Calc 2: Tutorial 5

November 18, 2019

1. Solve the following differential equations: (CHECK WITH LECTURER)

   (a) $x^2 \frac{dy}{dx} + xy = x + 1$

      i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} x^2 \frac{dy}{dx} + xy &= x + 1 \\ \frac{1}{x^2}\left[x^2 \frac{dy}{dx} + xy\right] &= \frac{1}{x^2}\left[x + 1\right] \\ \boldsymbol{\frac{dy}{dx} + \frac{y}{x}} &\boldsymbol{= \frac{1}{x} + \frac{1}{x^2}, p\left(x\right) = \frac{1}{x}} \end{aligned}$$

      ii. Find the integrating factor $\mu\left(x\right) = e^{\int p(x)dx}$.

$$\begin{aligned} \mu\left(x\right) &= e^{\int \frac{1}{x}dx} \\ &= e^{\ln x} \\ \boldsymbol{\mu\left(x\right)} &\boldsymbol{= x} \end{aligned}$$

      iii. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned} x\left[\frac{dy}{dx} + \frac{y}{x}\right] &= x\left[\frac{1}{x} + \frac{1}{x^2}\right] \\ x\frac{dy}{dx} + y &= 1 + \frac{1}{x} \\ \frac{d}{dx}\left[yx\right] &= 1 + \frac{1}{x} \\ \int \frac{d}{dx}\left[yx\right]dx &= \int 1 + \frac{1}{x}dx \\ yx &= x + \ln x + c \\ y &= 1 + \frac{\ln x + c}{x} \\ &= 1 + \frac{\ln x}{x} + \frac{c}{x} \end{aligned}$$

(b) $\frac{dy}{dx} + y\cot x = \csc x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

A. Already in correct standard form, $p(x) = \cot x$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int \cot x\, dx} \\ &= e^{\ln|\sin x|} \\ &= \sin x\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}\sin x\left[\frac{dy}{dx} + y\cot x\right] &= \sin x\,[\csc x] \\ \sin x\frac{dy}{dx} + y\sin x\cdot\frac{\cos x}{\sin x} &= \sin x\left[\frac{1}{\sin x}\right] \\ \sin x\frac{dy}{dx} + y\cos x &= 1 \\ \sin x\frac{dy}{dx} + y\cos x &= 1 \\ \frac{d}{dx}[y\sin x] &= 1 \\ \int\frac{d}{dx}[y\sin x]\,dx &= \int 1dx \\ y\sin x + c &= x + c \\ y &= \frac{x+c}{\sin x}\end{aligned}$$

(c) $x\frac{dy}{dx} = y - x^2e^{-x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} - x^2e^{-x} \\ \frac{dy}{dx} - \frac{y}{x} &= -x^2e^{-x}\end{aligned}$$

A. From the above, $p(x) = -\frac{1}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int -\frac{1}{x}dx} \\
&= e^{-\int \frac{1}{x}dx} \\
&= e^{-\ln x} \\
\mu(x) &= e^{\ln x^{-1}} \\
&= x^{-1} \\
&= \frac{1}{x}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
\frac{1}{x}\left(\frac{dy}{dx} - \frac{y}{x}\right) &= \frac{1}{x}\left(-x^2e^{-x}\right) \\
-\left(-x^{-2}y + x^{-1}\frac{dy}{dx}\right) &= -xe^{-x} \\
-\frac{dy}{dx}\left(x^{-1}y\right) &= -xe^{-x} \\
-\int \frac{dy}{dx}\left(x^{-1}y\right)dx &= \int -xe^{-x}dx \\
-x^{-2}y &= -\int xe^{-x}dx
\end{aligned}$$

iv. Before we can proceed, again, integrate by parts, if you don't remember, GG.com...jk,

$$\int u\,dv = uv - \int vdu$$

A. Let $u = x$, $dv = e^{-x}$

$$\begin{aligned}
\frac{du}{dx} &= 1 \\
du &= dx
\end{aligned}$$

$$\begin{aligned}
\int dv &= \int e^{-x} \\
v &= -e^{-x}
\end{aligned}$$

B. Lets find the answer

$$\int u\,dv = -x \cdot e^{-x} - \int e^{-x}dx$$

C. Again, we need to deal with integration, now use u substitution
Let $u = -x$, $du = -dx$

$$\begin{aligned} -\int e^{-x} dx &= -\int e^{u} du \\ &= -e^{u} \\ -\int e^{-x} dx &= -e^{-x} + c \end{aligned}$$

v. Substitute back in

$$\begin{aligned} -x^{-1} y &= \int x e^{-x} dx \\ &= -e^{-x} + c \\ y &= -e^{-x} \cdot -x - cx \\ &= x e^{-x} - cx \\ y &= x e^{-x} + cx \text{ Note: c is constant, sign doesn't matter for now} \end{aligned}$$

(d) $x \frac{dy}{dx} + 2y = \frac{\sin x}{x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x) dx}$.

$$\begin{aligned} \mu(x) &= e^{\int \frac{2}{x} dx} \\ &= e^{2 \ln x} \\ \mu(x) &= x^2 \end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x) y]$ and write it as such.

