Disc. Maths: C2 - Logic of Quan. Statements

October 23, 2019

1 Predicates and Quantified Statements

1. A predicate (AKA open statement)
   (a) contains variables
   (b) is not statement until the variables are "filled in"
2. Domain of predicate variable: Set of all values that can be "filled in" to the variables.
3. To obtain predicate, remove the nouns.
4. $x \epsilon A$ indicates x is an element of A.
5. $\{1, 2, 3\}$ refers to set containing only 1, 2 and 3. $\{1, 2, 3, ...\}$ indicates all positive integers.
6. 2 sets are equal only if they have same elements.
7. Symbols for sets:

Symbols	Set
R	All real numbers
R^+	All positive real numbers
Z	All integers
Z^{nonneg}/N	Set of non-negative integers (include 0) / natural numbers

8. $\{x \in D | P(x)\}$ reads as the set of all x in D such that $P(x)$ (is true, for a truth set).
   (a) x is random variable
   (b) D is the domain of x
   (c) $P(x)$ is a predicate
9. Notations:
   (a) $P(x) \implies Q(x)$: All truth set elements of $P(x)$ is in truth set of $Q(x)$.

i. Basically, if I promised $P(x)$, then I must deliver, $Q(x)$.

(b) $P(x) \iff Q(x)$: Same truth set

10. Quantifiers:

    (a) Quantities

    (b) Add to predicates, a substitute for "fixed values"

    (c) $\forall$: For all

        i. Example: $\forall x$ = For every x

        ii. Universal statement: $\forall x \in D, Q(x)$, is true if $Q(x)$ is true for all x (where x is part of domain D). It is false if there is a **counterexample**, or a value for x where $Q(x)$ is false.

    (d) $\exists$: There exists

        i. Example: $\exists x$ = At least one x

        ii. Existential statement: $\exists x \in D, Q(x)$, is true if at least one $Q(x)$ is true. Where x is part of D. It is false if all is false.

11. **Method of exhaustion** - proving true for every case. Good for finite domain.

12. $\ni$: "such that"

13. Rewriting

    (a) To informal:

        i. Replace all symbols

        ii. Remove quantifiers and make them "wordy"

    (b) The opposite for formal

14. Universal condition statements

    (a) A universal statement with a condition (if, then).

    (b) $\forall x$, if $P(x)$ then $Q(x)$.

15. Equivalent forms of universal statement

    (a) $\forall x \in U$,if $P(x)$ then $Q(x) \equiv \forall x \in D, Q(x)$.

    (b) Narrow U to be true statement domain of $P(x)$

16. Equivalent forms of existential statements

    (a) $\exists x \in U$ such that $P(x)$ and $Q(x)$ can be written as "$\exists x \in D$ such that $Q(x)$", provided D consists of all elements in U that make $P(x)$true.

17. Negation of quantified statements

    (a) $\sim (\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)$, and vice versa

(b) Basically something like DeMorgan's law.

(c) Negation of "all are" (universal statement) is $\equiv$ Some are not (Existential).

18. Negation of universal conditional statements

(a) Negate a "for all" conditional statement

(b) $\sim(\forall x, P(x) \to Q(x)) \equiv \exists x, P(x) \wedge \sim Q(x)$

$$\begin{aligned} \sim(\forall x, P(x) \to Q(x)) &\equiv \exists x, \sim(P(x) \to Q(x)) \\ &\equiv \exists x, \sim(\sim P(x) \vee Q(x)) \\ \sim(\forall x, P(x) \to Q(x)) &\equiv \exists x, P(x) \wedge \sim Q(x) \ [DeMorgan] \end{aligned}$$

(c) $\sim(\forall x, \text{if } P(x), \text{then} Q(x)) \equiv \exists x \, \text{such that } P(x) \, \text{and} \sim Q(x)$

19. Negations of Multiply Quantified Statements

(a) $\sim(\forall x, \exists y \ni P(x,y)) \equiv \exists x \forall y, \sim P(x,y)$

(b) $\sim(\exists x \ni \forall y, P(x,y)) \equiv \forall x, \exists y \ni \sim P(x,y)$

20. The Relation Among $\forall$, $\exists$, $\wedge$, and $\vee$

(a) $\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge ... \wedge Q(x_n)$

i. This reads: For all x in domain D, such that (not shown, but should be $\ni$) $Q(x)$ evaluates to true.

ii. The statement is equivalent to "All $Q(x)$ evaluates to true".

