(2 pts.)

Q-1 Following the hint, first step is to write ODE $y''+py'+qy=0$ for y_1 and y_2. Then

$$y_1''+py_1'+qy_1=0 \quad — \text{(I)}$$
$$y_2''+py_2'+qy_2=0 \quad — \text{(II)}$$

where p and q are variables.

Eliminating q from (I) & (II), we get (i.e. (I) $\times -y_2$ + (II) $\times y_1$ gives)

$$(y_1y_2''-y_2y_1'')+p(y_1y_2'-y_2y_1')=W'+pW=0$$

where W' is the derivative of W and W is Wronskian w.r.t. y_1 & y_2.

$$W'=\frac{d}{dx}(W)=\frac{d}{dx}(y_1y_2'-y_2y_1')$$
$$=\cancel{y_1'y_2'}+y_1y_2''-\cancel{y_2'y_1'}-y_2y_1''$$
$$=y_1y_2''-y_2y''$$

Here, $W'+pW=0$ is first order separable ODE

$$\int\frac{dW}{W}+\int p\,dx=0 \Rightarrow \int\frac{dW}{W}=-\int p\,dx$$

$$\Rightarrow \boxed{W(y_1,y_2)=C'e^{-\int_{x_0}^{x}p(t)\,dt}}$$

For $x=x_0$, we get $C'=W(y_1(x_0),y_2(x_0))$

$$\therefore W(y_1(x),y_2(x))=Ce^{-\int_{x_0}^{x}p(t)\,dt},\quad C=W(y_1(x_0),y_2(x_0))$$

Now, for $y_1(x)=e^{-x}\cos\omega x$, $y_2(x)=e^{-x}\sin\omega x$

So, ODE is given by $y''+py'+qy=0$ where $p=-(\lambda_1+\lambda_2)$, $q=\lambda_1\lambda_2$

and $\lambda_1=-1+\omega i$, $\lambda_2=-1-\omega i$ are roots of above ODE.

So, ODE is $y''-(-1+\omega i-1-\omega i)y'+((-1+\omega i)(-1-\omega i))y=0$

$$\Rightarrow y''+2y'+(1+\omega^2)y=0$$

Next, $y_1(x) = e^{-x}\cos\omega x \Rightarrow y_1'(x) = -e^{-x}\cos\omega x - \omega e^{-x}\sin\omega x$

$\Rightarrow y_1(0) = 1 \quad , y_1'(0) = -1$

& $y_2(x) = e^{-x}\sin\omega x \Rightarrow y_2'(x) = -e^{-x}\sin\omega x + \omega e^{-x}\cos\omega x$

$\Rightarrow y_2(0) = 0 \quad , y_2'(0) = \omega$

$\therefore W(y_1(x), y_2(x)) = Ce^{-\int_0^x 2\,dt} = Ce^{-2x} \quad (x_0 = 0)$

where $C = W(y_1(0), y_2(0)) = y_1(0)y_2'(0) - y_2(0)y_1'(0)$
$= \omega$

$\therefore \boxed{W(y_1(x), y_2(x)) = \omega e^{-2x}}$

(2pts.)

Q-2 Let the buoy is depressed y meter from its equilibrium position.
Then volume of water displaced when buoy is depressed y meter from its equilibrium position $= \pi \times (0.15)^2 \times y$ (Diameter = 30cm $\Rightarrow$ Radius = 15 cm = 0.15 m)

So, ODE will be

$my'' = -\pi \times 0.0225\, y \times k$, where $k = 9800$ nt. is the weight of water per cubic meter

$\Rightarrow y'' + \omega^2 y = 0$, where $\omega^2 = \dfrac{0.0225\,\pi \times k}{m}$

and period $\dfrac{2\pi}{\omega} = 3 \Rightarrow \omega = \dfrac{2\pi}{3} \Rightarrow \omega^2 = \dfrac{4\pi^2}{9}$

Hence, $\dfrac{4\pi^{\cancel{2}}}{9} = \dfrac{0.0225\,\cancel{\pi}\,k}{m}$

$\Rightarrow m = \dfrac{0.0225 \times 9800 \times 9}{4\pi} = 157.92$ nt.

