

Calc 2: Tutorial 5

November 18, 2019

1. Solve the following differential equations: (CHECK WITH LECTURER)

(a) $x^2 \frac{dy}{dx} + xy = x + 1$

- i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}x^2 \frac{dy}{dx} + xy &= x + 1 \\ \frac{1}{x^2} \left[x^2 \frac{dy}{dx} + xy \right] &= \frac{1}{x^2} [x + 1] \\ \frac{dy}{dx} + \frac{y}{x} &= \frac{1}{x} + \frac{1}{x^2}, p(x) = \frac{1}{x}\end{aligned}$$

- ii. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.

$$\begin{aligned}\mu(x) &= e^{\int \frac{1}{x} dx} \\ &= e^{\ln x} \\ \mu(x) &= x\end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x)y]$ and write it as such.

$$\begin{aligned}x \left[\frac{dy}{dx} + \frac{y}{x} \right] &= x \left[\frac{1}{x} + \frac{1}{x^2} \right] \\ x \frac{dy}{dx} + y &= 1 + \frac{1}{x} \\ \frac{d}{dx} [yx] &= 1 + \frac{1}{x} \\ \int \frac{d}{dx} [yx] dx &= \int 1 + \frac{1}{x} dx \\ yx &= x + \ln x + c \\ y &= 1 + \frac{\ln x + c}{x}\end{aligned}$$

(b) $\frac{dy}{dx} + y \cot x = \csc x$

- i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.
 A. Already in correct standard form, $p(x) = \cot x$
 ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int \cot x dx} \\ &= e^{\ln|\sin x|} \\ &= \sin x\end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}\sin x \left[\frac{dy}{dx} + y \cot x \right] &= \sin x [\csc x] \\ \sin x \frac{dy}{dx} + y \sin x \cdot \frac{\cos x}{\sin x} &= \sin x \left[\frac{1}{\sin x} \right] \\ \sin x \frac{dy}{dx} + y \cos x &= 1 \\ \sin x \frac{dy}{dx} + y \cos x &= 1 \\ \frac{d}{dx} [y \sin x] &= 1 \\ \int \frac{d}{dx} [y \sin x] dx &= \int 1 dx \\ y \sin x + c &= x + c \\ y &= \frac{x + c}{\sin x}\end{aligned}$$

(c) $x \frac{dy}{dx} = y - x^2 e^{-x}$

- i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} - x^2 e^{-x} \\ \frac{dy}{dx} - \frac{y}{x} &= -x^2 e^{-x}\end{aligned}$$

- A. From the above, $p(x) = -\frac{1}{x}$
 ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int -\frac{1}{x} dx} \\ &= e^{-\int \frac{1}{x} dx} \\ &= e^{-\ln x} \\ \mu(x) &= e^{\ln x^{-1}} \\ &= x^{-1} \\ &= \frac{1}{x}\end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}\frac{1}{x} \left(\frac{dy}{dx} - \frac{y}{x} \right) &= \frac{1}{x} (-x^2 e^{-x}) \\ - \left(-x^{-2}y + x^{-1} \frac{dy}{dx} \right) &= -x e^{-x} \\ - \frac{dy}{dx} (x^{-1}y) &= -x e^{-x} \\ - \int \frac{dy}{dx} (x^{-1}y) dx &= \int -x e^{-x} dx \\ -x^{-2}y &= - \int x e^{-x} dx\end{aligned}$$

- iv. Before we can proceed, again, integrate by parts, if you don't remember, GG.com...jk,

$$\int u dv = uv - \int v du$$

- A. Let $u = x$, $dv = e^{-x}$

$$\begin{aligned}\frac{du}{dx} &= 1 \\ du &= dx\end{aligned}$$

$$\begin{aligned}\int dv &= \int e^{-x} \\ v &= -e^{-x}\end{aligned}$$

- B. Lets find the answer

$$\int u dv = -x \cdot e^{-x} - \int e^{-x} dx$$

- C. Again, we need to deal with integration, now use u substitution

$$\text{Let } u = -x, du = -dx$$

$$\begin{aligned}- \int e^{-x} dx &= - \int e^u du \\ &= -e^u \\ - \int e^{-x} dx &= -e^{-x} + c\end{aligned}$$

v. Substitute back in

$$\begin{aligned}
 -x^{-1}y &= \int xe^{-x}dx \\
 &= -e^{-x} + c \\
 y &= -e^{-x} \cdot -x - cx \\
 &= xe^{-x} - cx \\
 y &= xe^{-x} + cx \text{ Note: } c \text{ is constant, sign doesn't matter for now}
 \end{aligned}$$

