Rubric of Quiz 3

1 (a) Let X_i denote a random variable such that

$$X_i = \begin{cases} 1 & \text{if the } i\text{th dog gets his own collar} \\ 0 & \text{otherwise} \end{cases}$$

The total number of dogs that got their own collar $W_N = X_1 + X_2 + \cdots + X_N$.

Now

$$\left. \begin{aligned} P_{X_i}(1) &= \frac{1}{N} \\ P_{X_i}(0) &= 1 - \frac{1}{N} \end{aligned} \right\} \text{ for } i = 1, 2, \ldots, N.$$

Hence $E[X_i] = 1 \times \frac{1}{N} = \frac{1}{N}$

Therefore

$$E[W_n] = E[X_1] + E[X_2] + \cdots + E[X_N]$$

$$= \underbrace{\frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N}}_{(N \text{ times})}$$

$$= N \times \frac{1}{N} = \boxed{1}$$

+8

b) Now $E[x_i^2] = 1^2 \times \frac{1}{N} = \frac{1}{N}$

Hence $Var[x_i] = E[x_i^2] - (E[x_i])^2$

$$= \frac{1}{N} - \frac{1}{N^2}$$

+3

Now, $Cov[x_i, x_j] = E[x_i x_j] - E[x_i]E[x_j]$

Now, $x_i x_j = 1$ if and only if $x_i = 1$ and $x_j = 1$
$= 0$ otherwise

So, $E[x_i x_j] = P_{x_i, x_j}(1, 1) = P_{x_i | x_j}(1|1) P_{x_j}(1)$

+8

Now $P_{x_i|x_j}(1|1) = \frac{1}{N-1}$

Hence $E[x_i x_j] = \frac{1}{N(N-1)}$

Therefore $Cov[x_i, x_j] = \frac{1}{N(N-1)} - \frac{1}{N^2}$

So, $Var[W_n] = Var[x_1] + \cdots + Var[x_n] + N(N-1) Cov[x_i, x_j]$

$$= N\left(\frac{1}{N} - \frac{1}{N^2}\right) + N(N-1)\left[\frac{1}{N(N-1)} - \frac{1}{N^2}\right]$$

$$= 1 - \frac{1}{N} + 1 - \left(\frac{N-1}{N}\right) = 1 - \cancel{\frac{1}{N}} + \cancel{1} - \cancel{1} + \cancel{\frac{1}{N}} = \boxed{1}$$

+4

(Total = 25 points)

2 a Let Y be the random variable which takes value $\lambda = 1,2,3,4$ for the selected type writer.

Then $P[Y=\lambda] = \frac{1}{4}$ for $\lambda = 1,2,3,4$
$= 0$ otherwise

+4

Let X_i be the number of misprint on page i and let X be the total number of misprints in the article.

+4 Then $X = X_1 + X_2 + X_3$

Now, for any i it is given that conditional distribution of X_i given $Y=\lambda$ is Poisson (λ).

Therefore $E[X_i \mid Y=\lambda] = \lambda$ for $\lambda = 1,2,3,4$

Now

$$E[X_i] = E[E[X_i \mid Y]] = \sum_{\lambda=1}^{4} E[X_i \mid Y=\lambda]\, P[Y=\lambda]$$

$$= \sum_{\lambda=1}^{4} \lambda \times \frac{1}{4} = \frac{1}{4}(1+2+3+4) = \frac{10}{4} = \frac{5}{2}$$

+8

Now, $E[X] = E[X_1] + E[X_2] + E[X_3]$

$$= 3 \times \frac{5}{2} = \frac{15}{2} = \boxed{7.5}$$

+4

(Total = 20 points)

Note:

Some students may write

$$E[X|Y] = E[X_1|Y] + E[X_2|Y] + E[X_3|Y]$$

Now $E[X|Y=\lambda] = E[X_1|Y=\lambda] + E[X_2|Y=\lambda] + E[X_3|Y=\lambda]$

$$= \lambda + \lambda + \lambda = 3\lambda$$

Now $E[X] = E[E[X|Y]]$

$$= \sum_{\lambda=1}^{4} E[X|Y=\lambda]\, P[Y=\lambda]$$

$$= \sum_{\lambda=1}^{4} 3\lambda \cdot \frac{1}{4} = \frac{3}{4}(1+2+3+4)$$

$$= \frac{3}{\cancel{4}_{2}} \times \cancel{10}^{5} = \frac{15}{2} = \boxed{7.5}$$

- Both are essentially same and both are acceptable.

2 b (6)

Let $U = \min(X,Y)$ and $V = \max(X,Y)$.

Let us define two events:

[redacted] $A = \{U \le u\}$, $B = \{V \le v\}$

Then CDF of U and V:

$$F_{U,V}(u,v) = P[U \le u, V \le v]$$

+4

$$= P[A \cap B] = P[B] - P[A^c \cap B]$$

(since $B = (A \cap B) \cup (A^c \cap B)$)

$$= P[V \le v] - P[U > u, V \le v] \quad \ldots\ldots (1)$$

Now $U = \min(X,Y) > u$ if and only if $X > u$ and $Y > u$

and $V = \max(X,Y) \le v$ if and only if $X \le v$ and $Y \le v$

Therefore

+4

$$P[U > u, V \le v] = P[X > u, Y > u, X \le v, Y \le v]$$

$$= P[u < X \le v, u < Y \le v]$$

Therefore from (1),

$$F_{U,V}(u,v) = P[V \le v] - P[U > u, V \le v]$$

$$= P[X \le v, Y \le v] - P[u < X \le v, u < Y \le v]$$

$$= P[X \le v]P[Y \le v] - P[u < X \le v]P[u < Y \le v]$$

(since X and Y are independent)

