Calc II - C3: Infinite Sequences & Series

December 24, 2019

1 Terms to describe $\{a_n\}$

1. A set of numbers
   (a) $a_n = 2n, n = \{1, 2, 3, ...\}$
   (b) $a_n = 1\left(2^{n-1}\right), N = \{1, 2, 3, ...\}$
   (c) $a_n = 1\left(-1\right)^{n-1}, N = \{1, 2, 3, ...\}$
   (d) $a_n = \frac{n}{n+1}$
   (e) $a_n = (-1)^n \left(\frac{n+1}{3^n}\right)$
2. Definition 3.2: Let $n = 1, 2, 3$
   (a) If $a_n \leq M$,
      i. $\{a_n\}$**bounded above** by M
      ii. M is **upper bound**
   (b) and vice versa ($a_n \geq M$, lower bound)
3. $\{a_n\}$
   (a) **positive** if $a_n \geq 0, \forall n$ and vice versa ($a_n \leq 0$).
   (b) **increasing** if $a_{n+1} \geq a_n$, $\forall n$, and vice versa ($a_{n+1} \leq 0$)
   (c) **monotonic =** all increasing/decreasing
   (d) **alternating** = $a_n \cdot a_{n+1} \leq 0$ (consecutive, opposite sign)
4. As $n \to \infty$
   (a) **convergent** if $\lim_{n\to\infty}$ is real number
   (b) **divergent** if $\lim_{n\to\infty}$ is $\pm\infty$

1.1 Example

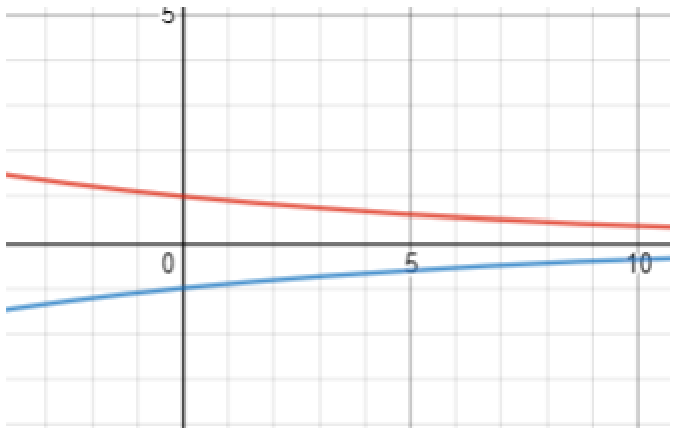
Describe the following sequences using the terms in definition above:

1. $\{n\} = \{1, 2, 3, 4, 5...\}$
   (a) Not bounded above
   (b) Bounded below, Lower bound = 1
   (c) Not bounded
   (d) Positive
   (e) Increasing
   (f) Monotonic
   (g) Not alternating
   (h) Divergent (from 1)
2. $\left\{\frac{n-1}{n}\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}...\right\}$
   (a) Not bounded above
   (b) Bounded below, Lower bound = 0
   (c) Not bounded
   (d) Positive
   (e) Increasing
   (f) Monotonic
   (g) Not alternating
   (h) Convergent (to 1)
3. $\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}...\right\}$
   (a) Bounded above, $\frac{1}{4}$
   (b) Bounded below, Lower bound = $-\frac{1}{2}$
   (c) Bounded
   (d) Not increasing / decreasing
   (e) Not monotonic
   (f) Alternating
   (g) Convergent (to 0)

