Chapter 5   Functions
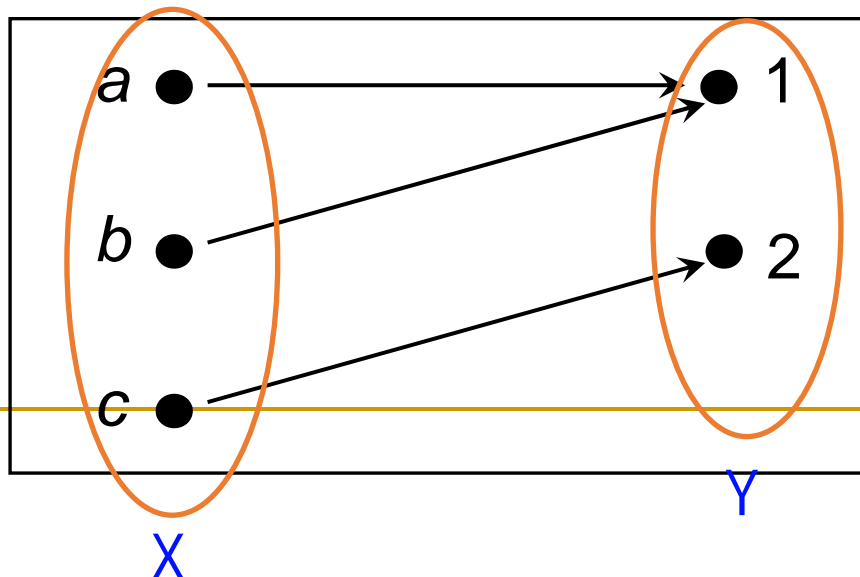
5.1 Introduction

- Functions are binary relations in which further restrictions are imposed on the pairs which can occur. For each input, one output

- A function from a set *A* to a set *B* is a binary relation in which every element of *A* is associated with a uniquely specified element of *B*. only one

- Functions are also called as mappings or transformation.

5.1 Introduction (cont)

- In digraph terms a function is a relation such that there is precisely one arc leaving every element of *A*.

- E.g. The digraph representing the function from {*a*, *b*, *c*} to {1, 2} containing the pairs (*a*, 1), (*b*, 1) and (*c*, 2).

f(x) -> y

5.1 Introduction (cont)

- Let f be a function from a set A to a set B.

  - For each $a \in A$, there exists a uniquely determined $y \in B$ with $(x, y) \in f$, write as $y = f(x)$, and refer to $f(x)$ as the image of x under f.

  - Write as $f : A \to B$ to indicate that f is a function which transform, or maps, each element of A to a uniquely determined element of B.

  - A is called the domain of f, and B is the codomain of f.

5.1 Introduction (cont)

- ❑ The range of *f* is the set of images of all the elements of *A* under *f*, denoted by *f*(*A*). Hence, $f(A) = \{f(x) : x \in A\}$.

- ❑ The Venn diagram provides a useful diagrammatic illustration of a function from a set *A* to a set *B*.
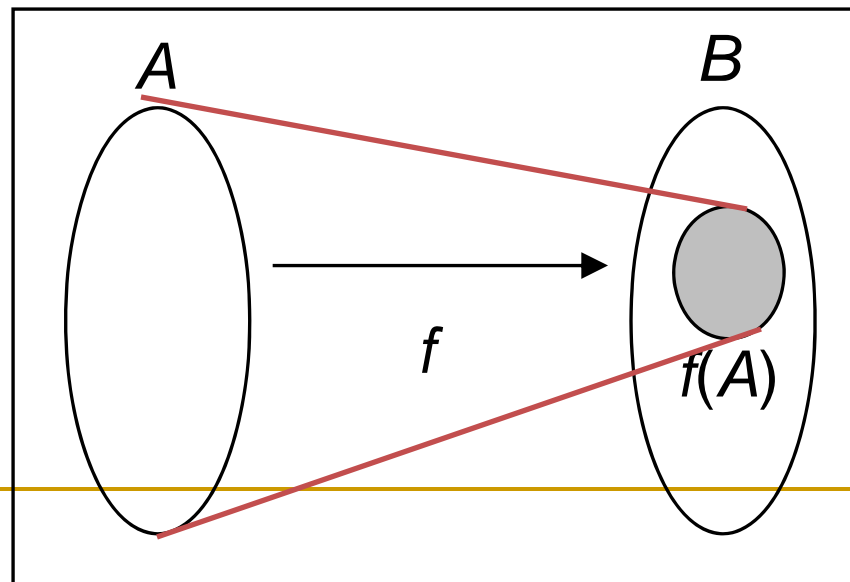
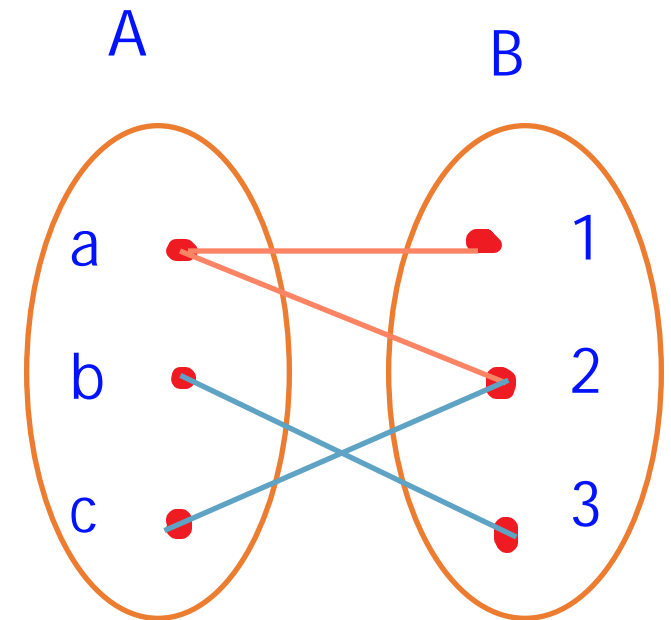codomain of f (everything)

range of f f(a)

