Calc 2: Tutorial 6

November 20, 2019

1 2^{nd} order homogeneous linear D.E.

1. Find the general solution of
   (a) $y'' - 6y' + 5y = 0$
      i. Rewrite
      $$r^2 - 6r + 5 = 0$$
      ii. Factorize
      $$\begin{aligned}(r-5)(r-1) &= 0\\ r &= 5, r = 1\\ y &= C_1 e^{5x} + C_2 e^{x}\end{aligned}$$
   (b) $4y'' - 4y' + 1 = 0$
      i. Rewrite
      $$4r^2 - 4r + 1 = 0$$
      ii. Factorize
      $$\begin{aligned}(2r-1)(2r-1) &= 0\\ (2r-1)^2 &= 0\\ r &= \frac{1}{2}\end{aligned}$$
      iii. Obtain equation
      $$\begin{aligned}y &= C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}\\ y &= e^{\frac{1}{2}x}(C_1 + C_2 x)\end{aligned}$$
   (c) $4y'' - 2y' + y = 0$
      i. Note:
         A. Complex roots
         B. Use general solution:
      $$\begin{aligned}y &= C_1 e^{\alpha x}\sin\beta x + C_2 e^{\alpha x}\cos\beta x\\ y &= e^{ax}(C_1 \sin\beta x + C_2\cos\beta x)\end{aligned}$$
      ii. Rewrite
      $$4r^2 - 2r + 1 = 0$$

iii. Factorize

$$\begin{aligned}
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(1)}}{2(4)} \\
&= \frac{2 \pm \sqrt{4 - 16}}{8} \\
&= \frac{2 \pm \sqrt{-12}}{8} \\
&= \frac{2 \pm \sqrt{-1}\sqrt{12}}{8} \\
&= \frac{2 \pm \sqrt{-1} \cdot 2\sqrt{3}}{8} \\
&= \frac{2 \pm 2i\sqrt{3}}{8} \\
&= \frac{1 \pm i\sqrt{3}}{4} \\
r &= \frac{1}{4} \pm \frac{\sqrt{3}}{4} i
\end{aligned}$$

iv. Obtain equation

$$y = e^{\frac{1}{4}} \left(C_1 \sin \frac{\sqrt{3}}{4} x + C_2 \cos \frac{\sqrt{3}}{4} x \right)$$

(d) $y'' - 2y' + 1 = 0$

i. Rewrite

$$r^2 - 2r + 1 = 0$$

ii. Factorize

$$\begin{aligned}
r^2 - 2r + 1 &= 0 \\
(r-1)(r-1) &= 0 \\
(r-1)^2 &= 0 \\
r &= 1
\end{aligned}$$

iii. Obtain equation

$$y = e^{x} (C_1 + C_2 x)$$

(e) $2y'' + 7y' - 4 = 0$

i. Rewrite

$$2r^2 + 7r - 4 = 0$$

ii. Factorize

$$\begin{aligned}
(2r-1)(r+4) &= 0 \\
r &= \frac{1}{2}, -4
\end{aligned}$$

iii. Obtain equation

$$y = C_1 e^{\frac{1}{2}x} + C_2 e^{-4x}$$

(f) $2y'' - 2y' + 5 = 0$

i. Rewrite

$$2r^2 - 2r + 5 = 0$$

ii. Factorize

$$a = 2, b = -2, c = 5$$

$$
\begin{aligned}
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
r &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(5)}}{2(2)} \\
&= \frac{2 \pm \sqrt{4 - 40}}{4} \\
&= \frac{2 \pm \sqrt{-36}}{4} \\
&= \frac{2 \pm \sqrt{-1}\sqrt{36}}{4} \\
&= \frac{2 \pm 6i}{4} \\
r &= \frac{1}{2} \pm 3i
\end{aligned}
$$

iii. Obtain equation

$$y = e^{\frac{1}{2}} (C_1 \sin 3x + C_2 \cos 3x)$$

2. Solve the following I.V.P.:

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0, y = 0$ and $\frac{dy}{dx} = 6$ when $x = 0$

