Tutorial 8

December 30, 2019

1. Determine whether the series is convergent or divergent. If it is convergent, find its sum:

   (a) $1 + 0.4 + 0.16 + 0.064 \ldots$

      i. $T_n = 0.4^n, n = 0, 1, 2, 3...$

      ii. $a = 1,\ r = 0.4$

      iii. The series is convergent

      $$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{1}{1-0.4} \\ S_\infty &= \frac{5}{3} \end{aligned}$$

   (b) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} ...$

      i. Calculations

      $$\begin{aligned} r_1 &= \frac{-\frac{10}{3}}{5} \\ &= -\frac{2}{3} \end{aligned}$$

      $$\begin{aligned} r_2 &= \frac{\frac{20}{9}}{-\frac{10}{3}} \\ &= -\frac{2}{3} \\ a &= 5 \\ r &= -\frac{2}{3} \end{aligned}$$

      ii. Convergent,

      $$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{5}{1-\left(-\frac{2}{3}\right)} \\ S_\infty &= 3 \end{aligned}$$

(c) $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

i. Convergent

ii. $a = 3$

iii. $r = \frac{1}{2}$

iv. Sum

$$\begin{aligned} S_\infty &= \frac{3}{1-\frac{1}{2}} \\ S_\infty &= 6 \end{aligned}$$

(d) $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$

$$\frac{(-6)^{n-1}}{5^{n-1}} = \left(-\frac{6}{5}\right)^{n-1}$$

i. $n^\infty = \infty$

ii. Divergent

(e) $\sum_{n=1}^{\infty} 8^{-n}3^{n+1}$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3^{n+1}}{8^n} &= \sum_{n=1}^{\infty} 3 \cdot \frac{3^n}{8^n} \\ &= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{8}\right)^n \\ &= \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{8}\right) \cdot \left(\frac{3}{8}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{9}{8}\left(\frac{3}{8}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{9}{8}\left(\frac{3}{8}\right)^{n-1} \end{aligned}$$

i. $a = \frac{9}{8}, r = \frac{3}{8}$

ii. Convergent, sum:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{\frac{9}{8}}{1-\frac{3}{8}} \\ &= \frac{9}{5} \end{aligned}$$

(f) $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

i. Use the comparison test

$$n^2 + 4n + 3 > n^2 + 4n > n^2$$
$$\frac{2}{n^2 + 4n + 3} < \frac{2}{n^2}$$

ii. $2\sum \frac{1}{n^2}$ is a convergent $p - series$.

iii. $\therefore \sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ is convergent.

iv. ALTERNATIVELY

A. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$ Note: partial fractions

B. Notice that this is a telescoping eries

$$\begin{aligned}\sum\left(\frac{1}{n+1}\right) - \sum\left(\frac{1}{n+3}\right) &= \left(\frac{1}{2} + \frac{1}{3} + \cancel{\frac{1}{4} + .. + \frac{1}{(n-1)+1} + \frac{1}{n+1}}\right) - \left(\cancel{\frac{1}{4} + \frac{1}{5}} + .\right.\\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{(n-1)+3} + \frac{1}{n+3}\end{aligned}$$

C. When $n \to \infty$

$$\begin{aligned}S_\infty &= \frac{1}{2} + \frac{1}{3} + \frac{1}{\infty} + \frac{1}{\infty}\\ &= \frac{1}{2} + \frac{1}{3}\\ &= \frac{5}{6}\end{aligned}$$

(g) $\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$

$$\begin{aligned}a_n &= \frac{1}{e^{2n}}\\ \lim \frac{1}{e^{2n}} &= \frac{1}{e^{2\infty}}\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

i. Therefore, it is convergent

(h) $\sum_{n=1}^{\infty} \frac{3}{n}$

$$\begin{aligned}3\sum \frac{1}{n} &= \sum \frac{1}{n}\\ &= \text{harmonic series, divergent}\end{aligned}$$