$$\begin{aligned} x^2 \left[\frac{dy}{dx} + \frac{2y}{x}\right] &= x^2 \left[\frac{\sin x}{x^2}\right] \\ x^2 \frac{dy}{dx} + 2xy &= \sin x \\ \frac{d}{dx}\left[x^2 y\right] &= \sin x \\ \int \frac{d}{dx}\left[x^2 y\right] dx &= \int \sin x \, dx + c \\ x^2 y &= -\cos x + c \\ y &= \frac{c - \cos x}{x^2} \end{aligned}$$

(e) $x\frac{dy}{dx} + 3y = 4x + 3$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{1}{x}\left[x\frac{dy}{dx} + 3y\right] = \frac{1}{x}\left[4x + 3\right]$$
$$\frac{dy}{dx} + \frac{3y}{x} = 4 + \frac{3}{x}$$

A. From above $p(x) = \frac{3}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int \frac{3}{x}dx} \\ &= e^{3\int \frac{1}{x}dx} \\ &= e^{3\ln x} \\ &= e^{\ln x^3} \\ \mu(x) &= x^3\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu(x)\, y\right]$ and write it as such.

$$\begin{aligned}x^3\left[\frac{dy}{dx} + \frac{3y}{x}\right] &= x^3\left[4 + \frac{3}{x}\right] \\ x^3\frac{dy}{dx} + 3x^2y &= 4x^3 + 3x^2 \\ \frac{d}{dx}\left[x^3y\right] &= 4x^3 + 3x^2 \\ \int \frac{d}{dx}\left[x^3y\right]dx &= \int \left(4x^3 + 3x^2\right)dx \\ x^3y &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\ x^3y &= x^4 + x^3 + c \\ y &= \frac{x^4}{x^3} + \frac{x^3}{x^3} + \frac{c}{x^3} \\ y &= x + 1 + \frac{c}{x^3}\end{aligned}$$

(f) $\frac{dy}{dx} = 2y + e^{3x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} = 2y + e^{3x}$$
$$\frac{dy}{dx} - 2y = e^{3x}, p(x) = -2$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int -2dx}\\ &= e^{\int -2dx}\\ &= e^{-\int 2dx}\\ \mu(x) &= e^{-2x}\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$\begin{aligned}e^{-2x}\left[\frac{dy}{dx}-2y\right] &= e^{-2x}\left[e^{3x}\right]\\ e^{-2x}\frac{dy}{dx}-2e^{-2x}y &= e^{x}\\ \frac{dy}{dx}\left[e^{-2x}y\right] &= e^{x}\\ \int\frac{dy}{dx}\left[e^{-2x}y\right]dx &= \int e^{x}dx\\ e^{-2x}y &= e^{x}+c\\ y &= \frac{e^{x}+c}{e^{-2x}}\\ &= e^{3x}+ce^{-(-2x)}\\ y &= e^{3x}+ce^{2x}\end{aligned}$$

(g) $x\frac{dy}{dx} = 2y + x^3\ln x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2y}{x}+x^2\ln x\\ \frac{dy}{dx}-\frac{2y}{x} &= x^2\ln x\\ \frac{dy}{dx}-\frac{2y}{x} &= x^2\ln x, p(x) = -\frac{2}{x}\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int p(x)dx}\\ \mu(x) &= e^{\int -2x^{-1}dx}\\ &= e^{-2\int \frac{1}{x}dx}\\ &= e^{\ln x^{-2}dx}\\ &= x^{-2}\\ \mu(x) &= \frac{1}{x^2}\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
\frac{1}{x^2}\left[\frac{dy}{dx}-\frac{2y}{x}\right] &= \frac{1}{x^2}\left[x^2\ln x\right] \\
x^{-2}\frac{dy}{dx}-2x^{-3}y &= \ln x \\
\frac{dy}{dx}\left[x^{-2}y\right] &= \ln x \\
\int\frac{dy}{dx}\left[x^{-2}y\right]dx &= \int\ln x\,dx
\end{aligned}$$

A. Time to use Calc 1 skills, Integration by parts

$$\int u\,dv = uv-\int vdu$$

Let $u=\ln x$ and $v'=1$
$\frac{du}{dx}=\frac{1}{x}, v=x$

B. Finally we get

$$\begin{aligned}
\int\ln x\,dx &= x\ln x-\int x\frac{1}{x}dx \\
&= x\ln x-\int 1dx \\
&= x\ln x-x+c
\end{aligned}$$

C. Continue plugging in

$$\begin{aligned}
\int\frac{dy}{dx}\left[x^{-2}y\right]dx &= x\ln x-x+c \\
x^{-2}y &= x\ln x-x+c \\
y &= x^3\ln x-x^3+cx^2
\end{aligned}$$

(h) $x\frac{dy}{dx}-3y=x^4$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
x\frac{dy}{dx}-3y &= x^4 \\
\frac{dy}{dx}-\frac{3y}{x} &= x^3
\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int -\frac{3}{x}dx} \\
&= e^{-3\int \frac{1}{x}dx} \\
&= e^{-3\ln x} \\
&= e^{\ln x^{-3}} \\
\mu(x) &= x^{-3}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\, y]$ and write it as such.

