# Chapter 6
# Order Relations and Structures

# 6.1 Partially Ordered Sets

- A relation $R$ on a set $A$ is called a partial order if $R$ is reflexive, antisymmetric, and transitive.
- The set $A$ with the partial order $R$ is called a partially ordered set, or poset, denoted by $(A, R)$.

# 6.1 Partially Ordered Sets (cont)

- E.g. $(A, R) \to (A, R)$
  - Let $A$ be a collection of subsets of a subset of a set $S$. The relation $\subseteq$ of set inclusion is a partial order on $A$, so $(A, \subseteq)$ is a poset.
  - Let $Z^+$ be the set of positive integers. The usual relations $\leq$ (less than or equal to) and $\geq$ (greater or equal to) are partial orders on $Z^+$, but the relations $<$ (less than) and $>$ (greater than) are not partial order since they are not reflexive.

# 6.1 Partially Ordered Sets (cont)

- The relation of divisibility ($a\ R\ b$ if and only if $a|b$) is a partial order on $Z^+$ but $R$ is not partial order on Z since it is not antisymmetric, for example $-2|2$ and $2|-2$ but $-2 \neq 2$.

# 6.1 Partially Ordered Sets (cont)

- Let $R$ be a partial order on a set $A$, then the inverse relation $R^{-1}$ is also a partial order. The poset $(A, R^{-1})$ is called the dual of the poset $(A, R)$, and the partial order $R^{-1}$ is called the dual of the partial order $R$.
- The most familiar partial orders are the relations $\leq$ or $\geq$ on Z and R.
- In general, a partial order relation on a set often use the symbols $\leq$ or $\geq$ for $R$ (relation $R$). Do not mistake this to familiar relation $\leq$ on Z (integers) or R (real numbers).

# 6.1 Partially Ordered Sets (cont)

- Symbols such as $\leq_1$, $\leq'$, $\geq_1$, $\geq'$ can be used to denote partial orders.
-

| Poset | Dual Poset |
|---|---|
| $(A, \leq)$ | $(A, \geq)$ |
| $(A, \leq_1)$ | $(A, \geq_1)$ |
| $(B, \leq')$ | $(B, \geq')$ |

# 6.1 Partially Ordered Sets (cont)

- If $(A, \leq)$ is a poset, the elements $a$ and $b$ of $A$ are said to be comparable if

  $a \leq b$ or $b \leq a$.

  divides

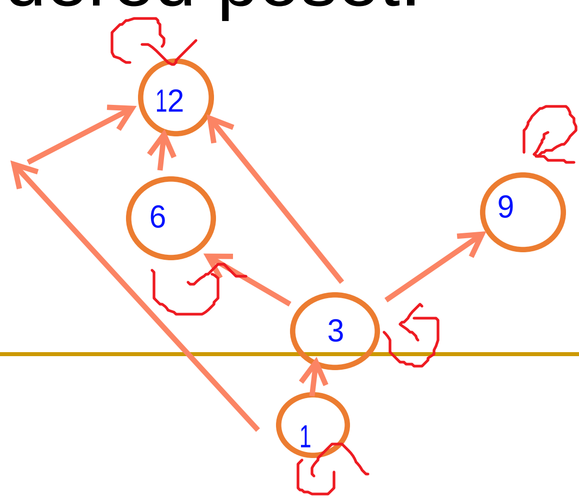
  - Consider $(A, \leq) = (Z^+, |)$

    2 and 6 are comparable since $2 \leq 6$ or $2|6$.

    2 and 7 are not comparable since $2 \nmid 7$ and $7 \nmid 2$.

# 6.1 Partially Ordered Sets (cont)

- If every pair of elements in a poset *A* is comparable, then *A* is a linearly ordered set, and the partial order is called a linear order. We also say that *A* is a chain.
  - $(Z^+, \leq)$ is linearly ordered poset.
  - $(A, |)$ where $A = \{1, 3, 6, 9, 12\}$ is not a linearly ordered poset.




not in a chain, it branches off

# 6.1 Partially Ordered Sets (cont)

- Theorem 1

  If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset, with partial order $\leq$ defined by

  $(a, b) \leq (a', b')$ if $a \leq a'$ in $A$ and $b \leq b'$ in $B$.

  - The symbol $\leq$ is being used to denote three distinct partial orders.
  - The partial order $\leq$ defined on the Cartesian product $A \times B$ is called the product partial order.

# 6.1 Partially Ordered Sets (cont)

- Let $A = \{1, 3, 5\}$, $B = \{2, 4, 8\}$, and $\leq_A$ means "less than or equal to", | means "divides", then $(A, \leq_A)$ and $(B, |)$ are posets.

  Hence, $(A \times B, \leq)$ is also a poset since

  $1 \leq_A 3$, $2|4$, $(1, 2) \leq (3, 4)$, also

  $3 \leq_A 5$, $4|8$, $(3, 4) \leq (5, 8)$.

  Hence $(1, 2) \leq (3, 4)$, $(3, 4) \leq (5, 8)$

  $\Rightarrow (1, 2) \leq (5, 8)$ because $1 \leq_A 5$, $2|8$.

# 6.1 Partially Ordered Sets (cont)

- $(A, \leq_A)$ and $(B, |)$ are linearly ordered but not $(A \times B, \leq)$ since some elements are not comparable. For examples,

  $(1, 4) \not\leq (3, 2)$ since $4 \nmid 2$ even $1 \leq_A 3$;

  $(3, 2) \not\leq (1, 4)$ since $3 \not\leq_A 1$ even $2|4$.

  So, $A$ and $B$ are linearly ordered $\not\Rightarrow$ $A \times B$ linearly ordered.

- If $(A, \leq)$ is a poset, we say $a < b$ if $a \leq b$ but $a \neq b$.

# 6.1 Partially Ordered Sets (cont)

- Suppose that ($A$, $\leq$) and ($B$, $\leq$) are posets, we define ($A \times B$, $\prec$) as

  ($a$, $b$)$\prec$($a$', $b$') if $a < a$' or if $a = a$' and $b \leq b$'.

  This ordering is called lexicographic, or "dictionary" order.
  - The ordering of the elements in the first coordinate dominates, except in case of "ties", when attention passes to the second coordinate.

