Calculus 1 C3: Differentiation

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1 Example

1. $10x$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} \\
&= \lim_{h \to 0} \frac{10(x+h) - 10x}{h} \\
&= \lim_{h \to 0} \frac{\cancel{10x} + 10h \cancel{- 10x}}{h} \\
&= \lim_{h \to 0} \frac{+10h}{h} \\
f'(x) &= 10
\end{aligned}
$$

2. $3x^2 + x + 1$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 + x + h + 1 - 3x^2 + x + 1}{h} \\
&= \lim_{h \to 0} \frac{3\left(x^2 + 2hx + h^2\right) + x + h + 1 - 3x^2 + x + 1}{h} \\
&= \lim_{h \to 0} \frac{\cancel{3x^2} + 6hx + 3h^2 + \cancel{x} + h \cancel{+1 - 3x^2 + x + 1}}{h} \\
&= \lim_{h \to 0} \frac{6hx + 3h^2 + h}{h} \\
&= \lim_{h \to 0} 6x + 3h + 1 \\
\boldsymbol{f'(x)} &\boldsymbol{= 6x + 1}
\end{aligned}
$$

3. $\sqrt{x}$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h \to 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}}\right) \\
f'(x) &= \frac{1}{2\sqrt{x}}
\end{aligned}$$

2 Example

1. Evaluate the following derivatives

   (a) $y = 8x^5$

   $$\begin{aligned} y &= 8x^5 \\ y' &= 40x^4 \end{aligned}$$

   (b) $y = 3\ln(x)$

   $$y' = \frac{3}{x}$$

   (c) $y = -7e^x$

   $$\begin{aligned} y &= -7e^x \\ \frac{dy}{dx} &= -7 * 1e^x \\ &= -7e^x \end{aligned}$$

   (d) $y = \frac{1}{2}cos(x)$

   $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(-sin(x)) \\ &= -\frac{1}{2}sin(x) \end{aligned}$$

   (e) $y = \pi x$

   $$\frac{dy}{dx} = \pi$$

(f) $y = \frac{1}{3x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{3}\frac{dy}{dx}\left(x^{-1}\right) \\ &= -\frac{1}{3}x^{-2} \\ &= -\frac{1}{3x^2} \end{aligned}$$

3 Example

1. Evaluate the derivatives

(a) $y = 2x^4 + \frac{3}{x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx}\left(2x^4\right) + \frac{dy}{dx}\left(\frac{3}{x^2}\right) \\ &= 8x^3 - \frac{6}{x^3} \end{aligned}$$

(b) $y = e^x - 5sin\left(x\right)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx}\left(e^x\right) - 5\frac{dy}{dx}\left(sin\left(x\right)\right) \\ &= e^x - 5\left(cos\left(x\right)\right) \\ &= e^x - 5cos\left(x\right) \end{aligned}$$

(c) $y = \frac{x^3+4}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx}\left(x^2\right) + \frac{dy}{dx}\left(\frac{4}{x}\right) \\ &= 2x - \frac{4}{x^2} \end{aligned}$$

4 Example

1. Evaluate the derivatives

(a) $y = x\ln x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx}\left(x\right)\cdot\left(\ln x\right) + x\cdot\frac{dy}{dx}\left(\ln x\right) \\ &= \ln\left(x\right) + x\cdot\frac{1}{x} \\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{= \ln\left(x\right) + 1} \end{aligned}$$

(b) $y = 2x^2 \sin x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}\left(2x^2\right) \cdot (\sin x) + 2x^2 \cdot \frac{dy}{dx}(\sin x) \\ &= 4x \cdot \sin(x) + 2x^2 \cdot (\cos(x)) \\ &= 4x \sin(x) + 2x^2 \cos(x)\end{aligned}$$

(c) $y = \left(2x^3 + 1\right)(x - 5)$

$$\begin{aligned}\frac{dy}{dx} &= \left(2x^3 + 1\right)(x - 5) \\ &= \frac{dy}{dx}\left(2x^3 + 1\right)(x - 5) + \left(2x^3 + 1\right)\frac{dy}{dx}(x - 5) \\ &= 6x^2 (x - 5) + \left(2x^3 + 1\right) 1 \\ &= 6x^3 - 30x^2 + 2x^3 + 1 \\ \frac{dy}{dx} &= 8x^3 - 30x^2 + 1\end{aligned}$$

