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MTH100 – GRADE IMPROVEMENT EXAMINATION 20220326

TIME: 1 HOUR **MAXIMUM MARKS: 40**

NB: You may use any known result (i.e. propositions and lemmas stated in lecture slides, and results from tutorial problems) without proof; however, it should be identified clearly. Other than the above, results from the textbook or other sources cannot be used without proof. Marks will depend on the correctness and completeness of your proofs. All questions have equal marks. Start each question on a fresh page (side).

1. Given the matrix A below.

a. Find an LU decomposition of A. (*3 marks*)

b. Use the LU decomposition of part a. to find a solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{v}$, where $\mathbf{v}$ is a general vector $\mathbf{v} = (a,b,c)$. **(NB: Do not use any other method. Your answer should be presented as a formula involving** a, b, c.) (*5 marks*)

c. Solve the system $A\mathbf{x} = \mathbf{v}$ for the vectors $\mathbf{v}_1 = (6, 6, 6)$ and $\mathbf{v}_2 = (12, -6, 18)$ using the formula of part b. or by any other method. (*2 marks*)

(NB: You will not receive marks unless your work is clearly shown.)

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

2. a) Show that the back-shift function $S: \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $S(\langle a_1, a_2, a_3, \ldots\rangle) = \langle a_2, a_3, a_4, \ldots \rangle$ is a linear operator. Describe its kernel. (*3 marks*)

b) Show that the function $T: C^{(1)}[\mathbb{R}] \to C[\mathbb{R}]$ given by $T(f(x)) = f(x) - f'(x)$ for all functions $f(x) \in C^{(1)}[\mathbb{R}]$ is a linear transformation. Describe its kernel. (*NB: Here $C^{(1)}[\mathbb{R}]$ denotes the vector space of all continuous real-valued functions defined on the real line which have a continuous first derivative, and $C[\mathbb{R}]$ denotes the space of all continuous real-valued functions on the real line.*) (*3 marks*)

c) Let $V = \mathbb{R}_2[t]$ and let $W = \{p(t) \in V: p(t) = tp'(t)\}$. Show that W is a subspace of V, and find a basis for W and hence its dimension, with brief explanation. (*4 marks*)

3. Let A be an n×n square matrix ($n \geq 2$) such that $A^2 = \mathbf{0}$, i.e. the zero matrix, PROVE or DISPROVE: rank (A) $\leq n/2$. (*NB*: **You must clearly write PROVE or DISPROVE at the top of your answer in capital letters.** *1 mark is reserved for this. If not done, your answer will not be considered.*)

4. Let $V = \mathbb{R}^{2\times 2}$ = vector space of 2×2 matrices with real entries, and consider the function $U: V \to V$ given by $U(A) = DA$, for all $A \in V$, where D is the fixed matrix $D = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$. Here c and d are **distinct positive** real numbers..

a) Show that U is a linear operator. (*2 marks*)

b) Determine the matrix of U with regard to any suitable ordered basis β of V. (***Remark: the choice of ordered basis is left to you, but should be clearly specified.***) (*5 marks*)

c) Determine the nullity and rank of U, with brief explanation. (*3 marks*)

SOLUTIONS & RUBRIC

FOLLOW. NOT NECESSARILY IN SAME ORDER.

RUBRIC WILL COME AT THE END OF THE SOLUTION.

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(3)

MTH100

Solutions and Rubrics for [redacted] Exams [redacted]

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List of Common Errors and Marks Deductions:

1. Using an undefined symbol. Please deduct 1/2 mark each time this is done.
2. Writing an equation in which the LHS and RHS are not comparable, for example, if the LHS is an $m \times n$ matrix and the RHS is a real number. Please deduct 1/2 mark each time this is done.
3. Writing a meaningless or completely illogical statement. Please deduct 1 mark for every meaningless statement.
4. Please deduct 1/2 mark for every calculation mistake.

Above deductions to be applied while checking answer-books, specially for proof-type questions.

Q(1)a) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[\substack{e_1 : R_2 \to R_2 - 2R_1 \\ e_2 : R_3 \to R_3 + R_1}]{} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix}$

$\xrightarrow[e_3 : R_3 \to R_3 + 2R_2]{} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

$= U$

$\begin{cases} f_3 : R_3 \to R_3 - 2R_2 \\ f_2 : R_3 \to R_3 - R_1 \\ f_1 : R_2 \to R_2 + 2R_1 \end{cases}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[f_3 : R_3 - 2R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\xrightarrow[f_2 : R_3 \to R_3 - R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

$\xrightarrow[f_1 : R_2 \to R_2 + 2R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = L$

$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = A$

b) $Ax = v$

$L\bar{y} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\Rightarrow y_1 = a, \quad 2y_1 + y_2 = b, \quad -y_1 - 2y_2 + y_3 = c$

