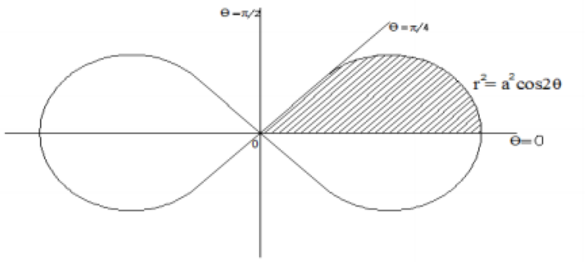
Worksheet-7 Date- 08/11/2022

Total-20 marks

Math-III

Q1. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing the order of integration. (5 marks)

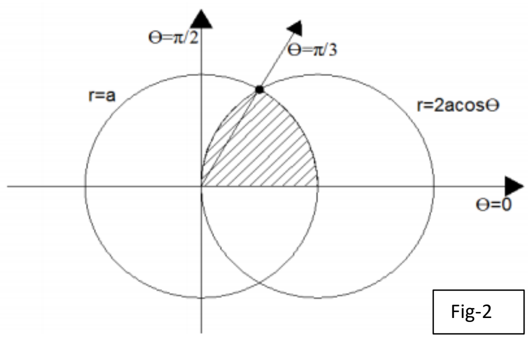
Q2. Find the area enclosed by the leminiscate (See Fig-1) $r^2 = a^2 \cos 2\theta$ by double integration. (5 marks)



Hint: The required area = 4*(Shaded region)

Fig-1

Q3. Find the common area (see Fig-2) to the circles $r = a, r = 2a \cos\theta$. (5 marks)



Hint: The required area = 2*(shaded region)

Fig-2

Q4. Let me present to you an argument that establishes the equality of the real number $\ln 5$ with $\infty - \infty$. Needless to say that the argument has a flaw. Can you figure out where does the argument go wrong? Give reason for your answer. (5 marks)

$$\ln 5 = \ln 1 + \ln 5 = \ln 1 - \ln\left(\frac{1}{5}\right) = \lim_{b \to \infty} \ln \frac{b^2+1-2b}{b^2+1} - \ln(1/5)$$

$$= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1}\right]_2^b$$

$$= \lim_{b\to\infty}\left[2\ln(x-1) - \ln(x^2+1)\right]_2^b$$

$$= \int_2^{\infty} \left(\frac{2}{x-1} - \frac{2x}{x^2+1} \right) dx$$

$$= \int_2^{\infty} \frac{2}{x-1}\, dx - \int_2^{\infty} \frac{2x}{x^2+1}\, dx$$

$$= \lim_{b \to \infty} [2 \ln(x-1)]_2^b - \lim_{b \to \infty} [\ln(x^2+1)]_2^b$$

$$= \infty - \infty$$

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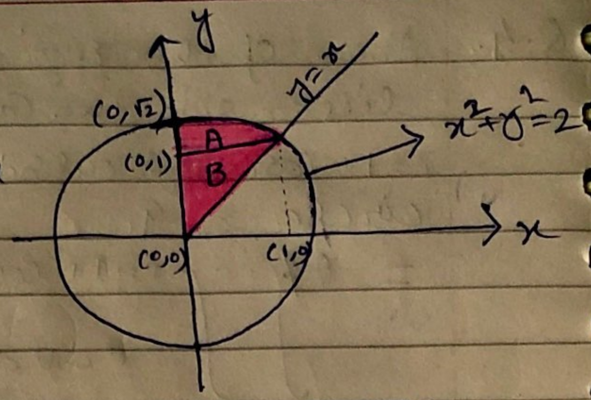
Rubric and Solution of WorkSheet - 7

Q.1. Given integral is:

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$



So, from the integral we

1 have, $x=0$, $x=1$ and $y=x$, $y=\sqrt{2-x^2}$
$(\Rightarrow x^2+y^2=(\sqrt{2})^2)$

By changing the order of integration, we have,

2 in region B: $y:=0$ to 1 and $x:=0$ to y. &
In region A: $y:=1$ to $\sqrt{2}$ and $x:=0$ to $\sqrt{2-y^2}$

$$\therefore\ I = \iint_B \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \iint_A \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy.$$

$$= \int_0^1\int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy.$$

2

$$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 (\sqrt{2}y - y)\,dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\,dy.$$

$$= \frac{(\sqrt{2}-1)}{2} + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2} + 1 - \sqrt{2} + \frac{1}{2}$$

$$= 1 - \frac{\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}}$$

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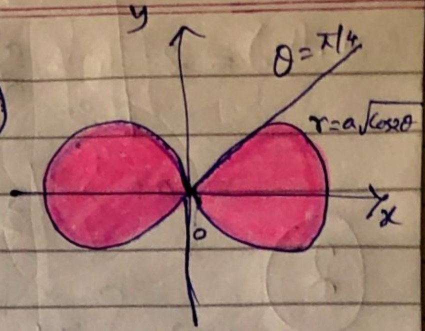
Q.2. The required area

$$= 4 \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$ (2)



$$= 4 \int_{\theta=0}^{\theta=\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2 \int_{\theta=0}^{\theta=\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

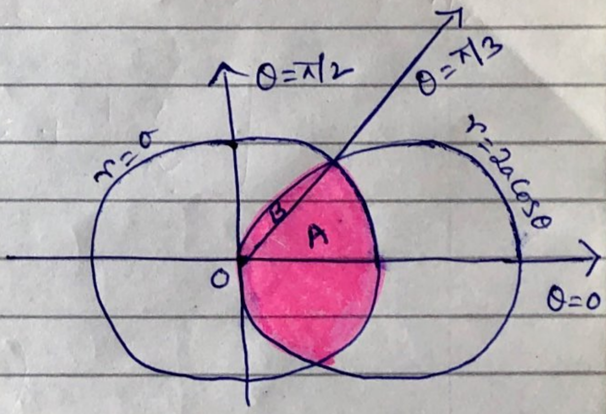
$$= a^2$$ (3)

Q.3. Given $r=a$, $r=2a\cos\theta$.

$\Rightarrow a = 2a\cos\theta$

$\Rightarrow \theta = \pi/3$

(2)



& when $r=0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$.

$\therefore$ The required area $= 2\left[\iint_A r\,dr\,d\theta + \iint_B r\,dr\,d\theta\right]$

(1) $$= 2\left[\int_0^{\pi/3}\int_0^{a} r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{2a\cos\theta} r\,dr\,d\theta\right]$$

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$$= 2\left[\int_0^{\pi/3} \frac{a^2}{2}\,d\theta + \int_{\pi/3}^{\pi/2} \frac{4a^2}{2}\cos^2\theta\,d\theta\right]$$

$$= a^2\left\{\pi/3 + 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}\right\}$$

2

$$= a^2\left\{\pi/3 + 2\left[(\pi/2 - \pi/3) + \left(0 - \sin\frac{2\pi}{3}\right)\right]\right\}.$$

$$= a^2\left\{\pi/3 + \frac{\pi}{3} - \frac{\sqrt{3}}{2}\right\}$$

$$= a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

Q.4. (Dun)

The problematic step, if we can call it "problematic", is the penultimate step.

To avoid this pathological situation, one must combine logarithms in the antiderivative **before** calculating the limit $b \to \infty$.

Since we didn't do that, we encountered the indeterminate form $\infty - \infty$.

Now, what's the way to evaluate this indeterminate form? Well, by combining logarithms indeed. Isn't it? That would yield the same answer in the end!