Rubric For Quiz 2

1 (a) Let X be the time interval between the receipt of two data packets.

Then $X \sim \exp\left(\frac{1}{30}\right)$ (if we use minute) or $\exp(2)$ (if they measure in hour)

Then $Y = X - 10$ where Y is the waiting time of Professor for a data packet to arrive.

$$P[Y > y \mid X > 10] = P[X - 10 > y \mid X > 10]$$

$$= P[X > 10 + y \mid X > 10]$$

Now $P[X > 10 + y \mid X > 10] = 1$ if $y < 0$

If $y > 0$, $P[X > 10 + y \mid X > 10]$

$$= \frac{P[X > 10 + y, X > 10]}{P[X > 10]}$$

$$= \frac{P[X > 10 + y]}{P[X > 10]}$$

$$= \frac{\int_{10+y}^{\infty} \frac{1}{30} e^{-\frac{1}{30}x} \, dx}{\int_{10}^{\infty} \frac{1}{30} e^{-\frac{1}{30}x} \, dx}$$

$$= \frac{\left[-e^{-\frac{1}{30}x}\right]_{10+y}^{\infty}}{\left[-e^{-\frac{1}{30}x}\right]_{10}^{\infty}} = \frac{e^{-\frac{1}{30}(10+y)}}{e^{-\frac{1}{30}(10)}}$$

$$= e^{-\frac{1}{30}y}$$

Thus Conditional CCDF will be: $\boxed{\begin{cases} e^{-\frac{1}{30}y} & \text{if } y \geq 0 \\ 1 & \text{if } y < 0 \end{cases}}$ +5

The Conditional CDF:

$$F_{Y|\{X>10\}}(y) = \begin{cases} 1 - e^{-\frac{1}{30}y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

The Conditional PDF:

$$\boxed{f_{Y|\{X>10\}}(y) = \begin{cases} \frac{1}{30}e^{-\frac{1}{30}y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}}$$ +2

In (a) total 7 points

Note: (1) If they use hour, they will get similar answer ($\frac{1}{30}$ will be replaced by 2)

(2) Technically in that case they should write $Y = X - \frac{10}{60} = \left(X - \frac{1}{6}\right)$ in hour

But even if they write $Y = X - 10$, they will get the $\boxed{\text{correct answer}}$.

Hence don't take off any point if they use 2 in the PDF, but $Y = X - 10$ as long as they come up with the correct answer.

(3) If they don't show the calculate the CCDF but write the answer stating that it follows from the memoryless property of exponential distribution then they will get 4 points (For ~~OR~~ CCDF & PDF together) (instead of 7)

(4)

(b) $E[Y \mid X > 10] = \int_0^\infty y \frac{1}{30} e^{-\frac{1}{30}y} dy$

They can either use integration by parts or use the formula for expectation of exponential distribution

$= \boxed{30 \text{ minutes}}$

+5

Note: If they use the PDF with unit in hours the answer will be $\frac{1}{2}$ hour and that will be correct.

(c) Let Z be the additional time professor has to wait,

then $Z = Y - 12 = X - 22$.

By the same way

$$f_{Z|\{X>22\}}(z) = \begin{cases} \frac{1}{30} e^{-\frac{1}{30}z} & \text{for } z \geq 0 \\ = 0 & \text{otherwise} \end{cases}$$

Then $E[Z \mid X > 22] = \int_0^\infty z \frac{1}{30} e^{-\frac{1}{30}z} dz$

$= \frac{1}{30} \int_0^\infty z e^{-\frac{1}{30}z} dz = \boxed{30 \text{ minutes.}}$

+3

Note: 1 In B and c, the answer will come as $\frac{1}{2}$ hour if they use PDF in hours.

