Discrete Math, CSE 121 : End-Sem Exam, Monsoon 2022

General Instructions:

(a) Maximum marks = 35; Duration: 2 hours.

(b) This is a closed-book exam. The exam paper is self-contained. Any attempt to use any source during the exam will be dealt with according to the Academic Dishonesty Policy of the institute.

(c) In every proof/derivation clearly state your assumptions and give details of each step.

(d) You will be evaluated for your attempt and approach. Therefore, you are encouraged to attempt the questions even if you can not complete an answer. The Academic Dishonesty Policy of the institute is equally applicable even for partial answers.

Questions:

1. What is wrong with this "proof?"– "Theorem:" *If n^2 is positive, then n is positive.* "Proof: *Suppose that n^2 is positive. Because the conditional statement "If n is positive, then n^2 is positive" is true, we can conclude that n is positive"*. [1 mark]

   Solution:Let $P(n)$ be "n is positive" and $Q(n)$ be "n^2 is positive." Then our hypothesis is $Q(n)$. The statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \to Q(n))$. From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \to Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of affirming the conclusion. A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative.

2. Show that if a, b, k, and m are integers such that $k \geq 1$, $m \geq 2$, and $a \equiv b(mod\ m)$, then $a^k \equiv b^k(mod\ m)$. [2 marks]

   Solution: Using the algebraic identity $a^k - b^k = (a-b)(a^{k-l} + a^{k-2}b + \ldots + ab^{k-2} + b^{k-l})$. Specifically, the hypothesis that $a \equiv b(mod\ m)$ means that $m|(a-b)$. Therefore by Theorem "Let a, b, and c be integers, where $a \neq 0$. Then if $a|b$, then $a|bc$ for all integers c", m divides the right-hand side of this identity, so $m|(a^k - b^k)$. This means precisely that $a^k \equiv b^k(mod\ m)$. (It is acceptable if they do not write the theorem explicitly.)

   One can also prove it by mathematical induction. Since $a \equiv b(mod\ m)$, Theorem "Let m be a positive integer. If $a \equiv b(mod\ m)$ and $c \equiv d(mod\ m)$, then $ac \equiv bd(mod\ m)$." implies that $aa \equiv bb(mod\ m)$, i.e., $a^2 = b^2(mod\ m)$. Invoking this Theorem again, since $a \equiv b(mod\ m)$ and $a^2 \equiv b^2(mod\ m)$, we obtain $a^3 \equiv b^3(mod\ m)$. After $k-1$ applications of this process, we obtain $a^k \equiv b^k(mod\ m)$, as desired.

3. Prove that for every graph G, $\chi(G) \leq \Delta(G) + 1$, where $\chi(G)$ and $\Delta(G)$ are chromatic number and maximum vertex degree of G, respectively. [2 marks]

   Solution: If $|G| = 1$, the statement is trivial. Assume that the result is true for $|G| = n$ and let G be a graph on $n+1$ vertices, labeled $1, 2, \ldots, n+1$. Let $H = G - 1$ i.e. the graph obtained by removing the vertex 1 from G. As H is $(\Delta(G)+1)$-colorable and $d(1) \leq \Delta(G)$, the vertex 1 can be given a color other than its neighbors.

4. Show that by removing two white squares and two black squares from a standard 8×8 chessboard (wherein no two adjacent squares have the same color) you can make it impossible to tile the remaining squares using dominoes (a set of two adjacent square tiles). [2 marks]

   Solution: Remove the two black squares adjacent to one of the white corners, and remove two white squares other than that corner. Then no domino can cover that white corner, because neither of the squares adjacent to it remains.

   (Correct solution to this question is to give a construction as described above. However, in the class it was announced that a solution where they show a tiling of the chessboard with dominos after removing any two white and black squares will be acceptable. In essence, both proof and disproof of the given statement are acceptable.)

5. A connected graph G is called chromatically k-critical if the chromatic number of G is k, but for every edge of G, the chromatic number of the graph obtained by deleting this edge from G is $k-1$. Show that if G is chromatically k-critical, then the degree of every vertex of G is at least $k-1$. [3 marks]

   Solution: We give a proof by contradiction. Suppose that G is chromatically k-critical but has a vertex v of degree $k-2$ or less. Remove from G one of the edges incident to v. By definition of "k-critical," the resulting graph can be colored with $k-1$ colors. Now restore the missing edge and use this coloring for all vertices except v. Because we had a proper coloring of the smaller graph, no two adjacent vertices have the same color. Furthermore, v has at most $k-2$ neighbors, so we can color v with an unused color to obtain a proper $(k-1)$-coloring of G. This contradicts the fact that G has chromatic number k. Therefore our assumption was wrong, and every vertex of G must have degree at least $k-1$.

6. Consider the recurrence relation $f_{n+2} = f_{n+1} + f_n$, $f_0 = 0, f_1 = 1$, when n is a non-negative integer. Show that $f_0f_1 + f_1f_2 + \ldots + f_{2n-1}f_{2n} = f_{2n}^2$, when n is a positive integer. [4 marks]