i. Rewrite

$$r^2 + 2r + 10 = 0$$

ii. Factorize

$$
\begin{aligned}
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
r &= -1 \pm 3i
\end{aligned}
$$

iii. Obtain equation

$$y = e^{-x} (C_1 \sin 3x + C_2 \cos 3x)$$

iv. Find C_1 and C_2

$$
\begin{aligned}
0 &= e^{-0} (C_1 \sin 0 + C_2 \cos 0) \\
&= C_2 \\
C_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
y' &= \frac{d}{dx}\left[e^{-x} (C_1 \sin 3x + C_2 \cos 3x)\right] \\
&= \frac{d}{dx}\left[e^{-x}\right] \cdot (C_1 \sin 3x + C_2 \cos 3x) + e^{-x} \cdot \frac{d}{dx}\left[(C_1 \sin 3x + C_2 \cos 3x)\right] \\
&= -e^{-x} \cdot (C_1 \sin 3x + C_2 \cos 3x) + e^{-x}\left(C_1 \frac{d}{dx}[\sin 3x] + C_2 \frac{d}{dx}[\cos 3x]\right) \\
&= -e^{-x} (C_1 \sin 3x + C_2 \cos 3x) + e^{-x} (3C_1 \cos 3x - 3C_2 \sin 3x)
\end{aligned}
$$

A. Substitute in the values given and C_2

$$
\begin{aligned}
6 &= -e^0 (C_1 \sin 0 + 0 \cos 0) + e^0 (3C_1 \cos 0 - 0 \sin 0) \\
&= 0 + 3C_1 (1) + 0 \\
6 &= 3C_1 \\
C_1 &= 2
\end{aligned}
$$

v. Obtain solution

$$
\begin{aligned}
y &= e^{-x} (2 \sin 3x + 0 \cos 3x) \\
y &= 2e^{-x} \sin 3x
\end{aligned}
$$

(b) $\frac{d^2y}{dx^2} - 9y = 0, y = 2$ and $\frac{dy}{dx} = -1$ when $x = 0$

i. Rewrite

$$r^2 - 9 = 0$$

ii. Factorize

$$\begin{aligned}(r-3)(r+3) &= 0 \\ r &= \pm 3\end{aligned}$$

iii. Obtain equation

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

iv. Find C_1 and C_2

$$y = 2, \frac{dy}{dx} = -1, x = 0$$

$$\begin{aligned}2 &= C_1 e^{3(0)} + C_2 e^{-3(0)} \\ C_1 + C_2 &= 2 \\ C_2 &= 2 - C_1\end{aligned}$$

$$\begin{aligned}y' &= \frac{d}{dx}\left[C_1 e^{3x} + C_2 e^{-3x}\right] \\ &= \frac{d}{dx}\left[C_1 e^{3x}\right] + \frac{d}{dx}\left[C_2 e^{-3x}\right] \\ &= C_1 \cdot 3e^{3x} + C_2(-3)e^{-3x} \\ &= C_1 \cdot 3e^{3x} - 3C_2 e^{-3x} \\ -1 &= C_1 \cdot 3e^0 - 3C_2 e^0 \\ -1 &= 3C_1 - 3C_2 \\ -1 &= 3C_1 - 3(2 - C_1) \\ -1 &= 3C_1 - 6 + 3C_1 \\ 5 &= 6C_1 \\ C_1 &= \frac{5}{6} \\ C_1 &= \frac{5}{6}\end{aligned}$$

A. Substitute in $C_2 = 2 - C_1$

$$\begin{aligned}C_2 &= 2 - \frac{5}{6} \\ &= \frac{7}{6}\end{aligned}$$

v. Find the solution for I.V.P.

$$\begin{aligned}y &= \frac{5}{6}e^{3x} + \frac{7}{6}e^{-3x} \\ y &= \frac{1}{6}\left(5e^{3x} + 7e^{-3x}\right)\end{aligned}$$