(i) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$

$$
\begin{aligned}
a_n &= \frac{(n+1)^2}{n\,(n+2)} \\
a_1 &= \frac{4}{3} = 1.333 \\
a_2 &= \frac{9}{16} = 1.123 \\
a_3 &= \frac{16}{15} = 1.066
\end{aligned}
$$

i. Divergent, $n - th$ term test.

ii. $\lim_{n\to\infty} a_n$

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+2n} \div n^2 \\
&= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{2}{n}} \\
&= \lim_{n\to\infty} \frac{1+0+0}{1+0} \\
&= 1\,(\neq 0)
\end{aligned}
$$

(j) $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n}$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^n}{6^n} + \frac{2^n}{6^n} &= \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n}\right) \\
&= \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n
\end{aligned}
$$

i. Since both $|r| \leq 1$, convergent.

ii. Sum

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{\frac{1}{2}}{1-\frac{1}{2}} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{\frac{1}{3}}{1-\frac{1}{3}} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^n}{6^n} + \frac{2^n}{6^n} &= 1 + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}
$$

(k) $\sum_{n=1}^{\infty} \frac{1}{5+2^{-n}}$

i. $\sum_{n=1}^{\infty} \frac{1}{5+\frac{1}{2^n}}$

ii. Step 1: Write out some terms

iii. Step 2: Find a formula for S_n

(l) $\sum_{n=1}^{\infty} \left(\frac{3}{n^2+4n+3} - 8^{-n}3^{n+1} \right)$

i. Solution

$$
\begin{aligned}
\sum \left(\frac{3}{n^2+4n+3} - 8^{-n}3^{n+1} \right) &= \frac{3}{2} \left(\sum \frac{3}{n^2+4n+3} \right) - \sum 8^{-n}3^{n+1} \\
&= \frac{3}{2} \left(\frac{5}{6} \right) - \frac{9}{5} \text{Note: } \frac{5}{6} \text{comes from Q1f,} \frac{9}{5} \text{comes from Q1e} \\
&= -\frac{11}{20}
\end{aligned}
$$

ii. Solution Errata: $-\frac{11}{20}$

2. Use the Divergence Test (n-th test for divergence) to show that the following series diverges.

Steps for Divergence Test

1. If $\lim_{n\to\infty} a_n = 0$, converges
2. Otherwise, diverges

(a) $\sum_{n=1}^{\infty} \tan^{-1} n$

$$
\begin{aligned}
f(x) &= \tan^{-1} x \\
\lim_{n\to\infty} \tan^{-1} x &= \lim_{n\to\infty} \tan^{-1} x \\
&= \frac{\pi}{2} \neq 0 \\
&= \mathit{diverges}
\end{aligned}
$$

i. Since $\lim_{n\to\infty} \tan^{-1} x$ diverges, $\sum_{n=1}^{\infty} \tan^{-1} n$ diverges.

(b) $\sum_{n=1}^{\infty} \frac{1-n^2}{4+n^2}$

$$\begin{aligned}
f\left(x\right) &= \frac{1-x^2}{4+x^2} \\
\lim_{n\to\infty} f\left(x\right) &= \lim_{n\to\infty} \frac{1-x^2}{4+x^2} \\
\lim_{n\to\infty} f\left(x\right) &= \lim_{n\to\infty} \frac{\frac{1}{x^2}-\frac{x^2}{x^2}}{\frac{4}{x^2}+\frac{x^2}{x^2}} \\
&= \lim_{n\to\infty} \frac{0-1}{0+1} \\
&= -1 \neq 0 \\
&= diverges
\end{aligned}$$

i. Since $\lim_{n\to\infty} \frac{1-x^2}{4+x^2}$ diverges, $\sum_{n=1}^{\infty} \frac{1-n^2}{4+n^2}$ diverges