2 Convergence and Divergence of Sequences

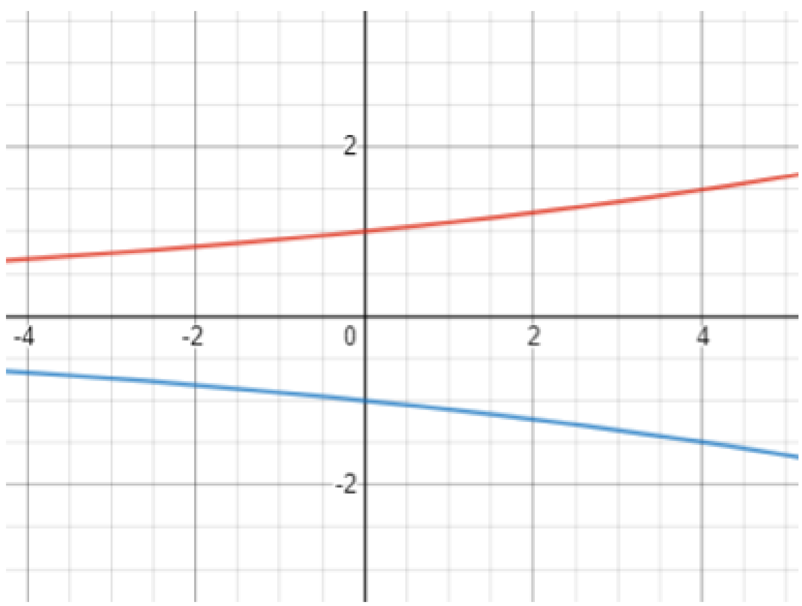
2.1 Definition

1. If $\lim_{n \to \infty} a_n = L$, **converges** (is **convergent**)



(a)

2. Else if L D.N.E. or $\pm\infty$, it **diverges** (is **divergent**)



(a)

2.2 Theorem

1. If $\lim_{n \to \infty} f(x) = L$ and $f(n) = a_n$, and n is an integer, then $\lim_{n \to \infty} a_n = L$

2. Put it simply (and slightly incorrectly), if the series converges, and $f(n)$ is a_n, then $f(\infty)$ is simply L.

2.3 Theorem

1. Every bounded & monotonic sequence is **convergent**.

2.4 Laws - Limit Laws for Convergent Sequences:

If $\{a_n\}$ and $\{b_n\}$are convergent sequences, and c is constant, then:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(c \cdot a_n) = c \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty} c = c$
4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n) \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p, p > 0, a_n > 0$

2.5 Technique - Rational functions

If a_n is rational function, to find $\lim_{n\to\infty} a_n$, divide both by highest power of n in **denominator.**

2.6 Example

Find the limit of the following sequences if exist:

1. $a_n = \frac{3n^2-n+4}{7+6n^2}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{n}{n^2} + \frac{4}{n^2}}{\frac{7}{n^2} + \frac{6n^2}{n^2}} \\
&= \frac{3}{6} \\
\boldsymbol{\lim_{n\to\infty} a_n} &= \boldsymbol{\frac{1}{2}}
\end{aligned}$$

2. $a_n = \frac{n}{n+1}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}} \\
&= \frac{1}{1} \\
\boldsymbol{\lim_{n\to\infty} a_n} &\boldsymbol{= 1}
\end{aligned}$$

3. $a_n = \frac{n^2-n}{n^2+1}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2-n}{n^2+1} \\
&= \lim_{n\to\infty} \frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{1 + \frac{1}{n^2}} \\
\boldsymbol{\lim_{n\to\infty} a_n} &\boldsymbol{= 1}
\end{aligned}$$

2.7 Example

By using L'Hospital's Rule, calculate the following:

1. $\lim_{n\to\infty} \frac{\ln n}{n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{n} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
\boldsymbol{\lim_{n\to\infty} \frac{\ln n}{n}} &\boldsymbol{= 0}
\end{aligned}$$

2. $\lim_{n\to\infty} \frac{e^{2n}}{n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{e^{2n}}{n} &= \lim_{n\to\infty} \frac{\frac{d}{dn}\left[e^{2n}\right]}{\frac{d}{dn}\left[n\right]} \\
&= \lim_{n\to\infty} \frac{2e^{2n}}{1} \\
&= \lim_{n\to\infty} 2e^{2n} \\
\lim_{n\to\infty} \frac{e^{2n}}{n} &= \infty
\end{aligned}$$

2.8 Theorem - Squeeze Theorem

1. If $a_n \leq b_n \leq c_n, n > n_0$
   (a) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$
   (b) Then $\lim_{n\to\infty} b_n = L$

2.8.1 Example

Evaluate:

1. $\lim_{n\to\infty} \left\{ \frac{\cos^2 n}{3^n} \right\}$

$$\begin{aligned}
-1 \leq \cos n \leq 1 \\
1 \leq \cos^2 n \leq 1 \\
\frac{1}{3} \leq \frac{\cos^2 n}{3^n} \leq \frac{1}{3} \\
\lim_{n\to\infty} \frac{1}{3} \leq \lim_{n\to\infty} \frac{\cos^2 n}{3^n} \leq \lim_{n\to\infty} \frac{1}{3} \\
0 \leq \lim_{n\to\infty} \frac{\cos^2 n}{3^n} \leq 0
\end{aligned}$$

