

Chapter 5 Functions

5.1 Introduction

5.2 Properties of Functions

5.3 Functions for Computer Science

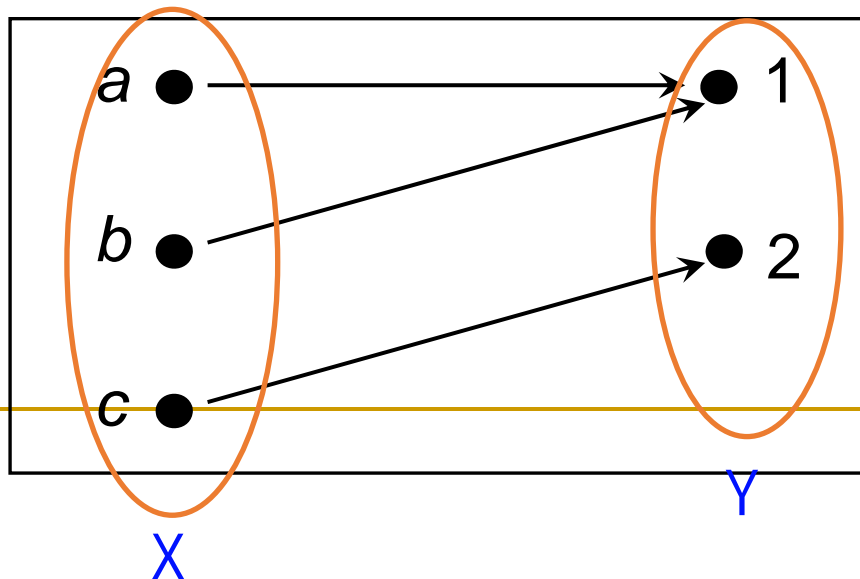
5.4 Permutations

5.1 Introduction

- Functions are binary relations in which further restrictions are imposed on the pairs which can occur. For each input, one output
- A function from a set A to a set B is a binary relation in which every element of A is associated with a uniquely specified element of B . only one
- Functions are also called as mappings or transformation.

5.1 Introduction (cont)

- In digraph terms a function is a relation such that there is **precisely one arc** leaving every element of A .
- E.g. The digraph representing the function from $\{a, b, c\}$ to $\{1, 2\}$ containing the pairs $(a, 1)$, $(b, 1)$ and $(c, 2)$.



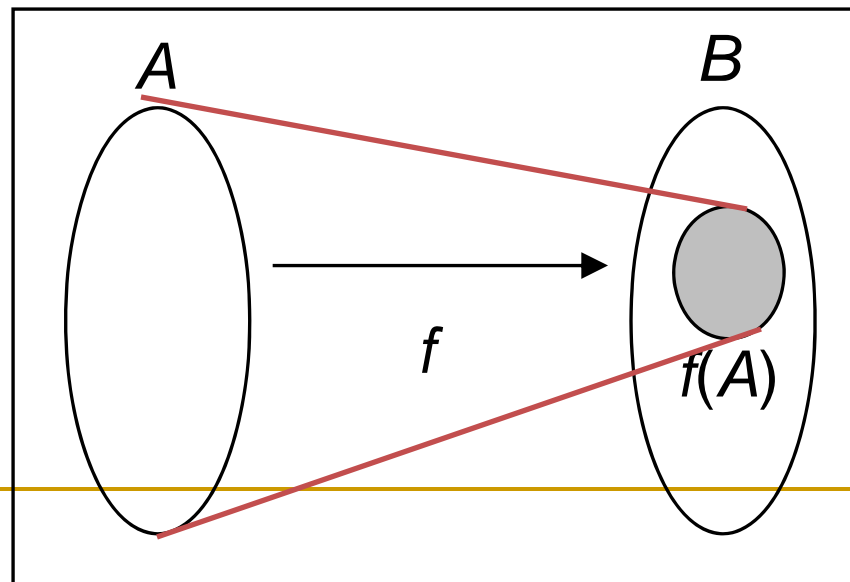
$$f(x) \rightarrow y$$

5.1 Introduction (cont)

- Let f be a function from a set A to a set B .
 - For each $a \in A$, there exists a uniquely determined $y \in B$ with $(x, y) \in f$, write as $y = f(x)$, and refer to $f(x)$ as the image of x under f .
 - Write as $f : A \rightarrow B$ to indicate that f is a function which transform, or maps, each element of A to a uniquely determined element of B .
 - A is called the domain of f , and B is the codomain of f .

5.1 Introduction (cont)

- The range of f is the set of images of all the elements of A under f , denoted by $f(A)$. Hence, $f(A) = \{f(x) : x \in A\}$.
- The Venn diagram provides a useful diagrammatic illustration of a function from a set A to a set B .



codomain of f (everything)

range of f $f(A)$

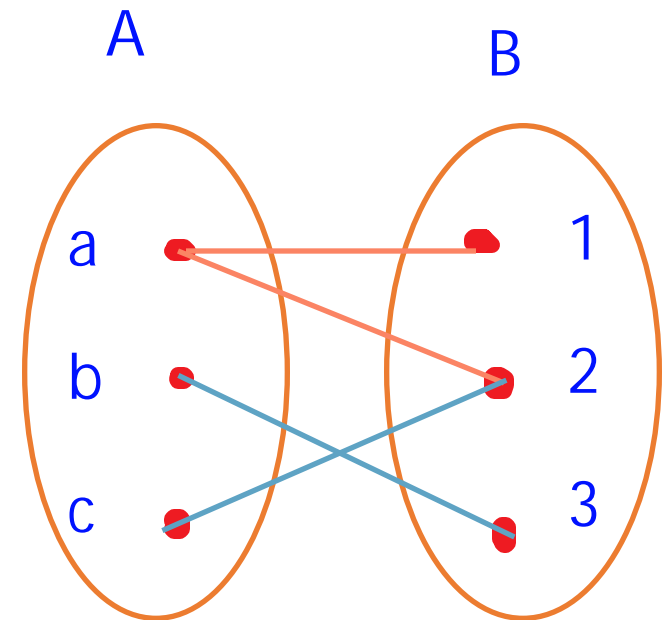
E.g.1

For each of the following relations between the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Determine whether the relation gives a function from A to B .