5

E.g.1

For each of the following relations between the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Determine whether the relation gives a function from A to B.

i. $\{(a, 1), (a, 2),(b, 3), (c, 2)\}$ No

ii. $\{(a, 1), (b, 2), (c, 1)\}$ Yes

iii. $\{(a, 1), (c, 2)\}$ Yes

A   B

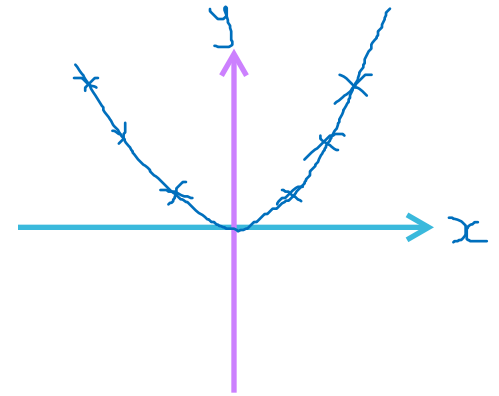a   1
b   2
c   3

E.g.2

Which of the following relations are functions?

i. The relation on Z given by

$\{(\overset{x}{x}, \overset{y}{x^2}) : x \in Z\}.$ $y=x^2$

ii. The relation on R given by

$\{(x, y) : x = y^2\}.$
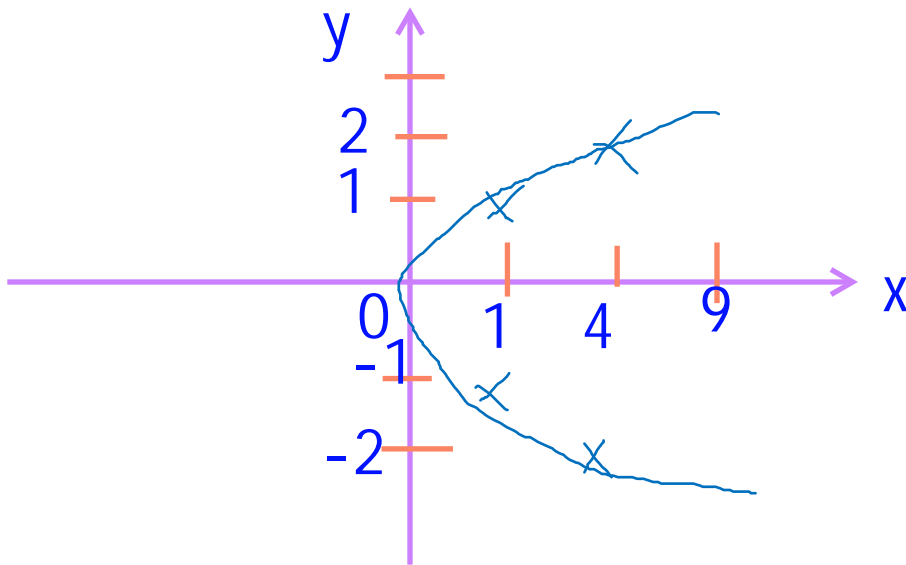
This is a f(x) from Z to Z

7

E.g.2

Which of the following relations are functions?

i.   The relation on Z given by

$\{(x, x^2) : x \in Z\}$.

E.g.2

Which of the following relations are functions?

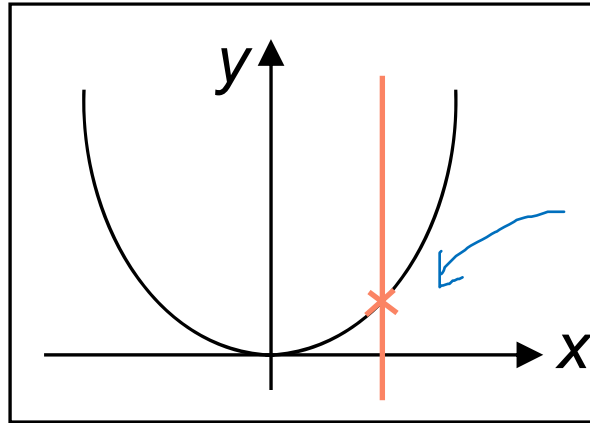ii.   The relation on R given by

$\{(x, y) : x = y^2\}$.

This is not a function from X to Y

5.1 Introduction (cont)

- When dealing with a function $f : A \rightarrow B$ where A and B are infinite sets of numbers, we can use the more traditional mathematical idea of graphing a function to give a geometric picture of the function.