(b) $\exists x \in D \ni Q(x) \equiv Q(x_1) \vee Q(x_2) \vee ... \vee Q(x_n)$

i. This reads: There exists x in domain D, such that $Q(x)$ evaluates to true.

ii. The statement is equivalent to "Some $Q(x)$ evaluates to true"

(c) $\forall x [P(x) \vee Q] \equiv \forall x P(x) \vee Q$ (Refer to example 18)

(d) $\forall x[P(x) \wedge Q(x)] \equiv \forall x P(x) \wedge \forall x Q(x)$ (Refer to example 19)

(e) $\forall$ distributes over $\wedge$, $\exists$ distributes over $\vee$. But NOT vice versa.

21. Variants of Universal and Conditional Statements

(a) Consider the statement of the form:

$$\forall x \in D, \text{if} P(x) \, \text{then} Q(x)$$

(b) Its contrapositive is the statement

$$\forall x \in D, \text{if} \sim Q(x) \, \text{then} \sim P(x)$$

(c) Its converse is the statement

$$\forall x \in D, \text{if} Q(x) \, \text{then} P(x)$$

(d) Its inverse is
$$\forall x \in D, \text{if} \sim P(x)\, \text{then} \sim Q(x)$$

22. Universal condition statement equivalence
    (a) Equivalent to contrapositive:
        i. For all x, if $P(x)$ then $Q(x) \equiv$ For all x, if not $Q(x)$ then not $P(x)$
    (b) NOT equivalent to:
        i. Converse: For all x, if $Q(x)$ then $P(x)$
        ii. Inverse: For all x, if not $P(x)$ then not $Q(x)$
23. Using diagrams to test for validity
    (a) Helpful and convincing
    (b) Steps
        i. Represent truth of premises in diagrams
        ii. Aalyze diagrams to see if they also apply to conclusion

1.1 Example

Consider:
$p(x)$: The number (x + 2) is an even integer.
$q(x, y)$: The numbers $y + 2$, $x–y$, and $x + 2y$ are even integers.
Domain for x is {4, 5} and domain for y is {2, 3}
Therefore,

1. $p(4) : (4 + 2) : 6$ is an even integer. (T)
2. $p(5) : (5 + 2) : 7$ is an even integer. (F)
3. $q(4, 2)$: The numbers $2+2 : 4$, $4-2 : 2$, and $4+2(2) : 8$ are even integers. (T)
4. $q(4, 3)$: The numbers $3+2 : 5$, $4-3 : 1$, and $4+2(3) : 10$ are even integers. (F)
5. $q(5, 2)$:The numbers $2+2 : 4$, $5-2 : 3$, and $5+2(2) : 9$ are even integers. (F)
6. $q(5, 3)$:The numbers $3+2 : 5$, $5-3 : 2$, and $5+2(3) : 11$ are even integers. (F)

1.2 Example

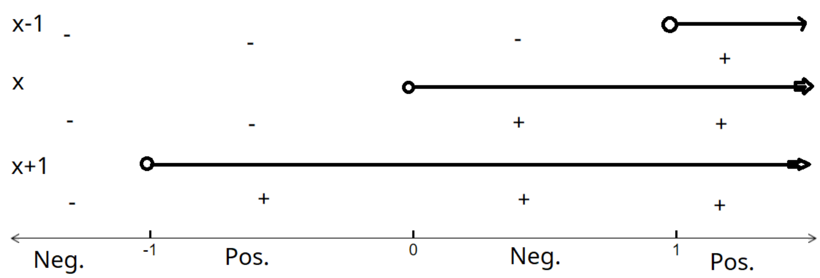
Find the truth set for the predicate below,

$$x > \frac{1}{x}, domain : R$$

1. Find a way to make it easier to compute

$$\begin{aligned} x &= \frac{1}{x} \\ x - \frac{1}{x} &= 0 \\ \frac{x^2 - 1}{x} &= 0 \\ \frac{(x+1)(x-1)}{x} &= 0 \\ \frac{(x+1)(x-1)}{x} &> 0 \end{aligned}$$