i. $\{(a, 1), (a, 2), (b, 3), (c, 2)\}$ No

ii. $\{(a, 1), (b, 2), (c, 1)\}$ Yes

iii. $\{(a, 1), (c, 2)\}$ Yes



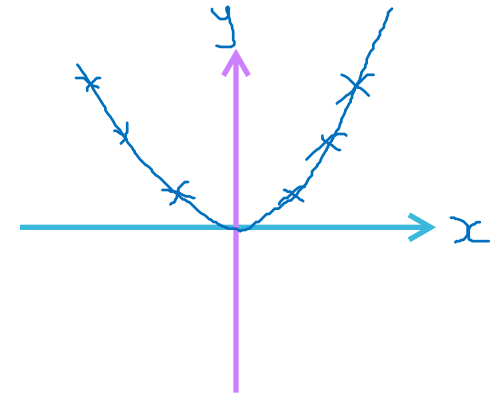
E.g.2

Which of the following relations are functions?



i. The relation on Z given by $\{(\overset{x}{x}, \overset{y}{x^2}) : x \in Z\}$. $y=x^2$

ii. The relation on R given by $\{(x, y) : x = y^2\}$.



This is a $f(x)$ from Z to Z

E.g.2

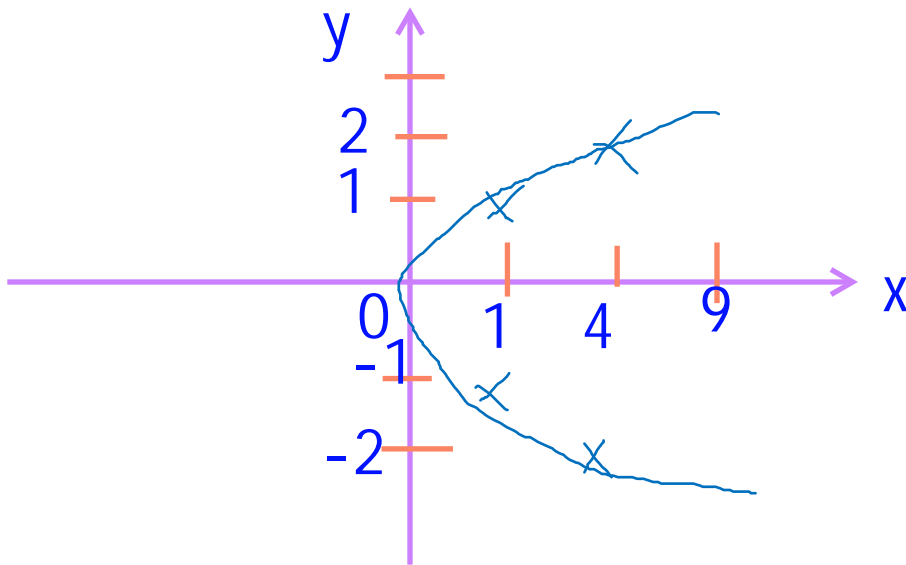
Which of the following relations are functions?

- i. The relation on Z given by $\{(x, x^2) : x \in Z\}$.

E.g.2

Which of the following relations are functions?

- ii. The relation on R given by $\{(x, y) : x = y^2\}$.



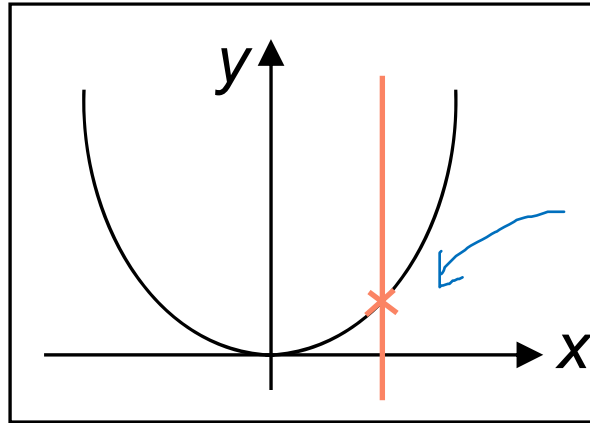
This is not a function from X to Y

5.1 Introduction (cont)

- When dealing with a function $f : A \rightarrow B$ where A and B are infinite sets of numbers, we can use the more traditional mathematical idea of graphing a function to give a geometric picture of the function.

