

Ginwidth=

Calc 1: Tutorial 8

August 19, 2019

1. Find the critical numbers of the function.

(a) $f(x) = x^3 + x^2 - x$

$$f(x) = x^3 + x^2 - x$$

- i. Critical numbers, when $\frac{dy}{dx} = 0$

$$f'(x) = 3x^2 + 2x - 1$$

- ii. When $f'(x) = 0$

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0$$

$$x = \frac{1}{3}, x = -1$$

- iii. Critical numbers: $\frac{1}{3}, -1$

(b) $g(\theta) = \theta + \sin \theta$

- i. Find $g'(\theta)$

$$g(\theta) = \theta + \sin \theta$$

$$g'(\theta) = 1 + \cos \theta$$

- ii. Find $g'(\theta) = 0$

$$1 + \cos \theta = 0$$

$$\theta = \cos^{-1}(-1)$$

$$= (\pi + 2\pi n), n \text{ is an integer}$$

- iii. Critical numbers: $(\pi + 2\pi n)$

2. Find the absolute maximum and absolute minimum values of f on the given interval

(a) $f(x) = x^3 - 3x + 1, [0, 3]$

i. Find the endpoints

A. $f(0) = 1$

B. $f(3) = 3^3 - 3(3) + 1 = 19$

ii. Find the formula for $\frac{dy}{dx}$

$$f'(x) = 3x^2 - 3$$

iii. Find all stationary points

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

iv. Find the values at the stationary points

$$f(1) = 1^3 - 3(1) + 1$$

$$= 2 - 3$$

$$= -1$$

$$f(-1) = (-1)^3 - 3(-1) + 1$$

$$= 3$$

v. Determine abs. max and abs.min points

A. Abs. max: 19

B. Abs. min = -1

(b) $f(x) = \frac{x}{x^2+4}, [0, 3]$

i. Find the endpoints

A. $f(0) = \frac{0}{0^2+4} = 0$

B. $f(3) = \frac{3}{3^2+4} = \frac{3}{13}$

ii. Find the formula for $\frac{dy}{dx}$

$$f'(x) = \frac{(x^2 + 4) \frac{d}{dx}[x] - x \frac{d}{dx}[x^2 + 4]}{(x^2 + 4)^2}$$

$$= \frac{(x^2 + 4) - x(2x)}{(x^2 + 4)^2}$$

$$= \frac{x^2 + 4 - 2x^2}{(x^2 + 4)^2}$$

$$f'(x) = \frac{-x^2 + 4}{(x^2 + 4)^2}$$

iii. Find all stationary points

$$\begin{aligned}\frac{-x^2 + 4}{(x^2 + 4)^2} &= 0 \\ -x^2 + 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2\end{aligned}$$

iv. Find the values at the stationary points

$$\begin{aligned}f(2) &= \frac{2}{2^2 + 4} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \\ f(-2) &= \frac{-(-2)^2 + 4}{((-2)^2 + 4)^2} \\ &= 0\end{aligned}$$

v. Determine abs. max and abs.min points

- A. Abs. max: $\frac{1}{4}$
- B. Abs. min = 0

(c) $f(x) = x - 2 \cos x, [-\pi, \pi]$

i. Find the endpoints

- A.
- B.

$$\begin{aligned}f(-\pi) &= -\pi - (-\pi) \cos(-\pi) \\ &= -\pi - (-\pi) \cos(-\pi) \\ &= -\pi - (-\pi)(-1) \\ &= -2\pi\end{aligned}$$

C.

$$\begin{aligned}f(\pi) &= \pi - (\pi) \cos(\pi) \\ &= \pi - (\pi) \cos(\pi) \\ &= \pi - (\pi)(-1) \\ &= 2\pi\end{aligned}$$

ii. Find the formula for $\frac{dy}{dx}$

$$\begin{aligned}f'(x) &= 1 - 2(-\sin x) \\ &= 1 + 2 \sin x\end{aligned}$$

iii. Find all stationary points

$$(\pi 2 + 2\pi n, 1), (3\pi 2 + 2\pi n, 1), (0 + 2\pi n, 0), (\pi + 2\pi n, 0)$$

