MTH100 **Monsoon Semester – 2021-22** **Section B**

Submission for Wednesday 23rd March 2022 – 15 minutes. Max Marks: 5

Instructions: *Open notes and textbook; consultation and use of calculators, computers and internet not allowed. You may use any* ***known*** *result. This includes all propositions and observations in the lecture slides, and results from tutorials. If you use any other result from any other source, including the textbook, you have to give a full proof of that result.*

Q. PROVE or DISPROVE: If U and A are n×n (square) matrices, such that U is invertible, then rank (UA) = rank (A). (5 *marks)*

Remark: You must clearly write PROVE or DISPROVE at the top of your answer. 1 mark is reserved for this. If not written, you will directly get 0 marks. For PROVE, you must give a general proof using known results; use of examples is not acceptable. For DISPROVE, you must give a concrete (numerical) counter-example.

SOLUTION & RUBRIC

RECALL: Standard results for matrices:

(1) For an $m \times n$ matrix A, $\text{rank}(A) \leq \min\{m, n\} \rightarrow$ Tut 09, Q1

(2) If the product AB is defined, then $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ $\rightarrow$ Tut 09, Q3

(3) ~~If A is~~ An $n \times n$ (square) matrix A is invertible if and only if $\text{rank}(A) = n \rightarrow$ consequence of Rank Theorem.

(PTO)

Answer: PROVE → 1 MARK

Proof: Consider:

$$\text{rank}(UA) \le \min\{\text{rank}(U), \text{rank}(A)\}$$

$$\le \min\{n, \text{rank}(A)\} \quad \to \text{by } 2 \;\; \to \text{by } 3$$

$$= \text{rank}(A) \quad \to \text{by } 1$$

i.e. $\text{rank}(UA) \le \text{rank}(A)$ (4)

OTOH, $A = U^{-1}(UA)$, so in a similar way,

$$\text{rank}(A) \le \min\{\text{rank}(U^{-1}), \text{rank}(UA)\}$$

$$= \min\{n, \text{rank}(UA)\}$$

$$= \text{rank}(UA)$$

i.e. $\text{rank}(A) \le \text{rank}(UA)$ (5)

From 4 and 5, $\text{rank}(UA) = \text{rank}(A)$, as required.

→ Proof: 4 marks. Very easy, no partial credit.