   (a) $\therefore$ According to Squeeze Theorem, $\lim_{n\to\infty} \frac{\cos^2 n}{3^n} = 0$

2. $\lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\}$ (Lecturer mention exam might come out)

$$\begin{aligned}
\frac{n!}{n^n} &= \frac{n\,(n-1)\,(n-2)\,...3 \cdot 2 \cdot 1}{n \cdot n \cdot n ... \cdot n} \\
&= \frac{n}{n} \cdot \frac{n-1}{n} \cdot ... \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \\
&= 1 \cdot \left(1 - \frac{1}{n}\right) \cdot ... \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq \lim_{n\to\infty} 1 \\
\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq \lim_{n\to\infty} 1 \\
0 \leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq 0
\end{aligned}$$

   (a) According to S.T., $\lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} = 0$

2.9 Theorem 3.3

1. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

2.9.1 Example

1. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if exists.

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

2.10 Theorem 3.4

1. Sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$, diverges for all other r.
2. Also,

$$\lim_{n\to\infty} = \begin{cases} 0 & , -1 < r < 1 \\ 1 & , r = 1 \end{cases}$$

2.10.1 Evaluate:

1. $\lim_{n\to\infty} (-1)^n = \infty, Divergent$
2. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0, convergent$
3. $\lim_{n\to\infty} 5^n = \infty, Divergent$

3 Infinite Series

3.1 Definition 3.4

1. **Infinite series** (or just **series**) $= a_1 + a_2 + a_3 + ... + a_n + ...$
    (a) $\sum a_n$ OR $\sum_{n=1}^{\infty} a_n$
2. Series = infinite sum of numbers
3. Sequence = numbers, 1-to-1 correspondence with positive integer.

3.1.1 Example

1. $\left\{\frac{1}{n}\right\} = \left\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\right\}$
    (a) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$
2. $\left\{\frac{(-1)^{n-1}}{2^{n-1}}\right\} = \left\{1, -\frac{1}{2}, \frac{1}{4}, ...\right\}$
    (a) $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{2^{n-1}}\right) = 1 - \frac{1}{2} + \frac{1}{4} - ...$

3.2 Definition 3.5 - Partial Sums, S_k

1. $k-th$ Partial sum, S_k of $\sum a_n$
    (a) $S_k = a_1 + a_2 + ... + a_k$
2. Sequence of partial sum of $\sum_{n=1}^{\infty} a_n$ is
    (a) $\{S_1, S_2...S_n, ...\}$

3.2.1 Example

Find the sequence of partial sum of the series $\sum_{n=1}^{\infty} n$

$$\begin{aligned}\sum_{n=1}^{\infty} n &= 1+2+3+... \\ S_1 &= 1 \\ S_2 &= 1+2=3 \\ S_3 &= ....\end{aligned}$$

1. Sequence of partial sum: $\left\{1, 3, 6, 10, \frac{n}{2}(n+1), ...\right\}$

4 Convergence & Divergence of Series

4.1 Definition

1. The series $\sum_{n=1}^{\infty} a_n$ **converges** if and only if the sequence $\{Sn\}$ is convergent and $\lim_{n\to\infty} S_n = s$ exists as a real number.
2. The number s is called the **sum** of the series, $s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$
3. If the sequence $\{Sn\}$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ **diverges**
4. To put it simply, series converges if and only if:
    (a) sequence is convergent AND
    (b) the value of function when approaching infinity is a real number.

4.1.1 Example

Show that the series $\sum_{n=1}^{\infty} n$ is divergent

1. $\{S_n\} = \{1, 2, 3, ..., \infty\}$ is divergent
2. Therefore, it is divergent.

4.2 Theorem 3.5

1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
   (a) Converse not generally true. Further investigation needed.

4.3 Theorem 3.6: Test for Divergence

1. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ D.N.E., the series $\sum_{n=1}^{\infty} a_n$ is divergent.
2. Put it simply, if the limit to infinity is not 0, then it diverges.

4.3.1 Example

Show that the following series are divergent.