(c) $\sum_{n=1}^{\infty} \frac{2}{5-2^{-n}}$

$$\begin{aligned}
\lim_{n\to\infty} f\left(x\right) &= \lim_{n\to\infty} \frac{2}{5-2^{-x}} \\
&= \lim_{n\to\infty} \frac{2}{5-\frac{1}{2^x}} \\
&= \frac{2}{5-\frac{1}{2^\infty}} \\
&= \frac{2}{5} \neq 0 \\
&= diverges
\end{aligned}$$

i. Since $\lim_{n\to\infty} \frac{2}{5-2^{-x}}$ diverges, $\sum_{n=1}^{\infty} \frac{2}{5-2^{-n}}$ diverges

(d) $\sum_{n=1}^{\infty} \frac{2n}{\ln(n+3)}$

$$\begin{aligned}
f\left(x\right) &= \frac{2x}{\ln\left(x+3\right)} \\
\lim_{n\to\infty} f\left(x\right) &= \lim_{n\to\infty} \frac{\frac{d}{dx}\left[2x\right]}{\frac{d}{dx}\left[\ln\left(x+3\right)\right]} \\
&= \lim_{n\to\infty} \frac{2}{\frac{1}{x+3}} \\
&= \lim_{n\to\infty} 2\left(x+3\right) \\
&= \lim_{n\to\infty} 2x+6 \\
\lim_{n\to\infty} f\left(x\right) &= \infty \\
&= diverges
\end{aligned}$$

3. Determine whether the following series converges or diverges, using the given test.

   (a) Direct Comparison Test

   Direct Comparison Test Steps
   Important: BOTH Positive, $x > 0$
   1. Find a formula, b_n that is easy to evaluate, and is bigger than a_n, the given formula, for ALL terms. Usually by dropping a few terms.
   2. Check if the series b_n is convergent/divergent

   i. $\sum_{n=1}^{\infty} \frac{1}{4+n^2}$

   A. **Drop the denominator**

$$\frac{1}{4+n^2} < \frac{1}{n^2}$$

   B. **Determine if b_n is convergent/divergent.**

   C. Since ALL terms in both sequence are positive, $\frac{1}{4+n^2} < \frac{1}{n^2}$ for ALL terms, and b_n converges, therefore, $\sum_{n=1}^{\infty} \frac{1}{4+n^2}$ will also **converge**.

   ii. $\sum_{n=1}^{\infty} \frac{1}{3n-5}$

   A. **Drop the denominator**

$$3n - 5 < 3n$$

$$\frac{1}{3n-5} > \frac{1}{3n}$$

$$\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$$

   B. Note: we cannot drop $\frac{1}{3}$ directly and make it $3n - 5 < 3n$, otherwise the statement will no longer be guaranteed.

   C. **Determine if b_n is convergent/divergent.** b_n is a harmonic series, and is always divergent

   D. Since ALL terms in both sequence are positive when $n > 0$, $\frac{1}{3n-5} > \frac{1}{n}$ for ALL terms, and b_n diverges, therefore,$\sum_{n=1}^{\infty} \frac{1}{3n-5}$ will also **diverge**.

   iii. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

   A. Since $\sin^2 n$ is always between 0 and 1, lets drop the numerator

$$\frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}}$$
$$\leq \frac{1}{n^{\frac{3}{2}}}$$

B. Determine if b_n is convergent. b_n is a $p-series$, with $p = \frac{3}{2}$. Since $p > 1$, b_n converges.

C. When $n > 0$, since ALL terms in both sequence are positive , and $\frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}}$ for all terms, and b_n converges, therefore, the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ also **converges**.

(b) Integral Test

i. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

A. Make it into a function

$$f(x) = \frac{1}{\sqrt[3]{x}}$$

B. Integrate the function

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt[3]{x}} dx \\
&= \lim_{b\to\infty} \int_1^b x^{-\frac{1}{3}} dx \\
&= \lim_{b\to\infty} \left[\frac{3}{2} x^{\frac{2}{3}}\right]_1^b \\
&= \lim_{b\to\infty} \frac{3}{2}\left(b^{\frac{2}{3}} - 1\right) \\
&= \infty
\end{aligned}$$

