Calc II - C2: Differential Eq. with Modelling Apps

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1 Introduction

1. Differential equations ($D.E.$): Equations containing derivatives.
2. In this chapter: Find $y = f(x)$ which satisfies ($D.E.$)

1.1 Definitions

Term	Definition
Differential equations (D.E.)	Equation containing derivatives
Ordinary differential equation (O.D.E)	D.E. with only ONE independent variable
Order	Highest derivative presentation
Degree	Greatest power of highest order derivative

1.2 Example

Differential Equation	Independent Variables	Dependent Variables	Order	Degree
$\frac{d^2y}{dx^2} + x^3y = \sin x$	x	y	2(Note: $\frac{d^2y}{dx^2}$)	1(Note:$\left(\frac{d^2y}{dx^2}\right)^1$)
$\frac{d^3y}{dx^3} + 4y\left(\frac{dy}{dx}\right)^2 = x\frac{d^2y}{dx^2} + e^x$	x	y	3	1
$\frac{du}{dx} + \frac{dv}{dx} = e^x$	x	u, v	1	1
$\left(\frac{d^4s}{dt^4}\right)^2 + \left(\frac{d^2s}{dt^2}\right)^5 + \frac{ds}{dt} = 0$	t	s	4	2

1.3 Solution of D.E.

Definition 2.5:

1. **Solution of DE:** Function f (free of derivatives), satisfies identically a D.E.

1.4 Example

Show that the function $f(x) = 2x^3 - 5x + C$ for any real number C, is the solution of the $D.E.$ $y' = 6x^2 - 5$.

1.4.1 Solution

1. Let $y = 2x^3 - 5x + C$
2. Differentiate y with respect to x, we have
$$y' = 6x^2 - 5$$
3. Substitute into D.E., we have
$$6x^2 - 5 = 6x^2 - 5$$
4. Thus, $y = f(x) = 2x^3 - 5x + C$ for any real number C is the solution of $y' = 6x^2 - 5$
5. **General solution:** $y = f(x) = 2x^3 - 5x + C$.
   (a) Every solution is in this form.
6. **Particular solution:** Assigning C to specific value.

1.5 Example

Show that every member of the family of function $y\left(t\right)=\frac{1+ce^{t}}{1-ce^{t}}$ is a solution of the D.E. $y\prime(t)=\left(y^{2}-1\right)/2$.

1. Let $y=\frac{1+ce^{t}}{1-ce^{t}}$
2. Find the derivative

$$
\begin{aligned}
y' &= \frac{\left(1-ce^{t}\right)\frac{d}{dx}\left[1+ce^{t}\right]-\left(1+ce^{t}\right)\frac{d}{dx}\left[1-ce^{t}\right]}{\left(1-ce^{t}\right)^{2}} \\
&= \frac{\left(1-ce^{t}\right)\left(ce^{t}\right)-\left(1+ce^{t}\right)\left(-ce^{t}\right)}{\left(1-ce^{t}\right)^{2}} \\
&= \frac{ce^{t}-ce^{2t}-\left(-ce^{t}-ce^{2t}\right)}{\left(1-ce^{t}\right)^{2}} \\
&= \frac{ce^{t}-ce^{2t}+ce^{t}+ce^{2t}}{\left(1-ce^{t}\right)^{2}} \\
y' &= \frac{2ce^{t}}{\left(1-ce^{t}\right)^{2}}
\end{aligned}
$$

3. Simplify the second equation, to make them equal

$$
\begin{aligned}
y' &= \frac{\left(y^{2}-1\right)}{2} \\
&= \frac{\left(\left(\frac{1+ce^{t}}{1-ce^{t}}\right)^{2}-1\right)}{2} \\
&= \frac{\left(\frac{1+ce^{t}}{1-ce^{t}}\right)^{2}-1}{2} \\
&= \frac{\frac{\left(1+ce^{t}\right)^{2}}{\left(1-ce^{t}\right)^{2}}-1}{2} \\
&= \frac{\frac{\left(1+ce^{t}\right)^{2}}{\left(1-ce^{t}\right)^{2}}-1}{2} \\
&= \frac{\left(1+ce^{t}\right)^{2}-\left(1-ce^{t}\right)^{2}}{2\left(1-ce^{t}\right)^{2}} \\
&= \frac{1+2ce^{t}+ce^{2t}-\left(1-2ce^{t}+ce^{2t}\right)}{2\left(1-ce^{t}\right)^{2}} \\
&= \frac{1+2ce^{t}+ce^{2t}-1+2ce^{t}-ce^{2t}}{2\left(1-ce^{t}\right)^{2}} \\
&= \frac{4ce^{t}}{2\left(1-ce^{t}\right)^{2}} \\
y' &= \frac{2ce^{t}}{\left(1-ce^{t}\right)^{2}}
\end{aligned}
$$

4. Conclusion, since both of the equations result in the same y', every member of the family of function $y\left(t\right)=\frac{1+ce^{t}}{1-ce^{t}}=\frac{u}{v}$ is a solution of $y'\left(t\right)$.

1.6 Example (Continue from 1.5)

Given $y(0)=2$ (initial condition), solve the initial value problem $y\prime=\frac{\left(y^{2}-1\right)}{2}$.

$$
\begin{aligned}
y\left(0\right) &= \frac{1+ce^{0}}{1-ce^{0}} \\
&= \frac{1+c}{1-c}
\end{aligned}
$$

$$
\begin{aligned}
2 &= \frac{1+c}{1-c} \\
2\left(1-c\right) &= 1+c \\
2-2c &= 1+c \\
2-1 &= c+2c \\
1 &= 3c \\
c &= \frac{1}{3}
\end{aligned}
$$

1. Substitute the c into the equation

$$
y\left(t\right) = \frac{1+\frac{1}{3}e^{t}}{1-\frac{1}{3}e^{t}}
$$

2 Modelling with Differential Equations

1. **Mathematical model:** Mathematical description of a real-world phenomenon.

