Calc II - C3: Infinite Sequences & Series

December 29, 2019

1 Terms to describe $\{a_n\}$

1. A set of numbers
   (a) $a_n = 2n, n = \{1, 2, 3, ...\}$
   (b) $a_n = 1\left(2^{n-1}\right), N = \{1, 2, 3, ...\}$
   (c) $a_n = 1\left(-1\right)^{n-1}, N = \{1, 2, 3, ...\}$
   (d) $a_n = \frac{n}{n+1}$
   (e) $a_n = (-1)^n \left(\frac{n+1}{3^n}\right)$
2. Definition 3.2: Let $n = 1, 2, 3$
   (a) If $a_n \leq M$,
      i. $\{a_n\}$**bounded above** by M
      ii. M is **upper bound**
   (b) and vice versa ($a_n \geq M$, lower bound)
3. $\{a_n\}$
   (a) **positive** if $a_n \geq 0, \forall n$ and vice versa ($a_n \leq 0$).
   (b) **increasing** if $a_{n+1} \geq a_n$, $\forall n$, and vice versa ($a_{n+1} \leq 0$)
   (c) **monotonic =** all increasing/decreasing
   (d) **alternating** $= a_n \cdot a_{n+1} \leq 0$ (consecutive, opposite sign)
4. As $n \to \infty$
   (a) **convergent** if $\lim_{n\to\infty}$ is real number
   (b) **divergent** if $\lim_{n\to\infty}$ is $\pm\infty$

1.1 Example

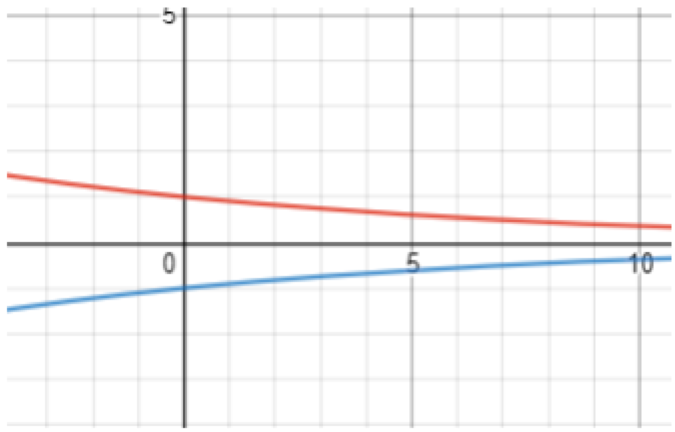
Describe the following sequences using the terms in definition above:

1. $\{n\} = \{1, 2, 3, 4, 5...\}$
   (a) Not bounded above
   (b) Bounded below, Lower bound = 1
   (c) Not bounded
   (d) Positive
   (e) Increasing
   (f) Monotonic
   (g) Not alternating
   (h) Divergent (from 1)
2. $\left\{\frac{n-1}{n}\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}...\right\}$
   (a) Not bounded above
   (b) Bounded below, Lower bound = 0
   (c) Not bounded
   (d) Positive
   (e) Increasing
   (f) Monotonic
   (g) Not alternating
   (h) Convergent (to 1)
3. $\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}...\right\}$
   (a) Bounded above, $\frac{1}{4}$
   (b) Bounded below, Lower bound = $-\frac{1}{2}$
   (c) Bounded
   (d) Not increasing / decreasing
   (e) Not monotonic
   (f) Alternating
   (g) Convergent (to 0)

2 Convergence and Divergence of Sequences

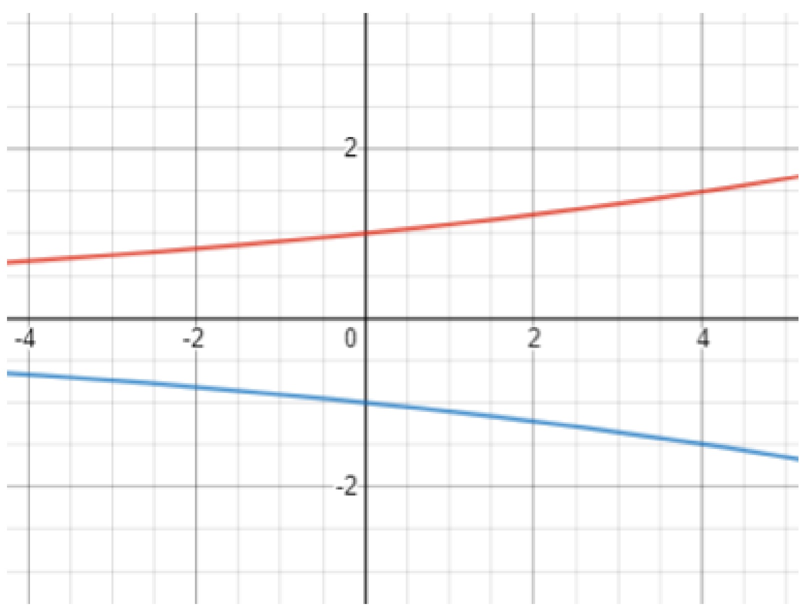
2.1 Definition

1. If $\lim_{n\to\infty} a_n = L$, **converges** (is **convergent**)



(a)

