Quiz 1 - Linear Algebra - CSE/ECE 344/544 Solutions

Instructions:

1. Correct Answer + Correct Justification: 2 Points for Q1

2. Correct Answer: 1 Point for Q1

3. Incorrect Answer: -1 Point (for each incorrectly answered T/F question)

4. Award marks for correct logic and for False statements in Q1; any contradictory example will work (rigorous proof not needed)

5. I represents the identity matrix

Q1-solutions
Follow the instructions while grading Q1

a. False

If the rank is less than n (i.e. $r < n$), a system of n linear equations in n variables can have infinitely many solutions.

b. False

If the columns of $\boldsymbol{A}$ span $\mathbb{R}^m$, then $\boldsymbol{Ax} = \boldsymbol{b}$ will always be consistent, however the converse is not true and the consistency of the equation does not imply any constraints on the columns of $\boldsymbol{A}$.

c. False

A linear relation of the type $\boldsymbol{v1} = \boldsymbol{v2} + \boldsymbol{v3}$ is still possible. Not being multiples does not imply that none of the vectors can be expressed as a linear combination of the others.

d. True

Assume $\boldsymbol{A}$ is an $n \times n$ matrix, $\boldsymbol{B}$ is an $n \times p$ matrix. $\boldsymbol{A}$ is a diagonal matrix. Let $\boldsymbol{A}$=$\{a_{ij}\}$. Then a_{ij}=0 when $i \neq j$. Let $\boldsymbol{b}_i$ denote the i^{th} column of $\boldsymbol{B}$. Then $\boldsymbol{B}$=[$\boldsymbol{b}_1$ $\boldsymbol{b}_2$ ... $\boldsymbol{b}_p$]. Thus, $\boldsymbol{AB}$= [$\boldsymbol{Ab}_1\boldsymbol{Ab}_2$...$\boldsymbol{Ab}_p$]. Due to property of $\boldsymbol{A}$, $\boldsymbol{Ab}_i =$ $[a_{11}b_{1i}\ a_{22}b_{2i}\ ...\ a_{nn}b_{ni}]^T$. Let $\boldsymbol{AB}$= $\boldsymbol{C} = \{c_{ij}\}$. Then $c_{ij} = a_{ii}b_{ij}$. Thus, i^{th} row of $\boldsymbol{C}$ is i^{th} row of $\boldsymbol{B}$ scaled by a_{ii}.

e. False

This is only possible if $\boldsymbol{B}$ is square and non-singular (invertible), in which case, $\boldsymbol{C}$=$\boldsymbol{B}^{-1}\boldsymbol{BD}$=$\boldsymbol{ID}$=$\boldsymbol{D}$.

f. False

The result of the multiplication is $\boldsymbol{A}^2 - \boldsymbol{AB} + \boldsymbol{BA} - \boldsymbol{B}^2$. However, for matrix multiplication, we cannot say that $\boldsymbol{AB} = \boldsymbol{BA}$. Hence, it will not simplify to $\boldsymbol{A}^2 - \boldsymbol{B}^2$.

g. True

Given, $\boldsymbol{AB} = \boldsymbol{BA}$, pre and post multiplying both sides with $\boldsymbol{A}^{-1}$

=> $\boldsymbol{A}^{-1}\boldsymbol{ABA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{BAA}^{-1}$

=> $(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{BA}^{-1}\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{AA}^{-1})$

=> $\boldsymbol{IBA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{BI}$ [Since $\boldsymbol{XX}^{-1} = \boldsymbol{I}$]

=> $\boldsymbol{BA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{B}$

h. True

Given, $\boldsymbol{x}$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ => $\boldsymbol{x}^T\boldsymbol{u}$= 0 and $\boldsymbol{x}^T\boldsymbol{v} = 0$

To Prove: $\boldsymbol{x}^T(\boldsymbol{u} - \boldsymbol{v}) = 0$

$\boldsymbol{x}^T(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{x}^T\boldsymbol{u} - \boldsymbol{x}^T\boldsymbol{v}$

$= 0 - 0$

$= 0$

i. True

Since the columns are orthonormal, $\boldsymbol{U}^{-1}\boldsymbol{U} = \boldsymbol{I} = \boldsymbol{U}^T\boldsymbol{U}$

=> $\boldsymbol{U}^T = \boldsymbol{U}^{-1}$

=> $\boldsymbol{UU}^T = \boldsymbol{UU}^{-1}$

=> $\boldsymbol{UU}^T = \boldsymbol{I}$

Hence, the columns of $\boldsymbol{U}^T$ are orthonormal => $\boldsymbol{U}$ has orthonormal rows.

j. False

This is only the case when $\boldsymbol{U}$ is a square matrix and can't be generalized.

Q2-solutions

Any Correct Proof: 2 Points (Partial marking to be followed)

To Prove: $(\boldsymbol{ABC})^{-1}$= $\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

We know that $\boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$

If the hypothesis is true, then $(\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{ABC})$ must be $\boldsymbol{I}$

Solving: $(\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{ABC}) = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{IBC} = \boldsymbol{C}^{-1}\boldsymbol{IC} = \boldsymbol{C}^{-1}\boldsymbol{C} = \boldsymbol{I}$

Hence, the hypothesis is true.

OR

Use the property that $(\boldsymbol{XY})^{-1} = \boldsymbol{Y}^{-1}\boldsymbol{X}^{-1}$

Thus,

$(\boldsymbol{ABC})^{-1} = ((\boldsymbol{AB})\boldsymbol{C})^{-1} = \boldsymbol{C}^{-1}(\boldsymbol{AB})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Q3-solutions

Any Correct Proof: 1 Point for each part (Partial Marking to be followed)

a. Given $\boldsymbol{P} = \boldsymbol{u}\boldsymbol{u}^T$

$=> \boldsymbol{P}^2 = \boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T = \boldsymbol{u}(\boldsymbol{u}^T\boldsymbol{u})\boldsymbol{u}^T$

$= \boldsymbol{u}\boldsymbol{I}\boldsymbol{u}^T$ (given $\boldsymbol{u}^T\boldsymbol{u} = \boldsymbol{I}$)

$= \boldsymbol{u}\boldsymbol{u}^T = \boldsymbol{P}$

Hence, $\boldsymbol{P}^2 = \boldsymbol{P}$

b. Given $\boldsymbol{P} = \boldsymbol{u}\boldsymbol{u}^T$

$=> \boldsymbol{P}^T = (\boldsymbol{u}\boldsymbol{u}^T)^T$

$= ((\boldsymbol{u}^T)^T)(\boldsymbol{u}^T)$

$= (\boldsymbol{u})(\boldsymbol{u}^T)$

$= \boldsymbol{u}\boldsymbol{u}^T = \boldsymbol{P}$

Hence, $\boldsymbol{P}^T = \boldsymbol{P}$

c. Given, $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{P}$

$=> \boldsymbol{Q}^2 = (\boldsymbol{I} - 2\boldsymbol{P})^2$

$= \boldsymbol{I} - 4\boldsymbol{P} + 4\boldsymbol{P}^2$ (Using the result from part a, $\boldsymbol{P}^2 = \boldsymbol{P}$)

$= \boldsymbol{I} - 4\boldsymbol{P} + 4\boldsymbol{P} = \boldsymbol{I}$

Hence, $\boldsymbol{Q}^2 = \boldsymbol{I}$