2.1 Models of Population Growth

1. **Assumption**: Population grows at rate proportional to size of population.
2. Variables
   (a) $t = time$ (the independent variable)
   (b) $P =$ the number of individuals in the population (dependent variable)
3. Rate of growth: $\frac{dP}{dt}$

$$
\begin{aligned}
&\frac{dP}{dt} \propto P \\
&\frac{dP}{dt} = kP, k = \text{ proportionality constant}
\end{aligned}
$$

2.1.1 Example

Show that any exponential function of the form $P\left(t\right) = Ce^{kt}$ is a solution of $\frac{dP}{dt} = kP$

1. Find the derivative

$$
\begin{aligned}
P\left(t\right) &= Ce^{kt} \\
P'\left(t\right) &= k\left(Ce^{kt}\right) \\
&= kP
\end{aligned}
$$

2.2 A Model for the Motion of A Spring

1. Consider the motion of an object with mass m at the end of a vertical spring.
2. If the spring is stretched (or compressed) x units from its natural length, then, by Hooke's Law, it exerts a force that is proportional to x:

$$
Restoring\, force = -kx
$$

   Where k is a positive constant (called the spring constant).
3. By Newton's Second Law ($F = ma$), we have

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &= -kx \\
\frac{d^2x}{dt^2} &= -\frac{k}{m}x
\end{aligned}
$$

2.2.1 Example

Find the nonzero values of k in function $x = \sin kt$ which satisfy the $D.E.$, $\frac{d^2x}{dt^2} = -9x$

1. $x = \sin kt$

$$\begin{aligned}
\frac{dx}{dt} &= k\cos kt \\
\frac{d^2x}{dt^2} &= k\left(-k\sin kt\right) \\
\frac{d^2x}{dt^2} &= -k^2\sin kt \\
&= -k^2x \\
-9x &= -k^2x \\
9 &= k^2 \\
k &= 3
\end{aligned}$$

2.3 Direction fields

1. Impossible to obtain an explicit formula for most differential equation
2. Another way to learn solution through graphical approach.
3. **Example**: $y' = F(x, y)$
   (a) $F(x, y) \equiv$ Slope of a solution curve at a point (x, y)
   (b) **Solution curve**: Graph of solution of ODE
   (c) **Direction/slope field:** Drawing of short line segments with slope $F(x, y)$ at several points.
      i. Indicates direction the solution curve is heading
      ii. Visualize general shape of solution curve.

2.3.1 Example

1. Sketch the direction field for the DE $y\prime = x^2 + y^2 - 1$
2. Use part (a) to sketch the solution curve that passes through the origin.
3. Solution:

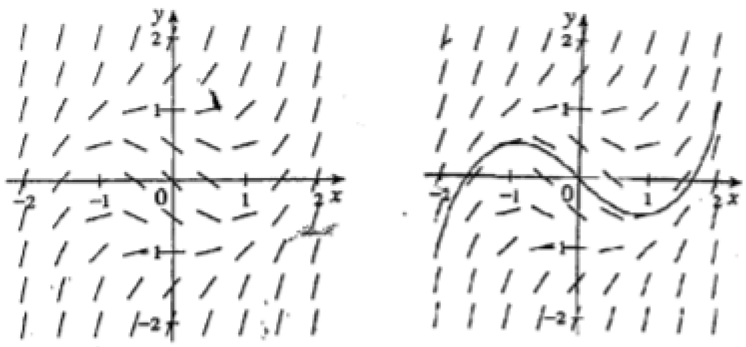
(a)

x	-2	-1	0	1	2
y	0	0	0	0	0
y'	3	0	-1	0	3

(b)

x	-2	-1	0	1	2
y	1	1	1	1	1
y'	4	1	0	1	4

(c) Notice that we orignally are required to find the other values, however, remember that $(-x)^2 = x^2$ and $(-y)^2 = y^2$. Therefore, we could omit inverse values since they are simply the same. Besides that, as you might see, all the slope are slowly increasing as the x and y increases. Therefore, we could start graphing it out.



(d)

2.3.2 Equilibrium solutions

1. An ODE of the form $y' = F(y)$ in which the independent variable is missing from the right side, is called **autonomous**.
2. For any autonomous equation $y\prime = F(y)$, if $F(c) = 0$, then a constant solution $y = c$ of the ODE is called an **equilibrium solution**.
3. Example: Which of the following DE are autonomous? Determine the equilibrium solution of each autonomous equation.
   (a) $\frac{dy}{dx} = 1 - y^2$
      i. Autonomous.

ii. Finding equilibrium solution

$$1 - y^2 = 0$$
$$y = 1, -1$$

(b) $\frac{dx}{dt} = 1 + t^3$

i. Not autonomous.

(c) $\frac{dP}{dt} = P\left(1 - \frac{P}{K}\right)$, K is a constant

i. Autonomous.

$$P\left(1 - \frac{P}{K}\right) = 0$$
$$P - \frac{P^2}{K} = 0$$
$$\frac{P^2}{K} = P$$
$$P^2 = PK$$
$$K = \frac{P^2}{P}$$
$$K = P$$
$$P = K$$

(d) $\frac{dy}{dx} = 2xy$

i. Not autonomous.

2.4 Euler's Method

1. Numerical process to generate table of approximate values of the function that solves the initial value problem $\frac{dy}{dx} = y' = F(x, y)$,and $y(x_0) = y_0$.

2. Iteration formula for Euler's method

   (a) $x_n = x_{n-1} + h$

   i. $TL; DR$: The next x value is simply the previous one + step size.

   (b) $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

   i. $TL; DR$: The next y value is simply the previous one + step size * the gradient of the previous step. Remember that $\frac{dy}{dx} = F(x, y)$ as per point 1.

   (c) h is the horizontal distance called the step size.

3. The smaller the step size, h, the better the approximation

2.4.1 Example

Use Euler's method to find approximate values for the solution of the initial value problem $F = \frac{dy}{dx} = x - y, y(0) = 1$ on the interval $[0, 1]$ using five steps of size $h = 0.2$.

1.

n	Xn	$y_n = y_{n-1} + 0.2(x_{n-1} - y_{n-1})$
0	0	1
1	.2	$1 + 0.2(-1) = 1 - 0.2 = 0.8$
2	.4	$0.8 + 0.2(.2 - .8) = 0.68$
3	.6	$0.68 + 0.2(.4 - .68) = 0.624$
4	.8	$0.624 + 0.2(.6 - 0.624) = 0.6192$
5	1	$0.6192 + 0.2(.8 - 0.6192) = 0.65536$

2.4.2 Example

Use Euler's method with $n = 5$ to approximate the solution of the initial value problem $\frac{dy}{dx} = -2xy^2, y(0) = 1$ on the interval $[0, 0.5]$. Find the actual solution of the initial value problem. Finally, sketch the graphs of the approximate solutions and the actual solution for $0 \leq x \leq 0.5$ on the same set of axes.