2. Utilize the number line system



3. Determine the range (where it is greater than 0, in this case)

$$Truth\,Set = \{x \in R | -1 < x < 0, x > 1\}$$

1.3 Example

Let $P(x)$: x is the factor of 8.
$Q(x)$: x is the factor of 4.
$R(x)$: $x < 5$, $x \neq 3$.
Domain for x is Z^+.

1.3.1 Answer

The truth set of:

$$\begin{aligned} P(x) &= \{1, 2, 4, 8\} \\ Q(x) &= \{1, 2, 4\} \\ R(x) &= \{1, 2, 4\} \end{aligned}$$

Therefore,

$$
\begin{aligned}
Q(x) &\implies P(x)\\
R(x) &\implies P(x)\\
Q(x) &\iff R(x)
\end{aligned}
$$

1.4 Example

Let $D = 1, 2, 3, 4, 5$, and consider the statement "$\forall x \in D, x \geq \frac{1}{x}$". Show that this statement is **true**.

1. Let $P(x)$ be $x \geq \frac{1}{x}$.

   (a)

x	$P(x)$	Result
1	$1 \geq \frac{1}{1}$	T
2	$2 \geq \frac{1}{2}$	T
3	$3 \geq \frac{1}{3}$	T
4	$4 \geq \frac{1}{4}$	T
5	$5 \geq \frac{1}{5}$	T

   (b) $\therefore$ Since $P(x)$is true for all $x \in D$, thus $\forall x \in D, x \geq \frac{1}{x}$ is a **true statement.**

2. Consider the statement "$\forall x \in R, x > \frac{1}{x}$" Find counterexamples to show that this statement is **false.**

   (a)

x	$x > \frac{1}{x}$	Result
1	$1 > \frac{1}{1}$	F

   (b) This is a counterexample.

   (c) The statement $\forall x \in R, x > \frac{1}{x}$ is false.

1.5 Example

1. Consider the statement $\exists x \in Z \ni x^2 = x$ ($\ni$ means such that)Show that this statement is true. EXTRA NOTE: The statement reads (There exists) x (in) $\mathbb{Z}$ (such that) $x^2 = x$.

   (a)

   $$
   \begin{aligned}
   x^2 - x &= 0\\
   x(x-1) &= 0\\
   x &= 0, 1
   \end{aligned}
   $$

   (b) Let $x = 0$

   $$
   \begin{aligned}
   0^2 &= 0\\
   x^2 &= x
   \end{aligned}
   $$

(c) Thus, $\exists x \in \mathbb{Z} \ni x^2 = x$ is true.

2. Let $D = \{5, 6, 7, 8, 9, 10\}$ and consider the statement $\exists x \in D \ni x^2 = x$. Show that this statement is **false**.

(a)

x	$x^2 = x$	Result
5	$5^2 : 25 = 5$	F
6	$6^2 : 36 = 6$	F
7	$7^2 : 49 = 7$	F
8	$8^2 : 64 = 8$	F

(b) $\therefore \exists x \in D \ni x^2 = x$ is a **false statement**.

1.6 Example

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol $\forall$ and $\exists$.

1. $\forall x \in R, x^2 \neq -1$

   (a) For all real numbers, its square cannot be -1.

2. $\exists x \in Z \ni x^2 = x$

   (a) The square of some integers is equal to itself.

1.7 Example

Rewrite each of the following statements formally. Use quantifiers and variables.