$$\begin{aligned}
x^{-3}\left[\frac{dy}{dx} - \frac{3y}{x}\right] &= x^{-3}\left[x^3\right] \\
x^{-3}\frac{dy}{dx} - 3x^{-4}y &= 1 \\
\frac{d}{dx}\left[x^{-3}y\right] &= 1 \\
\int \frac{d}{dx}\left[x^{-3}y\right] dx &= \int 1dx \\
x^{-3}y &= x + c \\
y &= \frac{x}{x^{-3}} + \frac{c}{x^{-3}} \\
y &= x^4 + cx^3
\end{aligned}$$

2. Solve the following I.V.P.:

   (a) $\frac{dy}{dx} = 9.8 - 0.196y, y(0) = 48$

      i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + 0.196y = 9.8$$

      ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int 0.196dx} \\
&= e^{0.196\int 1dx} \\
&= e^{0.196x}
\end{aligned}$$

      iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned}
e^{0.196x}\left[\frac{dy}{dx}+0.196y\right] &= e^{0.196x}\left[9.8\right]\\
e^{0.196x}\frac{dy}{dx}+e^{0.196x}0.196y &= 9.8e^{0.196x}\\
\frac{d}{dx}\left[ye^{0.196x}\right] &= 9.8e^{0.196x}\\
\int\frac{d}{dx}\left[ye^{0.196x}\right]dx &= \int\left(9.8e^{0.196x}\right)dx\\
ye^{0.196x} &= 9.8\cdot\left(\frac{1}{0.196}e^{0.196x}+c\right)\\
ye^{0.196x} &= \frac{9.8}{0.196}e^{0.196x}+9.8c\\
y &= \frac{9.8}{0.196\cdot e^{0.196x}}e^{0.196x}+\frac{9.8c}{e^{0.196x}}\\
&= \frac{9.8}{0.196}+\frac{9.8c}{e^{0.196x}}\\
y &= 50+\frac{9.8c}{e^{0.196x}}
\end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned}
y\left(0\right) &= 50+\frac{9.8c}{e^{0.196(0)}}\\
48 &= 50+\frac{9.8c}{1}\\
c &= \frac{48-50}{9.8}\\
&= -\frac{10}{49}
\end{aligned}$$

v. Substitute c back into the original equation

$$\begin{aligned}
y &= 50+\frac{9.8\left(-\frac{10}{49}\right)}{e^{0.196x}}\\
&= 50+\frac{9.8\left(-\frac{10}{49}\right)}{e^{0.196x}}\\
&= 50-\frac{2}{e^{0.196x}}\\
y &= 50-2e^{-0.196x}
\end{aligned}$$

(b) $x\frac{dy}{dx}+y=\frac{x}{x+1}, y\left(1\right)=1$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x+1}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\mu(x) = e^{\int \frac{1}{x}dx}$$
$$= e^{\ln x}$$
$$\mu(x) = x$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$x\left[\frac{dy}{dx} + \frac{y}{x}\right] = x\left[\frac{1}{x+1}\right]$$
$$x\frac{dy}{dx} + y = \frac{x}{x+1}$$
$$\frac{d}{dx}[xy] = \frac{x}{x+1}$$
$$\int \frac{d}{dx}[xy]\,dx = \int \frac{x}{x+1}dx$$
$$xy = \int x(x+1)^{-1}\,dx$$

A. On first sight, integration by parts seems to be pretty good, but if you try it out you're in for a world of pain. A combination of long division and u substitution works best in this case. Because:
1. Numerator and denominator same power

B. Do long division

$$\int \frac{x}{(x+1)}dx = \int 1 - \frac{1}{x+1}dx$$
$$= \int 1dx - \int \frac{1}{x+1}dx$$
$$\int \frac{x}{(x+1)}dx = x - \ln(x+1) + c$$

C. Move everything to the right place

$$xy = x - \ln(x+1) + c$$
$$y = \frac{x - \ln(x+1) + c}{x}$$
$$= 1 - \frac{1}{x}\ln(x+1) + \frac{c}{x}$$

iv. Plug in the values given, and solve for c

$$y(1) = 1 - \frac{1}{1}\ln(1+1) + \frac{c}{1}$$
$$1 = 1 - \ln(2) + c$$
$$c = \ln(2)$$

v. Substitute c back into the original equation

$$\begin{aligned}
y &= 1 - \frac{1}{x}\ln(x+1) + \frac{1}{x}\ln(2) \\
&= 1 - \frac{1}{x}\left(\ln(x+1) - \ln(2)\right) \\
&= 1 - \frac{1}{x}\ln\frac{x+1}{2} \\
&= 1 + \frac{1}{x}\ln\left(\frac{x+1}{2}\right)^{-1} \\
y &= 1 + \frac{1}{x}\ln\frac{2}{x+1}
\end{aligned}$$

(c) $x\frac{dy}{dx} + 3y = x^2 - 4x + 3, y(1) = 0$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1

$$\frac{dy}{dx} + \frac{3}{x}y = x - 4 + \frac{3}{x}$$

.

