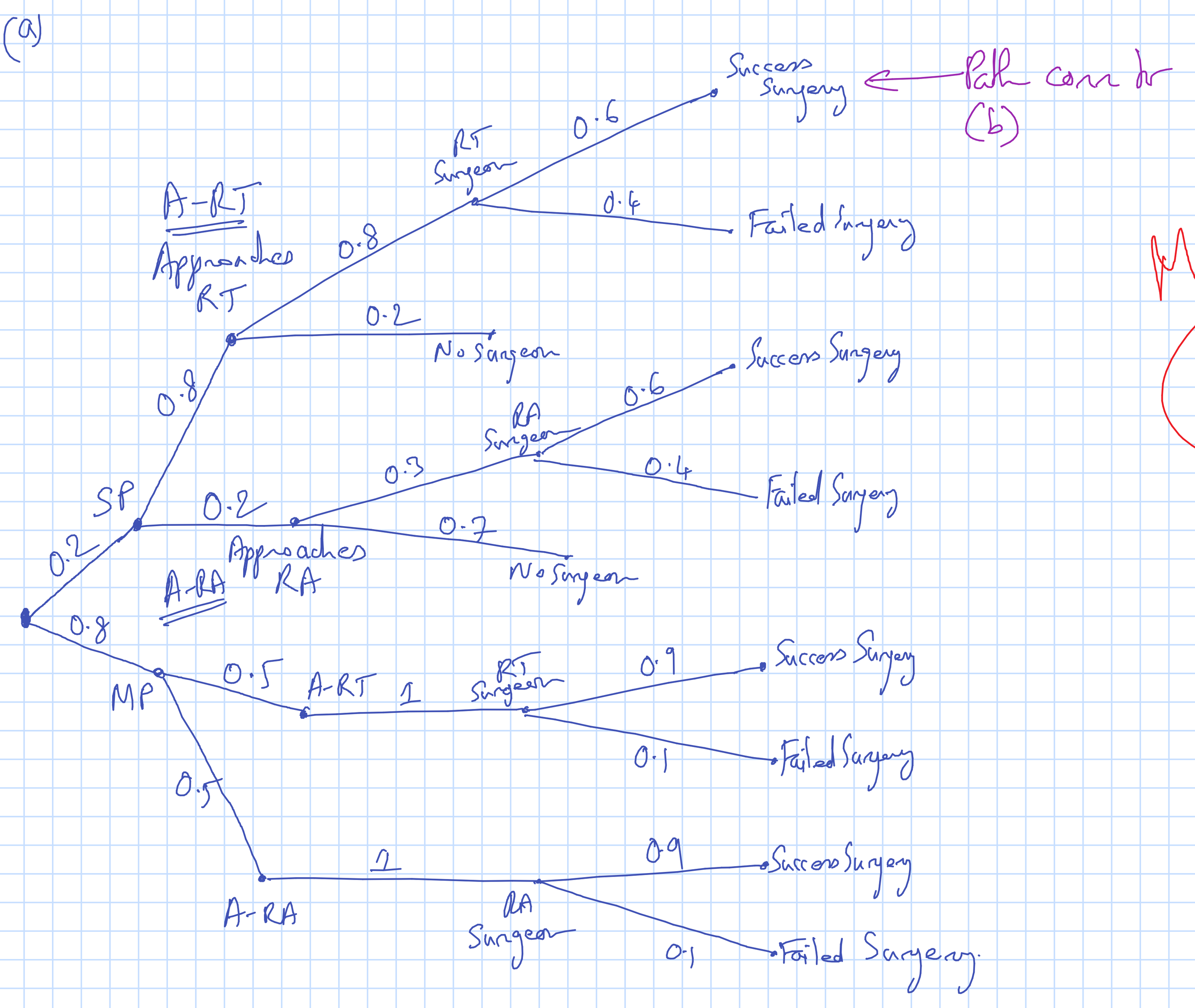
(a) Draw the tree diagram.
(b) Derive the conditional probability that a patient is accepted by a RT, given that the patient is a serious heart patient and had a successful surgery.
(c) Derive the probability that a patient accepted by a RT has a successful surgery.
(d) Derive the probability that a patient accepted by a RA has a successful surgery.

$P[SP] = 0.2$; $P[MP] = 0.8$.
SP: Serious patient. MP: Moderate patient.

(a)



Max 15

(b) $P\left[\text{Patient is accepted by RT} \,\middle|\, \text{Serious Heart Patient, Successful Surgery}\right]$

$$= \frac{P[\text{Patient is accepted by RT, Serious Heart Patient, Successful Surgery}]}{P[\text{Serious Heart Patient, Successful Surgery}]}$$

For the numerator use the top path in the tree diagram. The prob is $(0.2)(0.8)^2(0.6)$

Max 15

The denominator can be written as

$P[\text{Patient is accepted by RT, Serious Heart Patient, Successful Surgery}]$ } Top path in tree diagram

$+\ P[\text{Patient is accepted by RA, Serious Heart Patient, Successful Surgery}]$ } The second path from top that results in a successful surgery

$$= (0.2)(0.8)^2(0.6) + (0.2)^2(0.3)(0.6).$$

The prob of interest is

$$\frac{(0.2)(0.8)^2(0.6)}{(0.2)(0.8)^2(0.6) + (0.2)^2(0.3)(0.6)}.$$

(c) $P[\text{Successful Surgery} \mid \text{Patient accepted by RT}]$

Max 15

$$= \frac{P[\text{Successful Surgery, Patient accepted by RT}]}{P[\text{Patient accepted by RT}]}$$

$$= \frac{P[\text{Successful Surgery, Patient accepted by RT, Patient is Severe}] + P[\text{Successful Surgery, Patient accepted by RT, Patient is Moderate}]}{P[\text{Patient is Moderate, Patient accepted by RT}] + P[\text{Patient is Serious, Patient accepted by RT}]}$$

$$= \frac{(0.2)(0.8)^2(0.6) + (0.8)(0.5)(0.9)}{(0.8)(0.5) + (0.2)(0.8)^2}$$

(d) Replace RT above by RA.

$P[\text{Successful Surgery} \mid \text{Patient accepted by RA}]$

Max 15

$$= \frac{P[\text{Successful Surgery, Patient accepted by RA, Patient is Severe}] + P[\text{Successful Surgery, Patient accepted by RA, Patient is Moderate}]}{P[\text{Patient is Moderate, Patient accepted by RA}] + P[\text{Patient is Serious, Patient accepted by RA}]}$$

$$= \frac{(0.2)^2(0.3)(0.6) + (0.8)(0.5)(0.9)}{(0.2)^2(0.3) + (0.8)(0.5)}$$