1. Every real number is positive, negative, or zero.

   (a) $\forall x \in \mathbb{R} \ni x > 0 \cup x < 0 \cup x = 0$

2. Some real numbers are rational.

   (a) $\exists x \in \mathbb{R} \ni x = \frac{a}{b}, (a, b \in \mathbb{Z}, b \neq 0)$

1.8 Example

$\forall x \in \mathbb{R}$,if $x > 4$, then $x^2 > 16$, Informal:

1. For any real number greater than 4, its square must also be greater than 16.

1.9 Example

The square of any even integers is even. Formal way,

1. $\forall x \in \mathbb{Z}$,if $x = 2m, m \in \mathbb{Z}$, then $x^2 = 2n, n \in \mathbb{Z}$

1.10 Example

" $\forall$ polygons p, if p is a square, then p is a rectangle." is equivalent to

1. $\forall$ square polygons p, p is a rectangle,

1.11 Example

"$\exists$ a number n such that n is prime and n is even." is equivalent to

1. $\exists$ a prime number such that n is even.
2. $\exists$ an even number such that n is prime.

1.12 Example

Write negations for the following statements.

1. $\forall$ irrational numbers x, x is not an integer.
   (a) $\exists$ irrational numbers x, x is an integer.
2. $\exists\, x \in R$ such that x is rational.
   (a) $\forall x \in R$ such that x is irrational.
3. No politicians are honest.
   (a) Some politicians are not honest.
4. All dinosaurs are extinct
   (a) Some dinosaurs are not extinct.
5. Some exercises have answers.
   (a) All exercises don't have answers.

1.13 Example

Write the negations for the following statements.

1. $\forall$ animals x, if x is a cat then x has whiskers and x has claws.
   (a) $\forall x \in D, [p\,(x) \to (Q\,(x) \wedge R\,(x))]$
   (b) Negation: $\exists x \in D, p\,(x) \wedge (\sim Q\,(x) \vee \sim R\,(x))$
   (c) $\exists$ animals x, where x is a cat and x don't have whiskers or x don't have claws.
2. $\forall x \in R$, if $x > 3$ then $x^2 > 9$
   (a) $\exists x \in R$,where $x > 3$ and $x^2 \leq 9$.

1.14 Example

Negate the following statements and determine their truth values.

1. $r(x) : 2x + 1 = 5$
   $s(x) : x^2 = 9$
   $\exists\, x \ni [r\,(x) \implies s\,(x)]$

   (a) $\forall x \ni [r\,(x) \wedge \sim s\,(x)]$

   (b) Determine truth values

      i. When $x = 3$

      $$r\,(3) : 2\,(3) + 1 = 7\ (F)$$
      $$s\,(3) : 3^2 : 9 = 9\ (T)$$

      ii. $\forall x \ni [F \wedge \sim T] \equiv \forall x \ni c$

      iii. Therefore,

         A. the negation of $\exists\, x \ni [r\,(x) \implies s\,(x)]$ is a false statement.

2. $p(x) : x$ is odd.
   $q(x) : x^2–1$ is even.
   $\forall x \in Z$, if x is odd, then $\left(x^2 - 1\right)$ is even.

   (a) Determine negation

      i. $\exists x \in \mathbb{Z}$, x is odd, and $\left(x^2 - 1\right)$ is not even.

   (b) Determine final results

      i. Lets check the universal statement

         A. If x is an odd integer, then when divided by 2, it must leave a 0.5 behind. So,

         $$\frac{x}{2} = b + 0.5, \text{where b is an integer}$$
         $$x = 2b + 1$$

         B. Okay, so lets think about the second part, if $\left(x^2 - 1\right)$ is even, then it should be divisible without leaving a 0.5 behind.

         $$\begin{aligned} \frac{\left(x^2 - 1\right)}{2} &= \frac{x^2}{2} - \frac{1}{2} \\ &= x \cdot \frac{x}{2} - \frac{1}{2} \\ &= x \cdot (b + 0.5) - \frac{1}{2}, \text{where b is an integer} \\ &= bx + \frac{1}{2}x - \frac{1}{2} \end{aligned}$$

C. Lets substitute in the x we derived from earlier in part A. into part B. Remember that although x can be any number, it is the same number for both part A and part B.

$$\begin{aligned}\frac{\left(x^2-1\right)}{2} &= bx+\frac{1}{2}x-\frac{1}{2}\\ &= b\left(2b+1\right)+\frac{1}{2}\left(2b+1\right)-\frac{1}{2}\\ &= 2b^2+b+b+\frac{1}{2}-\frac{1}{2}\\ &= 2b^2+b+b\end{aligned}$$