(c) $\frac{d^2y}{dx^2} + 16y = 0, y = -10$ and $\frac{dy}{dx} = 3$ when $x = \frac{\pi}{2}$

i. Rewrite

$$r^2 + 16 = 0$$

ii. Factorize

$$\begin{aligned}r &= \frac{-0 \pm \sqrt{-4(16)}}{2} \\ &= \pm\frac{\sqrt{-1}\sqrt{64}}{2} \\ &= \pm\frac{8i}{2} \\ &= \pm 4i\end{aligned}$$

iii. Obtain equation

$$\begin{aligned} y &= e^{0x}\left(C_1 \sin 4 + C_2 \cos 4\right) \\ &= C_1 \sin 4x + C_2 \cos 4x \end{aligned}$$

iv. Find C_1 and C_2

$$\begin{aligned} -10 &= C_1 \sin 2\pi + C_2 \cos 2\pi \\ -10 &= C_2 \\ C_2 &= -10 \end{aligned}$$

$$\begin{aligned} y' &= 4C_1 \cos 4x - 4C_2 \sin 4x \\ 3 &= 4C_1 \cos 4\left(\frac{\pi}{2}\right) - C_2 \sin 4\left(\frac{\pi}{2}\right) \\ 3 &= 4C_1 \cos\left(2\pi\right) - C_2 \sin 2\pi \\ 3 &= 4C_1 \cos\left(2\pi\right) \\ C_1\left(1\right) &= \frac{3}{4} \\ C_1 &= \frac{3}{4} \end{aligned}$$

v. Find the solution for I.V.P.

$$\begin{aligned} y &= C_1 \sin 3x + C_2 \cos 3x \\ y &= \frac{3}{4}\sin 3x - 10\cos 3x \end{aligned}$$

(d) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0, y = -1$ and $\frac{dy}{dx} = 5$ when $x = -4$

i. Rewrite

$$r^2 + 14r + 49 = 0$$

ii. Factorize

$$\begin{aligned} r^2 + 14r + 49 &= 0 \\ \left(r+7\right)\left(r+7\right) &= 0 \\ \left(r+7\right)^2 &= 0 \\ r &= -7 \end{aligned}$$

iii. Obtain equation

$$\begin{aligned} y &= e^{-7x}\left(C_1 + C_2 x\right) \\ y &= C_1 e^{-7x} + C_2 e^{-7x} x \end{aligned}$$

iv. Find C_1 and C_2

A. Find the equations

$$\begin{aligned} -1 &= e^{-7(-4)}\left(C_1 + C_2\left(-4\right)\right) \\ -1 &= e^{28}\left(C_1 - 4C_2\right) \\ -1 &= C_1 e^{28} - 4C_2 e^{28} \\ C_1 e^{28} &= -1 + 4C_2 e^{28} \\ C_1 &= -e^{-28} + 4C_2 \end{aligned}$$

$$\begin{aligned} y' &= \frac{d}{dx}\left[e^{-7x}\left(C_1 + C_2 x\right)\right] \\ &= \frac{d}{dx}\left[C_1 e^{-7x} + C_2 e^{-7x} x\right] \\ &= -7C_1 e^{-7x} + C_2\left[-7e^{-7x} \cdot x + e^{-7x}\right] \\ y' &= -7C_1 e^{-7x} + C_2 e^{-7x} - 7C_2 x e^{-7x} \end{aligned}$$

B. Substitute the values in

$$y = -1 \, and \, \frac{dy}{dx} = 5 \, when \, x = -4$$

$$\begin{aligned}
5 &= -7C_1e^{-7(-4)} + C_2e^{-7(-4)} - 7C_2(-4)e^{-7(-4)} \\
&= -7C_1e^{28} + C_2e^{28} + 28C_2e^{28} \\
5 &= -7C_1e^{28} + 29Ce^{28} \\
5 &= -7\left(-e^{-28} + 4C_2\right)e^{28} + 29C_2e^{28} \\
5 &= 7 - 28C_2e^{28} + 29C_2e^{28} \\
-2 &= C_2e^{28} \\
C_2 &= -\frac{2}{e^{28}}
\end{aligned}$$

