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Calc 2: Tutorial 5

November 15, 2019

1. Solve the following differential equations: (CHECK WITH LECTURER)

    (a) $x^2 \frac{dy}{dx} + xy = x + 1$

    i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

    $$x^2 \frac{dy}{dx} + xy = x + 1$$

    $$\frac{1}{x^2}\left[x^2\frac{dy}{dx} + xy\right] = \frac{1}{x^2}[x+1]$$

    $$\boldsymbol{\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x} + \frac{1}{x^2}, p(x) = \frac{1}{x}}$$

    ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

    $$\mu(x) = e^{\int \frac{1}{x}dx}$$

    $$= e^{\ln x}$$

    $$\boldsymbol{\mu(x) = x}$$

    iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

    $$x\left[\frac{dy}{dx} + \frac{y}{x}\right] = x\left[\frac{1}{x} + \frac{1}{x^2}\right]$$

    $$x\frac{dy}{dx} + y = 1 + \frac{1}{x}$$

    $$\frac{d}{dx}[yx] = 1 + \frac{1}{x}$$

    $$\int \frac{d}{dx}[yx]\,dx = \int 1 + \frac{1}{x}dx$$

    $$yx = x + \ln x + c$$

    $$y = 1 + \frac{\ln x + c}{x}$$

    (b) $\frac{dy}{dx} + y\cot x = \csc x$

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i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
   A. Already in correct standard form, $p(x) = \cot x$
ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\mu(x) = e^{\int \cot x \, dx}$$
$$= e^{\ln|\sin x|}$$
$$= \sin x$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\sin x \left[\frac{dy}{dx} + y \cot x\right] = \sin x \, [\csc x]$$
$$\sin x \frac{dy}{dx} + y \sin x \cdot \frac{\cos x}{\sin x} = \sin x \left[\frac{1}{\sin x}\right]$$
$$\sin x \frac{dy}{dx} + y \cos x = 1$$
$$\sin x \frac{dy}{dx} + y \cos x = 1$$
$$\frac{d}{dx}[y \sin x] = 1$$
$$\int \frac{d}{dx}[y \sin x] \, dx = \int 1 dx$$
$$y \sin x + c = x + c$$
$$y = \frac{x + c}{\sin x}$$

(c) $x\frac{dy}{dx} = y - x^2 e^{-x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} = \frac{y}{x} - x^2 e^{-x}$$
$$\frac{dy}{dx} - \frac{y}{x} = -x^2 e^{-x}$$

   A. From the above, $p(x) = -\frac{1}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int -\frac{1}{x} dx}$$
$$= e^{-\int \frac{1}{x} dx}$$
$$= e^{-\ln x}$$
$$\mu(x) = e^{\ln x^{-1}}$$
$$= x^{-1}$$
$$= \frac{1}{x}$$

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iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right) = \frac{1}{x}\left(-x^2 e^{-x}\right)$$

$$-\left(-x^{-2}y + x^{-1}\frac{dy}{dx}\right) = -xe^{-x}$$

$$-\frac{dy}{dx}\left(x^{-1}y\right) = -xe^{-x}$$

$$-\int \frac{dy}{dx}\left(x^{-1}y\right)dx = \int -xe^{-x}dx$$

$$-x^{-2}y = -\int xe^{-x}dx$$

iv. Before we can proceed, again, integrate by parts, if you don't remember, GG.com...jk,

$$\int u\,dv = uv - \int v\,du$$

A. Let $u = x$, $dv = e^{-x}$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int dv = \int e^{-x}$$

$$v = -e^{-x}$$

B. Lets find the answer

$$\int u\,dv = -x \cdot e^{-x} - \int e^{-x}dx$$

C. Again, we need to deal with integration, now use u substitution

D. Let $u = -x$, $du = -dx$

$$-\int e^{-x}dx = -\int e^u du$$

$$= -e^u$$

$$-\int e^{-x}dx = -e^{-x} + c$$

v. Substitute back in

$$-x^{-1}y = \int xe^{-x}dx$$

$$= -e^{-x} + c$$

$$y = -e^{-x} \cdot -x - cx$$

$$= xe^{-x} - cx$$

$$y = xe^{-x} + cx \text{ Note: c is constant, sign doesn't matter for now}$$

(d) $x\frac{dy}{dx} + 2y = \frac{\sin x}{x}$

    i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

    ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int \frac{2}{x}dx}$$

$$= e^{2\ln x}$$

$$\mu(x) = x^2$$

    iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$x^2\left[\frac{dy}{dx} + \frac{2y}{x}\right] = x^2\left[\frac{\sin x}{x^2}\right]$$

$$x^2\frac{dy}{dx} + 2xy = \sin x$$

$$\frac{d}{dx}\left[x^2y\right] = \sin x$$

$$\int \frac{d}{dx}\left[x^2y\right]dx = \int \sin x \, dx + c$$

$$x^2y = -\cos x + c$$

$$y = \frac{c - \cos x}{x^2}$$

(e) $x\frac{dy}{dx} + 3y = 4x + 3$

    i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{1}{x}\left[x\frac{dy}{dx} + 3y\right] = \frac{1}{x}[4x + 3]$$

