(2) *Proving that Carmichael numbers are squarefree.*

(a) Show that a given nonsquarefree number n can be written in the form $n = p^\ell N$ for some prime p and integers N and ℓ with $\ell \geq 2$ and $\gcd(p, N) = 1$.

(b) Show that $(1 + pN)^{n-1} \not\equiv 1 \pmod{p^2}$.

(c) Deduce that Carmichael numbers are squarefree.

(a) Take a prime p such that $p^2 \mid n$. Then $p^\ell || n$ for some $\ell \geq 2$. So $n = p^\ell N$ say, where $N = n/p^\ell$ is coprime to p.

(b) Now $(1+pN)^{n-1} \equiv 1 + pN(n-1) \pmod{p^2} \equiv 1 - pN \pmod{p^2} \not\equiv 1 \pmod{p^2}$, by the Binomial Theorem and because $p^2 \mid n$ and $\gcd(N, p) = 1$.

(c) Now take $a = 1 + pN$. Then $a^{n-1} \not\equiv 1 \pmod{p^2}$ by (b), so $a^{n-1} \not\equiv 1 \pmod{n}$. Hence, as $\gcd(a, n) = 1$, n is not a Carmichael number.