Worksheet # 3 solution

Problem 1 : $y' + y'' = k$

Make change of variables $z = y' \Rightarrow z' = y''$

Then the ODE becomes $z + z' = k$

Solution :

$$z = e^{-t}\left(\int e^{t} k \, dt + c_1\right) = k + c_1 e^{-t}$$

$$z = y' = k + c_1 e^{-t}$$

$$\Rightarrow \quad y = kt - c_1 e^{-t} + c_2$$

Initial condition : $y(0) = y_0$, $y'(0) = v_0$

$$y_0 = -c_1 + c_2 \quad , \quad v_0 = k + c_1 \Rightarrow c_1 = v_0 - k$$

$$y_0 = -v_0 + k + c_2 \; ; \; c_2 = y_0 + v_0 - k$$

So, the final dependence of motion on the initial conditions is

$$y = kt + (k - v_0) e^{-t} + y_0 + v_0 - k$$

$$\boxed{y = y_0 + kt + (k - v_0)(e^{-t} - 1)}$$

Problem 2 : $y_1 = x^{3/2}$, $y_1' = \frac{3}{2}\sqrt{x}$, $y_1'' = \frac{3}{4\sqrt{x}}$

$y_2 = x^{-1/2}$, $y_2' = -\frac{1}{2} x^{-3/2}$, $y_2'' = \frac{3}{4} x^{-5/2}$

Substitute these to the given ODE

$$4x^2 y_1'' - 3y_1 = 3x^{3/2} - 3x^{3/2} = 0$$

$$4x^2 y_2'' - 3y_2 = 3x^{-1/2} - 3x^{-1/2} = 0$$

So they are two independent (one is not multiple of the other) solutions of a second-order ODE, consequently, they are a basis of solutions.

The general solution can be written as

$$y = c_1 y_1 + c_2 y_2 = c_1 x^{3/2} + c_2 x^{-1/2}$$

$y(1) = -3 = c_1 + c_2$; $y'(1) = 0 = \frac{3}{2} c_1 - \frac{1}{2} c_2$

$\Rightarrow$ $c_1 = -\frac{3}{4}$, $c_2 = -\frac{9}{4}$

$$\boxed{y(x) = -\frac{3}{4} x^{3/2} - \frac{9}{4} x^{-1/2}}$$

Problem 3: looking at the basis, we know that the characteristic polynomial can be factorized as

$$P(\lambda) = (\lambda - \sqrt{5})^2 = \lambda^2 - 2\sqrt{5}\lambda + 5$$

The corresponding ODE is

$$\boxed{y'' - 2\sqrt{5}y + 5 = 0}$$

Problem 4: By the Newton's law and Hooke's law,

$$my'' = -ky \Rightarrow y'' + \frac{k}{m}y = 0$$

its characteristic equation $\rightarrow$ $\lambda^2 + \frac{k}{m} = 0$

$$\Rightarrow \lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

(i) If we double the mass

$$\lambda^2 + \frac{k}{2m} = 0 \Rightarrow \lambda = \frac{\pm i}{\sqrt{2}}\sqrt{\frac{k}{m}} = \frac{\pm i}{\sqrt{2}}\omega_0$$

So, the frequency will be lower by a factor $\frac{1}{\sqrt{2}}$.

(ii) If we take a spring of twice the modulus

$$\lambda^2 + \frac{2k}{m} = 0 \Rightarrow \lambda = \pm i\sqrt{2}\sqrt{\frac{k}{m}} = \pm i\sqrt{2}\,\omega_0$$

the frequency will be higher by a factor $\sqrt{2}$

Problem 5 : $\dfrac{x^2}{x^2 \ln x} = \dfrac{1}{\ln x} \neq$ constant

$\Rightarrow$ x^2 and $x^2 \ln x$ are linearly independent.