MTH100 **Monsoon Semester – 2021-22** **Section B** (1)

Submission for Tuesday 15th March 2022 – 17 minutes. Max Marks: 5

Instructions: *Open notes and textbook; consultation and use of calculators, computers and internet not allowed. You may use any* ***known*** *result. This includes all propositions and observations in the lecture slides, and results from tutorials. If you use any other result from any other source, including the textbook, you have to give a full proof of that result.*

Given $V = \mathbb{R}^{2\times 2}$, i.e. the vector space of all 2×2 matrices with real entries, and its two subspaces $U = \left\{\begin{bmatrix} a & b \\ c & a \end{bmatrix} : a,b,c \in \mathbb{R}\right\}$ and $W = \left\{\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} : a,b \in \mathbb{R}\right\}$.

a) Find bases for U and W respectively. Justify your answer briefly. *(3 marks)*

b) Determine the dimensions of U + W and U ∩ W. Justify your answer briefly. *(2 marks)*

Rubric

a) Basis for U → 0.5 marks
(all 3 matrices must be shown, else 0 marks)
Justify (only if basis is correct):
0.5 marks for lin. indep.
0.5 marks for spanning
(Total: 1.5 for U)
Basis for W → 0.5 marks
(all 2 matrices must be shown, else 0 marks)
&

(PTO)

(Cont'd)

Justify (only if basis correct) :-

0.5 marks for lin. indep.

0.5 marks for spanning

(Total: 1.5 for W)

(b) Dimension of $U+W$ → 0.5 mark

Justify → 0.5 marks

Dimension of $U \cap W$ → 0.5 marks

Justify → 0.5 marks

(Remark: Minor variations in justification are acceptable).

SOLUTION Below

~~2~~ 3

a) A basis for U consists of the vectors (matrices)

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Clearly, if $X_1 = \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in U$, then

$$X_1 = a A_1 + b E_{12} + c E_{21}.$$

So these three matrices form a spanning set for U. Furthermore, if $a A_1 + b E_{12} + c E_{21} = \bar{0}$

then $$\begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = c = 0.$$

So they are also lin. indep.

* A basis for W consists of $A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

As before, if $X_2 = \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in W$, then

$X_2 = a A_{21} + b E_{22}$, so these matrices form a spanning set.

Also, if $a A_{21} + b E_{22} = \bar{0}$, then $\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Longrightarrow a = 0$ and $b = 0$.

So, they are also lin. indep.

(b) We see that the matrix $A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \in W$,

but $A_3 \notin U$.

Hence $U \underset{\neq}{\subset} U + W$.

But $\dim U = 3$, and since $\dim V = 2 \times 2 = 4$,

it follows that $U + W = V$, i.e. $\dim (U+W) = 4$.

Applying Proposition 20, we get that

$$\dim (U+W) = \dim U + \dim W - \dim (U \cap W)$$

or $$4 = 3 + 2 - \dim (U \cap W).$$

Hence, $\dim (U \cap W) = 5 - 4 = 1$

thm.