$$\begin{aligned}
C_1 &= -e^{-28} + 4C_2 \\
C_1 &= -e^{-28} + 4\left(-\frac{2}{e^{28}}\right) \\
&= -e^{-28} - 8e^{-28} \\
&= -\frac{9}{e^{28}}
\end{aligned}$$

v. Find the solution for I.V.P.V (Given answer is bolded)

$$\begin{aligned}
y &= -\frac{9}{e^{28}}e^{-7x} + -\frac{2}{e^{28}}e^{-7x}x \\
&= -9e^{-7x-28} - 2xe^{-7x-28} \\
\boldsymbol{y} &\boldsymbol{= -9e^{-7(x+4)} - 2xe^{-7(x+4)}} \\
&= e^{-7x-28}(-2x-9) \\
y &= e^{-7(x+4)}(-2x-9)
\end{aligned}$$

2 2^{nd} order non-homogeneous linear D.E.

1. Find General Solution for:

(a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}
r^2 - 2r + 1 &= 0 \\
(r-1)(r-1) &= 0 \\
(r-1)^2 &= 0 \\
r &= 1
\end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^x(C_1 + C_2x)$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned}
y_p &= Ae^{2x} \\
y_p' &= 2Ae^{2x} \\
y_p'' &= 4Ae^{2x}
\end{aligned}$$

B. Plug into the function

$$\begin{aligned}
\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= e^{2x} \\
4Ae^{2x} - 2\left(2Ae^{2x}\right) + Ae^{2x} &= e^{2x} \\
4Ae^{2x} - 4Ae^{2x} + Ae^{2x} &= e^{2x}
\end{aligned}$$

C. Find A

$$\begin{aligned} 4Ae^{2x} - 4Ae^{2x} + Ae^{2x} &= e^{2x} \\ Ae^{2x} &= e^{2x} \\ A &= 1 \end{aligned}$$

D. Find y_p

$$y_p = e^{2x}$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^x (C_1 + C_2 x) + e^{2x}$$

(b) $9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = x^2 + 2x + 3$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$9r^2 + 6r + 1 = 0$$

$$\begin{aligned} (3r+1)^2 &= 0 \\ r &= -\frac{1}{3} \end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^{-\frac{1}{3}x} (C_1 + C_2 x)$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned} y_p &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

B. Plug into the function

$$\begin{aligned} 9(2A) + 6(2Ax + B) + Ax^2 + Bx + C &= x^2 + 2x + 3 \\ Ax^2 + 12Ax + Bx + 18A + 6B + C &= x^2 + 2x + 3 \\ Ax^2 + (12A + B)x + 18A + 6B + C &= x^2 + 2x + 3 \end{aligned}$$

C. Find A, B and C by equating them

A

$$\begin{aligned} Ax^2 &= x^2 \\ A &= 1 \end{aligned}$$

B

$$\begin{aligned} 12A + B &= 2 \\ B &= 2 - 12(A) \\ &= 2 - 12(1) \\ B &= -10 \end{aligned}$$

C

$$\begin{aligned} 18A + 6B + C &= 3 \\ 18(1) + 6(-10) + C &= 3 \\ C &= 3 - 18 + 60 \\ C &= 45 \end{aligned}$$

D. Find y_p

$$y_p = Ax^2 + Bx + C$$
$$\boldsymbol{y_p = x^2 - 10x + 45}$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = (C_1 + C_2 x)\, e^{-\frac{1}{3}x} + x^2 - 10x + 45$$

(c) $\frac{d^2y}{dx^2} - 4y = 3e^{-2x}$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned} \frac{d^2y}{dx^2} - 4y &= 0 \\ r^2 - 4 &= 0 \\ (r-2)(r+2) &= 0 \\ r &= \pm 2 \end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^{2x} C_1 + e^{-2x} C_2$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned} y_p &= x\left[Ae^{-2x}\right] \\ y_p &= \boldsymbol{Axe^{-2x}} \\ y_p' &= Ae^{-2x} + (-2)\, Axe^{-2x} \\ &= Ae^{-2x} - 2Axe^{-2x} \\ y_p'' &= \frac{d}{dx}\left[Ae^{-2x} - 2Axe^{-2x}\right] \\ &= -2Ae^{-2x} - 2\left(Ae^{-2x} + (-2)\, Axe^{-2x}\right) \\ y_p'' &= -4Ae^{-2x} + 4Axe^{-2x} \end{aligned}$$

