ECE 351 DSP: Assignment 2

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Total: 30 points

Submission deadline: During class on 23.10.2024

A word on the notation: I shall represent finite duration causal signals as arrays. For example, $x[n] = [1, 2, 3]$ means $x[0] = 1$, $x[1] = 2$, and $x[2] = 3$, and $x[n] = 0$ for all other n.

Coding has to be done in Python. MATLAB codes will be marked zero.

1) Consider the LTI system given in Figure 1.
   a) Find $h[n]$.
   b) Assume the input signal given by

$$x[n] = \begin{cases} \frac{3}{2^n}, & \text{if } 0 \leq n \leq 51 \\ 0. & \text{otherwise} \end{cases}$$

   Write a python code to obtain the output using 16-point DFTs and overlap-and-save method. You can use commands like scipy.fft.fft to directly compute the individual 16-points DFTs.

[2+7=9 points]

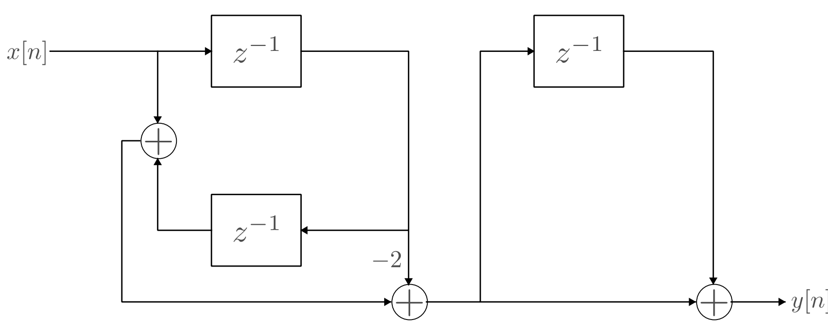

Fig. 1: Figure for Q.1

Solution: a) This is a cascade form representation. The system on the left has transfer function $1 - 2z^{-1} + z^{-2}$, and the next system has transfer function $1 + z^{-1}$. Hence, the transfer function of the complete system is

$$H(z) = (1 - 2z^{-1} + z^{-2})(1 + z^{-1}) = 1 - z^{-1} - z^{-2} + z^{-3}.$$

Hence, $h[n] = [1, -1, -1, 1]$. ∎

2) Consider the LTI system given in Figure 2.

   a) Find $h[n]$.

   b) Assume the input signal given by

$$x[n] = \begin{cases} \frac{(-1)^n}{3^n}, & \text{if } 0 \leq n \leq 39 \\ 0. & \text{otherwise} \end{cases}$$

   Write a python code to obtain the output using 8-point DFTs and overlap-and-add method. You can use commands like scipy.fft.fft to directly compute the individual 8-points DFTs.

[2+7=9 points]

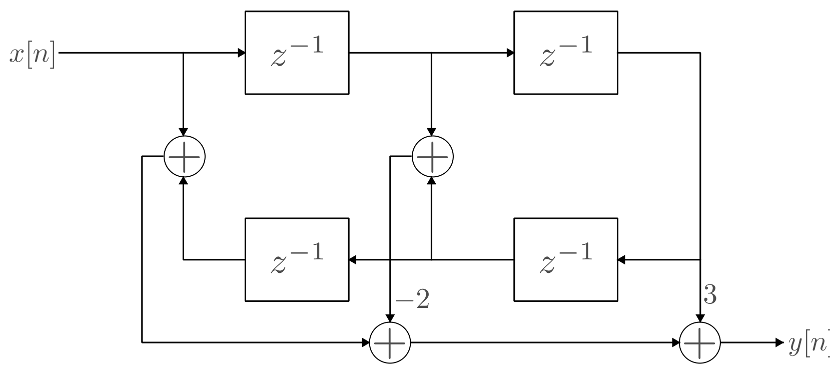

Fig. 2: Figure for Q.2

Solution: a) This is a direct form reporesentation, and by inspection, we have $h[n] = [1, -2, 3, -2, 1]$. ■

3) Consider the parallel form representation shown in Figure 3.

   a) Find the poles of this system.

   b) Is the system stable?