$\Rightarrow W = mg = 157.92 \times 9.8$ nt

$\Rightarrow \boxed{W = 1547.63 \text{ nt}}$

(1X2=2pts.)

(1pt)

Q-3 (a) Characteristic equation corresponding to (A) is:

$\lambda^2 + a\lambda + b = 0$

and $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \Rightarrow \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$

$\Rightarrow a = -(\lambda_1 + \lambda_2)$ and $b = \lambda_1\lambda_2$

Hence, by using this formulas one can find the constants a & b and then putting these values in

$$y'' + ay' + by = 0$$

we can get the ODE.

(1 pt.)

(b) 1. ODE has distinct real roots $\lambda_1 = 2.6$ & $\lambda_2 = -4.3$

So, ODE is given by : $y'' - (2.6 - 4.3)y' + (2.6 \times (-4.3))y = 0$

i.e. $\boxed{y'' + 1.7y' - 11.18y = 0}$

2. ODE has complex roots $\lambda_1 = -3.1 + 2.1i$
$\lambda_2 = -3.1 - 2.1i$

So, ODE is given by : $y'' - (-3.1 + 2.1i - 3.1 - 2.1i)y' + ((-3.1+2.1i)(-3.1-2.1i))y = 0$

i.e. $\boxed{y'' + 6.2y' + 14.02y = 0}$

(1pt.)

Q-4 ODE of a damped system is given by

$$my'' + cy' + ky = 0$$

where $m = 2500$ kg, $k = 2500$ kg/sec^2, c is damping constant.

For oscillation free ride, we must have

$$c^2 > 4mk$$

i.e. $c^2 > 4 \times (2500)^2 = (2 \times 2500)^2$

$\Rightarrow c > 2 \times 2500$

i.e. $\boxed{c > 5000}$

(2pts.)

Q-5 The force of inertia in Newton's second law is my'', where $m = 3$ kg is the mass of the water.

The dark ~~(blue)~~ portion of the water given in the figure, a column of height $2y$, is the portion that causes the restoring force of the vibration.

And this volume $= \pi \times (.02)^2 \times 2y$ $\left[\begin{array}{l} \text{Diameter} = 4\text{cm} \\ \Rightarrow \text{Radius} = 2\text{cm} = 0.02\text{ m} \end{array}\right]$

As weight of water per cubic meter is $V = 9800$ nt

$\Rightarrow$ weight of that dark volume of water $= \pi \times (.02)^2 \times 2y \times 9800$ nt

Hence, ODE will be $y'' + \omega^2 y = 0$, where $\omega^2 = \dfrac{\pi \times (0.02)^2 \times 2 \times 9800}{3}$

$\Rightarrow \omega^2 = 8.21$

$\Rightarrow \omega = 2.86$

Hence, the general solution will be

$$y = A\cos 2.86t + B\sin 2.86t$$

and the frequency is $\frac{\omega}{2\pi} = 0.456$ per second,

that is, water makes about $\underset{(\text{frequency} \times 60)}{\underset{\shortparallel}{27}}$ oscillations per minute.

(1pt.)

Q-6 $D^2 + aD + bI = 0$ — ①

So, $y'' + ay' + by = 0$ is corresponding homogeneous linear ODE.

As $e^{\mu x}$ and $e^{\lambda x}$ are solutions of ①, $\therefore$ by superposition principle

$y(x) = \dfrac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda}$ is also a solution of ①

Letting $\mu \to \lambda$, we will fix the λ and y then is regarded as a function of μ.

$\therefore$ By L' Hôpital's rule,

$$\frac{xe^{\mu x} - 0}{1} \to xe^{\mu x} \quad (\text{Differentiating } y \text{ w.r.t. } \mu)$$