2. Else if L D.N.E. or $\pm\infty$, it **diverges** (is **divergent**)



(a)

2.2 Theorem

1. If $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$, and n is an integer, then $\lim_{n\to\infty} a_n = L$

2. Put it simply (and slightly incorrectly), if the series converges, and $f(n)$ is a_n, then $f(\infty)$ is simply L.

2.3 Theorem

1. Every bounded & monotonic sequence is **convergent**.

2.4 Laws - Limit Laws for Convergent Sequences:

If $\{a_n\}$ and $\{b_n\}$are convergent sequences, and c is constant, then:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(c \cdot a_n) = c \lim_{n\to\infty} a_n$
3. $\lim_{n\to\infty} c = c$
4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n) \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p, p > 0, a_n > 0$

2.5 Technique - Rational functions

If a_n is rational function, to find $\lim_{n\to\infty} a_n$, divide both by highest power of n in **denominator.**

2.6 Example

Find the limit of the following sequences if exist:

1. $a_n = \frac{3n^2-n+4}{7+6n^2}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{n}{n^2} + \frac{4}{n^2}}{\frac{7}{n^2} + \frac{6n^2}{n^2}} \\
&= \frac{3}{6} \\
\boldsymbol{\lim_{n\to\infty} a_n} &\boldsymbol{= \frac{1}{2}}
\end{aligned}$$

2. $a_n = \frac{n}{n+1}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n+1}{n}} \\
&= \frac{1}{1} \\
\boldsymbol{\lim_{n\to\infty} a_n} &\boldsymbol{= 1}
\end{aligned}$$

3. $a_n = \frac{n^2-n}{n^2+1}$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2-n}{n^2+1} \\
&= \lim_{n\to\infty} \frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n^2}} \\
\boldsymbol{\lim_{n\to\infty} a_n} &\boldsymbol{= 1}
\end{aligned}$$

2.7 Example

By using L'Hospital's Rule, calculate the following:

1. $\lim_{n\to\infty} \frac{\ln n}{n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{n} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
\boldsymbol{\lim_{n\to\infty} \frac{\ln n}{n}} &\boldsymbol{= 0}
\end{aligned}$$

2. $\lim_{n\to\infty} \frac{e^{2n}}{n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{e^{2n}}{n} &= \lim_{n\to\infty} \frac{\frac{d}{dn}\left[e^{2n}\right]}{\frac{d}{dn}\left[n\right]} \\
&= \lim_{n\to\infty} \frac{2e^{2n}}{1} \\
&= \lim_{n\to\infty} 2e^{2n} \\
\lim_{n\to\infty} \frac{e^{2n}}{n} &= \infty
\end{aligned}$$

2.8 Theorem - Squeeze Theorem

1. If $a_n \leq b_n \leq c_n, n > n_0$
   (a) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$
   (b) Then $\lim_{n\to\infty} b_n = L$

2.8.1 Example

Evaluate:

1. $\lim_{n\to\infty} \left\{ \frac{\cos^2 n}{3^n} \right\}$

$$\begin{aligned} -1 &\leq \cos n \leq 1 \\ 1 &\leq \cos^2 n \leq 1 \\ \frac{1}{3} &\leq \frac{\cos^2 n}{3^n} \leq \frac{1}{3} \\ \lim_{n\to\infty} \frac{1}{3} &\leq \lim_{n\to\infty} \frac{\cos^2 n}{3^n} \leq \lim_{n\to\infty} \frac{1}{3} \\ 0 &\leq \lim_{n\to\infty} \frac{\cos^2 n}{3^n} \leq 0 \end{aligned}$$

   (a) $\therefore$ According to Squeeze Theorem, $\lim_{n\to\infty} \frac{\cos^2 n}{3^n} = 0$

2. $\lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\}$ (Lecturer mention exam might come out)

$$\begin{aligned} \frac{n!}{n^n} &= \frac{n\,(n-1)\,(n-2)\ldots 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \ldots \cdot n} \\ &= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \\ &= 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq \lim_{n\to\infty} 1 \\ \lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq \lim_{n\to\infty} 1 \\ 0 &\leq \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} \leq 0 \end{aligned}$$

   (a) According to S.T., $\lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\} = 0$

2.9 Theorem 3.3

1. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

2.9.1 Example

1. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if exists.

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

2.10 Theorem 3.4

1. Sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \leq 1$, diverges for all other r.

2. Also,

$$\lim_{n\to\infty} = \begin{cases} 0 & , -1 < r < 1 \\ 1 & , r = 1 \end{cases}$$

2.10.1 Evaluate:

1. $\lim_{n\to\infty} (-1)^n = \infty, Divergent$

2. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0, convergent$

3. $\lim_{n\to\infty} 5^n = \infty, Divergent$

3 Infinite Series

3.1 Definition 3.4

1. **Infinite series** (or just **series**) $= a_1 + a_2 + a_3 + ... + a_n + ...$

   (a) $\sum a_n$ OR $\sum_{n=1}^{\infty} a_n$

2. Series = infinite sum of numbers

3. Sequence = numbers, 1-to-1 correspondence with positive integer.

3.1.1 Example

1. $\left\{\frac{1}{n}\right\} = \left\{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\right\}$

