Rubrics for Mid Sem Exam

1

1. To give PMF of X, first note that the value set (Range) of X is the set of all positive integers

i.e. $S_X = \{1, 2, 3, \ldots\}$ → +1

Note: They can either write in words or in notation

Now,

$$P[X=k] = P[(\underbrace{TT\ldots T}_{(k \text{ times})}H) \cup (\underbrace{HH\ldots H}_{(k \text{ times})}T)] \rightarrow +2$$

$$= P[(TT\ldots TH)] + P[(HH\ldots HT)] \rightarrow +2$$

+2

$$= q^k p + p^k q \quad \text{where } q = 1-p$$

OR $= (1-p)^k p + p^k (1-p)$

They can write in either way

So, $P[X=k] = q^k p + p^k q$ for $k = 1, 2, \ldots$

$= 0$ otherwise

Total = 7 points

Now,



$$E[X] = \sum_{k=1}^{\infty} k\,P[X=k] \rightarrow +2$$

$$= \sum_{k=1}^{\infty} k\left[p^k q + q^k p\right] \rightarrow +2$$

$$= pq\left(\sum_{k=1}^{\infty} k p^{k-1} + \sum_{k=1}^{\infty} k q^{k-1}\right)$$

$$= pq\left(\sum_{k=1}^{\infty} \frac{d(p^k)}{dp} + \sum_{k=1}^{\infty} \frac{d(q^k)}{dq}\right)$$

$$= pq\left(\frac{d\left(\sum_{k=1}^{\infty} p^k\right)}{dp} + \frac{d\left(\sum_{k=1}^{\infty} q^k\right)}{dq}\right)$$

$$= pq\left[\frac{d\left(\frac{p}{1-p}\right)}{dp} + \frac{d\left(\frac{q}{1-q}\right)}{dq}\right]$$

$$= pq\left\{\frac{1(1-p) - p(-1)}{(1-p)^2} + \frac{1(1-q) - q(-1)}{(1-q)^2}\right\}$$

$$= pq\left[\frac{1}{q^2} + \frac{1}{p^2}\right] \quad +3$$

$$= \boxed{\frac{p}{q} + \frac{q}{p}} \longrightarrow +1$$

$\left(\underline{\text{or}} \quad \frac{p}{(1-p)} + \frac{(1-p)}{p} \quad \underline{\text{or}} \quad \frac{p^2+q^2}{pq}\right.$ : All are O.K.$\left.\right)$

Note: They have to find the sum and there are many ways. I have shown one way but they can use a different way. Some students can use formula and write the two sum directly and that is O.K.

Total = 8 points

Question 1:
Total = 15 points

2 (a) The required probability $= \dfrac{\binom{n-1}{m-1}}{\binom{n}{m}}$

$$= \frac{\dfrac{(n-1)!}{(m-1)!\,\cancel{(n-m)!}}}{\dfrac{n!}{m!\,\cancel{(n-m)!}}} = \boxed{\frac{m}{n}}$$

+3

(b) If N be the number of days a person must wait to get his first food stamp, then N is a geometric $\left(\frac{m}{n}\right)$ random Variable.

+6

$$\boxed{\begin{aligned} \text{Hence, } P[N=j] &= \left(1-\frac{m}{n}\right)^{j-1}\left(\frac{m}{n}\right) \quad \text{for } j=1,2,\ldots \\ &= 0 \qquad \text{otherwise} \end{aligned}}$$

Note:
Some students may exclude from waiting the day a person gets the first food stamp.

Hence they may write

$$P[N=j] = \left(1-\frac{m}{n}\right)^{j}\left(\frac{m}{n}\right) \quad \text{for } j=0,1,2,\ldots$$

$$= 0 \qquad \text{otherwise.}$$

(c) Let $N_k$ be the number of days a person must wait to get his kth food stamp then

$$S_{N_k} = \{k, k+1, k+2, \dots\}$$

and
$$P[N_k = j] = \binom{j-1}{k-1}\left(\frac{m}{n}\right)^{k-1}\left(1-\frac{m}{n}\right)^{j-k}\left(\frac{m}{n}\right)$$
$$= \binom{j-1}{k-1}\left(\frac{m}{n}\right)^{k}\left(1-\frac{m}{n}\right)^{j-k}$$

(Both are O.K.)

+6

for $j = k, k+1, \dots$

$= 0$ otherwise

Note: Some students may exclude from waiting the day a person gets his k-th food stamp.

