Worksheet #1 Solution

Problem 1.

(a)
$$y' + xe^{-x^2/2} = 0$$
$$\Rightarrow \frac{dy}{dx} + xe^{-x^2/2} = 0$$
$$\Rightarrow dy = -xe^{-x^2/2}\,dx$$
$$\Rightarrow \int dy = -\int xe^{-x^2/2}\,dx$$
$$\hookrightarrow u = -\frac{x^2}{2} \Rightarrow du = -x\,dx$$
$$\int e^u\,du = e^u = e^{-x^2/2}$$
$$\Rightarrow \boxed{y(x) = e^{-x^2/2} + C}$$

(b)
$$y' = 4e^{-x}\cos x \Rightarrow \frac{dy}{dx} = 4e^{-x}\cos x$$
$$\Rightarrow dy = 4e^{-x}\cos x\,dx \Rightarrow \int dy = \int 4e^{-x}\cos x\,dx$$
$$I = \int e^{-x}\cos x\,dx = -\cos x.e^{-x} - \int \sin x.e^{-x}\,dx$$
$$= -\cos x.e^{-x} + \sin x.e^{-x} - \int \cos x.e^{-x}\,dx$$
$$= e^{-x}(\sin x - \cos x) - I$$
$$\Rightarrow I = \frac{1}{2}e^{-x}(\sin x - \cos x)$$
$$\Rightarrow \boxed{y(x) = 2e^{-x}(\sin x - \cos x) + C}$$

Problem 2.

(a) Differentiating $y^2 - 4x^2 = C$, we get

$$2yy' - 8x = 0$$

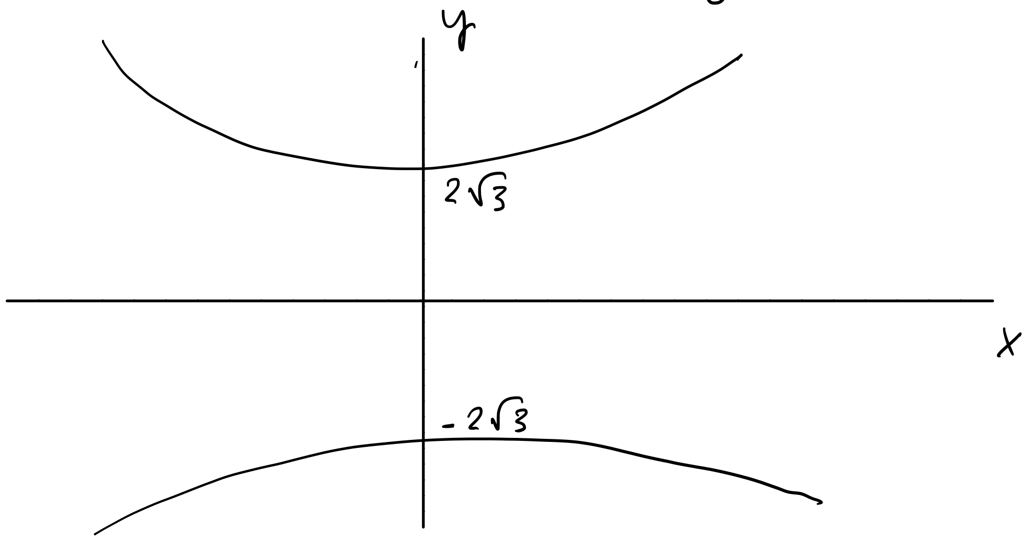
$$\Rightarrow yy' = 4x$$

(b) Putting $y = 4$ & $x = 1$ in $y^2 - 4x^2 = C$, we get

$$C = 16 - 4 = 12$$

So, particular solution is $y^2 = 12 + 4x^2$.

(c) $y^2 - 4x^2 = 12 \Rightarrow \dfrac{y^2}{12} - \dfrac{x^2}{3} = 1 \rightarrow$ hyperbola



Problem 3.

- $y = cx - c^2 \Rightarrow y' = c \Rightarrow c^2 = (y')^2$

Also, $c^2 = cx - y$

$= y'x - y$

So, $y'x - y = (y')^2 \Rightarrow (y')^2 - xy' + y = 0$

- $y = \frac{1}{4}x^2 \Rightarrow y' = \frac{x}{2}$

Then, $(y')^2 - xy' + y = \dfrac{x^2}{4} - \dfrac{x^2}{2} + \dfrac{x^2}{4} = 0$

Problem 4. Let $y(t)$ be the amount at time t. We know that Radioactive decay satisfies the ODE

$$\frac{dy}{dt} = -ky$$

which has solution $y(t) = y(0)e^{-kt}$.