(d) $y = e^x \left(x^2 + 7\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}(e^x) \cdot \left(x^2 + 7\right) + e^x \cdot \frac{dy}{dx}\left(x^2 + 7\right) \\ &= e^x \cdot \left(x^2 + 7\right) + e^x \cdot 2x \\ &= e^x \left(x^2 + 7\right) + 2xe^x \\ &= e^x \left(x^2 + 2x + 7\right)\end{aligned}$$

5 Proof & Examples

5.1 Proofs

1. $\frac{d}{dx}(\tan x) = \sec^2(x)$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\ &= \frac{\cos(x)\frac{d}{dx}(\sin x) - \sin(x)\frac{d}{dx}(\cos(x))}{(cosx)^2} \\ &= \frac{\cos(x) \cdot (\cos(x)) - \sin(x) \cdot (-\sin(x))}{cos^2 x} \\ &= \frac{\cos^2 x + \sin^2(x)}{\cos^2 x}; \text{note: } sin^2(x) + cos^2(x) = 1 \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

2. $\frac{d}{dx}(\cot x) = -csc^2(x)$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\tan^{-1}(x)\right) \\
&= \frac{d}{dx}\left(\frac{cos(x)}{sin(x)}\right) \\
&= \frac{d}{dx}\left(\frac{cos(x)}{sin(x)}\right) \\
&= \frac{\sin(x)\frac{d}{dx}(cos(x)) - \cos(x)\frac{d}{dx}(\sin(x))}{(sin(x))^2} \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)(\cos(x))}{(sin(x))^2} \\
&= \frac{-\left(\sin^2(x)\right) - cos^2(x)}{(sin(x))^2} \\
&= \frac{-\left(sin^2(x) + cos^2(x)\right)}{sin^2(x)} \\
&= -\frac{1}{sin^2(x)} \\
&= -\csc^2(x)
\end{aligned}$$

3. $\frac{d}{dx}(\sec x) = \sec x \tan x$ (REPROOF WITH SIMPLER METHOD)

$$\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{cos\,(x)}\right) \\
&= \frac{d}{dx}\left(\frac{1}{cos\,(x)}\right) \\
&= \frac{d}{dx}\left(\frac{\tan(x)}{\sin(x)}\right) \\
&= \frac{\sin(x)\,\frac{d}{dx}(\tan(x)) - \tan(x)\,\frac{d}{dx}(\sin(x))}{sin^2\,(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \tan(x)(\cos(x))}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \frac{\sin(x)}{\cos(x)}(\cos(x))}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x)\right) - \sin(x)}{\sin^2(x)} \\
&= \frac{\sin(x)\left(\sec^2(x) - 1\right)}{\sin^2(x)} \\
&= \left(\sec^2(x) - 1\right)\cdot\left(\frac{1}{sin\,(x)}\right) \\
&= \left(\tan^2(x)\right)\cdot\left(\frac{1}{\sin(x)}\right) \\
&= \tan(x)\cdot\frac{\cancel{\sin(x)}}{\cos(x)}\cdot\frac{1}{\cancel{\sin(x)}} \\
&= \tan(x)\cdot\frac{1}{cos\,(x)} \\
&= \tan(x)\sec(x)
\end{aligned}$$

$$\boldsymbol{\frac{d}{dx}(\sec x) = \sec(x)\tan(x)}$$

4. $\frac{d}{dx}\left(\csc\left(x\right)\right) = -\csc x \cot x$

$$\begin{aligned}
\frac{d}{dx}\left(\csc\left(x\right)\right) &= \frac{d}{dx}\left(\frac{1}{\sin\left(x\right)}\right) \\
&= \frac{\sin\left(x\right)\frac{d}{dx}\left(1\right) - \frac{d}{dx}\left(\sin\left(x\right)\right)}{\sin^2\left(x\right)} \\
&= \frac{\sin\left(x\right)\left(0\right) - \cos\left(x\right)}{\sin^2\left(x\right)} \\
&= \frac{-\cos\left(x\right)}{\sin^2\left(x\right)} \\
&= -\csc\left(x\right)\cot\left(x\right)
\end{aligned}$$