, n is an integer

iv. Find the values at the stationary points

$$\begin{aligned} f(2) &= \frac{2}{2^2 + 4} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} f(-2) &= \frac{-(-2)^2 + 4}{\left((-2)^2 + 4\right)^2} \\ &= 0 \end{aligned}$$

v. Determine abs. max and abs.min points

A. Abs. max: 3

B. Abs. min = -3

3. For the following functions

(a)

$$g(x) = 200 + 8x^3 + x^4$$

i. Find local max and local mins

$$\begin{aligned} g'(x) &= 24x^2 + 4x^3 \\ &= 4x^2(6 + x) \end{aligned}$$

$$\begin{aligned} g'(x) &= 0 \\ 4x^2(6 + x) &= 0 \\ x = 0, x &= -6 \end{aligned}$$

A. Find the coordinates

$$\begin{aligned} g(0) &= 200 + 8(0)^3 + (0)^4 \\ &= 200 \end{aligned}$$

$$\begin{aligned} g(-6) &= 200 + 8(-6)^3 + (-6)^4 \\ &= -232 \end{aligned}$$

B. Local max. and local min.

$$(0, 200), (-6, -232)$$

ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

A. Find the second derivative

$$\begin{aligned} g''(x) &= \frac{d}{dx} [24x^2 + 4x^3] \\ &= 48x + 12x^2 \end{aligned}$$

B. Find the inflection points

$$\begin{aligned} 0 &= 12x(4 + x) \\ x &= 0, x = -4 \end{aligned}$$

C. Find the coordinates

$$(0, 200)$$

$$\begin{aligned} g(-4) &= 200 + 8(-4)^3 + (-4)^4 \\ &= -56 \end{aligned}$$

$$(-4, -56)$$

iii.

$f'(x)$	dec.	0		inc.	0	inc.
x	-7	-6	-4	-1	0	1
$f''(x)$	c.up		0	c.down	0	c.up

$$(b) f(x) = 2x^3 - 7x^2 + 4x + 5$$

i. Find the local maximum and minimum points of the curve.

A. When $f'(x) = 0$

$$\begin{aligned} f'(x) &= 6x^2 - 14x + 4 \\ &= 2(3x^2 - 7x + 2) \\ &= 2(3x - 1)(x - 2) \\ 0 &= (3x - 1)(x - 2) \end{aligned}$$

$$x = \frac{1}{3}, x = 2$$

B. Coordinates

$$\begin{aligned} f\left(\frac{1}{3}\right) &= 2\left(\frac{1}{3}\right)^3 - 7\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right) + 5 \\ &= \frac{152}{27} \end{aligned}$$

$$\left(\frac{1}{3}, \frac{152}{27}\right)$$

$$\begin{aligned} f(2) &= 2(2)^3 - 7(2)^2 + 4(2) + 5 \\ &= 1 \end{aligned}$$

$$(2, 1)$$

- ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

$$\begin{aligned} f''(x) &= \frac{d}{dx} [6x^2 - 14x + 4] \\ &= 12x - 14 \end{aligned}$$

$$\begin{aligned} f''(x) &= 0 \\ 2(6x - 7) &= 0 \\ x &= \frac{7}{6} \end{aligned}$$

A. Coordinates

$$\begin{aligned} f\left(\frac{7}{6}\right) &= 6\left(\frac{7}{6}\right)^2 - 14\left(\frac{7}{6}\right) + 4 \\ &= -\frac{25}{6} \\ &\left(\frac{7}{6}, -\frac{25}{6}\right) \end{aligned}$$

- iii. Sketch the curve (use the table)

	$f'(x)$	increasing	0	decreasing	0	increasing
A.	x	0	$\frac{1}{3}$	$1\frac{1}{6}$	2	3
	$f''(x)$	-ve		0	+ve	

(c) $f(x) = \ln(1 + x^2)$

- i. Find the local maximum and minimum points of the curve.