# 6.1 Partially Ordered Sets (cont)

- If $(A, \leq)$ and $(B, \leq)$ are linearly ordered sets, then the lexicographic order $\prec$ on $A \times B$ is also a linear order.
- From previous example,

  $(1, 4) \prec (3, 2)$ since $1 \leq 3$,

  $(1, 4) \prec (1, 8)$ since $1 = 1$ and $4|8$.

# 6.1 Partially Ordered Sets (cont)

- Lexicographic ordering is easily extended to Cartesian products $A_1 \times A_2 \times \ldots \times A_n$ as follows:

  $a_1 < a_1'$ or

  $a_1 = a_1'$ and $a_2 < a_2'$ or

  $a_1 = a_1'$, $a_2 = a_2'$, and $a_3 < a_3'$ or …

  $a_1 = a_1'$, $a_2 = a_2'$, …, $a_{n-1} = a_{n-1}'$ and $a_n = a_n'$.

Thus the first coordinate dominates except in equality, in which case we consider the second coordinate. If equality holds again, pass to the next coordinate, and so on.

# 6.1 Partially Ordered Sets (cont)

- Let $S = \{a, b, \ldots, z\}$ be the ordinary alphabet, linearly ordered in the usual way, ($a \leq b$, $b \leq c$, …, $y \leq z$).

$$S^n = S \times S \times \ldots \times S \text{ (}n\text{ factors)}$$

can be identified with the set of all words having length $n$.

Then $park \prec part$, $help \prec hind$, $jump \prec mump$.

# 6.1 Partially Ordered Sets (cont)

- If $S$ is a poset, we can extend lexicographic order to $S^*$ in the following way.

  If $x = a_1a_2\ldots a_n$ and $y = b_1b_2\ldots b_k$ are in $S^*$ with $n \leq k$, we say that $x \prec y$ if $(a_1a_2\ldots a_n) \prec (b_1b_2\ldots b_n)$ in $S^n$ under lexicographic ordering of $S^n$.

  For example, *park* $\prec$ *part* $\Rightarrow$ *park* $\prec$ *partition*

  *help* $\prec$ *helping*, *park* $\prec$ *parking*

# 6.1 Partially Ordered Sets (cont)

- Theorem 2

  The digraph of a partial order has no cycle of length greater than 1.

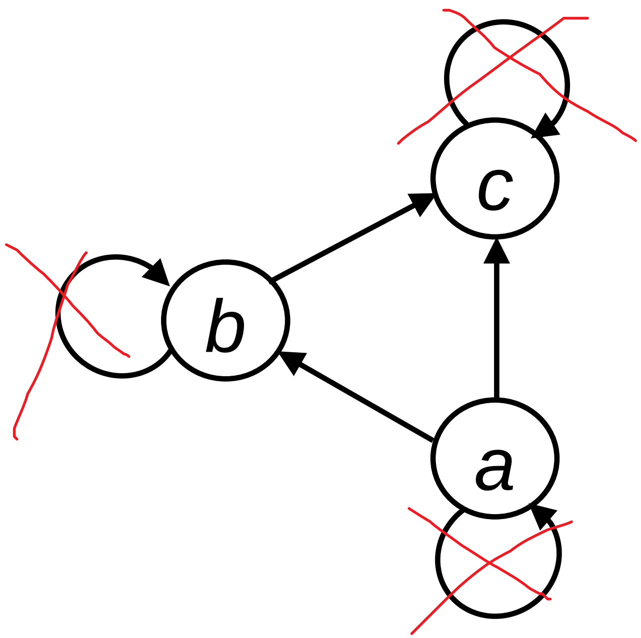
# 6.2 Hasse Diagram

- A simplification of digraph obtained by:

1. omitting all cycles of length 1;
2. omitting all edges that are implied by the transitive property;
3. drawing all the edges slanting upwards so that the arrow need not be drawn;
4. representing vertices by dots instead of circles.

# 6.2 Hasse Diagram (cont)

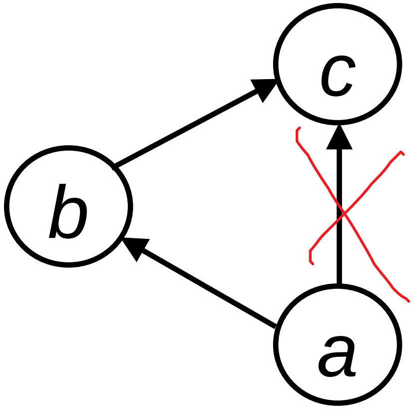
- Consider the digraph given:




Arrange from bottom to top

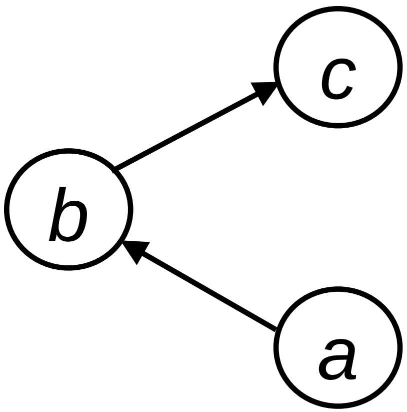
# 6.2 Hasse Diagram (cont)