C. Conclusion. Since $\int_1^{\infty} f(x)\,d = \infty$, it diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ is divergent.

ii. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

A. **Make it into a function**

$$f(x) = \frac{1}{x \ln x}$$

B. **Integrate the function**

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x \ln x} dx \\
&= \lim_{b\to\infty} \int_1^b \frac{1}{x \ln x} dx
\end{aligned}$$

Let $u = \ln x$

$$\begin{aligned}
\frac{du}{dx} &= \frac{1}{x} \\
du &= \frac{1}{x} dx
\end{aligned}$$

Subsitute inside

$$\begin{aligned}
\lim_{b\to\infty}\int_1^b \frac{1}{x\ln x}dx &= \lim_{b\to\infty}\int_1^b \frac{1}{u}du \\
&= \lim_{b\to\infty}\left[\ln u\right]_1^b \\
&= \lim_{b\to\infty}\left(\ln b - \ln 1\right) \\
&= \lim_{b\to\infty}\ln b \\
&= \infty
\end{aligned}$$

C. **Conclusion**. Since $\int_1^\infty f(x)\, d = \infty$, it diverges, then $\sum_{n=1}^{\infty}\frac{1}{n\ln n}$ is divergent.

(c) Limit Comparison Test

i. $\sum_{n=1}^{\infty}\frac{1+n^2}{1+n^4}$

A. Simplify this series, fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as,

$$b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$$

, $p-series$ with $p = 2 > 1$, converges.

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1+n^2}{1+n^4}\cdot n^2 \\
&= \lim_{n\to\infty}\frac{n^2+n^4}{1+n^4} \\
&= \lim_{n\to\infty}\frac{n^2+n^4}{1+n^4} \div \frac{n^4}{n^4} \\
&= \lim_{n\to\infty}\frac{n^{-2}+1}{n^{-4}+1} \\
c &= 1
\end{aligned}$$

B. Since $c > 0$, and the series b_n converges, $\sum_{n=1}^{\infty}\frac{1+n^2}{1+n^4}$ must **converge.**

ii. $\sum_{n=1}^{\infty}\frac{1}{n^3-n}$

A. Simplify this series, fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as,

$$b_n = \frac{1}{n^3}$$

, which is a $p-series$ with $p=3>1$, converges

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1}{n^3-n}\cdot n^3\\ &= \lim_{n\to\infty}\frac{n^3}{n^3-n}\div\frac{n^3}{n^3}\\ &= \lim_{n\to\infty}\frac{1}{1-\frac{1}{n^2}}\\ c &= 1>0\end{aligned}$$

B. Since $c>0$, and the series b_n converges, $\sum_{n=1}^{\infty}\frac{1}{n^3-n}$ must **converge.**

iii. $\sum_{n=1}^{\infty}\frac{n+7}{\sqrt[3]{n^7+n^2}}$

A. Incorrect (wrong way) solution

B. Simplify this series, fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as,

$$\begin{aligned}b_n &= \frac{n}{\sqrt[3]{n^7}}\\ &= \frac{n}{n^{\frac{7}{3}}}\\ b_n &= \frac{1}{n^{\frac{4}{3}}}\end{aligned}$$

, which is a $p-series$ with $p=\frac{4}{3}>1$, **converges**

C. Since $c>0$, and the series b_n converges, $\sum_{n=1}^{\infty}\frac{n+7}{\sqrt[3]{n^7+n^2}}$ must **converge.**

D. **Correct solution**

$$b_n=\frac{n}{\sqrt[3]{n^3}}, b_n\text{is a convergent}p-series$$

$$p=\frac{4}{3}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{\left(\frac{n+7}{\sqrt[3]{n^7+n^2}}\right)}{\frac{n}{\sqrt[3]{n^7}}} &= \lim_{n\to\infty}\left(\frac{n+7}{\sqrt[3]{n^7+n^2}}\cdot\frac{3\sqrt{n^7}}{n}\right)\\ &= \lim_{n\to\infty}\left(\frac{n+7}{n}\cdot\frac{\sqrt[3]{n^7}}{\sqrt[3]{n^7+n^2}}\right)\\ &= \lim_{n\to\infty}\left(\left(1+\frac{7}{n}\right)\cdot\sqrt[3]{\frac{1}{1+n^{-5}}}\right)\\ &= (1+0)\cdot\sqrt[3]{\frac{1}{1+0}}=1\,(converges)\end{aligned}$$

