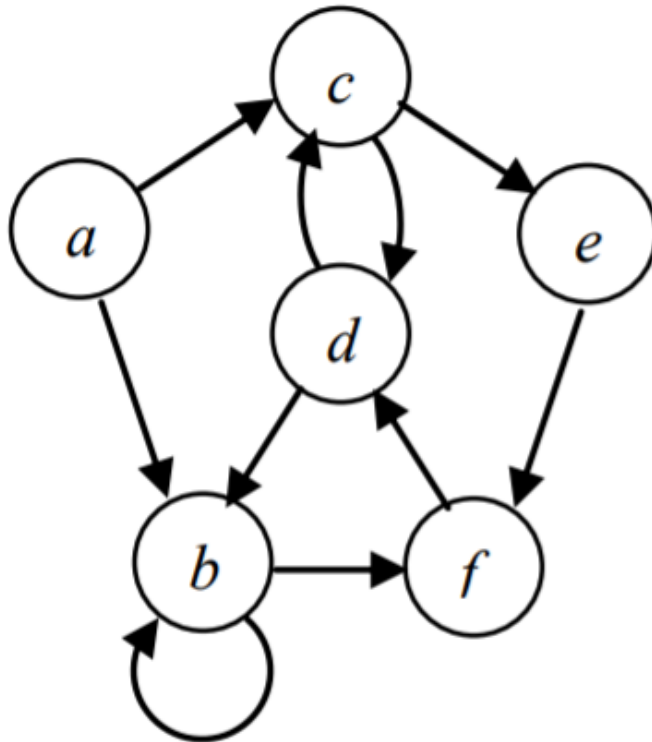


Tutorial 8

December 11, 2019

1. Let R be the relation whose digraph is given as follow:



- (a) List all paths of length 1.
- i. $R^1 = \{(a, c), (a, b), (b, b), (b, f), (c, e), (c, d), (d, c), (d, b), (e, f)\}$
- (b) List all paths of length 3 starting from vertex a .
- i. $R^3 = \{(a, b), (a, f), (a, d), (a, e)\}$
- (c) Find a cycle starting at vertex d . (**DOUBLE CONFIRM**)
- i. $\{d, c, d\}$

2. Determine whether the given relation on $A = \{1, 2, 3, 4\}$ is reflexive, ir-reflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers

(a) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$

i. Matrix-style

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

ii. **Reflexive**, matrix have all 1's on its main diagonal.

iii. **Not irreflexive**. $m_{11} = 1$

iv. **Symmetric**. In the matrix, if $m_{ij} = 1$, then $m_{ji} = 1$

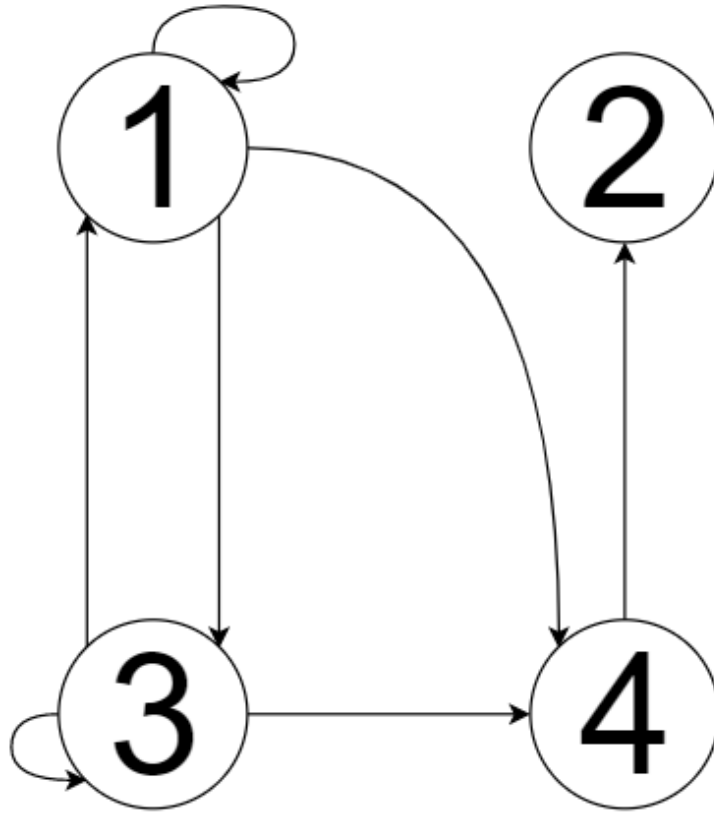
v. **Not antisymmetric**.

