Worksheet 2

October 6, 2021

1. Let $\{ r_1, r_2, \dots , r_{\Phi(m)} \}$ be a reduced residue system modulo m.
   Prove that: m divides $r_1 + r_2 + \dots + r_{\Phi(m)}$ for $m > 2$.

   Hint : Show $(r_i , m) = 1$, then $(m - r_i , m) = 1$ -----------(i)
   Show $\Phi(m)$ is even. -----------(ii)

2. Let $\pi(x)$ denote the number of primes that are less than or equal to the real number x.
   Thus,

$$\pi(x) = \begin{cases} 0 & x < 2 \\ 1 & 2 \le x < 3 \\ 2 & 3 \le x < 5 \\ . & \\ . & \\ . & \\ n & p_n \le x < p_{n+1} \end{cases}$$

   p_n denotes the n-th prime.

   Let $p_1, p_2, \dots , p_n$ denote primes $\le x$.

   Let $N = \{ p_1^{k_1} \dots p_n^{k_n} \mid k_1 \ge 0, \dots , k_n \ge 0 \}$ i.e. N consists of 1 and all positive integers whose prime factorization only uses $p_1, p_2, \dots , p_n$.

   1. Prove $\sum_{n \in N} \frac{1}{n} > ln(x)$
   2. Prove $\prod_{p \le x} (1 - \frac{1}{p})^{-1} = \sum_{n \in N} \frac{1}{n}$
   3. Use (1) and (2) to show

   $$\sum_{p \le x} ln(1 - \frac{1}{p})^{-1} > ln(ln(x))$$

   4. Use expansion of *-ln(1-x)* to get

   $$- ln(1 - x) \le x + x^2 \qquad for\, x \le \frac{1}{2}$$

   5. Use (3) and (4) to conclude

$$\sum_{p \le x} \frac{1}{p} + \sum_{p \le x} \frac{1}{p^2} > ln(ln(x))$$

6. Prove $\sum_{p \le x} \frac{1}{p^2} \le 1$
7. Use (5) and (6) to conclude

$$\sum_{p \le x} \frac{1}{p} > ln(ln(x)) - 1$$

8. Use (7) to obtain a new proof of infinitude of primes.