Set A → Q1, Set B → Q4, Set C → Q8, Set D → Q4

1a.

$$f(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \quad +\frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{4} \le x \le \frac{1}{2} \quad +1 \\ -x + \frac{3}{4} & \frac{1}{2} \le x \le \frac{3}{4} \quad +1 \\ 0 & \frac{3}{4} \le x \le 1 \quad +\frac{1}{2} \end{cases}$$

1b.

$u(0,t) = 0$ +1 for all time t

$u(1,t) = 0$ +1 for all time t

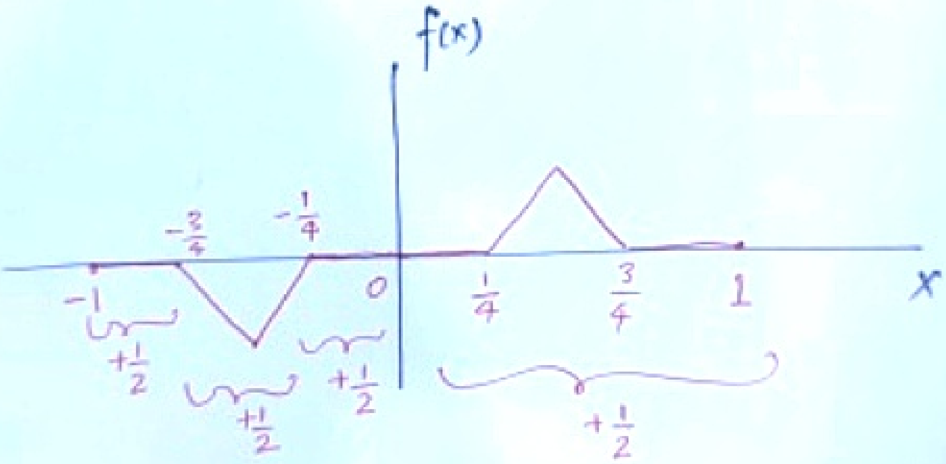
1c.

$u(x,0) = f(x)$ +1 for all x

$u_t(x,0) = 0$ +1

1d. odd +1

1e.



(1) (f) Since $f(x)$ has odd extension, so we can express $f(x)$ as a Fourier series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ —— (2)
(Marks)

Since $L=1$

$$= 2\int_0^1 f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_{\frac{1}{4}}^{1/2}\left(x-\frac{1}{4}\right)\sin(n\pi x)\,dx + 2\int_{1/2}^{3/4}\left(-x+\frac{3}{4}\right)\sin(n\pi x)dx$$

$$= I_1 + I_2 \text{ (say)}$$

$I_1 \rightarrow$

$$2\left[\left(x-\frac{1}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right)+\left(\frac{\sin(n\pi x)}{n^2\pi^2}\right)\right]^{1/2}$$

$$= \frac{-n\pi \cos\left(\frac{n\pi}{2}\right) + 4\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right)\right)}{2n^2\pi^2}$$ — (2) Marks

$I_2 \rightarrow$

$$\frac{-4\sin\left(\frac{3\pi n}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right) + \pi n \cos\left(\frac{n\pi}{2}\right)}{2n^2\pi^2}$$

$$= \frac{-4\sin\left(\frac{n\pi}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right) + \pi n \cos\left(\frac{n\pi}{2}\right)}{2n^2\pi^2}$$

— (2) Marks

$I_1 + I_2 \rightarrow$

$$\frac{-4\sin\left(\frac{n\pi}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

(g) $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

A_n is calculated in (f) ; that is

$$A_n = \frac{-4\sin\left(\frac{n\pi}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \quad \text{— (2) Marks}$$