1. $\sum_{n=1}^{\infty} \frac{n^2}{3+5n^2}$

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2}{3+5n^2} &= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{3}{n^2}+\frac{5n^2}{n^2}} \\ &= \lim_{n\to\infty} \frac{1}{\frac{3}{n^2}+5} \\ \lim_{n\to\infty} \frac{n^2}{3+5n^2} &= \frac{1}{5} \neq 0\end{aligned}$$

   (a) $\therefore$ *Divergent*

2. $\sum_{n=1}^{\infty} \frac{-n}{2n+3}$

$$\begin{aligned}\lim_{n\to\infty} \frac{-n}{2n+3} &= \lim_{n\to\infty} \frac{\frac{-n}{n}}{\frac{2n}{n}+\frac{3}{n}} \\ &= \lim_{n\to\infty} \frac{-1}{2+\frac{3}{n}} \\ \lim_{n\to\infty} \frac{-n}{2n+3} &= -\frac{1}{2} \\ \lim_{n\to\infty} \frac{-n}{2n+3} &\neq 0\end{aligned}$$

5 Special Series

5.1 Geometric Series

5.1.1 Definition

1. **Geometric series:** Series with the form $a + ar + ar^2 + ... + ar^n + ...$
2. Notation: $\sum_{n=1}^{\infty} ar^{n-1}$ OR $\sum_{n=0}^{\infty} ar^n$

3. **Convergent**, if
    (a) $|r| < 1$, AND
    (b) $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
    (c) Otherwise, its **divergent**

5.1.2 Example

Determine if the following series converge or diverge. If they converge give the value of the series.

1. $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$
    (a) $9^{-n+2}4^{n+1}$

$$\begin{aligned} 9^{-n+2}4^{n+1} &= \frac{9^2}{9^n}4\left(4^n\right) \\ &= \frac{9^2}{9^n}4\left(4^n\right) \\ &= 4 \cdot 9^2 \cdot \left(\frac{4}{9}\right)^2 \\ r &= \frac{4}{9} < 1, conv \end{aligned}$$

$$S_\infty = \frac{\left(9 \cdot 4^2\right)}{1 - \frac{4}{9}} = 259.2$$

2. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$
    (a) $\frac{(-4)^{3n}}{5^{n-1}}$

$$\begin{aligned} \frac{(-4)^{3n}}{5^{n-1}} &= \frac{(-4)^{3n}}{5^n \cdot 5^{-1}} \\ &= 5 \cdot \frac{\left(-4^3\right)^n}{5^n} \\ &= 5 \cdot \left(\frac{-4^3}{5}\right)^n \\ &= 5 \cdot \left(-\frac{4^3}{5}\right)^n \\ r &= -\frac{4^3}{5} \end{aligned}$$

    i. Since $|r| > 1$, the series is divergent

5.2 Telescoping Series

Definition 3.8: A telescoping series is a series whose sum can be found by exploiting the circumstance that nearly every term cancels with either a succeeding or preceding term.

5.2.1 Example

1. Given the telescoping series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} + ...$

   (a) Find the n-th partial sum S_n of the series for n = 1, 2, 3, 4

$$S_1 = \frac{1}{2}$$

$$\begin{aligned} S_2 &= \frac{1}{2} + \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} S_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} S_4 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \\ &= \frac{4}{5} \end{aligned}$$

   (b) Find S_n.

$$\begin{aligned} \sum \frac{1}{n\,(n+1)} &= \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1} \\ &= S_n \end{aligned}$$

   (c) Show that the series is convergent and find its sum.

$$\begin{aligned} S_\infty &= \lim_{n\to\infty} S_n \\ &= 1 - \frac{1}{\infty + 1} \\ &= 1, convergent \end{aligned}$$

i. Note: Sum is 1 because they cancel out each other for every term. So only the remaining term counts.

5.2.2 Example

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)}+\frac{1}{2^n}\right)$.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2}{n\left(n+1\right)}+\frac{1}{2^n}\right) &= 2\sum\frac{1}{n\left(n+1\right)}+\sum\frac{1}{2^n}\\ &= 2+\sum\left(\frac{1}{2}\right)^n\\ &= 2+\frac{\frac{1}{2}}{1-\frac{1}{2}}\\ \sum_{n=1}^{\infty}\left(\frac{2}{n\left(n+1\right)}+\frac{1}{2^n}\right) &= 3\end{aligned}$$

5.3 Harmonic Series (Not in finals, safe to ignore)

Definition 3.9: A series of the form $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$ is called a harmonic series. Always **divergent**.