5.1 Introduction (cont)

- The graph of the function $f : R \rightarrow R$ given by $f(x) = x^2$:

  only one
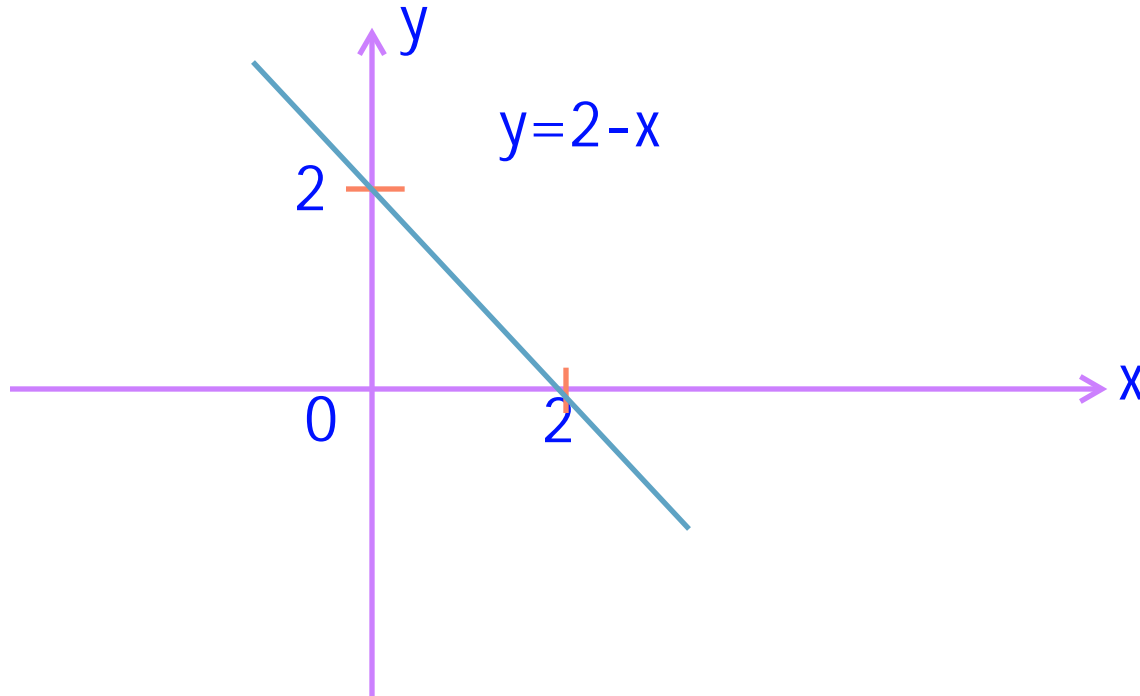
- The horizontal axis (x-axis) denotes the domain R and the vertical axis (y-axis) denotes the codomain R.

- The curve consists of those points (x, y) in Cartesian product $R \times R$ for which $y = f(x)$.

E.g.3

Sketch the graph of the function $f : R \to R$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.

$f(2)=2-2=0$

$f(3)=2-3=-1$

y=2-x

E.g.3

Sketch the graph of the function $f : R \rightarrow R$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.

E.g.3

Sketch the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.
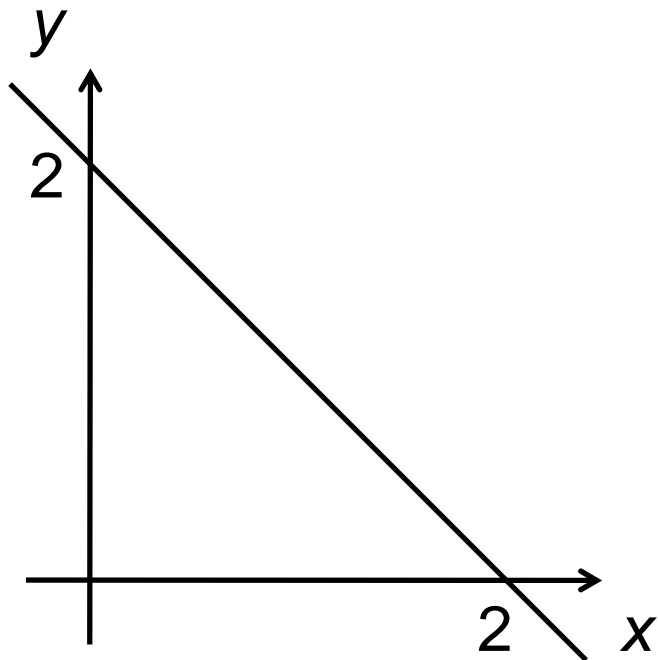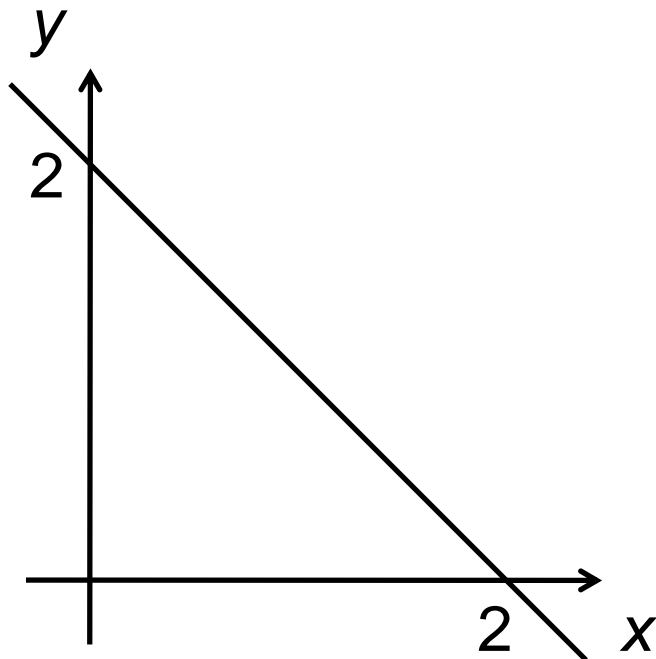
$x = 2,\ f(2) = 0$

