Calc 1:Tutorial 4

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1. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

   (a) $x^3 - 3x + 1 = 0, (0, 1)$

      i. Let $f(x) = x^3 - 3x + 1$

      ii. At $f(0.1)$,

      $$\begin{aligned} f(0) &= (0.1)^3 - 3(0.1) + 1 \\ &= 0.701 \end{aligned}$$

      iii. At $f(0.9)$,

      $$\begin{aligned} f(1) &= (0.9)^3 - 3(0.9) + 1 \\ &= -0.971 \end{aligned}$$

      iv. Since $f(0.1) > 0$ and $f(0.9) < 0$, by the intermediate value theorem we can conclude that there is a root in the interval $(0, 1)$

   (b) $x^2 = \sqrt{x+1}, (1, 2)$

      i. Solve for $f(x)$

      $$\begin{aligned} x^4 &= x + 1 \\ x^4 - x - 1 &= 0 \end{aligned}$$

         A. Let $f(x) = x^4 - x - 1$

      ii. At $f(1.1)$

      $$\begin{aligned} f(1,1) &= (1.1)^4 - (1.1) - 1 \\ &= -0.6359 \end{aligned}$$

      iii. At $f(1.9)$

      $$\begin{aligned} f(1.9) &= (1.9)^4 - (1.1) - 1 \\ &= 10.9321 \end{aligned}$$

iv. Since $f(1.1) < 0$ and $f(1.9) > 0$, by the intermediate value theorem we can conclude that there is a root in the interval $(1, 2)$.

2. Find the limit.

(a) $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$

$$\lim_{x\to 5^-} (x-5)^3 = -0$$

$$\lim_{x\to 5^-} \frac{e^x}{(x-5)^3} = -\infty$$

(b) $\lim_{x\to 5^+} \ln(x-5)$

$$\begin{aligned}\lim_{x\to 5^+} \ln(x-5) &= \ln\left(0^+\right)\\ &= +\infty\end{aligned}$$

(c) $\lim_{x\to\infty} \frac{3x+5}{x+4}$

$$\begin{aligned}\lim_{x\to\infty} \frac{3x+5}{x+4} &= \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{5}{x}}{\frac{x}{x}+\frac{4}{x}}\\ &= \lim_{x\to\infty} \frac{3+\frac{5}{x}}{1+\frac{4}{x}}\\ &= \frac{3+\cancel{\frac{5}{\infty}}}{1+\cancel{\frac{4}{\infty}}}\\ &= \frac{3}{1}\\ &= 3\end{aligned}$$

(d) $\lim_{t\to -\infty} \frac{t^2+2}{t^3+t^2-1}$

$$\begin{aligned}\lim_{t\to -\infty} \frac{t^2+2}{t^3+t^2-1} &= \lim_{t\to -\infty} \frac{\frac{t^2}{t^2}+\frac{2}{t^2}}{\frac{t^3}{t^2}+\frac{t^2}{t^2}-\frac{1}{t^2}}\\ &= \lim_{t\to -\infty} \frac{1+\frac{2}{t^2}}{t+1-\frac{1}{t^2}}\\ &= \frac{1+0}{-\infty+1-0}\\ &= 0\end{aligned}$$

(e) $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} &= \frac{\lim_{x\to\infty} x+2}{\lim_{x\to\infty} \sqrt{9x^2+1}} \\
&= \frac{\lim_{x\to\infty} 1+\frac{2}{x}}{\lim_{x\to\infty} \frac{\sqrt{9x^2+1}}{\sqrt{x^2}}} \\
&= \frac{\lim_{x\to\infty} 1+\frac{2}{x}}{\lim_{x\to\infty} \sqrt{9+\frac{1}{x^2}}} \\
&= \frac{1}{3}
\end{aligned}$$

(f) $\lim_{x\to\infty} \frac{\sin^2 x}{x}$

i. Squeeze theorem

$$\begin{aligned}
-1 < sin\left(x\right) &< 1 \\
0 < sin^2\left(x\right) &< 1 \\
0 < \frac{sin^2\left(x\right)}{x} &< \frac{1}{x} \\
\lim_{x\to\infty} 0 < \lim_{x\to\infty} \frac{sin^2\left(x\right)}{x} &< \lim_{x\to\infty} \frac{1}{x} \\
0 < \lim_{x\to\infty} \frac{sin^2\left(x\right)}{x} &< 0
\end{aligned}$$

A. $\therefore \lim_{x\to\infty} \frac{sin^2(x)}{x} = 0$

ii.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sin^2 x}{x} &= \frac{0}{\infty} \\
&= 0
\end{aligned}$$

(g) $\lim_{x\to\infty} \tan^{-1}\left(x^4 - x^2\right)$

$$\begin{aligned}
\lim_{x\to\infty} x^4 - x^2 &= \infty \\
\lim_{x\to\infty} \tan^{-1}\left(\infty\right) &= \frac{\pi}{2}\text{(from the graph of inverse tangent)}
\end{aligned}$$

(h) $\lim_{x\to\infty} e^{-x^2}$

$$\begin{aligned}
\lim_{x\to\infty} e^{-x^2} &= e^{-\infty^2} \\
&= \frac{1}{e^{\infty^2}} \\
&= 0
\end{aligned}$$