5.1 Introduction (cont)

- The graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$:

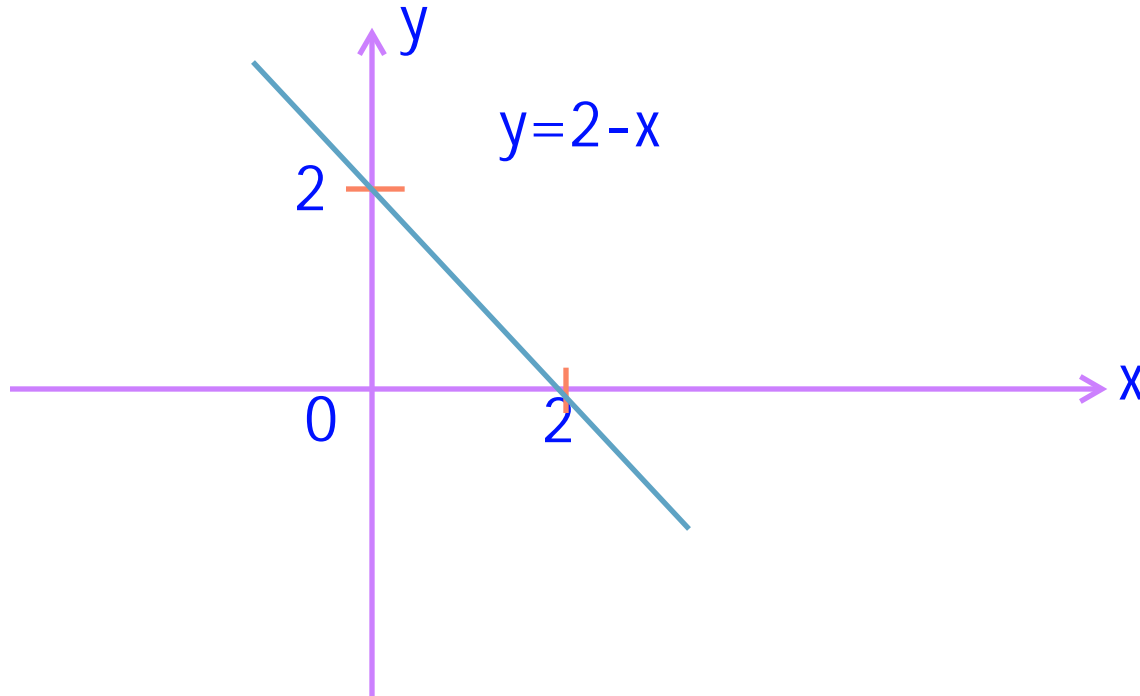


only one

- The horizontal axis (x-axis) denotes the domain \mathbb{R} and the vertical axis (y-axis) denotes the codomain \mathbb{R} .
- The curve consists of those points (x, y) in Cartesian product $\mathbb{R} \times \mathbb{R}$ for which $y = f(x)$.

E.g.3

Sketch the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.