   (a) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$

2. $\left\{\frac{(-1)^{n-1}}{2^{n-1}}\right\} = \left\{1, -\frac{1}{2}, \frac{1}{4}, ...\right\}$

   (a) $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{2^{n-1}}\right) = 1 - \frac{1}{2} + \frac{1}{4} - ...$

3.2 Definition 3.5 - Partial Sums, S_k

1. $k-th$ Partial sum, S_k of $\sum a_n$
   (a) $S_k = a_1 + a_2 + ... + a_k$
2. Sequence of partial sum of $\sum_{n=1}^{\infty} a_n$ is
   (a) $\{S_1, S_2 ... S_n, ...\}$

3.2.1 Example

Find the sequence of partial sum of the series $\sum_{n=1}^{\infty} n$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + ...$$
$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = ...$$

1. Sequence of partial sum: $\left\{1, 3, 6, 10, \frac{n}{2}(n+1), ...\right\}$

4 Convergence & Divergence of Series

4.1 Definition

1. The series $\sum_{n=1}^{\infty} a_n$ **converges** if and only if the sequence $\{Sn\}$ is convergent and $\lim_{n\to\infty} S_n = s$ exists as a real number.
2. The number s is called the **sum** of the series, $s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$
3. If the sequence $\{Sn\}$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ **diverges**
4. To put it simply, series converges if and only if:
   (a) sequence is convergent AND
   (b) the value of function when approaching infinity is a real number.

4.1.1 Example

Show that the series $\sum_{n=1}^{\infty} n$ is divergent

1. $\{S_n\} = \{1, 2, 3, ..., \infty\}$ is divergent
2. Therefore, it is divergent.

4.2 Theorem 3.5

1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
   (a) Converse not generally true. Further investigation needed.

4.3 Theorem 3.6: Test for Divergence

1. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ D.N.E., the series $\sum_{n=1}^{\infty} a_n$ is divergent.
2. Put it simply, if the limit to infinity is not 0, then it diverges.

4.3.1 Example

Show that the following series are divergent.

1. $\sum_{n=1}^{\infty} \frac{n^2}{3+5n^2}$

$$\lim_{n\to\infty} \frac{n^2}{3+5n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{3}{n^2}+\frac{5n^2}{n^2}}$$
$$= \lim_{n\to\infty} \frac{1}{\frac{3}{n^2}+5}$$
$$\lim_{n\to\infty} \frac{n^2}{3+5n^2} = \frac{1}{5} \neq 0$$

   (a) $\therefore$ *Divergent*

2. $\sum_{n=1}^{\infty} \frac{-n}{2n+3}$

$$\lim_{n\to\infty} \frac{-n}{2n+3} = \lim_{n\to\infty} \frac{\frac{-n}{n}}{\frac{2n}{n}+\frac{3}{n}}$$
$$= \lim_{n\to\infty} \frac{-1}{2+\frac{3}{n}}$$
$$\lim_{n\to\infty} \frac{-n}{2n+3} = -\frac{1}{2}$$
$$\lim_{n\to\infty} \frac{-n}{2n+3} \neq 0$$

5 Special Series

5.1 Geometric Series

5.1.1 Definition

1. **Geometric series:** Series with the form $a + ar + ar^2 + ... + ar^n + ...$
2. Notation: $\sum_{n=1}^{\infty} ar^{n-1}$ OR $\sum_{n=0}^{\infty} ar^n$

3. **Convergent**, if
   (a) $|r| < 1$, AND
   (b) $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
   (c) Otherwise, its **divergent**

5.1.2 Example

Determine if the following series converge or diverge. If they converge give the value of the series.

1. $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$
   (a) $9^{-n+2}4^{n+1}$

$$\begin{aligned} 9^{-n+2}4^{n+1} &= \frac{9^2}{9^n}4\left(4^n\right) \\ &= \frac{9^2}{9^n}4\left(4^n\right) \\ &= 4 \cdot 9^2 \cdot \left(\frac{4}{9}\right)^2 \\ r &= \frac{4}{9} < 1, conv \end{aligned}$$

$$S_\infty = \frac{\left(9 \cdot 4^2\right)}{1 - \frac{4}{9}} = 259.2$$

2. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$
   (a) $\frac{(-4)^{3n}}{5^{n-1}}$

$$\begin{aligned} \frac{(-4)^{3n}}{5^{n-1}} &= \frac{(-4)^{3n}}{5^n \cdot 5^{-1}} \\ &= 5 \cdot \frac{\left(-4^3\right)^n}{5^n} \\ &= 5 \cdot \left(\frac{-4^3}{5}\right)^n \\ &= 5 \cdot \left(-\frac{4^3}{5}\right)^n \\ r &= -\frac{4^3}{5} \end{aligned}$$

      i. Since $|r| > 1$, the series is divergent

5.2 Telescoping Series

Definition 3.8: A telescoping series is a series whose sum can be found by exploiting the circumstance that nearly every term cancels with either a succeeding or preceding term.