$x = 3,\ f(3) = -1$

5.2 Properties of Functions

- Let $f : A \rightarrow B$ be a function.

  - f is an <mark>injective</mark> (or <mark>one-to-one</mark>) function if

    $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all a_1, $a_2 \in A$;

    or $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ i.e. different inputs give different outputs.

  - f is <mark>everywhere defined</mark> if <mark>$Dom(f) = A$</mark>.

  - f is a <mark>surjective</mark> (or <mark>onto</mark>) function if the range of f coincides with the codomain of f, i.e. each element in the codomain is a value of the function; $f(A)=B$

    or for every $b \in B$, there exists an $a \in A$ with $b = f(a)$.

5.2 Properties of Functions (cont)

- *f* is a <mark>bijective</mark> function if *f* is both <mark>injective</mark> and <mark>surjective</mark>.

- *f* is a <mark>one-to-one correspondence</mark> between *A* and *B* if *f* is everywhere defined, injective, and surjective.

E.g.4

Decide which of the functions below is injective or surjective. Which are bijective?

i.
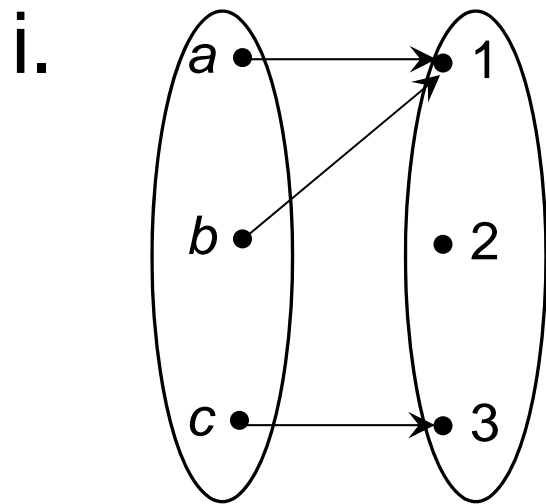
not injective
not surjective

ii.

Bijective

f(A) = B
f is surjective

f(a)=1=f(b)
But a≠b, f is not injective

2 ∈ B but 2∉ f(n). Is not surjective
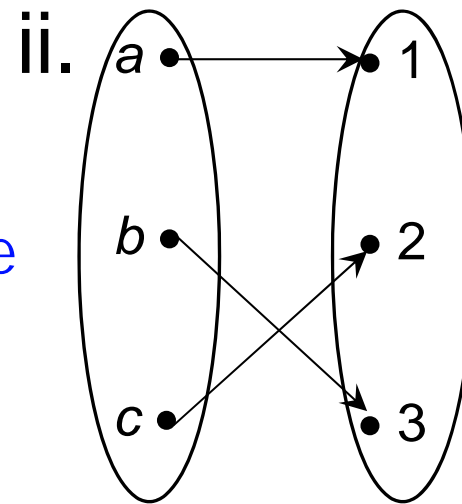
∴ f is not bijective

E.g.4 (cont)

iii.
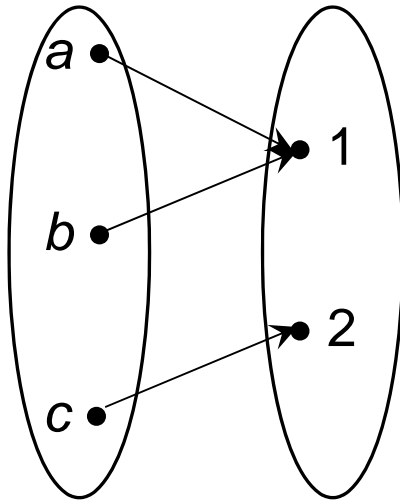
f(a)=1=f(b)
but a ≠ b. f is not injective.

f(A) = B
f is surjective.

iv.
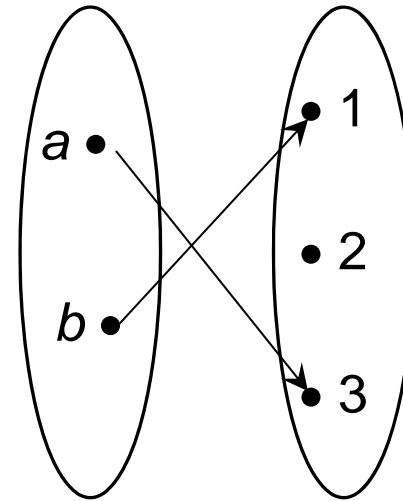
f is injective.

