Quiz 4

November 11, 2022

1. The Mongoldt function Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if} n = p^k \text{ where p is a prime and} k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = -\sum_{d|n} \mu(d) \log d$.

Hint: First show that $\sum_{d|n} \Lambda(d) = \log n$ and then apply the Möbius inversion formula.

Sol^n- If p is a prime, then -

$$\sum_{d|p^k} \Lambda(d) = \Lambda(1) + \Lambda(p) + \Lambda(p^2) + \cdots + \Lambda(p^k)$$

$$= 0 + \log p + \log p + \cdots + \log p$$

$$= k \log p = \log p^k.$$

Now if the prime factorization of a +ve integer n is given by - $n = p_1^{k_1} \cdots p_r^{k_r}$, then only non-zero terms in $\sum_{d|n} \Lambda(d)$ come from divisors d of the form $p_i^{s_i}$. Hence

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \left(\sum_{d|p_i^{k_i}} \Lambda(d) \right) = \sum_{i=1}^{r} \log p_i^{k_i} = \log n.$$

By Mobius inversion formula, we have.

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \mu(d) [\log n - \log d]$$

$$= \sum_{d|n} \mu(d) \log n - \sum_{d|n} \mu(d) \log d$$

$$= \log n \underbrace{\sum_{d|n} \mu(d)}_{=0} - \sum_{d|n} \mu(d) \log d = -\sum_{d|n} \mu(d) \log d. \quad \square$$