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int p(x)dx} \\
\mu(x) &= e^{\int \frac{3}{x}dx} \\
&= e^{3\ln x} \\
&= e^{\ln x^3} \\
\mu(x) &= x^3
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
\frac{dy}{dx} + \frac{3}{x}y &= x - 4 + \frac{3}{x} \\
x^3\left[\frac{dy}{dx} + \frac{3}{x}y\right] &= x^3\left[x - 4 + \frac{3}{x}\right] \\
x^3\frac{dy}{dx} + 3x^2y &= x^4 - 4x^3 + 3x^2 \\
\frac{d}{dx}\left[x^3y\right] &= x^4 - 4x^3 + 3x^2 \\
\int \frac{d}{dx}\left[x^3y\right]dx &= \int x^4 - 4x^3 + 3x^2dx \\
x^3y &= \frac{x^5}{5} - x^4 + x^3 + c \\
y &= \frac{x^2}{5} - x + 1 + \frac{c}{x^3}
\end{aligned}$$

iv. Plug in the values given, and solve for c

$$\begin{aligned} y(1) &= \frac{1^2}{5} - 1 + 1 + \frac{c}{1^3} \\ 0 &= \frac{1}{5} + c \\ c &= -\frac{1}{5} \end{aligned}$$

v. Substitute c back into the original equation

$$y = \frac{x^2}{5} - x + 1 - \frac{1}{5x^3}$$

3. **Separable Equations:**
   Solve the following differential equations:

   (a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

   i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{2x}}{4y^3} \\ 4y^3 \frac{dy}{dx} &= e^{2x} \end{aligned}$$

   ii. Integrate both sides

$$\begin{aligned} \int 4y^3 \frac{dy}{\cancel{dx}} \cancel{dx} &= \int e^{2x} dx \\ 4 \int y^3 dy &= \int e^{2x} dx \\ 4\left(\frac{y^4}{4} + c\right) &= \frac{1}{2} e^{2x} + c \\ y^4 + c &= \frac{1}{2} e^{2x} + c \end{aligned}$$

   iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

   A. DO NOT forget to include C, constant of integration

$$\begin{aligned} y^4 &= \frac{1}{2} e^{2x} + c \\ y &= \pm \sqrt[4]{\frac{1}{2} e^{2x} + c} \end{aligned}$$

   (b) $\frac{dy}{dx} = \frac{xy}{2 \ln y}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy}{2\ln y}\\ 2\ln y\frac{dy}{dx} &= xy\\ \frac{2}{y}\frac{dy}{dx}\ln y &= x\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int \frac{2}{y}\frac{dy}{dx}\ln y\,dx &= \int x\,dx\\ 2\int \frac{1}{y}\ln y\,dy &= \frac{x^2}{2}+c\\ 2\left(\frac{\ln^2 y}{2}+c\right) &= \frac{x^2}{2}+c \text{check below for elaboration}\\ \ln^2 y + c &= \frac{x^2}{2}+c\end{aligned}$$

A. Use $u - substitution$, let $u = \ln y$

$$\begin{aligned}\frac{du}{dy} &= \frac{1}{y}\\ du &= \frac{1}{y}dy\end{aligned}$$

$$\begin{aligned}\int \frac{1}{y}\ln y\,dy &= \int u\,du\\ &= \frac{u^2}{2}+c\\ &= \frac{(\ln y)^2}{2}+c\\ \int \frac{1}{y}\ln y\,dy &= \frac{\ln^2 y}{2}+c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

B.

$$\begin{aligned}\ln^2 y + c &= \frac{x^2}{2}+c\\ \ln y &= \pm\sqrt{\frac{x^2}{2}+c}\\ y &= e^{\pm\sqrt{\frac{x^2}{2}+c}}\end{aligned}$$

(c) $\frac{dx}{dt} + e^{t+x} = 0$

Before-we-start: This question is a lil' tricky. Usually, we have x as our parameter, but this has t as parameter. Therefore, we need to make it $x(t) = ...$ instead.

i. Write D.E. as $g(y)\, dy = f(x)\, dx$

$$\begin{aligned}\frac{dx}{dt} + e^{t+x} &= 0\\ \frac{dx}{dt} &= -e^{t+x}\\ &= -e^{t} \cdot e^{x}\\ \frac{1}{e^{x}} dx &= \frac{1}{e^{t}} dt\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\int e^{-x} dx &= \int e^{-t} dt\\ -e^{-x} + c &= -e^{-t} + c\\ e^{-x} &= e^{-t} + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}\ln e^{-x} &= \ln\left(e^{-t} + c\right)\\ -x &= \ln\left(e^{-t} + c\right)\\ x &= -\ln\left(e^{-t} + c\right)\end{aligned}$$

(d) $2\sqrt{xy}\frac{dy}{dx} = 1, x, y > 0$

i. Write D.E. as $g(y)\, dy = f(x)\, dx$

$$\begin{aligned}2\sqrt{xy}\frac{dy}{dx} &= 1\\ \sqrt{xy} dy &= \frac{1}{2} dx\\ \sqrt{x}\sqrt{y} dy &= \frac{1}{2} dx\\ y^{\frac{1}{2}} dy &= \frac{1}{2} x^{-\frac{1}{2}} dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}
\int y^{\frac{1}{2}}\, dy &= \int \frac{1}{2}x^{-\frac{1}{2}}\, dx \\
\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c &= \frac{1}{2}\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c\right) \\
\frac{2}{3}y^{\frac{3}{2}} + c &= \frac{1}{2}\left(2x^{\frac{1}{2}} + c\right) \\
\frac{2}{3}y^{\frac{3}{2}} + c &= x^{\frac{1}{2}} + c
\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y\left(x\right)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
\frac{2}{3}y^{\frac{3}{2}} &= x^{\frac{1}{2}} + c \\
y^{\frac{3}{2}} &= \frac{3}{2}\left(x^{\frac{1}{2}} + c\right) \\
y^{\frac{3}{2}} &= \frac{3}{2}\sqrt{x} + c
\end{aligned}$$