(d) $x \frac{dy}{dx} + 2y = \frac{\sin x}{x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
 \mu(x) &= e^{\int \frac{2}{x} dx} \\
 &= e^{2 \ln x} \\
 \mu(x) &= x^2
 \end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
 x^2 \left[\frac{dy}{dx} + \frac{2y}{x} \right] &= x^2 \left[\frac{\sin x}{x^2} \right] \\
 x^2 \frac{dy}{dx} + 2xy &= \sin x \\
 \frac{d}{dx} [x^2 y] &= \sin x \\
 \int \frac{d}{dx} [x^2 y] dx &= \int \sin x dx + c \\
 x^2 y &= -\cos x + c \\
 y &= \frac{c - \cos x}{x^2}
 \end{aligned}$$

(e) $x \frac{dy}{dx} + 3y = 4x + 3$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
 \frac{1}{x} \left[x \frac{dy}{dx} + 3y \right] &= \frac{1}{x} [4x + 3] \\
 \frac{dy}{dx} + \frac{3y}{x} &= 4 + \frac{3}{x}
 \end{aligned}$$

A. From above $p(x) = \frac{3}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int \frac{3}{x} dx} \\ &= e^{3 \int \frac{1}{x} dx} \\ &= e^{3 \ln x} \\ &= e^{\ln x^3} \\ \mu(x) &= x^3\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}x^3 \left[\frac{dy}{dx} + \frac{3y}{x} \right] &= x^3 \left[4 + \frac{3}{x} \right] \\ x^3 \frac{dy}{dx} + 3x^2 y &= 4x^3 + 3x^2 \\ \frac{d}{dx} [x^3 y] &= 4x^3 + 3x^2 \\ \int \frac{d}{dx} [x^3 y] dx &= \int (4x^3 + 3x^2) dx \\ x^3 y &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\ x^3 y &= x^4 + x^3 + c \\ y &= \frac{x^4}{x^3} + \frac{x^3}{x^3} + \frac{c}{x^3} \\ y &= x + 1 + \frac{c}{x^3}\end{aligned}$$

(f) $\frac{dy}{dx} = 2y + e^{3x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx} &= 2y + e^{3x} \\ \frac{dy}{dx} - 2y &= e^{3x}, p(x) = -2\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int -2 dx} \\ &= e^{\int -2 dx} \\ &= e^{-\int 2 dx} \\ \mu(x) &= e^{-2x}\end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
 e^{-2x} \left[\frac{dy}{dx} - 2y \right] &= e^{-2x} [e^{3x}] \\
 e^{-2x} \frac{dy}{dx} - 2e^{-2x}y &= e^x \\
 \frac{dy}{dx} [e^{-2x}y] &= e^x \\
 \int \frac{dy}{dx} [e^{-2x}y] dx &= \int e^x dx \\
 e^{-2x}y &= e^x + c \\
 y &= \frac{e^x + c}{e^{-2x}} \\
 &= e^{3x} + ce^{-(2x)} \\
 y &= e^{3x} + ce^{2x}
 \end{aligned}$$

(g) $x \frac{dy}{dx} = 2y + x^3 \ln x$

- i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{2y}{x} + x^2 \ln x \\
 \frac{dy}{dx} - \frac{2y}{x} &= x^2 \ln x \\
 \frac{dy}{dx} - \frac{2y}{x} &= x^2 \ln x, p(x) = -\frac{2}{x}
 \end{aligned}$$

- ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
 \mu(x) &= e^{\int p(x)dx} \\
 \mu(x) &= e^{\int -2x^{-1}dx} \\
 &= e^{-2 \int \frac{1}{x} dx} \\
 &= e^{\ln x^{-2}dx} \\
 &= x^{-2} \\
 \mu(x) &= \frac{1}{x^2}
 \end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left

side becomes the product rule $\frac{d}{dx} [\mu(x)y]$ and write it as such.

$$\begin{aligned}\frac{1}{x^2} \left[\frac{dy}{dx} - \frac{2y}{x} \right] &= \frac{1}{x^2} [x^2 \ln x] \\ x^{-2} \frac{dy}{dx} - 2x^{-3}y &= \ln x \\ \frac{dy}{dx} [x^{-2}y] &= \ln x \\ \int \frac{dy}{dx} [x^{-2}y] dx &= \int \ln x dx\end{aligned}$$

A. Time to use Calc 1 skills, Integration by parts

$$\int u dv = uv - \int v du$$

Let $u = \ln x$ and $v' = 1$

$$\frac{du}{dx} = \frac{1}{x}, v = x$$

B. Finally we get

$$\begin{aligned}\int \ln x dx &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c\end{aligned}$$

C. Continue plugging in

$$\begin{aligned}\int \frac{dy}{dx} [x^{-2}y] dx &= x \ln x - x + c \\ x^{-2}y &= x \ln x - x + c \\ y &= x^3 \ln x - x^3 + cx^2\end{aligned}$$

