Solutions and Rubrics for Mid-Semester Exam: M-I

Indraprastha Institute of Information Technology, Delhi

List of Common Errors and Marks Deductions:

1. Using an undefined symbol. Please deduct 1/2 mark each time this is done.
2. Writing an equation in which the LHS and RHS are not comparable, for example, if the LHS is an $m \times n$ matrix and the RHS is a real number. Please deduct 1/2 mark each time this is done.
3. Writing a meaningless or completely illogical statement. Please deduct 1 mark for every meaningless statement.
4. Please deduct 1/2 mark for every calculation mistake.

Question (1(a)-5 marks).

Let A and B be $m \times n$ matrices (where $m, n \in \mathbb{N}$) having columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}_1, \ldots, \mathbf{b}_n$, respectively. Suppose $\mathbf{b}_j = j^2\mathbf{a}_{j-1}$ for $j = 2, \ldots, n$ and $\mathbf{b}_1 = \mathbf{a}_n$. Find an $n \times n$ matrix E such that $AE = B$.

Answer.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ denote the columns of an $m \times n$ matrix C. Then

$$AE = [A\mathbf{c}_1 \quad A\mathbf{c}_2 \quad \cdots \quad A\mathbf{c}_n]$$

Thus $AC = B$ if and only if $A\mathbf{c}_j = \mathbf{b}_j$ holds for each $j = 1, \ldots, n$.

Since we only need to find one such matrix, it is enough to use the basic definition of the matrix product to observe that $A\mathbf{e}_n = \mathbf{a}_n = \mathbf{b}_1$ and that $Aj^2\mathbf{e_{j-1}} = j^2\mathbf{a}_{j-1} = \mathbf{b}_j$ for $j = 2, \ldots, n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix I_n.

Thus if

$$E = [\mathbf{e}_n \quad 2^2\mathbf{e}_1 \quad 3^2\mathbf{e}_2 \quad \cdots \quad n^2\mathbf{e}_{n-1}]$$

then

$$\begin{aligned}
AE &= A[\mathbf{e}_n \quad 2^2\mathbf{e}_1 \quad 3^2\mathbf{e}_2 \quad \cdots \quad n^2\mathbf{e}_{n-1}] \\
&= [A\mathbf{e}_n \quad 2^2A\mathbf{e}_1 \quad 3^2A\mathbf{e}_2 \quad \cdots \quad n^2A\mathbf{e}_{n-1}] \\
&= [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] \\
&= B
\end{aligned}$$

Rubric.

- 2 marks for arriving at a correct guess for E. Partial marks may be given depending upon how close a guess comes to being correct. This is left to the grader's judgement.
- 3 marks for showing that $AE = B$, either by using the definition of the matrix product from the textbook, or by calculations based on the row-column rule taught in high school. Please make sure to make the marks deductions listed on the first page of the document when reading the argument provided by the student.

Question (1(b)-5 marks).

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & 9 & 0 \end{bmatrix}$. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$. Solve the equations $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ by row reducing exactly one matrix.

Answer.

First we reduce the matrix $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2]$ to echelon form.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 5 \\ 0 & 9 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 9 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 4 & 0 & 0 & 0 & 5 \\ 0 & 9 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

As the augmented columns are not pivot columns, both equations represent consistent systems. We bring the matrix to RREF to find solutions.

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 5 \\ 0 & 9 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 5/4 \\ 0 & 9 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{9}R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 5/4 \\ 0 & 1 & 0 & 1/9 & 7/9 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Thus the solution of the system $A\mathbf{x} = \mathbf{b}_1$ is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 1/9 \\ x_3 &= 1 \end{aligned}$$

and the solution of the system $A\mathbf{x} = \mathbf{b}_2$ is

$$\begin{aligned} x_1 &= 5/4 \\ x_2 &= 7/9 \\ x_3 &= 1 \end{aligned}$$

Rubric.

- Please refer to list on first page of this document for marks deductions when reading the solution, particularly calculation errors. Do not give a zero if the answer is correct up to minor calculation mistakes.
- 2 marks for correctly identifying the matrix which has to be reduced to echelon form.
- 1 mark for reducing it to RREF.
- 1 mark for finding the solution of each of the two systems.

