Chapter 2

Logic of Quantified Statements

2.1 Predicates and Quantified Statements

2.2 Multiple Quantified Statements

2.3 Proving Validity of Arguments with Quantified Statements

Introduction

- Logical analysis of compound statements: simple statements joined by the connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, cannot be used to determine the validity in the majority.
- To determine the validity for the majority (such as "all" or "some"), it is necessary to separate the statements into parts.
- Predicate calculus: symbolic analysis of predicates and quantified statements.
- Statement calculus / propositional calculus: symbolic analysis of ordinary compound statements.

2.1 Predicates and Quantified Statements

- In logic, a predicate or open statement is a declarative sentence which
  i. contains one or more variables,
  ii. is not a statement, but
  iii. becomes a statement when the variables in it are replaced by certain allowable choices.
- The domain of a predicate variable: set of all values that may be substituted in place of the variables.

- The predicates can be obtained by removing nouns from the statement.
- E.g. Let P be “is a student at TAR UC” and Q be “is a student at”. Then P and Q are predicate symbols.
- The sentences “x is a student at TAR UC”, “x is a student at y” are symbolized as $P(x)$ and as $Q(x, y)$, respectively.
- x and y are predicate variables.

Example 1:

Consider

$p(x)$: The number $(x + 2)$ is an even integer.

$q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

Domain for x is $\{4, 5\}$ and

domain for y is $\{2, 3\}$

Therefore,

- $p(4)$:
- $p(5)$:

- $q(4, 2)$:
- $q(4, 3)$:
- $q(5, 2)$:
- $q(5, 3)$:

Notes:

- The sets in which predicate variables take their values may be in words or in symbols (set is denoted by upper-case letter and element of set is denoted by lower-case letter).
- E.g. $x \in A$ indicates that "x is an element of the set A" or "x is in A",

  $x \notin A$ indicates that "x is not in A".

- E.g. {1, 2, 3} refers to the set whose elements are 1, 2, and 3 while {1, 2, 3, …} indicates the set of all positive integers.
- 2 sets are equal if, and only if, they have exactly same elements.

- Some special symbolic names for certain sets of number:

Symbol	Set
R	set of all real numbers
R^+	set of positive real numbers
Z	set of all integers
$Z^{\mathbf{nonneg}}$ / N	set of non-negative integers: 0, 1, 2, … or natural numbers

Definition:

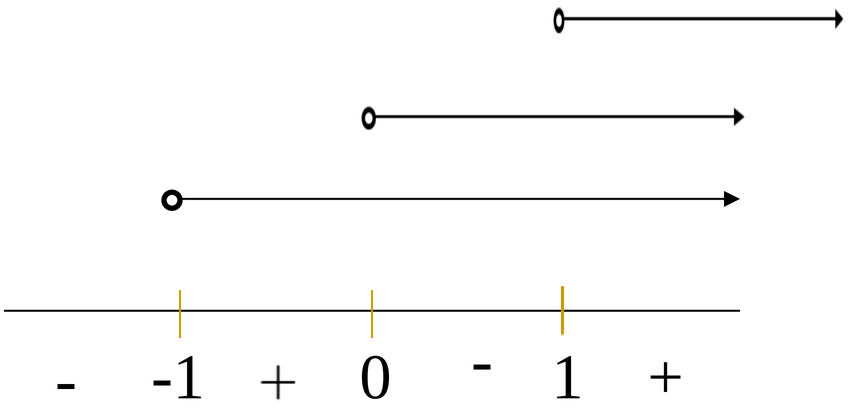
- If $P(x)$ is a predicate and x has domain D, the *truth set* of $P(x)$ is the set of all elements of D that make $P(x)$ true when substituted for x, denoted $\{ x \in D | P(x)\}$, read as "the set of all x in D such that $P(x)$".
- E.g. Let $P(x)$ be "x is a factor of 10" and suppose the domain of x is all positive integers. Then the truth set $P(x)$ is {1, 2, 5, 10}

Example 2:

Find the truth set for the predicate below,

$$x > \frac{1}{x}, \text{ domain: } R$$

$$x-\frac{1}{x}>0$$



Notation:

- Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.
- $P(x) \Rightarrow Q(x)$: every element in the truth set of $P(x)$ is in the truth set of $Q(x)$
- $P(x) \Leftrightarrow Q(x)$: $P(x)$ and $Q(x)$ have identical truth sets.

Example 3:

Let $P(x)$: x is the factor of 8.

$Q(x)$: x is the factor of 4.

$R(x)$: $x < 5$, $x \neq 3$.

Domain for x is Z^+.

Example 3:

$P(x)$: x is the factor of 8.