5.2 Example

1. Evaluate the following derivatives

   (a) $y = \frac{e^x}{\sin(x)}$

$$\begin{aligned}
y' &= \frac{\sin\left(x\right)\cdot\frac{d}{dx}\left(e^x\right) - e^x\frac{d}{dx}\left(\sin\left(x\right)\right)}{\sin^2\left(x\right)} \\
&= \frac{\sin\left(x\right)\cdot e^x - e^x\cos\left(x\right)}{\sin^2\left(x\right)} \\
&= e^x\frac{\left(\sin\left(x\right) - \cos\left(x\right)\right)}{\sin^2\left(x\right)}
\end{aligned}$$

   (b) $y = \frac{x}{\ln x}$

$$\begin{aligned}
y' &= \frac{\ln\left(x\right)\frac{d}{dx}\left(x\right) - x\frac{d}{dx}\left(\ln x\right)}{\left(\ln x\right)^2} \\
&= \frac{\ln\left(x\right) - x\frac{1}{x}}{\left(\ln x\right)^2} \\
&= \frac{\ln\left(x\right) - 1}{\left(\ln x\right)^2}
\end{aligned}$$

(c) $y = \frac{2x}{(x+2)^2}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x+2)^2 \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(x+2)^2}{(x+2)^4} \\
&= \frac{2(x+2)^2 - 2x \cdot 2(x+2)(1)}{(x+2)^4} \\
&= \frac{2(x+2)^2 - 4x(x+2)}{(x+2)^4} \\
&= \frac{2(x+2)((x+2) - 2x)}{(x+2)^4} \\
&= \frac{2(2-x)}{(x+2)^3}
\end{aligned}$$

(d) $y = \sec x$

$$\frac{dy}{dx} = \sec x \tan x$$

6 Example

1. Evaluate the derivatives.

(a) $y = (2x-3)^5$

$$\begin{aligned}
u &= 2x - 3 \\
y &= u^5 \\
\frac{dy}{dx} &= 2 \\
\frac{dv}{dx} &= 5u^4 \cdot \frac{dy}{dx}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= 5(2x-3)^4(2) \\
\frac{dy}{dx} &= 10(2x-3)^4
\end{aligned}$$

(b) $y = \ln\left(2x - x^5\right)$

$$\begin{aligned}
u &= 2x - x^5 \\
\frac{du}{dx} &= 2 - 5x^4
\end{aligned}$$

$$\begin{aligned}
y &= \ln(u) \\
\frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} \\
&= \frac{2 - 5x^4}{2x - x^5}
\end{aligned}$$

(c) $y = \sin\left(7x^2\right)$

$$\begin{aligned} u &= 7x^2 \\ \frac{du}{dx} &= 14x \end{aligned}$$

$$\begin{aligned} y &= \sin u \\ \frac{dy}{dx} &= \frac{du}{dx}\cos u \\ \boldsymbol{\frac{dy}{dx}} &\boldsymbol{= 14x\cos\left(7x^2\right)} \end{aligned}$$

(d) $y = 2e^{4x}$

$$\begin{aligned} \frac{dy}{dx} &= 2\frac{dy}{dx}e^{4x} \\ &= 2\frac{dy}{dx}\left(e^x\right)^4 \end{aligned}$$

$$\begin{aligned} u &= e^x \\ \frac{du}{dx} &= e^x \end{aligned}$$

$$\begin{aligned} y &= 2u^4 \\ \frac{dy}{dx} &= 8u^3 \cdot \frac{du}{dx} \\ &= 8\left(e^x\right)^3 \cdot e^x \\ &= 8e^{3x} \cdot e^x \\ &= 8e^{4x} \end{aligned}$$

(e) $y = \tan^7(x)$

$$\begin{aligned} y &= (\tan(x))^7 \\ u &= \tan(x) \\ \frac{du}{dx} &= \sec^2(x) \end{aligned}$$

$$\begin{aligned} y &= u^7 \\ \frac{dy}{dx} &= 7u^6 \cdot \frac{du}{dx} \\ &= 7\left(\tan x\right)^6 \cdot \sec^2(x) \\ &= 7\tan^6 x \cdot \sec^2(x) \end{aligned}$$

(f) $y = \sqrt{1+x^2}$

$$\begin{aligned} y &= \left(1+x^2\right)^{\frac{1}{2}} \\ u &= 1+x^2 \\ \frac{du}{dx} &= 2x \end{aligned}$$

$$\begin{aligned} y &= u^{\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{1+x^2}} \cdot 2x \\ &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$