5.2.1 Example

1. Given the telescoping series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} + ...$

   (a) Find the n-th partial sum S_n of the series for n = 1, 2, 3, 4

$$S_1 = \frac{1}{2}$$

$$\begin{aligned} S_2 &= \frac{1}{2} + \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} S_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} S_4 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \\ &= \frac{4}{5} \end{aligned}$$

   (b) Find S_n.

$$\begin{aligned} \sum \frac{1}{n\,(n+1)} &= \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \mathbf{\frac{1}{2}}\right) + \left(\mathbf{\frac{1}{2}} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1} \\ &= S_n \end{aligned}$$

   (c) Show that the series is convergent and find its sum.

$$\begin{aligned} S_\infty &= \lim_{n\to\infty} S_n \\ &= 1 - \frac{1}{\infty + 1} \\ &= 1, convergent \end{aligned}$$

i. Note: Sum is 1 because they cancel out each other for every term. So only the remaining term counts.

5.2.2 Example

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)}+\frac{1}{2^n}\right)$.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2}{n\left(n+1\right)}+\frac{1}{2^n}\right) &= 2\sum\frac{1}{n\left(n+1\right)}+\sum\frac{1}{2^n}\\ &= 2+\sum\left(\frac{1}{2}\right)^n\\ &= 2+\frac{\frac{1}{2}}{1-\frac{1}{2}}\\ \sum_{n=1}^{\infty}\left(\frac{2}{n\left(n+1\right)}+\frac{1}{2^n}\right) &= 3\end{aligned}$$

5.3 Harmonic Series (Not in finals, safe to ignore)

Definition 3.9: A series of the form $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$ is called a harmonic series. Always **divergent**.

5.3.1 Proof for divergence (Proof by contradiction)

1. Assume harmonic series converges to H.
2. Calculations

$$\begin{aligned}H &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\\ H &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+...\\ H &> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+...\\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\\ &= H\end{aligned}$$

3. Contradiction ($H > H$). So the harmonic series is not convergent.

5.4 $p-series$

Definition 3.10: A $p-series$ is a series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+...$

$\sum_{n=1}^{\infty}\frac{1}{n^P}=\begin{cases} converges & p>1\\ diverges & p\leq 1\end{cases}$

5.4.1 Example

Determine whether the given series converges or diverges.

1. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...$
   (a) $p=2, conv$
2. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+...$
   (a) $p=\frac{1}{2}, div$

6 Convergent Test for Series

6.1 The Integral Test

Theorem 3.8: Suppose f is a continuous, positive, **decreasing** function on $[1,\infty)$ and $a_n=f(n)$.

a. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

b. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

6.1.1 Example:

By using Integral Test, determine whether the following series converges or diverges.

1. $\sum_{n=1}^\infty \frac{5}{n+1}$

$$\begin{aligned}\int_1^\infty \frac{5}{x+1}dx &= 5\ln(x+1)_1^\infty \\ &= 5\ln\infty - 5\ln 2 \\ \int_1^\infty \frac{5}{x+1}dx &= \infty\end{aligned}$$

   (a) $\therefore Divergent$

2. $\sum_{n=1}^\infty e^{-n}$

$$\begin{aligned}\int_1^\infty e^{-x}dx &= \int_1^\infty e^{-x}dx \\ &= \left[-e^{-x}\right]_1^\infty \\ &= -e^{-\infty} - \left(-e^{-1}\right) \\ &= -e^{-\infty} + e^{-1} \\ \int_1^\infty e^{-x}dx &= e^{-1}\end{aligned}$$

   (a) $\therefore Convergent$

6.2 Comparison Test/Limit Comparison Test

Theorem 3.9: **Comparison Test**

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

a. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

b. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

6.2.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+1}$

$$a_n = \frac{3}{2n^2+4n+1}$$
$$2n^2+4n+1 > 2n^2 > n^2$$
$$\frac{3}{2n^2+4n+1} < \frac{3}{n^2}$$

   (a) $3\sum \frac{1}{n^2}$ is a convergent $p - series$.
   (b) $\sum_{n=1}^{\infty} \frac{3}{2n^2+4n+1}$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$$a_n = \frac{1}{2^n+1}$$
$$2^n+1 > 2^n$$
$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

   (a) Since $\sum \frac{1}{2^n}$ is a convergent geometric series
   (b) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent.

Theorem 3.10: **Limit Comparison Test**

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge. If c = 0, no conclusion.

6.2.2 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$
   (a) $a_n = \frac{1}{1+\sqrt{n}}, b_n = \frac{1}{\sqrt{n}}$
   (b) b_n is a divergent p-series, $p = \frac{1}{2}$
   (c) $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(\frac{1}{1+\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} \\ &= \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} \\ &= 1\end{aligned}$$

   (d) Therefore, $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$
   (a) $a_n = \frac{1}{2^n-1}, b_n = \frac{1}{2^n}$
   (b) b_n is a convergent geometric series.
   (c) $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n-1} \\ &= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} \\ &= 1\end{aligned}$$

   (d) Therefore, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

6.3 Alternating Series Test

Theorem 3.11: If the alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ where $b_n > 0$ 0 satisfies:

1. $b_{n+1} \leq b_n$ for all n (decreasing);
2. $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

Note: This test CANNOT be used to determine if the series is divergent.