Step 1: Delete all cycles of length 1

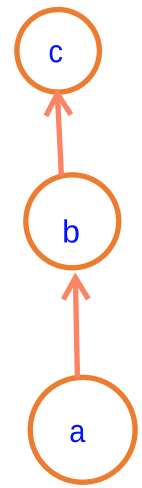


# 6.2 Hasse Diagram (cont)

Step 2: Eliminate all edges that are implied by the transitive property



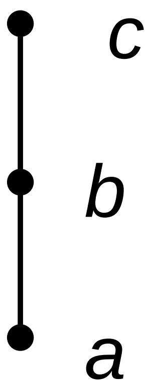

Stretch into 1 line -->




continue below

# 6.2 Hasse Diagram (cont)

Step 3: Hasse diagram obtained




"Perfect" diagram

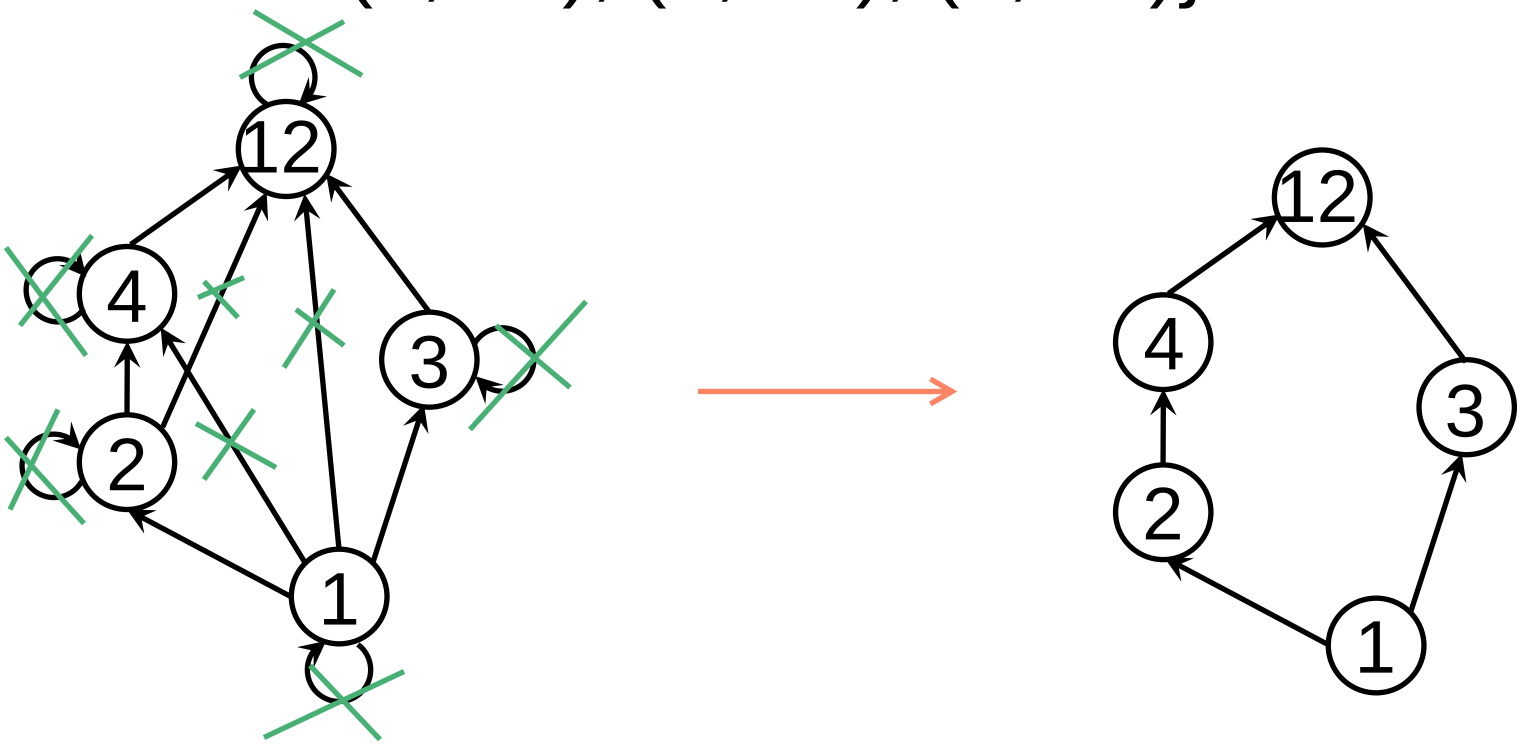
Can no longer be simplified

# E.g.1

Let $A$ = {1, 2, 3, 4, 12}. Consider the partial order of divisibility on $A$. That is, if $a$ and $b \in A$, $a \leq b$ if and only if $a|b$. Draw the Hasse diagram for the poset $(A, \leq)$.

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (1, 3), (1, 4), (1, 12), (2, 4), (2, 12), (3, 12), (4, 12)\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (1, 3), (1, 4), (1, 12), (2, 4), (2, 12), (3, 12), (4, 12)\}$$




Hasse Diagram

## E.g.2

Let $a \leq b$ if and only if $a|b$ and $a \geq b$ if and only if $a$ is a multiple of $b$ or $b|a$. Draw the Hasse diagrams of $(A, \leq)$ and $(A, \geq)$ for

i. $A = \{1, 2, 4, 8, 16\}$,

$(A, \leq)$: a $\quad a \leq b \iff a|b$

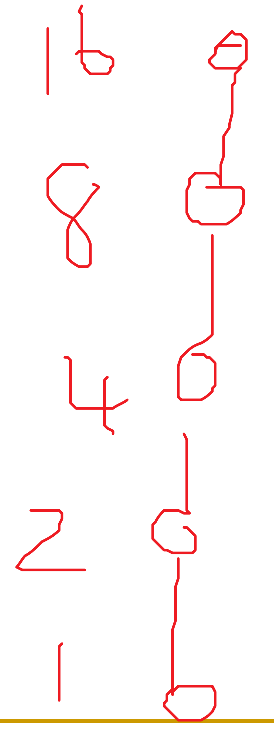
$R = \{(1, 1), (1, 2), (1, 4), (1, 8), (1, 16), (2, 2),$
$(2, 4), (2, 8), (2, 16), (4, 4), (4, 8),$
$(4, 16), (8, 8), (8, 16), (16, 16)\}$

$$a \leq b \iff a|b$$

$$1|2 \to 1 \leq 2$$

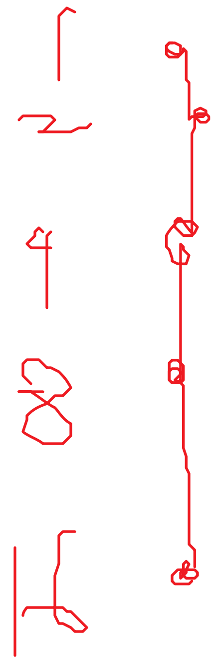
$$2|4 \to 2 \leq 4$$

$$4|8 \to 4 \leq 8$$



$(A, \geq)$:

$R = \{(1, 1), (2, 1), (2, 2), (4, 1), (4, 2), (4, 4),$
$(8, 1), (8, 2), (8, 4), (8, 8), (16, 1),$
$(16, 2), (16, 4), (16, 8), (16, 16)\}$