D. Now remember that from part B, b is an integer, so if we take

$$\frac{\left(x^2-1\right)}{2} = 2\left(integer\right)^2+integer+integer$$

E. The result does not have any fraction, and is an integer.

F. Since the result is an integer, with no fraction $\left(x^2-1\right)$, must be an even number, since only even numbers can be divided by 2 without leaving fractions behind. Hence, $\forall x \in Z$, if x is odd, then $\left(x^2-1\right)$ is even, is a true satement.

ii. The negation statement is false.

iii. The universal statement is true.

1.15 Example

Rewrite each of the following without using variables or the symbols ∀ or ∃.

1. ∀ colours C, ∃ an animal A such that A is coloured C.
   (a) For all colours C, there exists an animal A such that A is coloured C.
   (b) For all colour, there is an animal with the same colour.
   (c) **There is an animal that is coloured by all the colour.**
2. ∃ a book b such that ∀ people p, p has read b.
   (a) There exists a book b such that every people p, p has read b.
      i. **There is a book where everyone has read the book**

1.16 Example

Rewrite the following formally using quantifiers and variables.

1. Everybody trusts somebody.

(a) $\forall$ peoples $\in x$, $\exists$ people y, x trusts y.

2. Somebody trusts everybody.

   (a) $\exists$ people x, $\forall$ peoples y such that x trusts y.

1.17 Example (Check answer)

Negate the statements below.

1. $\exists$ a book b such that $\forall$ people p, p has read b.

   (a) $\forall$books b, there $\exists$ people p, such that p has not read b.

2. $\forall$ even integers n, $\exists$ an integer k such that $n = 2k$.

   (a) $\exists$ even integers n, $\forall$ integer k, $n \neq 2k$.

3. $\exists$ a person x such that $\forall$ people y, x loves y

   (a) $\forall$ people x, $\exists$ a people y, such that x do not love y.

4. $\forall x$, $\exists y \left[\left(P\left(x,y\right) \wedge Q\left(x,y\right)\right) \rightarrow R\left(x,y\right)\right]$

   (a)

$$\begin{aligned}\exists x, \forall y \sim \left[\sim \left(P\left(x,y\right) \wedge Q\left(x,y\right)\right) \vee R\left(x,y\right)\right] &\equiv \exists x, \forall y \sim \left[\sim \left(P\left(x,y\right) \wedge Q\left(x,y\right)\right) \vee R\left(x,y\right)\right] \\ &\equiv \exists x, \forall y \sim \left[\sim P\left(x,y\right) \vee \sim Q\left(x,y\right) \vee R\left(x,y\right)\right] \\ &\equiv \exists x, \forall y \left[P\left(x,y\right) \wedge Q\left(x,y\right) \wedge \sim R\left(x,y\right)\right]\end{aligned}$$

1.18 Example 18 - The Relation Among $\forall$, $\exists$, $\wedge$, and $\vee$

For the universe of natural numbers N, the assertion $\forall x \left[P\left(x\right) \vee Q\right]$ is equivalent to the infinite conjunction:

$$\left[P\left(1\right) \vee Q\right] \wedge \left[P\left(2\right) \vee Q\right] \wedge \left[P\left(3\right) \vee Q\right] \wedge ... \wedge \left[P\left(N\right) \vee Q\right]$$

which can be rearranged using the distributive laws to form:

$$\left[P\left(1\right) \wedge P\left(2\right) \wedge ... \wedge P\left(N\right)\right] \vee Q$$

which is equivalent to

$$\forall x P\left(x\right) \vee Q$$

1. Note:

   (a) The variable x in $P\left(x\right)$ for example 18 is bound by quantifiers, which Q is free (constant in a sense).