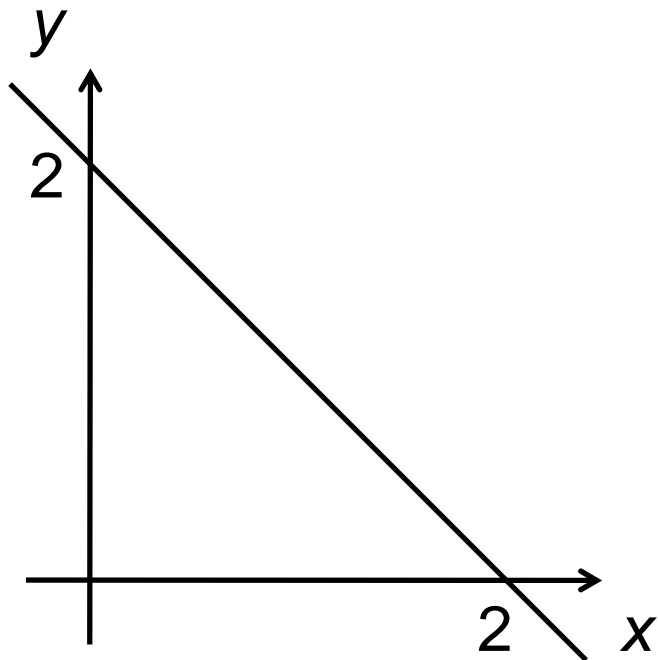


$$f(2) = 2 - 2 = 0$$

$$f(3) = 2 - 3 = -1$$

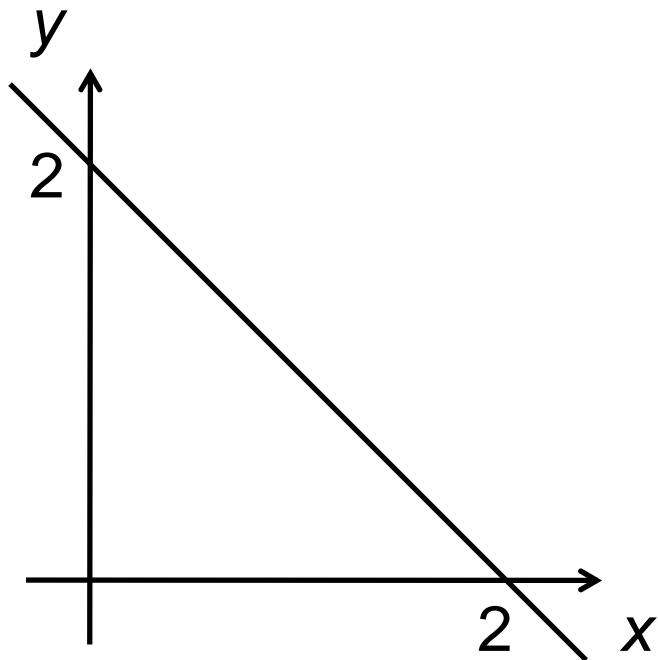
E.g.3

Sketch the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



E.g.3

Sketch the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



$$x = 2, f(2) = 0$$

$$x = 3, f(3) = -1$$

5.2 Properties of Functions

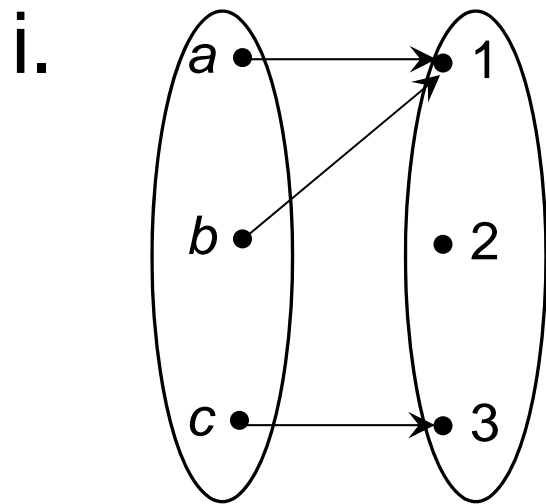
- Let $f: A \rightarrow B$ be a function.
 - f is an **injective** (or **one-to-one**) function if
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A;$$
or $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ i.e. different inputs give different outputs.
 - f is **everywhere defined** if $\text{Dom}(f) = A$.
 - f is a **surjective** (or **onto**) function if the range of f coincides with the codomain of f , i.e. each element in the codomain is a value of the function;
$$f(A) = B$$
or for every $b \in B$, there exists an $a \in A$ with
$$b = f(a).$$

5.2 Properties of Functions (cont)

- f is a **bijective** function if f is both **injective** and **surjective**.
- f is a **one-to-one correspondence** between A and B if f is everywhere defined, injective, and surjective.

E.g.4

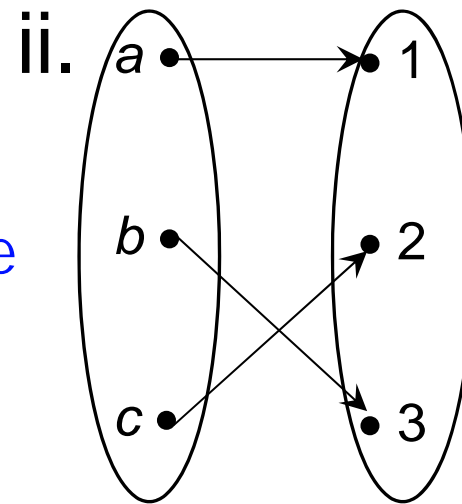
Decide which of the functions below is injective or surjective. Which are bijective?



not injective
not surjective

$f(a)=1=f(b)$
But $a \neq b$, f is not injective

$2 \in B$ but $2 \notin f(A)$. Is not surjective



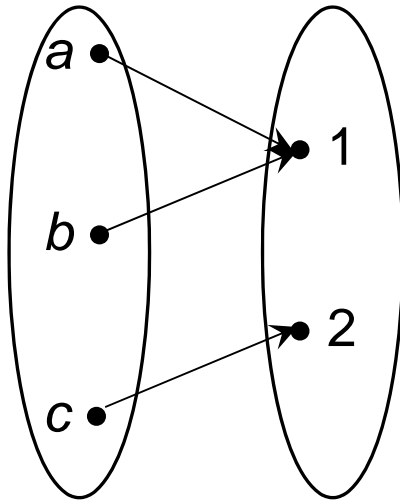
Bijjective

$f(A) = B$
 f is surjective

$\therefore f$ is not bijective

E.g.4 (cont)

iii.



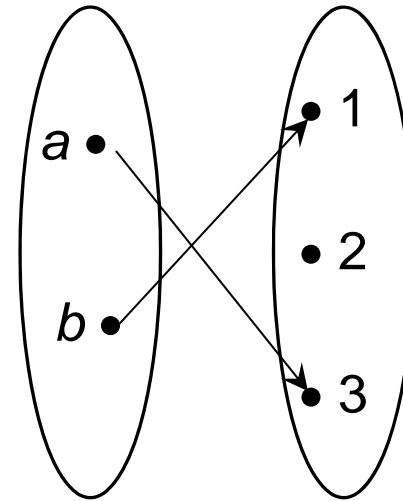
$$f(a) = 1 = f(b)$$

but $a \neq b$. f is not injective.

$$f(A) = B$$

f is surjective.

iv.



f is injective.