$Q(x)$: x is the factor of 4.

$R(x)$: $x < 5$, $x \neq 3$.

Quantifiers

- Added to predicates to obtained statements.
- Refers to quantities.
- The universal quantifier, $\forall$, denotes “for all”.
- The existential quantifier, $\exists$, denotes “there exists”.

Notes:

- $\forall$ *x* - for all *x*, for each *x*, for every *x*.
- $\exists$ *x* - for some *x*, for at least one *x*, there exists an *x* such that.

Universal

- Let $Q(x)$ be a predicate and D the domain of x.
- A Universal Statement , "$\forall x \in D, Q(x)$", is defined to be **true** if, and only if, $Q(x)$ is true for **every** x in D, and **false** if, and only if, $Q(x)$ is false for **at least one** x in D.
- A value for x for which $Q(x)$ is false is called a counterexample to the universal statement.

Existential

- Let $Q(x)$ be a predicate and D the domain of x.
- An Existential Statement, "$\exists x \in D, Q(x)$", is defined to be **true** if, and only if, $Q(x)$ is true for **at least one** x in D, and **false** if, and only if, $Q(x)$ is false for **every** x in D.

Example 4:

i) Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement "$\forall x \in D,\ x \geq \dfrac{1}{x}$". Show that this statement is true.

Let $P(x)$ be $x \geq \dfrac{1}{x}$.

Let $P(x)$ be $x \geq \frac{1}{x}$.

$P(1)$:

$P(2)$:

$P(3)$:

$P(4)$:

$P(5)$:

Since $P(x)$ is true for all $x \in D$, thus

$\forall x \in D, x \geq \frac{1}{x}$ is a true statement.

ii) Consider the statement "$\forall x \in R, x > \dfrac{1}{x}$"

Find counterexamples to show that this statement is false.

Let $P(x)$ be $x > \dfrac{1}{x}$.

Note:

- The technique used in (i) is called method of exhaustion – showing the truth of the predicate separately to each individual element of the domain.
- Only for finite predicate variable.

Example 5:

i) Consider the statement $\exists x \in Z \ni x^2 = x$ ($\ni$ means such that). Show that this statement is true.

Let $P(x)$ be $x^2 = x$.

ii) Let D = {5, 6, 7, 8, 9, 10} and consider the statement $\exists x \in D \ni x^2 = x$. Show that this statement is false.

Let $P(x)$ be $x^2 = x$.

$P(5)$:

$P(6)$:

$P(7)$:

$P(8)$:

$P(9)$:

$P(10)$:

Note:

- It is important to be able to translate either from formal into informal language.

Example 6:

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol $\forall$ and $\exists$.

i) $\forall x \in R, x^2 \neq -1$

ii) $\exists x \in Z \ni x^2 = x$

Example 7:

Rewrite each of the following statements formally. Use quantifiers and variables.

i) Every real number is positive, negative, or zero.

ii) Some real numbers are rational.

Universal Condition Statements

- $\forall x$, if P(x) then Q(x).

Example 8:

$\forall x \in R$, if $x > 4$, then $x^2 > 16$.

Informal way,

Example 9:

The square of any even integers is even.

Formal way,

Equivalent Forms of Universal Statements

- "$\forall x \in U$, if P(x) then Q(x)" can be written in the form "$\forall x \in D$, $Q(x)$" by narrowing U to be the domain D consisting of all values of the variable x that make $P(x)$ true.

Note:

- Conversely, $\forall x \in D$, $Q(x)$ can be rewritten as $\forall x$, if x is in D then $Q(x)$.

Example 10:

"$\forall$ polygons *p*, if *p* is a square, then *p* is a rectangle." is equivalent to

Equivalent Forms of Existential Statements

- "$\exists x \in U$ such that $P(x)$ and $Q(x)$" can be written as "$\exists x \in D$ such that $Q(x)$", provided D is taken to consist of all elements in U that make $P(x)$ true.

Example 11:

"$\exists$ a number n such that n is prime and n is even." is equivalent to

Negations of Quantified Statements

- The negation of the statement $\forall x$ in D, $Q(x)$ is logically equivalent to $\exists x$ in D such that $\sim Q(x)$.
- Symbolically,

  $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D$ such that $\sim Q(x)$.
- In word, the negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not").

- The negation of the statement $\exists x$ in D such that $Q(x)$ is logically equivalent to $\forall x$ in D, $\sim Q(x)$.
- Symbolically,

  $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$
- In word, the negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not").

Example 12:

Write negations for the following statements.

i) $\forall$ irrational numbers x, x is not an integer.

ii) $\exists\ x \in R$ such that x is rational.

iii) No politicians are honest.

iv) All dinosaurs are extinct.

v) Some exercises have answers.

