Discrete Mathematics, CSE 121 : Mid-Semester Exam, Monsoon 2023

General Instructions:

(a) Maximum marks = 25; Duration: 1 hours.

(b) This is a closed-book exam. The exam paper is self-contained. Any attempt to use any source during the exam will be dealt with according to the Academic Dishonesty Policy of the institute.

(c) In every proof/derivation clearly state your assumptions and give details of each step.

(d) You will be evaluated for your attempt and approach. Therefore, you are encouraged to attempt the questions even if you can not complete an answer. The Academic Dishonesty Policy of the institute is equally applicable even for partial answers.

Questions:

1. Given a set A, consider a relation $R \subseteq A \times A$. Using quantifiers express that R is a partial order. Recall that a partial order is a relation that is reflexive, antisymmetric, and transitive. [2 marks]

2. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ iff $a + d = b + c$. Show that R is an equivalence relation. Recall that an equivalence relation is reflexive, symmetric, and transitive. [3 marks]

3. Consider the real numbers $a_1, a_2, \ldots, a_n$ and their arithmetic mean A. Prove that $\exists i \in \{1, 2, \ldots, n\}$ such that $a_i \geq A$. [4 marks]

4. Prove that $a_{m,n} = m + n \; \forall (m, n) \in \mathbf{N} \times \mathbf{N}$ if it is defined recursively by $a_{0,0} = 0$ and the following:

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & if \ n = 0 \ and \ m > 0 \\ a_{m,n-1} + 1 & if \ n > 0 \end{cases}$$

   [5 marks]

5. Prove that it is always possible to tile a $2^n \times 2^n$ checkerboard with one square removed using right triominoes. [5 marks]

6. Prove that for all perfect squares n^2, if it is a multiple of 3 then so is n for all positive integers n. Using this fact and strong induction, prove that $\sqrt{3}$ is irrational. [1+5 marks]

Solutions Minor typos in the question paper were rectified during the exam.

1. $\forall a, b, c\,((a,a) \in R \wedge ((a,b) \in R \wedge (b,a) \in R \Rightarrow a = b) \wedge ((a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R))$.

2. For reflexivity, $((a,b),(a,b)) \in R$ because $a+b = b+a$. For symmetry, $((a,b),(c,d)) \in R$ if and only if $a+d = b+c$, which is equivalent to $c+b = d+a$, which is true if and only if $((c,d),(a,b)) \in R$. For transitivity, suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Thus we have $a+d = b+c$ and $c+e = d+f$. Adding, we obtain $a+d+c+e = b+c+d+f$. Simplifying, we have $a+e = b+f$, which tells us that $((a,b),(e,f)) \in R$.

3. Suppose that $a_i < A \forall i \in \{1, 2, \ldots, n\}$. Adding these inequalities, we see that $\frac{a_1+a_2+\ldots+a_n}{n} < A$, which contradicts the definition of A: $\frac{a_1+a_2+\ldots+a_n}{n} = A$. Thus a proof by contradiction.

4. The basis step requires that we show that this formula holds when $(m,n) = (0,0)$. The inductive step requires that we show that if the formula holds for all pairs smaller than (m,n) in the lexicographic ordering of $N \times N$, then it also holds for (m,n). For the basis step we have $a_{0,0} = 0 = 0 + 0$. For the inductive step, assume that $a_{m',n'} = m' + n'$ whenever (m',n') is less than (m,n) in the lexicographic ordering of $N \times N$. By the recursive definition, if $n = 0$ then $a_{m,n} = am - l, n + 1$; since $(m-1,n)$ is smaller than (m,n), the inductive hypothesis tells us that $a_{m-l,n} = m - 1 + n$, so $a_{m,n} = m - 1 + n + 1 = m + n$, as desired. Now suppose that $n > 0$, so that $a_{m,n} = a_{m,n-l} + 1$. Again we have $a_{m,n-l} = m+n-1$, so $a_{m,n} = m+n-1+1 = m+n$, and the proof is complete.