$u_t(x,0) = 0$

$$\Rightarrow u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = 0$$

$\Rightarrow B_n = 0$ for all n — (2) Marks

Q.2.
η
Set A

Set B → Q-9

a. $(e^{-2x}y')' + (e^{-2x} + \lambda e^{-2x})y = 0$ $\quad \frac{+1}{+1}$

$$\downarrow$$

$$y'' - 2y' + (\lambda+1)y = 0$$

b. $\lambda = -\alpha^2, \alpha > 0$

$$u^2 - 2u + (\lambda+1) = 0$$

$$u = \frac{2 \pm \sqrt{4 - 4(\lambda+1)}}{2}$$

$$= 1 \pm \sqrt{-\lambda}$$

+1

$\underline{\lambda = -\alpha^2}$ $\quad u = 1 \pm \sqrt{\alpha^2} = 1 \pm \alpha$

$$y = c_1 e^{(1+\alpha)x} + c_2 e^{(1-\alpha)x} \rightarrow \text{trivial}$$

$\underline{\lambda = 0}$ $\quad u = 1 \quad y = c_1 e^x + c_2 x e^x \rightarrow \text{trivial}$

+1
+2

$\underline{\lambda = \alpha^2}$ $\quad u = 1 \pm i\alpha$

$$y = e^x (c_1 \cos(\alpha x) + c_2 \sin(\alpha x))$$

$$y(0) = c_1 = 0 \Rightarrow y = c_2 e^x \sin(\alpha x)$$

$$y(1) = c_2 \sin(\alpha) = 0$$

$$\alpha = n\pi.$$

+1 → All eigenvalues are $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$

c. $\lambda_n \rightarrow y = c_2 e^x \sin(n\pi x)$ +1

$$e^x \sin(n\pi x)$$

d. $\omega(x) = e^{-2x}$ +2

Q.3. (Set A) (ISHANI CHOUDHARY)

a. $\begin{cases} s^2X - 2 = \frac{1}{2}(-X+Y) & +1 \\ s^2Y = \frac{1}{2}(X-Y) & +1 \end{cases}$

b. $\begin{cases} \left(s^2+\frac{1}{2}\right)X - \frac{Y}{2} = 2 \\ -\frac{1}{2}X + \left(s^2+\frac{1}{2}\right)Y = 0 \Rightarrow X = (2s^2+1)Y \end{cases}$

$= \frac{2s^2+1}{s^2(s^2+1)}$ +1

$\left(s^2+\frac{1}{2}\right)(2s^2+1)Y - \frac{Y}{2} = 2$

$Y(2s^4+2s^2) = 2 \Rightarrow Y = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2+s^4}$

+1

c. $X = \frac{2s^2+1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$

Num. $= (As+B)(s^2+1) + (Cs+D)s^2$

$= As^3 + As + Bs^2 + B + Cs^3 + Ds^2$

$= (A+C)s^3 + (B+D)s^2 + As + B$

$A+C=0,\ B+D=2,\ A=0,\ B=1$

$C=0 \quad D=1$

$X = \frac{1}{s^2} + \frac{1}{s^2+1}$

+1 +1

$Y = \frac{1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$

$A+C=0,\ B+D=0,\ A=0,\ B=1$

$C=0 \quad D=-1$

$Y = \frac{1}{s^2} - \frac{1}{s^2+1}$

+1 +1

d. $x(t) = \mathcal{L}^{-1}(X) = t + \sin(t)$ +1

$y(t) = \mathcal{L}^{-1}(Y) = t - \sin(t)$ +1

Q.4.

a. $s^2 Y + 5sY + 6Y = e^{-\frac{\pi}{2}s} \cancel{+} e^{-\pi s}\frac{s}{s^2+1}$ +1

$$Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s^2+5s+6} \cancel{+} e^{-\pi s}\frac{s}{(s^2+1)(s^2+5s+6)}$$ +1

b. +1 $\frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{1}{s+2} - \frac{1}{s+3}$

$A+B=0 \Rightarrow A=-B \Rightarrow A=1$

$3A+2B=1 \Rightarrow -B=1 \Rightarrow B=-1$

+$\frac{1}{2}$ $\frac{s}{(s^2+1)(s^2+5s+6)} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3}$