In summary,

Statement	When is it true?	When is it false?
$\exists x\, p(x)$	For some (at least one) *a* in the universe, $p(a)$ is true.	For every *a* in the universe, $p(a)$ is false.
$\forall x\, p(x)$	For every replacement *a* from the universe, $p(a)$ is true.	There is at least one replacement *a* from the universe for which $p(a)$ is false.

Negations of Universal Conditional Statements

- By the definition of the negation of a "for all" statement,
- $\sim(\forall x, P(x) \rightarrow Q(x))$

  $\equiv \exists x$ such that $\sim(P(x) \rightarrow Q(x))$ (1)

  but, the negation of an if-then statement is logically equivalent to an "and" statement,

  $\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x)$ (2)

  substitute (2) into (1),

$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x$ such that $P(x) \wedge \sim Q(x)$.

- Symbolically,

  $\sim(\forall x$, if $P(x)$ then $Q(x))$

  $\equiv \exists x$ such that $P(x)$ and $\sim Q(x)$.

Example 13:

Write the negations for the following statements.

i) $\forall$ animals x, if x is a cat then x has whiskers and x has claws.

Example 13:

ii) $\forall x \in R$, if $x > 3$ then $x^2 > 9$.

Example 14:

Negate the following statements and determine their truth values.

i) $r(x)$: $2x + 1 = 5$ $s(x)$: $x^2 = 9$

$\exists\, x\, [\, r(x) \rightarrow s(x)\,].$

Example 14:

ii) $p(x)$: x is odd. $q(x)$: $x^2 - 1$ is even.

$\forall x \in Z$, if x is odd, then $(x^2 - 1)$ is even.

2.2 Multiple Quantified Statements

- Contain more than 1 quantifier.

Example 15:

Rewrite each of the following without using variables or the symbols $\forall$ or $\exists$.

i) $\forall$ colours C, $\exists$ an animal A such that A is coloured C.

ii) $\exists$ a book b such that $\forall$ people p, p has read b.

Example 16:

Rewrite the following formally using quantifiers and variables.

i) Everybody trusts somebody.

ii) Somebody trusts everybody.

Negations of Multiply Quantified Statements

- The negation of $\forall x, \exists y \ni P(x,y)$ is logically equivalent to $\exists x \ni \forall y, \sim P(x,y)$.
- The negation of $\exists x \ni \forall y, P(x, y)$ is logically equivalent to $\forall x, \exists y \ni \sim P(x,y)$.

Example 17:

Negate the statements below.

i) $\exists$ a book b such that $\forall$ people p, p has read b.

ii) $\forall$ even integers n, $\exists$ an integer k such that $n = 2k$.

iii) $\exists$ a person x such that $\forall$ people y, x loves y.

iv) $\forall x, \exists y\ [(P(x, y) \wedge Q(x, y)) \rightarrow R(x, y)]$

The Relation Among $\forall$, $\exists$, $\wedge$, and $\vee$

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then

- "$\forall x \in D, Q(x)$" is logically equivalent to "$Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$"
- E.g. Let $Q(x)$ be "$x \cdot x = x$" and $D = \{0, 1\}$.

  Then $\forall x \in D, Q(x)$ can be rewritten as $\forall$ binary digits x, $x \cdot x = x$.

  This is equivalent to $0 \cdot 0 = 0$ and $1 \cdot 1 = 1$, symbolically, $Q(0) \wedge Q(1)$.

- "$\exists x \in D$ such that $Q(x)$" is logically equivalent to "$Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$"
- E.g. Let $Q(x)$ be "$x + x = x$" and $D = \{0, 1\}$.

  Then $\exists x \in D$ such that $Q(x)$ can be rewritten as $\exists$ a binary digit such that $x + x = x$.

  This is equivalent to $0 + 0 = 0$ or $1 + 1 = 1$, symbolically, $Q(0) \vee Q(1)$.

Example 18:

For the universe of natural numbers *N*, the assertion $\forall x\, [P(x) \vee Q]$ is equivalent to the infinite conjunction:

which can be rearranged using the distributive laws to form:

which is equivalent to $\forall x\, P(x) \vee Q$.

Notes:

- The variable x in $P(x)$ for example 18 is bound by quantifiers while the variable in Q is free.
- The universal quantifier, $\forall$ distributes over the logical connective $\wedge$ but not the existential quantifier, $\exists$.
- The existential quantifier, $\exists$ distributes over the logical connective $\vee$ but the universal quantifier, $\forall$ does not.