Question (2(a-e) - 10 marks). All subparts of this question carry equal marks.

In each part of this question, V is a vector space and W is a subset of V. Decide whether W is a subspace of V. Justify your answer with a short proof or counterexample.

(a) $V = \mathbb{R}(t)$, the set of *all* polynomials in t which have real coefficients (please note that the degrees of the polynomials are not bounded).

$W = \{p(t) = a_0 + \cdots + a_n t^n \mid a_{2k} = 0, \text{ if } k \in \mathbb{N} \text{ and } 2k \in \{0, \ldots, n\}\}$

(b) $V = \mathbb{R}^n$

$W = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n \geq 0\}$

(c) $V = \mathbb{R}^n$

$W = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \geq 0\}$

(d) $V = \mathbb{R}^\infty$, the set of all sequences indexed by $\mathbb{N}$

Fix $k \in \mathbb{N}$.

$W = \{(a_n) \mid a_1 + \cdots + a_k = 0\}$.

(e) $V = \mathbb{R}^{n \times n}$, the set of all $n \times n$ matrices having real entries

(*Note for Section A:* $\mathbb{R}^{n \times n}$ is the same as $M_n(\mathbb{R})$)

$W = \{A \mid A \text{ is in reduced row echelon form}\}$

Answer. (a) Yes. W is a subspace of V. This is justified as follows.

W is non-empty. We check that it is closed under vector addition and scalar multiplication.

Let $p, q \in W$. Then $p = a_0 + a_1 t + \cdots + a_n t^n$ for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$, and $q = b_0 + b_1 t + \cdots + b_m t^m$ for some $m \in \mathbb{N}$ and $b_0, \ldots, b_m \in \mathbb{R}$. Further the even numbered coefficients of both p and q are all zero. Without loss of generality, we may assume that $m \leq n$.

$$\begin{aligned} p + q &= a_0 + a_1 t + \cdots + a_n t^n + b_0 + b_1 t + \cdots + b_m t^m \\ &= \sum_{j=1}^{m} (a_j + b_j) t^j + \sum_{m+1}^{n} a_j t^j \end{aligned}$$

If $k \in \mathbb{N}$ and $2k \in \{0, \ldots, m\}$ then

$$a_{2k} + b_{2k} = 0 + 0 = 0$$

and if $k \in \mathbb{N}$ and $2k \in \{m+1, \ldots, n\}$ then

$$a_{2k} = 0.$$

Thererfore

$$p + q \in W$$

Since the choices of p and q were arbitrary, W is closed under vector addition.

Next, let $c \in \mathbb{R}$ and $p \in W$. Then $p = a_0 + a_1 t + \cdots + a_n t^n$ for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$ and $a_{2k} = 0$ whenever $k \in \mathbb{N}$ and $2k \in \{0, \ldots, n\}$.

Clearly $ca_{2k} = 0$ whenever $k \in \mathbb{N}$ and $2k \in \{0, \ldots, n\}$. Therefore

$$cp \in W$$

Since the choices of c and p were arbitrary we see that W is closed under scalar multiplication.

(b) No. W is not a subspace of V. This is justified by the following counterexample.

The vector $\mathbf{e}_1 = (1, 0, \ldots, 0) \in W$. However its scalar multiple $(-1, 0, \ldots, 0) \notin W$.

(c) Yes. W is a subspace of V, because $W = V$.

(d) Yes. W is a subspace of V. This is justified as follows.

W is non-empty. We check that it is closed under vector addition and scalar multiplication.

Let $(a_n), (b_n) \in W$. Then

$$\sum_{j=1}^{k} a_j + b_j = \sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j = 0 + 0 = 0$$

Therefore $(a_n) + (b_n) \in W$. Since the choices of (a_n) and (b_n) were arbitrary, W is closed under vector addition.

Next, let $c \in \mathbb{R}$ and $(a_n) \in W$. Then

$$\sum_{j=1}^{k} ca_j = c \sum_{j=1}^{k} a_j = 0$$

Hence $c(a_n) \in W$. Since the choices of c and (a_n) were arbitrary we see that W is closed under scalar multiplication.

