Homework 2

1. For an integer $k \geq 1$, if $c_k = \dfrac{p_k}{q_k}$ is the kth convergent of the simple continued fraction $< a_0, a_1, a_2, \cdots, a_n >$ and $a_0 > 0$, then show that

   (a)
   $$\frac{p_k}{p_{k-1}} =< a_k, a_{k-1}, \cdots, a_1, a_0 >,$$

   (b)
   $$\frac{q_k}{q_{k-1}} =< a_k, a_{k-1}, \cdots, a_2, a_1 > .$$

2. Write the infinite continued fraction expansion of

   (a) $\sqrt{2}$,

   (b) $\frac{1}{\sqrt{3}}$.

3. Evaluate the continued fraction

   (a) $< \dot{1}, \dot{2} >$,

   (b) $< 9, \dot{9}, \dot{1}8 >$.

4. Prove that of any two consecutive convergent of a irrational number ξ, at least one, a/b, satisfies the inequality
   $$\left| x - \frac{a}{b} \right| < \frac{1}{2b^2}.$$

5. The Pell numbers p_n and q_n are defined by
   $p_0 = 0$, $p_1 = 1$, $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$, and
   $q_0 = 1$, $q_1 = 1$, $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 2$.
   This gives two sequences
   $$0, 1, 2, 5, 12, 29, 70, \cdots$$
   $$1, 1, 3, 7, 17, 41, 99, \cdots$$
   If $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Show that the Pell numbers can be expressed as
   $$p_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \text{and } q_n = \frac{\alpha^n + \beta^n}{2}$$
   for $n \geq 0$.

6. For the Pell numbers, derive the relation below, where $n \geq 1$:

   (a) $p_{2n} = 2p_n q_n$.

   (b) $p_n + p_{n-1} = q_n$.

   (c) $2q_n^2 - q_{2n} = (-1)^n$.

   (d) $p_n + p_{n+1} + p_{n+3} = 3p_{n+2}$.

   (e) $q_n^2 - 2p_n^2 = (-1)^n$; hence q_n/p_n are the convergents of $\sqrt{2}$.