1.19 Example 19

For the universe of natural numbers N, the proposition $\forall x[P(x) \wedge Q(x)]$ can be expanded into an infinite conjunction:

$$[P(1) \wedge Q(1)] \wedge [P(2) \wedge Q(2)] \wedge ... \wedge [P(N) \wedge Q(N)]$$

which can be rearranged using associative and commutative laws to obtain:

$$P(1) \wedge P(2) \wedge ... \wedge P(N) \wedge Q(1) \wedge Q(2) \wedge ... \wedge Q(N)$$

which is equivalent to

$$\forall x P(x) \wedge \forall x Q(x)$$

1.20 Example

Let the universe be the integers,

1. $P(x)$: x is an even integer.
2. $Q(x)$: x is an odd integer.
3. Then $\exists x\, P(x) \wedge \exists x\, Q(x)$ is true but $\exists x\ [P(x) \wedge Q(x)]$ is false.
   (a) TRUE: There exists x such that x is an even integer, and there exists x such that x is an odd integer.
   (b) FALSE: There exists x such that x is an even integer AND an odd integer.
   (c) The point here is that $\exists$ is not distributable over $\wedge$.
4. Therefore $\exists x\, P(x) \wedge \exists x\, Q(x)$ and $\exists x\ [P(x) \wedge Q(x)]$ are not equivalent.
5. However, $\exists x\, P(x) \wedge \exists x\, Q(x)$ implies $\exists x\ [P(x) \wedge Q(x)]$ is valid.

1.21 Example

Write the contrapositive, converse and inverse for the following statements.

1. $\forall x \in R$, if $x > 3$, then $x^2 > 9$
   (a) Contrapositive: $\forall x \in R$, if not $x^2 > 9$, then not $x > 3$
   (b) Converse: $\forall x \in R$, if $x^2 > 9$, then $x > 3$
   (c) Inverse: $\forall x \in R$, if not $x > 3$, then not $x^2 > 9$
2. $\forall$ animals A, if A is a cat then A has whiskers and A has claws.
   $\forall x \in A$,if $P(x)$, then $Q(x)$
   (a) Contrapositive:
      i. $\forall x \in A$, if $\sim Q(x)$, then $\sim P(x)$

ii. $\forall$ animals A, if A do not has whiskers or A do not has claws, then A is not a cat.

(b) Converse:

i. $\forall x \in A$,if $Q(x)$, then $P(x)$.

ii. $\forall$ animals A, if A has whiskers and A has claws then A is a cat.

(c) Inverse:

i. $\forall x \in A$,if A is NOT a cat, then A DO NOT has whiskers OR A DO NOT has claws.

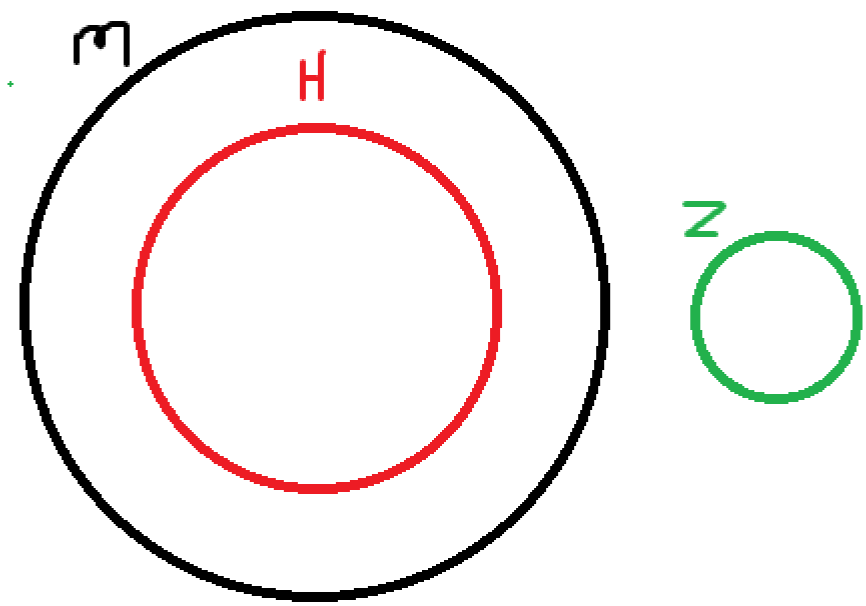
1.22 Example

Determine the validity of the following argument using diagrams.