6.3.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
   (a) $b_n = \frac{1}{n}$
   (b) $b_n = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right)$, decreasing
   (c) $\lim_{n\to\infty} b_n = 0$
   (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+3}$
   (a) $b_n = \frac{n^2}{n^2+3}$
   (b) $b_n = \left(\frac{1}{4}, \frac{4}{7}, ...\right)$ increase
   (c) This test is not suitable, run another test.

6.4 Absolute Convergence

Definition 3.11: A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

6.4.1 Example

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p - series\,(p = 2)$

Theorem 3.12: If $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

Definition 3.12: a series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if the series $\sum_{n=1}^{\infty} a_n$ is convergent but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is divergent.

6.4.2 Example

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is **conditionally convergent** because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating harmonic series which is convergent, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

6.4.3 Example

Test the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ for absolute convergence.

$$\left|\frac{\cos n}{n^2}\right| < \frac{1}{n^2}$$

1. Since $\sum \frac{1}{n^2}$ is a convergent $p-series$ $(p = 2)$, but comparison test, $\sum \left|\frac{\cos n}{n^2}\right|$ is absolutely convergent.

2. So, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

6.5 The Ratio Test

Theorem 3.13: Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1 & , \sum_{n=1}^{\infty} a_n \text{converges} \\ > 1 \text{or} \infty & , \sum_{n=1}^{\infty} a_n \text{diverges} \\ 1 & , \text{the Ratio Test inconclusive} \end{cases}$

6.5.1 Example

Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \frac{99^n}{n!}$

$$a_n = \frac{99^n}{n!}, a_{n+1} = \frac{99^{n+1}}{(n+1)!}$$

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{99^n} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{99^{n+1}}{99^n} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} 99 \cdot \frac{n!}{(n+1)\, n!} \text{note: still the same thing} \\ &= \lim_{n\to\infty} \frac{99}{n+1} = 0 \\ &< 1\, (convergent) \end{aligned}$$

2. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$$a_n = \frac{(2n)!}{(n!)^2}, a_{n+1} = \frac{(2\,(n+1))!}{((n+1)!)^2}$$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(2\,(n+1))!}{((n+1)!)^2} * \frac{(n!)^2}{(2n)!} \\
&= \lim_{n\to\infty}\frac{(2\,(n+1))!}{(2n)!} * \frac{(n!)^2}{((n+1)!)^2} \\
&= \lim_{n\to\infty}\frac{(2\,(n+1))!}{(2n)!} * \left(\frac{n!}{(n+1)!}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)\,\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \left(\frac{\cancel{n!}}{(n+1)\,\cancel{n!}}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)\,\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \left(\frac{\cancel{n!}}{(n+1)\,\cancel{n!}}\right)^2 \\
&= \lim_{n\to\infty}\frac{(2n+2)\,(2n+1)}{(n+1)\,(n+1)} \\
&= \lim_{n\to\infty}\frac{4n^2+6n+2}{n^2+2n+1} = 4\,(>1)
\end{aligned}
$$

(a) The series is divergent

3. $\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)!}$

(a) $a_n = \frac{n^2}{(2n-1)!}, a_{n+1} = \frac{(n+1)^2}{(2(n+1)-1)!} = \frac{(n+1)^2}{(2n+1)!}$

(b)

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)!} \cdot \frac{(2n-1)!}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{n^2} \cdot \frac{(2n-1)!}{(2n+1)!} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{(2n-1)!}{(2n+1)\,(2n+1-1)\,(2n+1-2)!} \text{note: expand the factorial} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{\cancel{(2n-1)!}}{(2n+1)\,(2n)\,\cancel{(2n-1)!}} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} \cdot \frac{1}{(2n+1)\,(2n)} \\
&= \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) \cdot \frac{1}{(2n+1)\,(2n)} \\
&= 1 \cdot 0 \\
&= 0 \\
&= convergent
\end{aligned}
$$

6.6 The Root Test

Theorem 3.14: Suppose we have the series $\sum_{n=1}^{\infty} a_n$.

Define, $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

Then, if $L < 1$, the series is absolutely convergent (and hence convergent).

If $L > 1$, the series is divergent.

$L = 1$, the test is inconclusive.

Example: Determine whether the following series converges or diverges:

1. $\sum_{n=1}^{\infty} \left(\frac{4n-3n^2}{5n^2+2}\right)^n$

   (a) $a_n = \left(\frac{4n-3n^2}{5n^2+2}\right)^n$

$$\sqrt[n]{\left(\frac{4n-3n^2}{5n^2+2}\right)^n} = \frac{4n-3n^2}{5n^2+2}$$
$$\lim_{n\to\infty}\left|\frac{4n-3n^2}{5n^2+2}\right| = \frac{3}{5} < 1$$

   (b) The series is convergent

2. $\sum_{n=1}^{\infty} \frac{n^n}{4^{1+3n}}$

$$\begin{aligned}\sqrt{\frac{n^n}{4^{1+3n}}} &= \frac{n}{4^{\frac{1+3n}{n}}}\\ &= \frac{n}{4^{\frac{1}{n}+3}}\end{aligned}$$

$$\begin{aligned}\left|\lim_{n\to\infty}\frac{n}{4^{\frac{1}{n}+3}}\right| &= \frac{\infty}{4^3}\\ &= \infty\end{aligned}$$