(d) Alternating Series Test

Theorem 3.11: If the alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ where $b_n > 0$ 0 satisfies:
1. $b_{n+1} \leq b_n$ for all n (decreasing);
2. $\lim_{n\to\infty} b_n = 0$
Then the series is convergent.
Note: This test CANNOT be used to determine if the series is divergent.

i. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+4}$

A. $a_n = \frac{(-1)^{n+1}}{n+4}, b_n = \frac{1}{n+4}$

B. $b_n = \left(\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, ...\right)$, decreasing

C. $\lim_{n\to\infty} b_n$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n+4} &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

D. $\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+4}$ is a **convergent** series.

ii. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

A. $a_n = \frac{\cos(n\pi)}{\sqrt{n}}$

$$\begin{aligned}\frac{\cos(n\pi)}{\sqrt{n}} &= \left(\frac{\cos(\pi)}{\sqrt{1}}, \frac{\cos(2\pi)}{\sqrt{2}}, \frac{\cos(3\pi)}{\sqrt{3}}, \frac{\cos(4\pi)}{\sqrt{4}}, ...\right) \\ &= \left(\frac{-1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, ...\right)\end{aligned}$$

B. $b_n = \frac{1}{\sqrt{n}}$

C. $\lim_{n\to\infty} b_n$

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\sqrt{n}} &= \lim_{n\to\infty} \frac{1}{\sqrt{\infty}} \\ &= 0\end{aligned}$$

D. $\therefore \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ is a **convergent** series

(e) Ratio Test

Theorem 3.13: Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1 & , \sum_{n=1}^{\infty} a_n \text{converges} \\ > 1 \text{or} \infty & , \sum_{n=1}^{\infty} a_n \text{diverges} \\ 1 & , \text{the Ratio Test inconclusive} \end{cases}$

(a) $\sum_{n=1}^{\infty} \frac{5^{n-1}}{4^{n+2}(n+1)^2}$

$$a_n = \frac{5^{n-1}}{4^{n+2}\left(n+1\right)^2}$$
$$a_{n+1} = \frac{5^n}{4^{n+3}\left(n+2\right)^2}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{5^n}{4^{\cancel{n+2}+1}\cancel{(n+2)^2}}\cdot\frac{\cancel{4^{n+2}}\cancel{(n+1)^2}}{5^{\cancel{n}-1}} \\
&= \lim_{n\to\infty}\frac{5}{4} \\
&= \frac{5}{4} > 1
\end{aligned}$$

i. Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, the series **diverges**

(b) $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{(n+1)!}{1\cdot 3\cdot 5\cdot \ldots\cdot(2n)}\cdot\frac{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}{n!}\right) \\
&= \lim_{n\to\infty}\left(\frac{(n+1)\cancel{n!}}{\cancel{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}\cdot(2n)}\cdot\frac{\cancel{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}}{\cancel{n!}}\right) \\
&= \lim_{n\to\infty}\frac{n+1}{2n} \\
&= \lim_{n\to\infty}\frac{1}{2}+\frac{1}{2n} \\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{2}
\end{aligned}$$

i. Since $\frac{1}{2} < 1$, $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}$ **converges**

