Worksheet - 10 Solutions

Q-1

$$iR + i'L = v(t)$$

Take Laplace transform

$$R\mathcal{L}(i) + L(-i(0) + s\mathcal{L}(i)) = \mathcal{L}(v(t))$$

$$(R+sL)\mathcal{L}(i) = \mathcal{L}(v(t))$$

$$\Rightarrow \mathcal{L}(i) = \frac{\mathcal{L}(v(t))}{R+sL}$$

$$v(t) = 20\cos t \; u(t-\pi)$$

$$\mathcal{L}(v(t)) = 20e^{-\pi s}\mathcal{L}(\cos(t+\pi))$$

$$\left[\text{Using } \mathcal{L}(f(t)\,u(t-a)) = e^{-as}\mathcal{L}(f(t+a))\right]$$

$$\Rightarrow \mathcal{L}(v(t)) = 20e^{-\pi s}\mathcal{L}(-\cos t) = -20e^{-\pi s}\mathcal{L}(\cos t)$$

$$= \frac{-20se^{-\pi s}}{s^2+1}$$

$$\Rightarrow \mathcal{L}(i) = \frac{-20se^{-\pi s}}{(s^2+1)(s+100)}$$

Use partial fraction

$$\frac{s}{(s^2+1)(s+100)} = \frac{As+B}{s^2+1} + \frac{C}{s+100}$$

$$\Rightarrow (As+B)(s+100) + C(s^2+1) = s$$

$$\Rightarrow \left.\begin{aligned} A + C &= 0 \\ 100A + B &= 1 \\ 100B + C &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A - 100B &= 0 \\ 100A + B &= 1 \quad \times 100 \\ \hline (10^4+1)A &= 10^2 \end{aligned}$$

$$\Rightarrow A = \frac{100}{10^4+1}$$

$$\Rightarrow C = \frac{-100}{10^4+1}$$

$$\Rightarrow B = \frac{1}{10^4+1}$$

So, $\mathcal{L}(i) = -20\, e^{-\pi s}\left(\frac{A\, s}{s^2+1} + \frac{B}{s^2+1} + \frac{C}{s+100}\right)$

$$\Rightarrow i(t) = -20\left(A\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}e^{-\pi s}\right) + B\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) + C\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s+100}\right)\right)$$

$$= -20\Big(A\cdot \cos(t-\pi)\,u(t-\pi) + B\,\sin(t-\pi)\,u(t-\pi)$$
$$+ C\, e^{-100(t-\pi)}\,u(t-\pi)\Big)$$

[Using, if $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}^{-1}(e^{-as}F(s)) = f(t-a)\,u(t-a)$

$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$, $\mathcal{L}(\sin t) = \frac{1}{s^2+1}$, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$]

So,

$$i(t) = \begin{cases} 0 & \text{if } 0<t<\pi \\ -20\left(-A\cos t - B\sin t + C e^{-100(t-\pi)}\right) & \text{if } t>\pi \end{cases}$$

Q-2.

$v(t)$ is the voltage across capacitor.

Current, $i(t)$ and charge, $q(t)$ are related by

$$i(t) = q' = \frac{dq}{dt}$$

So, voltage $v(t) = \frac{q(t)}{C}$. So, we need to find $q(t)$

Eq. will be

$$L\,i'(t) + R\,i(t) + \frac{q(t)}{C} = \delta(t)$$

$$\Rightarrow L\,q'' + R\,q' + \frac{1}{C}q(t) = \delta(t)$$

Taking Laplace transform

(Using $i(0) = q'(0) = 0$, $q(0) = 0$)

$$Ls^2 \mathcal{L}(q) + Rs\mathcal{L}(q) + \frac{1}{C}\mathcal{L}(q) = \mathcal{L}(\delta(t)) = e^{-0 \cdot s} = 1$$

$$\Rightarrow \quad \mathcal{L}(q) = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

$$= \frac{1}{s^2 + 8s + 116} = \frac{1}{(s+4)^2 + 10^2}$$

Now, $\mathcal{L}(\sin 10t) = \dfrac{10}{s^2 + 10^2}$

Using s-shifting Theorem

$$e^{at} f(t) = \mathcal{L}^{-1}(F(s-a))$$

$$\Rightarrow \quad q(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+4)^2 + 10^2}\right) = \frac{1}{10} e^{-4t} \sin 10t$$

$$\Rightarrow \quad v(t) = \frac{q(t)}{C}$$

$$= \frac{116}{10} e^{-4t} \sin 10t$$