2 ∈ B but 2 f(A)
f is not surjective

f is not bijective

E.g.5

Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$. Determine whether each of the following function is <mark>one-to-one</mark>, <mark>onto,</mark> and <mark>everywhere defined</mark>.   injective   surjective

i. $f : A \rightarrow B$; $f = \{(1, y), (2, z), (3, x)\}$

ii. $g: A \rightarrow D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

iii. $h: B \rightarrow C$; $h = \{(x, e), (y, e), (z, \pi)\}$

iv. $k : D \rightarrow B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$.

i. $f : A \rightarrow B$; $f = \{(1, y), (2, z), (3, x)\}$

A

B
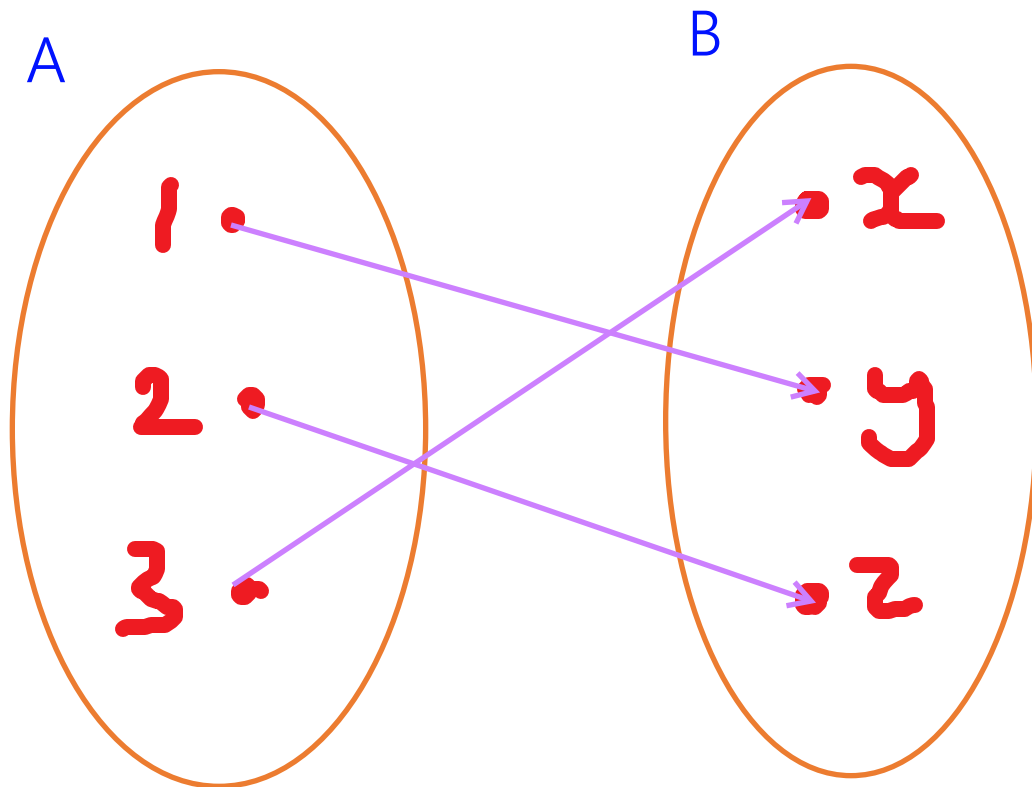
f is one-to-one
f is onto
f is everywhere defined

ii. $g: A \to D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

A   B
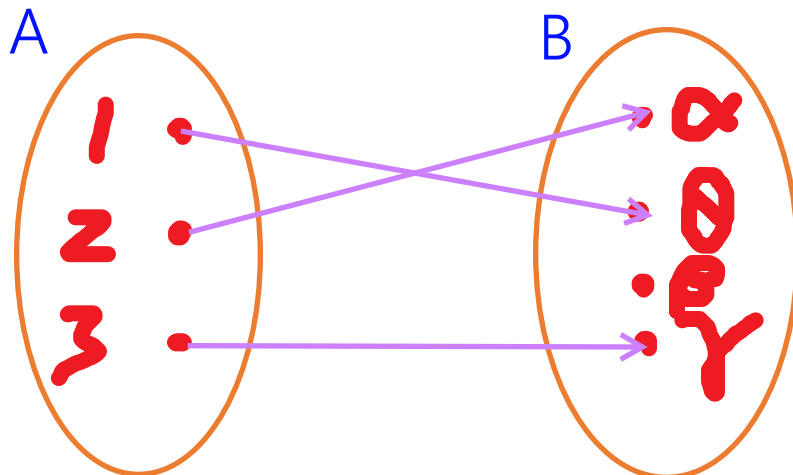
f is one-to-one
$\beta \in D$ but $\beta \notin f(n)$
f is not onto
f is everywhere defined