1. Approximate solution

(a)

n	Xn	$y_n = y_{n-1} + 0.1\left(-2x_{n-1}y_{n-1}^2\right)$
0	0	1
1	0.1	$1 + 0.1\left(-2\left(0\right)\left(1\right)^2\right) = 1$
2	0.2	$1 + 0.1\left(-2\left(0.1\right)\left(1\right)^2\right) = 0.98$
3	0.3	$0.98 + 0.1\left(-2\left(0.2\right)\left(0.98\right)^2\right) = 0.9416$
4	0.4	$0.9416 + 0.1\left(-2\left(0.3\right)\left(0.9416\right)^2\right) = 0.8884$
5	0.5	$0.8884 + 0.1\left(-2\left(0.4\right)\left(0.8884\right)^2\right) = 0.8253$

2. Actual solution (Method: Separation of variables)

$$\begin{aligned}\frac{dy}{dx} &= -2xy^2\\ dy &= -2xy^2dx\\ \frac{1}{y^2}dy &= -2xdx\\ \frac{1}{y^2}dy + 2xdx &= 0\end{aligned}$$

(a) Now integrate them

$$\begin{aligned}\int\left(\frac{1}{y^2}dy + 2xdx\right) &= 0\\ \int y^{-2}dy + \int 2xdx &= 0\\ -\frac{1}{y} + c + \frac{2x^2}{2} + c &= 0\\ -\frac{1}{y} + x^2 + c &= 0\end{aligned}$$

(b) Then when $x = 0, y = 1$, so:

$$\begin{aligned}-\frac{1}{y} + x^2 + c &= 0\\ -1 + 0^2 + c &= 0\\ c &= 1\end{aligned}$$

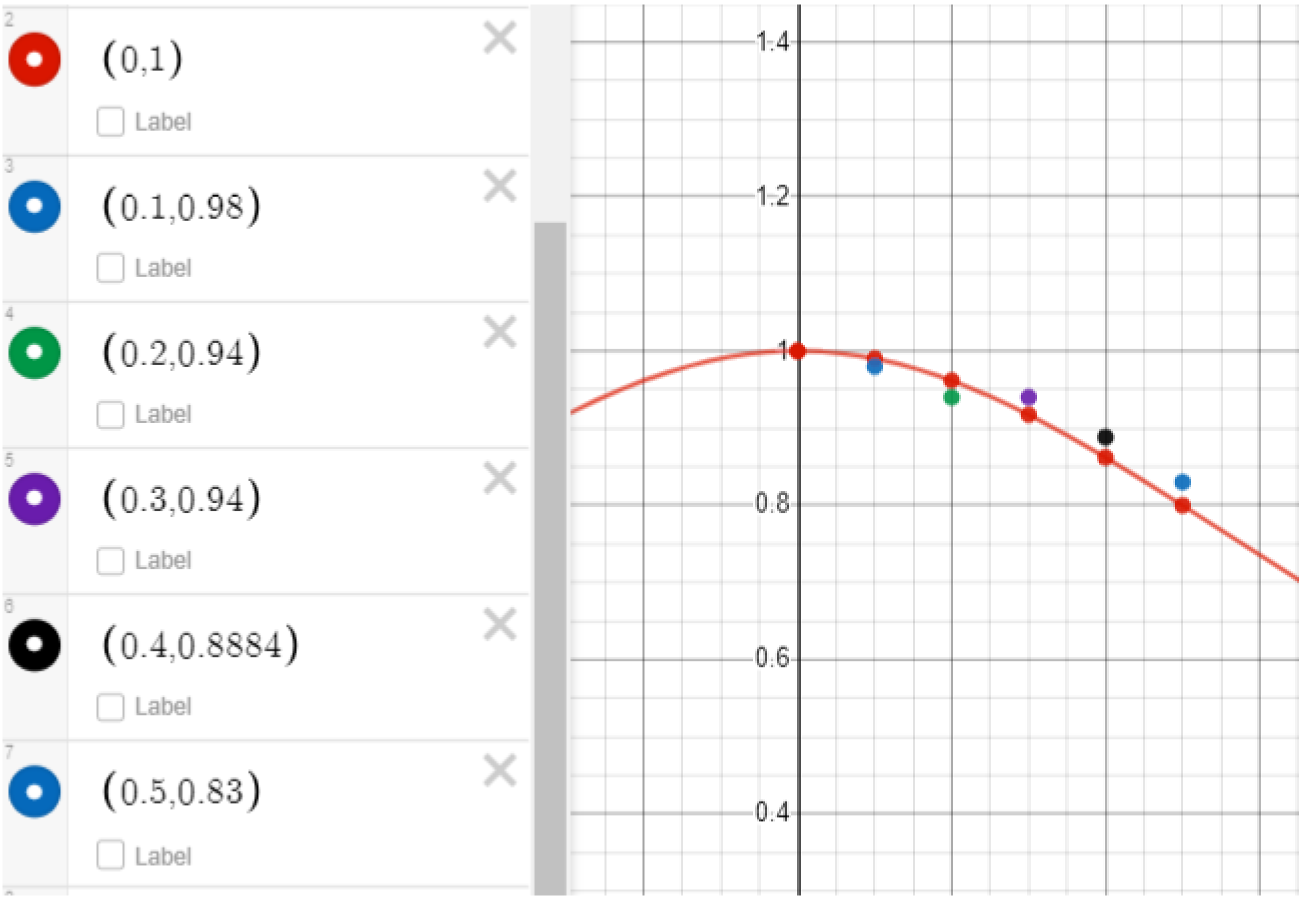
(c) Finally, we arrive at our equation

$$\begin{aligned}-\frac{1}{y} + x^2 + 1 &= 0\\ -\frac{1}{y} &= -\left(x^2 + 1\right)\\ \frac{1}{y} &= x^2 + 1\\ y &= \frac{1}{x^2 + 1}\end{aligned}$$

(d) Lets check our answers

x	$\frac{1}{x^2+1}$
0	1
0.1	0.99009901
0.2	0.96153846
0.3	0.91743119
0.4	0.86206897
0.5	0.8

i.

ii. 