~~$s^3: A+C+D=0$~~

$s = (As+B)(s^2+5s+6) + C(s^2+1)(s+3) + D(s^2+1)(s+2)$

+1$\frac{1}{2}$ +2

$s^3: A+C+D=0 \Rightarrow A=-C-D$

$s^2: 5A+B+3C+2D=0$ +$\frac{1}{2}$ $\boxed{A=\frac{2}{5}-\frac{3}{10}=\frac{4-3}{10}=\frac{1}{10}}$

$s: 6A+5B+C+D=1$

$1: 6B+3C+2D=0 \Rightarrow B=-\frac{1}{2}C-\frac{1}{3}D$ $\quad B=\frac{1}{5}-\frac{1}{10}=\frac{2-1}{10}$

$\Rightarrow -5C-5D-\frac{1}{2}C-\frac{1}{3}D+3C+2D=0$ +$\frac{1}{2}$ $\boxed{B=\frac{1}{10}}$

$C\left(-5-\frac{1}{2}+3\right)+D\left(-5-\frac{1}{3}+2\right)=0$

$C\left(\frac{-5}{2}\right)+D\left(\frac{-10}{3}\right)=0 \Rightarrow 15C+20D=0$

$\Rightarrow 3C+4D=0$

$\Rightarrow 3C=-4D$

$\Rightarrow -6C-6D-\frac{5}{2}C-\frac{5}{3}D+C+D=1$

$C\left(-6-\frac{5}{2}+1\right)+D\left(-6-\frac{5}{3}+1\right)=1$

$\boxed{C=-\frac{2}{5}}$ +$\frac{1}{2}$ $\quad C\left(\frac{-15}{2}\right)+D\left(\frac{-20}{3}\right)=1 \Rightarrow 45C+40D=-6$

$\Rightarrow 15\cdot -4D+40D=-6$

$3C=-\cancel{4}^{2}\cdot\frac{3}{\cancel{10}_{5}}$ $\quad \boxed{D=\frac{6}{20}=\frac{3}{10}}$ +$\frac{1}{2}$ $\Leftarrow \Rightarrow -20D=-6$

$=-\frac{6}{5}$

c.

$$Y(s) = e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2}-\frac{1}{s+3}\right) + e^{-\pi s}\left(A\frac{s}{s^2+1}+\frac{B}{s^2+1}+\frac{C}{s+2}+\frac{D}{s+3}\right)$$

$\frac{1}{10}$ $\frac{1}{10}$ $\frac{4}{5}$ $\frac{3}{10}$

$$y(t) = \mathcal{L}^{-1}(Y)$$

$$= u\left(t-\frac{\pi}{2}\right)e^{-2\left(t-\frac{\pi}{2}\right)} - u\left(t-\frac{\pi}{2}\right)e^{-3\left(t-\frac{\pi}{2}\right)}$$

$$+\left[A\,u(t-\pi)\cos(t-\pi) + B\,u(t-\pi)\sin(t-\pi) + C\,u(t-\pi)e^{-2(t-\pi)} + D\,u(t-\pi)e^{-3(t-\pi)}\right]$$

$+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$

$5C + 5D = -e^{\pi s}$

Q.5.

a. $\mathcal{L}(f)(s-a)$ or $F(s-a)$

$f(s-a)$ ✗

b. $\mathcal{L}\left(\int_0^t f(x)\,dx\right)(s) = \dfrac{\mathcal{L}(f)(s)}{s} = \dfrac{F(s)}{s}$

c. $\mathcal{L}\left(f(t)\,u(t-a)\right)(s) = e^{-as}\,\mathcal{L}\left(f(t+a)\right)$

d. $\mathcal{L}(f*g)(s) = \mathcal{L}(f)(s)\,\mathcal{L}(g)(s)$

$= F(s)\,G(s)$

e. $\mathcal{L}\left(t f(t)\right)(s) = \cancel{-\mathcal{L}(f')(s)}$

$= -\mathcal{L}(f)'(s)$

$= -F'(s)$

Q.6.

