Worksheet-5

Math-III

Total Marks- 20

Total Timing-30 mins

Date – 12/10/2022

Q1. Find all the local maxima, local minima, and saddle points of $f(x, y) = x^3 - 3y^2 + 3xy - 3y$.

(10)

Q2. Use the method of Lagrange multipliers to find the minimum value of $4x + 9y$, subject to the constraints $xy = 4$, $x > 0$, $y > 0$. (10)

Worksheet-5 Solution + Rubric

M-III

Q.1. Solution:

$$\text{Given } f(x,y) = x^3 - 3y^2 + 3xy - 3y$$

$\therefore f_x = 3x^2 + 3y$ & $f_y = -6y + 3x - 3$ (2)

Now, (At extremum) $f_x = 0 \Rightarrow y = -x^2$ & $f_y = 0 \Rightarrow -2y + x - 1 = 0$

$\Rightarrow -2(-x^2) + x - 1 = 0$

$\Rightarrow 2x^2 + x - 1 = 0$

$\Rightarrow (2x - 1)(x+1) = 0$

$\therefore x = 1/2$ or -1

$\therefore y = -\frac{1}{4}$ or -1

$\therefore$ Critical points are: $(1/2, -1/4)$ & $(-1, -1)$ (2)

Now, $f_{xx} = 6x$

$f_{xy} = 3$

$f_{yy} = -6$

At $(1/2, -1/4)$: $f_{xx} f_{yy} - f_{xy}^2$

$= 6x \times (-6) - 3^2 = -\cancel{6}^{3} \times \frac{1}{\cancel{2}} \times 6 - 9$

$= -27 < 0$.

At $(-1,-1)$: $f_{xx} f_{yy} - f_{xy}^2$

$= 6x \times (-6) - 3^2 = 6(-1)(-6) - 9$

$= 36 - 9 = 27 > 0$. (1)

As at, $(-1,-1)$, $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} = -6 < 0$.

So, the critical point $(-1,-1)$ is a local maxima. (2)

But at $(1/2, -1/4)$; $f_{xx}f_{yy} - f_{xy}^2 = -27 < 0$.

So, the critical point $(1/2, -1/4)$ is a saddle point. (3)

Q.2. Solution:

~~Given, $f(x,y) = 4x + 9y$.~~

~~and $xy = 4$.~~

~~Given $f(x,y) = 4x + 9y$ and $xy = 4$, $x > 0$, $y > 0$.~~

~~Let,~~

Let, $f(x,y) = 4x + 9y$ & $g(x,y) = xy - 4$
$(x > 0, y > 0)$

$\therefore \nabla f = 4\hat{i} + 9\hat{j}$ and $\nabla g = y\hat{i} + x\hat{j}$. (1)

From the method of Lagrange's multipliers (1)

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow 4\hat{i} + 9\hat{j} = (\lambda y)\hat{i} + (\lambda x)\hat{j}$$

(1)

$\therefore y = \frac{4}{\lambda}$ and $x = \frac{9}{\lambda}$.

But, $xy = 4$ and $x > 0, y > 0$

$\Rightarrow \frac{36}{\lambda^2} = 4 \Rightarrow \lambda = \pm 3$. (1)

But, as $x > 0, y > 0 \Rightarrow \lambda = 3$ and $x = 3$ & $y = \frac{4}{3}$. (1)

Hence the minimum value at $(3, 4/3)$ is

$= 4 \times 3 + 9 \times \frac{4}{3} = 12 + 12 = 24$. (5)