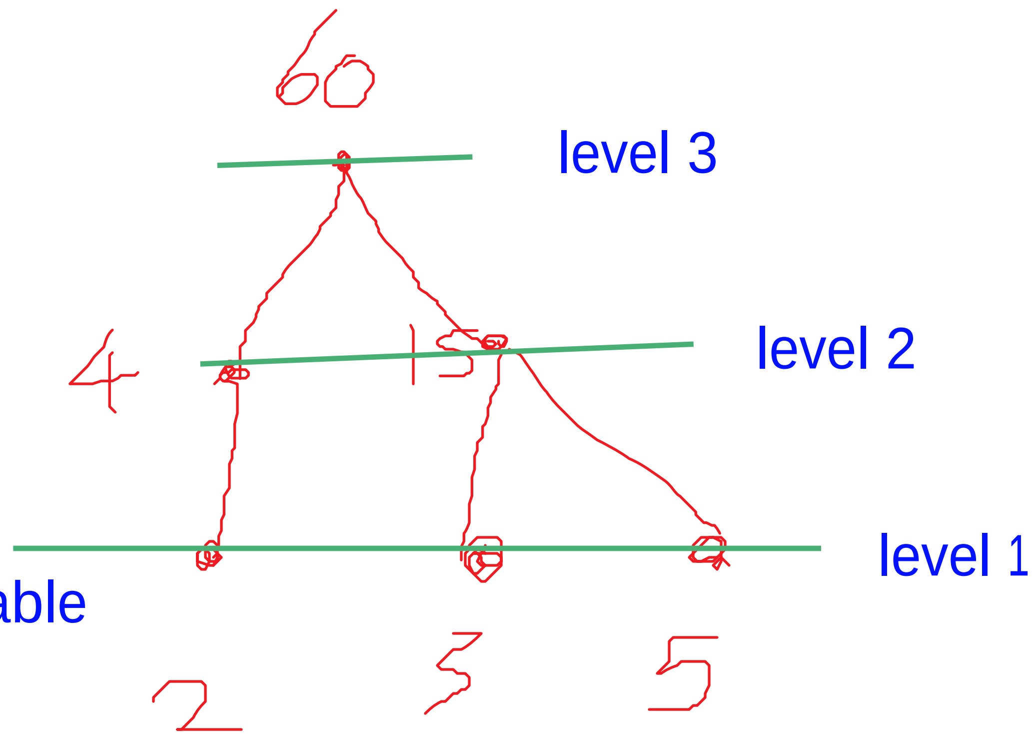


# E.g.2 (cont)

ii. $A = \{2, 3, 4, 5, 15, 60\}$.

## $(A, \leq)$:

$R$ = {(2, 2), (2, 4), (2, 60), (3, 3), (3, 15),
(3, 60), (4, 4), (4, 60), (5, 5), (5, 15),
(5, 60), (15, 15), (15, 60), (60, 60)}

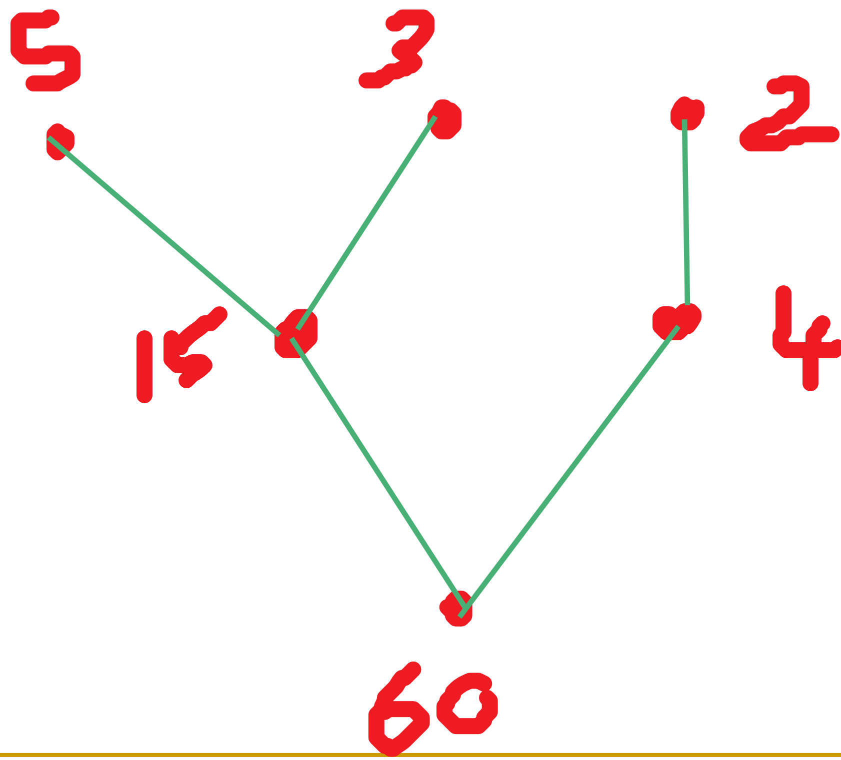



split 3
because not comparable

note:
no need to draw
green arrow and write level

$(A, \geq)$:

$R = \{(2, 2), (4, 2), (4, 4), (3, 3), (5, 5), (15, 3), (15, 5), (15, 15), (60, 2), (60, 3), (60, 4), (60, 5), (60, 15), (60, 60)\}$



# Notes:

- E.g.2 (i) is a finite linearly ordered set.
- If ($A$, $\leq$) is a poset and ($A$, $\geq$) is the dual poset, then the Hasse diagram of ($A$, $\geq$) is just the Hasse diagram of ($A$, $\leq$) turned upside down.

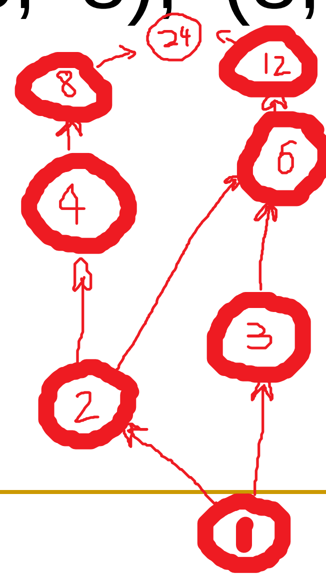
# E.g.3

Let $D_n$ denotes the set of positive divisor of $n$. Draw the Hasse diagrams of the posets $(D_{24}, |)$ and $(D_{30}, |)$.

