MTH-204: Worksheet 9

12 April, 2023

Consider the Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,\ n = \text{constant}$$

Any solution of this equation is called Legendre's function. Let $k = n(n+1)$.

1. Substitute $y = \sum_{m=0}^{\infty} a_m x^m$ in Legendre's equation and arrive at (1)

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0$$

2. Find a general formula for a_s by equating $x^0, x^1 \&\ x^n,\ n \geq 2$ coefficients equal to zero. (1)

3. Show that series solution is of the form (1)

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1(x)$ contains only even powers of x while $y_2(x)$ contains only odd powers of x.

4. Find first four terms of $y_1(x)$ & $y_2(x)$. (2)

5. Recall that $k = n(n+1)$. Show that when n is a non-negative even integer $y_1(x)$ reduces to a polynomial and when n is a non-negative odd integer $y_2(x)$ reduces to a polynomial. (2)

6. For $k = n(n+1)$ & $n \geq 0$ an integer, if we choose highest coefficient a_n of polynomial $y_1(x)$ or $y_2(x)$ (depending on if n is even or odd), find $p_0(x),\ p_1(x),\ p_2(x),\ p_3(x),\ p_4(x)$ & $p_5(x)$ where $p_n(x)$ is n^{th} Legendre polynomial. Note that (3)

$$p_n(x) = \begin{cases} y_1(x) & \text{if } n \text{ even} \\ y_2(x) & \text{if } n \text{ odd} \end{cases}$$

and highest coefficient $a_n = \frac{(n!)^2}{2^n (n+1)!}$.