B. Plug into the function

$$\begin{aligned} \frac{d^2y}{dx^2} - 4y &= 3e^{-2x} \\ \left(-4Ae^{-2x} + 4Axe^{-2x}\right) - 4\left(Axe^{-2x}\right) &= 3e^{-2x} \end{aligned}$$

C. Find A

$$\begin{aligned} \left(-4Ae^{-2x} + 4Axe^{-2x}\right) - 4\left(Axe^{-2x}\right) &= 3e^{-2x} \\ -4Ae^{-2x} &= 3e^{-2x} \\ -4A &= 3 \\ A &= -\frac{3}{4} \end{aligned}$$

D. Find y_p

$$y_p = -\frac{3}{4} e^{-2x}$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$\begin{aligned} y = e^{2x} C_1 + e^{-2x} C_2 - \frac{3}{4} e^{-2x} \\ e^{2x} C_1 + \left(C_2 - \frac{3}{4}\right) e^{-2x} \end{aligned}$$

(d) $3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 9\cos 2x - 23\sin 2x$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y &= 0\\ 3r^2 + 8r + 5 &= 0\\ (3r+5)(r+1) &= 0\\ r &= -\frac{5}{3}, -1\end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^{-\frac{5}{3}x}C_1 + e^{-x}C_2$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned}y_p &= A\sin(2x) + B\cos(2x)\\ y_p' &= \frac{d}{dx}\left[A\sin(2x) + B\cos(2x)\right]\\ &= A(2\cos 2x) - 2B\sin(2x)\\ y_p' &= 2A\cos 2x - 2B\sin(2x)\\ y'' &= -4A\sin 2x - 4B\cos 2x\end{aligned}$$

B. Plug into the function

$$\begin{aligned}3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5\left(A\sin(2x) + B\cos(2x)\right) &= 9\cos 2x - 23\sin 2x\\ 3(-4A\sin 2x - 4B\cos 2x) + 8\left(2A\cos 2x - 2B\sin(2x)\right) + 5A\sin(2x) + 5B\cos(2x) &= 9\cos 2x - 23\sin 2x\\ -12A\sin 2x - 12B\cos 2x + 16A\cos 2x - 16B\sin 2x + 5A\sin 2x + 5B\cos 2x &= 9\cos 2x - 23\sin 2x\\ -12A\sin 2x + 5A\sin 2x - 16B\sin 2x - 12B\cos 2x + 5B\cos 2x + 16A\cos 2x &= 9\cos 2x - 23\sin 2x\\ -7A\sin 2x - 16B\sin 2x - 7B\cos 2x + 16A\cos 2x &= 9\cos 2x - 23\sin 2x\\ (-7B + 16A)\cos 2x + (-7A - 16B)\sin 2x &= 9\cos 2x - 23\sin 2x\end{aligned}$$

C. Find $A\&B$ (by equating coefficients)

$$\begin{aligned}-7B + 16A &= 9\\ 16A &= 9 + 7B\\ A &= \frac{9+7B}{16}\end{aligned}$$

$$\begin{aligned}-7A - 16B &= 23\\ -7\left(\frac{9+7B}{16}\right) - 16B &= 23\\ \frac{-63-49B}{16} - 16B &= 23\\ -63 - 49B - 256B &= 368\\ 305B &= 305\\ B &= 1\end{aligned}$$

$$\begin{aligned}A &= \frac{9+7(1)}{16}\\ &= \frac{16}{16}\\ A &= 1\end{aligned}$$

D. Find y_p

$$y_p = \sin(2x) + \cos(2x)$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{-\frac{5}{3}x}C_1 + e^{-x}C_2 + \sin(2x) + \cos(2x)$$