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$(D_{24}, |)$:

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 8),$
$(1, 12), (1, 24), (2, 2), (2, 4), (2, 6), (2, 8),$
$(2, 12), (2, 24), (3, 3), (3, 6), (3, 12),$
$(3, 24), (4, 4), (4, 8), (4, 12), (4, 24), (6, 6),$
$(6, 12), (6, 24), (8, 8), (8, 24), (12, 12),$
$(12, 24), (24, 24)\}$

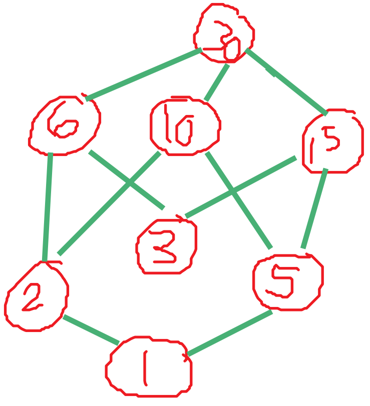



simplified version

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$(D_{30}, |)$:

$R = \{(1, 1), (1, 2), (1, 3), (1, 5), (1, 6), (1, 10), (1, 15), (1, 30), (2, 2), (2, 6), (2, 10), (2, 30), (3, 3), (3, 6), (3, 15), (3, 30), (5, 5), (5, 10), (5, 15), (5, 30), (6, 6), (6, 30), (10, 10), (10, 30), (15, 15), (15, 30), (30, 30)\}$




Its a 3D cube!

# 6.2 Hasse Diagram (cont)

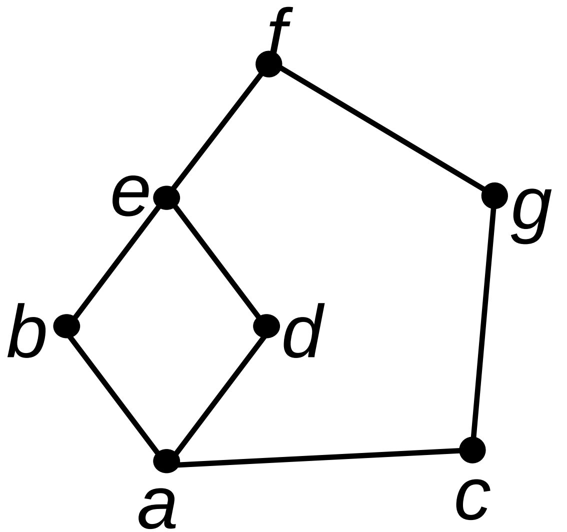
- If $A$ is a poset with partial order $\leq$, sometimes need to find a linear order $\prec$ for the set $A$ that will merely be an extension of the given partial order in the sense that if $a \leq b$, then $a \prec b$.
- The process of constructing a linear order such as $\prec$ is called a topological sorting.

# 6.2 Hasse Diagram (cont)

- The problem might arise when we have to enter a finite poset *A* into a computer.
  - The elements of *A* must be entered in some order, and we might want them entered so that the partial order is preserved.
  - If $a \leq b$, then $a$ is entered before $b$.
  - A topological sorting $\prec$ will give an order of entry of the elements that meets this condition.

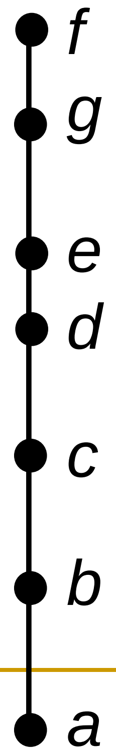
# 6.2 Hasse Diagram (cont)

- E.g. Refer to the following Hasse diagram.



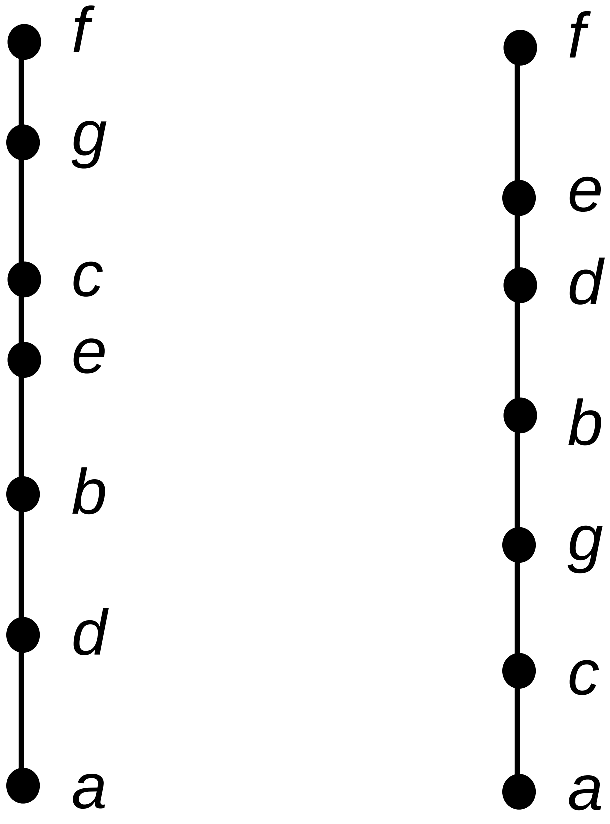
# 6.2 Hasse Diagram (cont)

The partial order $\prec$ whose Hasse diagram shown below is clearly a linear order, i.e. every pair in $\leq$ is also in the order $\prec$, so $\prec$ is a topological sorting of the partial order $\leq$.



# 6.2 Hasse Diagram (cont)

- Below are two other solutions to this problem.



# 6.2 Hasse Diagram (cont)

- Let $(A, \leq)$ and $(A', \leq')$ be posets and let $f : A \to A'$ be a one-to-one corresponding between $A$ and $A'$. The function $f$ is called an isomorphism from $(A, \leq)$ to $(A', \leq')$ if, for any $a$ and $b$ in $A$,

$$a \leq b \text{ if and only if } f(a) \leq' f(b).$$

# 6.2 Hasse Diagram (cont)

- If $f : A \rightarrow A'$ is an isomorphism, then $(A, \leq)$ and $(A', \leq')$ are isomorphic posets.
  - Let $A$ be the set $Z^+$ of positive integers, and let $\leq$ be the usual partial order on $A$. Let $A'$ be the set of positive even integers, and let $\leq'$ be the usual partial order on $A'$. Then the function $f : A \rightarrow A'$ is given by $f(a) = 2a$.

    Since $f$ is one-to-one, onto, and everywhere defined, $f$ is one-to-one corresponding.

    Also, $f(a) = 2a$, $f(b) = 2b$,

    so $a \leq b$ if and only if $f(a) \leq' f(b)$.

    Thus $f$ is an isomorphism.