5.3.1 Proof for divergence (Proof by contradiction)

1. Assume harmonic series converges to H.
2. Calculations

$$\begin{aligned}H &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\\ H &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+...\\ H &> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+...\\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\\ &= H\end{aligned}$$

3. Contradiction ($H>H$). So the harmonic series is not convergent.

5.4 $p-series$

Definition 3.10: A $p-series$ is a series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+...$

$\sum_{n=1}^{\infty}\frac{1}{n^P}=\begin{cases}converges & p>1\\ diverges & p\leq 1\end{cases}$

5.4.1 Example

Determine whether the given series converges or diverges.

1. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$
   (a) $p = 2, conv$
2. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ...$
   (a) $p = \frac{1}{2}, div$

6 Convergent Test for Series

6.1 The Integral Test

Theorem 3.8: Suppose f is a continuous, positive, **decreasing** function on $[1, \infty)$ and $a_n = f(n)$.

a. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

b. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

6.1.1 Example:

By using Integral Test, determine whether the following series converges or diverges.

1. $\sum_{n=1}^\infty \frac{5}{n+1}$

$$\begin{aligned} \int_1^\infty \frac{5}{x+1}dx &= 5\ln(x+1)_1^\infty \\ &= 5\ln\infty - 5\ln 2 \\ \int_1^\infty \frac{5}{x+1}dx &= \infty \end{aligned}$$

   (a) $\therefore Divergent$

2. $\sum_{n=1}^\infty e^{-n}$

$$\begin{aligned} \int_1^\infty e^{-x}dx &= \int_1^\infty e^{-x}dx \\ &= \left[-e^{-x}\right]_1^\infty \\ &= -e^{-\infty} - \left(-e^{-1}\right) \\ &= -e^{-\infty} + e^{-1} \\ \int_1^\infty e^{-x}dx &= e^{-1} \end{aligned}$$

   (a) $\therefore Convergent$

6.2 Comparison Test/Limit Comparison Test

Theorem 3.9: **Comparison Test**

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

a. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

b. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

6.2.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+1}$

$$a_n = \frac{3}{2n^2+4n+1}$$
$$2n^2+4n+1 > 2n^2 > n^2$$
$$\frac{3}{2n^2+4n+1} < \frac{3}{n^2}$$

   (a) $3\sum \frac{1}{n^2}$ is a convergent $p - series$.

   (b) $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+1}$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$$a_n = \frac{1}{2^n+1}$$
$$2^n+1 > 2^n$$
$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

   (a) Since $\sum \frac{1}{2^n}$ is a convergent geometric series

   (b) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent.

Theorem 3.10: **Limit Comparison Test**

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge. If c = 0, no conclusion.

6.2.2 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

   (a) $a_n = \frac{1}{1+\sqrt{n}}, b_n = \frac{1}{\sqrt{n}}$

   (b) b_n is a divergent p-series, $p = \frac{1}{2}$

   (c) $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(\frac{1}{1+\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} \\
&= 1
\end{aligned}
$$

   (d) Therefore, $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

   (a) $a_n = \frac{1}{2^n-1}, b_n = \frac{1}{2^n}$

   (b) b_n is a convergent geometric series.

   (c) $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n-1} \\
&= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} \\
&= 1
\end{aligned}
$$

   (d) Therefore, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

6.3 Alternating Series Test

Theorem 3.11: If the alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ where $b_n > 0$ 0 satisfies:

1. $b_{n+1} \leq b_n$ for all n (decreasing);
2. $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

Note: This test CANNOT be used to determine if the series is divergent.

6.3.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
   (a) $b_n = \frac{1}{n}$
   (b) $b_n = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right)$, decreasing
   (c) $\lim_{n\to\infty} b_n = 0$
   (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+3}$
   (a) $b_n = \frac{n^2}{n^2+3}$
   (b) $b_n = \left(\frac{1}{4}, \frac{4}{7}, ...\right)$ increase
   (c) This test is not suitable, run another test.

