Test 1: Math 1 (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

January 21st

Duration: 60 minutes **Maximum Marks:** 10

Question 1 (5 marks).

(a) (2.5 marks) Let $c \in \mathbb{R}$ be a fixed scalar. Define $T : \mathbb{P}_2 \to \mathbb{P}_3$ by

$$T(a_0 + a_1x + a_2x^2) = \int_c^x (a_0 + a_1t + a_0t^2)\,\mathrm{dt}$$

Show that T is a 1-1 linear transformation.

(b) (2.5 marks) Find the matrix of T with respect to the ordered bases $\mathcal{B} = \{1, x - c, (x - c)^2\}$ of $\mathbb{P}_2$, and $\mathcal{C} = \{1, x - c, x^2 - c^2, x^3 - c^3\}$ of $\mathbb{P}_3$.

Solution.

(a) Let $p(x), q(x) \in \mathbb{P}_2$. We know (from high school) that

$$\int_c^x (p(t) + q(t))\,\mathrm{dt} = \int_c^x p(t)\,\mathrm{dt} + \int_c^x q(t)\,\mathrm{dt}$$

Therefore $T(p(x) + q(x)) = T(p(x)) + T(q(x))$. Similarly if $p(x) \in \mathbb{P}_2$ and $\alpha \in \mathbb{R}$ then

$$\int_c^x \alpha p(t)\,\mathrm{dt} = \alpha \int_c^x p(t)\,\mathrm{dt}$$

Therefore $T(\alpha p(x)) = \alpha T(p(x))$.

As T respects addition and scalar multiplication, it is a linear transformation. Next we show that T is 1-1.

First method:

T is 1-1 if and only if $\ker T = \{0\}$. Therefore we will show that $\ker T = \{0\}$.

Let $p(x) = a_0 + a_1x + a_2x^2 \in \ker T$. Then

$$\int_c^x (a_0 + a_1t + a_2t^2)\,\mathrm{dt} = 0.$$

Therefore

$$a_0 t\Big|_c^x + \frac{1}{2}a_1 t^2\Big|_c^x + \frac{1}{3}a_2 t^3\Big|_c^x = 0 \implies a_0(x-c) + \frac{1}{2}a_1(x^2-c^2) + \frac{1}{3}a_2(x^3-c^3) = 0$$

A polynomial is zero only if all its coefficients are zero. Therefore

$$a_0 = \frac{1}{2}a_1 = \frac{1}{3}a_2 = 0 \implies a_0 = a_1 = a_2 = 0 \implies p(x) = 0$$

As the choice of $p(x)$ was arbitrary, it follows that $\ker T = \{0\}$.

Second method:

We show that $T(p(x)) = T(q(x)) \implies p(x) = q(x)$.

Let $p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2$ be polynomials such that $T(p(x)) = T(q(x))$. Then

$$\int_c^x (a_0 + a_1 t + a_2 t^2)\,\mathrm{dt} = \int_c^x (b_0 + b_1 t + b_2 t^2)\,\mathrm{dt}$$

Therefore

$$a_0 t\Big|_c^x + \frac{1}{2}a_1 t^2\Big|_c^x + \frac{1}{3}a_2 t^3\Big|_c^x = b_0 t\Big|_c^x + \frac{1}{2}b_1 t^2\Big|_c^x + \frac{1}{3}b_2 t^3\Big|_c^x$$

Hence

$$(a_0 - b_0)(x-c) + \frac{1}{2}(a_1 - b_1)(x^2 - c^2) + \frac{1}{3}(a_2 - b_2)(x^3 - c^3) = 0.$$

A polynomial is zero only if all its coefficients are zero. Therefore

$$a_0 - b_0 = \frac{1}{2}(a_1 - b_1) = \frac{1}{3}(a_2 - b_2) = 0 \implies a_0 = b_0, a_1 = b_1, a_2 = b_2 \implies p(x) = q(x).$$

Third Method:

Compute the matrix of T with respect to fixed bases for $\mathbb{P}_2$ and $\mathbb{P}_3$. For example, you may choose the bases in part (b) of this question.

Regardless of what you choose, the matrix you obtain will have linearly independent columns. For the choice of bases in part (b), this is clear because every column is a pivot column (swap twice to get the zero row to the bottom).

The same holds true if we choose the natural bases $\mathcal{B}' = \{1, x, x^2\}$ and $\mathcal{C}' = \{1, x, x^2, x^3\}$. (Of course the most convenient choice of bases for this computation is $\mathcal{B}' = \{1, 2x, 3x^2\}$ and $\mathcal{C}' = \{1, x-c, x^2-c^2, x^3-c^3\}$, but I leave this part to you.)

Let us verify that the matrix of the transformation has linearly independent columns when $\mathcal{B}' = \{1, x, x^2\}$ and $\mathcal{C}' = \{1, x, x^2, x^3\}$.

$$T(1) = \int_c^x \mathrm{dt} = x - c \implies [T(1)]_{\mathcal{C}'} = (-c, 1, 0, 0)$$

$$T(x) = \int_c^x t\,\mathrm{dt} = \frac{1}{2}(x^2 - c^2) \implies [T(x)]_{\mathcal{C}'} = (-c^2/2, 0, 1/2, 0)$$

$$\begin{aligned} T(x^2) &= \int_c^x t^2\,\mathrm{dt} \\ &= \frac{1}{3}(x^3 - c^3) \end{aligned}$$

Therefore

$$[T(x^2)]_{\mathcal{C}} = (-c^3/3, 0, 0, 1/3)$$

Thus the matrix of T with respect to bases $\mathcal{B}'$ and $\mathcal{C}'$ is

$$[T]_{\mathcal{B}',\mathcal{C}'} = [[T(1)]_{\mathcal{C}'} \quad [T(x)]_{\mathcal{C}'} \quad [T(x^2)]_{\mathcal{C}'}] = \begin{bmatrix} -c & -c^2/2 & -c^3/3 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Clearly, this matrix has linearly independent columns. As the matrix transformation on coordinate vectors is the same as the action of T on the corresponding vectors, the transformation T must be 1-1.