(e) That's pretty close!

2.4.3 Example

Consider the initial-value problem $\frac{dy}{dx} = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. Use Euler's method to obtain an approximation to $y(2.5)$ using $h = 0.1$.

1.

N	x_n	$y_n = y_{n-1} + 0.1\left(0.1\sqrt{y_{n-1}} + 0.4x_{n-1}^2\right)$
0	2	4
1	2.1	$y_n = 4 + 0.1\left(0.1\sqrt{4} + 0.4(2)^2\right) = 4.18$
2	2.2	$y_n = 4.18 + 0.1\left(0.1\sqrt{4.18} + 0.4(2.1)^2\right) = 4.3768$
3	2.3	$y_n = 4.3768 + 0.1\left(0.1\sqrt{4.3768} + 0.4(2.2)^2\right) = 4.5913$
4	2.4	$y_n = 4.5913 + 0.1\left(0.1\sqrt{4.5913} + 0.4(2.3)^2\right) = 4.8243$
5	2.5	$y_n = 4.8243 + 0.1\left(0.1\sqrt{4.8243} + 0.4(2.4)^2\right) = 5.0767$

2.5 First Order Linear Differential Equations

2.5.1 Definition

An n-th order linear Ordinary Differential Equation (O.D.E.) has the form of:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

1. A linear O,D,E, is said to be homogenous (or alike) if $f(x) = 0$, and vice versa.

Important properties of linear O.D.E.

1. Dependent variables & derivatives are ONLY **power of one**.
2. Non-linear O.D.E. = ...O.D.E. that is not linear
3. Coefficient of derivatives and y (A.K.A the multiplier) must be function of independent variable x ONLY.
4. Function on RHS must all be functions of independent variable x.
5. Constant can be considered funciton of x, e.g. $a(x) = 4$

Example

D.E.	Order	Linear/Non-linear	Homogeneous/Non-homogenous (only for linear D.E.)
$x^2\frac{dy}{dx}+(\sin x)\,y=0$	1	Linear	Homogeneous
$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$	2	Non-linear	N/A
$\frac{d^4y}{dx^4}+x^2\frac{d^3y}{dx^3}-x^3\frac{dy}{dx}=xe^x$	4	Linear	Non-homogenous
$\frac{d^2y}{dx^2}+\left(\frac{d^2}{dx^2}\right)^2+6y=x$	2	Non-linear	N/A

2.5.2 Definition

A first order linear D.E. is in the form:

$$a_1(x)\frac{dy}{dx}+a_0(x)\,y=f(x)\,,where\,a_1(x)\neq 0$$

1. Dividing by $a_1(x)$, we have

$$\frac{dy}{dx}+p(x)\,y=q(x)$$

   (a) Where $p(x)$ and $q(x)$are continuous functions.

2. **Standard form of F.O.D.E.:** $\frac{dy}{dx}+p(x)\,y=q(x)$

Solution Process

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
2. Find the integrating factor $\mu(x)=e^{\int p(x)dx}$.
   (a) Note: For this step, the $+c$ is omitted when integrating, else in the end you'll get an extra c somewhere as a constant. That would be a waste of time.
3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution $y(t)$

2.5.3 Example 1

Solve $\frac{dy}{dx}+3y=e^{2x}$.

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
2. Find the integrating factor $\mu(x)=e^{\int p(x)dx}$.

$$\frac{dy}{dx}+3y=e^{2x},p(x)=3$$
$$\mu(x)=e^{\int 3dx}$$
$$\mu(x)=e^{3x}$$

3. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$e^{3x}\left(\frac{dy}{dx}+3y\right)=e^{3x}\left(e^{2x}\right)$$
$$e^{3x}\frac{dy}{dx}+3e^{3x}y=e^{3x}\left(e^{2x}\right)$$
$$\frac{d}{dx}\left(e^{3x}y\right)=e^{5x}\quad\left(\text{if you didn't notice, }\frac{d}{dx}(ab)=a'b+ab'\right)$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\int\frac{d}{dx}\left(e^{3x}y\right)dx=\int e^{5x}dx$$
$$e^{3x}y=\frac{1}{5}e^{5x}+c$$

5. Solve for the solution $y(t)$

$$y=\frac{1}{5}e^{2x}+ce^{-3x}$$

2.5.4 Example 2

Solve $x\frac{dy}{dx}+y=x^3$

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} x\frac{dy}{dx}+y&=x^3\\ \frac{x\frac{dy}{dx}+y}{x}&=\frac{x^3}{x}\\ \frac{dy}{dx}+\frac{y}{x}&=x^2, p\left(x\right)=\frac{1}{x} \end{aligned}$$

2. Find the integrating factor $\mu\left(x\right)=e^{\int p(x)dx}$.

$$\begin{aligned} \mu\left(x\right)&=e^{\int\frac{1}{x}dx}\\ \mu\left(x\right)&=e^{\ln x}\\ &=x \end{aligned}$$

3. Multiply everythin in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned} x\left(\frac{dy}{dx}+\frac{y}{x}\right)&=x\left(x^2\right)\\ x\frac{dy}{dx}+y&=x^3\\ \frac{d}{dx}\left[xy\right]&=x^3 \end{aligned}$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\begin{aligned} \int\frac{d}{dx}\left[xy\right]dx&=\int x^3dx\\ xy&=\frac{x^4}{4}+c \end{aligned}$$

5. Solve for the solution $y\left(t\right)$

$$y=\frac{1}{4}x^3+\frac{c}{x}$$

2.5.5 Example 3

Solve $x\frac{dy}{dx}+2y=x^2-x+1,\ y\left(1\right)=\frac{1}{2}$.

1. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} \frac{1}{x}\left(x\frac{dy}{dx}+2y\right)&=\frac{1}{x}\left(x^2-x+1\right)\\ \frac{dy}{dx}+\frac{2}{x}y&=x-1+\frac{1}{x}, p\left(x\right)=\frac{2}{x} \end{aligned}$$

2. Find the integrating factor $\mu\left(x\right)=e^{\int p(x)dx}$.

$$\begin{aligned} \mu\left(x\right)&=e^{\int\frac{2}{x}dx}\\ \mu\left(x\right)&=e^{\ln x^2}\\ \mu\left(x\right)&=x^2 \end{aligned}$$

3. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned} x^2\left(\frac{dy}{dx}+\frac{2}{x}y\right)&=x^2\left(x-1+\frac{1}{x}\right)\\ x^2\frac{dy}{dx}+x^2\frac{2}{x}y&=x^3-x^2+x\\ x^2\frac{dy}{dx}+2xy&=x^3-x^2+x\\ \frac{d}{dx}\left(x^2y\right)&=x^3-x^2+x \end{aligned}$$

4. Integrate both sides, make sure you properly deal with the constant of integration.

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int x^3 - x^2 + x dx$$
$$x^2 y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c$$

5. Solve for the solution $y(t)$

$$y = \frac{1}{x^2}\left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c\right)$$
$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{c}{x^2}$$

   (a) When $y(1) = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c$$
$$c = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$$
$$c = \frac{1}{12}$$

   (b) The solution

$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{1}{12x^2}$$

2.5.6 Example

Solve the equation $(x-2)\frac{dy}{dx} - y = (x-2)^3$

$$\frac{dy}{dx} - \frac{y}{(x-2)} = \frac{(x-2)^3}{(x-2)}$$
$$\frac{dy}{dx} - \frac{y}{(x-2)} = (x-2)^2, p(x) = -\frac{1}{(x-2)}$$

1. Find the integrating factor

$$\mu(x) = e^{\int -(x-2)^{-1} dx}$$
$$= e^{-\ln(x-2)}$$
$$= e^{\ln(x-2)^{-1}}$$
$$\mu(x) = \frac{1}{x-2}$$

2. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x) y]$ and write it as such.

$$\frac{dy}{dx} - \frac{y}{(x-2)} = (x-2)^2$$
$$\frac{1}{x-2}\left(\frac{dy}{dx} - \frac{y}{(x-2)}\right) = \frac{1}{x-2}(x-2)^2$$
$$\frac{1}{x-2}\frac{dy}{dx} - \frac{1}{(x-2)}\frac{y}{(x-2)} = x-2$$
$$(x-2)^{-1}\frac{dy}{dx} - y(x-2)^{-2} = x-2$$
$$\int \frac{d}{dx}\left[(x-2)^{-1} y\right] dx = \int x - 2dx$$
$$\frac{y}{x-2} = \frac{x^2}{2} - 2x + c$$
$$y = \left(\frac{x^2}{2} - 2x + c\right)(x-2)$$

2.6 Seperable Equations

Definition 2.8 A separable D.E. is any D.E. that we can write in the form of

$$g(y)\frac{dy}{dx} = f(x)$$

Solution Process

1. Write D.E. as $g(y)\,dy = f(x)\,dx$
2. Integrate both sides

$$\int g(y)\,dy = \int f(x)\,dx + C$$

3. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)
   (a) DO NOT forget to include C, constant of integration

2.6.1 Example

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{y+1} \\ (y+1)\,dy &= 2xdx \\ \int (y+1)\,dy &= \int 2xdx \\ \frac{y^2}{2} + y &= \frac{2x^2}{2} + c \\ y &= x^2 - \frac{1}{2}y^2 + c \end{aligned}$$

2.6.2 Example

Solve $\frac{dy}{dx} = \frac{3y-1}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3y-1}{x} \\ \int \frac{1}{3y-1}dy &= \int \frac{1}{x}dx \\ \frac{1}{3}\ln(3y-1) &= \ln x + c \\ \ln(3y-1) &= 3(\ln x + c) \\ \ln(3y-1) &= \ln x^3 + \ln A, A = e^{3c} \\ 3y - 1 &= Ax^3 \\ y &= \frac{A}{3}x^3 + \frac{1}{3} \end{aligned}$$

2.6.3 Example

Solve the differential equation $y^2\frac{dy}{dx} = x^2 + 1$ given that $y = 1$ when $x = 2$

$$\begin{aligned} y^2\frac{dy}{dx} &= x^2 + 1 \\ y^2 dy &= \left(x^2 + 1\right)dx \\ \int y^2 dy &= \int x^2 + 1dx \\ \frac{y^3}{3} &= \frac{x^3}{3} + x + c \\ y^3 &= x^3 + 3x + c \\ y &= \sqrt[3]{x^3 + 3x + c} \end{aligned}$$

2.6.4 Example

Solve $\frac{dy}{dx} = e^{3x-5y}$

$$\begin{aligned}
\frac{dy}{dx} &= e^{3x-5y} \\
&= \frac{e^{3x}}{e^{5y}} \\
e^{5y} dy &= e^{3x} dx \\
\int e^{5y} dy &= \int e^{3x} dx \\
\frac{1}{5} e^{5y} &= \frac{1}{3} e^{3x} + c \\
e^{5y} &= \frac{5}{3} e^{3x} + c
\end{aligned}$$

2.6.5 Example

Solve $4ydx - (1+x)\,dy = 0, y(0) = 1$

$$\begin{aligned}
4ydx &= (1+x)\,dy \\
\int \frac{1}{1+x} dx &= \int \frac{1}{4y} dy \\
\ln(1+x) &= \frac{1}{4} \ln y + c
\end{aligned}$$

1. $y(0) = 1$

$$\begin{aligned}
\ln(1) &= \frac{1}{4} \ln 1 + c \\
c &= 0
\end{aligned}$$

2. Solve the rest of the equation

$$\begin{aligned}
\ln(1+x) &= \frac{1}{4} \ln y \\
1 + x &= y^{\frac{1}{4}} \\
y &= (1+x)^4
\end{aligned}$$

2.7 Second Order Differential Equation

A second order O.D.E. has the form

$$a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)\, y = f(x)$$

1. If $a_2(x), a_1(x)$and $a_0(x)$ are constants, then we have a second order D.E. with constant coefficients.

Definition 2.9

1. Second Order Linear D.E.
   (a) $ay'' + by' + cy = f(x)$ where a, b, c are constants and $a \neq 0$.
   (b) **Homogeneous:** $f(x) = 0$, and vice versa

2.7.1 Second Order Homogeneous Linear D.E.

1. **Form**: $ay'' + by' + cy = 0$
2. **General solution**:
   (a) $y = C_1 y_1 + C_2 y_2$, where $y_1 \neq k y_2$ for any constant k, if $y_1$and y_2 are the solutions.
3. We assume all solutions to the D.E. will be of the form $y = e^{rx}$, r is a constant.

$$\begin{aligned}
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2 e^{rx}
\end{aligned}$$

4. Plug these into the D.E.

$$\begin{aligned}
ar^2 e^{rx} + bre^{rx} + ce^{rx} &= 0 \\
e^{rx}\left(ar^2 + br + c\right) &= 0
\end{aligned}$$

(a) Since $e^{rx} \neq 0$,

$$ar^2 + br + c = 0$$

i. Like, if $AB = 0$, either A or B must be 0.