$\Rightarrow y_2 = b - 2a \qquad \Rightarrow y_3 = c + a + 2(b - 2a) = c + 2b - 3a$

$$U\bar{x} = \bar{y} \Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c+2b-3a \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 + 3x_3 = a$$

$$-3x_2 - 3x_3 = b-2a$$

$$2x_3 = c + 2b - 3a$$

$$\Rightarrow x_3 = \frac{c+2b-3a}{2}$$

$$x_2 = \frac{-b+2a - 3x_3}{3}$$

$$= \frac{-b+2a}{3} - \left(\frac{c+2b-3a}{2}\right) = \frac{13a-8b-3c}{6}$$

Now, $x_1 = \dfrac{a - x_2 - 3x_3}{2} = \dfrac{a - \left(\frac{13a-8b-3c}{6}\right) - 3\left(\frac{c+2b-3a}{2}\right)}{2}$

$$= \frac{6a-13a+8b+3c-9c-18b+27a}{6 \times 2}$$

$$= \frac{20a-10b-6c}{12} = \frac{10a-5b-3c}{6}$$

$$\bar{x} = \left[\frac{10a-5b-3c}{6}, \frac{13a-8b-3c}{6}, \frac{c+2b-3a}{2}\right]$$

(c) For $v_1 = (6,6,6)$ we have $\bar{x}_1 = \left[\dfrac{60-30-18}{6}, \dfrac{78-48-18}{6}, \dfrac{6+12-18}{2}\right]$

$$= [2, 2, 0]$$

For. $v_2 = [12, -6, 18]$, $\bar{x}_2 = [16, 25, -15]$

Rubric:

a) 2 marks for calculating U correctly and 1 mark for calculating L correctly. If steps not shown cut 50%.

b) 2 marks for calculating the solution of $Ly = u$ correctly and 3 marks for calculating the solution of $Ux = y$ correctly. If steps not shown, cut 50%.

c) 1 mark for each correct solution. If steps not shown, cut 50%.

Q3. PROVE (4)

Proof: Put rank$(A) = r$, nullity$(A) = k$, so that $r + k = n$ (1), by Rank The

Suppose that $\bar{w} \in$ ~~Range(A)~~ $COL(A)$,

so $\bar{w} = A\bar{v}$ for some $\bar{v} \in \mathbb{R}^n$.

Then $A\bar{w} = A(A\bar{v}) = A^2\bar{v} = \bar{0}\bar{v}$

$= \bar{0}$ zero vector zero matrix

(2)

So, $\bar{w} \in Nul(A)$

$\Rightarrow$ ~~Nul(A)~~ $Col(A) \subseteq Nul\,A$ (3)

Hence, $k \geq r$. (4)

So now, suppose BWOC that $r > \frac{n}{2}$.

$\therefore r + k \geq 2r$, using (4)

$> 2\,n/2 = n \Rightarrow\Leftarrow$ to (1)

Result follows.

RUBRIC

PROVE $\rightarrow$ 1 mark

Proof $\rightarrow$ 9 marks for a correct proof. ~~Ma~~ It could be a minor variation of above.

NO PARTIAL CREDIT.

Marks may be cut as per general guidelines ~~at the start~~ on page 3.

(5)

Q 4 a) Let $A, B \in V$ and let $c \in \mathbb{R}$.

Then (i) $U(A+B) = D(A+B) = DA + DB$

$= U(A) + U(B)$ → additivity

and (ii) $U(cA) = D(cA) = c(DA)$

$= c\,U(B)$ → homogeneity

(b) We consider the ordered basis

$$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

Then $U(E_{11}) = DE_{11} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ d \\ 0 \end{bmatrix}_\beta \qquad (1)$$

$U(E_{12}) = DE_{12} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ 0 \\ d \end{bmatrix}_\beta \qquad (2)$$

$U(E_{21}) = DE_{21} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} d & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ c \\ 0 \end{bmatrix}_\beta \qquad (3)$$

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Q 4 (b) – cont'd

Finally,

$$U(E_{22}) = D E_{22} = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 \\ d \\ 0 \\ c \end{bmatrix}_\beta \quad (4)$$

Taking the answers of (1) to (4) as columns, we get

$$[U]_\beta = \begin{bmatrix} c & 0 & d & 0 \\ 0 & c & 0 & d \\ d & 0 & c & 0 \\ 0 & d & 0 & c \end{bmatrix} \quad (5)$$

Answer: NB: Obviously the answer would be different if a different β is chosen.

(C) Suppose $A \in \operatorname{Ker}(U)$.

Then, $U(A) = \bar{O} \Rightarrow DA = \bar{O}$

$\Rightarrow D^{-1}(DA) = D^{-1}\bar{O} \Rightarrow IA = \bar{O} \Rightarrow A = \bar{O}$

(NB: Since $\det D = c^2 - d^2 \neq 0$, D is invertible)

$\therefore$ nullity$(U) = 0$ and so rank $(U) = 4$, by Rank Theorem.