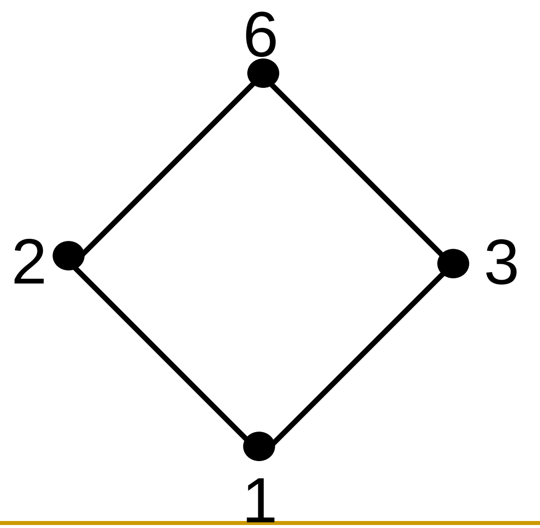
# 6.2 Hasse Diagram (cont)

- Theorem 1 Principle of Correspondence

  If the elements of $B$ have any property relating to one another or to other elements of $A$, and if this property can be defined entirely in terms of the relation $\leq$, then the elements of $B'$ must possess exactly the same property, defined in terms of $\leq'$.

# 6.2 Hasse Diagram (cont)

- Two finite isomorphic posets must have the same Hasse diagrams.
  - Let $A = \{1, 2, 3, 6\}$ and let $\leq$ be the relation | (divides). The Hasse diagram for $(A, \leq)$ is given as follows:



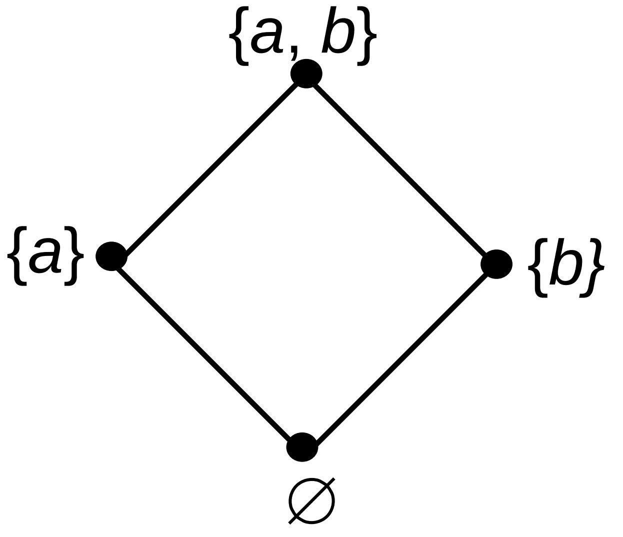
# 6.2 Hasse Diagram (cont)

Let $A' = \wp(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, and let $\leq'$ be set containment, $\subseteq$.

If $f : A \to A'$ is defined by $f(1) = \emptyset$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a, b\}$, then $f$ is one-to-one corresponding.

Since $x|y$ if and only if $f(x) \subseteq f(y)$, $f$ is order preserving. And if each label $a \in A$ of the Hasse diagram is replaced by $f(a)$ and the Hasse diagram for $(A', \leq')$ is obtained.

# 6.2 Hasse Diagram (cont)




Thus the function *f* is an isomorphism.

# 6.3 Extremal Elements of Partially Ordered Sets

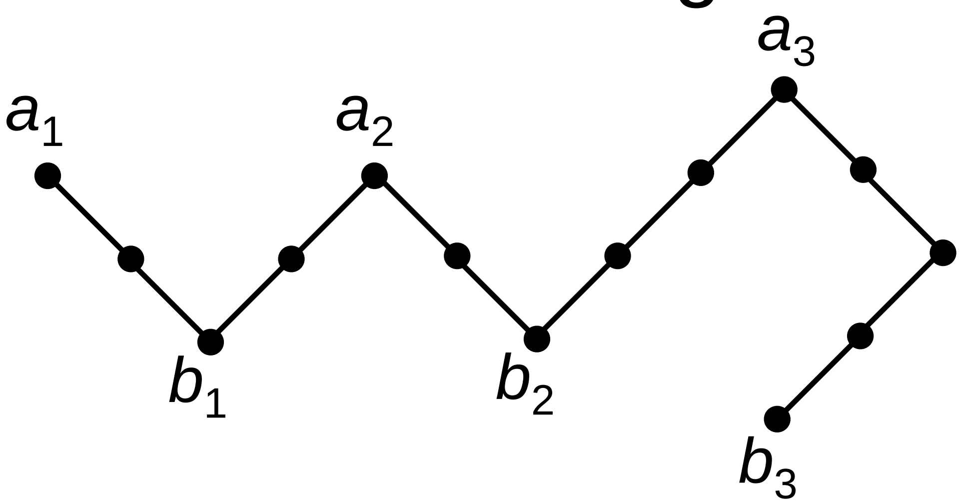
- Consider a poset $(A, \leq)$.
  - An element $a \in A$ is called a maximal element of $A$ if there is no element $c$ in $A$ such that $a < c$.
  - An element $b \in A$ is called a minimal element of $A$ if there is no element $c$ in $A$ such that $c < b$.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- If $(A, \leq)$ is a poset and $(A, \geq)$ is its dual poset,
  - an element $a \in A$ is a maximal element of $(A, \geq) \Leftrightarrow a$ is a minimal element of $(A, \leq)$.
  - an element $a \in A$ is a minimal element of $(A, \geq) \Leftrightarrow a$ is a maximal element of $(A, \leq)$.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Consider the following Hasse diagram.




  - The elements $a_1$, $a_2$, and $a_3$ are maximal elements of $A$, and the elements $b_1$, $b_2$, and $b_3$ are minimal elements.
  - Since there is no line between $b_2$ and $b_3$, neither $b_2 \leq b_3$ nor $b_3 \leq b_2$.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Let $A$ be the poset of nonnegative real numbers with the usual partial order $\leq$. Then 0 is a minimal element and there are no maximal elements of $A$.
- The poset Z with the usual partial order $\leq$ has no maximal elements and has no minimal elements.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Theorem 1

  Let $A$ be a finite nonempty poset with partial order $\leq$. Then $A$ has at least one maximal element and at least one minimal element.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- By using the concept of a minimal element, we can give an algorithm for finding a topological sorting of a given finite poset $(A, \leq)$.
  - If $a \in A$ and $B = A - \{a\}$, then $B$ is also a poset under the restriction of $\leq$ to $B \times B$.
  - Assume a linear array name SORT that produced is ordered by increasing index, that is SORT[1] $\prec$ SORT[2] $\prec$ …
  - The relation $\prec$ on $A$ defined in this way is a topological sorting of $(A, \leq)$

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Algorithm for finding a topological sorting of a finite poset $(A, \leq)$:

  Step 1: Choose a minimal element of $A$.