(e) No. W is not a subspace of V. This is justified by the following counterexample.

$I_n \in W$, but $2I_n \notin W$. Therefore W is not closed under scalar multiplication.

Rubric.

- 1 mark for answering Yes/No correctly, in each part.
- For parts (a) and (d), award 1/2 mark each for the verification of closure under vector addition and scalar multiplication in W.
- For part (c), award 1 mark for identifying W as $\mathbb{R}^n$.
- For parts (b) and (e) award 1 mark for an appropriate counterexample.

Question (3). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \geq 0, a + c = b + d = 1$ and $A \neq I_2$.

Let $P = \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix}$.

(a) (7 marks) Show that P is invertible. Find P^{-1} and $P^{-1}AP$.

(b) (3 marks) Find a formula for A^n.

Answer.

First Solution:

Part (a):

Recall that a general 2×2 - matrix is invertible if and anly if $xw - yz \neq 0$ where

$$X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

In that case,

$$X^{-1} = \frac{1}{xw - yz} \begin{bmatrix} w & -y \\ -z & x \end{bmatrix} \tag{$*$}$$

Thus P is invertible if and only if $-b - c \neq 0$.

Now $-b - c = 0 \implies b = c = 0 \implies A = I_2$, a contradiction. Therefore

$$P \text{ is invertible.} \tag{$**$}$$

Using $(*)$, we get

$$P^{-1} = \frac{1}{-b - c} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \tag{$***$}$$

We now compute $P^{-1}AP$.

$$\begin{aligned} P^{-1}AP &= \frac{1}{-b-c} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix} \\ &= \frac{1}{-b-c} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \begin{bmatrix} b(a+c) & a-b \\ c(b+d) & c-d \end{bmatrix} \\ &= \frac{1}{-b-c} \begin{bmatrix} -1 & -1 \\ -c & b \end{bmatrix} \begin{bmatrix} b & a-b \\ c & c-d \end{bmatrix} \\ &= \frac{1}{-b-c} \begin{bmatrix} -b-c & b-a+d-c \\ 0 & -c(a-b)+b(c-d) \end{bmatrix} \\ &= \frac{1}{-b-c} \begin{bmatrix} -b-c & 0 \\ 0 & (-c-b)(a-b) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix} \end{aligned} \tag{1}$$

Part (b):

From (1),

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & a+d-1 \end{bmatrix} = D, \tag{2}$$

say.

$\therefore$ from (2), $A = PDP^{-1}$ and so:

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= \underbrace{PDP^{-1}PDP^{-1}\ldots PDP^{-1}}_{n \text{ factors}} \\ &= PD^nP^{-1} \end{aligned} \tag{3}$$

$$= P \begin{bmatrix} 1 & 0 \\ 0 & (a+d-1)^n \end{bmatrix} P^{-1} \tag{4}$$

Substituting for P and P^{-1}, we get:

$$A^n = \frac{1}{-b-c} \begin{bmatrix} -b-c(a+d-1)^n & -b+b(a+d-1)^n \\ -c+c(a+d-1)^n & -c-b(a+d-1)^n \end{bmatrix}$$

Second Solution:

Part (a):

By the Invertible Matrix Theorem, P is invertible if and only if $\operatorname{Col} P = \mathbb{R}^2$.

Clearly P is not the zero matrix. Therefore $\dim \operatorname{Col} P$ is either 1 or 2.

$\dim \operatorname{Col} P = 1 \implies (b, c)$ is a linear multiple of $(1, -1) \implies c = -b \implies b = c = 0 \implies A = I_2$, a contradiction. Therefore P is invertible.

The remaining part of the solution is the same as the first solution.

Rubric. Part (a):

- Please ensure that the marks deductions are made for any mistakes that are mentioned in the list on the first page of this document.
- 2 marks for arguing that P is not invertible if and only if $c = -b$, either by using linear dependence of matrix columns or determinants
- 3 marks for arguing that $-b - c = 0$ leads to a contradiction. You may give partial credit if the student makes a recognizable attempt towards this.
- 1 mark for calculating P^{-1} correctly, as in $(* * *)$
- 1 mark for arriving at the expression for $P^{-1}AP$ in (1) (note that $a - b = a + d - 1$, so this expression may differ in the student's answer)

Part (b):

- 1 mark for arriving at $A^n = PD^nP^{-1}$
- 1 mark for calculating D^n correctly
- 1 mark for substituting the values of P and P^{-1} and obtaining the answer

Solve ONE of the following two problems (either 5(a) or 5 (b)).