(e) $\frac{dy}{dx} = \frac{xy}{x+2}$

i. Write D.E. as $g\left(y\right) dy = f\left(x\right) dx$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xy}{x+2} \\
\frac{1}{xy}dy &= \frac{1}{x+2}dx \\
\frac{1}{x}\cdot\frac{1}{y}dy &= \frac{1}{x+2}dx \\
\frac{1}{y}dy &= \frac{x}{x+2}dx \\
y^{-1}dy &= \frac{x}{x+2}dx
\end{aligned}$$

ii. Integrate both sides

$$\int y^{-1}dy = \int \frac{x}{x+2}dx$$

A. For first part,

$$\int y^{-1}dy = \ln y + c$$

B. For second part, ideally you want to use long division + u-substitution. Just something learned from above.

$$\begin{aligned}\int \frac{x}{x+2}dx &= \int 1 - \frac{2}{x+2}dx \\ &= \int 1dx - 2\int \frac{1}{x+2}dx\end{aligned}$$

Let $u = x + 2$,

$$\begin{aligned}\frac{du}{dx} &= 1 \\ du &= dx\end{aligned}$$

$$\begin{aligned}\int 1dx - 2\int \frac{1}{x+2}dx &= x - 2\int \frac{1}{u}du + c \\ &= x - 2\ln u + c \\ &= x - 2\ln(x+2) + c\end{aligned}$$

C. Find the final integration

$$\int y^{-1}dy = \int \frac{x}{x+2}dx$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}\ln y + c &= x - 2\ln(x+2) + c \\ \ln y &= x - \ln(x+2)^2 + c \\ y &= e^{x-\ln(x+2)^2+c} \\ &= \frac{e^{x+c}}{e^{\ln(x+2)^2}} \\ &= \frac{e^x \cdot e^c}{(x+2)^2} \\ y &= \frac{Ae^x}{(x+2)^2}, A = e^c\end{aligned}$$

(f) $y\left(x^2-1\right)\frac{dy}{dx} = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}y\left(x^2-1\right)\frac{dy}{dx} &= 1 \\ y\frac{dy}{dx} &= \frac{1}{(x^2-1)} \\ ydy &= \frac{1}{(x^2-1)}dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}
\int y dy &= \int \frac{1}{(x^2-1)} dx \\
\frac{y^2}{2} + c &= -\int \frac{1}{1-x^2} dx \\
\frac{y^2}{2} + c &= -\tanh^{-1}(x) + c \\
&= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c \\
\frac{y^2}{2} + c &= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c
\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
\frac{y^2}{2} + c &= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c \\
y^2 &= \ln(x-1) - \ln(x+1) + c \\
y^2 &= \ln\frac{x-1}{x+1} + c \\
y &= \sqrt{\ln\frac{x-1}{x+1} + c}
\end{aligned}$$

B. Let $A = e^c$

$$\begin{aligned}
y &= \sqrt{\ln\frac{x-1}{x+1} + \ln A} \\
&= \sqrt{\ln\frac{A(x-1)}{x+1}}
\end{aligned}$$

(g) $\frac{dx}{dt} = \frac{4\sin t + 6\cos 2t}{x}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{4\sin t + 6\cos 2t}{x} \\
xdx &= 4\sin t + 6\cos 2t\, dt
\end{aligned}$$

ii. Integrate both sides

$$\int xdx = \int 4\sin t + 6\cos 2t\,dt$$
$$\frac{x^2}{2} + c = 4\left(-\cos t + c\right) + 6\int \cos 2t\,dt$$
$$= 4\left(-\cos t + c\right) + 6\left(\frac{1}{2}\sin\left(2t\right) + c\right)$$
$$\frac{x^2}{2} + c = -4\cos t + 3\sin\left(2t\right) + c$$

A. To integrate $\cos 2t$, let $u = 2t$

$$\frac{du}{dt} = 2$$
$$du = 2dt$$

$$\int \cos 2tdt = \frac{1}{2}\int \cos u\,du$$
$$= \frac{1}{2}\left(\sin u + c\right)$$
$$\int \cos 2tdt = \frac{1}{2}\sin\left(2t\right) + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y\left(x\right)$)

A. DO NOT forget to include C, constant of integration

$$\frac{x^2}{2} + c = -4\cos t + 3\sin\left(2t\right) + c$$
$$\frac{x^2}{2} = -4\cos t + 3\sin 2t + c$$