6

Hence

$$F_{U,V}(u,v) = F_X(v)F_Y(v) - \left(F_X(v) - F_X(u)\right)\left(F_Y(v) - F_Y(u)\right)$$

$$\Rightarrow F_{U,V}(u,v) = \cancel{F_X(v)F_Y(v)} - \cancel{F_X(v)F_Y(v)} + F_X(u)F_Y(v) + F_X(v)F_Y(u) - F_X(u)F_Y(u)$$

$$\Rightarrow \boxed{F_{U,V}(u,v) = F_X(v)F_Y(u) + F_X(u)F_Y(v) - F_X(u)F_Y(u)}$$

+8

Now the joint PDF is:

$$f_{U,V}(u,v) = \frac{\partial^2 F_{U,V}(u,v)}{\partial u \partial v}$$

$$= \frac{\partial}{\partial u}\left[f_X(v)F_Y(u) + F_X(u)f_Y(v) - 0\right]$$

$$= \boxed{f_X(v)f_Y(u) + f_X(u)f_Y(v)}$$

+4

(Total = 20 points)

(3) Let S be the number of heads in 10,000 independent tosses of a fair Coin. (7)

Then $S = X_1 + X_2 + \cdots + X_{10000}$

where each X_i is a Bernoulli $\left(\frac{1}{2}\right)$ random variable.

+4

$E[X_i] = \frac{1}{2}$ $\quad Var[X_i] = \frac{1}{4}$, Hence $E[S] = \frac{1}{2} \times 10000$
$= 5000$

1% of 5000 = 50

Now $P\left[\,|S-5000| < 50\,\right]$

$$= P\left[\frac{|S-5000|}{\sqrt{10,000 \times \frac{1}{4}}} < \frac{50}{\sqrt{10,000 \times \frac{1}{4}}}\right]$$

$$= P\left[\frac{|S-5000|}{50} < 1\right]$$

$$= P\left[\,|Z| < 1\,\right]$$

(By Central limit theorem where $Z \sim N(0,1)$)

+8

$$= P[-1 < Z < 1]$$

$$= \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)]$$

$$= 2\Phi(1) - 1 = (.8413) \times 2 - 1$$

$$= 1.6826 - 1 = \boxed{.6826}$$

$P[S > 5100]$

$= P\left[\dfrac{S-5000}{\sqrt{10{,}000 \times \frac{1}{4}}} > \dfrac{5100-5000}{\sqrt{10{,}000 \times \frac{1}{4}}}\right]$

$= P\left[Z > \dfrac{100}{50}\right]$ (By central Limit theorem)

+8

$= P[Z > 2] = 1 - \Phi(2)$

$= 1 - .9772 = \boxed{.0228}$

(Total = 20 points)

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4 $\phi_Y(s) = \frac{1}{(1-s)}$

(a) $\frac{d[\phi_Y(s)]}{ds} = (-1)(1-s)^{-1-1}(-1) = \frac{1}{(1-s)^2}$

$\frac{d^2[\phi_Y(s)]}{ds^2} = \frac{2}{(1-s)^3}$

$\frac{d^3[\phi_Y(s)]}{ds^3} = \frac{6}{(1-s)^4}$

+7

So, $E[Y] = \left.\frac{d[\phi_Y(s)]}{ds}\right|_{s=0} = \boxed{1}$

$E[Y^2] = \left.\frac{d^2[\phi_Y(s)]}{ds^2}\right|_{s=0} = \boxed{2}$

$E[Y^3] = \left.\frac{d^3[\phi_Y(s)]}{ds^3}\right|_{s=0} = \boxed{6}$

(b) Given that $\phi_Y(s) = \frac{1}{(1-s)}$

$\phi_V(s) = \frac{1}{(1-s)^4}$

and Y and V are independent.

Hence if $W = Y + V$,

then $\phi_W(s) = \phi_Y(s)\,\phi_V(s) = \frac{1}{(1-s)} \times \frac{1}{(1-s)^4}$

$$= \frac{1}{(1-s)^5}$$

+2

$$\frac{d[\phi_W(s)]}{ds} = \frac{5}{(1-s)^6}$$

and $$\frac{d^2[\phi_W(s)]}{ds^2} = \frac{30}{(1-s)^7}$$

Hence $$E[W^2] = \left.\frac{d^2[\phi_W(s)]}{ds^2}\right|_{s=0}$$

$$= \boxed{30}$$

+6

Total = 15 points

Note: • Some student may expand the MGF in Binomial expansion and calculate the moments from there.

(a) $\phi_Y(s) = \frac{1}{(1-s)} = 1 + s + s^2 + s^3 + \cdots$

So, $\left.\frac{d^n(\phi_Y(s))}{ds^n}\right|_{s=0} = (\text{Coefficient of } s^n) \times n!$

So, $E[Y] = 1$

$E[Y^2] = (1) \times 2! = 2$

$E[Y^3] = (1) \times 3! = 6$

(b) $\phi_W(s) = \frac{1}{(1-s)^5} = 1 + 5s + 15s^2 + 35s^3 + \cdots$

So, $E[W^2] = 15 \times 2! = \boxed{30}$

This answer will be O.K.

- Some students may identify the distribution of the random variable from the MGF and get the answer from there.

e.g.: $\phi_Y(s) = \frac{1}{1-s}$ will imply that

Y is exponential $(\lambda = 1)$ RV

So, $E[Y] = 1$, $E[Y^2] = 2$, $E[Y^3] = 3! = 6$

$\phi_W(s) = \frac{1}{(1-s)^5}$ So, W is Erlang $(n=5, \lambda=1)$

So, $E[W] = 5$, $Var[W] = 5$

So, $E[W^2] = (E[W])^2 + Var(W) = 5^2 + 5 = 30$

This method will be O.K. too.