   (a) The series is divergent

6.6.1 Example, exapnd the following:

1. $(x + \frac{1}{x})^5$

$$\begin{aligned}\left(x+\frac{1}{x}\right)^5 &= \sum_{k=0}^{\infty}\binom{5}{k}x^{5-k}\left(\frac{1}{x}\right)^k\\ &= \binom{5}{0}x^5\left(\frac{1}{x}\right)^0+\binom{5}{1}x^4\left(\frac{1}{x}\right)^1+\binom{5}{2}x^3\left(\frac{1}{x}\right)^2+\binom{5}{3}x^2\left(\frac{1}{x}\right)^3+\binom{5}{4}x^1\left(\frac{1}{x}\right)^4+\binom{5}{5}x^0\left(\right.\\ &= x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}\end{aligned}$$

2. $(2-x)^4$

   (a) $2^5 - 5\left(2^4 x\right) + 10\left(2^3\right)x^3 + 5\left(2^1 x^4\right) - x^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

   (b) $161 + 72\sqrt{5}$

3. $\left(2+\sqrt{5}\right)^4$

   (a) Start by writing the terms in pascal triangle: $1, 4, 6, 4, 1$

   (b) Continue by adding values into the equation

$$\left(2+\sqrt{5}\right)^4 = 1\left(2\right)^4 + 4\left(2\right)^3\left(\sqrt{5}\right) + 6\left(2\right)^2\left(\sqrt{5}\right)^2 + 4\left(2\right)\left(\sqrt{5}\right)^3 + 1\left(\sqrt{5}\right)^4$$

6.7 Binomial Series

Theorem: **Binomial Theorem**

If n is any positive integer, then

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\ &= a^n + na^{n-1}b + \frac{n\left(n-1\right)}{2!}a^{n-2}b^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}a^{n-3}b^3 + ... + nab^{-1} + b^n\end{aligned}$$

Where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ (binomial coefficient)

Pascal's Triangle – collection of the binomial coefficients.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

6.7.1

Definition 3.14: **Binomial Series**

If n is any real number and $|x| < 1$, then

$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^n$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$ is called a Binomial Series.

6.7.2 Example: Expand the following functions as a power series:

1. $\frac{1}{(1+x)^2}$

$$
\begin{aligned}
\frac{1}{(1+x)^2} &= (1+x)^{-2} \\
&= 1 + (-2)\,x + \frac{-2\,(-2-1)}{2!}x^2 + \frac{-2\,(-2-1)\,(-2-2)}{3!}x^3 + \ldots \\
&= 1 - 2x + 3x^2 - 4x^3 + \ldots.
\end{aligned}
$$

2. $\sqrt{1+x}$

$$
\begin{aligned}
\sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\
&= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}=1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 \\
&= 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots
\end{aligned}
$$

6.8 Power Series

Definition 3.13: A power series is a series of the form $\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots$ where x is a variable and the c_n's are constants called the **coefficients** of the series.

More generally, a series of the form $\sum c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \ldots$ is called a **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a.

Theorem 3.15: For a given power series $\sum c_n\,(x-a)^n$ there are only three possibilities:

i. The series converges only at the point $x = a$.

ii. The series converges for all x.

iii. The series converges if $|x-a|<R$ and diverges if $|x-a|>R$, where the positive number R is called the **radius of convergence** of the power series. The set of points at which the series converges is called the **interval of convergence**.

Example: Determine the radius of convergence and interval of convergence for the following power series:-

1. $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

$$a_n = \sum_{n=0}^{\infty} \frac{n\left(x+2\right)^n}{3^{n+1}}$$
$$a_{n+1} = \sum_{n=0}^{\infty} \frac{\left(n+1\right)\left(x+2\right)^{n+1}}{3^{n+2}}$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\left(n+1\right)\left(x+2\right)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\left(x+2\right)^n} \\ &= \frac{1}{3}\left(\frac{n+1}{n}\right)\left(x+2\right)\end{aligned}$$

(a) The series converges if lim <1

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{1}{3}\left(\frac{n+1}{n}\right)\left(x+2\right)\right| &< 1 \\ \lim_{n\to\infty}\left|\frac{1}{3}\left(x+2\right)\right| &< 1\end{aligned}$$

(b) Radius of convergence

i. Since x is not dependent on n, we can factor it out and make it become

$$\begin{aligned}\frac{1}{3}\left|\left(x+2\right)\right| &< 1 \\ \left|\left(x+2\right)\right| &< 3\end{aligned}$$

ii. Therefore, the radius of convergence is $R=3$

(c) Interval of convergence

i.

$$\begin{aligned}-1 < \frac{1}{3}\left(x+2\right) &< 1 \\ -1 < \frac{x+2}{3} &< 1 \\ -3 < x+2 &< 3 \\ -5 < x &< 1\end{aligned}$$