Hence they may write
$$P[N_k = j] = \binom{j}{k-1}\left(\frac{m}{n}\right)^{k-1}\left(1-\frac{m}{n}\right)^{j-k+1}\left(\frac{m}{n}\right)$$
$$= \binom{j}{k-1}\left(\frac{m}{n}\right)^{k}\left(1-\frac{m}{n}\right)^{j-k+1}$$

for $j = k-1, k, k+1, \dots$

$= 0$ otherwise

(d) Let $N_1$ be the number of food stamps in 45 days.

Then $N_1$ is Binomial $\left(45, \frac{m}{n}\right)$ random Variable.

Hence

$$P[N_1 = j] = \binom{45}{j}\left(\frac{m}{n}\right)^{j}\left(1-\frac{m}{n}\right)^{45-j} \quad \text{for } j = 0, 1, 2, \dots, 45$$

$$= 0 \quad \text{otherwise}$$

+6

(e) $P[Z_i = 1] = P[X_i = 1, Y_i = 1]$

$$= P[Y_i = 1 \mid X_i = 1]\, P[X_i = 1]$$

$$= \frac{\binom{n-2}{m-2}}{\binom{n-1}{m-1}} \times \frac{m}{n}$$

$$= \frac{\cancel{(n-2)!}}{\cancel{(m-2)!}\,\cancel{(n-m)!}} \times \frac{\overset{(m-1)}{\cancel{(m-1)!}}\,\cancel{(n-m)!}}{\underset{(n-1)}{\cancel{(n-1)!}}} \times \frac{m}{n}$$

$$= \boxed{\frac{(m-1)\,m}{(n-1)\,n}}$$

+4

(f) $P[X = x \mid Z = 5]$

$$= \frac{P[X = x, Z = 5]}{P[Z = 5]} \quad \begin{matrix} \to +4 \\ \to +4 \end{matrix}$$

(for each calculation)

Continued to [redacted] next page

Now $P[X = x , Z = 5]$

$= P[$ The first person got $x$ food stamps in 45 days and out of those $x$ days, there are 5 days where the second person also got food stamps, in the $(x-5)$ days only the first person got the food stamp but not the second person$]$

$$= \binom{45}{x}\binom{x}{5} P[X_i = 1 , Y_i = 1]^5 \; P[X_i = 1 , Y_i = 0]^{x-5} \left(1 - P[X_i = 1]\right)^{45-x}$$

Now $P[X_i = 1, Y_i = 0] = P[Y_i = 0 \mid X_i = 1] \, P[X_i = 1]$

$$= \frac{\binom{n-2}{m-1}}{\binom{n-1}{m-1}} \times \frac{m}{n}$$

$$= \frac{\cancel{(n-2)!}}{\cancel{(m-1)!}\,\cancel{(n-m-1)!}} \times \frac{\cancel{(m-1)!}\,\overset{(n-m)}{\cancel{(n-m)!}}}{\underset{(n-1)}{\cancel{(n-1)!}}} \times \frac{m}{n}$$

$$= \frac{(n-m)\,m}{(n-1)\,n}$$

So, [redacted] $P[X = x , Z = 5]$

$$= \binom{45}{x}\binom{x}{5} \left\{\frac{(m-1)m}{(n-1)n}\right\}^5 \left\{\frac{(n-m)\,m}{(n-1)\,n}\right\}^{x-5} \left(1 - \frac{m}{n}\right)^{45-x}$$

Now $P[Z=5] = \binom{45}{5}\left(P[X_i=1, Y_i=1]\right)^5 \times \left(1 - P[X_i=1, Y_i=1]\right)^{45-5}$

$$= \binom{45}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^5 \left\{1 - \frac{(m-1)m}{(n-1)n}\right\}^{40}$$

Thus,

$P[X=x \mid Z=5]$

$$= \frac{\binom{45}{x}\binom{x}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^5 \left\{\frac{(n-m)m}{(n-1)n}\right\}^{x-5}\left(1-\frac{m}{n}\right)^{45-x}}{\binom{45}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^5\left\{1-\frac{(m-1)m}{(n-1)n}\right\}^{40}}$$

$$= \frac{\binom{45}{x}\binom{x}{5}\left\{\frac{(n-m)m}{(n-1)n}\right\}^{x-5}\left(1-\frac{m}{n}\right)^{45-x}}{\binom{45}{5}\left\{1-\frac{(m-1)m}{(n-1)n}\right\}^{40}}$$

for $x = 5, 6, 7, \ldots, 45$

$= 0$ otherwise.