Question ((5a) - 10 marks). Prove or disprove:

If $\{\overline{v}_1, \ldots, \overline{v}_n\}$ is a basis for a vector space V, then so is $\{\alpha_1\overline{v}_1, \ldots, \alpha_n\overline{v}_n\}$ where the α_i are non-zero scalars.

Answer.

Let $S = \{v_1, \ldots, v_n\}$, and let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, where $\alpha_i \neq 0$ for each i.

Let $S' = \{\alpha_1 v_1, \ldots, \alpha_n v_n\}$. Yes, S' is a basis for V.

First Solution:

We show that S' is a linearly independent subset of V.

Suppose

$$c_1\alpha_1 v_1 + \cdots + c_n\alpha_n v_n = 0,$$

where $c_1, \ldots, c_n \in \mathbb{R}$. Since $v_1, \ldots, v_n$ are linearly independent, it follows that

$$c_i\alpha_i = 0, \forall i = 1, \ldots, n.$$

Since the α_i are non-zero, this can only hold true if $c_i = 0, \forall i = 1, \ldots, n$.

By the Basis Theorem on p. 259 of the course textbook, S' is a basis for V.

Second Solution:

We show that S' spans V.

Let $v \in V$. Since $v \in \operatorname{Span} S$, there exist scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} c_i v_i$$

Hence

$$v = \sum_{i=1}^{n} \frac{c_i}{\alpha_i}\alpha_i v_i \in \operatorname{Span} S'.$$

Therefore $V = \operatorname{Span} S'$.

By the Basis Theorem on p. 259 of the course textbook, S' is a basis for V.

Third Solution:

We show that S' is linearly independent and spans V.

Let $v \in V$. Since $v \in \operatorname{Span} S$, there exist scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} c_i v_i$$

Hence

$$v = \sum_{i=1}^{n} \frac{c_i}{\alpha_i}\alpha_i v_i \in \operatorname{Span} S'.$$

Therefore $V = \operatorname{Span} S'$.

Suppose

$$c_1\alpha_1 v_1 + \cdots + c_n\alpha_n v_n = 0,$$

where $c_1, \ldots, c_n \in \mathbb{R}$. Since $v_1, \ldots, v_n$ are linearly independent, it follows that

$$c_i\alpha_i = 0, \forall i = 1, \ldots, n.$$

Since the α_i are non-zero, this can only hold true if $c_i = 0, \forall i = 1, \ldots, n$. Hence S' is linearly independent.

Therefore S' is a basis for V.

Rubric.

- Please ensure that marks deductions are for any mistakes of the type mentioned on the first page of this document.
- 1 mark for selecting prove. No marks for the remaining solution if the student selects disprove.
- For the first solution: award 5 marks for showing linear independence. and 4 marks for citing the appropriate results to argue that this is enough. One such result is cited in the proof above. Students may cite other results.
- For the second solution: award 5 marks for the spanning argument, and 4 marks for citing the appropriate results to argue that this is enough. One such result is cited in the proof above. Students may cite other results.
- For the third solution: 4.5 marks for linear independence, 4.5 marks for spanning.

Question ((5b)- 10 marks). Prove or disprove:

If $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is a basis for a vector space V, then so is $\{\bar{v}_1, \bar{v}_1 + \bar{v}_2, \ldots, \bar{v}_1 + \bar{v}_2 + \ldots + \bar{v}_n\}$.

Answer.

Let $S = \{v_1, \ldots, v_n\}$, and let $S' = \{v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_n\}$. Yes, S' is a basis for V.

First Solution:

We show that S' is a linearly independent subset of V.