6.4 Absolute Convergence

Definition 3.11: A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

6.4.1 Example

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p - series\,(p = 2)$

Theorem 3.12: If $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

Definition 3.12: a series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if the series $\sum_{n=1}^{\infty} a_n$ is convergent but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is divergent.

6.4.2 Example

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is **conditionally convergent** because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating harmonic series which is convergent, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

6.4.3 Example

Test the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ for absolute convergence.

$$\left|\frac{\cos n}{n^2}\right| < \frac{1}{n^2}$$

1. Since $\sum \frac{1}{n^2}$ is a convergent $p-series$ $(p = 2)$, but comparison test, $\sum \left|\frac{\cos n}{n^2}\right|$ is absolutely convergent.
2. So, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

6.5 The Ratio Test

Theorem 3.13: Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1 & , \sum_{n=1}^{\infty} a_n \text{converges} \\ > 1 \text{or} \infty & , \sum_{n=1}^{\infty} a_n \text{diverges} \\ 1 & , \text{the Ratio Test inconclusive} \end{cases}$

6.5.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{99^n}{n!}$

$$a_n = \frac{99^n}{n!}, a_{n+1} = \frac{99^{n+1}}{(n+1)!}$$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{99^n} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{99^n} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} 99 \cdot \frac{n!}{(n+1)\, n!} \text{note: still the same thing} \\ &= \lim_{n\to\infty} \frac{99}{n+1} = 0 \\ &< 1\, (convergent) \end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$$a_n = \frac{(2n)!}{(n!)^2}, a_{n+1} = \frac{(2\,(n+1))!}{((n+1)!)^2}$$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(2\,(n+1))!}{((n+1)!)^2} * \frac{(n!)^2}{(2n)!} \\
&= \lim_{n\to\infty}\frac{(2\,(n+1))!}{(2n)!} * \frac{(n!)^2}{((n+1)!)^2} \\
&= \lim_{n\to\infty}\frac{(2\,(n+1))!}{(2n)!} * \left(\frac{n!}{(n+1)!}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)\,\cancel{(2n)}!}{\cancel{(2n)}!} \cdot \left(\frac{\cancel{n}!}{(n+1)\,\cancel{n}!}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)\,\cancel{(2n)}!}{\cancel{(2n)}!} \cdot \left(\frac{\cancel{n}!}{(n+1)\,\cancel{n}!}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)}{(n+1)\,(n+1)} \\
&= \lim_{n\to\infty}\frac{4n^2+6n+2}{n^2+2n+1} = 4\,(>1)
\end{aligned}
$$

(a) The series is divergent

3. $\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)!}$

(a) $a_n = \frac{n^2}{(2n-1)!}, a_{n+1} = \frac{(n+1)^2}{(2(n+1)-1)!} = \frac{(n+1)^2}{(2n+1)!}$

(b)

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)!} \cdot \frac{(2n-1)!}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{n^2} \cdot \frac{(2n-1)!}{(2n+1)!} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{(2n-1)!}{(2n+1)\,(2n+1-1)\,(2n+1-2)!} \text{note: expand the factorial} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{\cancel{(2n-1)!}}{(2n+1)\,(2n)\,\cancel{(2n-1)!}} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{1}{(2n+1)\,(2n)} \\
&= \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) \cdot \frac{1}{(2n+1)\,(2n)} \\
&= 1\cdot 0 \\
&= 0 \\
&= \mathit{convergent}
\end{aligned}
$$

6.6 The Root Test

Theorem 3.14: Suppose we have the series $\sum_{n=1}^{\infty} a_n$.

Define, $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

Then, if $L < 1$, the series is absolutely convergent (and hence convergent).

If $L > 1$, the series is divergent.

$L = 1$, the test is inconclusive.

Example: Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \left(\frac{4n-3n^2}{5n^2+2}\right)^n$

   (a) $a_n = \left(\frac{4n-3n^2}{5n^2+2}\right)^n$

$$\sqrt[n]{\left(\frac{4n-3n^2}{5n^2+2}\right)^n} = \frac{4n-3n^2}{5n^2+2}$$
$$\lim_{n\to\infty}\left|\frac{4n-3n^2}{5n^2+2}\right| = \frac{3}{5} < 1$$