(Note: They don't have to simplify the answer)

(g) $P[X = x \mid Z = 5, Y = 10]$

$$= \frac{P[X = x, Z = 5, Y = 10]}{P[Z = 5, Y = 10]} \begin{matrix} \rightarrow +6 \\ \rightarrow +4 \end{matrix}$$

(For each Calculation)

Now

$P[X = x, Z = 5, Y = 10]$

$= P[$ The first person gets food stamp on $x$ days, out of these $x$ days, there are 5 days where both person get food stamp and in the $(x-5)$ days, only the first person gets the food stamp but not the second person, and out of $(45-x)$ days, there are 5 days where the second person gets the food stamp but not the first person and in the $45 - x - 5 = 40 - x$ days none of them gets any food stamp $]$

$$= \binom{45}{x}\binom{x}{5}\binom{45-x}{5}\left(P[X_i=1, Y_i=1]\right)^{5} \times \left(P[X_i=1, Y_i=0]\right)^{x-5} \times \left(P[X_i=0, Y_i=1]\right)^{5}\left(P[X_i=0, Y_i=0]\right)^{45-x-5}$$

Now

$$P[X_i=0\,,Y_i=1] = P[Y_i=1\mid X_i=0]\;P[X_i=0]$$

$$= \frac{\binom{n-2}{m-1}}{\binom{n-1}{m}} \times \frac{\binom{n-1}{m}}{\binom{n}{m}}$$

$$= \frac{\cancel{(n-2)!}}{\cancel{(m-1)!}\,\cancel{(n-m-1)!}} \times \frac{\overset{m}{\cancel{m!}}\,\cancel{(n-m-1)!}}{\underset{(n-1)}{\cancel{(n-1)!}}} \times \frac{\cancel{(n-1)!}}{\cancel{(n-m-1)!}\;m!} \times \frac{\cancel{m!}\,\overset{(n-m)}{\cancel{(n-m)!}}}{\underset{n}{\cancel{n!}}}$$

$$= \frac{(n-m)\,m}{(n-1)\,n}$$

Also

$$P[X_i=0\,,Y_i=0] = \frac{\binom{n-2}{m}}{\binom{n}{m}}$$

$$= \frac{\cancel{(n-2)!}}{\cancel{m!}\,\cancel{(n-m-2)!}} \times \frac{\cancel{m!}\,\overset{(n-m)(n-m-1)}{\cancel{(n-m)!}}}{\underset{n(n-1)}{\cancel{n!}}}$$

$$= \frac{(n-m)(n-m-1)}{n(n-1)}$$

Therefore

$P[X=x, Z=5, Y=10]$

$$= \binom{45}{x}\binom{x}{5}\binom{45-x}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^{5}\left\{\frac{(n-m)m}{(n-1)n}\right\}^{x-5} \times$$

$$\times \left\{\frac{(n-m)m}{(n-1)n}\right\}^{5}\left\{\frac{(n-m)(n-m-1)}{n(n-1)}\right\}^{40-x}$$

$$= \binom{45}{x}\binom{x}{5}\binom{45-x}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^{5}\left\{\frac{(n-m)m}{(n-1)n}\right\}^{x} \times$$

$$\times \left\{\frac{(n-m)(n-m-1)}{n(n-1)}\right\}^{40-x}$$

Now $P[Z=5, Y=10] = P[$ The second person gets 10 food stamps and out of those 10 days, there are 5 days where the first person also gets food stamp $]$

$$= \binom{45}{10}\binom{10}{5} P[X_i=1, Y_i=1]^{5}\, P[Y_i=1, X_i=0]^{5} \times$$

$$\times P[Y_i=0]^{45-10}$$

$$= \binom{45}{10}\binom{10}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^{5}\left\{\frac{(n-m)m}{(n-1)n}\right\}^{5}\left(1-\frac{m}{n}\right)^{35}$$

Therefore

$$P[X=x \mid Z=5, Y=10]$$

$$= \frac{\binom{45}{x}\binom{x}{5}\binom{45-x}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^5\left\{\frac{(n-m)m}{(n-1)n}\right\}^x \times \left\{\frac{(n-m)(n-m-1)}{n(n-1)}\right\}^{40-x}}{\binom{45}{10}\binom{10}{5}\left\{\frac{(m-1)m}{(n-1)n}\right\}^5\left\{\frac{(n-m)m}{(n-1)n}\right\}^5\left(1-\frac{m}{n}\right)^{35}}$$

$$= \frac{\binom{45}{x}\binom{x}{5}\binom{45-x}{5}\left\{\frac{(n-m)m}{(n-1)n}\right\}^{x-5}\left\{\frac{(n-m)(n-m-1)}{n(n-1)}\right\}^{40-x}}{\binom{45}{10}\binom{10}{5}\left(1-\frac{m}{n}\right)^{35}}$$

for $x = 5, 6, 7, \ldots, 40$

$= 0$ otherwise.