Suppose

$$\sum_{i=1}^{n} c_i \left(\sum_{j=1}^{i} v_j \right) = 0$$

where $c_1, \ldots, c_n \in \mathbb{R}$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{i} c_i v_j = 0.$$

By rearranging terms we obtain

$$\sum_{i=1}^{n} \left(\sum_{j=i}^{n} c_j \right) v_i = 0$$

Since $v_1, \ldots, v_n$ are linearly independent, it follows that

$$\sum_{j=i}^{n} c_j = 0 \text{ for } i = 1, \ldots, n. \tag{1}$$

Since the α_i are non-zero, this can only hold true if $c_i = 0, \forall i = 1, \ldots, n$.

Thus $(c_1, \ldots, c_n)$ is a solution of the linear system $A\mathbf{x} = 0$ where

$$A = [\mathbf{e}_1 \quad \mathbf{e}_1 + \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_1 + \cdots + \mathbf{e}_n]$$

Since A is upper triangular and has 1s on the diagonal, A is invertible. So by the Invertible Matrix Theorem, the system $A\mathbf{x} = 0$ has only the trivial solution.

Alternatively, it can also be easily seen that equation (1) can be solved by back substitution to show that $c_n = 0, c_{n-1} = 0, \ldots, c_1 = 0$.

Thus S' is a linearly independent set. By the Basis Theorem on p. 259 of the course textbook, S' is a basis for V.

Second Solution:

We show that S' spans V.

Let $v \in V$. Since $v \in \operatorname{Span} S$, there exist scalars $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} c_i v_i$$

Put $d_n = c_n$ and

$$d_j = c_j - c_{j+1}, \text{ for } j = 1, \ldots, n-1.$$

Then

$$
\begin{aligned}
\sum_{j=1}^{n} d_j \left(\sum_{i=1}^{j} v_j \right) &= c_n \sum_{i=1}^{n} v_i + \sum_{j=1}^{n-1} \left((c_j - c_{j+1}) \sum_{i=1}^{j} v_j \right) \\
&= \sum_{i=1}^{n} \left(c_n + \sum_{j=i}^{n-1} (c_j - c_{j+1}) \right) v_i \\
&= \sum_{i=1}^{n} c_i v_i \\
&= v
\end{aligned}
$$

Hence $v \in \operatorname{Span} S'$.

By the Basis Theorem on p. 259 of the course textbook, S' is a basis for V.

Third Solution:
We show that S' is linearly independent and spans V.

Please refer to first and second solution for proofs of linear independence and spanning.

Rubric.

- Please ensure that marks deductions are for any mistakes of the type mentioned on the first page of this document.
- 1 mark for selecting prove. No marks for the remaining solution if the student selects disprove.
- For the first solution: award 5 marks for showing linear independence, and 4 marks for citing the appropriate results to argue that this is enough. One such result is cited in the proof above. Students may cite other results.
- For the second solution: award 5 marks for the spanning argument, and 4 marks for citing the appropriate results to argue that this is enough. One such result is cited in the proof above. Students may cite other results.
- For the third solution: 4.5 marks for linear independence, 4.5 marks for spanning.

4.1

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Question (4). Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) (4 marks) Show that: If $\exists c \in \mathbb{R} \setminus \{0\}$ such that $f(x) = cx, \forall x \in \mathbb{R}$, then the graph of f is a proper nontrivial subspace of $\mathbb{R}^2$.

(The graph of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as the set $\{(x, y) \mid y = f(x)\}$.)

(b) (1 mark) What is the converse of the statement that is to be proved in part (a)?

(c) (5 marks) Is the converse that you stated in part (b) true? Justify with a short proof or an appropriate counterexample.

(*Note for Section B:* You may assume the following statement without proof -

Any proper nontrivial subspace of $\mathbb{R}^2$ is of the form $\text{Span}\{\bar{v}\}$ where v is a non-zero vector in $\mathbb{R}^2$.)

Answer:

a) We first show that W = graph of $f = \{(x, f(x)) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

(i) Clearly, $(0,0) \in W$ since $(0, f(0)) = (0, c.0) = (0,0)$.

Alternatively, we can see that W is non-empty since $(1, f(1)) = (1, c) \in W$.

(ii) Closure under addition.