Note: beta is due to definition of D earlier

iii. $h: B \to C$; $h = \{(x, e), (y, e), (z, \pi)\}$

B   C
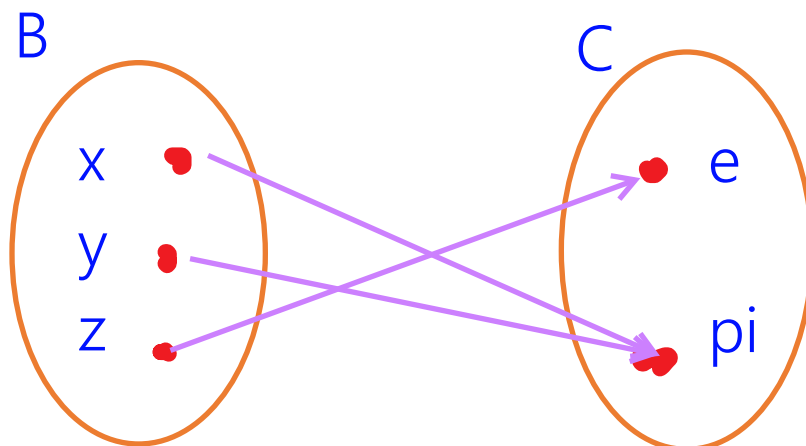
$f(x) = e = f(y), but\, x \neq y$
f is not one-to-one
$f(B) = C$
f is onto
f is everywhere defined

$A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$

iv. $k : D \rightarrow B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$

D

B

Change f to A

theta

alpha

beta

gamma

x

y

z
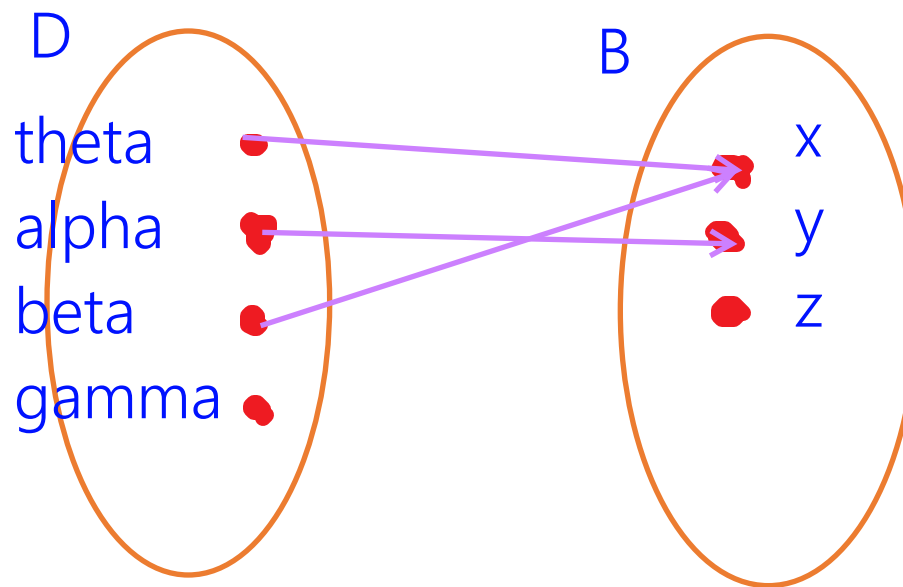
$f(\theta) = x = f(\beta), but\, \theta \neq \beta$
f is not one-to-one

$Z \in B, but\, Z \notin f(D)$

f is not onto,
$f(\gamma)$ is not defined,
f is NOT everywhere defined

E.g.6

Show that the function $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = x^2$ is neither injective nor surjective.
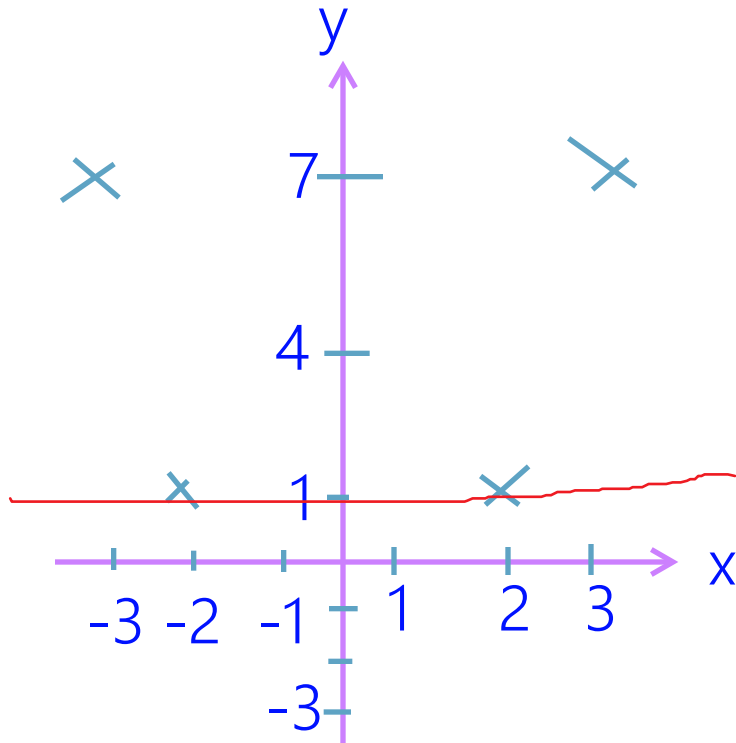
$h(1) = 1 = h(-1)$, but $1 \neq -1$