(h) $x \frac{dy}{dx} - 3y = x^4$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}x \frac{dy}{dx} - 3y &= x^4 \\ \frac{dy}{dx} - \frac{3y}{x} &= x^3\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int -\frac{3}{x} dx} \\ &= e^{-3 \int \frac{1}{x} dx} \\ &= e^{-3 \ln x} \\ &= e^{\ln x^{-3}} \\ \mu(x) &= x^{-3}\end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx} [\mu(x)y]$ and write it as such.

$$x^{-3} \left[\frac{dy}{dx} - \frac{3y}{x} \right] = x^{-3} [x^3]$$

$$x^{-3} \frac{dy}{dx} - 3x^{-4}y = 1$$

$$\frac{d}{dx} [x^{-3}y] = 1$$

$$\int \frac{d}{dx} [x^{-3}y] dx = \int 1 dx$$

$$x^{-3}y = x + c$$

$$y = \frac{x}{x^{-3}} + \frac{c}{x^{-3}}$$

$$y = x^4 + cx^3$$

2. Solve the following I.V.P.:

(a) $\frac{dy}{dx} = 9.8 - 0.196y, y(0) = 48$

- i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + 0.196y = 9.8$$

- ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned} \mu(x) &= e^{\int 0.196 dx} \\ &= e^{0.196 \int 1 dx} \\ &= e^{0.196x} \end{aligned}$$

- iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left

side becomes the product rule $\frac{d}{dx} [\mu(x)y]$ and write it as such.

$$\begin{aligned}
 e^{0.196x} \left[\frac{dy}{dx} + 0.196y \right] &= e^{0.196x} [9.8] \\
 e^{0.196x} \frac{dy}{dx} + e^{0.196x} 0.196y &= 9.8e^{0.196x} \\
 \frac{d}{dx} [ye^{0.196x}] &= 9.8e^{0.196x} \\
 \int \frac{d}{dx} [ye^{0.196x}] dx &= \int (9.8e^{0.196x}) dx \\
 ye^{0.196x} &= 9.8 \cdot \left(\frac{1}{0.196} e^{0.196x} + c \right) \\
 ye^{0.196x} &= \frac{9.8}{0.196} e^{0.196x} + 9.8c \\
 y &= \frac{9.8}{0.196 \cdot e^{0.196x}} e^{0.196x} + \frac{9.8c}{e^{0.196x}} \\
 &= \frac{9.8}{0.196} + \frac{9.8c}{e^{0.196x}} \\
 y &= 50 + \frac{9.8c}{e^{0.196x}}
 \end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}
 y(0) &= 50 + \frac{9.8c}{e^{0.196(0)}} \\
 48 &= 50 + \frac{9.8c}{1} \\
 c &= \frac{48 - 50}{9.8} \\
 &= -\frac{10}{49}
 \end{aligned}$$

v. Substitute c back into the original equation

$$\begin{aligned}
 y &= 50 + \frac{9.8 \left(-\frac{10}{49} \right)}{e^{0.196x}} \\
 &= 50 + \frac{9.8 \left(-\frac{10}{49} \right)}{e^{0.196x}} \\
 &= 50 - \frac{2}{e^{0.196x}} \\
 y &= 50 - 2e^{-0.196x}
 \end{aligned}$$

(b) $x \frac{dy}{dx} + y = \frac{x}{x+1}, y(1) = 1$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x+1}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int \frac{1}{x} dx} \\ &= e^{\ln x} \\ \mu(x) &= x\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}x \left[\frac{dy}{dx} + \frac{y}{x} \right] &= x \left[\frac{1}{x+1} \right] \\ x \frac{dy}{dx} + y &= \frac{x}{x+1} \\ \frac{d}{dx}[xy] &= \frac{x}{x+1} \\ \int \frac{d}{dx}[xy] dx &= \int \frac{x}{x+1} dx \\ xy &= \int x(x+1)^{-1} dx\end{aligned}$$

A. On first sight, integration by parts seems to be pretty good, but if you try it out you're in for a world of pain. A combination of long division and u substitution works best in this case. Because:

1. Numerator and denominator same power

B. Do long division

$$\begin{aligned}\int \frac{x}{(x+1)} dx &= \int 1 - \frac{1}{x+1} dx \\ &= \int 1 dx - \int \frac{1}{x+1} dx \\ \int \frac{x}{(x+1)} dx &= x - \ln(x+1) + c\end{aligned}$$

C. Move everything to the right place

$$\begin{aligned}xy &= x - \ln(x+1) + c \\ y &= \frac{x - \ln(x+1) + c}{x} \\ &= 1 - \frac{1}{x} \ln(x+1) + \frac{c}{x}\end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}y(1) &= 1 - \frac{1}{1} \ln(1+1) + \frac{c}{1} \\ 1 &= 1 - \ln(2) + c \\ c &= \ln(2)\end{aligned}$$

v. Substitute c back into the original equation

$$\begin{aligned}
 y &= 1 - \frac{1}{x} \ln(x+1) + \frac{1}{x} \ln(2) \\
 &= 1 - \frac{1}{x} (\ln(x+1) - \ln(2)) \\
 &= 1 - \frac{1}{x} \ln \frac{x+1}{2} \\
 &= 1 + \frac{1}{x} \ln \left(\frac{x+1}{2} \right)^{-1} \\
 y &= 1 + \frac{1}{x} \ln \frac{2}{x+1}
 \end{aligned}$$