(e) $y'' + 2y' + 5y = x^2$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$r^2 + 2r + 5 = 0$$

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= -1 \pm 2i \end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^{-x}\left(C_1 \sin 2x + C_2 \cos 2x\right)$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

B. Plug into the function

$$\begin{aligned} 2A + 2\left(2Ax + B\right) + 5\left(Ax^2 + Bx + C\right) &= x^2 \\ 2A + 4Ax + 2B + 5Ax^2 + 5Bx + 5C &= x^2 \\ 5Ax^2 + \left(4A + 5B\right)x + 2A + 2B + 5C &= x^2 + 0x + 0 \end{aligned}$$

C. Find A, B, C

$$\begin{aligned} 5A &= 1 \\ A &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} 4A + 5B &= 0 \\ 4\left(\frac{1}{5}\right) + 5B &= 0 \\ 5B &= -\frac{4}{5} \\ B &= -\frac{4}{25} \end{aligned}$$

$$\begin{aligned} 2A + 2B + 5C &= 0 \\ 2\left(\frac{1}{5}\right) + 2\left(-\frac{4}{25}\right) + 5C &= 0 \\ \frac{2}{5} - \frac{8}{25} + 5C &= 0 \\ 5C &= -\frac{2}{5} + \frac{8}{25} \\ 5C &= -\frac{2}{25} \\ C &= -\frac{2}{125} \end{aligned}$$

D. Find y_p

$$y_p = \frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125}$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{-x}\left(C_1 \sin 2x + C_2 \cos 2x\right) + \frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125}$$

(f) $y'' - y' + 9y = 3 \sin 3x$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$
\begin{aligned}
r^2 - r + 9 &= 0 \\
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(9)}}{2} \\
&= \frac{1 \pm \sqrt{1-36}}{2} \\
&= \frac{1 \pm \sqrt{-35}}{2} \\
&= \frac{1 \pm \sqrt{-1}\sqrt{35}}{2} \\
&= \frac{1 \pm \sqrt{35}i}{2} \\
r &= \frac{1}{2} \pm \frac{\sqrt{35}}{2}i
\end{aligned}
$$

ii. Find the complementary function, y_h

$$y_h = e^{\frac{1}{2}x}\left(C_1 \sin\frac{\sqrt{35}}{2}x + C_1 \cos\frac{\sqrt{35}}{2}x\right)$$

iii. Find the particular solution y_p

A. Find the derivatives

From $3\sin 3x$, we predict y will be:

$$
\begin{aligned}
y &= A\sin 3x + B\cos 3x \\
y' &= 3A\cos 3x - 3B\sin 3x \\
y'' &= -9A\sin 3x - 9B\cos 3x
\end{aligned}
$$

B. Plug into the function

$$
\begin{aligned}
y'' - y' + 9y &= 3\sin 3x \\
(-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) + 9(A\sin 3x + B\cos 3x) &= 3\sin 3x \\
-9A\sin 3x - 9B\cos 3x - 3A\cos 3x + 3B\sin 3x + 9A\sin 3x + 9B\cos 3x &= 3\sin 3x \\
3B\sin 3x - 3A\cos 3x &= 3\sin 3x \\
B\sin 3x - A\cos 3x &= \sin 3x + 0\cos 3x
\end{aligned}
$$

C. Find A, B

$$B\sin 3x - A\cos 3x = \sin 3x + 0\cos 3x$$

$$B = 1$$

$$A = 0$$

D. Find y_p

$$y_p = \cos 3x$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{\frac{1}{2}x}\left(C_1 \sin\frac{\sqrt{35}}{2}x + C_1 \cos\frac{\sqrt{35}}{2}x\right) + \cos 3x$$