(c) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

i. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2(n+1)+1}\left((n+1)+1\right)}\cdot\frac{4^{2n+1}(n+1)}{(-10)^{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-10)^{\cancel{n}+1}}{4^{\cancel{2n+1}+2(1)}\left((n+1)+1\right)}\cdot\frac{\cancel{4^{2n+1}}(n+1)}{\cancel{(-10)^{n}}}\right| \\ &= \lim_{n\to\infty} -\frac{5(n+1)}{8(n+2)} \\ &= \lim_{n\to\infty} -\frac{\frac{5n}{n}+\frac{5}{n}}{\frac{8n}{n}+\frac{16}{n}} \\ &= \lim_{n\to\infty} -\frac{5+\frac{5}{n}}{8+\frac{16}{n}} \\ &= -\frac{5}{8} < 1\end{aligned}$$

ii. Since $\frac{1}{2} < 1$, $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$ **converges**

(a) The Root Test

Theorem 3.14: Suppose we have the series $\sum_{n=1}^{\infty} a_n$.
Define, $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}$
Then, if $L < 1$, the series is absolutely convergent (and hence convergent).
If $L > 1$, the series is divergent.
$L = 1$, the test is inconclusive.

i. $\sum_{n=1}^{\infty}\frac{(-6)^n}{n}$

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-6)^n}{n}\right|} &= \lim_{n\to\infty}\left|\frac{(-6)^n}{n}\right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\frac{6}{n^{\frac{1}{n}}} \\ &= \lim_{n\to\infty}\frac{6}{n^{\frac{1}{n}}} \\ &= \frac{6}{1} \\ &= 6 > 1\end{aligned}$$

A. The series $\sum_{n=1}^{\infty}\frac{(-6)^n}{n}$ **is divergent.**

ii. $\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{4n^3+1}\right)^n$

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{5n-3n^3}{4n^3+1}\right)^n\right|} &= \lim_{n\to\infty} \left|\left(\frac{5n-3n^3}{4n^3+1}\right)^n\right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left|\left(\frac{5n-3n^3}{4n^3+1}\right)^n\right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{5n-3n^3}{4n^3+1} \\
&= \lim_{n\to\infty} \frac{5n-3n^3}{4n^3+1} \times \frac{n^3}{n^3} \\
&= \lim_{n\to\infty} \frac{\frac{5}{n^2}-3}{4+\frac{1}{n^3}} \\
&= -\frac{3}{4}
\end{aligned}$$

A. Since $L < 1$, the series is absolutely convergent, and hence convergent.

4. Use Ratio Test to determine whether the given series converges absolutely.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

$$\sum_{n=1}^{\infty} \left|\frac{(-3)^n}{n!}\right| = \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} &= \lim_{n\to\infty} \frac{3 \cdot \cancel{3^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{3^n}} \\
&= \lim_{n\to\infty} \frac{3}{n+1} \\
&= 0 < 1
\end{aligned}$$

i. Since $L < 1$, the series is absolutely convergent, hence **convergent**.

(b) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{(-3)^n}{n^3}\right| = \lim_{n\to\infty}\left|\frac{3^n}{n^3}\right|$$
$$\lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)^3}\cdot\frac{n^3}{3^n}\right)$$
$$= \left(\lim_{n\to\infty}\frac{\sqrt[3]{3}n}{n+1}\right)^3$$
$$= \left(\lim_{n\to\infty}\frac{\sqrt[3]{3}}{1+\frac{1}{n}}\right)^3$$
$$= 3 > 0$$

i. Since $L > 1$, the series is divergent

ii. Given solution,

$$\lim_{n\to\infty}\left|\frac{(-3)^n}{n^3}\right| = \lim_{n\to\infty}\left|\frac{3^n}{n^3}\right|$$
$$\lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)^3}\cdot\frac{n^3}{3^n}\right)$$
$$= \lim_{n\to\infty} 3\frac{n^3}{(n+1)^3}$$
$$= 3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3$$
$$= 3\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^3$$
$$= 3\,(1)$$
$$= 3 > 0$$

(c) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

i. $|\sin n| < 1$

ii. $\left|\frac{\sin n}{n^3}\right| < \frac{1}{n^3}$

(d) $\sum \frac{1}{n^3}$ is a convergent p-series.

(e) Hence, $\sum \frac{\sin n}{n^3}$ is convergent by direct comparison test.