MTH 204: Worksheet 4 solutions

1.

$$u(x,y) = c$$

$$\Rightarrow u_x dx + u_y dy = 0$$

$$\Rightarrow y' = \frac{dy}{dx} = -\frac{u_x}{u_y}$$

Orthogonal trajectory : $y' = \frac{u_y}{u_x}$ — ①

$$v(x,y) = c^*$$

$$\Rightarrow y' = -\frac{v_x}{v_y} \quad \text{— ②}$$

Now by ① & ② $\quad u_y = -v_x \quad \& \quad u_x = v_y$

$$u = e^x \sin y$$

$$\Rightarrow u_x = e^x \sin y \quad \& \quad u_y = e^x \cos y$$

$$v_y = u_x = e^x \sin y \Rightarrow v = -e^x \cos y + f(x)$$

$$\Rightarrow v_x = -e^x \cos y + f'(x)$$

$$= -u_y = -e^x \cos y$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow f(x) = c$$

$$\Rightarrow v(x,y) = -e^x \cos y + c = c^*$$

$$\Rightarrow e^x \cos y = c_1$$

(2.) Consider $y' = f(x)$. Then general solution is

$$y = \int f(x)\,dx + C$$
$$= g(x) + c$$

where c is arbitrary constant. So, for different values of c, all solution curves are parallel and hence congruent to each other.

OTs of family $y = g(x) + c$ will satisfy ODE $y' = -\frac{1}{f(x)}$.

The general solution is
$$y = -\int \frac{1}{f(x)}\,dx + C'$$
$$= h(x) + c'$$

Therefore, OTs are also congruent to each other.

(3.) $$25x^2 + 36y^2 = c \qquad (y > 0)$$

$$\Rightarrow 50x + 72yy' = 0$$

$$\Rightarrow y' = -\frac{50x}{72y} = \frac{-25x}{36y}$$

For orthogonal trajectory,

$$y' = \frac{36y}{25x}$$

$$\Rightarrow \frac{dy}{y} = \frac{36}{25}\frac{dx}{x}$$

$$\Rightarrow \log y = \frac{36}{25}\log x + \log c$$

$$\Rightarrow y = c\,x^{36/25} \qquad (c > 0)$$

4.

$$v = \frac{1}{dv/dt}$$

$$\Rightarrow v\,dv = dt$$

$$\Rightarrow \frac{v^2}{2} = t + \frac{v_0^2}{2}$$

$$\Rightarrow v = \sqrt{2t + v_0^2}$$

$$\frac{dx}{dt} = \sqrt{2t + v_0^2}$$

$$\Rightarrow x = x_0 + \frac{1}{3}(2t + v_0^2)^{3/2}$$

5.

$$(x-5)y' = y$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x-5}$$

$$\Rightarrow \log y = \log(x-5) + \log c$$

$$\Rightarrow y = c(x-5)$$

$$y(5) = a = 0$$

Hence, the IVP has a solution iff $a = 0$

This does not ~~present~~ contradict our present theorem

Since, $y' = \frac{y}{x-5} = f(x,y)$, $y(5) = a$

$f(x,y)$ is not continuous at all points (x,y) in some rectangle

$R: \ |x-5| < k_1 \ , \ |y-a| < k_2$

(6.) $y' = 3y^3 \ , \ y(1) = 1$

$R: \ |x-1| < a \ , \ |y-1| < b$

$f(x,y) = 3y^3 \leq 3(b+1)^3 = K$

$\alpha = \frac{b}{K} = \frac{b}{3(b+1)^3}$

$$\frac{d\alpha}{db} = \frac{1}{3(b+1)^3} - \frac{b}{(b+1)^4} = 0$$

$\Rightarrow \quad b+1 = 3b \quad \Rightarrow \quad b = \frac{1}{2}$

$\therefore \quad \alpha_{opt} = \frac{4}{81}$

$\frac{dy}{y^3} = 3\,dx \quad \Rightarrow \quad \frac{y^{-2}}{-2} = 3x + c$

$\Rightarrow \quad y^2 = \frac{-1}{6x+2c}$

$y(1) = 1 \quad \Rightarrow \quad 1 = \frac{-1}{6+2c} \quad \Rightarrow \quad c = -\frac{7}{2}$

$\therefore \quad y^2 = \frac{-1}{6x-7}$

So, solution exists if $|x-1| < \frac{1}{6}$

(7.) $y'' = k\sqrt{1+(y')^2}$

$k = 1, \quad y(-1) = y(1) = 0$

Put $\quad Y = y' \ , \ Y' = y'' = (1+Y^2)^{1/2}$

$\Rightarrow \quad \frac{dY}{(1+Y^2)^{1/2}} = dx$

$$\Rightarrow \sinh^{-1} Y = x + c$$

$$\Rightarrow Y = \sinh(x+c)$$

$$\Rightarrow y(x) = \int Y\,dx + c_1 = \cosh(x+c) + c_1$$

$$y(1) = \cosh(1+c) + c_1 = 0$$

$$y(-1) = \cosh(-1+c) + c_1 = 0$$

$$\Rightarrow c = 0 \quad \& \quad c_1 = -\cosh 1$$

So, $y(x) = \cosh x - \cosh 1$