   (b) The series is convergent

2. $\sum_{n=1}^{\infty} \frac{n^n}{4^{1+3n}}$

$$\sqrt{\frac{n^n}{4^{1+3n}}} = \frac{n}{4^{\frac{1+3n}{n}}} = \frac{n}{4^{\frac{1}{n}+3}}$$

$$\left|\lim_{n\to\infty}\frac{n}{4^{\frac{1}{n}+3}}\right| = \frac{\infty}{4^3} = \infty$$

   (a) The series is divergent

6.6.1 Example, exapnd the following:

1. $\left(x+\frac{1}{x}\right)^5$

$$\left(x+\frac{1}{x}\right)^5 = \sum_{k=0}^{\infty}\binom{5}{k}x^{5-k}\left(\frac{1}{x}\right)^k$$
$$= \binom{5}{0}x^5\left(\frac{1}{x}\right)^0 + \binom{5}{1}x^4\left(\frac{1}{x}\right)^1 + \binom{5}{2}x^3\left(\frac{1}{x}\right)^2 + \binom{5}{3}x^2\left(\frac{1}{x}\right)^3 + \binom{5}{4}x^1\left(\frac{1}{x}\right)^4 + \binom{5}{5}x^0\left(\right.$$
$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

2. $(2-x)^4$

   (a) $2^5 - 5\left(2^4 x\right) + 10\left(2^3\right)x^3 + 5\left(2^1 x^4\right) - x^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

   (b) $161 + 72\sqrt{5}$

3. $\left(2+\sqrt{5}\right)^4$

   (a) Start by writing the terms in pascal triangle: $1, 4, 6, 4, 1$

   (b) Continue by adding values into the equation

$$\left(2+\sqrt{5}\right)^4 = 1\left(2\right)^4 + 4\left(2\right)^3\left(\sqrt{5}\right) + 6\left(2\right)^2\left(\sqrt{5}\right)^2 + 4\left(2\right)\left(\sqrt{5}\right)^3 + 1\left(\sqrt{5}\right)^4$$

6.7 Binomial Series

Theorem: **Binomial Theorem**

If n is any positive integer, then

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\ &= a^n + na^{n-1}b + \frac{n\left(n-1\right)}{2!}a^{n-2}b^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}a^{n-3}b^3 + ... + nab^{-1} + b^n\end{aligned}$$

Where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ (binomial coefficient)

Pascal's Triangle – collection of the binomial coefficients.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

6.7.1

Definition 3.14: **Binomial Series**

If n is any real number and $|x| < 1$, then

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \text{ is called a Binomial Series.}$$

6.7.2 Example: Expand the following functions as a power series:

1. $\frac{1}{(1+x)^2}$

$$\begin{aligned}\frac{1}{(1+x)^2} &= (1+x)^{-2} \\ &= 1 + (-2)\,x + \frac{-2\,(-2-1)}{2!}x^2 + \frac{-2\,(-2-1)\,(-2-2)}{3!}x^3 + ... \\ &= 1 - 2x + 3x^2 - 4x^3 + ...\end{aligned}$$

2. $\sqrt{1+x}$

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}=1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 \\ &= 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + ...\end{aligned}$$

6.8 Power Series

Definition 3.13: A power series is a series of the form $\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots$where x is a variable and the c_n's are constants called the **coefficients** of the series.

More generally, a series of the form $\sum c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \ldots$ is called a **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a**.**

Theorem 3.15: For a given power series $\sum c_n\,(x-a)^n$ there are only three possibilities:

i. The series converges only at the point $x = a$.

ii. The series converges for all x.

iii. The series converges if $|x-a|<R$ and diverges if $|x-a|>R$, where the positive number R is called the **radius of convergence** of the power series. The set of points at which the series converges is called the **interval of convergence**.

Example: Determine the radius of convergence and interval of convergence for the following power series:-

1. $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

$$a_n = \sum_{n=0}^{\infty} \frac{n\,(x+2)^n}{3^{n+1}}$$
$$a_{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)\,(x+2)^{n+1}}{3^{n+2}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\,(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\,(x+2)^n}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)(x+2)$$

(a) The series converges if lim <1

$$\lim_{n\to\infty}\left|\frac{1}{3}\left(\frac{n+1}{n}\right)(x+2)\right|<1$$
$$\lim_{n\to\infty}\left|\frac{1}{3}(x+2)\right|<1$$