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} \cdot (2x) \\ &= \frac{2x}{1+x^2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 0 \\ \frac{2x}{1+x^2} &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

A. Coordinates

$$\begin{aligned} f(0) &= \ln 1 \\ &= 0 \end{aligned}$$

$$(0, 0)$$

- ii. Determine the interval(s) where the curve is concave upward and the interval(s) where it is concave downward. Find the coordinates of the inflection point.

$$\begin{aligned} f''(x) &= \frac{d}{dx} \left[\frac{2x}{1+x^2} \right] \\ &= \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} \\ &= \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} \\ &= \frac{2((1+x^2) - 2x^2)}{(1+x^2)^2} \\ &= \frac{2(1-x^2)}{(1+x^2)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= 0 \\ \frac{2(1-x^2)}{(1+x^2)^2} &= 0 \\ (1-x^2) &= 0 \\ -(x-1)(x+1) &= 0 \\ x &= \pm 1 \end{aligned}$$

A. Coordinates

$$\begin{aligned} f(1) &= 2(1)^3 - 7(1)^2 + 4(1) + 5 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(-1) &= 2(-1)^3 - 7(-1)^2 + 4(-1) + 5 \\ &= -8 \end{aligned}$$

$$(1, 4), (-1, -8)$$

- iii. Sketch the curve

	$f'(x)$	decr.	0	incr.	
A.	x	-2	-1	0	1
	$f''(x)$	c.down	0	c.up	0
					c.down

4. $f(x) = \frac{x}{(x-1)^3}$

(a) Find the vertical and horizontal asymptotes.

i. Vertical asymp.

$$(x-1)^3 = 0$$

$$x-1 = 0$$

$$x = 1$$

ii. Horizontal asymp.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x}{(x-1)^3} &= \frac{\infty}{(\infty-1)^3} \\ &= \frac{\infty}{\infty^3} \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x}{(x-1)^3} &= \frac{-\infty}{(-\infty-1)^3} \\ &= 0\end{aligned}$$

(b) Find the intervals of increase or decrease.

$$\begin{aligned}f'(x) &= \frac{d}{dx} \left[\frac{x}{(x-1)^3} \right] \\ &= \frac{(x-1)^3 - x \cdot 3(x-1)^2}{(x-1)^6} \\ &= \frac{(x-1)^2(x-1-3x)}{(x-1)^6} \\ &= -\frac{(1+2x)}{(x-1)^4} \\ -\frac{(1+2x)}{(x-1)^4} &= 0 \\ -1-2x &= 0 \\ x &= -\frac{1}{2}\end{aligned}$$

(c) Find the local maximum and minimum values

i. Local maximum: $(-0.5, -0.148)$

ii. Local minimum: None

(d) Find the intervals of concavity and the inflection points.

$$\begin{aligned}
 f''(x) &= \frac{d}{dx} \left[-\frac{(1+2x)}{(x-1)^4} \right] \\
 &= -\frac{(x-1)^4(2) - (1+2x)4(x-1)^3}{(x-1)^8} \\
 &= \frac{2(x-1)^4 - 4(1+2x)(x-1)^3}{(x-1)^8} \\
 &= \frac{2(x-1) - 4(1+2x)}{(x-1)^5} \\
 &= \frac{2x-2-4-8x}{(x-1)^5} \\
 &= \frac{-6-6x}{(x-1)^5} \\
 f''(x) &= \frac{-6(1+x)}{(x-1)^5}
 \end{aligned}$$

$$\begin{aligned}
 \frac{-6(1+x)}{(x-1)^5} &= 0 \\
 -6(1+x) &= 0 \\
 x &= -1
 \end{aligned}$$

$$\begin{aligned}
 f(-1) &= \frac{-1}{(-1-1)^3} \\
 &= \frac{-1}{(-2)^3} \\
 &= \frac{1}{8} \\
 &\left(-1, \frac{1}{8}\right)
 \end{aligned}$$

(e) Use the information from parts (a)-(d) to sketch the graph

	$f'(x)$	incr.	0	dec.
i.	x	-2	-1	0
	$f(x)$		$\frac{1}{8}$	
	$f''(x)$	c.down	0	c.up