Worksheet 9 (Solution)

1. (a) We prove using induction on $k \geq 1$. For $k = 1$

$$\begin{aligned} < a_0, a_1, \cdots, a_n > &= a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}} \\ &= a_0 + \frac{1}{< a_1, a_2, \cdots, a_n >} \\ &=< a_0, < a_1, a_2, \cdots, a_n >> . \end{aligned}$$

Let $k \geq 2$, suppose that

$$< a_0, a_1, \cdots, a_n >=< a_0, a_1, \cdots, a_{k-2}, < a_{k-1}, a_k, \cdots, a_n >>$$

By induction hypothesis

$$< a_0, a_1, \cdots, a_n > =< a_0, a_1, \cdots, a_{k-2}, \alpha_{k-1} >$$

where

$$\begin{aligned} \alpha_{k-1} &=< a_{k-1}, a_k, \cdots, a_n > \\ &= a_{k-1} + \frac{1}{< a_k, a_{k+1}, \cdots, a_n >} \end{aligned}$$

This implies

$$\begin{aligned} < a_0, a_1, \cdots, a_n > &=< a_0, a_1, \cdots, a_{k-2}, a_{k-1} + \frac{1}{< a_k, a_{k+1}, \cdots, a_n >} > \\ &=< a_0, a_1, \cdots, a_{k-2}, a_{k-1}, < a_k, a_{k+1}, \cdots, a_n > > \end{aligned}$$

(b) Do same as the case when k=1 in part (a).

2. (a)

$$
\begin{aligned}
.23 &= \frac{23}{100} = 0 + \frac{1}{\frac{100}{23}} \\
&= 0 + \frac{1}{4 + \dfrac{8}{23}} = 0 + \frac{1}{4 + \dfrac{1}{\dfrac{23}{8}}} \\
&= 0 + \frac{1}{4 + \dfrac{1}{2 + \dfrac{7}{8}}} = 0 + \frac{1}{4 + \dfrac{1}{2 + \dfrac{1}{\dfrac{8}{7}}}} \\
&= 0 + \frac{1}{4 + \dfrac{1}{2 + \dfrac{1}{1 + \dfrac{1}{7}}}}
\end{aligned}
$$

Thus $.23 = [0, 4, 2, 1, 7]$.

(b)

$$
\begin{aligned}
\frac{233}{177} &= 1 + \frac{56}{177} = 1 + \frac{1}{\frac{177}{56}} \\
&= 1 + \frac{1}{3 + \dfrac{9}{56}} = 1 + \frac{1}{3 + \dfrac{1}{\dfrac{56}{9}}} \\
&= 1 + \frac{1}{3 + \dfrac{1}{6 + \dfrac{2}{9}}} = 1 + \frac{1}{3 + \dfrac{1}{6 + \dfrac{1}{\dfrac{9}{2}}}} \\
&= 1 + \frac{1}{3 + \dfrac{1}{6 + \dfrac{1}{4 + \dfrac{1}{2}}}}
\end{aligned}
$$

Thus $\frac{233}{177} = [1, 3, 6, 4, 2]$.

3.

$$\begin{aligned}\frac{p}{q} &= [3,7,15,1]\\ &= 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}}\\ &= 3+\cfrac{1}{7+\cfrac{1}{16}}\\ &= 3+\frac{16}{113}\\ &= 3.14159292\end{aligned}$$

$$\pi = 3.14285714$$

So the value of p/q is approximately equal to π.

4. (a) For a proper fraction $\frac{p}{q}$, the continued fraction is

$$\frac{p}{q} = [0, a_1, \cdots, a_n].$$

For example

$$\frac{29}{67} = [0,2,3,4,2]$$

(b)

$$\begin{aligned}\frac{67}{29} &= 2+\frac{9}{29} = 2+\cfrac{1}{\cfrac{29}{9}}\\ &= 2+\cfrac{1}{3+\cfrac{2}{9}} = 2+\cfrac{1}{3+\cfrac{1}{\cfrac{9}{2}}}\\ &= 2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2}}}.\end{aligned}$$

$$\frac{67}{29} = [2,3,4,2]$$

(c) Compare (a), and (b), we conclude that for a fraction $\frac{p}{q}$ where $p > q$ if

$$\frac{q}{p} = [0, a_1, a_2, \cdots, a_n],$$

then

$$\frac{p}{q} = [a_1, a_2, \cdots, a_n].$$