$\therefore h$ is not injective

$h(x) \geq 0$

$-1 \notin h(\mathbb{Z})$

but $-1 \in \mathbb{Z}$

h is not surjective

E.g.7

$\mathbb{R} \quad \mathbb{R}$

Show that the function $k : \mathrm{R} \to \mathrm{R}$ given by $k(x) = 4x + 3$ is bijective.
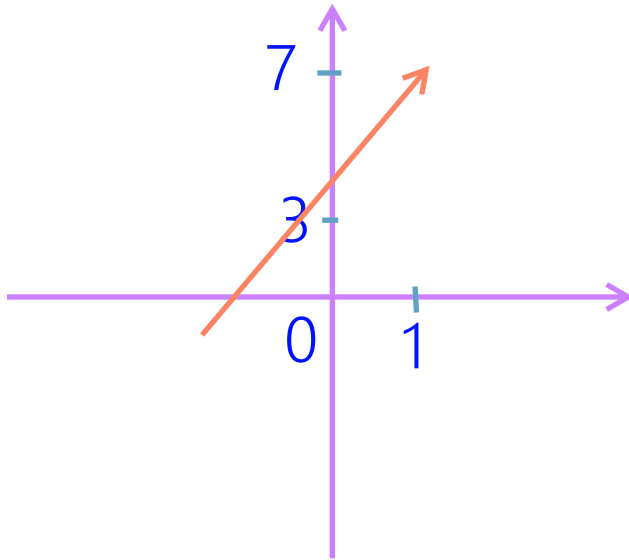
$$k(a) = k(b)$$
$$4a + 3 = 4b + 3$$
$$4a = 4b$$
$$a = b$$

$\therefore k$ is injective.

Let $c \in \mathbb{R}$, $k(x) = c$

$$4x + 3 = c$$
$$4x = c - 3$$
$$x = \frac{c - 3}{4}$$

This mean for any value in $\mathbb{R}$, we can find a value in x.

$$k(\mathbb{R}) = \mathbb{R}$$

$\therefore k$ is surjective.
$\therefore k$ is **bijective.**

5.3 Functions for Computer Science

- Let A be a subset of the universal set $U = \{u_1, u_2, u_3, ..., u_n\}$. The characteristics function of A is defined as a function from U to $\{0, 1\}$ by the following:

  - $f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$

  - If $A = \{4, 7, 9\}$ and $U = \{1, 2, 3, ..., 10\}$, then $f_A(2) = 0$, $f_A(4) = 1$, $f_A(7) = 1$, and $f_A(12)$ is undefined.

  - f_A is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- A Family of mod-n functions, one for each positive integer n, $f_n(m) = m$ (mod n), is a function from the nonnegative integers to the set $\{0, 1, 2, 3, ..., n-1\}$.

  - For a fixed n, any nonnegative integer z can be written as $z = kn + r$ with $0 \leq r < n$.

    Then $f_n(z) = r$, which can also written as $z \equiv r$ (mod n).

  - Each member of the mod function family is everywhere defined and onto, but not one-to-one.

5.3 Functions for Computer Science (cont)

- The floor function, which is defined for rational numbers as $f(q) = \lfloor q \rfloor$ is the largest integer less than or equal to q.
  - E.g. $\lfloor 1.5 \rfloor = 1, \lfloor -3 \rfloor = -3, \lfloor -2.7 \rfloor = -3$

- The ceiling function, which is defined for rational numbers as $c(q) = \lceil q \rceil$ is the smallest integer greater than or equal to q.
  - E.g. $\lceil 1.5 \rceil = 2, \lceil -3 \rceil = -3, \lceil -2.7 \rceil = -2$

5.3 Functions for Computer Science (cont)

- Many common algebraic functions are used in computer science, often with domains restricted to subsets of integers.

  - Any polynomial with integer coefficients, *p*, can be used to define a function on Z as follows:

    If $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ and $z \in$ Z, then *f(z)* is the value of *p* evaluated at *z*.

5.3 Functions for Computer Science (cont)

- Let $A = B = Z^+$ and let $f : A \rightarrow B$ be defined by $f(z) = 2^z$, which is called as base 2 exponential function. Other bases can be used to define similar functions.

- Let $A = B = R$ and let $f_n : A \rightarrow B$ be defined for each positive integer $n > 1$ as $f_n(x) = \log_n x$, the logarithm to the base n of x. Usually bases 2 and 10 are used.

5.3 Functions for Computer Science (cont)

- The domains and codomains in the function need not be sets of numbers.

  - Let A be a finite set and define $l: A \rightarrow Z$ as $l(w)$ is the length of the string w.

  - Let B be a finite subset of the universal set U and define $pow(B)$ to be the power set of B. Then pow is a function from V, the power set of U, to the power set of V.

5.3 Functions for Computer Science (cont)

- Let $A = B =$ the set of 2×2 matrices with real number entries and let $t(\mathbf{M}) = \mathbf{M}^T$, the transpose of $\mathbf{M}$. Then t is everywhere defined, onto, and one-to-one.

- For elements of $Z^+ \times Z^+$, $g(z_1, z_2)$ is defined to be GCD (z_1, z_2) and $m(z_1, z_2)$ to be LCM (z_1, z_2). Then g and m are a function from $Z^+ \times Z^+$ to Z^+.

5.3 Functions for Computer Science (cont)

❑ A Boolean function is a set from *A* to *B*, where *B* = {True, False}.

Let *P*(*x*): *x* is even and *Q*(*y*): *y* is odd.

Then *P* and *Q* are functions from Z to *B*.

The predicate *R*(*x*, *y*): *x* is even or *y* is odd is a Boolean function of two variables from Z × Z to *B*.