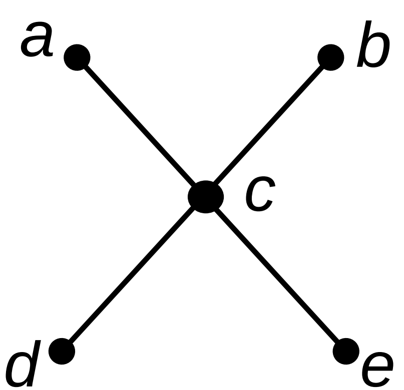
  Step 2: Make a next entry of SORT and replace $A$ with $A - \{a\}$.

  Step 3: Repeat steps 1 and 2 until $A = \{\ \}$.

  End of algorithm.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

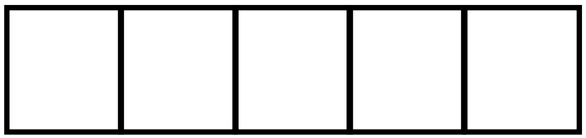
- Let $A = \{a, b, c, d, e\}$, and let the Hasse diagram of a partial order $\leq$ on $A$ be as shown below.



# 6.3 Extremal Elements of Partially Ordered Sets (cont)

A minimal element of this poset is the vertex labelled *d* (could also have chosen *e*). Put *d* in SORT [1] and show the following Hasse diagram of $A - \{d\}$.

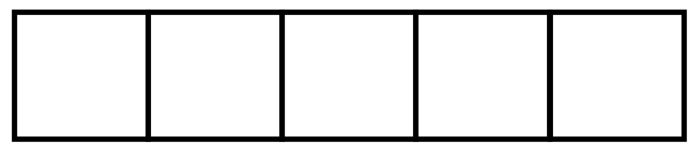
SORT



# 6.3 Extremal Elements of Partially Ordered Sets (cont)

A minimal element of the new $A$ is $e$, so $e$ becomes SORT [2], and $A - \{e\}$ is shown below.

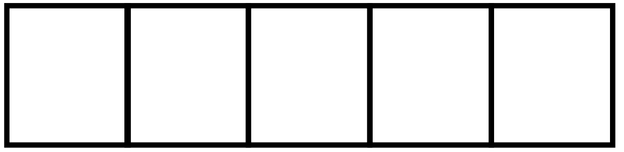
SORT



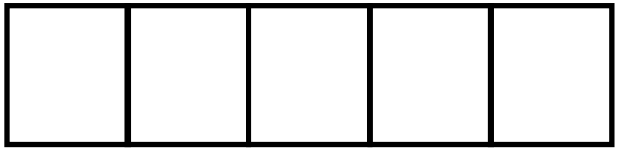
# 6.3 Extremal Elements of Partially Ordered Sets (cont)

The process continues until we have exhausted *A* and filled SORT.

SORT



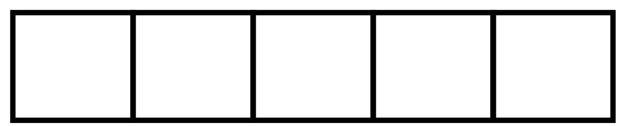
SORT



# 6.3 Extremal Elements of Partially Ordered Sets (cont)

The completed array SORT and the Hasse diagram of the poset corresponding to SORT is shown below. This is a topological sorting of $(A, \leq)$.

SORT



# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- An element $a \in A$ is called a greatest element of $A$ if $x \leq a$ for all $x \in A$.
- An element $a \in A$ is called a least element of $A$ if $a \leq x$ for all $x \in A$.
- An element $a$ of $(A, \leq)$ is a greatest (or least) element $\Leftrightarrow$ it is a least (or greatest) element of $(A, \geq)$.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Let $A$ be the poset of nonnegative real numbers with the usual partial order $\leq$. Then 0 is the least element and there is no greatest element.
- Let $S = \{a, b, c\}$ and the power set $A = \wp(S)$. Consider the poset $(A, \subseteq)$.

$A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

The empty set is a least element of $A$ and the set $S$ is a greatest element of $A$.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- The poset Z with usual partial order $\leq$ has neither a least nor greatest element.
- Theorem 2

  A poset has at most one greatest element and at most one least element.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- The greatest element of a poset, if exists, is denoted by *I* and is often called the unit element. The least element, if exists, is denoted by *0* and is often called the zero element.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Consider a poset $A$ and a subset $B$ of $A$.
  - An element $a \in A$ is called an upper bound of $B$ of $b \leq a$ for all $b \in B$.
  - An element $a \in A$ is called a lower bound of $B$ of $a \leq b$ for all $b \in B$.
- A subset $B$ of a poset may or may not have upper or lower bounds (in $A$). Moreover, an upper or lower bound of $B$ may or may not belong to $B$ itself.

# 6.3 Extremal Elements of Partially Ordered Sets (cont)

- Let $A$ be a poset and $B$ is a subset of $A$.
  - An element $a \in A$ is called a least upper bound of $B$, (LUB($B$)), if $a$ is an upper bound of $B$ and $a \leq a'$, whenever $a'$ is an upper bound of $B$.

    Thus $a$ = LUB($B$) if $b \leq a$ for all $b \in B$, and if whenever $a' \in A$ is also an upper bound of $B$, then $a \leq a'$.

# 6.4 Least Upper Bound and Greatest Lower Bound

- Let $A$ be a poset and $B$ is a subset of $A$.
  - An element $a \in A$ is called a least upper bound of $B$, (LUB($B$)), if $a$ is an upper bound of $B$ and $a \leq a'$, whenever $a'$ is an upper bound of $B$.

    Thus $a = \text{LUB}(B)$ if $b \leq a$ for all $b \in B$, and if whenever $a' \in A$ is also an upper bound of $B$, then $a \leq a'$.