2. $\sum_{n=0}^{\infty} \frac{2^n (4x-8)^n}{n}$

$$a_n = \frac{2^n (4x-8)^n}{n}, a_{n+1} = \frac{2^{n+1} (4x-8)^{n+1}}{n+1}$$

(a) Use the ratio test to find the limit L

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{2^{n+1} (4x-8)^{n+1}}{n+1} \cdot \frac{n}{2^n (4x-8)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2^{\cancel{n}+1} (4x-8)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{2^n (4x-8)^n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{2n (4x-8)}{n+1} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{2n(4x-8)}{n}}{\frac{n}{n}+\frac{1}{n}} \right| \\ &= \left| \frac{2 (4x-8)}{1} \right| \\ &= 2 \left|(4x-8)\right| \end{aligned}$$

(b) For the series to converge, $L < 1$

$$\begin{aligned} 2 \left|(4x-8)\right| &< 1 \\ \left|(4x-8)\right| &< \frac{1}{2} \end{aligned}$$

(c) Therefore, the radius of convergence is $R = \frac{1}{2}$

(d) Next, find the interval of convergence

$$\begin{aligned} -\frac{1}{2} < 4(x-2) < \frac{1}{2} \\ -\frac{1}{8} < x-2 < \frac{1}{8} \\ \frac{15}{8} < x < \frac{17}{8} \end{aligned}$$

(e) Therefore, the interval of convergence is $\frac{15}{8} < x < \frac{17}{8}$

6.8.1 Representations of Functions as Power Series

We say that the power series is a representation of a function on the interval of convergence.

A power series about $x = 0$ is a series of the form $\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$

When $c_i = 1, \forall i$, we have a geometric series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$

If $|x| < 1$, the series converges and the sum is $\frac{1}{1-x}$. Thus we have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x} \text{for} - 1 < x < 1$$

In above equation, we express the function $f(x) = \frac{1}{1-x}$ as the sum of the power series. So the power series is a representation of $f(x) = \frac{1}{1-x}$

6.8.2 Example: Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum \left(-x^2\right)^n$$

Series is convergent when

$$\begin{aligned} -1 &< -x^2 < 1 \\ -1 &< x^2 < 1 \\ 0 &< x < 1 \end{aligned}$$

6.8.3 Example: Find a power series representation for $\frac{1}{x+2}$

1. Solution

$$\begin{aligned} \frac{1}{x+2} &= \frac{1}{1-(-(x+1))} \\ &= \sum_{n=0}^{\infty} (-(x+1))^n \end{aligned}$$

2. Series is convergent when

$$\begin{aligned} -1 &< -(x+1) < 1 \\ -1 &< x+1 < 1 \\ -2 &< x < 0 \end{aligned}$$

6.8.4 Differentiation and Integration of Power Series

The differentiation and integration of power series are used to obtain power series representations.

Theorem 3.16: **Term-by-term Differentiation and Integration**

If the power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Is differentiable on the interval of convergence and

i. $f\prime(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$

ii.$\int f(x)\, dx$

$$\begin{aligned}\int f(x)dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}\end{aligned}$$

The radii of convergence of the power series in above equations are both R

6.8.5 Example

Find the power series representations for the functions. By using the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

1. $\frac{1}{(1-x)^2}$

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{d}{dx}\left[\sum x^n\right] \\ &= \sum \frac{d}{dx}(x^n) \\ &= \sum nx^{n-1}\end{aligned}$$

2. $\tan^{-1} x$ (hint: $\tan^{-1} x = \int \frac{1}{1+x^2} dx$)

$$
\begin{aligned}
\tan^{-1} x &= \int \frac{1}{1+x^2} dx \\
&= \int \sum \left(-x^2\right) dx \\
&= \sum \int (-1)^n \left(x^{2n}\right) dx \\
&= \sum \int (-1)^n \left(x^{2n}\right) dx \\
&= \sum (-1)^n \frac{x^{2n+1}}{2n+1} dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \\
&= \tan^{-1} x
\end{aligned}
$$

7 Taylor & Maclaurin Series

In previous section we were able to find power series representations for some specific functions. Now we consider the following questions: **Which functions have power series representations? How to find such representations?**

Without taking anything away from the process we looked at in the previous section, we come up with a more general method for writing a power series representation.

The concept of a **Taylor series** was formulated by the Scottish mathematician James Gregory and formally introduced by the English mathematician Brook Taylor in 1715. Who was Brook Taylor? He was the one invented Integration by part: $\int udv = uv - \int vdu$ If the **Taylor series is centered at zero**, then that series is also called a **Maclaurin series**, named after the Scottish mathematician Colin Maclaurin.

> **Theorem 3.17:** If f has a power series representation at a, that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$,
>
> Then $c_n = \frac{f^{(n)}(a)}{n!}$, where (n) represent the number of $'$ (or level of derivative)

7.1 Taylor Series (Intro)

Thus, we have a power series representation for the function $f(x)$ about $x = a$ is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...
\end{aligned}
$$

which is called the **Taylor Series** of the function f at a (or about a)

7.2 MacLaurin Series (Intro)

If $a = 0$, then the series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \\ &= f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + ... \end{aligned}$$

is called the **Maclaurin Series.**

7.3 Limit of Compounded Interest

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828459045...$$

7.3.1 Proof

1.