$$2 \in B \text{ but } 2 \notin f(A)$$

f is not surjective

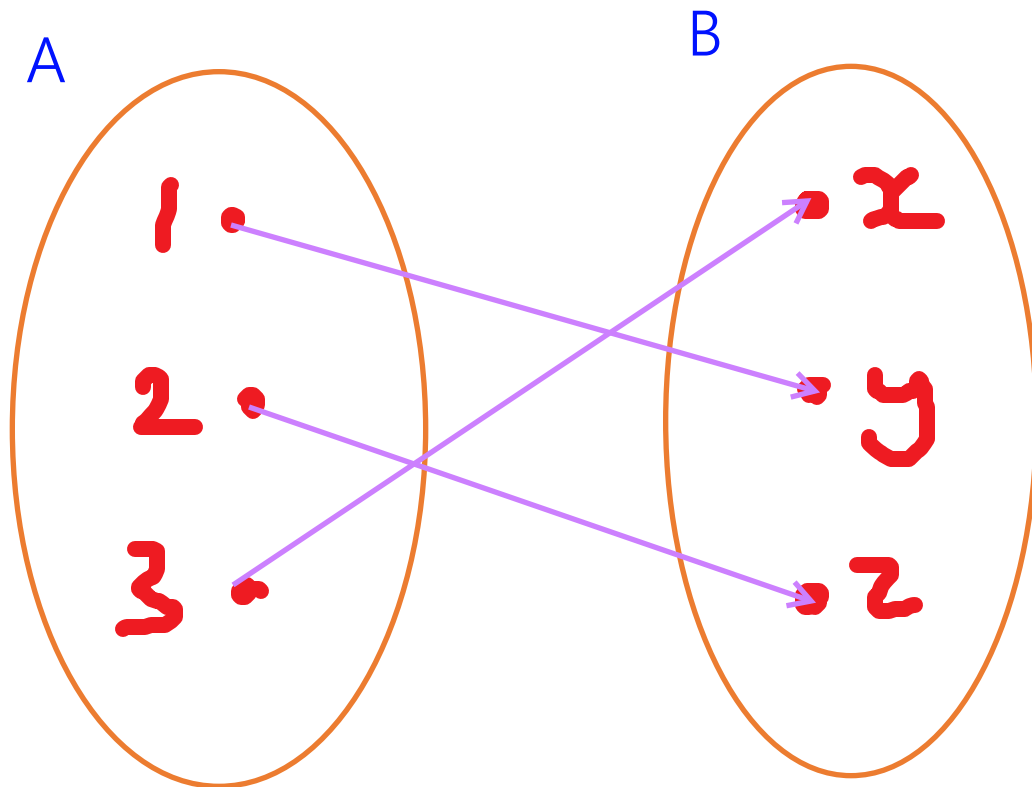
f is not bijective

E.g.5

Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$. Determine whether each of the following function is **one-to-one**, **onto**, and **everywhere defined**. injective surjective

- i. $f: A \rightarrow B; f = \{(1, y), (2, z), (3, x)\}$
- ii. $g: A \rightarrow D; g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$
- iii. $h: B \rightarrow C; h = \{(x, e), (y, e), (z, \pi)\}$
- iv. $k: D \rightarrow B; k = \{(\theta, x), (\alpha, y), (\beta, x)\}$.

i. $f: A \rightarrow B; f = \{(1, y), (2, z), (3, x)\}$



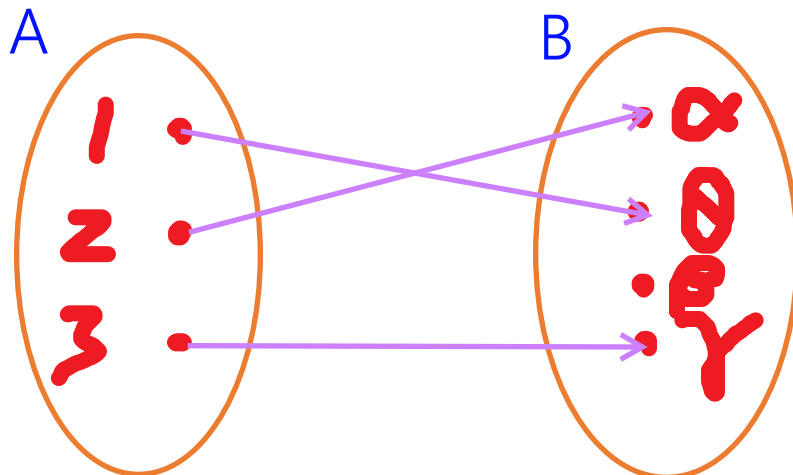
f is one-to-one

f is onto

f is everywhere defined

ii. $g: A \rightarrow D; g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

Change f to g

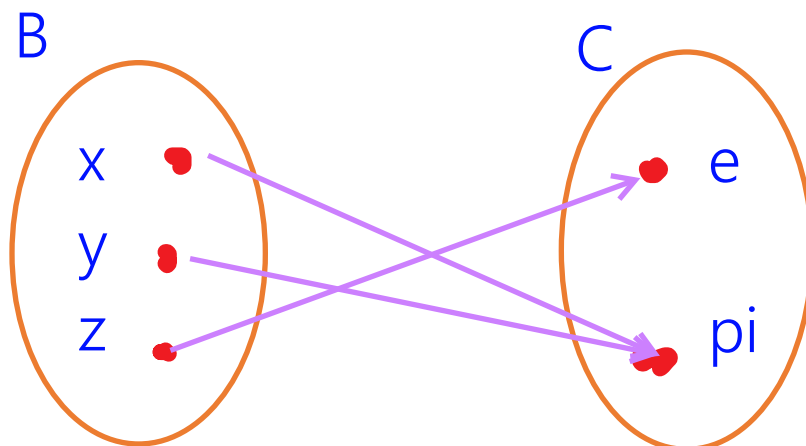


f is one-to-one
 $\beta \in D$ but $\beta \notin f(n)$
 f is not onto
 f is everywhere defined