(h) $\frac{dy}{dx} = e^{-y}\left(2x - 4\right)$

i. Write D.E. as $g\left(y\right)dy = f\left(x\right)dx$

$$\frac{1}{e^{-y}}dy = \left(2x - 4\right)dx$$
$$e^{y}dy = \left(2x - 4\right)dx$$

ii. Integrate both sides

$$\int e^{y}dy = \int \left(2x - 4\right)dx$$
$$e^{y} + c = \frac{2x^2}{2} - 4x + c$$
$$e^{y} + c = x^2 - 4x + c$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$
\begin{aligned}
e^y + c &= x^2 - 4x + c \\
e^y &= x^2 - 4x + c \\
\ln e^y &= \ln\left(x^2 - 4x + c\right) \\
y &= \ln\left(x^2 - 4x + c\right)
\end{aligned}
$$

(i) $\sec x \frac{dy}{dx} = e^{y+\sin x}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$
\begin{aligned}
\cos x \frac{dx}{dy} &= \frac{1}{e^{y+\sin x}} \\
\cos x dx &= e^{-y} \cdot e^{-\sin x} \\
\frac{\cos x}{e^{-\sin x}} dx &= e^{-y} dy
\end{aligned}
$$

ii. Integrate both sides

$$
\begin{aligned}
\int e^{-y} dy &= \int \cos x \cdot e^{\sin x} dx \\
-e^{-y} + c &= \int \cos x \cdot e^{\sin x} dx \\
&= \int \cos x \cdot e^{\sin x} dx \\
e^y &= -e^{\sin x} + c \text{ (check below)}
\end{aligned}
$$

A. Integrating $\int \cos x \cdot e^{\sin x} dx$ probably needs $u-$integration. Let $u = \sin x$

$$
\begin{aligned}
\frac{du}{dx} &= -\cos x \\
du &= -\cos x\, dx
\end{aligned}
$$

$$
\begin{aligned}
\int \cos x \cdot e^{\sin x} dx &= \int \cos x \cdot e^{\sin x} dx \\
&= -\int e^u du \\
&= -e^u + c \\
&= -e^{\sin x} + c \\
\int \cos x \cdot e^{-\sin x} dx &= c - e^{\sin x}
\end{aligned}
$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned} e^{-y} &= c - e^{\sin x} \\ \ln e^{-y} &= \ln\left(c - e^{\sin x}\right) \\ y &= -\ln\left(c - e^{\sin x}\right) \end{aligned}$$

(j) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$2y - 4\,dy = 3x^2 + 4x - 4\,dx$$

ii. Integrate both sides

$$\begin{aligned} \int 2y - 4\,dy &= \int 3x^2 + 4x - 4\,dx \\ y^2 - 4y + c &= x^3 + 2x^2 - 4x + c \end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned} y^2 - 4y &= x^3 + 2x^2 - 4x + c \\ y^2 - 4y - \left(x^3 + 2x^2 - 4x + c\right) &= 0 \end{aligned}$$

B. $a = 1, b = -4, c = -\left(x^3 + 2x^2 - 4x + c\right)$

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-(x^3 + 2x^2 - 4x + c))}}{2(1)} \\ &= \frac{4 \pm \sqrt{4^2 + 4(x^3 + 2x^2 - 4x + c)}}{2} \\ &= 2 \pm \frac{\sqrt{4(4 + x^3 + 2x^2 - 4x + c)}}{2} \\ &= 2 \pm \frac{2\sqrt{4 + x^3 + 2x^2 - 4x + c}}{2} \\ &= 2 \pm \sqrt{4 + x^3 + 2x^2 - 4x + c} \end{aligned}$$

4. Solve the following I.V.P.:

(a) $y' = \frac{y \cos x}{1+y^2}, y(0) = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}\frac{dy}{dx} &= \frac{y\cos x}{1+y^2}\\ \left(1+y^2\right)dy &= y\cos x\,dx\\ \frac{\left(1+y^2\right)}{y}dy &= \cos x\,dx\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}\frac{\left(1+y^2\right)}{y}dy &= \cos x\,dx\\ \int \frac{1}{y}+\frac{y^2}{y}dy &= \int \cos x\,dx\\ \int \frac{1}{y}dy+\int ydy &= \sin x + c\\ \ln y + \frac{y^2}{2} + c &= \sin x + c\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$). Note in some cases (like this, it is impossible at our current level)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}\ln y + \frac{y^2}{2} + c &= \sin x + c\\ \ln y + \frac{y^2}{2} &= \sin x + c\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}\ln 1 + \frac{1^2}{2} &= \sin(0) + c\\ \frac{1}{2} &= 0 + c\\ c &= \frac{1}{2}\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\ln y + \frac{y^2}{2} = \sin x + \frac{1}{2}$$

(b) $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0, y(0) = 1$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned} x + 2y\sqrt{x^2+1}\frac{dy}{dx} &= 0 \\ 2y\sqrt{x^2+1}\frac{dy}{dx} &= -x \\ 2y\frac{dy}{dx} &= -\frac{x}{\sqrt{x^2+1}} \\ 2ydy &= -\frac{x}{\sqrt{x^2+1}}dx \end{aligned}$$

ii. Integrate both sides

$$\begin{aligned} &\int 2ydy = \int -\frac{x}{\sqrt{x^2+1}}dx \\ &y^2 + c = -\int x\left(x^2+1\right)^{-\frac{1}{2}}\,dx \\ &y^2 + c = -\left(x^2+1\right)^{\frac{1}{2}} + c \text{ (Check below)} \end{aligned}$$

A. Integrate $-\int x\left(x^2+1\right)^{-\frac{1}{2}}\,dx$, I'm thinking about u-substitution. Let $u = x^2+1$

$$\begin{aligned} \frac{du}{dx} &= 2x \\ du &= 2x\,dx \end{aligned}$$

$$\begin{aligned} -\int x\left(x^2+1\right)^{-\frac{1}{2}}\,dx &= -\frac{1}{2}\int\left(x^2+1\right)^{-\frac{1}{2}}\cdot 2x\,dx \\ &= -\frac{1}{2}\int u^{-\frac{1}{2}}du \\ &= -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) \\ &= -u^{\frac{1}{2}} + c \\ -\int x\left(x^2+1\right)^{-\frac{1}{2}}\,dx &= -\left(x^2+1\right)^{\frac{1}{2}} + c \end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned} y^2 + c &= -\left(x^2+1\right)^{\frac{1}{2}} + c \\ y^2 &= -\left(x^2+1\right)^{\frac{1}{2}} + c \end{aligned}$$

iv. Substitute in the values given to find C, $y(0) = 1$

$$\begin{aligned}
1^2 &= -\left((0)^2 + 1\right)^{\frac{1}{2}} + c \\
1 &= -1 + c \\
c &= 2
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$y^2 = -\left(x^2 + 1\right)^{\frac{1}{2}} + 2$$

(c) $\frac{dy}{dt} = te^y, y(1) = 0$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\frac{1}{e^y} dy = t\,dt$$

ii. Integrate both sides

$$\begin{aligned}
\int e^{-y} dy &= \int t\,dt \\
-e^{-y} + c &= \frac{t^2}{2} + c
\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
-e^{-y} + c &= \frac{t^2}{2} + c \\
-e^{-y} &= \frac{t^2}{2} + c - c \\
e^{-y} &= -\frac{t^2}{2} + c \text{ (Const. - cons. = cons. (even 0))} \\
-y \ln e &= \ln\left(-\frac{t^2}{2} + c\right) \\
y &= -\ln\left(-\frac{t^2}{2} + c\right) \\
&= \ln \frac{1}{-\frac{t^2}{2} + c} \\
y &= \ln \frac{1}{-\frac{1}{2}t^2 + c}
\end{aligned}$$

iv. Substitute in the values given to find C

$$0 = \ln \frac{1}{-\frac{1}{2}1^2 + c}$$

$$\begin{aligned}
\ln \frac{1}{-\frac{1}{2} + c} &= 0 \\
\frac{1}{-\frac{1}{2} + c} &= e^0 \\
-\frac{1}{2} + c &= 1 \\
c &= 1 + \frac{1}{2} \\
c &= \frac{3}{2}
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
y &= \ln \frac{1}{-\frac{t^2}{2} + \frac{3}{2}} \\
&= \ln \frac{1}{\frac{-t^2+3}{2}} \\
y &= \ln \frac{2}{3 - t^2}
\end{aligned}$$

(d) $t\left(t-1\right)\frac{dx}{dt} = x\left(x+1\right), x\left(2\right) = 2$

i. Write D.E. as $g\left(y\right)dy = f\left(x\right)dx$

$$\begin{aligned}
t\left(t-1\right)\frac{dx}{dt} &= x\left(x+1\right) \\
\frac{dx}{x\left(x+1\right)} &= \frac{dt}{t\left(t-1\right)}
\end{aligned}$$

ii. Integrate both sides

$$\int \frac{1}{x\left(x+1\right)}dx = \int \frac{1}{t\left(t-1\right)}dt$$

$$\int \frac{1}{x\left(x+1\right)}dx = \int \frac{1}{t\left(t-1\right)}dt$$

A. For both parts, we need to use partial fractions, lets start with $\frac{1}{x(x+1)}$

$$\begin{aligned}
\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\
&= \frac{A(x+1)+Bx}{x(x+1)} \\
\frac{x(x+1)}{x(x+1)} &= A(x+1)+Bx \\
1 &= A(x+1)+Bx
\end{aligned}$$

When $x = 0$

$$\begin{aligned}
1 &= A(1)+0 \\
A &= 1
\end{aligned}$$

When $x = -1$

$$\begin{aligned}
1 &= A(-1+1)+B(-1) \\
1 &= -B \\
B &= -1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\
&= \frac{1}{x} - \frac{1}{x+1}
\end{aligned}$$

Start integrating:

$$\begin{aligned}
\int \frac{1}{x} - \frac{1}{x+1} dx &= \int \frac{1}{x} dx - \int \frac{1}{x+1} dx \\
&= \ln(x) - \ln(x+1) + c
\end{aligned}$$

B. Lets integrate the other one, $\int \frac{1}{t(t-1)} dt$
Again, partial fractions first.

$$\begin{aligned}
\frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1} \\
1 &= A(t-1)+Bt
\end{aligned}$$

When $t = 1$

$$B = 1$$

When $t = 2, B = 1$

$$\begin{aligned} 1 &= A(2-1) + 1(2) \\ 1 &= A + 2 \\ A &= -1 \end{aligned}$$

Substitute back in

$$\begin{aligned} \frac{1}{t(t-1)} &= \frac{A}{t} + \frac{B}{t-1} \\ &= -\frac{1}{t} + \frac{1}{t-1} \end{aligned}$$

Integrate:

$$\begin{aligned} \int -\frac{1}{t} + \frac{1}{t-1} dt &= \int -\frac{1}{t} dt + \int \frac{1}{t-1} dt \\ &= -\ln(t) + \ln(t-1) + c \end{aligned}$$

C. Combine the entire equation

$$\begin{aligned} \int \frac{1}{x(x+1)} dx &= \int \frac{1}{t(t-1)} dt \\ \ln(x) - \ln(x+1) + c &= -\ln(t) + \ln(t-1) + c \end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
\ln(x) - \ln(x+1) + c_1 &= -\ln(t) + \ln(t-1) + c_2 \\
\ln(x) - \ln(x+1) + c_1 - c_1 &= -\ln(t) + \ln(t-1) + c_2 - c_1 \\
\ln(x) - \ln(x+1) &= -\ln(t) + \ln(t-1) + c \\
\ln\frac{x}{x+1} &= \ln\frac{(t-1)}{t} + c \\
\ln\frac{x}{x+1} &= \ln\frac{(t-1)}{t} + c \\
\ln\frac{x}{x+1} &= \ln\frac{(t-1)}{t} + \ln e^c \\
\ln\frac{x}{x+1} &= \ln\frac{e^c(t-1)}{t} \\
\frac{x}{x+1} &= \frac{e^c(t-1)}{t} \\
1 - \frac{1}{x+1} &= \frac{e^c(t-1)}{t} \\
-\frac{1}{x+1} &= \frac{e^c(t-1)}{t} - 1 \\
\frac{1}{x+1} &= -\frac{e^c(t-1)}{t} + 1 \\
x+1 &= -\frac{t}{t+e^c(t-1)} \\
x &= -\frac{t}{t+e^c(t-1)} - 1 \\
&= -\frac{t-(t+e^c(t-1))}{t+e^c(t-1)} \\
&= -\frac{-e^c(t-1)}{t+e^c(t-1)} \\
&= \frac{e^c(t-1)}{t+e^c(t-1)} \\
&= \frac{c(t-1)}{t+c(t-1)}
\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
x &= \frac{c(t-1)}{t+c(t-1)} \\
2 &= \frac{c(2-1)}{2+c(2-1)} \\
2 &= \frac{c}{2+c} \\
4+2c &= c \\
c &= -\frac{4}{3}
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
x &= \frac{-\frac{4}{3}(t-1)}{t+-\frac{4}{3}(t-1)} \\
&= \frac{-\frac{4}{3}t+\frac{4}{3}}{t-\frac{4}{3}t+\frac{4}{3}} \\
&= \frac{\frac{-4t+4}{3}}{\frac{t+4}{3}} \\
x &= \frac{4(1-t)}{t-4}
\end{aligned}$$

(e) $\frac{dx}{dt} = e^{x+t}, x(0) = a$

i. Write D.E. as $g(y)\,dy = f(x)\,dx$

$$\begin{aligned}
\frac{dx}{dt} &= e^{x+t} \\
\frac{dx}{dt} &= e^{x}e^{t} \\
e^{-x}dx &= e^{t}dt
\end{aligned}$$

ii. Integrate both sides

$$\begin{aligned}
\int e^{-x}dx &= \int e^{t}dt \\
-e^{-x}+c &= e^{t}+c
\end{aligned}$$

iii. Try to change implicit solution into explicit solution (in terms of $y = y(x)$)

A. DO NOT forget to include C, constant of integration

$$\begin{aligned}
-e^{-x} &= e^{t} + c - c \\
-e^{-x} &= e^{t} + c - c \\
\ln\left(-e^{-x}\right) &= \ln\left(e^{t} + c\right) \\
\ln(-1) + \ln\left(e^{-x}\right) &= \ln\left(e^{t} + c\right) \\
\ln(-1) - x &= \ln\left(e^{t} + c\right) \\
-x &= \ln\left(e^{t} + c\right) - \ln(-1) \\
x &= -\ln\frac{\left(e^{t} + c\right)}{-1} \\
&= -\ln\left(-e^{t} - c\right) \\
&= -\ln\left(-e^{t} + c\right) \\
&= -\ln\left(-e^{t} + c\right)
\end{aligned}$$

iv. Substitute in the values given to find C

$$\begin{aligned}
a &= -\ln\left(-e^{0} + c\right) \\
&= -\ln(-1 + c) \\
a &= -\ln(-1 + c) \\
\ln(-1 + c) &= -a \\
-1 + c &= e^{-a} \\
c &= e^{-a} + 1
\end{aligned}$$

v. Substitute C back into the equation, to find the solution for the I.V.P.

$$\begin{aligned}
x &= -\ln\left(-e^{t} + e^{-a} + 1\right) \\
x &= -\ln\left(1 + e^{-a} - e^{t}\right)
\end{aligned}$$