Note: They don't have to simplify the answer.

(f) $P[X_{15}=1 \mid Z=5]$

$$= \frac{P[X_{15}=1, Z=5] \rightarrow +4}{P[Z=5] \rightarrow +4}$$

(for each calculation)

Now $P[X_{15}=1, Z=5]$

$= P[X_{15}=1, Z=5,$ The second person got a food stamp on the 15th day$]$

$+ P[X_{15}=1, Z=5,$ The second person ~~got~~ didn't get a food stamp on the 15th day$]$

$$= \binom{44}{4}\left[\frac{(m-1)m}{(n-1)n}\right]^4 \underbrace{\left[\frac{(m-1)m}{(n-1)n}\right]}_{15\text{th day}} \cdot \underbrace{\left(1-P[X_i=1, Y_i=1]\right)^{40}}_{(\text{Remaining } 40 \text{ days})}$$

$$+ \binom{44}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5 \underbrace{P[X_{15}=1, Y_{15}=0]}_{15\text{th day}} \underbrace{\left(1-P[X_i=1, Y_i=1]\right)^{39}}_{(\text{Remaining } 39 \text{ days})}$$

$$= \binom{44}{4}\left[\frac{(m-1)m}{(n-1)n}\right]^5 \left[1-\frac{(m-1)m}{(n-1)n}\right]^{40}$$

$$+ \binom{44}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5 \left[\frac{(n-m)m}{(n-1)n}\right]\left[1-\frac{(m-1)m}{(n-1)n}\right]^{39}$$

Now we have found

$$P[Z=5] = \binom{45}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left[1-\frac{(m-1)m}{(n-1)n}\right]^{40}$$

Therefore,

$P[X_{15}=1 | Z=5]$

They can simply write this expression and not simplify.

$$= \frac{\binom{44}{4}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left[1-\frac{(m-1)m}{(n-1)n}\right]^{40} + \binom{44}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left[\frac{(n-m)m}{(n-1)n}\right]\left[1-\frac{(m-1)m}{(n-1)n}\right]^{39}}{\binom{45}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left[1-\frac{(m-1)m}{(n-1)n}\right]^{40}}$$

$$= \frac{\binom{44}{4}\left[1-\frac{(m-1)m}{(n-1)n}\right] + \binom{44}{5}\frac{(n-m)m}{(n-1)n}}{\binom{45}{5}\left[1-\frac{(m-1)m}{(n-1)n}\right]}$$

and $P[X_{15}=0 | Z=5] = \frac{P[X_{15}=0, Z=5]}{P[Z=5]}$

$$= \frac{\binom{44}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left(1-\frac{m}{n}\right)\left[1-\frac{(m-1)m}{(n-1)n}\right]^{39}}{\binom{45}{5}\left[\frac{(m-1)m}{(n-1)n}\right]^5\left[1-\frac{(m-1)m}{(n-1)n}\right]^{40}}$$

$$= \frac{\binom{44}{5}\left(1 - \frac{m}{n}\right)}{\binom{45}{5}\left[1 - \frac{(m-1)m}{(n-1)n}\right]}$$

(Note: Instead of calculating $P[X_{15}=0 \mid Z=5]$, they can simply write, $P[X_{15}=0 \mid Z=5] = 1 - P[X_{15}=1 \mid Z=5]$
$= \text{-----}$

and that will be O.K.)

Note: In question 2, for each probability calculation, they should get some partial credit.

(i) $E[X_{15} | Z_{15} = 1]$

$= 1 \times P[X_{15} = 1 | Z_{15} = 1]$

$= 1 \times 1 = \boxed{1}$

+3

(They can directly write the answer)

(j) $E[X_{15} | Z_{15} = 0]$

$= P[X_{15} = 1 | Z_{15} = 0]$

$$= \frac{P[X_{15} = 1, Z_{15} = 0]}{P[Z_{15} = 0]}$$

$$= \frac{P[X_{15} = 1, Y_{15} = 0]}{P[X_{15} = 1, Y_{15} = 0] + P[X_{15} = 0, Y_{15} = 1] + P[X_{15} = 0, Y_{15} = 0]}$$

+5

$$= \frac{P[X_{15} = 1, Y_{15} = 0]}{1 - P[X_{15} = 1, Y_{15} = 1]}$$

$$= \frac{\frac{(n-m)m}{(n-1)n}}{1 - \frac{(m-1)m}{(n-1)n}} = \boxed{\frac{(n-m)m}{(n-1)n - (m-1)m}}$$