(b) Radius of convergence

i. Since x is not dependent on n, we can factor it out and make it become

$$\frac{1}{3}\left|(x+2)\right|<1$$
$$\left|(x+2)\right|<3$$

ii. Therefore, the radius of convergence is $R=3$

(c) Interval of convergence

i.

$$-1<\frac{1}{3}(x+2)<1$$
$$-1<\frac{x+2}{3}<1$$
$$-3<x+2<3$$
$$-5<x<1$$

2. $\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$

$$a_n = \frac{2^n (4x-8)^n}{n}, a_{n+1} = \frac{2^{n+1} (4x-8)^{n+1}}{n+1}$$

(a) Use the ratio test to find the limit L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{2^{n+1} (4x-8)^{n+1}}{n+1} \cdot \frac{n}{2^n (4x-8)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2^{\cancel{n}+1} (4x-8)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{2^n (4x-8)^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{2n (4x-8)}{n+1} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{2n(4x-8)}{n}}{\frac{n}{n}+\frac{1}{n}} \right| \\
&= \left| \frac{2 (4x-8)}{1} \right| \\
&= 2 \left|(4x-8)\right|
\end{aligned}$$

(b) For the series to converge, $L < 1$

$$\begin{aligned}
2 \left|(4x-8)\right| &< 1 \\
\left|(4x-8)\right| &< \frac{1}{2}
\end{aligned}$$

(c) Therefore, the radius of convergence is $R = \frac{1}{2}$

(d) Next, find the interval of convergence

$$\begin{aligned}
-\frac{1}{2} &< 4(x-2) < \frac{1}{2} \\
-\frac{1}{8} &< x-2 < \frac{1}{8} \\
\frac{15}{8} &< x < \frac{17}{8}
\end{aligned}$$

(e) Therefore, the interval of convergence is $\frac{15}{8} < x < \frac{17}{8}$

6.8.1 Representations of Functions as Power Series

We say that the power series is a representation of a function on the interval of convergence.

A power series about $x = 0$ is a series of the form $\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$

When $c_i = 1, \forall i$, we have a geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$

If $|x| < 1$, the series converges and the sum is $\frac{1}{1-x}$. Thus we have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x} \text{for} - 1 < x < 1$$

In above equation, we express the function $f(x) = \frac{1}{1-x}$ as the sum of the power series. So the power series is a representation of $f(x) = \frac{1}{1-x}$

6.8.2 Example: Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum \left(-x^2\right)^n$$

Series is convergent when

$$\begin{aligned} -1 &< -x^2 < 1 \\ -1 &< x^2 < 1 \\ 0 &< x < 1 \end{aligned}$$

6.8.3 Example: Find a power series representation for $\frac{1}{x+2}$

1. Solution

$$\begin{aligned} \frac{1}{x+2} &= \frac{1}{1-(-(x+1))} \\ &= \sum_{n=0}^{\infty} (-(x+1))^n \end{aligned}$$

2. Series is convergent when

$$\begin{aligned} -1 &< -(x+1) < 1 \\ -1 &< x+1 < 1 \\ -2 &< x < 0 \end{aligned}$$

6.8.4 Differentiation and Integration of Power Series

The differentiation and integration of power series are used to obtain power series representations.

Theorem 3.16: **Term-by-term Differentiation and Integration**

If the power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Is differentiable on the interval of convergence and
i. $f\prime(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$
ii.$\int f(x)\, dx$

$$\begin{aligned}\int f(x)dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}\end{aligned}$$

The radii of convergence of the power series in above equations are both R

6.8.5 Example

Find the power series representations for the functions. By using the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

1. $\frac{1}{(1-x)^2}$

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{d}{dx}\left[\sum x^n\right] \\ &= \sum \frac{d}{dx}(x^n) \\ &= \sum nx^{n-1}\end{aligned}$$

2. $\tan^{-1} x$ (hint: $\tan^{-1} x = \int \frac{1}{1+x^2} dx$)

$$\begin{aligned}
\tan^{-1} x &= \int \frac{1}{1+x^2} dx \\
&= \int \sum \left(-x^2\right) dx \\
&= \sum \int (-1)^n \left(x^{2n}\right) dx \\
&= \sum \int (-1)^n \left(x^{2n}\right) dx \\
&= \sum (-1)^n \frac{x^{2n+1}}{2n+1} dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \\
&= \tan^{-1} x
\end{aligned}$$