MTH 373/573: Scientific Computing | IIIT-Delhi, Monsoon 2022 | Kaushik Kalyanaraman

# Solutions to Quiz 3

## Problem 1: Positive definiteness of normal equations matrix — 5 points

Suppose that $A \in \mathbb{R}^{m \times n}$, $m \geq n$ is of rank $n$. Prove that the matrix $A^T A$ is positive definite.

Given that $A$ is full rank, this means that for any $0 \neq x \in \mathbb{R}^n$, we have that $Ax \neq 0$. Thus, $\|Ax\|_2^2 = (Ax)^T (Ax) = x^T A^T A x > 0 \implies A^T A$ is positive definite. □

## Problem 2: Nonsingularity of normal equations matrix — 5 points

Suppose that $A \in \mathbb{R}^{m \times n}$ $m \geq n$ is of rank $n$. Prove that the matrix $A^T A$ is nonsingular.

Given for any $0 \neq x \in \mathbb{R}^n$, $Ax \neq 0$. Let $y := A^T A x$. Then $x^T y = \|Ax\|_2^2 \neq 0 \implies y \neq 0$. Hence, for any nonzero $x$, $A^T A x \neq 0$ meaning that $A^T A$ is nonsingular. □

## Problem 3: Orthogonal vectors — 5 points

(a) Given two vectors $u$ and $v$, what is the definition of $u$ being orthogonal to $v$? (1 point)

Two vectors $u$ and $v$ are orthogonal if their innerproduct is 0, that is $\langle u, v \rangle = u^T v = 0 = v^T u = \langle v, u \rangle$ (since the standard innerproduct is symmetric).

(b) Prove that if two nonzero vectors are orthogonal to each other, then they must also be linearly independent. (4 points)

Consider a linear combination of two orthogonal vectors $u$ and $v$ with scalars $\alpha$ and $\beta$: $\alpha u + \beta v$. To check if $u$ and $v$ are independent, we need to check that the only linear combination of $u$ and $v$ that is 0 is the one with $\alpha = 0 = \beta$. Let $\alpha u + \beta v = 0$. Taking innerproduct of both sides with $u$, we obtain $\langle \alpha u, u \rangle + \langle \beta v, u \rangle = \langle 0, u \rangle \implies \alpha \langle u, u \rangle + \beta \langle v, u \rangle = \alpha + \beta \times 0 = 0$ since the innerproduct of the zero vector with any vector is 0. Thus, $\alpha = 0$. Likewise, $\beta = 0$. Hence $u$ and $v$ are linearly independent. □

## Problem 4: Orthogonal matrix norm preservation — 5 points

Show that multiplication by an orthogonal matrix preserves the Euclidean norm of a vector.

Let $x \in \mathbb{R}^n$ be any vector and $Q \in \mathbb{R}^{n \times n}$ any orthogonal matrix. Then $\|Qx\|_2^2 = (Qx)^T (Qx) = x^T Q^T Q x = x^T x = \|x\|_2^2$ since $Q^T Q = I \in \mathbb{R}^{n \times n}$ the identity matrix by definition. □

## Problem 5: Orthogonal projector 5 points

If $A$ is a matrix with linearly independent columns, show that $A\left(A^T A\right)^{-1} A^T$ is an orthogonal projector onto span$(A)$.

To check that $P := A\left(A^T A\right)^{-1} A^T$ is an orthogonal projector, we need to verify that $P^2 = P$, $P^T = P$ and that span$(P) = $ span$(A)$.

$$\begin{aligned} P^2 &= A\left(A^T A\right)^{-1} A^T A \left(A^T A\right)^{-1} A^T = A\left(A^T A\right)^{-1} \left(A^T A\right) \left(A^T A\right)^{-1} A^T \\ &= A\left(\left(A^T A\right)^{-1}\left(A^T A\right)\right)\left(A^T A\right)^{-1} A^T = A\left(A^T A\right)^{-1} A^T = P. \end{aligned}$$

$$P^T = \left(A\left(A^T A\right)^{-1} A^T\right)^T = \left(A^T\right)^T \left(\left(A^T A\right)^{-1}\right)^T A^T = A\left(\left(A^T A\right)^T\right)^{-1} A^T = A\left(A^T A\right)^{-1} A^T = P.$$

Finally, for any $x$, we have that span $P := \{Px\} = \left\{A\left(A^T A\right)^{-1} A^T x\right\} = \left\{A\left(\left(A^T A\right)^{-1} A^T x\right)\right\} = \{Ay\} =:$ span$(A)$.