11/30/2022

SOLUTION

MTH210 – SUBMISSION_20221201

TIME: 15 minutes MARKS: 5

No consultation – open notes – books and internet not allowed.

The k-cube (or hypercube) is the graph Q_k, whose vertices are ordered k-tuples (for $k \geq 1$) with entries from $\{0,1\}$, in which two vertices are adjacent if and only if they differ in exactly one position.

a. Find the number of vertices in the k-cube. (*0.5 marks*)
b. Find the degrees of the vertices of the k-cube. (*1 mark*)
c. Find the number of edges in the k-cube. (*1 mark*)
d. Is the k-cube bipartite (YES/NO) ? (*2.5 marks*)

For b., c., and d., justify your answer briefly.

ID: Wednesday, November 30, 2022 NAME: GROUP:

a. Number of vertices $= 2^k$

Each vertex is a k-tuple, each of whose entries can be chosen in 2 ways (0 or 1); this can be done in 2^k ways.

b. If u is a vertex, then each adjacent vertex differs in exactly one position. This one position can be chosen in k ways. $\therefore |N(u)| = k$.

Remark: Hence, Q_k is k-regular.

c. The number of edges $= k.2^{k-1}$.

Using the degree sum formula, (Prop. 30)

$2e(Q_k) = \sum d(u) = k.2^k$, so

$e(Q_k) = k.2^{k-1}$

d. YES — Q_k is bipartite.

For any vertex u, define parity(u) = sum of the entries of the k-tuple, u.

Let $O = \{u \in V(Q_k) : \text{parity}(u) \text{ is odd}\}$

$E = \{u \in V(Q_k) : \text{parity}(u) \text{ is even}\}$.

Since any vertex adjacent to u differs from u in exactly one position, the parity of all neighbours is different from the parity of u, or, any edge of Q_k joins vertices of different parity. Thus, $V(Q_k) = O \dot{\cup} E$ is a bipartition, with all edges joining vertices in one partite set with vertices in the other partite set. Thus, Q_k is bipartite.

Alternate Proof: Since any path in Q_k must join vertices with ~~odd parity / to even~~ alternating parities, any cycle in Q_k must be even. Result now follows from Proposition 32.