RUBRIC FOR Q4. (7)

4 a) 1 mark for additivity
1 mark for homogeneity

4 (b) 0.5 marks if the matrix is a 4×4 matrix
0.5 marks if β is explicitly specified.
1 mark for showing the steps - this may be given even if β is not explicitly specified if steps are meaningful.
3 marks for a final correct matrix.

4 (c) Rank = 4 $\rightarrow$ 0.5 marks (*)
Nullity = 0 $\rightarrow$ 0.5 mark (**)
Justify $\rightarrow$ 1 mark for each
(Total = 0.5 + 0.5 + 1 + 1 = 3)

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Q 2(a)

(i) Let $\langle a_n \rangle, \langle b_n \rangle \in \mathbb{R}^\infty$

Then: $S(\langle a_n \rangle + \langle b_n \rangle) = S\langle a_n + b_n \rangle$

$= \langle a_2 + b_2, a_3 + b_3, \ldots \rangle$ ①

OTOH, $S\langle a_n \rangle + S\langle b_n \rangle = \langle a_2, a_3, \cdots \rangle$

$+ \langle b_2, b_3, \cdots \rangle = \langle a_2 + b_2, a_3 + b_3, \cdots \rangle$ ②

Comparing ① and ②, we get additivity.

If $c \in \mathbb{R}$, then $S(c\langle a_n \rangle) = S\langle c a_n \rangle$

$= \langle c a_2, c a_3, \cdots \rangle$ ③

OTOH, $c\, S\langle a_n \rangle = c\langle a_2, a_3, \cdots \rangle$

$= \langle c a_2, c a_3, \cdots \rangle$ ④

Comparing ③ and ④, we get homogeneity.

~~Let~~ If $\langle a_n \rangle \in \operatorname{Ker} S$, then $a_2 = a_3 = \cdots$

$= a_n = \cdots = 0$.

So, $\operatorname{Ker} S = \operatorname{Span}\langle e_1 \rangle$, where $\langle e_1 \rangle$ is the sequence $\langle 1, 0, 0, \cdots \rangle$

(b) Let $f(x), g(x) \in C^{(1)}[\mathbb{R}]$ and let $c \in \mathbb{R}$.

Then: $T(f(x) + g(x)) = f(x) + g(x)$

$$- [f(x) + g(x)]' = f(x) + g(x)$$

$$- (f'(x) + g'(x)) = [f(x) - f'(x)] +$$

$$[g(x) - g'(x)] = T(f(x)) + T(g(x)) \quad (1)$$

Again, $T(cf(x)) = cf(x) - [cf(x)]'$

$$= cf(x) - cf'(x) = c(f(x) - f'(x))$$

$$= cT(f(x)) \quad (2)$$

(1) and (2) show additivity and homogeneity, respectively.

Recall that if $f(x) = e^{x}$, then $f'(x) = f(x)$.

Now, $Tf = 0 \Rightarrow f(x) - f'(x) = 0$

$\Rightarrow f(x) = f'(x)$.

So $\ker T = \operatorname{Span}\{e^{x}\}$.

<u>NB</u>: This can be shown more rigorously by solving the differential equation

$$\frac{dy}{dx} = y$$

Q2(c) If $O(t)$ represents the zero polynomial, then $O(t) = O(t) - t\,O'(t)$ so $O(t) \in W$.

If $p(t), q(t) \in W$, then $p(t) = t p'(t)$ and $q(t) = t q'(t)$, so

$$p(t) + q(t) = t p'(t) + t q'(t) = t\left[(p+q)(t)\right]' \quad ①$$

and if $c \in \mathbb{R}$, and $p(t) \in W$,

Then ~~(cp)'(t)~~ $[c p(t)]' = c\, p'(t) = (cp)(t)$. ②

① and ② prove closure, so W is a subspace by Prop. 8.

Finally, suppose $p(t) = a + bt + ct^2 \in W$

Then $p(t) = t p'(t) \Rightarrow a + bt + ct^2$

$$= t(b + 2ct) = bt + 2ct^2$$

$\Rightarrow a = 0$ and $c = 2c \Rightarrow c = 0$.

$\therefore p(t) = bt$ for some $b \in \mathbb{R}$

$\therefore \dim W = 1$ and $f(t) = t$ is a basis vector (polynomial) for W.

Q2. RUBRIC

(a) and (b) 1 mark for additivity
1 mark for homogeneity
1 mark for kernel
(a vague or descriptive answer is not acceptable — should be in mathematical terminology — however, since it's an observation, justification need not be given.

(c) 0.5 marks for zero
1 mark
1 marks } additive and scalar multiplication closure
0.5 marks for dimension
1 mark for basis of which 0.5 for reasoning and 0.5 mark for the basis.
The basis has to consist of a single fixed polynomial,
an answer like "all polynomials only having a single term which is of degree 1" is not acceptable.