MTH 211: End-semester exam

Maximum marks: 80 **Time:** 2 hours

Instructions. Be sure to show your work and explain your reasoning for full credit. No calculators, phones, notes, etc. are allowed. Please write your answers in the sequence of questions asked (**-2 points** for not following this rule).

1. ((4×5) points) The following are **true/false** questions. If you think the statement is true, give a proof, stating any theorems you need. If false, provide a concrete counterexample or give a proof.
   (a) If p_1, p_2, p_3 are consecutive primes then $2p_1p_2p_3 + 1$ is a prime number.
   (b) The quadratic congruence $x^2 \equiv a \pmod 4$ is solvable if and only if $a \equiv 1 \pmod 4$.
   (c) There are only finitely many integral solutions of the equation $x^2 - 16y^2 = 40$.
   (d) 3 is a quadratic residue of 31 and non-residue of 23.
   (e) The Euler phi function is completely multiplicative.
2. ((5×3) points) The following problems test your knowledge of theorems and definitions: State the theorem or definition requested; be sure to include any necessary hypotheses, and details.
   (a) State one primality test covered in class.
   (b) State Fermat's last theorem.
   (c) State Wilson's theorem.
3. ((6×3) points) Each of the following questions can be answered using only a minimal amount of pencil/paper calculations if *approached with the right method*. If you get stuck in a messy hand computation, you are on the wrong track. Answers arrived at by brute force methods, trial and error, or guessing won't earn credits.
   (a) Find the last two decimal digits of 413^{402}.
   (b) Find the fundamental solution of the following Pell's equations using $\sqrt{18} = \langle 4, 4, 8, 4, 8, 4, 8, \cdots \rangle$
      i. $x^2 - 18y^2 = -1$
      ii. $x^2 - 18y^2 = 1$.
   (c) Find a positive integer n such that $3^2|n$, $4^2|n+1$, and $5^2|n+2$.
4. ((6×2) points) Short proofs.
   (a) Let $1 \le k \le n$. Then

i. $\langle a_0, a_1, \cdots, a_n \rangle = \langle a_0, a_1, \cdots, a_{k-1}, \langle a_k, a_{k+1}, \cdots, a_n \rangle \rangle$.

ii. $\langle a_0, a_1, \cdots, a_n \rangle = a_0 + \frac{1}{\langle a_1, a_2, \cdots, a_n \rangle}$.

(b) Prove that if $n = a^2 + b^2$ for some $a, b \in \mathbb{Q}$, then $n = c^2 + d^2$ for some $c, d \in \mathbb{Z}$.

5. (5 points) Find the number α with the continued fraction expansion $[1, 2, 3, 2, 3, \cdots]$.

6. (Bonus question, 5 points) Write names of 5 mathematicians whose work we studied in this course.