Half-life of 3.6 days implies $y(3.6) = \frac{y(0)}{2}$

So, $\frac{y(0)}{2} = y(0)e^{-k3.6}$

$$\Rightarrow e^{-3.6k} = \frac{1}{2} \Rightarrow 3.6k = \ln(2)$$

$$\Rightarrow k = \frac{\ln(2)}{3.6} = 0.19$$

(a) After 1 day, $y(1) = 1e^{-0.19} = 0.83$ grams

(b) After 1 year, $y(365) = 1.e^{-0.19\times 365} = 3\times 10^{-31}$ grams.

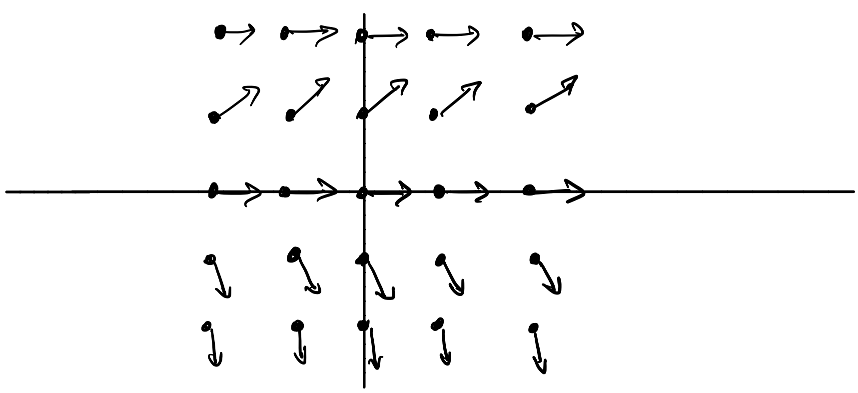
Problem 5. $y' = 2y - y^2$

point	slope
(0,0)	0
(0,1)	1
(0,2)	0
(1,0)	0
(-1,2)	0
(-2,-2)	-8

point	slope
(1,1)	1
(1,2)	0
(2,0)	0
(2,1)	1
(-1,-1)	-3
(-2,1)	1

point	slope
(2,2)	0
(-1,0)	0
(-2,0)	0
(1,-1)	-3
(-1,-2)	-8
(-2,2)	0

point	slope
(1,-2)	-8
(2,-1)	-3
(2,-2)	-8
(-1,1)	1
(-2,-1)	-3



Problem 6. $x_0 = 0,\quad y_0 = 1,\quad h = 0.1,\quad f(x,y) = y$

$x_1 = x_0 + h = 0.1,\quad y_1 = y_0 + f(x_0, y_0)h = 1 + 1 \times 0.1 = 1.1$

$x_2 = x_1 + h = 0.2,\quad y_2 = y_1 + f(x_1, y_1)h = 1.1 + 1.1 \times 0.1 = 1.21$

$x_3 = x_2 + h = 0.3,\quad y_3 = y_2 + f(x_2, y_2)h = 1.21 + 1.21 \times 0.1 = 1.331$

Problem 7.

$$y^3 y' + x^3 = 0$$

$$\Rightarrow y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3 = -\frac{1}{(y/x)^3}$$

Let $u = y/x \Rightarrow y = ux \Rightarrow y' = u + u'x$

$$\Rightarrow u + u'x = -\frac{1}{u^3} \Rightarrow u'x = -\frac{1}{u^3} - u = -\frac{u^4+1}{u^3}$$

$$\Rightarrow \frac{u^3}{u^4+1}du = -\frac{dx}{x} \Rightarrow \int \frac{u^3}{u^4+1}du = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{4}\ln(u^4+1) = -\ln(x) + C$$

$$\Rightarrow (u^4+1)^{1/4} = \frac{C}{x} \Rightarrow u^4 + 1 = \frac{C}{x^4}$$

$$\Rightarrow \left(\frac{y}{x}\right)^4 + 1 = \frac{C}{x^4} \Rightarrow \boxed{y^4 + x^4 = C}$$

Problem 8.

$$y' = -Ay\log(y)$$

$$\Rightarrow \frac{dy}{y\log(y)} = -A\,dt \Rightarrow \int \frac{dy}{y\log(y)} = \int -A\,dt$$

$$\Rightarrow \log(\log(y)) = -At + C$$

$$\Rightarrow \log(y) = Ce^{-At}$$

$$\Rightarrow \boxed{y = e^{Ce^{-At}}}$$