Let $\bar{w}_1, \bar{w}_2 \in W$. Then $\bar{w}_1 = (x_1, cx_1)$ and $\bar{w}_2 = (x_2, cx_2)$ for some $x_1, x_2 \in \mathbb{R}$.

But then $\bar{w}_1 + \bar{w}_2 = (x_1 + x_2, cx_1 + cx_2)$
$= (x_1 + x_2, c(x_1 + x_2)) = (x_1 + x_2, f(x_1 + x_2))$
$\in W$, as required.

Q4(a) – continued

4.2

(iii) Closure under scalar multiplication.
Suppose $\alpha \in \mathbb{R}$, and $\bar{w}_1 \in W$, as before.
Then, $\alpha\bar{w}_1 = \alpha(x_1, cx_1) = (\alpha x_1, c(\alpha x_1))$
$= (\alpha x_1, f(\alpha x_1)) \in W$,
as reqd.
Finally, W is non-trivial, since
$(1, c) = (1, f(1)) \neq \bar{0} \in W$, as noted before. Also, W is proper, since
$(1, c+1) \notin W$.
This completes (a)

Rubric for (a)

0.5 marks for showing that $\bar{0} \in W$, or, alternatively, W is non-empty.

1 mark for additive closure

1 mark for scalar multiplication closure

0.5 marks for showing that W is non-trivial

1 mark for showing that W is proper, i.e. $W \neq \mathbb{R}^2$

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4 marks in all.

Q4 – continued.

b) and c) will be considered together. There are two ~~alt~~ alternative correct answers, which will be considered separately.

Method A :–

b) Converse is: If the graph of f is a proper non-trivial subspace of $\mathbb{R}^2$, then there exists a non-zero $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$

c) The converse is NOT TRUE.

A counter-example is the function $f(x) = 0$. The graph of f is the x-axis in $\mathbb{R}^2$, which is a proper non-trivial subspace. However, there is clearly no non-zero c s.t. $f(x) = cx = 0$ for all $x \in \mathbb{R}$.

Rubric for Method A :

b) 1 mark for stating converse correctly.

c) 1 mark for stating NOT TRUE.

4 marks for the counter-example.

Q4 - continued

Method B:

b) Converse is: If the graph of f is a proper non-trivial subspace of $\mathbb{R}^2$, other than the x-axis, then there exists a non-zero $c \in \mathbb{R}$ s.t. $f(x) = cx$ for all $x \in \mathbb{R}$.

(Note: The above is a stronger statement than A).

c) Converse is TRUE.

Proof: Let W = graph of f.
Since W is non-trivial, by the given remark, $W = \text{Span}\{\bar{w}\}$, $\bar{w} \neq \bar{0}$.
Suppose $(x_1, y_1) \in W$, with $x_1 = 0$.
Then, $W = \{\alpha(0, y_1) : \alpha \in \mathbb{R}\}$ = y-axis, which cannot be the graph of a function.
Suppose $(x_1, y_1) \in W$ with $y_1 = 0$.
Then, $W = \{\alpha(x_1, 0) : \alpha \in \mathbb{R}\}$ = x-axis, which is not possible as stated in B.
$\therefore (x_1, y_1) \in W$, $x_1 \neq 0$, $y_1 \neq 0$.
Put $c = y_1/x_1$, so $c \neq 0$, and $y_1 = cx_1$.
In particular, $(1, c) = \frac{1}{x_1}(x_1, y_1) \in W$.
So, $W = \{\alpha(1, c) : \alpha \in \mathbb{R}\}$, i.e. $(1, c)$ is a basis element for W.

4.5

Q4 - completed

Now, for any $x \in \mathbb{R}$,
since $(1, c)$ is the only basis element for W,
$(x, f(x)) = x(1, c) = (x, cx)$,
uniquely.
$\therefore f(x) = cx$ for all $x \in \mathbb{R}$,
as required.

Rubric for Method B.
b) 1 mark for stating converse correctly, the underlined phrase must be clear.
c) 1 mark for TRUE.
4 marks for the above proof:-
1 mark each for excluding y-axis and x-axis. 1 mark for defining c. 1 mark for showing $f(x) = cx$.