Note: beta is due to definition of D earlier

iii. $h: B \rightarrow C; h = \{(x, e), (y, e), (z, \pi)\}$

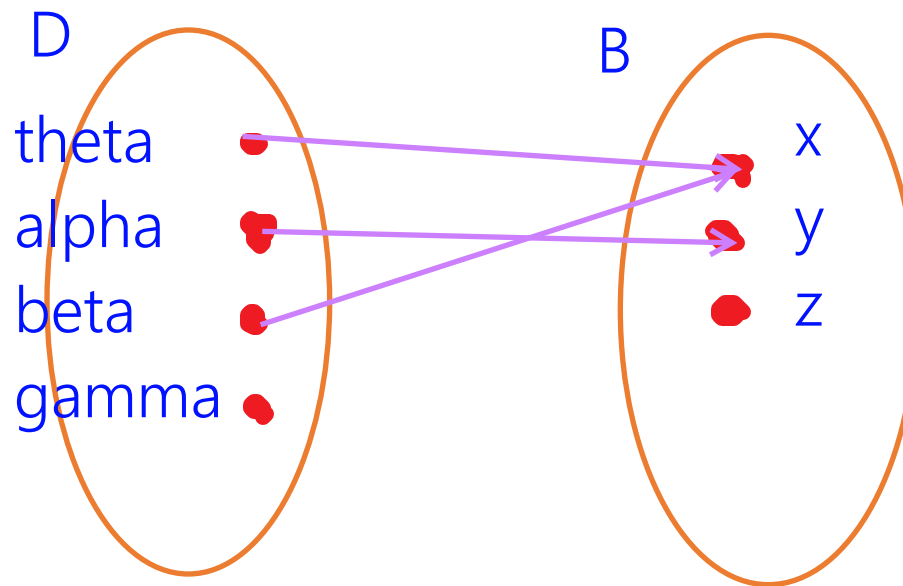
Change f to h



$f(x) = e = f(y)$, but $x \neq y$
 f is not one-to-one
 $f(B) = C$
 f is onto
 f is everywhere defined

$A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$

iv. $k : D \rightarrow B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$



Change f to A

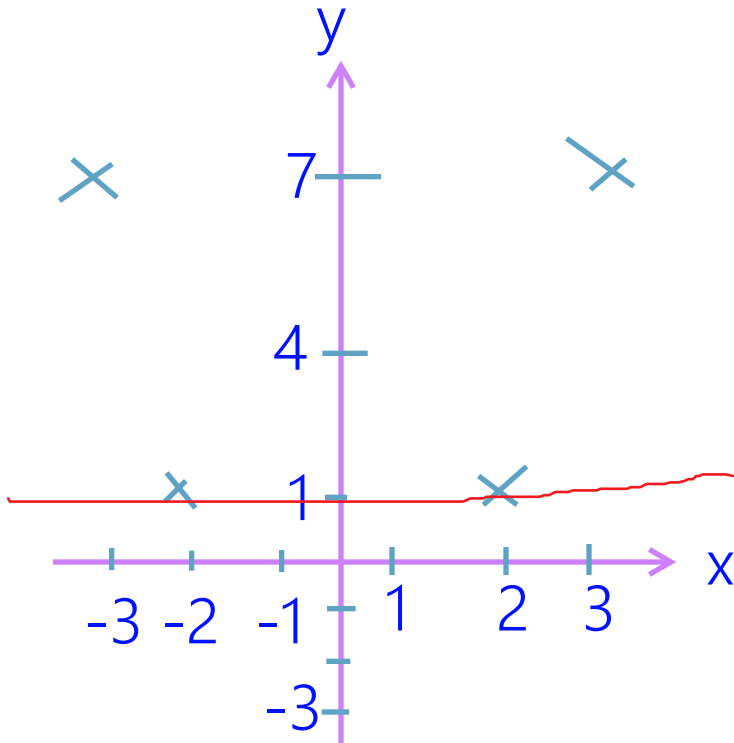
$f(\theta) = x = f(\beta)$, but $\theta \neq \beta$
 f is not one-to-one

$z \in B$, but $z \notin f(D)$

f is not onto,
 $f(\gamma)$ is not defined,
 f is NOT everywhere defined

E.g.6

Show that the function $h : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $h(x) = x^2$ is neither injective nor surjective.



$$h(1) = 1 = h(-1), \text{ but } 1 \neq -1$$

$\therefore h$ is not injective

$$h(x) \geq 0$$

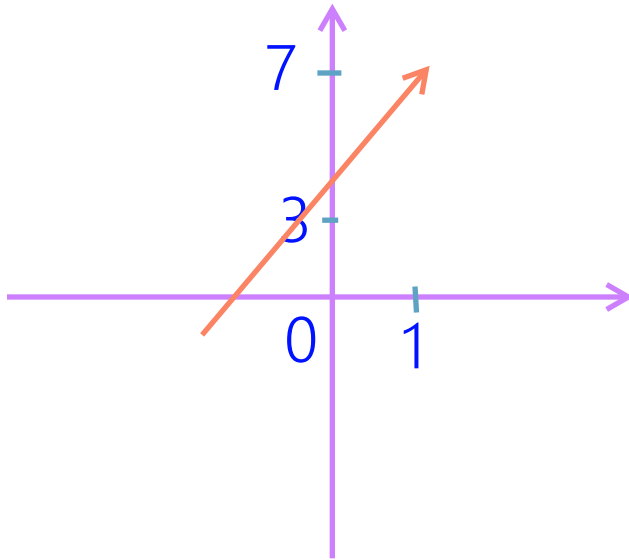
$$-1 \notin h(\mathbb{Z})$$

$$\text{but } -1 \in \mathbb{Z}$$

h is not surjective

E.g.7

Show that the function $k : \mathbb{R} \rightarrow \mathbb{R}$ given by $k(x) = 4x + 3$ is bijective.



$$\mathbb{R} \quad \mathbb{R}$$

$$k(a) = k(b)$$

$$4a + 3 = 4b + 3$$

$$4a = 4b$$

$$a = b$$

$\therefore k$ is injective.