vi. **Transitive**. In the matrix, $(M_R)_{\odot}^2 = M_R$

$$\begin{aligned} M_R \odot M_R &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \end{aligned}$$

(b) $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$

i. Digraph-style



- ii. **Not reflexive.** $(4, 4) \notin R$.
 - iii. **Not irreflexive.** $(1, 1) \in R$
 - iv. **Not symmetric.** $(1, 4) \in R$ but $(4, 1) \notin R$
 - v. **Not asymmetric.** $(1, 3) \in R$ and $(3, 1) \in R$
 - vi. **Not antisymmetric.** $(1, 3), (3, 1) \in R$, but $3 \neq 1$, or just **have loops**
 - vii. **Not transitive.** $(1, 4), (4, 2) \in R$ but $(1, 2) \notin R$
- (c) $R = \emptyset$
- i. Tikam style
 - ii.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= M^T$$

- iii. R is not reflexive since $(1, 1) \notin R$,
 - iv. R is irreflexive since $(1, 1), (2, 2), (3, 3), (4, 4) \notin R$
 - v. R is symmetric
 - vi. R is asymmetric. No loops
 - vii. R is antisymmetric
 - viii. R is transitive
- (d) $R = \{(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)\}$
- i. Tikam style version 2
 - ii. Not reflexive. $(2, 2) \notin R$
 - iii. R is not irreflexive since $(1, 1), (3, 3)$
 - iv. Not symmetric. $(1, 2) \in R$ but $(2, 1) \notin R$.
 - v. Not asymmetric. $(1, 3) \in R$ and $(3, 1) \in R$
 - vi. Not antisymmetric. $(1, 3) \in R$ and $(3, 1) \in R$ but $1 \neq 3$
 - vii. R is transitive.
3. Let $A = \{w, x, y, z\}$. Determine whether the relation R whose matrix M_R is given below is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers

Determining transitivity through matrix multiplication:

Basically, the original M_r tells you about the number of $1 - step$ paths from x to x . By multiplying each row of M_r by each column of M_r , you will find how many $2 - step$ paths there are. So say you multiply 1st row times first column, you'll get the $2 - step$ paths for $(1, 1)$ (note: $(row, column)$). For a matrix to be transitive, all elements with a $1 - step$ path must also have a $2 - step$ path. Note: Since we only care about "presence" of links, not "how many", its safe to just write '1' if there is to speed up calculation.

$$(a) \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- i. **Not reflexive.** $(1, 1) \notin R$
- ii. **Irreflexive.** The main diagonal are all 0's.
- iii. **Symmetric.** If $m_{ij} = 1$, then $m_{ji} = 1$
- iv. **Not asymmetric.** $m_{12} = m_{21} = 1$
- v. **Not antisymmetric.** $m_{12} = m_{21} = 1$ but $1 \neq 2$.

vi. **Not Transitive.**

$$\begin{aligned} (M_R)_{\odot}^2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{aligned}$$

$$M_{R23} = 1 \text{ but } M_{R32} = 0.$$

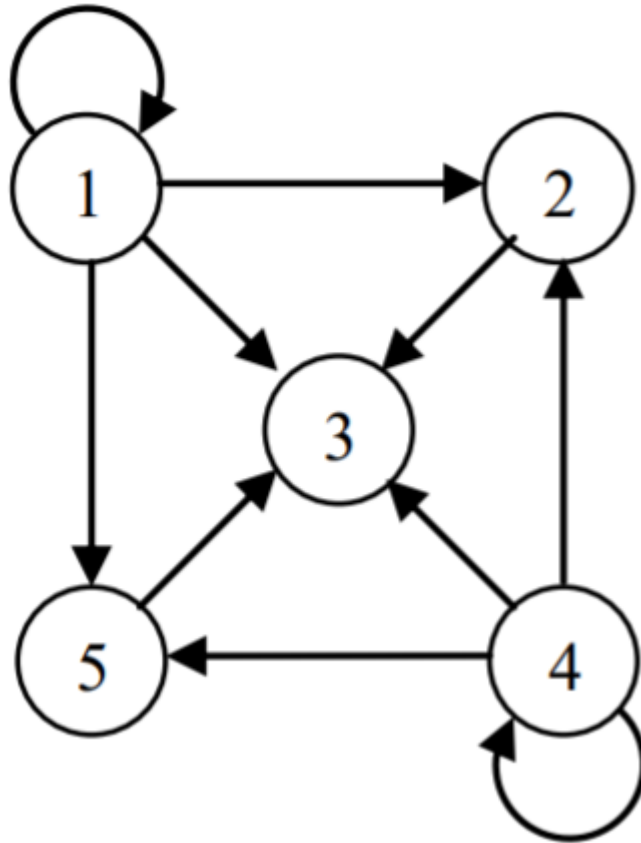
$$(b) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- i. **Not reflexive, not irreflexive.** $M_{00} = 0$, but $M_{33} = 1$
- ii. **Not symmetric, not asymmetric.** $M_{10} = 1$, but $M_{01} = 0$.
 $M_{32} = 1, M_{23} = 1$.
- iii. **Not antisymmetric.** $M_{32} = 1, M_{23} = 1$. But $2 \neq 3$.
- iv. **Not transitive.**