Let $F(s) = \ln\left(1+\frac{\omega^2}{s^2}\right)$

$$F'(s) = \frac{1}{1+\frac{\omega^2}{s^2}} \cdot \omega^2 \cdot \frac{-2}{s^3} = -2\omega^2 \frac{1}{s(s^2+\omega^2)}$$

+2

$$= -2\omega^2\left(\frac{A}{s} + \frac{Bs+C}{s^2+\omega^2}\right)$$

$$As^2 + A\omega^2 + Bs^2 + Cs = 1$$

$$\Rightarrow A+B=0, \quad C=0, \quad A\omega^2 = 1$$

$$\Rightarrow B = -\frac{1}{\omega^2} \qquad A = \frac{1}{\omega^2}$$

$$F'(s) = -2\omega^2\left(\frac{1}{\omega^2 s} - \frac{1}{\omega^2}\cdot\frac{s}{s^2+\omega^2}\right)$$

$$= -\frac{2}{s} + \frac{2s}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}(F')(t) = -2 + 2\cos(\omega t)$$

Now, $\mathcal{L}(t f(t))(s) = -F'(s)$ — +1

$$\Rightarrow \mathcal{L}^{-1}(F')(t) = -t f(t)$$

$$\Rightarrow -2+2\cos(\omega t) = -t f(t)$$

$$\Rightarrow f(t) = \frac{-2+2\cos(\omega t)}{-t}$$

$$= \frac{2-2\cos(\omega t)}{t}$$

+2

$$= \frac{4}{t}\sin^2\left(\frac{\omega t}{2}\right)$$

Q.7. For $n=1$, $2P_2 = 3xP_1 - P_0$

$$2P_2 = 3x^2 - 1 \Rightarrow P_2 = \frac{3}{2}x^2 - \frac{1}{2} \quad +1$$

For $n=2$, $3P_3 = 5xP_2 - 2P_1$

$$3P_3 = 5x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - 2x = \frac{15}{2}x^3 - \frac{9}{2}x$$

$$\Rightarrow P_3 = \frac{5}{2}x^3 - \frac{3}{2}x \quad +1$$

For $n=3$, $4P_4 = 7xP_3 - 3P_2$

$$4P_4 = 7x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - 3\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$= \frac{35}{2}x^4 - \frac{21}{2}x^2 - \frac{9}{2}x^2 + \frac{3}{2} = \frac{35}{2}x^4 - 15x^2 + \frac{3}{2}$$

$$P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \quad +1$$

For $n=4$, $5P_5 = 9xP_4 - 4P_3$

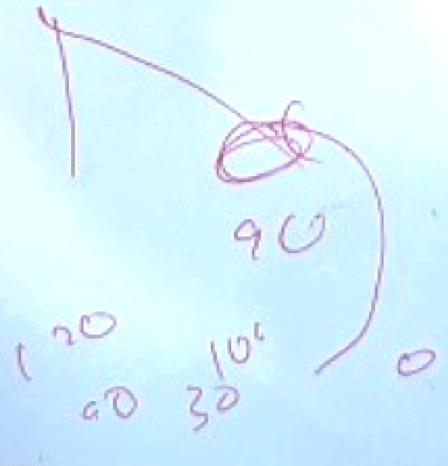
$$= 9x\left(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}\right) - 4\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$

$$= \frac{315}{8}x^5 - \frac{135}{4}x^3 + \frac{27}{8}x - 10x^3 + 6x$$

$$= \frac{315}{8}x^5 - \frac{175}{4}x^3 + \frac{75}{8}x$$

$$\Rightarrow P_5 = \underbrace{\frac{63}{8}}_{7.875}x^5 - \underbrace{\frac{35}{4}}_{8.75}x^3 + \frac{15}{8}x \quad +2$$

1.875



Q.8. Let $y = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)$

$= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots$

+2

$y' = a_0 r x^{r-1} + a_1 (r+1) x^r + \cdots$

$y'' = a_0 r(r-1) x^{r-2} + a_1 (r+1) r x^{r-1} + \cdots$

$(x^2 - x) y'' - x y' + y = 0$

$\Rightarrow x^2 y'' - x y'' - x y' + y = 0$

$\Rightarrow a_0 r(r-1) x^r + \ldots - a_0 r(r-1) x^{r-1} + \ldots - a_0 r x^r + \ldots + a_0 x^r + \ldots = 0$

smallest coefficient is x^{r-1}. So, we set

$a_0 r(r-1) = 0$ +1

Assuming $a_0 \neq 0$ $r = 0, 1$
+1 +1

Method-2 (Frobenius Shortcut Method)