3. Find the vertical and horizontal asymtotes of

   (a) $y = \frac{1}{x-1}$

      i. $x = -\infty$

$$\begin{aligned}\lim_{x\to-\infty} \frac{1}{x-1} &= \frac{1}{-\infty - 1} \\ &= 0\end{aligned}$$

      ii. $x = +\infty$

$$\begin{aligned}\lim_{x\to+\infty} \frac{1}{x-1} &= \frac{1}{+\infty - 1} \\ &= 0\end{aligned}$$

      iii. Hence, the only **horizontal asymtote** is $y = 0$

      iv. Vertical asymtotes, when denominator $= 0$

$$\begin{aligned}x - 1 &= 0 \\ x &= 1\end{aligned}$$

      v. Hence, the only **vertical asymtote** is $x = 1$

   (b) $y = \frac{x}{x-1}$

      i. Horizontal asymtote

         A. $x = -\infty$

$$\begin{aligned}\lim_{x\to-\infty} \frac{x}{x-1} &= \lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} \\ &= \lim_{x\to-\infty} \frac{1}{1 - \frac{1}{x}} \\ &= 1\end{aligned}$$

         B. $x = +\infty$

$$\begin{aligned}\lim_{x\to+\infty} \frac{x}{x-1} &= \lim_{x\to+\infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} \\ &= \lim_{x\to+\infty} \frac{1}{1 - \frac{1}{x}} \\ &= 1\end{aligned}$$

      ii. Hence, the only **horizontal asymtote** is $y = 1$

      iii. Vertical asymtote, when denominator $= 0$

$$\begin{aligned}x - 1 &= 0 \\ x &= 1\end{aligned}$$

iv. Hence, the only **vertical asymtote** is $x = 1$

(c) $y = \frac{1}{(x-1)^2}$

i. Horizontal asymtote

ii. $x = -\infty$

$$\begin{aligned}
\lim_{x \to -\infty} \frac{1}{(x-1)^2} &= \lim_{x \to -\infty} \frac{1}{x^2 - 2x + 2} \\
&= \lim_{x \to -\infty} \frac{1}{1 - \frac{1}{x}} \\
&= 1
\end{aligned}$$

iii. $x = +\infty$

$$\begin{aligned}
\lim_{x \to +\infty} \frac{x}{x-1} &= \lim_{x \to +\infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} \\
&= \lim_{x \to +\infty} \frac{1}{1 - \frac{1}{x}} \\
&= 1
\end{aligned}$$

iv. Hence, the only **horizontal asymtote** is $y = 1$

v. Vertical asymtote, when denominator = 0

$$\begin{aligned}
x - 1 &= 0 \\
x &= 1
\end{aligned}$$

vi. Hence, the only **vertical asymtote** is $x = 1$

(d) $y = \frac{x^2}{(x-1)(x-3)}$

i. $x = -\infty$

$$\begin{aligned}
\lim_{x \to -\infty} \frac{x^2}{(x-1)(x-3)} &= \lim_{x \to -\infty} \frac{x^2}{x^2 - 4x + 3} \\
&= \lim_{x \to -\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{4x}{x^2} + \frac{3}{x^2}} \\
&= \lim_{x \to -\infty} \frac{1}{1 - \frac{4}{x} + \frac{3}{x^2}} \\
&= 1
\end{aligned}$$

ii. $x = +\infty$

$$\begin{aligned}
\lim_{x\to+\infty} \frac{x^2}{(x-1)(x-3)} &= \lim_{x\to+\infty} \frac{x^2}{x^2-4x+3} \\
&= \lim_{x\to+\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{3}{x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{1-\frac{4}{x}+\frac{3}{x^2}} \\
&= 1
\end{aligned}$$

iii. Hence, the only **horizontal asymtote** is $y = 1$

iv. Vertical asymtote, when denominator $= 0$

$$\begin{aligned}
(x-1)(x-3) &= 0 \\
x &= 1, 3
\end{aligned}$$

v. Hence, the vertical asymtotes are $x = 1, x = 3$

4. Find the derivative of the function using the definition of derivative.

(a) $f(x) = 5 - 4x + 3x^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{5-4(x+h)+3(x+h)^2-5+4x-3x^2}{h} \\
&= \lim_{h\to 0} \frac{5-4x-4h+3\left(x^2+2hx+h^2\right)-5+4x-3x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5-5-4x+4x+3x^2-3x^2}+3h^2+6hx-4h}{h} \\
&= \lim_{h\to 0} \frac{3h^2+6hx-4h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3h+6x-4)}{\cancel{h}} \\
&= \lim_{h\to 0} (3h+6x-4)
\end{aligned}$$

$$\boldsymbol{f'(x) = 6x - 4}$$

(b) $f(x) = \frac{3}{x-2}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{x+h-2} - \frac{3}{x-2}}{h} \\
&= \lim_{h\to 0} \left(\frac{3}{x+h-2} * \frac{x-2}{x-2} - \frac{3}{x-2} * \frac{x+h-2}{x+h-2} \right) * \frac{1}{h} \\
&= \lim_{h\to 0} \frac{3x - 6 - 3x - 3h + 6}{(x+h-2)(x-2)} * \frac{1}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3x - 3x} - 3\cancel{h}\cancel{+6-6}}{(x+h-2)(x-2)} * \frac{1}{\cancel{h}} \\
&= \lim_{h\to 0} \frac{-3}{(x+h-2)(x-2)} \\
&= \frac{-3}{(x-2)(x-2)}
\end{aligned}
$$

$$
\boldsymbol{f'(x) = -\frac{3}{(x-2)^2}}
$$

(c) $f(x) = \sqrt{3x+1}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} * \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \lim_{h\to 0} \frac{3(x+h) + 1 - (3x+1)}{h\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \lim_{h\to 0} \frac{\cancel{3x - 3x + 1 - 1} + 3h}{h\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}}
\end{aligned}
$$

$$
\boldsymbol{f'(x) = \frac{3}{2\sqrt{3x+1}}}
$$