(c) $x \frac{dy}{dx} + 3y = x^2 - 4x + 3, y(1) = 0$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1

$$\frac{dy}{dx} + \frac{3}{x}y = x - 4 + \frac{3}{x}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
 \mu(x) &= e^{\int p(x)dx} \\
 \mu(x) &= e^{\int \frac{3}{x}dx} \\
 &= e^{3 \ln x} \\
 &= e^{\ln x^3} \\
 \mu(x) &= x^3
 \end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
 \frac{dy}{dx} + \frac{3}{x}y &= x - 4 + \frac{3}{x} \\
 x^3 \left[\frac{dy}{dx} + \frac{3}{x}y \right] &= x^3 \left[x - 4 + \frac{3}{x} \right] \\
 x^3 \frac{dy}{dx} + 3x^2y &= x^4 - 4x^3 + 3x^2 \\
 \frac{d}{dx} [x^3y] &= x^4 - 4x^3 + 3x^2 \\
 \int \frac{d}{dx} [x^3y] dx &= \int x^4 - 4x^3 + 3x^2 dx \\
 x^3y &= \frac{x^5}{5} - x^4 + x^3 + c \\
 y &= \frac{x^2}{5} - x + 1 + \frac{c}{x^3}
 \end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}y(1) &= \frac{1^2}{5} - 1 + 1 + \frac{c}{1^3} \\0 &= \frac{1}{5} + c \\c &= -\frac{1}{5}\end{aligned}$$

v. Substitute c back into the original equation

$$y = \frac{x^2}{5} - x + 1 - \frac{1}{5x^3}$$

3. Separable Equations:

Solve the following differential equations:

(a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^{2x}}{4y^3} \\4y^3 \frac{dy}{dx} &= e^{2x}\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int 4y^3 \frac{dy}{dx} dx &= \int e^{2x} dx \\4 \int y^3 dy &= \int e^{2x} dx \\4 \left(\frac{y^4}{4} + c \right) &= \frac{1}{2} e^{2x} + c \\y^4 + c &= \frac{1}{2} e^{2x} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}y^4 &= \frac{1}{2} e^{2x} + c \\y &= \pm \sqrt[4]{\frac{1}{2} e^{2x} + c}\end{aligned}$$

(b) $\frac{dy}{dx} = \frac{xy}{2 \ln y}$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\frac{dy}{dx} = \frac{xy}{2 \ln y}$$

$$2 \ln y \frac{dy}{dx} = xy$$

$$\frac{2}{y} \frac{dy}{dx} \ln y = x$$

ii. Integrate both sides

$$\int \frac{2}{y} \frac{dy}{dx} \ln y dx = \int x dx$$

$$2 \int \frac{1}{y} \ln y dy = \frac{x^2}{2} + c$$

$$2 \left(\frac{\ln^2 y}{2} + c \right) = \frac{x^2}{2} + c \text{ check below for elaboration}$$

$$\ln^2 y + c = \frac{x^2}{2} + c$$

A. Use u - substitution, let $u = \ln y$

$$\frac{du}{dy} = \frac{1}{y}$$

$$du = \frac{1}{y} dy$$

$$\int \frac{1}{y} \ln y dy = \int u du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{(\ln y)^2}{2} + c$$

$$\int \frac{1}{y} \ln y dy = \frac{\ln^2 y}{2} + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

B.

$$\ln^2 y + c = \frac{x^2}{2} + c$$

$$\ln y = \pm \sqrt{\frac{x^2}{2} + c}$$

$$y = e^{\pm \sqrt{\frac{x^2}{2} + c}}$$

(c) $\frac{dx}{dt} + e^{t+x} = 0$

Before-we-start: This question is a lil' tricky. Usually, we have x as our parameter, but this has t as parameter. Therefore, we need to make it $x(t) = \dots$ instead.

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dx}{dt} + e^{t+x} &= 0 \\ \frac{dx}{dt} &= -e^{t+x} \\ &= -e^t \cdot e^x \\ \frac{1}{e^x} dx &= \frac{1}{e^t} dt\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-x} dx &= \int e^{-t} dt \\ -e^{-x} + c &= -e^{-t} + c \\ e^{-x} &= e^{-t} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\ln e^{-x} &= \ln(e^{-t} + c) \\ -x &= \ln(e^{-t} + c) \\ x &= -\ln(e^{-t} + c)\end{aligned}$$