1. All human beings are mortal.
2. Zeus is not mortal.
3. $\therefore$ Zeus is not a human being.

1.22.1 Solution

1. Let H: set of human beings
2. M: set of those who are mortal
3. Z: Zeus



1.23 Example

Determine the validity of the following argument using diagrams.

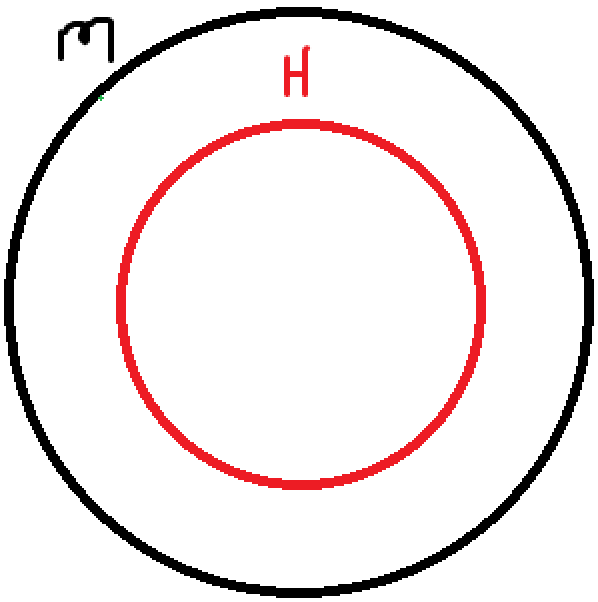
1. All human beings are mortal.
2. Felix is mortal.
3. Felix is a human being.

1.23.1 Solution

1. H: set of human beings
2. M: set of those who are mortal
3. F: Felix

Major Premise (the predicate of the conclusion, kinda major because they are what we want)

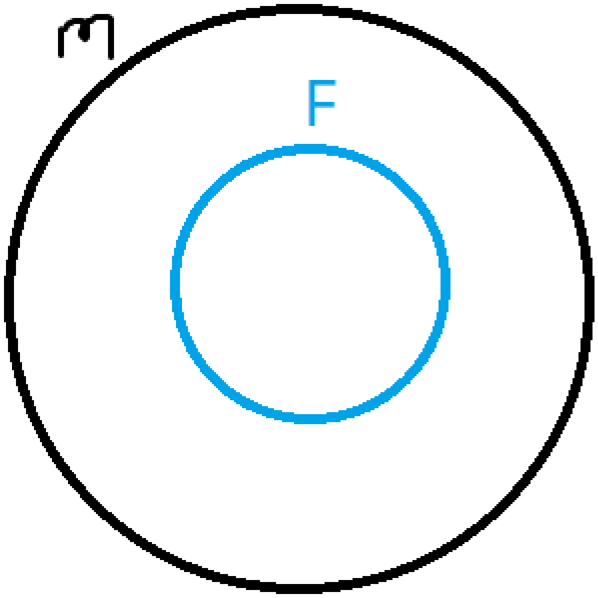
1. The disc of H falls entirely inside the disc of M.



2.

Minor Premise (the subject of the conclusion, kinda minor because if they lead to something)

1. F falls inside the disc of M.

2. 

Possible conclusions

1. Therefore, Felix is not a human being as F falls outside of the disc of H.

2. Therefore, Felix is a human being as F falls inside the disc of H.

Conclusion There is a contradiction between the conclusions, hence the argument is **invalid**.

1.24 Example

Determine the validity of the following argument using diagrams.