Let $c \in \mathbb{R}$, $k(x) = c$

$$4x + 3 = c$$

$$4x = c - 3$$

$$x = \frac{c - 3}{4}$$

This mean for any value in \mathbb{R} , we can find a value in x .

$$k(\mathbb{R}) = \mathbb{R}$$

$\therefore k$ is surjective.

$\therefore k$ is **bijective**.

5.3 Functions for Computer Science

- Let A be a subset of the universal set $U = \{u_1, u_2, u_3, \dots, u_n\}$. The characteristics function of A is defined as a function from U to $\{0, 1\}$ by the following:
 - $f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$
 - If $A = \{4, 7, 9\}$ and $U = \{1, 2, 3, \dots, 10\}$, then $f_A(2) = 0$, $f_A(4) = 1$, $f_A(7) = 1$, and $f_A(12)$ is undefined.
 - f_A is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- A Family of mod- n functions, one for each positive integer n , $f_n(m) = m \pmod{n}$, is a function from the nonnegative integers to the set $\{0, 1, 2, 3, \dots, n-1\}$.

- For a fixed n , any nonnegative integer z can be written as $z = kn + r$ with $0 \leq r < n$.

Then $f_n(z) = r$, which can also be written as $z \equiv r \pmod{n}$.

- Each member of the mod function family is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- The floor function, which is defined for rational numbers as $f(q) = \lfloor q \rfloor$ is the largest integer less than or equal to q .
 - E.g. $\lfloor 1.5 \rfloor = 1, \lfloor -3 \rfloor = -3, \lfloor -2.7 \rfloor = -3$
- The ceiling function, which is defined for rational numbers as $c(q) = \lceil q \rceil$ is the smallest integer greater than or equal to q .
 - E.g. $\lceil 1.5 \rceil = 2, \lceil -3 \rceil = -3, \lceil -2.7 \rceil = -2$

5.3 Functions for Computer Science (cont)

- Many common algebraic functions are used in computer science, often with domains restricted to subsets of integers.
 - Any **polynomial** with integer coefficients, p , can be used to define a function on \mathbb{Z} as follows:
If $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ and $z \in \mathbb{Z}$, then $f(z)$ is the value of p evaluated at z .

5.3 Functions for Computer Science (cont)

- Let $A = B = \mathbb{Z}^+$ and let $f : A \rightarrow B$ be defined by $f(z) = 2^z$, which is called as base 2 exponential function. Other bases can be used to define similar functions.
- Let $A = B = \mathbb{R}$ and let $f_n : A \rightarrow B$ be defined for each positive integer $n > 1$ as $f_n(x) = \log_n x$, the logarithm to the base n of x . Usually bases 2 and 10 are used.

5.3 Functions for Computer Science (cont)

- The domains and codomains in the function need not be sets of numbers.
 - Let A be a finite set and define $l: A \rightarrow \mathbb{Z}$ as $l(w)$ is the length of the string w .
 - Let B be a finite subset of the universal set U and define $pow(B)$ to be the power set of B . Then pow is a function from V , the power set of U , to the power set of V .

5.3 Functions for Computer Science (cont)

- Let $A = B$ = the set of 2×2 matrices with real number entries and let $t(\mathbf{M}) = \mathbf{M}^T$, the transpose of \mathbf{M} . Then t is everywhere defined, onto, and one-to-one.
- For elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$, $g(z_1, z_2)$ is defined to be $\text{GCD}(z_1, z_2)$ and $m(z_1, z_2)$ to be $\text{LCM}(z_1, z_2)$. Then g and m are a function from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to \mathbb{Z}^+ .

5.3 Functions for Computer Science (cont)

- A Boolean function is a set from A to B , where $B = \{\text{True}, \text{False}\}$.

Let $P(x)$: x is even and $Q(y)$: y is odd.

Then P and Q are functions from \mathbb{Z} to B .

The predicate $R(x, y)$: x is even or y is odd is a Boolean function of two variables from $\mathbb{Z} \times \mathbb{Z}$ to B .

5.4 Permutations

- A **bijection** from a **set A** to **itself**.
 - E.g. Let $A = \{1, 2, 3\}$. Then all the permutations of A are

$$\begin{array}{l} I_A = \begin{pmatrix} \overset{A}{1} & 2 & 3 \\ \underset{A}{1} & 2 & 3} \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{matrix} (1,2) \\ (2,1) \\ (1,3) \end{matrix} \\ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{array}$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

5.4 Permutations (cont)

- Theorem 1

If $A = \{a_1, a_2, \dots, a_n\}$ is a set containing n elements, then there are $n!$ permutations on A .

- The composition of two permutations, $p_1 \circ p_2$ is another permutation, usually referred to as the product of these permutations.