$$\begin{aligned} (M_R)_{\odot}^2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

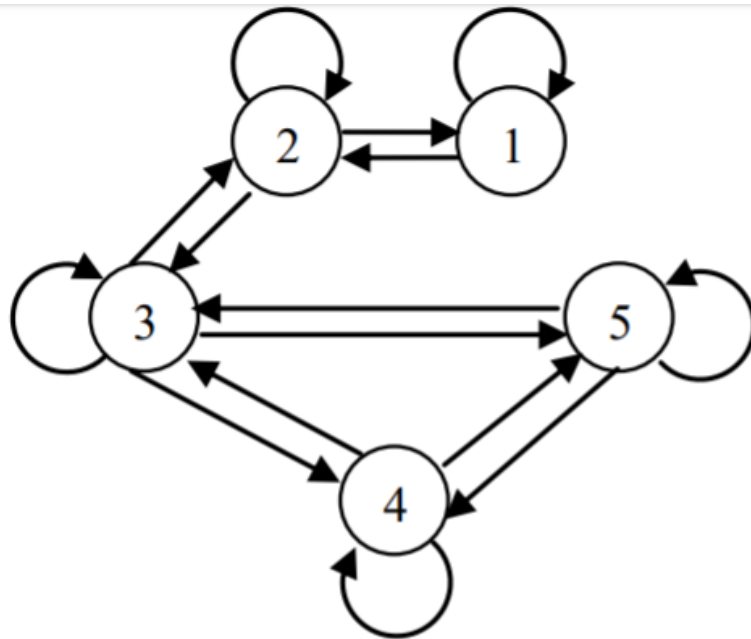
$$A. \text{ Since } (M_R)_{21} = 1 \text{ but } (M_R)_{21}^2 = 0.$$

4. Let $A = \{1, 2, 3, 4, 5\}$. Determine whether the relation R whose digraph is given below is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive. Explain your answers.



(a)

- i. **Not reflexive.** $(2, 2) \notin R$.
- ii. **Not irreflexive.** $(1, 1) \in R$
- iii. **Not symmetric.** $(2, 3) \in R$, but $(3, 2) \notin R$
- iv. **Not asymmetric.** $(1, 1), (4, 4) \in R$
- v. **Antisymmetric.** For all (a, b) , if aRb then $a = b$.
- vi. **Transitive.** For all (a, b, c) , if aRb , then aRc .



(b)

5. Let $A = \{a, b, c\}$. Determine whether the relation R whose matrix M_R is given is an equivalence relation. If yes, find A/R .

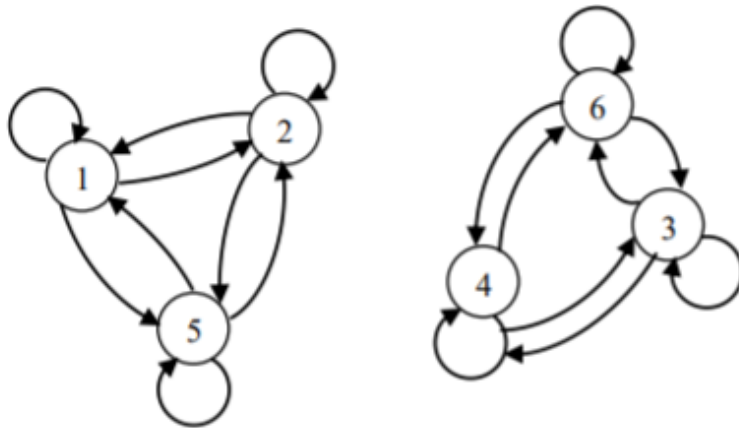
(a)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- i. Equivalence relation
- ii. $A/R = \{\{b, c\}, \{a\}\}$