   Solution: We prove this using the principle of mathematical induction. The basis step is for $n = 1$, and in that case the statement $f_0f_1 + f_1f_2 = f_2^2$

   is true since $01 + 11 = 1^2$. Next we assume the inductive hypothesis, that $f_0f_1 + f_1f_2 + \ldots + f_{2n-1}f_{2n} = f_{2n}^2$, and prove that $f_0f_1 + f_1f_2 + \ldots + f_{2n-1}f_{2n} + f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} = f_{2n+2}^2$. Which is as the following:

$$f_0f_1 + f_1f_2 + \ldots + f_{2n-1}f_{2n} + f_{2n}f_{2n+1}$$
$$= f_{2n}^2 + f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} \text{ (using inductive hypothesis)}$$
$$= f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1}f_{2n+2} \text{ (by factoring)}$$
$$= f_{2n}f_{2n+2} + f_{2n+1}f_{2n+2} \text{ (by definition of Fibonacci numbers)}$$
$$= (f_{2n} + f_{2n+1})f_{2n+2} \text{ (by factoring)}$$
$$= (f_{2n+2})f_{2n+2} = f_{2n+2}^2 \text{ (by definition of Fibonacci numbers)}$$

7. Given a graph $G = (V, E)$. The complement of G is defined as $G^c = (V', E')$ such that $V' = V$ and $E' = \{(u, v) : u \neq v, (u, v) \notin E\}$. Let G be a graph on 9 vertices. Then, prove that either $K_4 \subseteq G$ or $K_3^c \subseteq G$. [5 marks]

   Solution: This proof has to be exhaustive, i.e. by cases. Giving an instance is not a proof.

   Assume that $|V| = 9$. Then, we need to consider three cases.

Case I. There is a vertex a with $d(a) \leq 4$. Then, $|(N(a))^c| = |V\ N(a)| \geq 4$. If all vertices in $(N(a))^c$ are pairwise adjacent, then $K_4 \subseteq G$. Otherwise, there are two non-adjacent vertices, say $b, c \in (N(a))^c$. In that case a, b, c induces the graph K_3^c.

Case II. There is a vertex a with $d(a) \geq 6$. If the graph induced by $N(a)$ has a K_3^c subgraph, we are done. Otherwise, using Ramsay's number $r(3,3) = 6$ implies that the graph induced by $N(a)$ has a K_3 with vertices, say, b, c, d. In that case a, b, c, d induces the graph K_4^c.

Case III. Each vertex has degree 5. This case is not possible as $\sum d(v)$ should be even.

8. Prove the theorem "If p is prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 (mod\ p)$. Furthermore, for every integer a we have $a^p \equiv a (mod\ p)$". [5 marks]

   Solution: This is Fermat's little theorem. The proof is in the tutorial slides uploaded herewith.

9. A Hamilton path in a directed graph is a directed path that goes through each vertex exactly once. Consider a simple directed graph $G = (V, E)$ such that if $u, v \in V,\ u \neq v$ then exactly one of $(u, v) \in E$ and $(v, u) \in E$ is true. Prove or disprove that G has a Hamilton path. [5 marks]

   Solution: We prove this by induction on the number of vertices. The described graph is a tournament. If a tournament has just one vertex, the claim is true – the path containing just the single vertex is Hamiltonian. Now assume we know the claim is true for all tournaments on n vertices, and consider a tournament G on $n + 1$ vertices. Let v be any vertex in G. If we delete v (and all edges with v as an endpoint), the remaining tournament on n edges must have a Hamiltonian path by the inductive hypothesis. Label the vertices in this path $v_1, v_2, \ldots v_n$. In the original tournament G, consider the possible orientations of the edges incident to v:

   There are three cases:

   Case 1: If vv_1 is an edge (i.e. the edge containing v and v_1 is oriented towards v_1), then there is a Hamiltonian path with vertex order $v, v_1, \ldots, v_n$.

   Case 2: If v_nv is an edge, there is a Hamiltonian path with vertex order $v_1, \ldots, v_n, v$.

   Case 3: If Case 1 and Case 2 do not hold, as you look through the edges incident to v in order (starting with the edge containing v_1, then the edge containing v_2, etc...) there must come a point where the edges switch from pointing towards v to pointing away from v. That is, there is at least one number $1 \leq i \leq n - 1$ for which v_iv is an edge and vv_{i+1} is an edge. Then there is a Hamiltonian path with vertex order $v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_n$. See Figure 1.

   Please note that this is a case of either A or B but not both such as either friend or enemy but not both. This is different from A or B. Essentially, this is an application of Ramsey numbers, which was taught in this course.

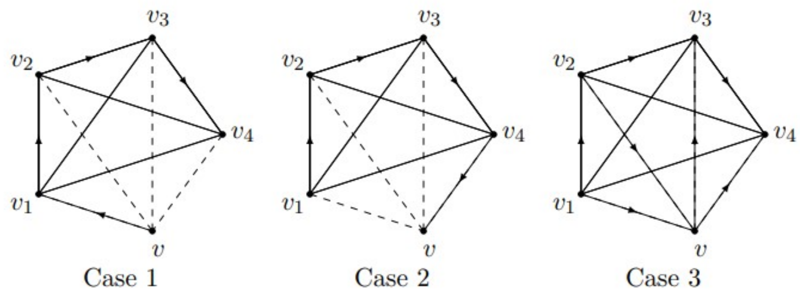

Figure 1: Three cases

10. Consider the procedure **MST** given in Figure 2. Prove or disprove that it produces a minimum spanning tree. [6 marks]

Solution: This is Prim's algorithm. The proof is in the tutorial slides uploaded herewith. Notice that some people have even disproved this. Give a 0 straightforward.

procedure MST (G: weighted connected undirected graph with n vertices)
$T :=$ a minimum-weight edge
for $i := 1$ **to** $n - 2$
  $e :=$ an edge of minimum weight incident to a vertex in T and not forming a
    simple circuit in T if added to T
  $T := T$ with e added
return T {T is a minimum spanning tree of G}

Figure 2: Minimum Spanning Tree Algorithm