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + ... \\ &= 1 + 1 + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + ... \end{aligned}$$

2. When $n \to \infty$,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... \\ &= e \end{aligned}$$

3. Expanding similarly,

$$\begin{aligned} \left(1 + \frac{x}{n}\right)^n &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\ &= e^x \end{aligned}$$

7.4 Example

Find the Taylor Series for e^x at $a = 2$, and the Maclaurin Series for e^x and e^{-x}.
Taylor Series:

$$f = e^x, f' = e^x, f'' = e^x, ...$$

1. $f(2) = e^2, f'(2) = e^2, f''(2) = e^2, ...$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$
$$f(x) = e + \frac{e^2}{1!}(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + ...$$

2. MacLaurin Series, $a = 0$

$$f = e^x, f' = e^x, f'' = e^x, ...$$
$$f(0) = 1, f'(0) = 1, f''(0) = 1, ..$$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$
$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$
$$= e^x$$

3. For e^{-x}, use $f(-x)$

		n=5	n=7	n=10	n=12
x	e^x	Maclaurin expansion			
-3	0.049787068367864	-0.65	-0.07142857	0.053325893	0.049997463
-2	0.135335283236613	0.06666667	0.13015873	0.135379189	0.135336433
-1	0.367879441171442	0.36666667	0.36785714	0.367879464	0.367879441
-0.5	0.606530659712633	0.60651042	0.60653057	0.60653066	0.60653066
0.5	1.648721270700130	1.64869792	1.64872117	1.648721271	1.648721271
1	2.718281828459050	2.71666667	2.71825397	2.718281801	2.718281828
2	7.389056098930650	7.26666667	7.38095238	7.388994709	7.389054567
3	20.085536923187700	18.4	19.8464286	20.07966518	20.08521256

7.5 Convergence of exponential series:

1. It converges for all x.
2. It converges faster when x is approaching 0, slower when x is away from 0.
3. It converges faster when n is bigger.

7.6 Example

Find the Maclaurin Series for $\sin x$. Then use it to find the Maclaurin Series for $\sin(3x^2)$.

Derivatives	$f^{(n)}(x)$
$f(x) = \sin x$	$f(0) = \sin 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f''''(x) = \sin x$	$f(0) = \sin 0$

$$
\begin{aligned}
f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \\
f(x) &= 0 + \frac{1}{1!}(x) + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 - ... \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \\
\sin\left(3x^2\right) &= \left(3x^2\right) - \frac{\left(3x^2\right)^3}{3!} + \frac{\left(3x^2\right)^5}{5!} - ...
\end{aligned}
$$

7.7 Important Maclaurin Series

We collect in the following table, some important Maclaurin Series.

Series	Range
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x^2 + x^3 + ...$	$(-1, 1)$
$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + ...$	$(-1, 1)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{3!} + ...$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$	$[-1, 1]$

8 Applications of Taylor & Maclaurin Series

8.1 Example

1. Evaluate $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

$$
\begin{aligned}
\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)}{x^3} \\
&= \lim_{x\to 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + ...}{x^3} \\
&= \lim_{x\to 0} \left(\frac{1}{3!} - \frac{x^2}{5!} + ...\right) \\
&= \frac{1}{3!} \\
\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \frac{1}{6}
\end{aligned}
$$

2. The above example showed us that Maclaurin series can provide a useful alternative to L'Hospital's Rule to evaluate limit.

8.2 Example

Use series to approximate the definite integral to within the indicated accuracy: $\int_0^1 e^{-x^2} dx$ (3 decimal places).

$$
\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \\
e^{-x^2} &= 1 + \frac{\left(-x^2\right)}{1!} + \frac{\left(-x^2\right)^2}{2!} + \frac{\left(-x^2\right)^3}{3!} + ... \\
&= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + ... \\
\int_0^1 e^{-x^2} &= \int_0^1 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + ...dx \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + ...\right]_0^1 \\
&= 0.743
\end{aligned}
$$

9 Taylor Polynomials and Maclaurin Polynomials

1. If $f(x)$ has n derivatives at $x = a$, we form a polynomial

$$
\begin{aligned}
T_n(x) &= \sum_{n=0}^{\infty} \frac{f^i(a)}{i!} (x-a)^i \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n
\end{aligned}
$$

   (a) which called the **Taylor Polynomial** of the degree n at $x = a$

2. If $a = 0$, then the polynomial

$$
\begin{aligned}
T_n(x) &= \sum_{n=0}^{\infty} \frac{f^i(0)}{i!} (x)^i \\
&= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + ... + \frac{f^n(0)}{n!} x^n
\end{aligned}
$$

   (a) is called the **Maclaurin Polynomial.**

9.1 Example

Find the first three non-zero terms in the Taylor Series for $f(x) = e^x \cos x$ about $x = a$.

$$g(x) = e^x = e^a + e^a (x-a) + \frac{e^a}{2} (x-a)^2 + ...$$
$$h(x) = \cos x = \cos a - \sin a (x-a) - \frac{\cos a}{2} (x-a)^2 + ...$$
$$f(x) = e^x \cos x = g(x) \cdot h(x)$$
$$= \left(e^a + e^a (x-a) + \frac{e^a}{2} (x-a)^2 \right) \left(\cos a - \sin a (x-a) - \frac{\cos a}{2} (x-a)^2 \right)$$