2 If in b, they don't use the conditional PDF but write the answer saying that by memoryless property they got it, give them 4 points

3 If in c, they use "similarly" or by memoryless property they got it, then give them 3 points.

Next assume $X \sim \text{Uniform}(0, 40)$

(a) $Y = X - 10$

$$P[Y > y \mid X > 10] = P[X > y+10 \mid X > 10]$$

If $y < 0$, $P[X > y+10 \mid X > 10] = 1$

If $y > 30$, $P[X > y+10 \mid X > 10] = 0$

If $0 \leq y \leq 30$, $P[X > y+10 \mid X > 10]$

$$= \frac{P[X > y+10, X > 10]}{P[X > 10]}$$

$$= \frac{P[X > y+10]}{P[X > 10]}$$

$$= \frac{\int_{y+10}^{40} \frac{1}{40}\,dx}{\int_{10}^{40} \frac{1}{40}\,dx} = \frac{\frac{(40-y-10)}{40}}{\frac{(40-10)}{40}}$$

$$= \frac{30-y}{30} = 1 - \frac{y}{30}$$

Thus CCDF : $\boxed{\begin{cases} 1 - \frac{y}{30} & \text{if } 0 \leq y \leq 30 \\ 1 & \text{if } y < 0 \\ 0 & \text{if } y > 30 \end{cases}}$

+5

7

So, CDF, $F_{Y|\{X>10\}}(y) = 0$ if $y<0$
$= \frac{y}{30}$ if $0 \le y \le 30$
$= 1$ if $y > 30$

Then PDF, $\boxed{f_{Y|\{X>10\}}(y) = \frac{1}{30} \text{ if } 0 \le y \le 30;\; = 0 \text{ otherwise}}$ +2

In part (a) Total = 7 points

(b) $E[Y \mid X>10] = \int_0^{30} y \cdot \frac{1}{30}\, dy = \left[\frac{y^2}{2\times 30}\right]_0^{30}$

$= \boxed{15 \text{ minutes.}}$ +5

(c) If $Z = Y - 12 = X - 22$,
in the same way,

$f_{Z|\{X>22\}}(z) = \frac{1}{18}$ if $0 \le z \le 18$
$= 0$ otherwise

Then $E[Z \mid X>22] = \int_0^{18} z \cdot \frac{1}{18}\, dz = \left[\frac{z^2}{2\times 18}\right]_0^{18}$

$= \boxed{9 \text{ minutes.}}$ +3

(In Question (1) : Total = 30 points)

(2) Let X be the time for one way trip of the engineer.

Then $X \sim N(\mu, \sigma^2)$ where $\mu = 24$ minutes and $\sigma = 3$ minutes.

(a) $P[X > 30] = P\left[\frac{X-24}{3} > \frac{30-24}{3}\right]$

$= P[Z > 2]$ (where $Z \sim N(0,1)$)

$= 1 - \Phi(2) = \boxed{.0228}$

+5 +1 Total = +6

(b) $P[X > 15] = P\left[\frac{X-24}{3} > \frac{15-24}{3}\right]$

$= P[Z > -3] = 1 - \Phi(-3) = \Phi(3) = .9987$

+5 +1 Total = +6

Thus 99.87% of time, he will be late for work.

(c) $P[X > 26] = P\left[\frac{X-24}{3} > \frac{26-24}{3}\right]$

$= P\left[Z > \frac{2}{3}\right] = P[Z > .67]$

$= 1 - \Phi(.67) = 1 - .7486 = \boxed{.2514}$

+5 +1

Total = +6

(d) We need to find l such that

$$P[X > l] = .15$$

$$\Rightarrow P\left[\frac{X-24}{3} > \frac{l-24}{3}\right] = .15$$

$$\Rightarrow P\left[Z > \frac{l-24}{3}\right] = .15$$

$$\Rightarrow P\left[Z \leq \frac{l-24}{3}\right] = .85$$

Now it is given $\Phi(1.04) = .85$

+4

Total = +6

Hence $\frac{l-24}{3} = 1.04 \Rightarrow l-24 = 3 \times 1.04$

$\Rightarrow l - 24 = 3.12 \quad \Rightarrow l = 24 + 3.12 = \boxed{27.12 \text{ minutes}}$

+2

Thus above 27.12 minutes, we have the slowest 15 percent of the trips.

(e) From (a) we have $P[X \geq 30] = .0228$

Thus probability that two of the next three trips will take atleast half an hour

(using Binomial distribution)

$$= \binom{3}{2} (.0228)^2 (.9772)$$

$$= \boxed{3 (.0228)^2 (.9772)}$$

+6

Question 2

Total = 30 points

10

(3) Let X be the distance (in miles) traveled by the team before they meet you.

+5

- With probability $\frac{1}{2}$, you will not repair your car at all and the team will have to drive 100 miles. In that case $X = 100$ and $P[X = 100] = \frac{1}{2}$

+10

- Now suppose that you have fixed the car in a fraction of an hour equal to t.