(d) $2\sqrt{xy} \frac{dy}{dx} = 1, x, y > 0$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}2\sqrt{xy} \frac{dy}{dx} &= 1 \\ \sqrt{xy} dy &= \frac{1}{2} dx \\ \sqrt{x} \sqrt{y} dy &= \frac{1}{2} dx \\ y^{\frac{1}{2}} dy &= \frac{1}{2} x^{-\frac{1}{2}} dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int y^{\frac{1}{2}} dy &= \int \frac{1}{2} x^{-\frac{1}{2}} dx \\ \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c &= \frac{1}{2} \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \right) \\ \frac{2}{3} y^{\frac{3}{2}} + c &= \frac{1}{2} \left(2x^{\frac{1}{2}} + c \right) \\ \frac{2}{3} y^{\frac{3}{2}} + c &= x^{\frac{1}{2}} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\frac{2}{3} y^{\frac{3}{2}} &= x^{\frac{1}{2}} + c \\ y^{\frac{3}{2}} &= \frac{3}{2} \left(x^{\frac{1}{2}} + c \right) \\ y^{\frac{3}{2}} &= \frac{3}{2} \sqrt{x} + c\end{aligned}$$

(e) $\frac{dy}{dx} = \frac{xy}{x+2}$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy}{x+2} \\ \frac{1}{xy} dy &= \frac{1}{x+2} dx \\ \frac{1}{x} \cdot \frac{1}{y} dy &= \frac{1}{x+2} dx \\ \frac{1}{y} dy &= \frac{x}{x+2} dx \\ y^{-1} dy &= \frac{x}{x+2} dx\end{aligned}$$

ii. Integrate both sides

$$\int y^{-1} dy = \int \frac{x}{x+2} dx$$

A. For first part,

$$\int y^{-1} dy = \ln y + c$$

- B. For second part, ideally you want to use long division + u-substitution. Just something learned from above.

$$\begin{aligned}\int \frac{x}{x+2} dx &= \int 1 - \frac{2}{x+2} dx \\ &= \int 1 dx - 2 \int \frac{1}{x+2} dx\end{aligned}$$

Let $u = x + 2$,

$$\begin{aligned}\frac{du}{dx} &= 1 \\ du &= dx\end{aligned}$$

$$\begin{aligned}\int 1 dx - 2 \int \frac{1}{x+2} dx &= x - 2 \int \frac{1}{u} du + c \\ &= x - 2 \ln u + c \\ &= x - 2 \ln(x+2) + c\end{aligned}$$

- C. Find the final integration

$$\int y^{-1} dy = \int \frac{x}{x+2} dx$$

- iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

- A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\ln y + c &= x - 2 \ln(x+2) + c \\ \ln y &= x - \ln(x+2)^2 + c \\ y &= e^{x - \ln(x+2)^2 + c} \\ &= \frac{e^{x+c}}{e^{\ln(x+2)^2}} \\ &= \frac{e^x \cdot e^c}{(x+2)^2} \\ y &= \frac{Ae^x}{(x+2)^2}, A = e^c\end{aligned}$$

(f) $y(x^2 - 1) \frac{dy}{dx} = 1$

- i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}y(x^2 - 1) \frac{dy}{dx} &= 1 \\ y \frac{dy}{dx} &= \frac{1}{(x^2 - 1)} \\ y dy &= \frac{1}{(x^2 - 1)} dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int y dy &= \int \frac{1}{(x^2 - 1)} dx \\ \frac{y^2}{2} + c &= - \int \frac{1}{1 - x^2} dx \\ \frac{y^2}{2} + c &= - \tanh^{-1}(x) + c \\ &= -\frac{1}{2} \ln(x + 1) + \frac{1}{2} \ln(x - 1) + c \\ \frac{y^2}{2} + c &= -\frac{1}{2} \ln(x + 1) + \frac{1}{2} \ln(x - 1) + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\frac{y^2}{2} + c &= -\frac{1}{2} \ln(x + 1) + \frac{1}{2} \ln(x - 1) + c \\ y^2 &= \ln(x - 1) - \ln(x + 1) + c \\ y^2 &= \ln \frac{x - 1}{x + 1} + c \\ y &= \sqrt{\ln \frac{x - 1}{x + 1} + c}\end{aligned}$$

B. Let $A = e^c$

$$\begin{aligned}y &= \sqrt{\ln \frac{x - 1}{x + 1} + \ln A} \\ &= \sqrt{\ln \frac{A(x - 1)}{x + 1}}\end{aligned}$$

(g) $\frac{dx}{dt} = \frac{4 \sin t + 6 \cos 2t}{x}$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dx}{dt} &= \frac{4 \sin t + 6 \cos 2t}{x} \\ x dx &= 4 \sin t + 6 \cos 2t dt\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int x dx &= \int 4 \sin t + 6 \cos 2t dt \\ \frac{x^2}{2} + c &= 4(-\cos t + c) + 6 \int \cos 2t dt \\ &= 4(-\cos t + c) + 6 \left(\frac{1}{2} \sin(2t) + c \right) \\ \frac{x^2}{2} + c &= -4 \cos t + 3 \sin(2t) + c\end{aligned}$$