+1 for simplification

(6 points)

(Total 6 points in (j))

3 (a) P( a rally has less than 3 hits)

= P( a rally has either 0 hit or 1 hit or 2 hits)

= P( 0 hit) + P( 1 hit) + P( 2 hits)

= P[ Payer 1 misses] + P[ Payer 1 hits, Player 2 misses]

+ P[ Payer 1 hits, Player 2 hits, Payer 1 misses]

(+6 points)

$= .3 + (.7)(.5) + (.7)(.5)(.3)$

$= .3 + .35 + .105$

$= \boxed{.755}$

(Note: If they don't simplify to decimal or fraction, take off 2 points)

(b) Let $H_i$ be the event : Hit by Player $i$
$M_i$ be the event : Miss by Player $i$ for $i=1,2$

Then,

P [a rally ends with a miss by Player 1]

$= P[M_1 \cup H_1 H_2 M_1 \cup H_1 H_2 H_1 H_2 M_1 \cup \ldots\ldots]$

$= P[M_1] + P[H_1 H_2 M_1] + P[H_1 H_2 H_1 H_2 M_1] + \ldots\ldots$

$$= (.3) + (.7)(.5)(.3) + (.7)(.5)(.7)(.5)(.3) + \cdots$$

$$= (.3)\left[1 + (.35) + (.35)^2 + \cdots\right]$$

$$= (.3)\frac{1}{1-(.35)} = \frac{.3}{.65} = \frac{\cancel{30}^{6}}{\cancel{65}_{13}} = \boxed{\frac{6}{13}}$$

$$= \boxed{.4615}$$

+10

(They can leave the answer either in fraction or decimal : Both are O.K.)

(Note: If they don't simplify to decimal or fraction, take off 2 points)

(c) Let $L$ be the length of a rally (which includes all hits and the final miss)

Then $L$ is a discrete RV with value set $S_L = \{1, 2, 3, \ldots\}$

Now

$P[L=2K] = P[$ K hits by Player 1, K-1 hits by Player 2, Final miss by Player 2 $]$

$= (.7)^K (.5)^{K-1}(.5) = (.7)^K(.5)^K$

for $K = 1, 2, \ldots$

$P[L=2K-1] = P[$ (K-1) hits by Player 1, (K-1) hits by Player 2, Final miss by Player 1 $]$

$= (.7)^{K-1}(.5)^{K-1}(.3)$ for $K=1,2,\ldots$

Note: Some students may not write the above in general terms.
They may try to guess the pattern by looking at some initial terms like:

$P[L=1] = .3$, $P[L=2] = (.7)(.5)$, $P[L=3] = (.7)(.5)(.3)$

$P[L=4] = (.7)(.5)(.7)(.5)$

and write the expression of expectation as a series.
As long as the series is correct, it is O.K.

Hence

$$\boxed{E[L] = \sum_{K=1}^{\infty}(2K)(.7)^K(.5)^{K-1}(.5) + \sum_{K=1}^{\infty}(2K-1)(.7)^{K-1}(.5)^{K-1}(.3)}$$

+9

OR

$$= \boxed{\sum_{k=1}^{\infty} (2k)(.7)^k(.5)^k + \sum_{k=1}^{\infty} (2k-1)(.7)^{k-1}(.5)^{k-1}(.3)}$$

- Both the expressions are O.K.
- If they write first few terms and not the general terms of the two series, please take off 2 points

(Question: 3 : Total = 25 points)

- Now the above terms can be simplified. (not part of the exam)

$$E[L] = 2\times(.35)\sum_{k=1}^{\infty} k(.35)^{k-1} + 2(.3)\sum_{k=1}^{\infty} k(.35)^{k-1}$$
$$-(.3)\sum_{k=1}^{\infty}(.35)^{k-1}$$

$$= 2\times(.35)\frac{1}{(1-.35)^2} + 2\times(.3)\frac{1}{(1-.35)^2} - (.3)\frac{1}{(1-.35)}$$

$$= \frac{2[.35+.3]}{(.65)^2} - \frac{.3}{(.65)} = \frac{2\times\cancel{(.65)}}{\cancel{(.65)^2}(.65)} - \frac{.3}{(.65)}$$

$$= \frac{2-.3}{(.65)} = \frac{1.7}{.65} = \frac{\cancel{170}\ 34}{\cancel{65}\ 13} = \boxed{\frac{34}{13}}$$

(I did not ask for this simplification in the Quiz but the students should know these steps.)