$$\frac{dy}{dx} + \frac{3y}{x} = 4 + \frac{3}{x}$$

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A. From above $p(x) = \frac{3}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int \frac{3}{x}dx}$$
$$= e^{3\int \frac{1}{x}dx}$$
$$= e^{3\ln x}$$
$$= e^{\ln x^3}$$
$$\mu(x) = x^3$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$x^3\left[\frac{dy}{dx} + \frac{3y}{x}\right] = x^3\left[4 + \frac{3}{x}\right]$$
$$x^3\frac{dy}{dx} + 3x^2y = 4x^3 + 3x^2$$
$$\frac{d}{dx}\left[x^3y\right] = 4x^3 + 3x^2$$
$$\int \frac{d}{dx}\left[x^3y\right]dx = \int\left(4x^3 + 3x^2\right)dx$$
$$x^3y = \frac{4x^4}{4} + \frac{3x^3}{3} + c$$
$$x^3y = x^4 + x^3 + c$$
$$y = \frac{x^4}{x^3} + \frac{x^3}{x^3} + \frac{c}{x^3}$$
$$y = x + 1 + \frac{c}{x^3}$$

(f) $\frac{dy}{dx} = 2y + e^{3x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} = 2y + e^{3x}$$
$$\frac{dy}{dx} - 2y = e^{3x}, p(x) = -2$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int -2dx}$$
$$= e^{\int -2dx}$$
$$= e^{-\int 2dx}$$
$$\mu(x) = e^{-2x}$$

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iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$e^{-2x}\left[\frac{dy}{dx} - 2y\right] = e^{-2x}\left[e^{3x}\right]$$

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = e^x$$

$$\frac{dy}{dx}\left[e^{-2x}y\right] = e^x$$

$$\int \frac{dy}{dx}\left[e^{-2x}y\right]dx = \int e^x dx$$

$$e^{-2x}y = e^x + c$$

$$y = \frac{e^x + c}{e^{-2x}}$$

$$= e^{3x} + ce^{-(-2x)}$$

$$y = e^{3x} + ce^{2x}$$

(g) $x\frac{dy}{dx} = 2y + x^3 \ln x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} = \frac{2y}{x} + x^2 \ln x$$

$$\frac{dy}{dx} - \frac{2y}{x} = x^2 \ln x$$

$$\frac{dy}{dx} - \frac{2y}{x} = x^2 \ln x, p(x) = -\frac{2}{x}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int p(x)dx}$$

$$\mu(x) = e^{\int -2x^{-1}dx}$$

$$= e^{-2\int \frac{1}{x}dx}$$

$$= e^{\ln x^{-2}dx}$$

$$= x^{-2}$$

$$\mu(x) = \frac{1}{x^2}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as

such.

$$\frac{1}{x^2}\left[\frac{dy}{dx} - \frac{2y}{x}\right] = \frac{1}{x^2}\left[x^2 \ln x\right]$$

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \ln x$$

$$\frac{dy}{dx}\left[x^{-2}y\right] = \ln x$$

$$\int \frac{dy}{dx}\left[x^{-2}y\right] dx = \int \ln x \, dx$$

A. Time to use Calc 1 skills, Integration by parts

$$\int u \, dv = uv - \int v \, du$$

B. Let $u = \ln x$ and $v' = 1$
C. $\frac{du}{dx} = \frac{1}{x} + c, v = x + c$
D. Finally we get

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x}$$

(h) $x\frac{dy}{dx} - 3y = x^4$

   i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

   ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

   iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

2. Solve the following I.V.P.:

   (a) $\frac{dy}{dx} = 98 - 0.196y, y(0) = 48$

   (b) $x\frac{dy}{dx} + y = \frac{x}{x+1}, y(1) = 1$

   (c) $x\frac{dy}{dx} + 3y = x^2 - 4x + 3, y(1) = 0$

3. **Separable Equations:**
Solve the following differential equations:

   (a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

   (b) $\frac{dy}{dx} = \frac{xy}{2\ln y}$

   (c) $\frac{dx}{dt} + e^{t+x} = 0$

(d) $2\sqrt{xy}\frac{dy}{dx} = 1, x, y > 0$

(e) $\frac{dy}{dx} = \frac{xy}{x+2}$

(f) $y\left(x^2 - 1\right)\frac{dy}{dx} = 1$

(g) $\frac{dx}{dt} = \frac{4\sin t + 6\cos 2t}{x}$

(h) $\frac{dy}{dx} = e^{-y}\left(2x - 4\right)$

(i) $\sec x\frac{dy}{dx} = e^{y+\sin x}$

(j) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$

4. Solve the following I.V.P.:

(a) $y' = \frac{y\cos x}{1+y^2}, y\left(0\right) = 1$

(b) $x + 2y\sqrt{x^2 + 1}\frac{dy}{dx} = 0, y\left(0\right) = 1$

(c) $\frac{dy}{dt} = te^y, y\left(1\right) = 0$

(d) $t\left(t - 1\right)\frac{dx}{dt} = x\left(x + 1\right), x\left(2\right) = 2$

(e) $\frac{dx}{dt} = e^{x+t}, x\left(0\right) = a$