A. To integrate $\cos 2t$, let $u = 2t$

$$\begin{aligned}\frac{du}{dt} &= 2 \\ du &= 2dt\end{aligned}$$

$$\begin{aligned}\int \cos 2t dt &= \frac{1}{2} \int \cos u du \\ &= \frac{1}{2} (\sin u + c) \\ \int \cos 2t dt &= \frac{1}{2} \sin(2t) + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\frac{x^2}{2} + c &= -4 \cos t + 3 \sin(2t) + c \\ \frac{x^2}{2} &= -4 \cos t + 3 \sin 2t + c\end{aligned}$$

(h) $\frac{dy}{dx} = e^{-y} (2x - 4)$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{1}{e^{-y}} dy &= (2x - 4) dx \\ e^y dy &= (2x - 4) dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^y dy &= \int (2x - 4) dx \\ e^y + c &= \frac{2x^2}{2} - 4x + c \\ e^y + c &= x^2 - 4x + c\end{aligned}$$

- iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$e^y + c = x^2 - 4x + c$$

$$e^y = x^2 - 4x + c$$

$$\ln e^y = \ln(x^2 - 4x + c)$$

$$y = \ln(x^2 - 4x + c)$$

(i) $\sec x \frac{dy}{dx} = e^{y+\sin x}$

- i. Write D.E. as $g(y) dy = f(x) dx$

$$\cos x \frac{dx}{dy} = \frac{1}{e^{y+\sin x}}$$

$$\cos x dx = e^{-y} \cdot e^{-\sin x}$$

$$\frac{\cos x}{e^{-\sin x}} dx = e^{-y} dy$$

- ii. Integrate both sides

$$\int e^{-y} dy = \int \cos x \cdot e^{\sin x} dx$$

$$-e^{-y} + c = \int \cos x \cdot e^{\sin x} dx$$

$$= \int \cos x \cdot e^{\sin x} dx$$

$$e^y = -e^{\sin x} + c \text{ (check below)}$$

A. Integrating $\int \cos x \cdot e^{\sin x} dx$ probably needs u -integration.

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x dx$$

$$\int \cos x \cdot e^{\sin x} dx = \int \cos x \cdot e^{\sin x} dx$$

$$= \int e^u du$$

$$= e^u + c$$

$$= e^{\sin x} + c$$

$$\int \cos x \cdot e^{-\sin x} dx = c - e^{\sin x}$$

- iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$e^{-y} = c - e^{\sin x}$$

$$\ln e^{-y} = \ln (c - e^{\sin x})$$

$$y = -\ln (c - e^{\sin x})$$

(j) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$

- i. Write D.E. as $g(y) dy = f(x) dx$

$$2y - 4 dy = 3x^2 + 4x - 4 dx$$

- ii. Integrate both sides

$$\int 2y - 4 dy = \int 3x^2 + 4x - 4 dx$$

$$y^2 - 4y + c = x^3 + 2x^2 - 4x + c$$

- iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

$$y^2 - 4y - (x^3 + 2x^2 - 4x + c) = 0$$

B. $a = 1, b = -4, c = -(x^3 + 2x^2 - 4x + c)$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-(x^3 + 2x^2 - 4x + c))}}{2(1)}$$

$$= \frac{4 \pm \sqrt{4^2 + 4(x^3 + 2x^2 - 4x + c)}}{2}$$

$$= 2 \pm \frac{\sqrt{4(4 + x^3 + 2x^2 - 4x + c)}}{2}$$

$$= 2 \pm \frac{2\sqrt{4 + x^3 + 2x^2 - 4x + c}}{2}$$

$$= 2 \pm \sqrt{4 + x^3 + 2x^2 - 4x + c}$$

4. Solve the following I.V.P.:

(a) $y' = \frac{y \cos x}{1+y^2}, y(0) = 1$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dy}{dx} &= \frac{y \cos x}{1 + y^2} \\ (1 + y^2) dy &= y \cos x dx \\ \frac{(1 + y^2)}{y} dy &= \cos x dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\frac{(1 + y^2)}{y} dy &= \cos x dx \\ \int \frac{1}{y} + \frac{y^2}{y} dy &= \int \cos x dx \\ \int \frac{1}{y} dy + \int y dy &= \sin x + c \\ \ln y + \frac{y^2}{2} + c &= \sin x + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$). Note in some cases (like this, it is impossible at our current level)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\ln y + \frac{y^2}{2} + c &= \sin x + c \\ \ln y + \frac{y^2}{2} &= \sin x + c\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}\ln 1 + \frac{1^2}{2} &= \sin(0) + c \\ \frac{1}{2} &= 0 + c \\ c &= \frac{1}{2}\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\ln y + \frac{y^2}{2} = \sin x + \frac{1}{2}$$