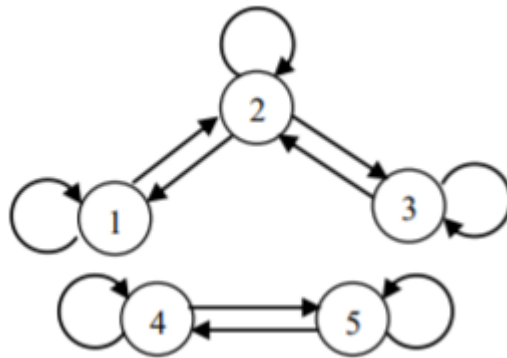
(b)
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- i. Not an equivalent relation

6. Determine whether the relation R whose digraph is given as below is an equivalence relation. If yes, find A/R .



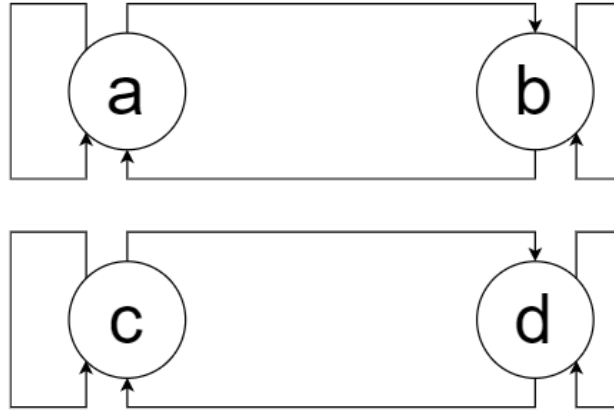
(a)



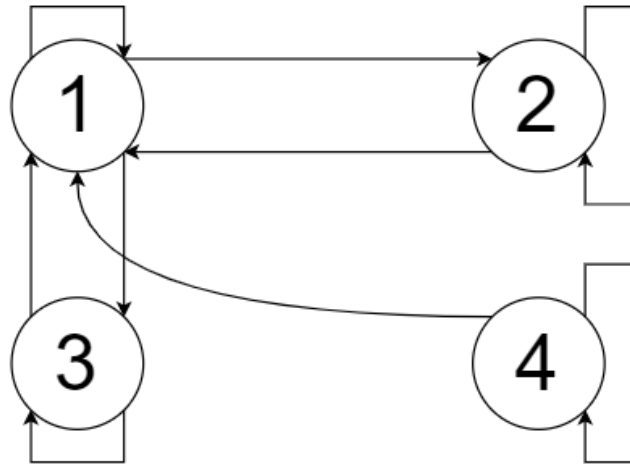
(b)

7. Determine whether the following relation R on the set A is an equivalence relation. If yes, find A/R

(a) $A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$



- i.
 - ii. Equivalence relation
 - iii. $A/R = \{\{a, b\}, \{c, d\}\}$
- (b) $A = \{1, 2, 3, 4\}, R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$



- i.
 - ii. Not equivalence relation
 - A. R is reflexive
 - B. R is not symmetric $(1, 4) \in A$ but $(4, 1) \notin A$
 - C. R is not transitive since $(3, 1), (1, 2) \in A$ but $(3, 2) \notin A$
- (c) $A = \{2, 3, 5, 6, 8\}, xRy \text{ iff } 3|(x-y).$
- i. $R = \{(2, 3), (3, 2), (2, 5), (5, 2), (6, 3), (3, 6), (8, 2), (2, 8), (8, 5), (5, 8)\}$
 - ii. R is reflexive
 - iii. R is symmetric

- iv. R is not transitive. $(2, 3), (3, 6) \in R$ but $(2, 6) \notin R$
 - v. Not equivalence relation
- (d) $A = \{1, 2, 3, 4, 5\}$, $xRy \iff x = y \pmod{2}$
- i. $R = y \pmod{2}$
 - ii. $R = \{(1, 1), (1, 3), (1, 5)\}$
 - iii. R is not reflexive, $(2, 2) \notin R$
 - iv. R is not symmetric, $(1, 3) \in R$ but not $(3, 1)$.
 - v. R is not transitive.
 - vi. Not equivalent relation