(b)

$$T(1) = \int_c^x \mathrm{dt} = x - c \implies [T(1)]_{\mathcal{C}} = (0, 1, 0, 0)$$

$$T(x-c) = \int_c^x (t-c)\,\mathrm{dt} = \frac{1}{2}(x^2 - c^2) - c(x-c) \implies [T(x-c)]_{\mathcal{C}} = (0, -c, 1/2, 0)$$

$$\begin{aligned} T((x-c)^2) &= \int_c^x (t^2 - 2ct + c^2)\,\mathrm{dt} \\ &= \frac{1}{3}(x^3 - c^3) - c(x^2 - c^2) + c^2(x - c) \end{aligned}$$

Therefore

$$[T((x-c)^2)]_{\mathcal{C}} = (0, c^2, -c, 1/3)$$

Thus the matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{B},\mathcal{C}} = [[T(1)]_{\mathcal{C}} \quad [T(x-c)]_{\mathcal{C}} \quad [T((x-c)^2)]_{\mathcal{C}}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -c & c^2 \\ 0 & 1/2 & -c \\ 0 & 0 & 1/3 \end{bmatrix}$$

Rubric.

(a)
- 1/2 mark for correctly showing that T respects vector addition
- 1/2 mark for correctly showing that T respect scalar multiplication
- First method of 1-1: 1 mark for choosing to show that the kernel of T is trivial, 1/2 mark for finishing the argument correctly
- Second method: 1/2 mark for choosing to show that $T(p(x)) = T(q(x))$ implies $p(x) = q(x)$, 1/2 mark for correctly calculating the required integrals, 1/2 mark for concluding that the coefficients of the polynomials must be equal
- Third method: 1 mark for choosing to show that the matrix of T has linearly independent columns. 1/2 mark for reasonably completing the argument (either by referring to part (b) or by computing the matrix with respect to some other choice of bases.)

(b)
- 1 mark for stating the correct formula or for substituting correct values in the formula, i.e. the step
$$[T]_{\mathcal{B},\mathcal{C}} = [[T(1)]_{\mathcal{C}} \quad [T(x-c)]_{\mathcal{C}} \quad [T((x-c)^2)]_{\mathcal{C}}]$$
- 1/2 mark for computing the required integrals correctly
- 1 mark for correctly identifying the coefficients of $T(1), T(x-c)$ and $T((x-c)^2)$ with respect to basis $\mathcal{C}$.

Question 2 (5 marks).

(a) (2.5 marks) Let Q be an invertible 2×2 matrix. Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(A) = QAQ^{-1} - A$$

Show that T is a linear transformation and find the kernel of T. Is T a 1-1 transformation? Justify your answer.

(b) (2.5 marks) Find the dimension of the kernel of T for the following choices of Q:

(i) $Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Solution.

(a) Let $A, B \in M_{2\times 2}(\mathbb{R})$.

$$\begin{aligned} T(A+B) &= Q(A+B)Q^{-1} - (A+B) \\ &= (QA+QB)Q^{-1} - A - B \\ &= QAQ^{-1} + QBQ^{-1} - A - B \\ &= T(A) + T(B) \end{aligned}$$

Let $A \in M_{2\times 2}(\mathbb{R}), c \in \mathbb{R}$.

$$\begin{aligned} T(cA) &= QcAQ^{-1} - cA \\ &= c(QAQ^{-1} - A) \\ &= cT(A) \end{aligned}$$

As T respects addition and scalar multiplication, it is a linear transformation.

$$\begin{aligned} \ker T &= \{A \in M_{2\times 2}(\mathbb{R}) \mid T(A) = 0\} \\ &= \{A \in M_{2\times 2}(\mathbb{R}) \mid QAQ^{-1} - A = 0\} \\ &= \{A \in M_{2\times 2}(\mathbb{R}) \mid QA = AQ\} \end{aligned}$$

Clearly $\ker T \neq \{0\}$, because $T(I) = QIQ^{-1} - I = 0$, so $I \in \ker T$. Hence T is not 1-1.

(b) (i) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{aligned} T(A) &= QAQ^{-1} - A \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Hence T is the zero transformation, and $\ker T = M_{2\times 2}(\mathbb{R})$. Therefore $\dim \ker T = 4$.

(ii) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{aligned} T(A) &= QAQ^{-1} - A \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & 2b/3 \\ 3c/2 & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 0 & -b/3 \\ c/2 & 0 \end{bmatrix} \end{aligned}$$

Thus $A \in \ker T \iff b = c = 0$. Hence

$$\ker T = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore $\dim \ker T = 2$.

Rubric.

(a)
- 1/2 mark for correctly showing that T respects vector addition
- 1/2 mark for correctly showing that T respect scalar multiplication
- 1/2 mark for using the correct definition of the kernel and substituting the appropriate terms into the definition
- 1/2 mark for answering that T is not 1-1
- 1/2 mark for correct justification.

(b)
- 1 mark for finding the correct dimension in (i).
- 1 mark for finding the correct dimension in (ii)
- 1/2 mark for correct justification in part (ii)