5. **Characteristic/auxiliary equation:** $ar^2 + br + c = 0$ for any $D.E.$ $ay'' + by' + c = 0$

(a) Since it is quadratic, we will have two roots, r_1 and r_2.

(b) Can be obtained with $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, roots can be either

i. Real, distinct roots $r_1 \neq r_2 \left(b^2 - 4ac > 0\right)$

ii. Repeated real roots, $r_1 = r_2$ $\left(b^2 - 4ac = 0\right)$

iii. Complex roots, $r_{1,2} = \alpha \pm \beta i$ $\left(b^2 - 4ac < 0\right)$

6. Note: In the following, C_1 and C_2 are arbitrary constants.

Real, distinct roots, $r_1 \neq r_2$ We have $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$,

1. **General solution**: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ (<< MUST REMEMBER, NOT GIVEN IN EXAMS)

Example

1. Find the general solution of

(a) $y'' - 3y' - 10y = 0$

$$\begin{aligned} r^2 - 3r - 10 &= 0 \\ (r-5)(r+2) &= 0 \\ r_1 = 5, r_2 &= -2 \\ y &= C_1 e^{5x} + C_2 e^{-2x} \end{aligned}$$

(b) $2y'' - 5y' - 3y = 0$

$$\begin{aligned} 2r^2 - 5r - 3 &= 0 \\ (2r+1)(r-3) &= 0 \\ r = -\frac{1}{2}, r &= 3 \\ y &= C_1 e^{-\frac{1}{2}x} + C_2 e^{3x} \end{aligned}$$

2. Solve the following $I.V.P.$ $y'' + 11y' + 24 = 0$, given $y(0) = 0$, $y'(0) = -7$.

$$\begin{aligned} r^2 + 11r + 24 &= 0 \\ (r+8)(r+3) &= 0 \\ r_1 = -8, r_2 &= -3 \\ y &= C_1 e^{-8x} + C_2 e^{-3x} \\ y(0) = 0 &\rightarrow C_1 e^0 + C_2 e^0 = 0 \rightarrow C_1 + C_2 = 0 \quad Ⓡ \\ y'(0) = -7 &\rightarrow y' = -8C_1 - 3C_2 = -7 \quad Ⓢ \end{aligned}$$

(a) Solve Ⓡ and Ⓢ

$$C_1 = -C_2$$

$$\begin{aligned} -8C_1 + 3C_1 &= -7 \\ -5C_1 &= -7 \\ C_1 &= \frac{7}{5} \end{aligned}$$

$$C_2 = -\frac{7}{5}$$

(b) Find the final equation

$$y = \frac{7}{5} e^{-8x} - \frac{7}{5} e^{-3x}$$

2.7.1.2 Repeated real root, $r_1 = r_2$

1. General solution: $y = C_1 e^{rx} + C_2 x e^{rx}$ OR $y = e^{rx}(C_1 + C_2 x)$

Example 1:

1. Find the general solution for
   (a) $y'' - 6y' + 9y = 0$
   (b) $y'' - 10y' + 25y = 0$
2. Answer
   (a)

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \\ r &= 3 \\ y &= e^{3x}\left(C_1 + C_2 x\right) \end{aligned}$$

   (b)

$$\begin{aligned} r^2 - 10r + 25 &= 0 \\ (r-5)(r-5) &= 0 \\ (r-5)^2 &= 0 \\ r &= 5 \end{aligned}$$

$$y = e^{5x}\left(C_1 + C_2 x\right)$$

Example 2:

1. Question: Solve the following I.V.P. $y'' - 5y' + 4 = 0$, given $y(0) = 12, y'(0) = -3$

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ r &= 2 \\ y &= e^{2x}\left(C_1 + C_2 x\right) \end{aligned}$$

   (a) $y(0) = 12$

$$\begin{aligned} y(0) &= 12 \\ e^0\left(C_1 + C_2 0\right) &= 12 \\ C_1 &= 12 \end{aligned}$$

   (b) Substitute into the equation, then solve using $y'(0) = -3$

$$\begin{aligned} y &= e^{2x}\left(12 + C_2 x\right) \\ y'(0) &= 2e^{2x} \cdot \left(12 + C_2 x\right) + e^{2x}\left(C_2\right) \\ -3 &= 2 \cdot (12) + C_2 \\ -3 &= 24 + C_2 \\ C_2 &= -27 \end{aligned}$$

   (c) Find the final equation

$$y = e^{2x}\left(12 - 27x\right)$$

2.7.1.3 Complex roots, $r_{1,2} = \alpha \pm \beta i$

1. $y_1 = e^{(\alpha+\beta i)x}, y_2 = e^{(\alpha+\beta i)x}$
2. General solution:

$$\begin{aligned} y &= C_1 e^{\alpha x}\sin\beta x + C_2 e^{\alpha x}\cos\beta x \\ y &= e^{\alpha x}\left(C_1 \sin\beta x + C_2 \cos\beta x\right) \end{aligned}$$

Example 1

1. Question
   (a) Find general solution of:
      i. $y'' - 10y' + 41y = 0$
      ii. $y'' + 4y' + 13y = 0$
2. Answer

(a) $r^2 - 10r + 41 = 0$

$$\begin{aligned} r &= \frac{10 \pm \sqrt{10^2 - 4(1)(41)}}{2(1)} \\ &= \frac{10 \pm \sqrt{-64}}{2} \\ &= \frac{10 \pm \sqrt{64}\sqrt{-1}}{2} \\ &= \frac{10 \pm 8i}{2} \\ &= 5 \pm 4i \end{aligned}$$

