

Problem 1.

Critical points:

$$y_1' = 0 \Rightarrow y_2 = 0$$

$$y_2' = 0 \Rightarrow -y_1 + \frac{1}{2}y_1^2 = 0 \Rightarrow y_1(-1 + \frac{1}{2}y_1) = 0$$

$$\Rightarrow y_1 = 0 \text{ \& } y_1 = 2$$

Two critical points: $(0,0)$ & $(2,0)$

Let $f(y_1, y_2) = y_2$ & $g(y_1, y_2) = -y_1 + \frac{1}{2}y_1^2$

$$\frac{\partial f}{\partial y_1} = 0, \frac{\partial f}{\partial y_2} = 1 \quad \& \quad \frac{\partial g}{\partial y_1} = -1 + y_1, \frac{\partial g}{\partial y_2} = 0$$

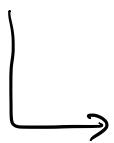
 $(0,0)$

$$y_1' = y_2$$

$$y_2' = -y_1$$

$$\Rightarrow Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y \Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$



center (stable)

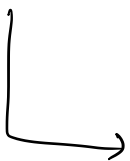
 $(2,0)$

$$y_1' = y_2$$

$$y_2' = y_1 - 2$$

$$\Rightarrow Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$



saddle (unstable)

Problem 2.

$$Y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t}$$

H. $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\lambda = 1$ $y = x \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1$ $y = -x \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Y_H = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$Y_0(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \Rightarrow Y_0^{-1} = \frac{1}{-2} \begin{pmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

$$\begin{aligned}
 \text{So, } U'(t) &= Y_0^{-1} \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t} \\
 &= \frac{1}{2} \begin{pmatrix} e^{-t} & e^{-t} \\ e^t & -e^t \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t} \\
 &= \frac{1}{2} \begin{pmatrix} -2e^{-t} \\ 4e^t \end{pmatrix} e^{3t} = \begin{pmatrix} e^{2t} \\ 2e^{4t} \end{pmatrix}
 \end{aligned}$$

$$U(t) = \begin{pmatrix} e^{2t}/2 \\ e^{4t}/2 \end{pmatrix}$$

$$Y_p(t) = \frac{1}{2} \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{4t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2e^{3t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$$

$$Y(t) = Y_H + Y_p = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$$

Problem 3.

$$Y' = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} Y + \begin{pmatrix} 0.6t \\ -2.5t \end{pmatrix}$$

$$\underline{\underline{H.}} \quad \lambda = 2 \Rightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\lambda = 5 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_H = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{2t}$$

$$\text{Let } Y_p(t) = U + Vt \Rightarrow Y_p' = V$$

$$V = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} (U + Vt) + \begin{pmatrix} 0.6 \\ -2.5 \end{pmatrix} t$$

$$\Rightarrow \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} V + \begin{pmatrix} 0.6 \\ -2.5 \end{pmatrix} = 0 \Rightarrow V = \begin{pmatrix} -0.43 \\ 1.12 \end{pmatrix}$$

$$\& \quad V = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} U \Rightarrow U = \begin{pmatrix} -0.241 \\ 0.534 \end{pmatrix}$$

$$\text{So, } Y(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -0.241 \\ 0.534 \end{pmatrix} + \begin{pmatrix} -0.43 \\ 1.12 \end{pmatrix} t$$

Problem 4.

let us define

$$y_1 = y$$

$$y_2 = y_1'$$

$$\text{then } y_2' - 9y_1 + y_1^3 = 0$$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = 9y_1 - y_1^3 = y_1(3 - y_1)(3 + y_1) \end{cases}$$

critical points are $(0,0)$, $(3,0)$, $(-3,0)$

$$f(y_1, y_2) = y_2$$

$$g(y_1, y_2) = 9y_1 - y_1^3$$

$$\frac{\partial f}{\partial y_1} = 0, \quad \frac{\partial f}{\partial y_2} = 1$$

$$\frac{\partial g}{\partial y_1} = 9 - 3y_1^2, \quad \frac{\partial g}{\partial y_2} = 0$$

$$\underline{\underline{(0,0)}}$$

$$Y' = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix} Y$$

$$\Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$$

↳ saddle (unstable)

$$\underline{\underline{(3,0)}}$$

$$\begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix}$$

$$\Rightarrow \lambda^2 + 18 = 0 \Rightarrow \lambda = \pm i\sqrt{18}$$

↳ center (stable)

$$\underline{\underline{(-3,0)}}$$

$$\begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix}$$

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↳ center (stable)