ECE 351 DSP: Assignment 1

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Total: 30 points

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A word on the notation: I shall represent finite duration causal signals as arrays. For example, $x[n] = [1, 2, 3]$ means $x[0] = 1$, $x[1] = 2$, and $x[2] = 3$, and $x[n] = 0$ for all other n.

Coding has to be done in Python. MATLAB codes will be marked zero.

1) Let $z = x + jy$. Are the following regions on the complex plane ROC of some z-transform? Justify.

   a) $2 < x^2 + y^2 < 3$.
   b) $2x^2 + 3y^2 > 5$.
   c) $2x^2 + 2y^2 < 1$.

[1.5+1.5+1.5=4.5 points]

Solution: a) Yes. This is the annular region between the circles centred at the origin with radii $\sqrt{2}$ and $\sqrt{3}$ respectively. b) No. This region represents the exterior of an ellipse. c) Yes. This region is the interior of the circle centred at the origin with radius $\frac{1}{\sqrt{2}}$. ■

2) Consider the LTI system with impulse response

$$h[n] = \begin{cases} \frac{n^n}{3^{1+2n}}, & \text{if } n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

   a) Which of the three possible shapes does the ROC of $h[n]$ have?
   b) Does the ROC include the unit circle?

[2+4=6 points]

Solution: a) We note that $h[n] = 0$ for $n < 0$ and hence the ROC is the outside of a circle centred at origin.

b) To check this, we need to check if the LTI system is stable, i.e., if $\sum_{n=-\infty}^{\infty} |h[n]|$ converges. To do so, we employ the root test, and note that $(|h[n]|)^{\frac{1}{n}} = \frac{1}{9}n3^{-\frac{1}{n}}$. So, $\lim_{n\to\infty}(|h[n]|)^{\frac{1}{n}} = \infty > 1$, and hence $\sum_{n=-\infty}^{\infty} |h[n]|$ diverges. So the unit circle is not contained within the ROC. ■

3) Consider the LTI system with the difference equation

$$y[n] = 5y[n-1] - 8y[n-2] + 4y[n-3] + x[n] + x[n-1] + x[n-2] - 4x[n-4].$$

a) Find the transfer function $H(z)$.
b) Write a Python code to print and plot the poles and zeros of $H(z)$.
c) Compute $h[n]$.
   Note: You can get the poles of $H(z)$ from the python code in step b). The poles are integers, so do round-off the poles returned by the python code to the nearest integer.
d) Does the ROC of $h[n]$ include the unit circle?

[2+3+6+1=12 points]

Solution: a) By taking z-transform, the difference equation becomes $Y(z)(1 - 5z^{-1} + 8z^{-2} - 4z^{-3}) = X(z)(1 + z^{-1} + z^{-2} - 4z^{-4})$. Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + z^{-2} - 4z^{-4}}{1 - 5z^{-1} + 8z^{-2} - 4z^{-3}}.$$

c) From the Python code, there is a second order pole at 2 and a first order pole at one. Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + z^{-2} - 4z^{-4}}{(1 - 2z^{-1})^2(1 - z^{-1})}.$$

Notice that the denominator polynomial has degree lower than the numerator polynomial. So, by Euclidean division, we have $H(z) = 2 + z^{-1} + U(z)$, where

$$U(z) = \frac{-1 + 10z^{-1} - 10z^{-2}}{(1 - 2z^{-1})^2(1 - z^{-1})}.$$

So,

$$\frac{U(z)}{z} = \frac{-z^2 + 10z - 10}{(z-2)^2(z-1)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}.$$

Using the method of partial fractions, we have $A = (z-1)\frac{U(z)}{z}|_{z=1} = -1$. Also, $C = (z-2)^2\frac{U(z)}{z}|_{z=2} = 6$. To get B, we use

$$\begin{aligned}
B &= \frac{\mathrm{d}}{\mathrm{d}z}(z-2)^2\frac{U(z)}{z}|_{z=2} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\frac{-z^2 + 10z - 10}{z-1}|_{z=2} \\
&= \frac{(z-1)(-2z+10) - (-z^2 + 10z - 10)}{(z-1)^2}|_{z=2} \\
&= 0.
\end{aligned}$$

So, we have

$$H(z) = 2 + z^{-1} - \frac{z}{z-1} + \frac{6z}{(z-2)^2}.$$

Hence, using the formulas given in class, $h[n] = v[n] - u[n] + 3n2^n u[n-1]$, where $v[n] = [2, 1]$.

d) Note that $H(z)$ has poles at both $z = 1$, and $z = 2$. Then, the ROC cannot contain these points. So, it cannot contain the unit circle. ■

4) Consider the signal $x[n] = [1, 2, 3, 4, 5]$. Define $Y(k) = X(\frac{\pi}{2}k)$ for $k = 0, 1, 2, 3$, where $X(\omega)$ is the DTFT of $x[n]$. Now consider the sequence $y[n]$, for $n = 0, 1, 2, 3$, obtained by taking the 4-point IDFT of $Y(k), k = 0, 1, 2, 3$. What is $y[n]$?

[3 points]

Solution: Note that the 4-point IDFT will produce $y[n] = x_p[n] = \sum_{l=-\infty}^{\infty} x[n - 4l]$. Note that when $n = 0, 1, 2, 3$, the only signals in the above sum that have non-zero values are $x[n]$ and $x[n+4]$. Adding these two, we get $y[n] = [6, 2, 3, 4]$. ■

5) Consider the signal $x[n] = \cos\big(2\sqrt{3}\pi n\big)$.

a) Is $x[n]$ periodic?

b) Consider the LTI system with impulse response $h[n] = u[n]$. Find the output $y[n]$ obtained by passing $x[n]$ through this system.

[1+3.5=4.5 points]

Solution: a) No, because $\sqrt{3}$ is irrational.

b) Here, $H(z) = \frac{1}{1-z^{-1}}$, and so

$$\begin{aligned} H(\omega) &= \frac{1}{1 - e^{-j\omega}} \\ &= \frac{1}{1 - \cos\omega + j\sin\omega}. \end{aligned}$$

So,

$$\begin{aligned} |H(\omega)| &= \frac{1}{\sqrt{(1-\cos\omega)^2 + \sin^2\omega}} \\ &= \frac{1}{\sqrt{2 - 2\cos\omega}} \\ &= \frac{1}{\sqrt{4\sin^2\frac{\omega}{2}}} \\ &= \frac{1}{2\sin\frac{\omega}{2}}, \end{aligned}$$

and

$$\begin{aligned}
\angle H(\omega) &= \pi - \tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right) \\
&= \pi - \tan^{-1}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}}\right) \\
&= \pi - \tan^{-1}\cot\frac{\omega}{2} \\
&= \pi - \frac{\pi}{2} + \frac{\omega}{2} \\
&= \frac{\pi}{2} + \frac{\omega}{2}.
\end{aligned}$$

So, $y[n] = |H(2\sqrt{3}\pi)|\cos\bigl(2\sqrt{3}\pi n + \angle H(2\sqrt{3}\pi)\bigr) = \frac{1}{2\sin\left(\sqrt{3}\pi\right)}\cos\bigl(2\sqrt{3}\pi n + \frac{\pi}{2} + \sqrt{3}\pi\bigr).$ ■