5.4 Permutations

- A bijection from a set *A* to itself.
  - E.g. Let *A* = {1, 2, 3}. Then all the permutations of *A* are

$$I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

5.4 Permutations (cont)

- Theorem 1

  If $A = \{a_1, a_2, \ldots, a_n\}$ is a set containing n elements, then there are $n!$ permutations on A.

- The composition of two permutations, $p_1 \circ p_2$ is another permutation, usually referred to as the product of these permutations.

5.4 Permutations (cont)

- Let b_1, b_2, …, b_r be r distinct elements of the set $A = \{a_1, a_2, …, a_n\}$. The permutation p: $A \rightarrow A$ defined by

  $p(b_1) = b_2$

  $p(b_2) = b_3$

  $\vdots$

  $p(b_{r-1}) = b_r$

  $p(b_r) = b_1$

  $p(x) = x$, if $x \in A$, $x \notin \{b_1, b_2, …, b_r\}$

  is called a cyclic permutation of length r, or a cycle of length r, and denoted by (b_1, b_2, …, b_r).

5.4 Permutations (cont)

■ The cyclic permutation p can be written by starting with any b_i, $1 \leq i \leq r$, and moving in a clockwise direction.
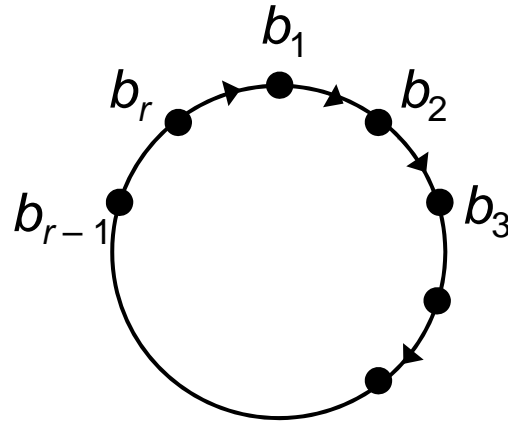
■ E.g. Let $A = \{1, 3, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

5.4 Permutations (cont)

- The notation for a cycle does not indicate the number of elements in the set A. Thus (3, 2, 1, 4) could be a permutation of the set {1, 2, 3, 4} or of {1, 2, 3, 4, 5, 6, 7, 8}.

- The cycle on a set A is of length 1 if and only if it is the identity permutation, I_A.

5.4 Permutations (cont)

- Two cycles of a set *A* are said to be disjoint if no element of *A* appears in both cycles.

    - E.g. Let *A* = {1, 2, 3, 4, 5, 6}. Then the cycles (1, 2, 5) and (3, 4, 6) are disjoint whereas (1, 2, 5) and (2, 4, 6) are not.

- Theorem 2

    A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length ≥ 2.

5.4 Permutations (cont)

- When a permutation is written as a product of disjoint cycles, the product is unique except for the order of the cycles.

E.g.8

Let $A = \{1, 2, 3\}$ and $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Find

i. p_1^{-1}

ii. $p_3 \circ p_2$

E.g.9

Let $A = \{1, 2, 3, 4, 5, 6\}$. Compute

i. $(4, 1, 3, \circ 5)\ (5, 6, 3)$

ii. $(5, 6, \circ 3)\ (4, 1, 3, 5).$

E.g.10

Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$

of the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles.

5.4.1   Even and Odd Permutations

- A cycle of length 2 is called a transposition, that is $p = (a_i, a_j)$, where $p(a_i) = a_j$ and $p(a_j) = a_i$.

- If $p = (a_i, a_j)$ is a transposition of A, then $p \circ p = I_A$, the identity permutation of A.

- Every cycle can be written as a product of transpositions. In fact
$(b_1, b_2, \ldots, b_{k+1})$
$= (b_1, b_{k+1}) \circ (b_1, b_k) \circ \ldots \circ (b_1, b_3) \circ (b_1, b_2)$.

  - $(1, 2, 3, 4, 5) = (1, 5) \circ (1, 4) \circ (1, 3) \circ (1, 2)$

5.4.1   Even and Odd Permutations (cont)

- Corollary 1

Every permutation of a finite set with at least two elements can be written as a product of transpositions which need not be disjoint.

- Every cycle can be written as a product of transpositions in many different ways.

  - $(1, 2, 3) = (1, 3) \circ (1, 2) = (2, 1) \circ (2, 3)$

    $= (1, 3) \circ (3, 1) \circ (1, 3) \circ (1, 2) \circ (3, 2) \circ (2, 3)$

5.4.1   Even and Odd Permutations (cont)

- Theorem 3

  A permutation of a finite set can be written as a product of an even number of transpositions, then it can never be written as a product of an odd number of transpositions, and conversely.

5.4.1 Even and Odd Permutations (cont)

- A permutation of a finite set is called even if it can be written as a product of an even number of transpositions, and it is called odd if it can be written as a product of an odd number of transpositions.

  - The product of two even permutations is even.

  - The product of two odd permutations is even.

  - The product of an even and an odd permutation is odd.

5.4.1 Even and Odd Permutations (cont)

- Theorem 4

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set with n elements, $n \geq 2$. There are $\dfrac{n!}{2}$ even permutations and $\dfrac{n!}{2}$ odd permutations.

E.g.11

Write the permutation in E.g.10 as a product of transpositions and determine whether it is an even or an odd permutation.

E.g.12

Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

is even or odd.