1. No polynomial functions have horizontal asymptote.

2. This function has a horizontal asymptote.

3. This function is not a polynomial.

1.24.1 Assign symbols

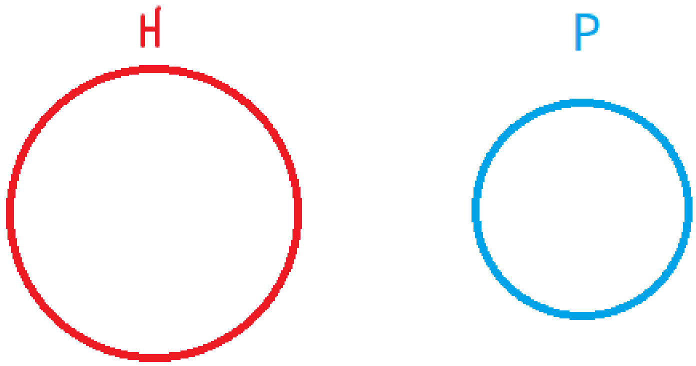
Let:

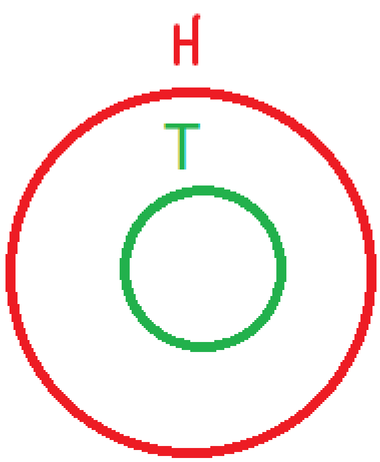
1. P : Set of polynomial functions

2. H : Set of functions with horizontal asymptotes

3. T : This function. (Conclusion)

1.24.2 Check the diagrams

1. Major premise - No polynomial functions have horizontal asymptote
   (a) The P and H are separated
      i. 

2. Minor premise - This function has a horizontal asymptote
   (a) The T is inside H
      i. 

3. Conclusion - This function is not a polynomial
   (a) The only possible conclusion is disc T falls outside of disc P. This function is not a polynomial.
   (b) Therefore, the cargument is valid.

1.25 Example

Determine the validity of the following argument using diagrams.

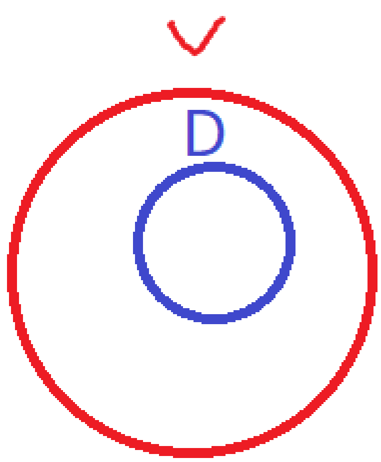
1. All **discrete mathematics** students can tell a valid argument from an invalid one.
2. All **thoughtful** people can tell a valid argument from an invalid one.
3. $\therefore$ All **discrete mathematics** students are **thoughtful**.

1.25.1 Make the terms and signs

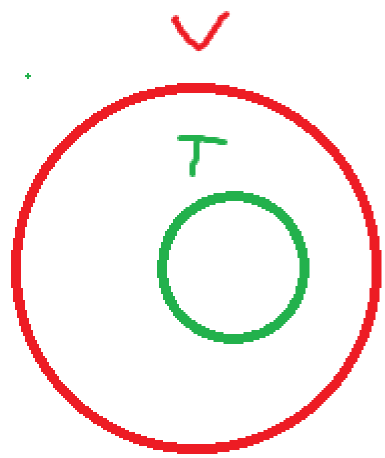
Let

1. D : Discrete mathematics students
2. T : Thoughtful people
3. V : Can tell a valid argument from invalid one.

1.25.2 Determine minor and major premises

1. Minor premise - All discrete mathematics students can tell a valid argument from an invalid one.
   (a) Disc D is inside disc V.
      i. 

2. Major premise - All **thoughtful** people can tell a valid argument from an invalid one.
   (a) Disc T inside disc V



i.

1.25.3 Conclusion - All discrete mathematics students are thoughtful.

1. Possible conclusions (ASK lecturer do we need the second sentence)

   (a) Disc D falls inside disc T. All discrete mathematics students are thoughtful

   (b) Disc D falls inside disc V but NOT disc T. All discrete mathematics are NOT thoughtful.

   (c) Disc D intersects with disc T. Some discrete mathematics student are thoughtful.

   (d) Disc Tfalls inside disc D. All thoughtful people are discrete maths students.

2. Since the possible conclusions contradict each other, the argument is **invalid**.