(b) $x + 2y\sqrt{x^2 + 1} \frac{dy}{dx} = 0, y(0) = 1$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}x + 2y\sqrt{x^2 + 1} \frac{dy}{dx} &= 0 \\2y\sqrt{x^2 + 1} \frac{dy}{dx} &= -x \\2y \frac{dy}{dx} &= -\frac{x}{\sqrt{x^2 + 1}} \\2y dy &= -\frac{x}{\sqrt{x^2 + 1}} dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int 2y dy &= \int -\frac{x}{\sqrt{x^2 + 1}} dx \\y^2 + c &= -\int x (x^2 + 1)^{-\frac{1}{2}} dx \\y^2 + c &= -(x^2 + 1)^{\frac{1}{2}} + c \text{ (Check below)}\end{aligned}$$

A. Integrate $-\int x (x^2 + 1)^{-\frac{1}{2}} dx$, I'm thinking about u -substitution.
Let $u = x^2 + 1$

$$\begin{aligned}\frac{du}{dx} &= 2x \\du &= 2x dx\end{aligned}$$

$$\begin{aligned}-\int x (x^2 + 1)^{-\frac{1}{2}} dx &= -\frac{1}{2} \int (x^2 + 1)^{-\frac{1}{2}} \cdot 2x dx \\&= -\frac{1}{2} \int u^{-\frac{1}{2}} du \\&= -\frac{1}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \\&= -u^{\frac{1}{2}} + c \\-\int x (x^2 + 1)^{-\frac{1}{2}} dx &= -(x^2 + 1)^{\frac{1}{2}} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}y^2 + c &= -(x^2 + 1)^{\frac{1}{2}} + c \\y^2 &= -(x^2 + 1)^{\frac{1}{2}} + c\end{aligned}$$

iv. Substitute in the values given to find C , $y(0) = 1$

$$\begin{aligned}1^2 &= -\left((0)^2 + 1\right)^{\frac{1}{2}} + c \\1 &= -1 + c \\c &= 2\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$y^2 = -(x^2 + 1)^{\frac{1}{2}} + 2$$

(c) $\frac{dy}{dt} = te^y, y(1) = 0$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\frac{1}{e^y} dy = t dt$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-y} dy &= \int t dt \\-e^{-y} + c &= \frac{t^2}{2} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}-e^{-y} + c &= \frac{t^2}{2} + c \\-e^{-y} &= \frac{t^2}{2} + c - c \\e^{-y} &= -\frac{t^2}{2} + c \text{ (Const. - cons. = cons. (even 0))} \\-y \ln e &= \ln \left(-\frac{t^2}{2} + c \right) \\y &= -\ln \left(-\frac{t^2}{2} + c \right) \\&= \ln \frac{1}{-\frac{t^2}{2} + c} \\y &= \ln \frac{1}{-\frac{1}{2}t^2 + c}\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
 0 &= \ln \frac{1}{-\frac{1}{2}1^2 + c} \\
 \ln \frac{1}{-\frac{1}{2} + c} &= 0 \\
 \frac{1}{-\frac{1}{2} + c} &= e^0 \\
 -\frac{1}{2} + c &= 1 \\
 c &= 1 + \frac{1}{2} \\
 c &= \frac{3}{2}
 \end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
 y &= \ln \frac{1}{-\frac{t^2}{2} + \frac{3}{2}} \\
 &= \ln \frac{1}{\frac{-t^2+3}{2}} \\
 y &= \ln \frac{2}{3-t^2}
 \end{aligned}$$

(d) $t(t-1) \frac{dx}{dt} = x(x+1), x(2) = 2$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}
 t(t-1) \frac{dx}{dt} &= x(x+1) \\
 \frac{dx}{x(x+1)} &= \frac{dt}{t(t-1)}
 \end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}
 \int \frac{1}{x(x+1)} dx &= \int \frac{1}{t(t-1)} dt \\
 \int \frac{1}{x(x+1)} dx &= \int \frac{1}{t(t-1)} dt
 \end{aligned}$$

A. For both parts, we need to use partial fractions, lets start

with $\frac{1}{x(x+1)}$

$$\begin{aligned}\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\ &= \frac{A(x+1) + Bx}{x(x+1)} \\ \frac{x(x+1)}{x(x+1)} &= A(x+1) + Bx \\ 1 &= A(x+1) + Bx\end{aligned}$$

When $x = 0$

$$\begin{aligned}1 &= A(1) + 0 \\ A &= 1\end{aligned}$$

When $x = -1$

$$\begin{aligned}1 &= A(-1+1) + B(-1) \\ 1 &= -B \\ B &= -1\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\ &= \frac{1}{x} - \frac{1}{x+1}\end{aligned}$$

Start integrating:

$$\begin{aligned}\int \frac{1}{x} - \frac{1}{x+1} dx &= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx \\ &= \ln(x) - \ln(x+1) + c\end{aligned}$$