5.4 Permutations (cont)

- Let b_1, b_2, \dots, b_r be r distinct elements of the set $A = \{a_1, a_2, \dots, a_n\}$. The permutation $p: A \rightarrow A$ defined by

$$p(b_1) = b_2$$

$$p(b_2) = b_3$$

$$\vdots$$

$$p(b_{r-1}) = b_r$$

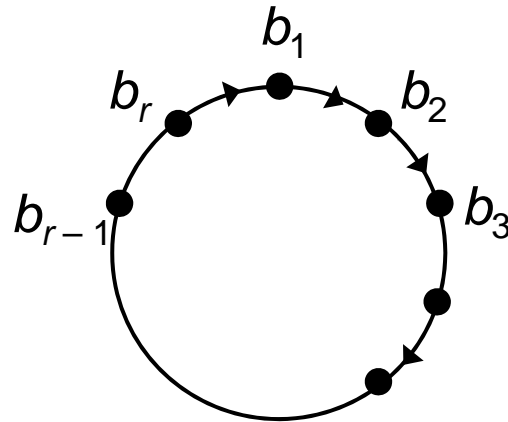
$$p(b_r) = b_1$$

$$p(x) = x, \text{ if } x \in A, x \notin \{b_1, b_2, \dots, b_r\}$$

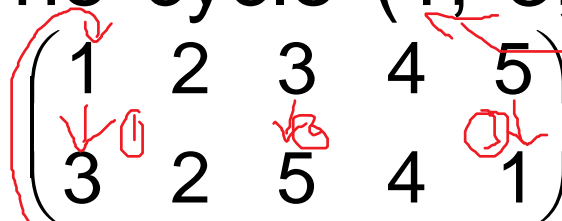
is called a cyclic permutation of length r , or a cycle of length r , and denoted by (b_1, b_2, \dots, b_r) .

5.4 Permutations (cont)

- The cyclic permutation p can be written by starting with any b_i , $1 \leq i \leq r$, and moving in a clockwise direction.



- E.g. Let $A = \{1, 3, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation



5.4 Permutations (cont)

- The notation for a cycle does not indicate the number of elements in the set A . Thus $(3, 2, 1, 4)$ could be a permutation of the set $\{1, 2, 3, 4\}$ or of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
- The cycle on a set A is of length 1 if and only if it is the identity permutation, I_A .

5.4 Permutations (cont)

- Two cycles of a set A are said to be disjoint if no element of A appears in both cycles.
- E.g. Let $A = \{1, 2, 3, 4, 5, 6\}$. Then the cycles $(1, 2, 5)$ and $(3, 4, 6)$ are disjoint whereas $(1, 2, 5)$ and $(2, 4, 6)$ are not.
- Theorem 2
A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length ≥ 2 .

5.4 Permutations (cont)

- When a permutation is written as a product of disjoint cycles, the product is unique except for the order of the cycles.

E.g.8

Let $A = \{1, 2, 3\}$ and $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Find

i. p_1^{-1} 1 go to 2, 2 go to 3, 3 go to 1 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

ii. $p_3 \circ p_2$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
cycle form

E.g.9

Let $A = \{1, 2, 3, 4, 5, 6\}$. Compute

i. $(4, 1, 3, 5) \circ (5, 6, 3)$ STA

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix} = (1, 3, 4) \circ (5, 6) \neq$$

ii. $(5, 6, 3) (4, 1, 3, 5).$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 3 & 1 & 4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4) \circ (3, 6)$$

E.g.10

Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$

of the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles.

$$(1, 3, 6) \circ (2, 4, 5) \circ (7, 8)$$

5.4.1 Even and Odd Permutations

- A cycle of length 2 is called a transposition, that is $p = (a_i, a_j)$, where $p(a_i) = a_j$ and $p(a_j) = a_i$.
- If $p = (a_i, a_j)$ is a transposition of A , then $p \circ p = I_A$, the identity permutation of A .
- Every cycle can be written as a product of transpositions. In fact

$$(b_1, b_2, \dots, b_{k+1}) \\ = (b_1, b_{k+1}) \circ (b_1, b_k) \circ \dots \circ (b_1, b_3) \circ (b_1, b_2).$$

$$\square (1, 2, 3, 4, 5) = (1, 5) \circ (1, 4) \circ (1, 3) \circ (1, 2)$$

5.4.1 Even and Odd Permutations (cont)

■ Corollary 1

Every permutation of a finite set with at least two elements can be written as a product of transpositions which need not be disjoint.

■ Every cycle can be written as a product of transpositions in many different ways.

$$\begin{aligned} \square (1, 2, 3) &= (1, 3) \circ (1, 2) = (2, 1) \circ (2, 3) \text{ even} \\ &= (1, 3) \circ (3, 1) \circ (1, 3) \circ (1, 2) \circ (3, 2) \circ (2, 3) \end{aligned}$$

5.4.1 Even and Odd Permutations (cont)

- Theorem 3

A permutation of a finite set can be written as a product of an even number of transpositions, then it can never be written as a product of an odd number of transpositions, and conversely.

5.4.1 Even and Odd Permutations (cont)

- A permutation of a finite set is called even if it can be written as a product of an even number of transpositions, and it is called odd if it can be written as a product of an odd number of transpositions.
 - The product of two even permutations is even.
 - The product of two odd permutations is even.
 - The product of an even and an odd permutation is odd.

5.4.1 Even and Odd Permutations (cont)

■ Theorem 4

Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite set with n elements, $n \geq 2$. There are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations.

E.g.11

Write the permutation in E.g.10 as a product of transpositions and determine whether it is an even or an odd permutation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$$

$$= (1,3,6) \circ (2,4,5) \circ (7,8)$$

$$= (1,6) \circ (1,3) \circ (2,5) \circ (2,4) \circ (7,8)$$

5 transpositions

therefore, it is an odd permutation

E.g.12

Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

is even or odd.

$$= (1,2,4,7) \circ (3,5,6)$$

$$= (1,7) \circ (1,4) \circ (1,2) \circ (3,6) \circ (3,5)$$

5 permutations

therefore, it is an odd

permutations