So, Comparing our differential Eqn with

$$y'' + \frac{F(x)}{x} y' + \frac{G(x)}{x^2} y = 0.$$

$\Rightarrow \frac{F(x)}{x} = \frac{-x}{x^2 - x} = \frac{-1}{x-1} \Rightarrow F(x) = \frac{-x}{x-1} = \frac{x}{1-x}$

$= \sum_{m=0}^{\infty} a_m x^m$

and

$\frac{G(x)}{x^2} = \frac{1}{x^2 - x} \Rightarrow G(x) = \frac{x}{x-1}$

$= \sum_{m=0}^{\infty} b_m x^m$

then the indicial Eqn is

$$[r(r-1) + a_0 r + b_0] c_0 = 0.$$

Where $y(x) = x^r \sum_{m=0}^{\infty} c_m x^m$, $c_0 \neq 0$

Here $a_0 = 0, b_0 = 0$

Q.9. a.

$$2y'' + \sin(y) = 0$$

$$y = y_1 \Rightarrow y_1' = y' = y_2$$

$$y' = y_2 \Rightarrow y'' = y_2' = -\frac{\sin(y)}{2} = \frac{-\sin(y_1)}{2}$$

$$\begin{cases} y_1' = y_2 & +1 \\ y_2' = -\frac{1}{2}\sin(y_1) & +1 \end{cases}$$

b. $\underbrace{y_2 = 0 \quad \overset{+1}{} \quad \sin(y_1) = 0}$ $\Rightarrow y_1 = n\pi$

$\underbrace{(n\pi, 0)}_{+1}$. $n = 0, \pm 1, \pm 2, \ldots$ integers $\in \mathbb{Z}$

c. $f(y_1, y_2) = y_2$ $\qquad g(y_1, y_2) = -\frac{1}{2}\sin(y_1)$

$$\left.\begin{array}{l} f(y_1, y_2) \approx f(a,b) + \frac{\partial f}{\partial y_1}(a,b)(y_1 - a) + \frac{\partial f}{\partial y_2}(a,b)(y_2 - b) \\ g(y_1, y_2) \approx \ldots \end{array}\right\} +1$$

at $\underset{\downarrow\ \ \downarrow}{(n\pi, 0)}$ (a, b)

$$\left.\begin{array}{l} f(y_1, y_2) \approx 0 + 1(y_2 - 0) = y_2 \\ g(y_1, y_2) \approx 0 + \frac{1}{2}(-1)^{n+1}(y_1 - n\pi) \end{array}\right\} \begin{array}{l} +1 \\ +1 \end{array}$$

or

$$\begin{cases} +1 \quad y_1' = y_2 \\ +1 \quad y_2' = \frac{1}{2}(-1)^{n+1}(y_1 - n\pi) \end{cases} \Leftrightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}(-1)^{n+1} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}(-1)^{n+1} \cdot n\pi \end{pmatrix}$$

d. $n \to$ even $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$ $\qquad \lambda^2 + \frac{1}{2} = 0 \Rightarrow \lambda = \pm\frac{i}{\sqrt{2}}$ } +1

$(n\pi, 0) \longrightarrow$ center. $+\frac{1}{2}$

n-odd $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ $\qquad \lambda^2 - \frac{1}{2} = 0 \Rightarrow \lambda = \pm\frac{1}{\sqrt{2}}$ } +1

$(n\pi, 0) \longrightarrow$ saddle. $+\frac{1}{2}$

set-A-9
B = 7
C = 3
D - 8

10. (Set A)

(a) True

(b) False

(c) False

(d) False

(e) True

(f) False

(g) True

(h) True

(i) True

(j) True