Example 19:

For the universe of natural numbers *N*, the proposition $\forall x\ [P(x) \wedge Q(x)]$ can be expanded into an infinite conjunction:

which can be rearranged using associative and commutative laws to obtain:

which is equivalent to $\forall x\ P(x) \wedge \forall x\ Q(x)$.

Example 20:

Let the universe be the integers,

$P(x)$: x is an even integer.

$Q(x)$: x is an odd integer.

Then $\exists x\, P(x) \wedge \exists x\, Q(x)$ is true but

$\exists x\, [P(x) \wedge Q(x)]$ is false.

Therefore $\exists x\, P(x) \wedge \exists x\, Q(x)$ and

$\exists x\, [P(x) \wedge Q(x)]$ are not equivalent.

However, $\exists x\, [P(x) \wedge Q(x)]$ implies

$\exists x\, P(x) \wedge \exists x\, Q(x)$ is valid.

Variants of Universal and Conditional Statements

- Consider the statement of the form

  $\forall x \in D$, if $P(x)$ then $Q(x)$,

1. Its *contrapositive* is the statement

   $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.

2. Its *converse* is the statement

   $\forall x \in D$, if $Q(x)$ then $P(x)$.

3. Its *inverse* is the statement

   $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$,

Example 21:

Write the contrapositive, converse and inverse for the following statements.

i) $\forall x \in R$, if $x > 3$, then $x^2 > 9$.

ii) $\forall$ animals *A*, if *A* is a cat then *A* has whiskers and *A* has claws.

Notes:

- A universal condition statement is logically equivalent to its contrapositive.

  $\forall x \in D$, if $P(x)$ then $Q(x)$

  $\equiv \forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$

- A universal condition statement is *NOT* logically equivalent to its converse.

  $\forall x \in D$, if $P(x)$ then $Q(x)$

  $\neq \forall x \in D$, if $Q(x)$ then $P(x)$

- A universal condition statement is *NOT* logically equivalent to its inverse.

  $\forall x \in D$, if $P(x)$ then $Q(x)$

  $\neq \forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$

Using Diagrams to Test for Validity

- Helpful and convincing in many situations.
- To test the validity of an argument diagrammatically,

i. represent the truth of both premises with diagrams

ii. analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

Example 22:

Determine the validity of the following argument using diagrams.

All human beings are mortal.

Zeus is not mortal.

$\therefore$ Zeus is not a human being.

Example 22:

Let H: set of human beings

M: set of those who are mortal

Z: Zeus

The two diagrams fit together in only one way, as shown below.

Therefore, Zeus is not a human being as Z falls outside the disc of H. Hence, the argument is valid.

Example 23:

Determine the validity of the following argument using diagrams.

All human beings are mortal.

Felix is mortal.

$\therefore$ Felix is a human being.

Let *H*: set of human beings

M: set of those who are mortal

F: Felix

Major Premise

The disc of *H* falls entirely inside the disc of *M*.

Minor Premise

F falls inside the disc of *M.*

The possible conclusions are

(1) Therefore Felix is not a human being as *F* falls outside the disc of *H*.

(2) Therefore Felix is a human being as *F* falls inside the disc of *H*.

There is a contradiction between the conclusions, therefore the argument is invalid.

Example 24:

Determine the validity of the following argument using diagrams.

No polynomial functions have horizontal asymptote.

This function has a horizontal asymptote.

$\therefore$ This function is not a polynomial.

Let *P*: set of polynomial functions

H: set of functions with horizontal asymptote

T: this particular function

Major Premise

The discs of *P* and *H* are separated.

Minor Premise

T falls inside the disc of *H*.

The possible conclusion is

Therefore this function is not a polynomial function as *T* falls outside the disc of *P*.

Hence, the argument is valid.

Example 25:

Determine the validity of the following argument using diagrams.

All discrete mathematics students can tell a valid argument from an invalid one.

All thoughtful people can tell a valid argument from an invalid one.

$\therefore$ All discrete mathematics students are thoughtful.

Let D: set of discrete mathematics students

V: set of students who can tell a valid argument from an invalid one

T: set of thoughtful people

Major Premise

The disc of D falls entirely inside the disc of V.

Minor Premise

The disc of T falls entirely inside the disc of V.

The possible conclusions are

(1) All thoughtful people are discrete mathematics students as the disc of *T* falls entirely inside the disc of *D*.

(2) All discrete mathematics students are thoughtful as the disc of *D* falls entirely inside the disc of *T*.

(3) Some discrete mathematics students are thoughtful as there is an intersection between the two discs.

(4) All discrete mathematics students are not thoughtful as both of the discs are separated.

Since the possible conclusions are contradict to each other, thus the argument is invalid.