(b) $r^2 + 4r + 13 = 0$

$$\begin{aligned} r &= \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} \\ &= \frac{-4 \pm 6i}{2(1)} \\ &= \frac{-4 \pm 6i}{2} \\ &= -2 \pm 3i \end{aligned}$$

(c) Find the final general solution

$$y = e^{-2x}\left(C_1 \sin 3x + C_2 \cos 3x\right)$$

Example 2

1. Solve the following I.V.P.

   (a) $y'' - 8y' + 17 = 0$, given $y(0) = -4, y'(0) = -1$

2. Answer

   (a) Find r

$$\begin{aligned} r^2 - 8r + 17 &= 0 \\ r &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(17)}}{2} \\ &= \frac{8 \pm \sqrt{64 - 4(68)}}{2} \\ &= 4 \pm \frac{\sqrt{-208}}{2} \\ &= 4 \pm \frac{\sqrt{-1}\sqrt{208}}{2} \\ &= 4 \pm \frac{\sqrt{208}i}{2} \\ &= 4 \pm ci \end{aligned}$$

   (b) Substitute into the equation

$$\begin{aligned} y &= e^{4x}\left(C_1 \sin x + C_2 \cos x\right) \\ y(0) &= -4 \\ -4 &= e^0\left(C_1 \sin 0 + C_2 \cos 0\right) \\ -4 &= C_2 \\ C_2 &= -4 \end{aligned}$$

$$\begin{aligned} y &= e^{4x}\left(C_1 \sin x - 4\cos x\right) \\ y' &= 4e^{4x}\left(C_1 \sin x - 4\cos x\right) + e^{4x}\left(C_1 \cos x + 4\sin x\right) \\ -1 &= 4e^0\left(C_1 \sin 0 - 4\cos 0\right) + e^0\left(C_1 \cos 0 + 4\sin 0\right) \\ -1 &= -16 + (C_1) \\ C_1 &= 15 \end{aligned}$$

   (c) Answer

$$y = e^{4x}\left(15 \sin x - 4\cos x\right)$$

2.7.2 Second Order Non-Homogeneous Linear D.E.

1. Form: $ay'' + by' + cy = f(x)$
2. Solution process (the method of undetermined coefficients)
   (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)
   (b) Find the complementary function, y_h
   (c) Find the particular solution y_p
   (d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

2.7.2.1 The Method of Undetermined Coefficients

1. Determine form of y_p
2. Substitute into $ay'' + by' + cy = f(x)$
3. Notes (ALL THESE NOT GIVEN, HAVE TO MEMORIZE):
   (a) This method is limited to non-homogeneous, linear, D.E. when $f(x)$ is a:
      i. constant (like below, e^{2x})

$f(x)$	y_p guess
a	A

      ii. exponential, polynomial, or sin/cosine function

$f(x)$	y_p guess
ae^{ax}	Ae^{ax}
$a\sin(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$a\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$a\sin(\beta x) + a\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$a\sin(\beta x) + b\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$

      iii. Finite sums

$f(x)$	y_p guess
$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$	$Ax^n + Bx^{n-1} + ... + Nx + P$

      iv. Product of above

$f(x)$	y_p guess
$e^{ax}\sin(\beta x)$ or $e^{ax}\cos(\beta x)$	$e^{ax}[A\sin(\beta x) + B\cos(\beta x)]$
$e^{ax}\left[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0\right]$	$e^{ax}\left[Ax^n + Bx^{n-1} + ... + Nx + P\right]$
$\left[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0\right]\sin(\beta x)$	$\left[Ax^n + Bx^{n-1} + ... + Nx + P\right]\sin(\beta x) +$ $\left[Ax^n + Bx^{n-1} + ... + Nx + P\right]\cos(\beta x)$
$\left[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0\right]\cos(\beta x)$	$\left[Ax^n + Bx^{n-1} + ... + Nx + P\right]\sin(\beta x) +$ $\left[Ax^n + Bx^{n-1} + ... + Nx + P\right]\cos(\beta x)$

   (b) y_p can be the sum of the forms inside the table.
      i. *E.g.* : $f(x) = 2x + e^{3x} \rightarrow y_p = Ax + B + Ce^{3x}$
   (c) **If y_p have any terms duplicated in** y_h, we have to multiply y_p by x until no duplicate term exists.
      i. Say $f(x) = \sin 2x$, $y_h = C_1 \sin 2x + C_2 \cos 2x$.
      ii. If we suggest $y_p = A\sin 2x + B\cos 2x$, this is duplicated in y_h **(barring the constant aside)**. So, we will have to multiply by x.
      iii. This makes $y_p = x[A\sin 2x + B\cos 2x]$, which is not duplicated, and valid.
      iv. Reason we want to do this is to prevent important constants (like A) from cancelling each other. So, we need to **guess** something else. (Remember we are guessing according to the table)

Example Find the general solution of $y'' + 6y' + 9y = e^{2x}$

1. Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned} r^2 + 6r + 9 &= 0 \\ (r+3)^2 &= 0 \\ r &= -3 \end{aligned}$$

2. Find the complementary function, y_h

$$y_h = e^{-3x}(C_1 + C_2 x)$$

3. Find the particular solution y_p

(a) Find the derivatives

$$\begin{aligned} y_p &= Ae^{2x} \\ y_p' &= 2Ae^{2x} \\ y_p'' &= 4Ae^{2x} \end{aligned}$$

(b) Plug into the function

$$\begin{aligned} 4Ae^{2x} + 6\left(2Ae^{2x}\right) + 9\left(Ae^{2x}\right) &= e^{2x} \\ 25Ae^{2x} &= e^{2x} \end{aligned}$$

(c) Find A

$$\begin{aligned} 25Ae^{2x} &= e^{2x} \\ A &= \frac{1}{25} \end{aligned}$$

(d) Find y_p

$$y_p = \frac{1}{25}e^{2x}$$

4. Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = e^{-3x}\left(c_1 + c_2 x\right) + \frac{1}{25}e^{2x}$$