[2+1=3 points]

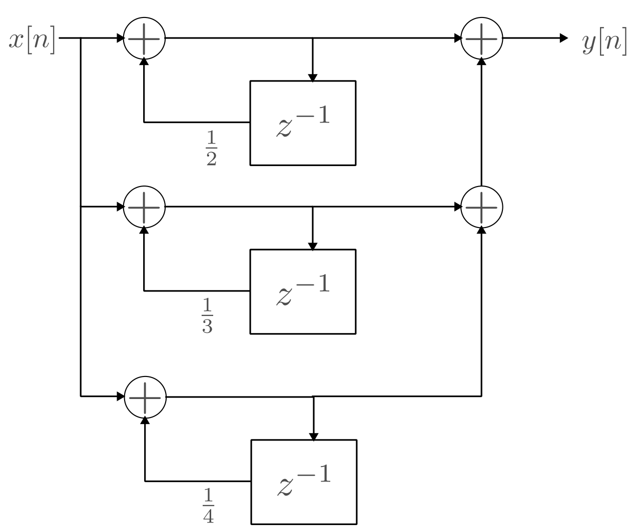

Fig. 3: Figure for Q.3

Solution: a) This is a parallel form representation, and by inspection we see that the poles are at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

b) All the poles are strictly inside the unit circle, and hence the system is stable. ■

4) Let $x[n]$ be an N length sequence, and define the $2N$-length sequence $y[n]$ by

$$y[n] = \begin{cases} x[\frac{n}{2}], & \text{if } n \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Express the $2N$-point DFT $Y(k), 0 \le k \le 2N-1$, of $y[n]$ in terms of the N-point DFT $X(k), 0 \le k \le N-1$, of $x[n]$.

[4 points]

Solution: Given any $k = 0, 1, \ldots, 2N-1$, we have

$$\begin{aligned} Y(k) &= \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}kn} \\ &\overset{(a)}{=} \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}km} \\ &= X(k), \end{aligned}$$

where (a) follows using the definition of $y[n]$. Thus, for $k = 0, 1, \ldots, N-1$, we have $Y(k) = X(k)$. For $k = N, N+1, \ldots, 2N-1$, we have by periodicity of DFT that $X(k) = X(k-N)$. Hence, we have

$$Y(k) = \begin{cases} X(k), & 0 \le k \le N-1 \\ X(k-N), & N \le k \le 2N-1. \end{cases}$$

■

5) Let N be a power of 3 (i.e., number like 9, 27, 81, etc.), and let $x[n]$ be a signal of length N. Suppose we want to compute the N-point DFT of $x[n]$ using a radix-3 decimation in frequency FFT algorithm. Write down the three $\frac{N}{3}$-length signals $f_1[n], f_2[n], f_3[n]$ you will use in the first phase of the algorithm, and how there respective $\frac{N}{3}$-point DFTs $F_1(k), F_2(k), F_3(k)$ will relate to the DFT of $x[n]$.

[5 points]

Solution: Define the notation $W_l = e^{-j\frac{2\pi}{l}}$. Now, since N is divisible by 3, we have for any $k = 0, 1, \ldots, N-1$

$$\begin{aligned} X(k) &= \sum_{n=0}^{N/3-1} x[n] W_N^{kn} + \sum_{n=0}^{N/3-1} x[n+N/3] W_N^{k(N/3+n)} + \sum_{n=0}^{N/3-1} x[n+2N/3] W_N^{k(2N/3+n)} \\ &= \sum_{n=0}^{N/3-1} \left(x[n] + x[n+N/3] e^{-j\frac{2\pi}{3}k} + x[n+2N/3] e^{-j\frac{4\pi}{3}k} \right) W_N^{kn} \end{aligned}$$

Hence, for $k = 0, 1, \ldots, N/3-1$, we have

$$X(3k) = \sum_{n=0}^{N/3-1} \Big(x[n] + x[n+N/3] + x[n+2N/3] \Big) W_{N/3}^{kn},$$

$$X(3k+1) = \sum_{n=0}^{N/3-1} \left(x[n] - \frac{1+j\sqrt{3}}{2} x[n+N/3] + \frac{-1+j\sqrt{3}}{2} x[n+2N/3] \right) W_{N/3}^{kn},$$

and

$$X(3k+2) = \sum_{n=0}^{N/3-1} \left(x[n] + \frac{-1+j\sqrt{3}}{2} x[n+N/3] - \frac{1+j\sqrt{3}}{2} x[n+2N/3] \right) W_{N/3}^{kn}.$$

Hence, we shall define for $n = 0, 1, \ldots, N/3-1$,

$$f_1[n] = x[n] + x[n+N/3] + x[n+2N/3],$$

$$f_2[n] = x[n] - \frac{1+j\sqrt{3}}{2} x[n+N/3] + \frac{-1+j\sqrt{3}}{2} x[n+2N/3],$$

and

$$f_3[n] = x[n] + \frac{-1+j\sqrt{3}}{2} x[n+N/3] - \frac{1+j\sqrt{3}}{2} x[n+2N/3],$$

and hence, $X(3k) = \sum_{n=0}^{N/3-1} f_1[n] W_{N/3}^{kn}$, $X(3k+1) = \sum_{n=0}^{N/3-1} f_2[n] W_{N/3}^{kn}$, and $X(3k+2) = \sum_{n=0}^{N/3-1} f_3[n] W_{N/3}^{kn}$, for $k = 0, 1, \ldots, N/3-1$. ■