B. Lets integrate the other one, $\int \frac{1}{t(t-1)} dt$
Again, partial fractions first.

$$\begin{aligned}\frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1} \\ 1 &= A(t-1) + Bt\end{aligned}$$

When $t = 1$

$$B = 1$$

When $t = 2, B = 1$

$$1 = A(2 - 1) + 1(2)$$

$$1 = A + 2$$

$$A = -1$$

Substitute back in

$$\begin{aligned}\frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1} \\ &= -\frac{1}{t} + \frac{1}{t-1}\end{aligned}$$

Integrate:

$$\begin{aligned}\int -\frac{1}{t} + \frac{1}{t-1} dt &= \int -\frac{1}{t} dt + \int \frac{1}{t-1} dt \\ &= -\ln(t) + \ln(t-1) + c\end{aligned}$$

C. Combine the entire equation

$$\begin{aligned}\int \frac{1}{x(x+1)} dx &= \int \frac{1}{t(t-1)} dt \\ \ln(x) - \ln(x+1) + c &= -\ln(t) + \ln(t-1) + c\end{aligned}$$

- iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}\ln(x) - \ln(x+1) + c_1 &= -\ln(t) + \ln(t-1) + c_2 \\ \ln(x) - \ln(x+1) + c_1 - c_1 &= -\ln(t) + \ln(t-1) + c_2 - c_1 \\ \ln(x) - \ln(x+1) &= -\ln(t) + \ln(t-1) + c\end{aligned}$$

$$\ln \frac{x}{x+1} = \ln \frac{(t-1)}{t} + c$$

$$\ln \frac{x}{x+1} = \ln \frac{(t-1)}{t} + c$$

$$\ln \frac{x}{x+1} = \ln \frac{(t-1)}{t} + \ln e^c$$

$$\ln \frac{x}{x+1} = \ln \frac{e^c(t-1)}{t}$$

$$\frac{x}{x+1} = \frac{e^c(t-1)}{t}$$

$$1 - \frac{1}{x+1} = \frac{e^c(t-1)}{t}$$

$$-\frac{1}{x+1} = \frac{e^c(t-1)}{t} - 1$$

$$\frac{1}{x+1} = -\frac{e^c(t-1)}{t} + 1$$

$$x+1 = -\frac{t}{t+e^c(t-1)}$$

$$x = -\frac{t}{t+e^c(t-1)} - 1$$

$$= -\frac{t - (t+e^c(t-1))}{t+e^c(t-1)}$$

$$= -\frac{-e^c(t-1)}{t+e^c(t-1)}$$

$$= \frac{e^c(t-1)}{t+e^c(t-1)}$$

$$= \frac{c(t-1)}{t+c(t-1)}$$

iv. Substitute in the values given to find C

$$\begin{aligned}x &= \frac{c(t-1)}{t+c(t-1)} \\2 &= \frac{c(2-1)}{2+c(2-1)} \\2 &= \frac{c}{2+c} \\4+2c &= c \\c &= -\frac{4}{3}\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}x &= \frac{-\frac{4}{3}(t-1)}{t+\frac{4}{3}(t-1)} \\&= \frac{-\frac{4}{3}t+\frac{4}{3}}{t-\frac{4}{3}t+\frac{4}{3}} \\&= \frac{\frac{-4t+4}{3}}{\frac{t+4}{3}} \\x &= \frac{4(1-t)}{t-4}\end{aligned}$$

(e) $\frac{dx}{dt} = e^{x+t}, x(0) = a$

i. Write D.E. as $g(y) dy = f(x) dx$

$$\begin{aligned}\frac{dx}{dt} &= e^{x+t} \\\frac{dx}{dt} &= e^x e^t \\e^{-x} dx &= e^t dt\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-x} dx &= \int e^t dt \\-e^{-x} + c &= e^t + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C , constant of integration

$$\begin{aligned}
 -e^{-x} &= e^t + c - c \\
 -e^{-x} &= e^t + c - c \\
 \ln(-e^{-x}) &= \ln(e^t + c) \\
 \ln(-1) + \ln(e^{-x}) &= \ln(e^t + c) \\
 \ln(-1) - x &= \ln(e^t + c) \\
 -x &= \ln(e^t + c) - \ln(-1) \\
 x &= -\ln \frac{(e^t + c)}{-1} \\
 &= -\ln(-e^t - c) \\
 &= -\ln(-e^t + c) \\
 &= -\ln(-e^t + c)
 \end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
 a &= -\ln(-e^0 + c) \\
 &= -\ln(-1 + c) \\
 a &= -\ln(-1 + c) \\
 \ln(-1 + c) &= -a \\
 -1 + c &= e^{-a} \\
 c &= e^{-a} + 1
 \end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
 x &= -\ln(-e^t + e^{-a} + 1) \\
 x &= -\ln(1 + e^{-a} - e^t)
 \end{aligned}$$