The team will have driven $100t$ miles by then.

If you start traveling at the same speed, you will meet them in the middle of the remaining distance. So, $X = \frac{100t + 100}{2} = 50t + 50$

Now $P(X < 50t + 50) = P(\text{you have fixed in a time} < t)$

$= \frac{1}{2} t$ (since it is uniform)

+5

Therefore

$$\boxed{\begin{aligned} F_X(x) = P[X \le x] &= 0 \quad \text{if } x < 50 \\ &= \frac{x-50}{100} \quad \text{if } 50 \le x < 100 \\ &= 1 \quad \text{if } x \geq 100 \end{aligned}}$$

$50t + 50 = x$
$\Rightarrow t = \frac{x-50}{50}$
$\Rightarrow \frac{1}{2} t = \frac{x-50}{100}$

Note: $F_X(x)$ is discontinuous at $x = 100$

Question 3 (20 points)

11

4

(a) $$f_{X,Y}(x,y) = \begin{cases} c\,e^{x+y} & \text{if } x, y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{-\infty}^{0}\int_{-\infty}^{0} c\,e^{x+y}\,dx\,dy = 1$$

$$\Rightarrow c\int_{-\infty}^{0}\int_{-\infty}^{0} e^{x}\,e^{y}\,dx\,dy = 1$$

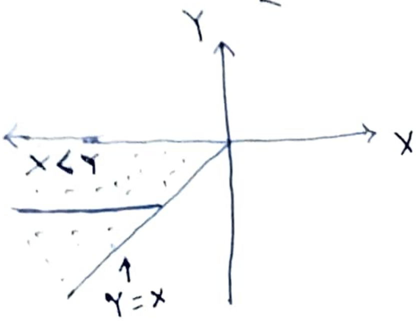
$$\Rightarrow c\int_{-\infty}^{0} e^{y}\left[e^{x}\right]_{-\infty}^{0} dy = 1 \Rightarrow c\int_{-\infty}^{0} e^{y}(1-0)\,dy = 1$$

$$\Rightarrow c\left[e^{y}\right]_{-\infty}^{0} = 1 \Rightarrow c(1-0) = 1 \Rightarrow \boxed{c = 1}$$

+6

12

(b) $$P[X<Y] = \int_{-\infty}^{0}\int_{-\infty}^{y} e^{x+y}\,dx\,dy \quad (c=1)$$



$$= \int_{-\infty}^{0} e^{y}\left\{\int_{-\infty}^{y} e^{x}\,dx\right\} dy$$

$$= \int_{-\infty}^{0} e^{y}\left[e^{x}\right]_{-\infty}^{y} dy = \int_{-\infty}^{0} e^{y}(e^{y}-0)\,dy$$

$$= \int_{-\infty}^{0} e^{2y}\,dy = \left[\frac{e^{2y}}{2}\right]_{-\infty}^{0} = \left(\frac{1}{2}-0\right) = \boxed{\frac{1}{2}}$$

+6

Note: Some students may just say that since the region $(X<Y)$ is one half of the region where PDF is non zero and the function $f_{X,Y}(x,y) = e^{x+y}$ is symmetric, then $P[X<Y] = \frac{1}{2}(\text{Total probability}) = \frac{1}{2}$

This answer is acceptable as long as they mention the property of $f_{X,Y}(x,y)$.

That way ~~[illegible]~~ they are showing their understanding that probability here is Volume not area.

If they donot say anything about the function, take off 2 points.

(c)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{0} e^{x+y}\,dy$$

$$= e^x \int_{-\infty}^{0} e^y\,dy = e^x \left[e^y\right]_{-\infty}^{0} = e^x(1-0) = e^x$$

+4

$$\Rightarrow \quad f_X(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{0} e^{x+y}\,dx = e^y \int_{-\infty}^{0} e^x\,dx$$

$$= e^y \left[e^x\right]_{-\infty}^{0} = e^y(1-0) = e^y$$

+4

$$\Rightarrow \quad f_Y(y) = \begin{cases} e^y & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Question 4: Total = 20 points

Note: Some student may compute just one integral to get one marginal and then by symmetry write the other marginal. That is acceptable as long as they compute one integral correctly and write both the answers.