# 6.4 Least Upper Bound and Greatest Lower Bound (cont)

- An element $a \in A$ is called a greatest lower bound of $B$, (GLB($B$)), if $a$ is a lower bound of $B$ and $a' \leq a$, whenever $a'$ is a lower bound of $B$.

  Thus $a$ = GLB($B$) if $a \leq b$ for all $b \in B$, and if whenever $a' \in A$ is also a lower bound of $B$, then $a' \leq a$.

# 6.4 Least Upper Bound and Greatest Lower Bound (cont)

- Upper bounds in $(A, \leq)$ correspond to lower bounds in $(A, \geq)$ (for the same set of elements), and lower bounds in $(A, \leq)$ correspond to upper bounds in $(A, \geq)$. Similar statements hold for greatest lower bounds and least upper bounds.
- Theorem 3

  Let $(A, \leq)$ be a poset. Then a subset $B$ of $A$ has at most one LUB and at most one GLB.

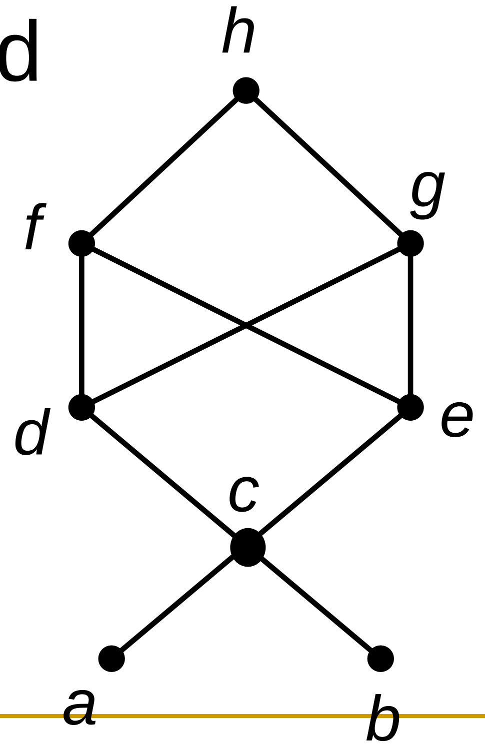
# 6.4 Least Upper Bound and Greatest Lower Bound (cont)

- In a finite poset $A$, as viewed from the Hasse diagram of $A$. Let $B = \{b_1, b_2, \ldots, b_r\}$. If $a = \text{LUB}(B)$, then $a$ is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by upward paths. Similarly if $a = \text{GLB}(B)$, then $a$ is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by downward paths.

# E.g.4

Consider the poset $A = \{a, b, c, d, e, f, g, h\}$, whose Hasse diagram is shown.

Let $B_1 = \{a, b\}$ and $B_2 = \{c, d, e\}$ be subsets of $A$. Find



# E.g.4 (cont)

i. upper and lower bounds of $B_1$ and $B_2$;

ii. all least upper bounds and greatest lower bounds of $B_1$ and $B_2$.

# E.g.4 (cont)

i. upper and lower bounds of $B_1$ and $B_2$;

upper bounds of $B_1$ =

lower bounds of $B_1$ =

upper bounds of $B_2$ =

lower bounds of $B_2$ =

# E.g.4 (cont)

ii. all least upper bounds and greatest lower bounds of $B_1$ and $B_2$.

LUB of $B_1$ =

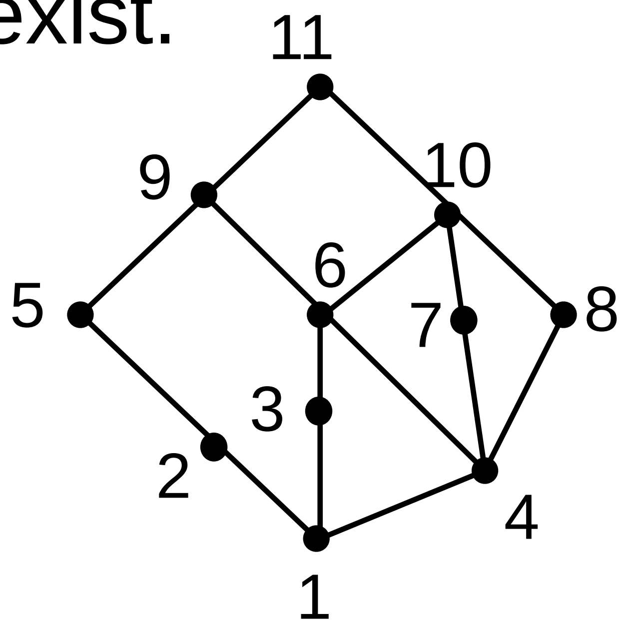
GLB of $B_1$ =

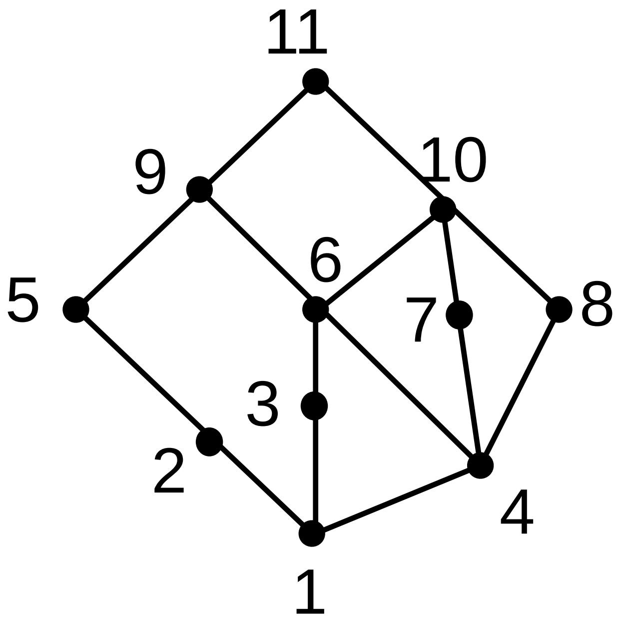
LUB of $B_2$ =

GLB of $B_2$ =

# E.g.5

Let $A = \{1, 2, 3, 4, 5, \ldots, 11\}$ be the poset whose Hasse diagram is shown below. Find the LUB and GLB of $B = \{6, 7, 10\}$, if they exist.






Exploring all upward paths from 6, 7, and 10

$\Rightarrow$ LUB ($B$) =

Exploring all downward paths from 6, 7, and 10 $\Rightarrow$ GLB ($B$) =