MTH 371: Mid Semester Exam
Instructor: Monika Arora

December 3, 2021

Instructions

- Show all your work to score full marks. Detailed answers are a must.
- You can use a calculator. No phones or other electronic devices may be used.
- In all the questions where process or distribution is to be identified write all the corresponding parameters. Incomplete information will lead to deduction of marks.
- Wherever needed, define an appropriate random variable and then solve the question.

Questions

1. Complete the following sentences

   (a) (1 point) Suppose that the amount of time one spends in a bank is exponentially distributed with mean ten minutes, that is, $\lambda = 1/10$. The probability that a customer will spend more than fifteen minutes in the bank is ....

   (b) (1 point) Let $\{X_i\}$ are jointly Normal with mean zero and covariance matrix (K). The distribution of $S_n/\sqrt{n}$ is .... The mean and variance of of variance of $S_n/\sqrt{n}$ is ......, ...respectively.

   (Here, $S_n = X_1 + X_2 + \ldots + X_n$.)

   (c) (1 point) The transition probability matrix of a discrete time Markov chain with $S = \{1, 2, 3\}$ is given by
   $$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$
   Let $A = \{1\}$. The vector of hitting probabilities from state space to set A is ....

2. Let there be two independent Poisson processes $\{N_1(t); t > 0\}$, $\{N_2(t); t > 0\}$ with rate $\mu_1 > 0$ and $\mu_2 > 0$ respectively. Answer the following questions

   (a) (2.5 points) Let there be a combined process $N(t) = N_1(t) + N_2(t)$, what is the distribution of $N_1(t)$ given $N(t)$.
   (Hint : $N(t)$ will also be a Poisson process with $(\mu_1 + \mu_2)t$.)

   (b) (2.5 points) What is the probability that the first arrival of the combined process comes from process 1.

3. Let $W(t)$ be a standard Brownian motion where $t\epsilon[0, \infty)$.

   (a) (2 points) Find $P(0 < W(1) + W(2) < 2, 3W(1) - 2W(2) > 0)$.

   (b) (4 points) Let $U(t)$ be another standard Brownian motion which is independent of $W(t)$ and let $-1 \leq \rho \leq 1$. Define a random process $X(t)$ given by $X(t) = \rho W(t) + \sqrt{(1 - \rho^2)}U(t)$. Find the covariance and correlation of $X(t)$ and $W(t)$.

4. Harry's mother has hidden a jar of Christmas cookies from him. It is kept in a drawer where he can reach by climbing a ladder. There are n steps in the ladder. He can climb up one step with probability p or can get back down one step with probability $1 - p$. He keeps on climbing up till he reaches the last step of the ladder and get the jar or his mother sees him. If his mother see him he starts getting down one step at a time and reaches the floor. While climbing up, if he hears a noise he starts getting down with the fear of his mother's arrival. Once the noise fades away he again starts climbing up. Assume that the time is discrete and the process gets over if he gets the jar. Answer the following questions

   (a) (1 point) How will you model the process, explain.

   (b) (1 point) Construct a one step transition probability matrix.

   (c) (1.5 points) If $p = 0.6$ and currently he is at step 5 and in total there are 15 steps in the ladder. What is the chance that he will be able to steal the jar. What will be the chance if the ladder is very large.

   (d) (1 point) If $p = 0.6$ and currently he is at step 1 and in total there are 15 steps in the ladder. What is the chance that he will ever be able to reach about half way (step 7) to the ladder.

   (e) (2.5 points) If initially he was on the floor (step 0) then find $P(X_1 = 1|X_0 = 0)$, $P(X_1 = 2)$ and $P(X_1 = 1, X_2 = 2, X_3 = 3)$.

5. It is well known to the Public Health Service that Harry's Restaurant gives only casual attention to matters of cleanliness; consequently, the Public Health Service periodically closes the restaurant and mandates a cleanup. Independent periods of operation and closure form two independent renewal processes. The open periods are iid with means μ_0 , and the closed periods are iid with means μ_1. Lengths of open periods and closed periods are independent of each other. Answer the following questions

   (a) (1.5 points) What is the long run proportion of time that Happy Harry's Restaurant is open to its adoring public.

   (b) (1.5 points) In the long run, what is the average number of times the restaurant remained open.