Example

1. Question: Suggest the form of y_p of $ay'' + by' + cy = f(x)$ if
   (a) $f(x) = 2x^3$
      i. $y_p = Ax^3 + Bx^2 + Cx + D$
   (b) $f(x) = 4e^{-3x}$
      i. $y_p = Ae^{-3x}$
   (c) $f(x) = 2(\sin 2x + \cos 2x)$
      i. $A \sin 2x + B \cos 2x$
   (d) $f(x) = \left(9x^2 - 3\right)e^{4x}$
      i. $Y_p = e^{4x}\left(Ax^2 + Bx + C\right)$
   (e) $f(x) = e^{4x} \sin 2x$
      i. $y_p = e^{4x}(A \sin 2x + B \cos 2x)$
   (f) $f(x) = \sin x + \cos 2x$
      i. $y_p = A \sin x + B \cos x + C \sin 2x + D \cos 2x$
   (g) $f(x) = e^{2x} - e^{4x}$
      i. $y_p = Ae^{2x} + Be^{4x}$

Example 1

1. Question: Find the general solution of $y'' + 4y = \sin x$
2. Answer
   (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned} y'' + 4y &= 0 \\ r^2 + 4 &= 0 \\ r &= 0 \pm 2i \leftarrow (remember : \alpha \pm \beta i) \end{aligned}$$

   (b) Find the complementary function, y_h. Note: repeated real roots

$$y_h = e^{0x}\left(C_1 \cos 2x + C_2 \sin 2x\right)$$

   (c) Find the particular solution y_p

$$\begin{aligned} f(x) &= \sin x \\ y_p &= A \sin x + B \cos x \\ y_p' &= A \cos x - B \sin x \\ y_p'' &= -A \sin x - B \cos x \\ -A \sin x - B \cos x + 4(A \sin x + B \cos x) &= \sin x \\ \sin x(-A + 4A) + \cos x(-B + 4B) &= \sin x \\ \sin x(3A) + \cos x(3B) &= \sin x \end{aligned}$$

i. Compare $\sin x\,(3A)$ with $\sin x$

$$\begin{aligned}\sin x\,(3A) &= \sin x\\ 3A &= 1\\ A &= \frac{1}{3}\end{aligned}$$

ii. Compare $\cos x\,(3B)$ with 0

$$\begin{aligned}\cos x\,(3B) &= 0\\ B &= 0\end{aligned}$$

iii. Find y_p

$$\begin{aligned}y_p &= \frac{1}{3}\sin x + 0\cos x\\ y_p &= \frac{1}{3}\sin x\end{aligned}$$

(d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{3}\sin x$$

Example 2

1. Question
   (a) Find the general solution of $y'' - 4y' - 12y = 2x^3 - x + 3$
2. Answer
   (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

$$\begin{aligned}r^2 - 4r - 12 &= 0\\ r &= 6, -2\end{aligned}$$

(b) Find the complementary function, y_h(Note, non-repeated real roots)

$$y_h = C_1e^{6x} + C_2e^{-2x}$$

(c) Find the particular solution y_p

$$\begin{aligned}y_p &= Ax^3 + Bx^2 + Cx + D\\ y_p' &= 3Ax^2 + 2Bx + c\\ y_p'' &= 6Ax + 2B\\ (6Ax + 2B) - 4\left(3Ax^2 + 2Bx + C\right) - 12\left(Ax^3 + Bx^2 + Cx + D\right) &= 2x^3 - x + 3\\ x^3(-12A) + x^2(-12A - 12B) + x(6A - 8B - 12C) + 2B - 4C - 12D &= 2x^3 - x + 3\end{aligned}$$

i. $-12A = 2$

$$A = -\frac{1}{6}$$

ii. $-12\left(-\frac{1}{6}\right) - 12B = 0$

$$B = \frac{1}{6}$$

iii. $6\left(-\frac{1}{6}\right) - 8\left(\frac{1}{6}\right) - 12C = -1$

$$C = -\frac{1}{9}$$

iv. $2\left(\frac{1}{6}\right) - 4\left(-\frac{1}{9}\right) - 12D = 3$

$$D = -\frac{5}{27}$$

(d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$ (REMEMBER THIS PART)

$$y = C_1e^{6x} + C_2e^{-2x} - \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{1}{9}x - \frac{5}{27}$$

Example 3

1. Solve the following I.V.P.
2. $y'' + y = 4x + 10\sin x$, given $y(\pi) = 0, y'(\pi) = 2$

   (a) Solve the homogeneous D.E. $ay'' + by' + cy = 0$ (find the r)

   $$\begin{aligned} r^2 + 1 = 0 \rightarrow r &= 0 \pm i \\ y_h &= C_1 \sin x + C_2 \cos x \end{aligned}$$

   (b) Find the complementary function, y_h(Note, non-repeated real roots)

   $$y_h = C_1 \sin x + C_2 \cos x$$

   (c) Find the particular solution y_p

   $$\begin{aligned} y_p &= (A_x + B) + (Cx\sin x + Dx\cos x) \\ y_p' &= A + (C\sin x + Cx\cos x + D\cos x - Dx\sin x) \\ &= A + (C\sin x + D\cos x) + (Cx\cos x - Dx\sin x) \\ y_p'' &= C\cos x - D\sin x + (C\cos x - Cx\sin x - D\sin x - Dx\cos x) \\ &= 2C\cos x - 2D\sin x - Cx\sin x - Dx\cos x \end{aligned}$$

   i. $2C\cos x - 2D\sin x - Cx\sin x - Dx\cos x = 4x + 10\sin x$

   $$A = 4, B = 0, C = 0, D = -5$$

   $$y_p = 4x - 5x\cos x$$

   (d) Find the non-homogeneous D.E. general solution, $y = y_h + y_p$

   $$y = C_1 \sin x + C_2 \cos x + 4x - 5x\cos x$$

   (e) Find C_1 and C_2

   $$\begin{aligned} y(\pi) &= 0 \\ -C_2 + 4\pi + 5\pi &= 0 \\ C_2 &= 9\pi \end{aligned}$$

   $$\begin{aligned} y &= C_1 \sin x + 9\pi\cos x + 4x - 5x\cos x \\ y' &= C_1 \cos x - 9\pi\sin x + 4 - 5\cos x + 5x\sin x \\ y'(\pi) &= -C_1 + 4 + 5 = 2 \\ C_1 &= 7 \end{aligned}$$

   (f) Find the solution

   $$y = 7\sin x + 9\pi\cos x + 4x - 5x\cos x$$