5. This problem was amply discussed in the class. This is Example 14 of Section 5.1 from the book. Reproducing the solution here:

   For the basis step, show that $P(n)$ is true when $n = 1$, where $P(n)$ is the statement to be proved.

   For the inductive step, refer to the the discussion in the book as shown in Figure 1.

   Note that, drawing the illustrative figure is not important and no marks will be either awarded for just drawing the figure, or for that matter, no marks will be deducted for not drawing the illustrative figure. The arguments for the proof alone will be evaluated.

6. The first part will be proved by contradiction/contrapositive: supposing that n is not a multiple of 3, that is $n = 3t + 1$ or $n = 3t + 2$, n^2 will not be a multiple of 3.

   For the second part, let $P(n)$ represent that there is no positive integer b such that $\sqrt{3} = n/b$. For the basis step, $P(1)$ is true because $\sqrt{3} > 1 \geq 1/b$ for all positive integers b. For the inductive step, assume that $P(j)$ is true for all $j \leq k$, where k is a positive integer. We need to prove that $P(k+1)$ is true.

Assume the contrary: $\sqrt{3} = (k+1)/b$ for some positive integer b. Squaring both sides and multiplying them by b^2, we have $3b^2 = (k+1)^2$. Which implies that $(k+1)^2$ is a multiple of 3. Therefore, $k+1$ is a multiple of 3 as well. Therefore we can write $k+1 = 3t$ for some positive integer t. Substituting, we have $3b^2 = 9t^2$, so $b^2 = 3t^2$. By the same reasoning as before, b is a multiple of 3, so $b = 3s$ for some positive integer s. Then we have $\sqrt{3} = (k+1)/b = (3t)/(3s) = t/s$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete.

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(k)$ is true for the positive integer k; that is, it is the assumption that every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right triominoes. It must be shown that under the assumption of the inductive hypothesis, $P(k+1)$ must also be true; that is, any $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

To see this, consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions. This is illustrated in Figure 6. No square has been removed from three of these four checkerboards. The fourth $2^k \times 2^k$ checkerboard has one square removed, so we now use the inductive hypothesis to conclude that it can be covered by right triominoes. Now temporarily remove the square from each of the other three $2^k \times 2^k$ checkerboards that has the center of the original, larger checkerboard as one of its corners, as shown in Figure 7. By the inductive hypothesis, each of these three $2^k \times 2^k$ checkerboards with a square removed can be tiled by right triominoes. Furthermore, the three squares that were temporarily removed can be covered by one right triomino. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard can be tiled with right triominoes.

We have completed the basis step and the inductive step. Therefore, by mathematical induction $P(n)$ is true for all positive integers n. This shows that we can tile every $2^n \times 2^n$ checkerboard, where n is a positive integer, with one square removed, using right triominoes. ◀

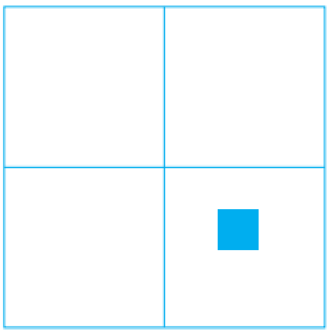

FIGURE 6 **Dividing a $2^{k+1} \times 2^{k+1}$ Checkerboard into Four $2^k \times 2^k$ Checkerboards.**

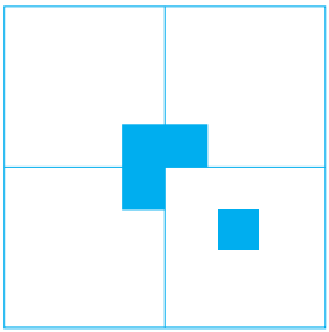

FIGURE 7 **Tiling the $2^{k+1} \times 2^{k+1}$ Checkerboard with One Square Removed.**

Figure 1: