Chapter 4
Relations and Digraphs

4.1 Product Sets and Partitions

- An ordered pair (a, b) is a listing of the objects a and b in a prescribed order, with a appearing first and b appearing second. Thus an ordered pair is merely a sequence of length 2.
  - If $A = B = \mathrm{R}$, the set of all real numbers, then $\mathrm{R} \times \mathrm{R}$, also denoted by R^2, is the set of all points in the plane. The ordered pair (a, b) gives the coordinates of a point in the plane.
  - The ordered pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

4.1 Product Sets and Partitions

- If A and B are two nonempty sets, the product set or Cartesian product $A \times B$ is defined as the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Thus

  $$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

  - The elements of $A \times B$ can be arranged in a convenient tabular array as shown below.

$A \backslash B$	b_1	b_2	..
a_1	(a_1, b_1)	(a_1, b_2)	...
a_2	(a_2, b_1)	(a_2, b_2)	...
:	:	:	:::

E.g.1

Let A = {1, 2, 3} and B = {r, s}, find $A \times B$ and $B \times A$.

$A \times B =$ {(1, r),(1, s),(2, r),(2, s),(3, r),(3, s)}

$B \times A =$ {(r, 1),(s, 1), ...(s, 3)}

4.1 Product Sets and Partitions (cont)

- Theorem 1

  For any two finite, nonempty sets A and B, $|A \times B| = |A| \times |B|$.

4.1 Product Sets and Partitions (cont)

- A marketing research firm classifies a person according to the following two criteria:

  Gender: male (m); female (f)

  Highest level of education completed: elementary school (e); high school (h); college (c); graduate school (g)

4.1 Product Sets and Partitions (cont)

Let $S = \{m, f\}$ and $L = \{e, h, c, g\}$.

The product set $S \times L$ contains all the categories into which the population is classified.

Thus the classification (f, g) represents a female who has completed graduate school. There are eight (2×4) categories in this classification scheme.

4.1 Product Sets and Partitions (cont)

- The Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ of then nonempty sets $A_1, A_2, \ldots, A_m$ is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$, where $a_i \in A_i$, $i = 1, 2, \ldots, m$. Thus

$$A_1 \times A_2 \times \cdots \times A_m$$
$$= \{(a_1, a_2, \ldots, a_m) | \; a_i \in A_i, i = 1, 2, \ldots, m\}.$$

4.1 Product Sets and Partitions (cont)

- If A_1 has n_1 elements, A_2 has n_2 elements, ..., and A_m has n_m elements, then

  $A_1 \times A_2 \times \cdots \times A_m$ has $n_1 \times n_2 \times \ldots \times n_m$ elements.

4.1 Product Sets and Partitions (cont)

- A software firm provides the following three characteristics for each program that it sells:

  Language: FORTRAN (f); PASCAL (p); LISP (l)

  Memory: 2 meg (2); 4 meg (4); 8 meg (8)

  Operating system: UNIX (u); DOS (d)

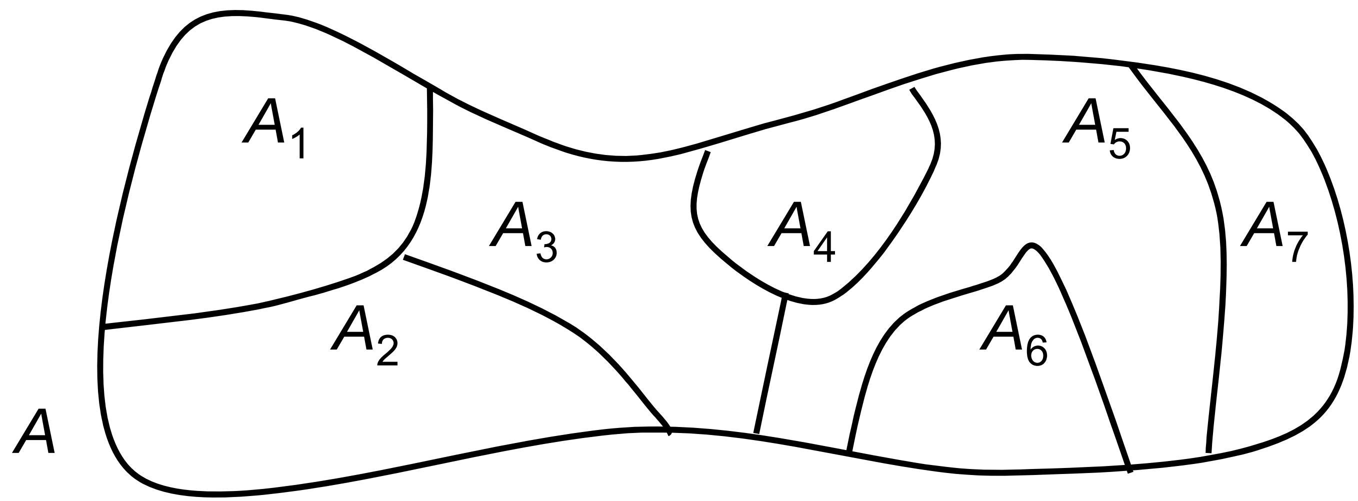
  Let $L = \{f, p, l\}$, $M = \{2, 4, 8\}$ and $O = \{u, d\}$. Then the Cartesian product $L \times M \times O$ contains all the categories that describe a program. There are 18 ($3 \times 3 \times 2$) categories in this classification scheme.

4.1 Product Sets and Partitions (cont)

- A partition or quotient set of nonempty set A is a collection P of nonempty subsets of A such that
  - each element of A belongs to one of the sets in P;
  - if A_1 and A_2 are distinct elements of P, then $A_1 \cap A_2 = \varnothing$.

4.1 Product Sets and Partitions (cont)

- The sets in P are called blocks or cells of the partitions.
  - Figure below shows a partition $P = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ into seven blocks



4.1 Product Sets and Partitions (cont)

- Let

  Z = set of integers,

  A_1 = set of all even integers, and

  A_2 = set of all odd integers.

  Then $\{A_1, A_2\}$ is a partition of Z.

$$A_1 \cup A_1 = \mathbb{Z}$$

$$A_1 \cap A_1 = \varnothing$$

Note:

- Since the members of a partition of a set are subsets of A, we see that the partition is a subset of $\wp(A)$, the power set of A. That is, partitions can be considered as particular kinds of subsets of $\wp(A)$.

E.g.2

Let $A = \{a, b, c, d, e, f, g, h\}$. Consider the following subsets of A:

$A_1 = \{a, b, c, d\}$,

$A_2 = \{a, c, e, f, g, h\}$,

$A_3 = \{a, e, c, g\}$,

$A_4 = \{b, d\}$,

$A_5 = \{f, h\}$.

Then {A2, A4} , {A3, A4, A5} are partitions of A

4.2 Relations and Digraph

- Let A and B be nonempty sets. A relation R from A to B is a subset of $A \times B$.
- If $R \subseteq A \times B$ and $(a, b) \in R$, then a is related to b by R, and write as $a\ R\ b$. If a is not related to b by R, then $a \not R\ b$. $(a,b) \notin \mathbb{R}$
- When A and B are equal, $R \subseteq A \times A$ is a relation on A, instead of a relation from A to A.
    - Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$.

      Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B.

4.2 Relations and Digraph (cont)

- Let A and B be sets of real numbers and define the following relation R (equals) from A to B:

  $a\ R\ b$ if and only of $a = b$.

- Let A be the set of all possible inputs to a given computer program, and let B be the set of all possible outputs from the same program. Define the following relation R from A to B:

  $a\ R\ b$ if and only if b is the output produced by the program when input a is used.

4.2 Relations and Digraph (cont)

- Let $A = Z^+$, the set of all positive integers. Define the following relation R on A:

  $a\ R\ b$ if and only if a divides b. $\frac{b}{a} \in Z$

  Then $4\ R\ 12$ but $5\ \not R\ 7$.

$$\frac{12}{4} = 3 \in Z \qquad \frac{7}{5} = \notin Z$$

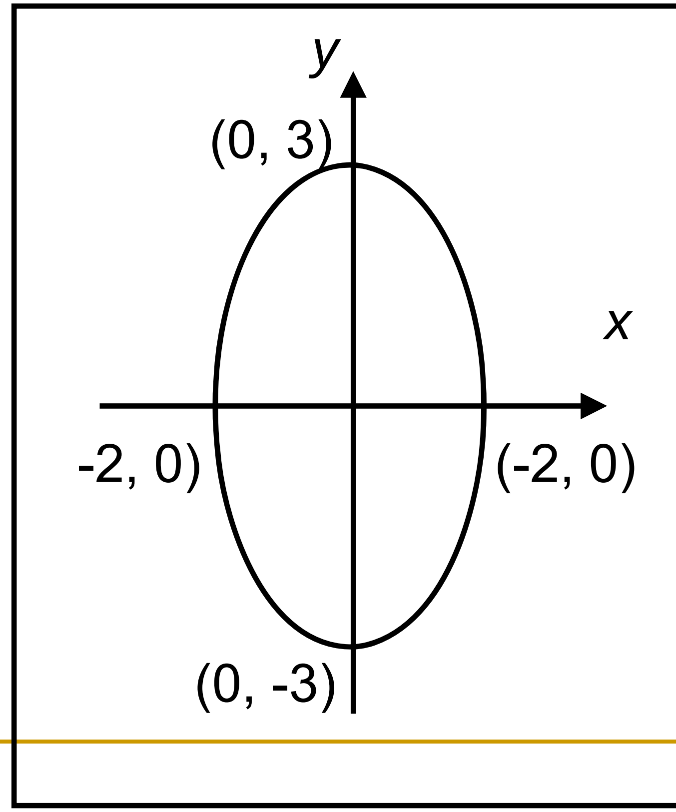
4.2 Relations and Digraph (cont)

- Let A = R, the set of real numbers. We define the following relation R on A:

  $x\ R\ y$ if and only if x and y satisfy the equation

  $$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

  Then set R consists of all points on the ellipse shown.

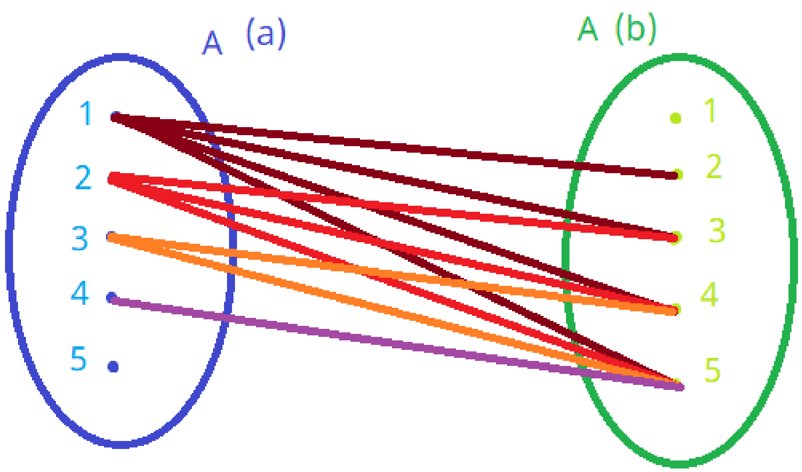


E.g.3

Let A = {1, 2, 3, 4, 5}. Define the following relation R (less than) on A:

$a\ R\ b$ if and only if $a < b$.

List the ordered pairs that belong to R.



R = {(1, 2),(1, 3)...,(1, 5),(2, 2), ...(2, 5),(3, 4), ...,(3, 5),(4, 5)}

E.g.4

An airline services the five cities c_1, c_2, c_3, c_4, and c_5. Table below gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is \$100, while the cost of going from c_4 to c_2 is \$200.

From / To	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

E.g.4 (cont)

We now define the following relation R on the set of cities $A = \{ c_1, c_2, c_3, c_4, c_5\}$:

> $c_i \ R \ c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to \$180.

Find R.

R = {(c1, c2), ...,(c1, c4),(c2, c4),(c3, c1),(c3, c2), (c4, c3),(c4, c5),(c5, c2),(c5, c4)}

4.2.1 Sets Arising from Relations

- Let R $\subseteq$ *A* $\times$ *B* be a relation from *A* to *B*. The domain of *R*, denoted by Dom (*R*), is the set of elements in *A* that are related to some element in *B*. In other words, Dom (*R*), a subset of *A*, is the set of all first elements in the pairs that make up *R*.
- The range of *R*, denoted by Ran (*R*), is the set of elements in *B* that are second elements of pairs in *R*, that is, all elements in *B* that are related to some element in *A*.
  - Refer to the previous example, Dom (*R*) = [-2, 2] and Ran (*R*) = [-3, 3].

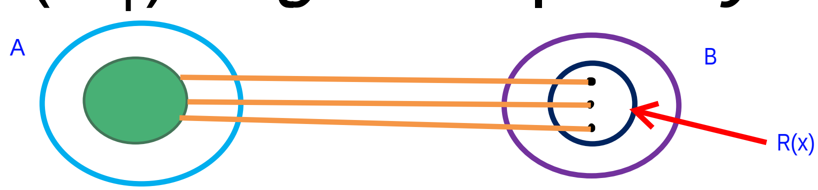
4.2.1 Sets Arising from Relations (cont)

- If R is a relation from A to B and $x \in A$, $R(x)$, the R-relative set of x, is defined to be the set of all y in B with the property that x is R-related to y. Thus, in symbol

$$R(x) = \{y \in B \mid x\ R\ y\}.$$

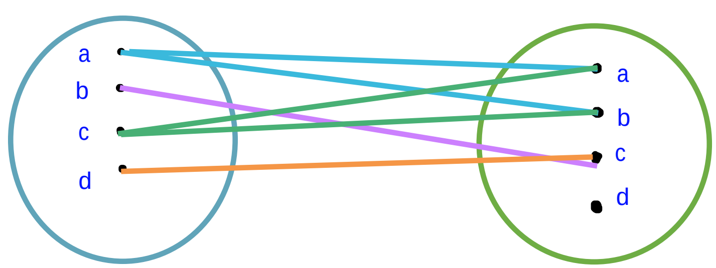
- If $A_1 \subseteq A$, the R-related set of A_1, is the set of all y in B with the property that x is R-related to y for some x in A_1. That is

$$R(A_1) = \{y \in B \mid x\ R\ y \text{ for some } x \text{ in } A_1\}.$$



4.2.1 Sets Arising from Relations (cont)

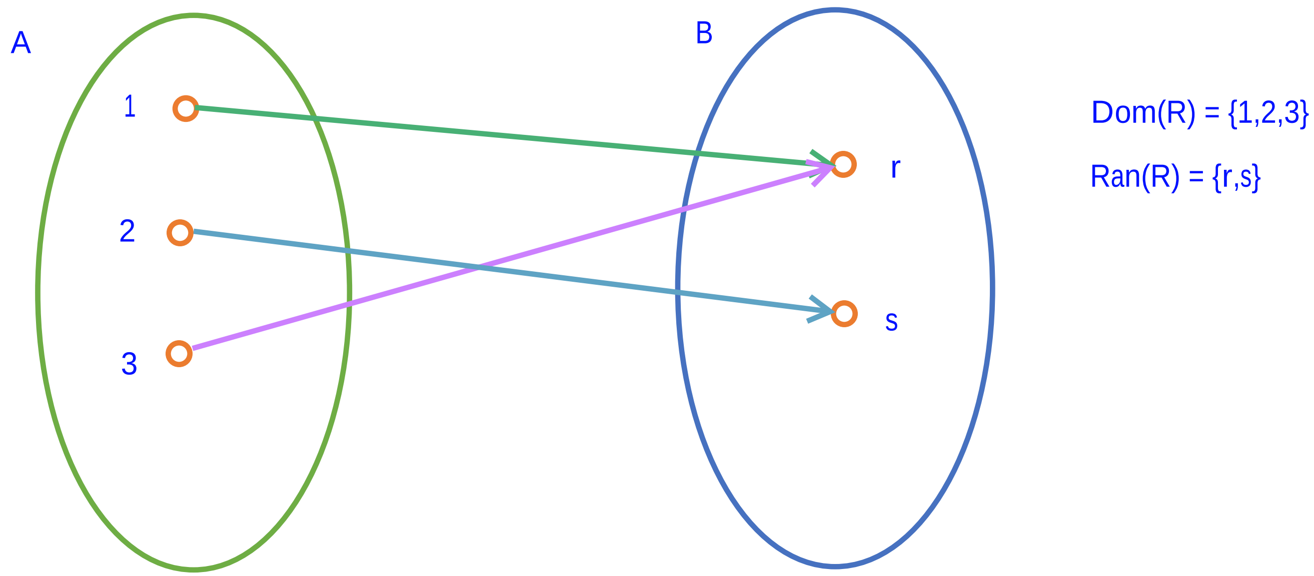
- Let $A = \{a, b, c, d\}$ and let
  $R = \{(a, a), (a, b), (b, c), (c, a), (d, c), (c, b)\}$
  Then $R(c) = \{a, b\}$, $R(d) = \{c\}$,
  and if $A_1 = \{c, d\}$, then $R(A_1) = \{a, b, c\}$.
- From the preceding definitions, $R(A_1)$ is the union of the sets $R(x)$, where $x \in A_1$.



E.g.5

Let A = {1, 2, 3}, B = {r, s}, and

R = {(1, r), (2, s), (3, r)} is a relation from A to B. Determine Dom (R) and Ran (R).



E.g.6

List the Dom (R) and Ran (R) in E.g.3.

Dom(R) = {1,2,3}

Ran(R) = {r,s}

4.2.1 Sets Arising from Relations (cont)

- Theorem 1

  Let R be a relation from A to B, and let A_1 and A_2 be subsets of A. Then

  - If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$.
  - $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$. AKA: R(A + B) = R(A) + R(B)
  - $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$. AKA: R(A AND B) = R(A) AND R(B)

- Theorem 2

  Let R and S be relations from A to B. If $R(a) = S(a)$ for all a in A, then $R = S$.

If R & S contains same relations, then R is the same as S

E.g.7

Let A = Z, R be "≤",

A_1 = {0, 1, 2}, and A_2 = {9, 13}.

Show that $R(A_1 \cap A_2) \neq R(A_1) \cap R(A_2)$.

$A_1 \cap A_2 = \varnothing$

$R\ (A_1 \cap A_2) = \varnothing$

$R\ (A_1) = \{0, 1, 2, \ldots\}$

$R\ (A_2) = \{9, 10, 11, 12, 13, \ldots\}$

$R\ (A_1) \cap R\ (A2) = \{9, 10, 11, 12, \ldots\}$ 6= 0

$\therefore$ $R\ (A_1 \cap A_2)$ 6= $R\ (A_1) \cap R\ (A_2)$

$R\ (A_1 \cap A_2) \leq R\ (A_1) \cap R\ (A_2)$

E.g.8

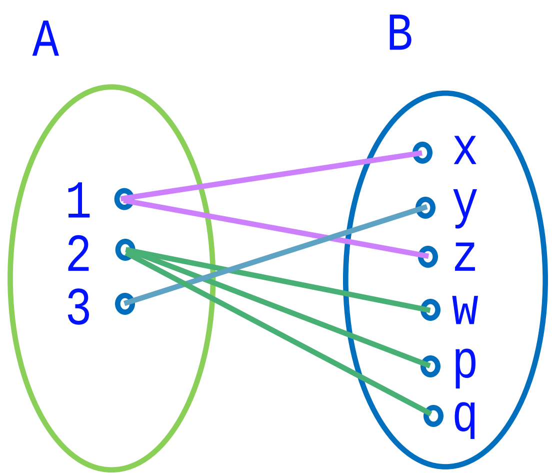
Let A = {1, 2, 3} and B = {x, y, z, w, p, q}, and consider the relation R = {(1, x), (1, z), (2, w), (2, p), (2, q), (3, y)}.

And let A_1 = {1, 2} and A_2 = {2, 3}.

Show that

LHS $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$. RHS

LHS $R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$. RHS



$$A_1 \cup A_2 = \{1,2,3\}$$

$$\Rightarrow R(A_1 \cup A_2) = \{x,y,z,w,p,q\}$$

$$R(A_1) = \{x,z,w,p,q\} \qquad R(A_2) = \{y,w,p,q\}$$

$$\Rightarrow R(A_1) \cup R(A_2) = \{x,y,z,w,p,q\}$$

Conclusion: $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

$$A_1 \cap A_2 = \{2\}$$

$$\Rightarrow R(A_1 \cap A_2) = \{w,p,q\}$$

$R(A_1) = \{x,z,w,p,q\}$ $R(A_2) = \{w,p,q,y\}$

$$\Rightarrow R(A_1) \cap R(A_2) = \{ w, p, q\}$$

$$\therefore R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$$

4.2.2 The Matrix of a Relation

- If $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets containing m and n elements, respectively, and R is a relation from A to B, we represent R by the $m \times n$ matrix $\mathbf{M}_R = [m_{ij}]$, which is defined by

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

$M_{ij} = 1$

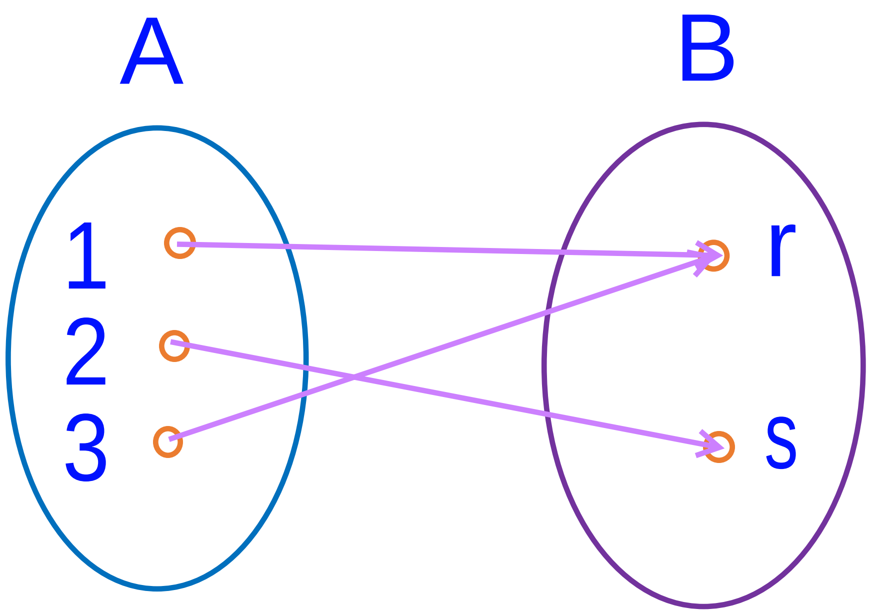
$M_{ij} = 0$

- The matrix $\mathbf{M}_R$ is called the matrix of R, which provides an easy way to check whether R has a given property.

E.g.9

Let $A = \{1, 2, 3\}$, $B = \{r, s\}$,

and $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B. Find $\mathbf{M}_R$.



$$M_R = A \begin{array}{c c} & \begin{array}{cc} & B \\ r & s \end{array} \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}\end{array}$$

E.g.10

Let A = {x, y, z}, B = {p, q, r, s}, and R is a relation from A to B.

Consider the matrix

$$\mathbf{M}_R = \begin{array}{c} \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Find R.

R = {(x,p), (x,s), (y,q), (y,r), (z,p), (z,r)}

4.2.3 The Digraphs of a Relation

- Directed graph or digraph of a relation *R* on *A* is a geometrical representation of *R*
- Step:
  - Draw a small circle, called vertex, for each element of *A* and label the vertices with the corresponding elements of *A*.
  - Draw an arrow, called edge, from vertex a_i to vertex a_j if and only if $a_i \, R \, a_j$.

4.2.3 The Digraphs of a Relation (cont)

- If R is a relation on A, the edges in the digraph of R correspond exactly to the pairs in R, and the vertices correspond exactly to the elements of the set A.
- If R is a relation on a set A and $a \in A$, then
  - the in-degree of a (relative to the relation R) is the number of $b \in A$ such that $(b, a) \in R$.
  - the out-degree of a is the number of $b \in A$ such that $(a, b) \in R$.

4.2.3 The Digraphs of a Relation (cont)

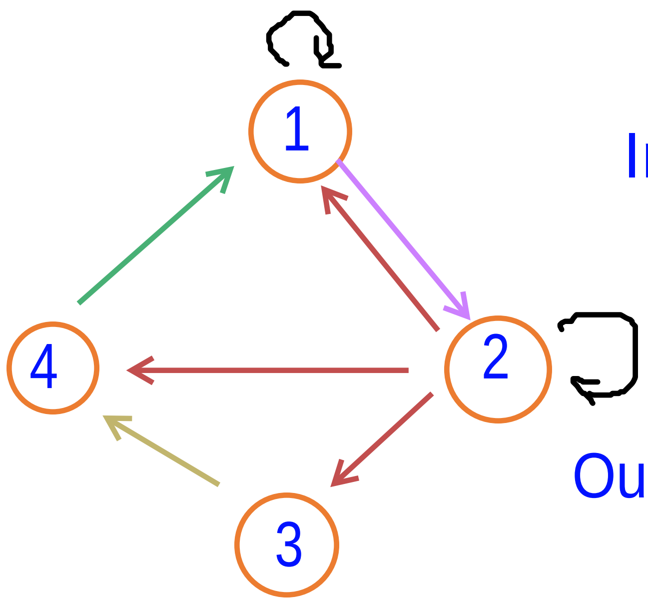
- In terms of digraph, in-degree of a vertex is the number of edges terminating at the vertex; out-degree is then number of edged leaving the vertex, or out-degree of *a* is $|R(a)|$.
- The sum of all in-degrees must equal to the sum of all out-degrees.

4.2.3 The Digraphs of a Relation (cont)

- A directed edge which drawn from vertex a to a such that $(a, a) \in R$ is called a loop at vertex a.
- If R is a relation on a set A, and B is a subset of A, the restriction of R to B is $R \cap B \times B$.

E.g.11

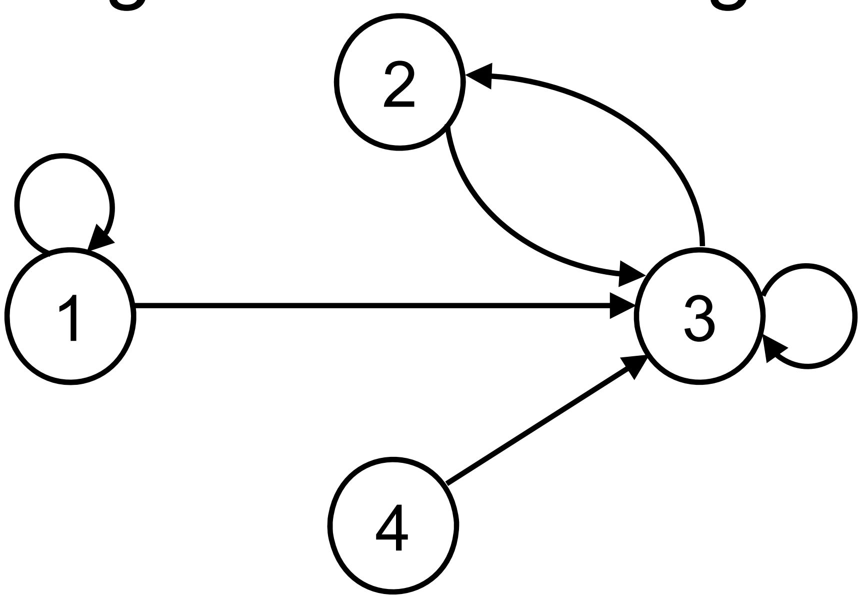
Let A = {1, 2, 3, 4}, R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)}. Draw the digraph of R. Hence, give the in-degree and out-degree of each vertex.



	1	2	3	4
In-degree	3	2	1	2
Out-degree	2	4	1	1

E.g.12

Find the relation determined by the following digraph and give its matrix. Then give the in-degree and out-degree of each vertex.



$R = \{(1,1), (1,3), (2,3),$
$(3,2), (3,3), (4,3)\}$

$R = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3)\}$

$$\mathbf{M}_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Vertex	1	2	3	4
In-degree	1	1	4	0
Out-degree	2	1	2	1

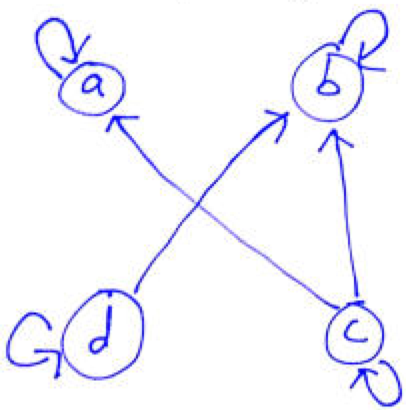
E.g.13

Let $A = \{a, b, c, d\}$ and let R be the relation on A that has the matrix

$$\mathbf{M}_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

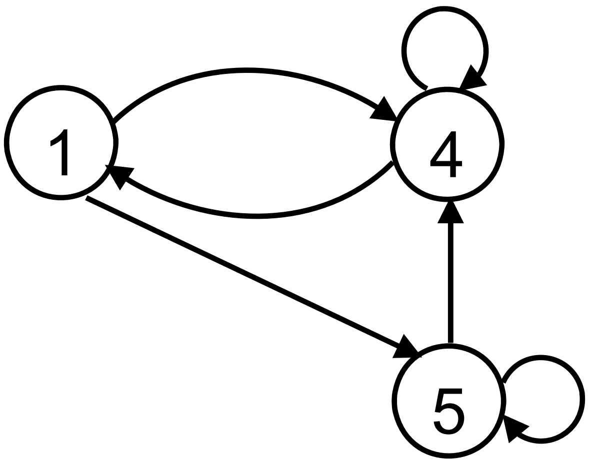
$R = \{(a, a), (b, b), (c, a), (c, b), (c, c), (d, b), (d, d)\}$



Vertex	a	b	c	d
in-degree	2	3	1	1
out-degree	1	1	3	2

E.g.14

Let A = {1, 4, 5} and let R be given by the digraph shown below. Find $\mathbf{M}_R$ and R.



$$M_R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$R = \{(1,4), (1,5), (4,1), (4,4), (5,4), (5,5)\}$$

E.g.15

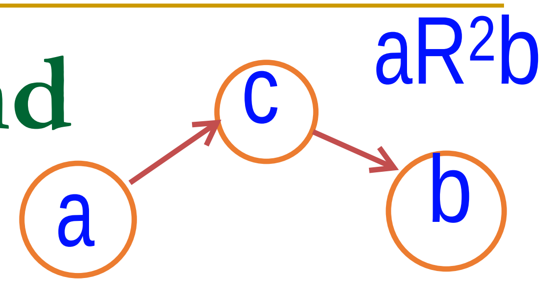
Let $A = \{a, b, c, d, e, f\}$ and

$R = \{(a, a), (a, c), (b, c), (a, e), (b, e), (c, e)\}$.

Let $B = \{a, b, c\}$. Find the restriction of R to B.

$R \cap (B \times B) = \{(a, a), (a, c), (b, c)\}$

4.3 Paths in Relations and Digraphs



- Suppose that R is a relation on a set A. A path of length n in R from a to b is a finite sequence π: a, x_1, x_2, ..., x_{n-1}, b, beginning with a and ending with b, such that

$$a\ R\ x_1,\ x_1\ R\ x_2,\ \ldots,\ x_{n-1}\ R\ b.$$

- A path of length n involves $n + 1$ elements of A, although they are not necessarily distinct.

4.3 Paths in Relations and Digraphs (cont)

- A path can be easily visualized with the aid of the digraph of a relation. It appears as a geometric path or succession of edges, where the indicated directions of the edges are followed.
- The length of a path is the number of edges in the path, where the vertices need not all be distinct.

4.3 Paths in Relations and Digraphs (cont)

- A path that begins and ends at the same vertex is called a cycle.
- It is clear that the paths of length 1 can be identified with the ordered pairs (x, y) that belong to R.
- If n is a fixed positive integer, a relation R^n on A is defined as follows:

  $x \; R^n \; y$ means that there is a path of length n from x to y in R.

4.3 Paths in Relations and Digraphs (cont)

- A relation R^{∞} on A, sometimes called the connectivity relation for R is defined as

  $x \; R^{\infty} \; y$ means that there is some path in R from x to y.

4.3 Paths in Relations and Digraphs (cont)

- Let A be the set of all living human beings, and let R be the relation of mutual acquaintance.

  That is, $a\ R\ b$ means that a and b know one another. Then $a\ R^2\ b$ means that a and b have an acquaintance in common.

  In general, $a\ R^n\ b$ if a knows someone x_1, who knows x_2, …, who knows x_{n-1}, who knows b.

4.3 Paths in Relations and Digraphs (cont)

Finally, $a\ R^{\infty}\ b$ means that some chain of acquaintances exists that begins at a and ends at b. It is interesting (and unknown) whether two Americans, say, are related by R^{∞}.

4.3 Paths in Relations and Digraphs (cont)

- Let A be a set of U.S. cities, and let $x\ R\ y$ if there is a direct flight from x to y on at least one airline.

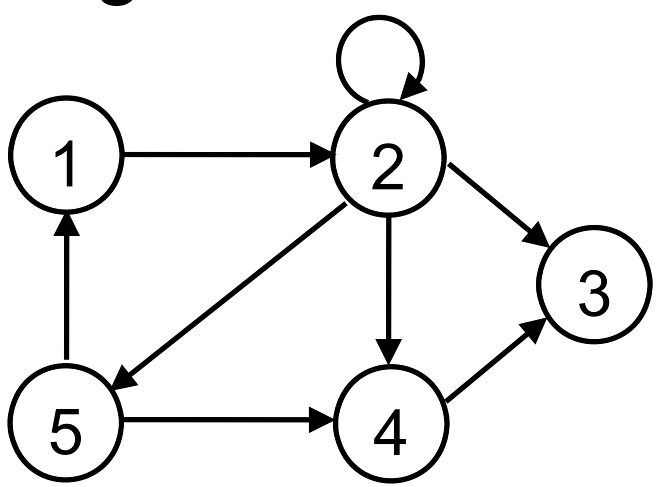
  Then x and y are related by R^n if one can book a flight from x to y having exactly $n - 1$ intermediate stops, and $x\ R^\infty\ y$ if one can get from x to y by plane.

4.3 Paths in Relations and Digraphs (cont)

- $R^n(x)$ consists of all vertices that can be reached from x by means of a path in R of length n. The set $R^\infty(x)$ consists of all vertices that can be reached from x by some path in R.

E.g.16

Consider the following digraph. List a path of length 1, 3 and 4, respectively.



Path of length

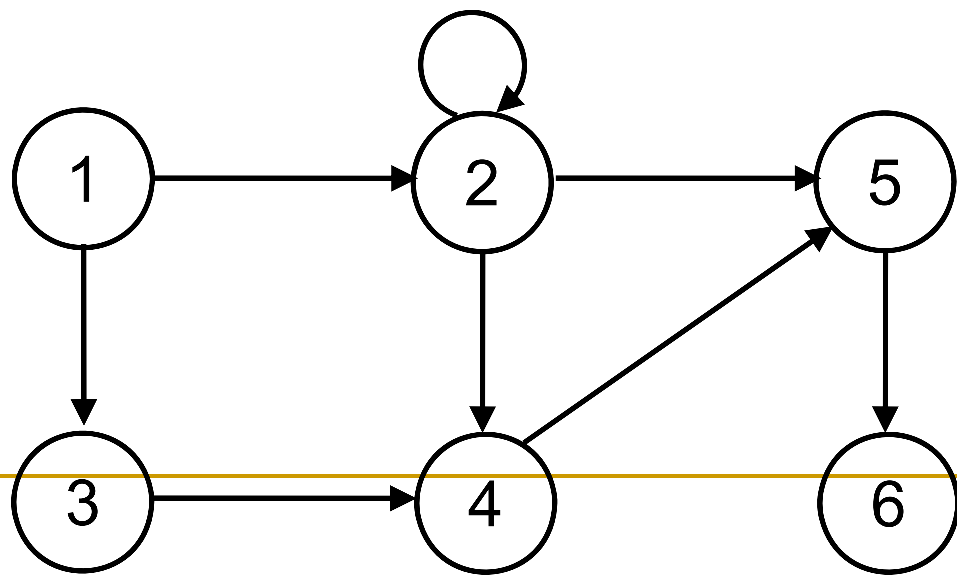
1: 2,2

3: 1,2,4,3

4: 1,2,5,4,3

E.g.17

Let A = {1, 2, 3, 4, 5, 6}. Let R be the relation whose digraph is shown below. Construct the digraph of R^2 on A (i.e. a line connects two vertices if and only if they are R^2-related, that is, if an only if there is a path of length two connecting those vertices).



$1R2$ and $2R2 \Rightarrow 1R^22$; $1R2$ and $2R4 \Rightarrow 1R^24$;

...

$3R4$ and $4R5 \Rightarrow 3R^25$; $4R5$ and $5R6 \Rightarrow 4R^26$;

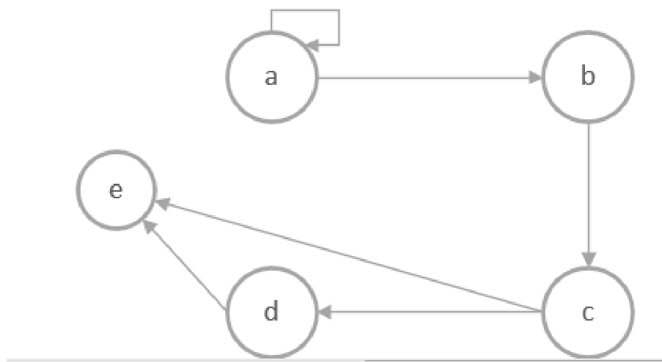
$R^2=\{(1,2), (1,4), (1,5), (2,2), (2,4), (2,5), (2,6), (3,5), (4,6)\}$

E.g.18

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$.

Compute R^2 and R^∞.



$R^2 = \{(a,a), (a,b), (a,c), (b,e), (b,d), (c,e)\}$

$R^\infty = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,e), (c,d), (d,e)\}$

4.3 Paths in Relations and Digraphs (cont)

- $\mathbf{M}_R$ can be used to compute R^2 or R^∞ when $|R|$ is large.
- Theorem 1

  Let R be a relation on a finite set $A = \{a_1, a_2, \ldots, a_n\}$, and let $\mathbf{M}_R$ be the $n \times n$ matrix representing R, then $\mathbf{M}_{R^2} = \mathbf{M}_R \odot \mathbf{M}_R$, usually denoted as $(\mathbf{M}_R)^2_\odot$.
- Similarly, $\mathbf{M}_{R^3} = \mathbf{M}_R \odot (\mathbf{M}_R \odot \mathbf{M}_R) = (\mathbf{M}_R)^3_\odot$.

Note: $\odot$ = NOT Hadamard product (just to denote special multiplication)

4.3 Paths in Relations and Digraphs (cont)

- Theorem 2

  For $n \geq 2$ and R is a relation on a finite set A,

  $\mathbf{M}_{R^n} = \mathbf{M}_R \odot \mathbf{M}_R \odot \ldots \odot \mathbf{M}_R$. ($n$ factors)

- Since $R^\infty = R \cup R^2 \cup R^3 \cup \ldots = \bigcup_{n=1}^{\infty} R^n$

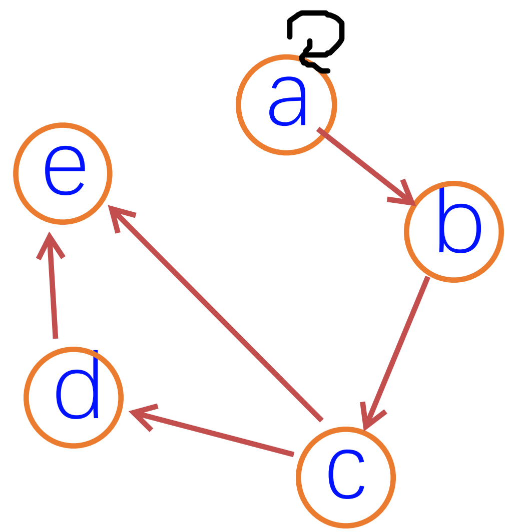
  and $\mathbf{M}_{R \cup S} = \mathbf{M}_R \vee \mathbf{M}_S$, then

  $\mathbf{M}_{R^\infty} = \mathbf{M}_R \vee (\mathbf{M}_R)^2_\odot \vee (\mathbf{M}_R)^3_\odot \vee \ldots$

E.g.19

The matrix of relation R on set A in E.g.18 is

$$\mathbf{M}_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Compute $\mathbf{M}_{R^2}$.

$$\mathbf{M}_{R^2} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

R^2={(a,a), (a,b), (a,c), (b,d), (b,e), (c,e)}

4.3 Paths in Relations and Digraphs (cont)

- The reachability relation R^* of a relation R on a set A that has n elements is defined as follows:

  $x\ R^*\ y$ means that $x = y$ or $x\ R^\infty\ y$.

  - y is reachable from x if either y is x or there is some path from x to y.
  - $\mathbf{M}_{R^*} = \mathbf{M}_{R^\infty} \vee \mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Thus

    $\mathbf{M}_{R^*} = \mathbf{I}_n \vee \mathbf{M}_R \vee (\mathbf{M}_R)^2_{\odot} \vee (\mathbf{M}_R)^3_{\odot} \vee \ldots$

4.3 Paths in Relations and Digraphs (cont)

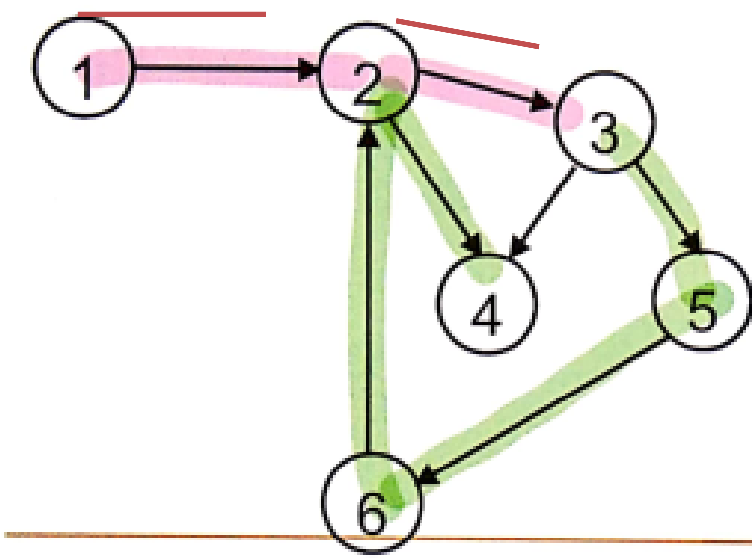
- Let π_1: a, x_1, x_2, …, x_{n-1}, b be a path in the relation R of length n from a to b, and let π_2: b, y_1, y_2, …, y_{m-1}, c be a path in R of length m from b to c. Then the composition of π_1 and π_2 is the path a, x_1, x_2, …, x_{n-1}, b, y_1, y_2, …, y_{m-1}, c of length $n + m$, which is denoted by $\pi_2 \circ \pi_1$, a path from a to c.

E.g.20

Consider the relation whose digraph is given as follows and the paths

π_1: 1, 2, 3 and π_2: 3, 5, 6, 2, 4.

Find the composition $\pi_2 \circ \pi_1$.



$\pi_2 \circ \pi_1 = 1, 2, 3, 5, 6, 2, 4.$

from vertex 1 to vertex 4 of length $2+4=6$.

4.4 Properties of Relations

4.4.1 Reflexive and Irreflexive Relations

- A relation R on a set A is reflexive:
  - If $(a, a) \in R$ for all $a \in R$, that is, if $a\ R\ a$ for all $a \in A$, i.e. every element $a \in A$ is related to itself.
  - The matrix of R must have all 1's on its main diagonal.
  - The digraph R has a cycle of length 1, or loop at every vertex.
  - Dom (R) = Ran $(R) = A$.

4.4.1 Reflexive and Irreflexive Relations (cont)

- A relation *R* on a set *A* is irreflexive:
  - If $a \not R a$ for every $a \in A$, i.e. no element is related to itself.
  - The matrix of *R* must have 0's on its main diagonal.
  - The digraph of *R* has no cycles of length 1.

no loops

4.4.1 Reflexive and Irreflexive Relations (cont)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

- E.g.
  - Let $\Delta = \{(a, a) \mid a \in A\}$, so that Δ is a relation of equality on the set A.

    Then Δ is reflexive, since $(a, a) \in \Delta$ for all $a \in A$.
  - Let $R = \{(a, b) \in A \times A \mid a \neq b\}$, so that R is the relation of inequality on the set A.

    Then R is Irreflexive, since $(a, a) \notin R$ for all $a \in A$.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

4.4.1 Reflexive and Irreflexive Relations (cont)

- Let A = {1, 2, 3}, and let R = {(1, 1), (1, 2)}. Then R is not reflexive since $(2, 2) \notin R$ and $(3, 3) \notin R$.

  Also, R is not irreflexive, since $(1, 1) \in R$.
- Let A be a nonempty set. Let $R = \varnothing \subseteq A \times A$, the empty relation.

  R = empty set is a subset of set of all ordered pairs from A to A

  Then R is not reflexive, since $(a, a) \notin R$ for all $a \in A$ (the empty set has no elements). However, R is irreflexive.

4.4.1 Reflexive and Irreflexive Relations (cont)

- Using the equality relation Δ on a set A:

  R is reflexive if and only if $\Delta \subseteq R$, and R is irreflexive if and only if $\Delta \cap R = \varnothing$.

  Each element is related to itself

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations

- A relation R on a set A is symmetric:
  - If whenever $a\ R\ b$, then $b\ R\ a$; R is not symmetric if for some a and $b \in A$ with $a\ R\ b$ but $b \not R\ a$.
  - The matrix $\mathbf{M}_R = [m_{ij}]$ of R satisfies the property that

    if $m_{ij} = 1$, then $m_{ji} = 1$.

    Moreover, if $m_{ij} = 0$, then $m_{ji} = 0$.

    Thus $\mathbf{M}_R$ is a symmetric matrix ($\mathbf{M}_R = \mathbf{M}_R^{\mathrm{T}}$).

If the first element is related to second element, then second element must be related to first element

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

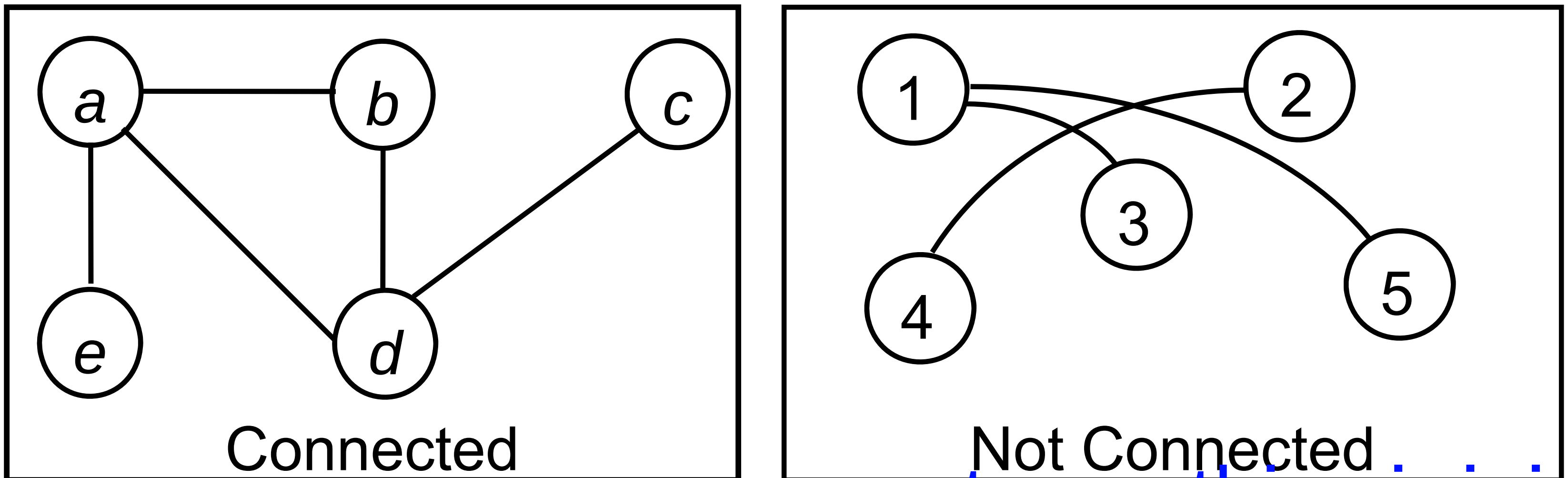
- If the two vertices in a digraph of *R* are connected by an edge, they must always be connected in both directions, i.e. from vertex *i* to vertex *j* and from vertex *j* to vertex *i*.
- The two edges connecting the vertices may be replaced with one undirected edge, or a “two way street”. The resulting diagram is called the graph of the symmetric relation.

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

- An undirected edge between a and b, in the graph of symmetric relation R, corresponds to a set $\{a, b\}$ such that $(a, b) \in R$ and $(b, a) \in R$. Sometimes such a set $\{a, b\}$ is referred as an undirected edge of relation R and a and b are called adjacent vertices.
- A symmetric relation R on a set A is called connected if there is a path from any element of A to any other element of A, i.e. the graph of R is all in one piece.

Undirected edge = lines with no arrows

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)



not everything is joined

- E.g. Let A be a set of people and
  let $R = \{(x, y) \in A \times A \mid x \text{ is a cousin of } y\}$.

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

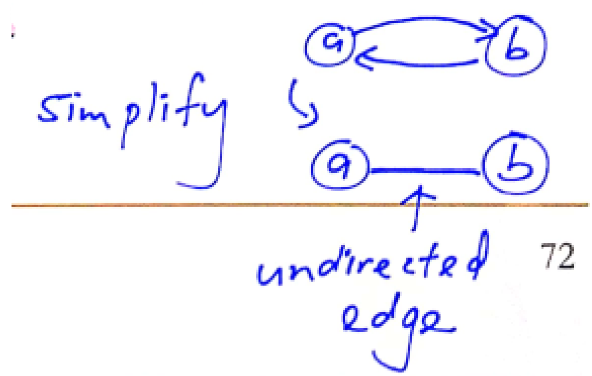
- A relation R on a set A is asymmetric:
  - If whenever a R b, then b R a; R is not asymmetric if for some a and $b \in A$ with both a R b and b R a.
  - The matrix $\mathbf{M}_R = [m_{ij}]$ of R satisfies the property that

    if $m_{ij} = 1$, then $m_{ji} = 0$. a to b true. b to a false.
  - The main diagonal of the matrix $\mathbf{M}_R$ are 0's, i.e. $m_{ii} = 0$ for all i.

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

- The digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*. This is true for any *i* and *j*, and in particular if *i* equals *j*.

  Thus there is no cycle of length 1, and all edges are "one-way streets".



4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

- A relation R on a set A is antisymmetric:
  - If whenever $a\ R\ b$ and $b\ R\ a$, then $a = b$; R is not antisymmetric if for a and b in A, $a \neq b$, and both $a\ R\ b$ and $b\ R\ a$.
  - The matrix $\mathbf{M}_R = [m_{ij}]$ of R satisfies the property

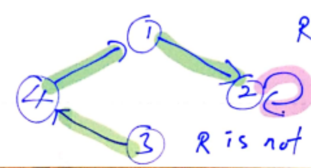
    if $i \neq j$, then $m_{ij} = 0$ or $m_{ji} = 0$.

if they are related, then they are same

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

- The contrapositive of this definition is that R is antisymmetric if whenever $a \neq b$, then $a \not R\, b$ or $b \not R\, a$.
- In the digraph of R, for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i. When $i = j$, no condition is imposed.
- Thus there may be cycles of length 1, but again all edges are "one way".
- E.g. Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$.

R is antisymmetric



R is not asymmetric

R is not symmetric

4.4.2 Symmetric, Asymmetric, and Antisymmetric Relations (cont)

- A property fails to hold for a relation *R* in general if we can find one situation where the property does not hold.

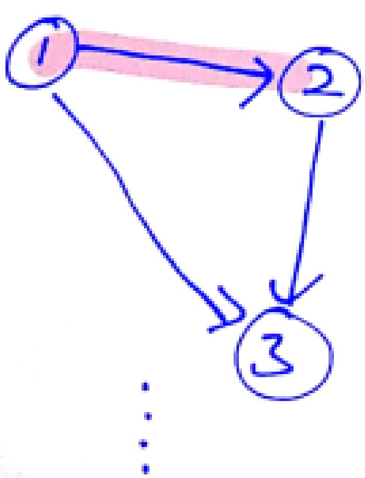
  If we can find counterexample, then the property fails

E.g.21

Let A = Z, the set of integers, and let

$R = \{(a, b) \in A \times A \mid a < b\}$

so that R is the relation less than. Is R symmetric, asymmetric, or antisymmetric?



R is not symmetric

$1 R 2$ but $2 \not R 1$.

$1 < 2$ but $2 \not< 1$

R is asymmetric

R is antisymmetric.

$a < b$ but none of the case of $b < a$

($2 < 3$ but $3 \nless 2$)

$\Rightarrow$ ($2 < 3$ but $3 \nless 2$) $2 R 3$ but $3 \not R 2$

$\Rightarrow$ R is not symmetric.

whenever $a < b$, $b \nless a$

R is asymmetric.

If $a \neq b$, either $a \nless b$ or $b \nless a$

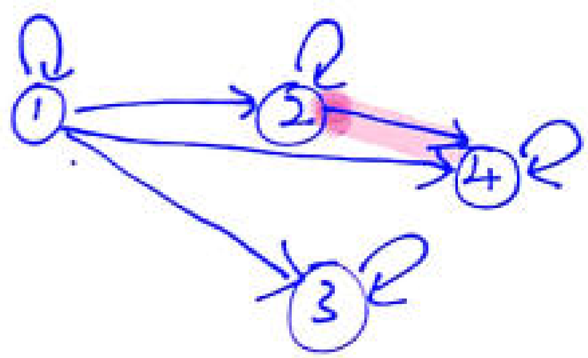
R is antisymmetric.

E.g.22

Let $A = Z^+$, the set of positive integers, and let

$$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}.$$

Is R symmetric, asymmetric, or antisymmetric?



$a \mid b$ but sometimes $b \nmid a$

$(2 \mid 4$ but $4 \nmid 2)$

$\Rightarrow$ $\Rightarrow (2,4) \in R$ but $(4,2) \notin R$

R is not symmetric.

$2/2$ and $2/2$ but $2=2$

$(2,2) \in R$ R is not asymmetric.

whenever a/b and b/a, then $a=b$

R is antisymmetric.

E.g.23

Determine whether the relation R whose matrix $\mathbf{M}_R$ is given is symmetric, asymmetric, or antisymmetric.

i. $\begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$

Symmetric. ($m_{ij}=m_{ji}$)
Not asymmetric. ($m_{ij}=m_{ji}$)
Not antisymmetric.
(If $i \neq j$, then m_{ij} or $m_{ji} = 0$

ii. $\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$

Symmetric
Not asymmetric. ($m_{ij}=m_{ji}$)
Not antisymmetric.
(If $i \neq j$, then m_{ij} or $m_{ji} = 0$

E.g.23

Determine whether the relation R whose matrix $\mathbf{M}_R$ is given is symmetric, asymmetric, or antisymmetric.

i. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\mathbf{M}_R = (\mathbf{M}_R)^T$

$\Rightarrow$ Symmetric.
Not asymmetric.
Not antisymmetric.

E.g.23

Determine whether the relation R whose matrix $\mathbf{M}_R$ is given is symmetric, asymmetric, or antisymmetric.

ii. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$M_R = (M_R)^T$

R is symmetric

R is not asymmetric

R is not anti-symmetric

E.g.23 (cont)

iii. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

iv. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

R is not symmetric
(Not all L->R reflected diagonals match)
R is not asymmetric
(Some L->R reflected diagonals match)
R is anti-symmetric

R is not symmetric
R not asymmetric
R is not antisymmetric

E.g.23 (cont)

v. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

R is not symmetric
R is not asymmetric
R is antisymmetric

vi. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

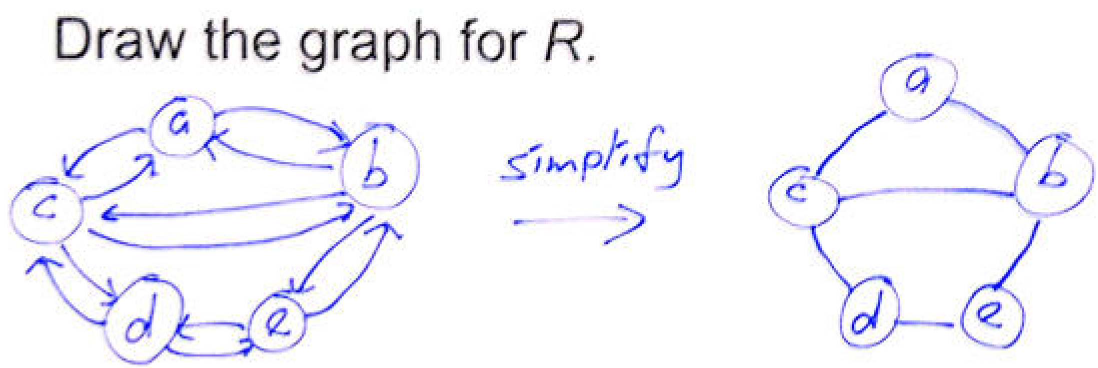
R is not symmetric
R is not asymmetric
R is not antisymmetric

E.g.24

Let $A = \{a, b, c, d, e\}$ and let R be the symmetric relation by

$R = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c)\}$.

Draw the graph for R.



4.4.3 Transitive Relations

- A relation R on a set A is transitive if whenever $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$.
- A relation on A is not transitive if there exist a, b, and c in A so that $a\ R\ b$ and $b\ R\ c$, but $a \not R\ c$.
- The matrix $\mathbf{M}_R = [m_{ij}]$ for a transitive relation R has the property

  if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$.

  i.e. $(\mathbf{M}_R)^2_{\odot}$ has a 1 in position i, k.

4.4.3 Transitive Relations (cont)

- The transitivity of R means that if $(\mathbf{M}_R)^2_\odot$ has a 1 in any position, then $\mathbf{M}_R$ have a 1 in the same position.

  In particular, if $(\mathbf{M}_R)^2_\odot = \mathbf{M}_R$, then R is transitive.

  - E.g. $\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

    is the matrix of a transitive relation R since $(\mathbf{M}_R)^2_\odot = \mathbf{M}_R$. R is transitive

4.4.3 Transitive Relations (cont)

- R is a transitive relation if and only if in the digraph of R if there is a directed edge from one vertex a to another vertex b, and if there exists a directed edge from vertex b to vertex c, then there must exists a directed edge from vertex a to vertex c.

4.4.3 Transitive Relations (cont)

- Theorem 1

  A relation R is transitive if and only if it satisfies the following property:

  If there is a path of length greater than 1 from vertex a to vertex b, there is a path of length 1 from a to b (a is related to b). Algebraically stated, R is transitive if and only if $R^n \subseteq R$ for all $n \geq 1$.

4.4.3 Transitive Relations (cont)

- Theorem 2

  Let R be a relation on a set A. Then

a) Reflexivity of R means that $a \in R(a)$ for all a in A.

b) Symmetry of R means that $a \in R(b)$ if and only if $b \in R(a)$.

c) Transitivity of R means that if $b \in R(a)$ and $c \in R(b)$, then $c \in R(a)$.

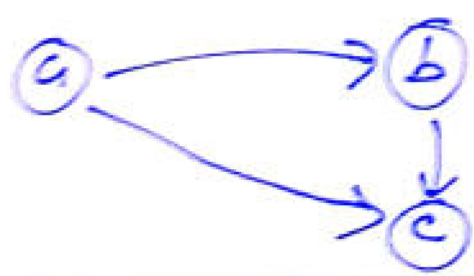
E.g.25

Let A = Z, the set of integers and let R be the relation less than. Is R transitive?

$a < b$, $b < c$, $\Rightarrow a < c$

aRb, bRc $\Rightarrow aRc$

$\therefore$ R is transitive.



E.g.26

Let $A = Z^+$ and let R be the relation

$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$.

Is R transitive?

$a \mid b \quad , \quad b \mid c \quad \Rightarrow \quad a \mid c$

$\frac{b}{a} \in \mathbb{Z} \qquad \frac{c}{b} \in \mathbb{Z} \qquad \frac{c}{a} \in \mathbb{Z}$

$\frac{b}{a} = h \qquad \frac{c}{b} = k \qquad \frac{c}{a} = \frac{kha}{a} = kh \in \mathbb{Z}$

$b = ah \qquad c = kb = kha.$

96

$aRb, \ bRc \Rightarrow aRc$

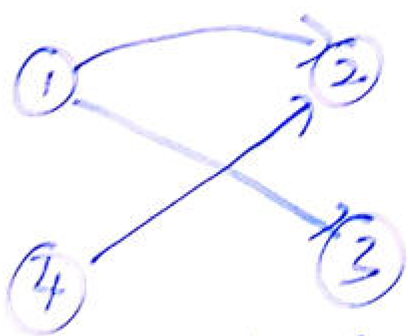
$\therefore$ R is transitive.

E.g.27

Let A = {1, 2, 3, 4} and let

R = {(1, 2), (1, 3),(4, 2)}.

Is R transitive?



R is transitive.

No path of length 2.

4.5 Equivalence Relations

- A relation R is called an equivalence relation if it is reflexive, symmetric and transitive. R is reflexive, symmetric, transitive
- E.g.
  - Let A be the set of all triangles in the plane and let R be the relation on A defined as follows: R is an equivalence relation.

    $R = \{(a, b) \in A \times A \mid a \text{ is congruent to } b\}.$
  - Let $A = \{1, 2, 3, 4\}$ and let

    $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}.$

simplify [1] [3]

E.g.28

Let A = Z, the set of integers, and let R be defined by $a\ R\ b$ if and only if $a \leq b$. Is R an equivalence relation?

==> R is reflexive

$1 \leq 2$ but 2 NOT ≤ 1

1 R 2 but 2 NOT R 1

==> R is NOT symmetric.

$\therefore$ R is NOT an equivalence relation

E.g.28

Let $A = Z$, the set of integers, and let R be defined by $a\ R\ b$ if and only if $a \leq b$. Is R an equivalence relation?

$a\ R\ a$ ($a \le a$) for every integer a

$\Rightarrow$

E.g.29

Let A = Z and let

$R = \{(a, b) \in A \times A \mid a$ and b yield the same remainder when divided by 2}.

In this case, we call 2 the modulus and write $a \equiv b$ (mod 2), read as "a is congruent to b mod 2."

Show that congruence mod 2 is an equivalent relation.

$a\ R\ a$ for every integer a

(a and a yield the same remainder when divided by 2)

$\Rightarrow$

Note:

- We may generalize the relation defined in the above example as follows:

  $R = \{(a, b) \in A \times A \mid a \equiv b \bmod n\}$.

  That is, $a \equiv b \pmod{n}$ if and only if a and b yield the same remainder when divided by n.

  Thus, congruence mod n is an equivalent relation.

- If $a \equiv b \pmod{n}$, then $a = qn + r$ and $b = pn + r$ and $a - b$ is a multiple of n. Thus $a \equiv b \pmod{n}$ if and only if $n \mid (a - b)$.

4.5 Equivalence Relations (cont)

- Theorem 1

  Let P be a partition of a set A. Define the relation R on A as follows:

  > a R b if and only if a and b are members of the same block.

  Then R is equivalence relation.

4.5 Equivalence Relations (cont)

- Proof:

1)If $a \in A$, then a is in the same block as itself, so $a\ R\ a$.

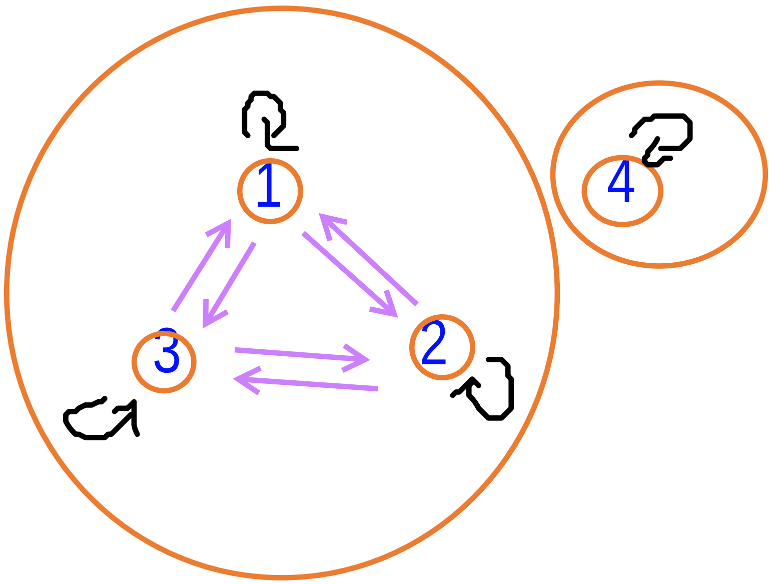
2)If $a\ R\ b$, then a and b are in the same block, so $b\ R\ a$.

3)If $a\ R\ b$ and $b\ R\ c$, then a, b, and c must lie in the same block of P.

- Since R is reflexive, symmetric, and transitive, R is an equivalence relation. R will be called the equivalence relation determined by P.

E.g.30

Let A = {1, 2, 3, 4} and consider the partition P = {{1, 2, 3}, {4}} of A. Find the equivalent relation R on A determined by P.



R={(1,1), (2,2), (3,3),
(1,2), (2,1),
(2,3), (3,2),
(1,3), (3,1),
(4,4)}

4.5 Equivalence Relations (cont)

- Lemma 1 Lemma = Subsidiary theorem

  Let R be an equivalent relation on a set A, and let $a \in A$ and $b \in A$. Then

  $a\ R\ b$ if and only if $R(a) = R(b)$.

- Theorem 2

  Let R be an equivalent relation on A, and let P be the collection of all distinct relatives sets $R(a)$ for a in A. Then P is a partition of A, and R is the equivalence relation determined by P.

4.5 Equivalence Relations (cont)

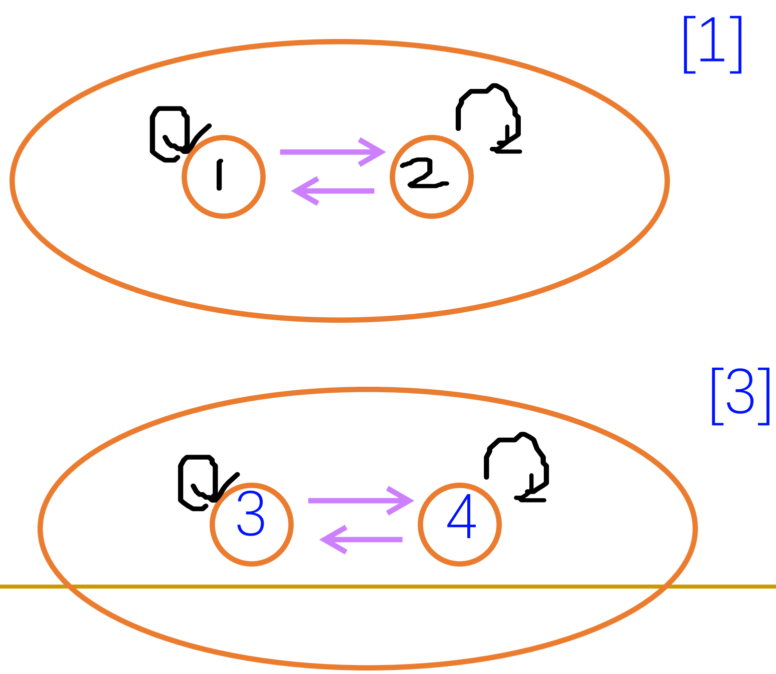
- If R is an equivalence relation on A, then the sets $R(a)$ are traditionally called equivalence classes of R. Some authors denote the class $R(a)$ by $[a]$.
- The partition constructed in Theorem 2 therefore consists of all equivalence classes of R, and denoted by A/R.

E.g.31

Let A = {1, 2, 3, 4} and let

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)}.

Determine A/R.



$A/R = \{\{1,2\}, \{3,4\}\}$

E.g.32 $A = \mathbb{Z}, aRb \Leftarrow), a \equiv b \pmod 2)$

Let R be the equivalent relation defined in E.g.29. Determine A/R.

$$A = \mathbb{Z}, aRb \Leftarrow), a \equiv b \pmod 2)$$

$$A/R = \{\{\text{even integers}\}, \{\text{odd integers}\}\}$$

$$\begin{aligned} R(0) &= \{..., -4, -2, 0, 2, 4, ...\} \\ &= \text{set of even integers} \end{aligned}$$

$$\begin{aligned} R(1) &= \{..., -3, -1, 1, 3, ...\} \\ &= \text{set of odd integers} \end{aligned}$$

4.6 Computer Representation of Relations and Digraphs

- We will discuss two methods for storing the data for a relation or digraph in a computer: Matrix of the relation and linked list.

4.6.1 Using Matrix of the Relation

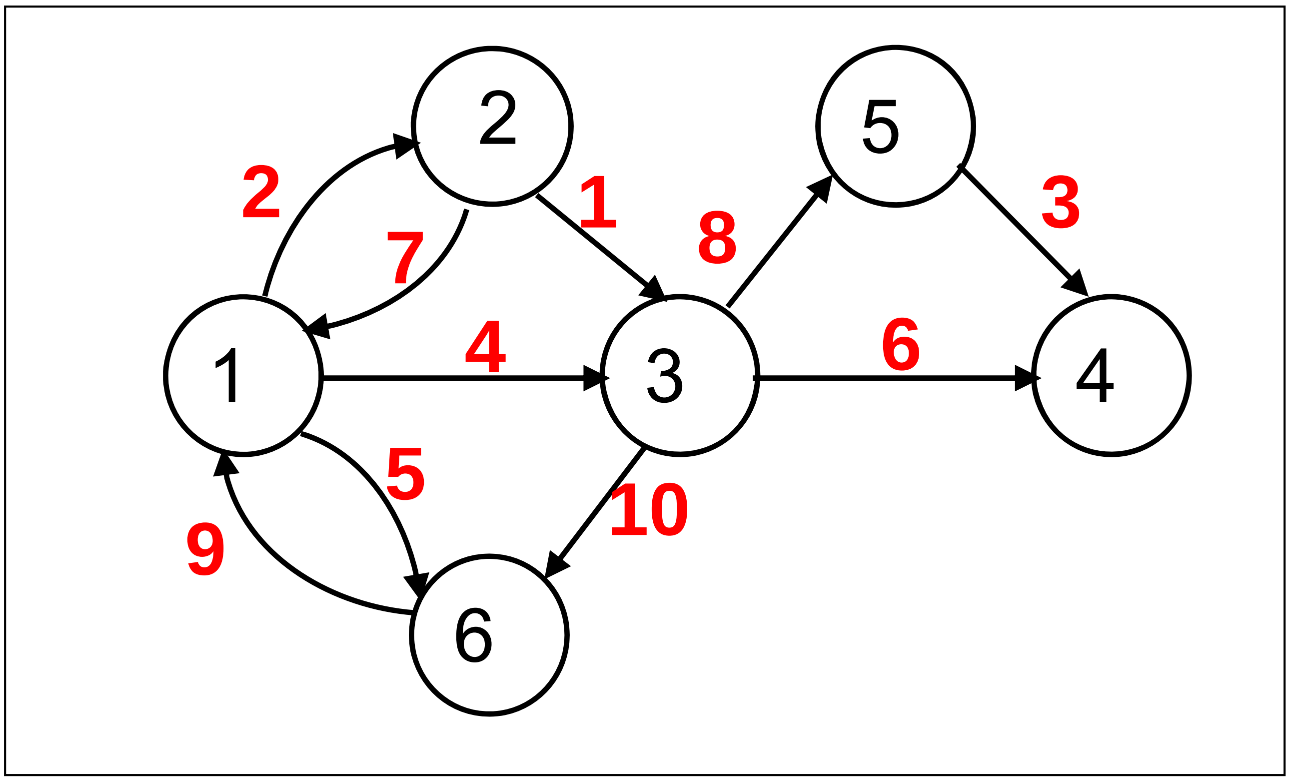
- A straightforward way data structure for a relation R on A would be an $n \times n$ array having 0's and 1's stored in each location.
  - E.g. If $A = \{1, 2\}$ and $R = \{(1, 1), (1, 2), (2, 2)\}$, then $\mathbf{M}_R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- These data could be represented by a two-dimensional array MAT, where MAT[1, 1] = 1, MAT[1, 2] = 1, MAT[2, 1] = 0, and MAT[2, 2] = 1.

4.6.2 Using Linked Lists

- The data can be represented by two arrays, TAIL and HEAD, giving the beginning vertex and end vertex, respectively, for all arrows.
- If we wish to make these edge data into a linked list, we will also need an array NEXT of pointers from each edge to the next edge.

4.6.2 Using Linked Lists (cont)

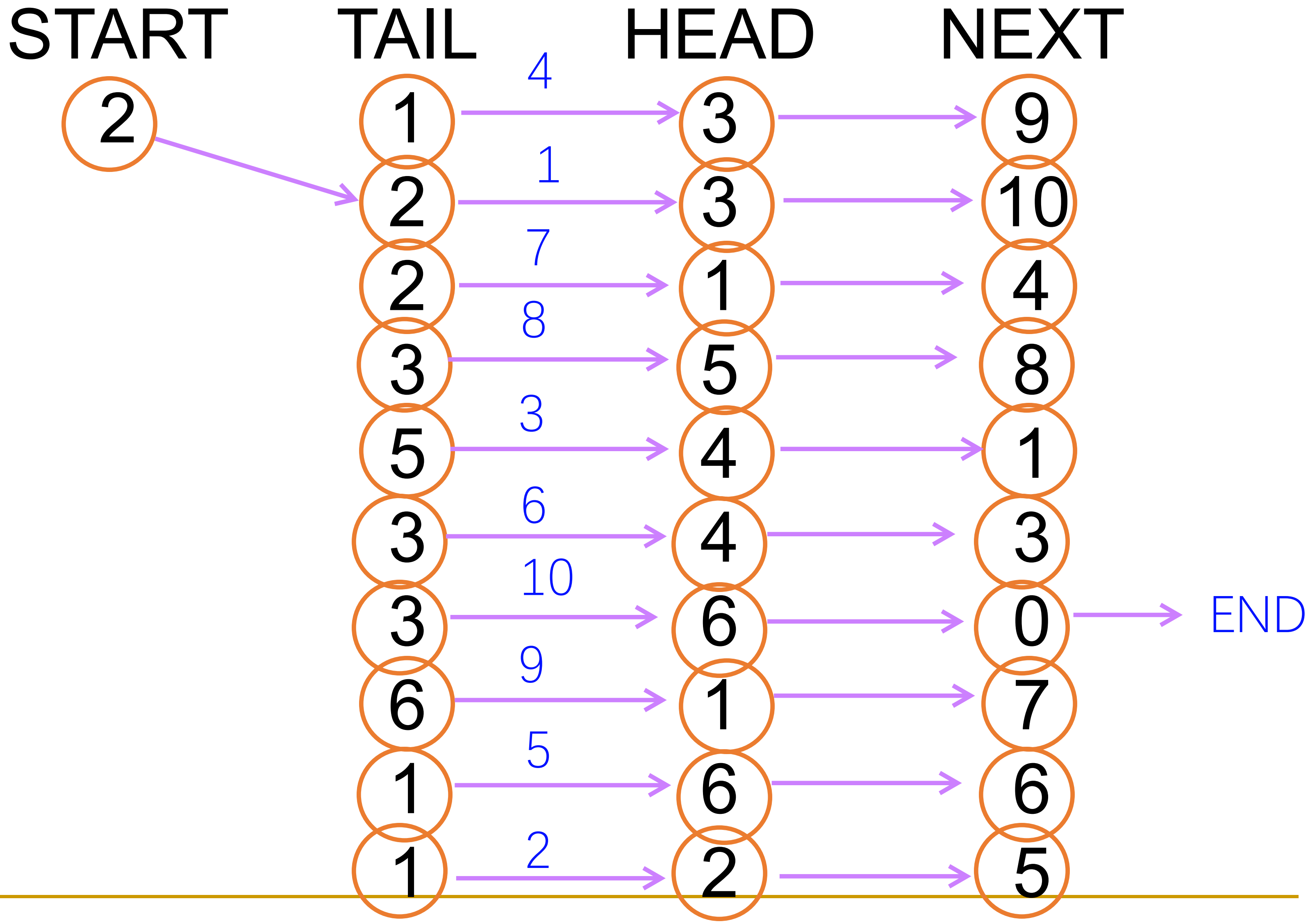
- E.g. Consider the relation whose digraph is shown below.



4.6.2 Using Linked Lists (cont)

- The vertices are the integers 1 through 6 and we arbitrarily number the edges as shown. If we wish to store the digraph in linked-list form so that the logical order coincides with the numbering of edges, we can use a scheme such as the following.

4.6.2 Using Linked Lists (cont)



4.6.2 Using Linked Lists (cont)

- This scheme and the numerous equivalent variations of it have important disadvantages. In many algorithms, it is efficient to locate a vertex and then immediately begin to investigate the edges that begin or end with this vertex. This is not possible in general with the storage mechanism shown above, so we now give a modification of it.
- Use an additional linear array VERT having one position for each vertex in the digraph.

4.6.2 Using Linked Lists (cont)

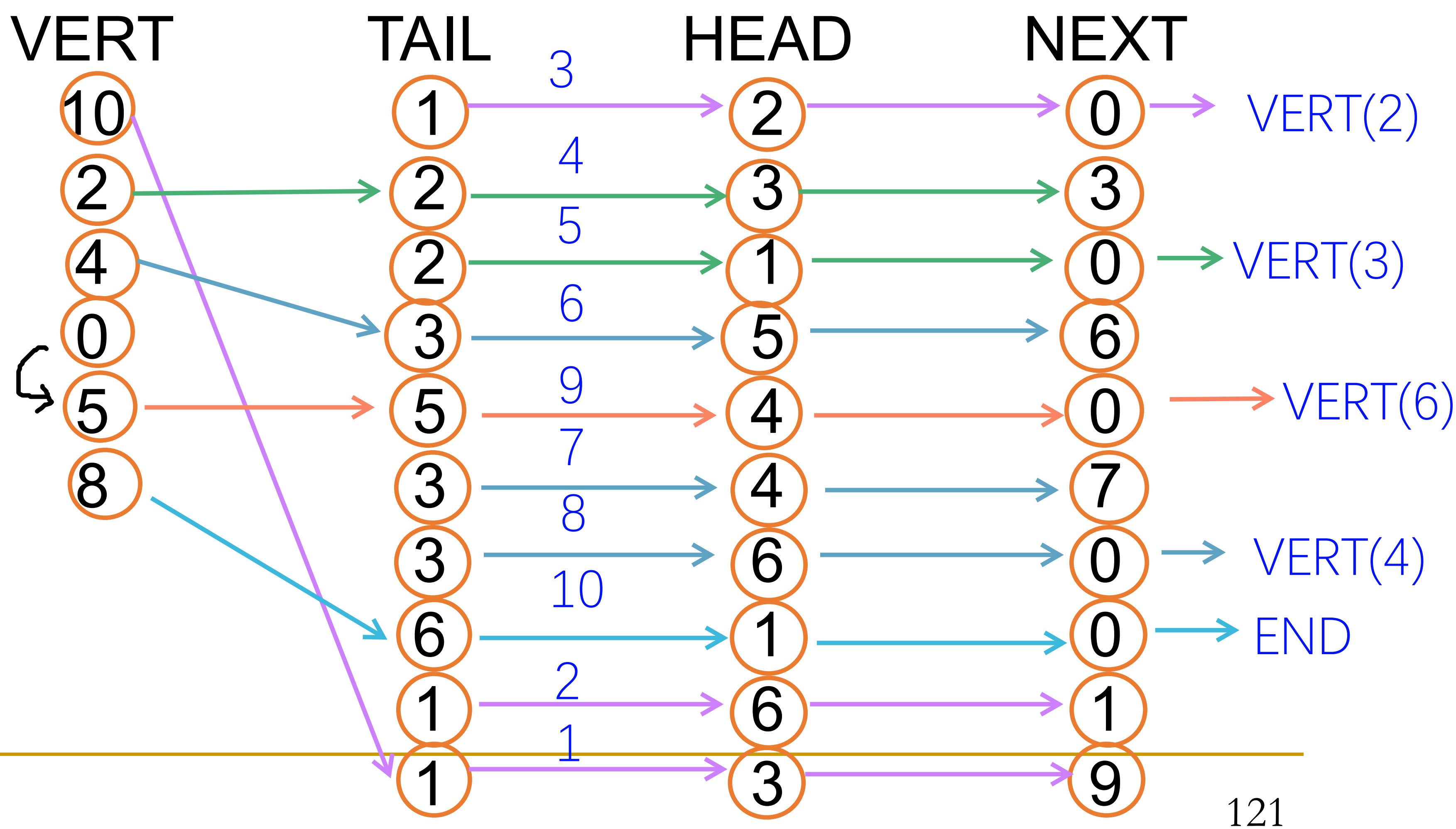
- For each vertex *I*, VERT[*I*] is the index, in TAIL and HEAD, of the first edge we wish to consider leaving vertex *I*. In the above digraph, the first edge could be taken to be the edge with the smallest number labelling it.
- Thus VERT, like NEXT, contains pointers to edges. For each vertex *I*, we must arrange the pointers in NEXT so that they link together all edges leaving *I*, starting with the edge pointed to by VERT[*I*]. The last of these edges is made to point to zero in each case.

4.6.2 Using Linked Lists (cont)

- In a sense, the data arrays TAIL and HEAD really contain several linked lists of edges, one list for each vertex.

4.6.2 Using Linked Lists (cont)

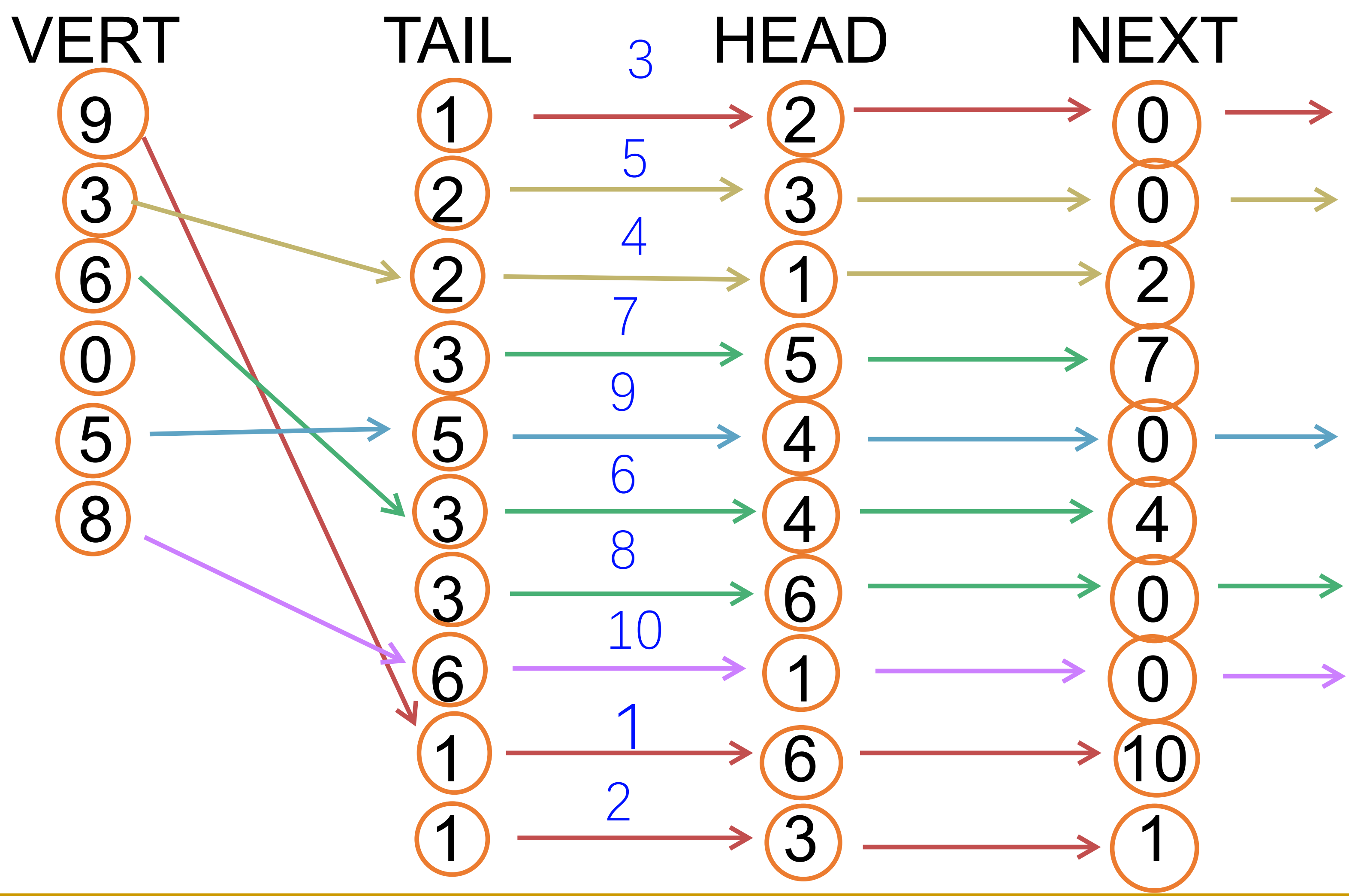
- This method is shown below for the previous digraph.



4.6.2 Using Linked Lists (cont)

- 0 indicates end of linked list.
- Figure below shows an alternative to describe the same digraph. The order of the edges leaving each vertex can be chosen arbitrarily.

4.6.2 Using Linked Lists (cont)



4.7 Operations on Relations

- The complement of R, $\overline{R}$, is referred to as the complementary relation, a relation from A to B that can be expressed as

  $a\ \overline{R}\ b$ if and only if $a \not R\ b$.

- The matrix of $\overline{R}$, $\mathbf{M}_{\overline{R}} = \overline{\mathbf{M}_R}$ is obtained from $\mathbf{M}_R$ by interchanging every 1 and 0 in $\mathbf{M}$.

4.7 Operations on Relations (cont)

- Let R and S be relations from a set A to a set B. The intersection $R \cap S$ means that $a\ R\ b$ and $a\ S\ b$.
  - $\mathbf{M}_{R \cap S} = \mathbf{M}_R \wedge \mathbf{M}_S$
- The union $R \cup S$ means that $a\ R\ b$ or $a\ S\ b$.
  - $\mathbf{M}_{R \cup S} = \mathbf{M}_R \vee \mathbf{M}_S$

4.7 Operations on Relations (cont)

- The inverse of R, R^{-1} is a relation from B to A (reverse order from R) defined by

  $b\ R^{-1}\ a$ if and only if $a\ R\ b$.

  - $(R^{-1})^{-1} = R$
  - Dom (R^{-1}) = Ran (R) and Ran (R^{-1}) = Dom (R)
  - $\mathbf{M}_{R^{-1}} = (\mathbf{M}_R)^T$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.7 Operations on Relations (cont)

- E.g. Let A = R. Let R be the relation $\leq$ on A and S be $\geq$.
  - Then the complement of R is $>$ and the complement of S is $<$.
  - $R^{-1} = S$ and $S^{-1} = R$.
  - $R \cap S$ is the relation of equality. $R \cup S$ is the universal relation in which any a is related to any b.

E.g.33

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.

Let $R = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a)\}$ and $S = \{(1, b), (2, c), (3, b), (4, b)\}$.

Compute

i. $\overline{R}$;

ii. $R \cap S$;

iii. $R \cup S$;

iv. R^{-1}.

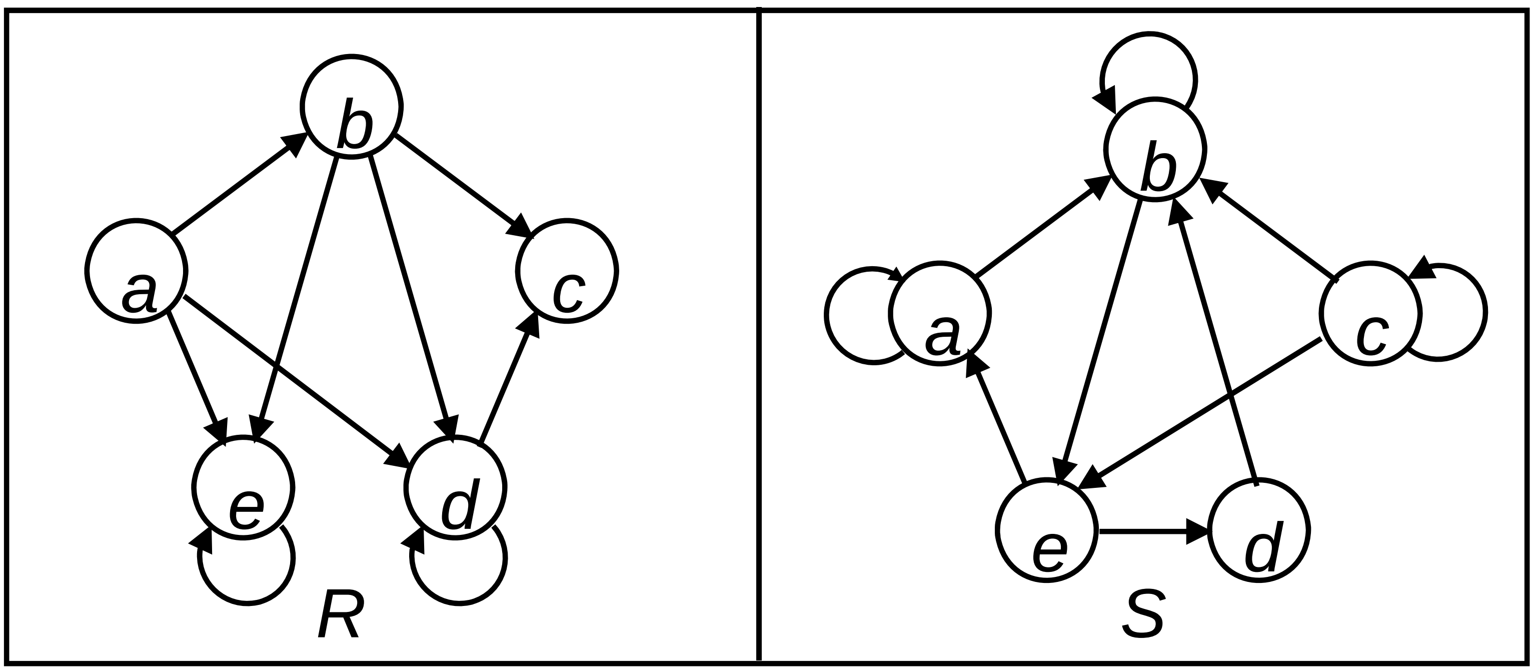
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$

$R = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a)\}$
$S = \{(1, b), (2, c), (3, b), (4, b)\}$

$$\bar{R} = \{(1, c), (2, a), (3, a), (3, c), (4, b), (4, c)\}$$
$$R \cap S = \{(1, b), (2, c), (3, b)\}$$
$$R \cup S = \{(1, a), (1, b), (2, b), (2, c), (3, b), (4, a), (4, b)\}$$
$$R^{-1} = \{(a, 1), (b, 1), (b, 2), (c, 2), (b, 3), (a, 4)\}$$

E.g.34

Let A = {a, b, c, d, e} and let R and S be two relations on A whose corresponding digraphs are shown below. Find $\overline{R}$, R^{-1}, and $R \cap S$.



$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

R = {(a, b), (a, d), (a, e), (b, c), (b, d), (b, e), (d, c), (d, d), (e, e)}

S = {(a, a), (a, b), (b, b), (b, e), (c, c), (c, b), (c, e), (d, b), (e, a), (e, d)}

$$\bar{R} = \{(a,a)\,,(a,c)\,,(b,a)\,,(b,b)\,,\\ (c,a)\,,(c,b)\,,(c,c)\,,(c,d,)\,,(c,e)\,,\\ (d,a)\,,(d,b)\,,(d,e)\,,(e,a)\,,(e,b)\,,(e,c)\,(e,d)\}$$

$$R^{-1} = \{(b,a)\,,(d,a)\,,(e,a)\,,(c,b)\,,(d,b)\,,(e,b)\,,(c,d)\,,(d,d)\,,(e,e)\}$$

$$R \cap S = \{(a,b)\,,(b,e)\}$$

E.g.35

Let A = {1, 2, 3} and let R and S be relations on A. Suppose that the matrices of R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $\mathbf{M}_{\overline{R}}$,$\mathbf{M}_{R^{-1}}$, $\mathbf{M}_{R \cap S}$ and $\mathbf{M}_{R \cup S}$.

$$\mathbf{M}_{\overline{R}} = \overline{\mathbf{M}}_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^{-1}} = (M_R)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} M_{R\cap S} &= M_R \wedge M_s \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} M_{R\cup S} &= M_R \vee M_s \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

4.7 Operations on Relations (cont)

- Theorem 1

  Suppose that R and S are relations from A to B.

  a) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

  b) If $R \subseteq S$, then $\overline{S} \subseteq \overline{R}$.

  c) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and

  $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

  d) $\overline{R \cap S} = \overline{R} \cup \overline{S}$ and $\overline{R \cup S} = \overline{R} \cap \overline{S}$.

4.7 Operations on Relations (cont)

- Theorem 2

  Let R and S be relations on A.

  a) If R is reflexive, so is R^{-1}.

  b) If R and S are reflexive, then so are $R \cap S$ and $R \cup S$.

  c) R is reflexive if and only if $\overline{R}$ is irreflexive.

4.7 Operations on Relations (cont)

- Theorem 3

  Let R be a relation on A. Then

  a) R is symmetric if and only if $R = R^{-1}$.

  b) R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta$. $= \{(a,a), (b,b), (c,c), \ldots\}$

  c) R is asymmetric if and only if $R \cap R^{-1} = \varnothing$.

4.7 Operations on Relations (cont)

- Theorem 4

  Let R and S be relations on A.

  a) If R is symmetric, so are R^{-1} and $\overline{R}$.

  b) If R and S are symmetric, so are $R \cap S$ and $R \cup S$.

4.7 Operations on Relations (cont)

- Theorem 5

  Let R and S be relations on A.

  a) $(R \cap S)^2 \subseteq R^2 \cap S^2$. $S^2 = S \times S = \{(x_1, x_2) : x_1, x_2 \in S\}$

  b) If R and S are transitive, so is $R \cap S$.

  c) If R and S are equivalence relations, so is $R \cap S$.

4.8.1 Reflexive Closures, Symmetric Closure

- A relation R on A may happen to be lack of some of the important relational properties such as reflexivity, symmetry and transitivity.
- If R does not possess a particular property, we may wish to add pairs to R until we get a relation that does have the require property.

4.8.1 Reflexive Closures, Symmetric Closure (cont)

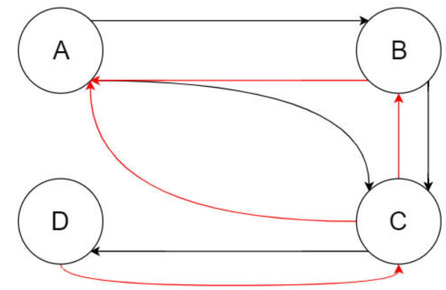
- We need to find the smallest relation R_1 on A that contains R and possesses the property we desire. Such R_1, if exists, is called the closure of R with respect to the property on the question.
- E.g.
  - Suppose R is a relation on A and is not reflexive. This can only occur because some pairs of the diagonal relation Δ are not in R. Thus $R_1 = R \cup \Delta$ is the smallest reflexive relation on A containing R; that is, the reflexive closure of R is $R \cup \Delta$.

4.8.1 Reflexive Closures, Symmetric Closure (cont)

- Suppose R is not symmetric. Then there must exist pairs (x, y) in R such that (y, x) is not in R. Since $(y, x) \in R^{-1}$, we must add all pairs from R^{-1}. So $R \cup R^{-1}$ is the smallest symmetric relation containing R; that is, $R \cup R^{-1}$ is the symmetric closure of R.

  The graph of the symmetric closure of R is simply the digraph of R with all edges made bidirectional.

E.g.36



If $A = \{a, b, c, d\}$ and

$R = \{(a, b), (a, c), (b, c), (c, d)\}$.

Find the symmetric closure of R.

$$R^{-1} = \{(b,a), (c,a), (c,b), (d,c)\}$$

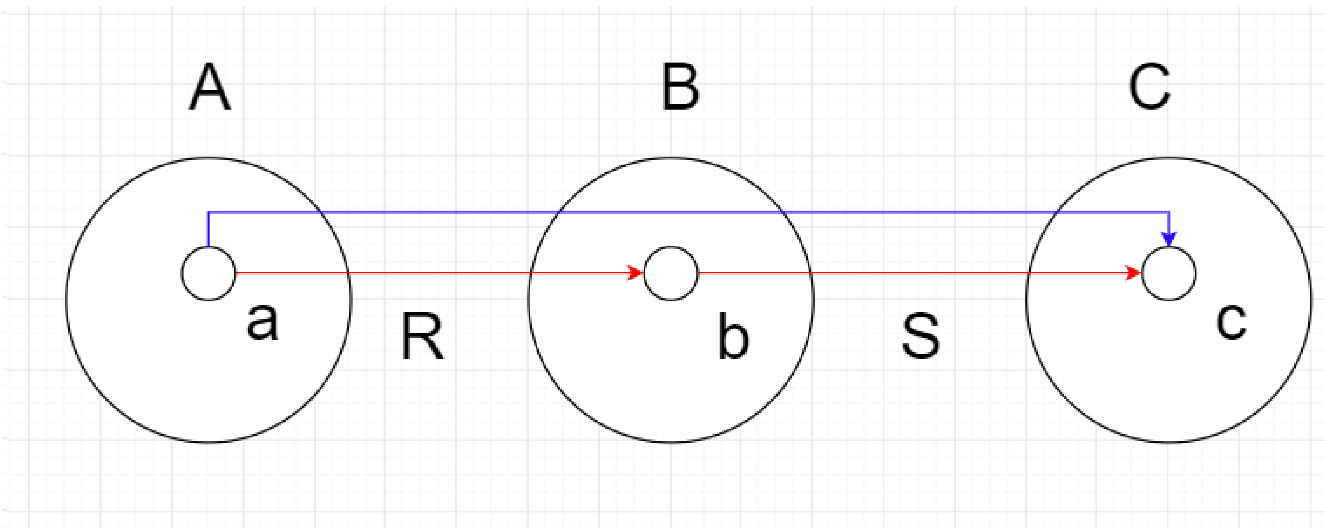
$$\begin{aligned} R \cup R^{-1} &= \{(a,b), (b,a), (c,a), (a,c), (b,c), (c,b), (c,d), (d,c)\} \\ &= \text{symmetric closure of R} \end{aligned}$$

4.8.2 Composition

- Suppose that *A*, *B*, and *C* are sets, *R* is a relation from *A* to *B*, and *S* is a relation from *B* to *C*. The composition of *R* and *S*, written $S \circ R$, is a relation from *A* to *C* and is defined as follows:
- If $a \in A$ and $c \in C$, then $a\ (S \circ R)\ c$ if and only if for some $b \in B$, we have $a\ R\ b$ and $b\ S\ c$.

$$(S \circ R)(a) = S(b) = c$$

$$"S(R(a))"$$



4.8.2 Composition (cont)

- If R and S have Boolean matrices $\mathbf{M}_R$ and $\mathbf{M}_S$ with respective size $n \times p$ and $p \times m$, thus $\mathbf{M}_R \odot \mathbf{M}_S = \mathbf{M}_{S \circ R}$.
  - The matrix method is much more reliable when the number of pairs in R and S is large.

4.8.2 Composition (cont)

- Theorem 6

  Let R be a relation from A to B and let S be a relation from B to C. Then, if A_1 is any subset of A, then we have

  $$(S \circ R)(A_1) = S(R(A_1)).$$

4.8.2 Composition (cont)

- Theorem 7

  Let A, B, C, and D be sets. R is a relation from A to B, S is a relation from B to C, and T a relation from C to D. Then

  $$T \circ (S \circ R) = (T \circ S) \circ R.$$

4.8.2 Composition (cont)

- Theorem 8

  Let A, B, and C be sets. R is a relation from A to B, and S is a relation from B to C. Then

  $$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

- In general, $S \circ R \neq R \circ S$.

E.g.37

$$\begin{tikzcd} a \arrow[r, "R"] \arrow[rr, bend right, "S \circ R"'] & b \arrow[r, "S"] & c \end{tikzcd}$$

Let $A = \{1, 2, 3, 4\}$,

$R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$, and

$S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$.

Find $S \circ R$.

$$S \circ R = \{(1,3), (1,4), (1,1), (2,1), (3,3)\}$$

E.g.38

Let $A = \{a, b, c\}$ and let R and S be relations on A whose matrices are

$$\mathbf{M}_R = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}, \quad \mathbf{M}_S = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Find $\mathbf{M}_{S \circ R}$.

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

E.g.39

Let $A = \{a, b\}$,

$R = \{(a, a), (b, a), (b, b)\}$, and

$S = \{(a, b), (b, a), (b, b)\}$. Show that

$S \circ R \neq R \circ S$.

LHS RHS

$$LHS = S \circ R = \{(a,b), (b,b), (b,a)\}$$

$$RHS = R \circ S = \{(a,a), (a,b), (b,a), (b,a)\}$$

$$S \circ R \neq R \circ S$$

Note: for the last one, can have 2 options, so both have to list down

4.8.3 Transitive Closure

- Theorem 1

  Let R be a relation on a set A. Then R^∞ is the transitive closure of R.

- From a geometric point of view, R^∞ is called the connectivity relation, since it specifies which vertices are connected (by paths) to other vertices.

4.8.3 Transitive Closure (cont)

- If we include the relation Δ, then $R^{\infty} \cup \Delta$ is the reachability relation R^*.
- Theorem 2

  Let A be a set with $|A| = n$, and let R be a relation on A. Then

  $$R^{\infty} = R \cup R^2 \cup \ldots \cup R^n.$$

E.g.40

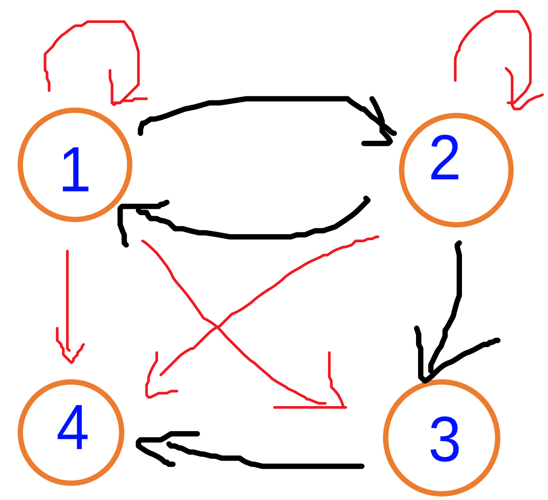
Let A = {1, 2, 3, 4}, and let

R = {(1, 2), (2, 3), (3, 4), (2, 1)}.

Find the transitive closure of R.

$$R^{\infty} = R \cup R^2 \cup R^3 \cup R^4$$

$$R^{\infty} = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$$



reds are add-ons to make it a "transitive closure"

4.9 Transitive Closure and Warshall's Algorithm

- Used to compute efficiently.
- Define $\mathbf{W}_0 = \mathbf{M}_R$, then we will have a sequence $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_n$ whose first term is $\mathbf{M}_R$ and last term is M_{R^∞} .

4.9 Transitive Closure and Warshall's Algorithm (cont)

- Procedure for computing $\mathbf{W}_k$ from $\mathbf{W}_{k-1}$:

  Step 1: Transfer to $\mathbf{W}_k$ all 1's in $\mathbf{W}_{k-1}$.

  Step 2: List the locations p_1, p_2, ..., in column k of $\mathbf{W}_{k-1}$, where the entry is 1, and the location q_1, q_2, ..., in row k of $\mathbf{W}_{k-1}$, where the entry is 1.

  Step 3: Put 1's in all positions p_i, q_j of $\mathbf{W}_k$ (if they are not already there).

E.g.41

Consider the relation R defined in E.g.40. Use Warshall's algorithm to compute the transitive closure of R.

Let $\mathbf{W}_0 = \mathbf{M}_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$C_1=2$
$R_1=2$
Add: (2,2)

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2=1,2$
$R_2=1,2,3$
Add: (1,1), (1,2), (1,3), (2,1), (2,2), (2,3)

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

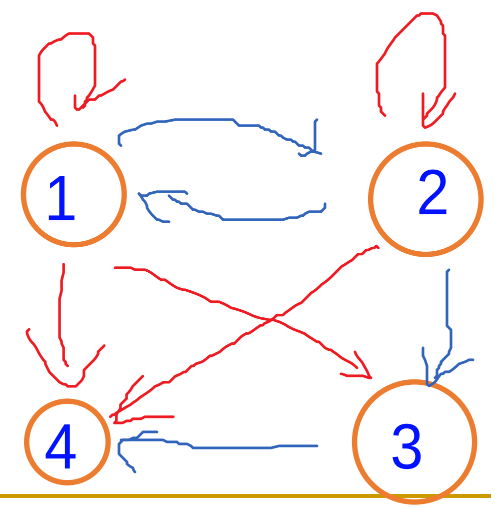
$C_3=1,2$
$R_3=4$
Add: (1,4), (2,4)

$$\Rightarrow \mathbf{W}_3 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

C_4=1,2,3
R_4= Nothing
Add: Nothing

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_{R^\infty} \text{ (Transitive closure)}$$

$\therefore$ The transitive closure of R is R^∞ :



4.9 Transitive Closure and Warshall's Algorithm (cont)

- Theorem 3

  If R and S are equivalence relations on A, then the smallest equivalence relation containing both R and S is $(R \cup S)^{\infty}$.

$$M_{R \cup S} = M_{(R \cup S)^{\infty}}$$

E.g.42

Let $A = \{1, 2, 3, 4, 5\}$,

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$, and

$S = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$.

Find the smallest equivalence relation containing R and S, and compute the partition of A that it produces.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Note: a lot below are same, no need to keep redraw

$$M_{R \cup S} = M_R \vee R_S$$

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

C_1=1,2
R_1=1,2

Add: (1,1), (1,2), (2,1), (2,2)
(P.S.: Add only if entry is 0

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

C_2=1,2
R_2=1,2

Add: (1,1), (1,2), (2,1), (2,2)

Column 2 = 1, 2

Row 2 = 1, 2

$\Rightarrow$ Add (1, 1), (1, 2), (2, 1), (2, 2) $\Rightarrow$ $\mathbf{W_2 = W_1}$

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

C_3=3,4

R_3=3,4

Add ….

W_3=W_2

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

C_4=3,4,5

R_4=3,4,5

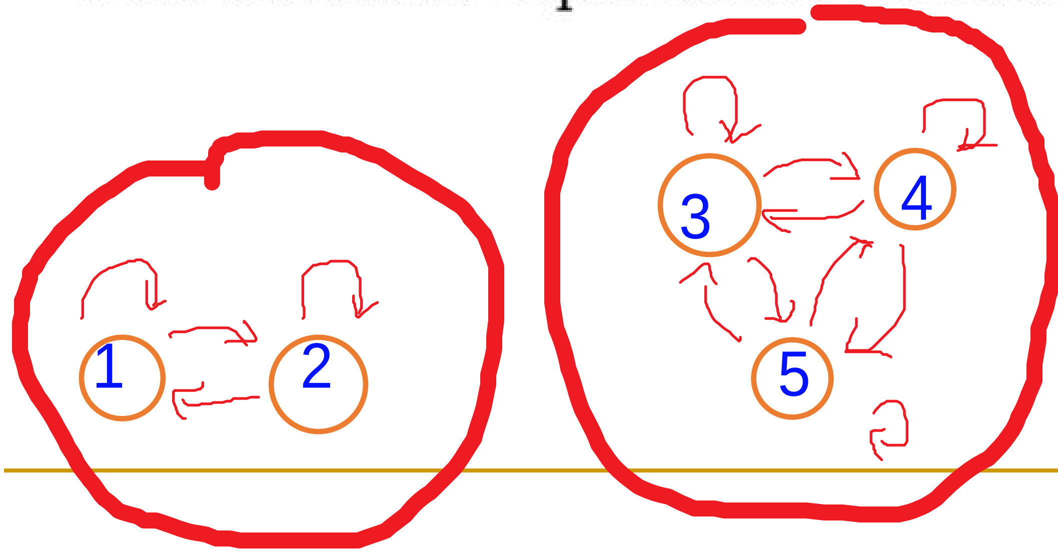
Add …

$$\Rightarrow \mathbf{W_4} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

C_5=3,4,5
R_5=3,4,5

Add: (3,3) ...

$$W_5 = W_4 = M_{(R \cup S)^\infty}$$
$$= \text{The smallest equivalence relation containing } R \text{ and } S$$



$$A|_{(R \cup S)^\infty} = \{\{1,2\},\{3,4,5\}\}$$