(g) $y'' + y' = 2x + 4 + 2e^x$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$
\begin{aligned}
y'' + y' &= 0 \\
r^2 + r &= 0 \\
r(r+1) &= 0 \\
r &= 0, -1
\end{aligned}
$$

ii. Find the complementary function, y_h

$$\begin{aligned} y_h &= C_1 e^{0x} + C_2 e^{-1x} \\ y_h &= C_1 + C_2 e^{-x} \end{aligned}$$

iii. Find the particular solution y_p

A. Find the derivatives (By prediction, find y, then find the rest of derivatives)

$$\begin{aligned} y &= Ax^2 + Bx + Ce^x \\ y' &= 2Ax + B + Ce^x \\ y'' &= 2A + Ce^x \end{aligned}$$

B. Plug into the function

$$\begin{aligned} 2A + Ce^x + 2Ax + B + Ce^x &= 2x + 4 + 2e^x \\ 2Ax + 2A + B + 2Ce^x &= 2x + 4 + 2e^x \end{aligned}$$

C. Find A, B and C

$$\begin{aligned} 2A &= 2 \\ A &= 1 \end{aligned}$$

$$\begin{aligned} 2A + B &= 4 \\ 2(1) + B &= 4 \\ B &= 4 - 2 \\ B &= 2 \end{aligned}$$

$$\begin{aligned} 2C &= 2 \\ C &= 1 \end{aligned}$$

D. Find y_p

$$y_p = x^2 + 2x + e^x$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = C_1 + C_2 e^{-x} + x^2 + 2x + e^x$$

(h) $y'' - 3y' - 18y = xe^{4x}$

i. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned} y'' - 3y' - 18y &= 0 \\ r^2 - 3r - 18 &= 0 \\ (r-6)(r+3) &= 0 \\ r &= 6, r = -3 \end{aligned}$$

ii. Find the complementary function, y_h

$$y_h = e^{6x} C_1 + e^{-3x} C_2$$

iii. Find the particular solution y_p

A. Find the derivatives

$$\begin{aligned} y &= (Bx + C) e^{4x} \\ y' &= \frac{d}{dx}\left[Bxe^{4x} + Ce^{4x}\right] \\ &= Be^{4x} + 4Bxe^{4x} + 4Ce^{4x} \\ y'' &= 4Be^{4x} + 4B\left(e^{4x} + 4xe^{4x}\right) + 16Ce^{4x} \\ &= 4Be^{4x} + 4Be^{4x} + 16Bxe^{4x} + 16Ce^{4x} \\ y'' &= 8Be^{4x} + 16Bxe^{4x} + 16Ce^{4x} \end{aligned}$$

B. Plug into the function

$$\begin{aligned}
8Be^{4x} + 16Bxe^{4x} + 16Ce^{4x} - 3\left(Be^{4x} + 4Bxe^{4x} + 4Ce^{4x}\right) - 18\left(Bxe^{4x} + Ce^{4x}\right) &= xe^{4x} \\
8Be^{4x} + 16Bxe^{4x} + 16Ce^{4x} - 3Be^{4x} - 12Bxe^{4x} - 12Ce^{4x} - 18Bxe^{4x} - 18Ce^{4x} &= xe^{4x} \\
16Bxe^{4x} - 12Bxe^{4x} - 18Bxe^{4x} - 12Ce^{4x} - 18Ce^{4x} + 16Ce^{4x} + 8Be^{4x} - 3Be^{4x} &= xe^{4x} \\
16Bxe^{4x} - 30Bxe^{4x} - 14Ce^{4x} + 5Be^{4x} &= xe^{4x} \\
xe^{4x}(-14B) - e^{4x}(14C + 5B) &= xe^{4x}
\end{aligned}$$

C. Find B and C

$$\begin{aligned}
-14B &= 1 \\
B &= -\frac{1}{14}
\end{aligned}$$

$$\begin{aligned}
14C + 5B &= 0 \\
14C &= -5\left(-\frac{1}{14}\right) \\
C &= \frac{5}{14} * \frac{1}{14} \\
&= \frac{5}{196}
\end{aligned}$$

D. Find y_p

$$y_p = \left(-\frac{1}{14}x + \frac{5}{196}\right)e^{4x}$$

iv. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{6x}C_1 + e^{-3x}C_2 + \left(-\frac{1}{14}x + \frac{5}{196}\right)e^{4x}$$