Calculus L2 - LIMITS AND CONTINUITY

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1 Example 2.1

1.

(a) $\lim_{x\to 2} 5 = 5$

(b) $\lim_{x\to 2}(4x-5)$

$$\begin{aligned}\lim_{x\to 2}(4x-5) &= 4(2)-5\\ &= 3\end{aligned}$$

(c) $\lim_{x\to 0}\left(2x^2+3x-5\right)$

$$\lim_{x\to 0}\left(2x^2+3x-5\right) = -5$$

(d) $\lim_{x\to 0}(x+1)(x-1)$

$$\lim_{x\to 0}(x+1)(x-1) = -1$$

(e) $\lim_{x\to 2}\left(\frac{x-1}{x^2-1}\right)$

$$\boldsymbol{\lim_{x\to 2}\left(\frac{x-1}{x^2-1}\right) = \frac{1}{3}}$$

(f) $\lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right)$

$$\begin{aligned}\lim_{x\to 2}\left(\frac{x-1}{x^2-1}\right) &= \lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right)\\ &= \frac{0}{0}\,(indeterminant)\end{aligned}$$

$$\begin{aligned}\lim_{x\to 2}\left(\frac{x-1}{x^2-1}\right) &= \lim_{x\to 1}\left(\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}\right)\\ &= \lim_{x\to 1}\left(\frac{1}{(x+1)}\right)\end{aligned}$$

$$\boldsymbol{\lim_{x\to 2}\left(\frac{x-1}{x^2-1}\right) = \frac{1}{2}}$$

(g) $\lim_{x\to 3}\left(\frac{x^2-x-6`}{x-3}\right)$

$$\lim_{x\to 3}\left(\frac{x^2-x-6`}{x-3}\right)=\frac{0}{0}\,(indeterminant)$$

$$\begin{aligned}\lim_{x\to 3}\left(\frac{x^2-x-6`}{x-3}\right)&=\lim_{x\to 3}\left(\frac{\cancel{(x-3)}\,(x+2)}{\cancel{x-3}}\right)\\&=\lim_{x\to 3}(x+2)\\&=5\end{aligned}$$

(h) $\lim_{x\to -1}\left(\frac{x^2+4x+3}{x+1}\right)$

$$\lim_{x\to -1}\left(\frac{x^2+4x+3}{x+1}\right)=\frac{0}{0}\,(indeterminate)$$

$$\begin{aligned}\lim_{x\to -1}\left(\frac{x^2+4x+3}{x+1}\right)&=\lim_{x\to -1}\left(\frac{(x+3)\,\cancel{(x+1)}}{\cancel{x+1}}\right)\\&=-1+3\\&=2\end{aligned}$$

(i) $\lim_{x\to 8}\left(\sqrt{x+1}\right)$

$$\begin{aligned}\lim_{x\to 8}\left(\sqrt{x+1}\right)&=\sqrt{8+1}\\&=\sqrt{9}\\&=3\end{aligned}$$

(j) $\lim_{x\to 1}\left(\frac{7+6x}{3x-2}\right)^4$

$$\begin{aligned}\lim_{x\to 1}\left(\frac{7+6x}{3x-2}\right)^4&=\left(\frac{7+6\,(1)}{3\,(1)-2}\right)^4\\&=\left(\frac{13}{1}\right)^4\\&=28561\end{aligned}$$

(k) $\lim_{x \to 0} \left(\frac{x}{1-\sqrt{1+x}} \right)$

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{x}{1-\sqrt{1+x}} \right) &= \lim_{x \to 0} \left(\frac{x}{1-\sqrt{1+x}} \right) \cdot \left(\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}} \right) \\
&= \lim_{x \to 0} \left(\frac{x\left(1+\sqrt{1+x}\right)}{1^2-(1+x)} \right) \\
&= \lim_{x \to 0} \left(\frac{\cancel{x}\left(1+\sqrt{1+x}\right)}{-\cancel{x}} \right) \\
&= \lim_{x \to 0} \left(-\left(1+\sqrt{1+x}\right) \right) \\
&= -2
\end{aligned}$$

(l) $\lim_{x \to 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right)$

$$\begin{aligned}
\lim_{x \to 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right) &= \lim_{x \to 0} \left(\frac{\sqrt{1+x^2}-1}{x^2} \right) * \left(\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} \right) \\
&= \lim_{x \to 0} \left(\frac{1+x^2-1^2}{x^2\left(\sqrt{1+x^2}+1\right)} \right) \\
&= \lim_{x \to 0} \left(\frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{1+x^2}+1\right)} \right) \\
&= \lim_{x \to 0} \left(\frac{1}{\sqrt{1+x^2}+1} \right) \\
&= \frac{1}{\sqrt{1+0^2}+1} \\
&= \frac{1}{2}
\end{aligned}$$

2 Example 2.2

Since $\lim_{x \to 0} (u(x))$is between the functions

$$\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) \leq \lim_{x \to 0} u(x) \leq \lim_{x \to 0} \left(1 + \frac{x^2}{2}\right)$$
$$1 \leq u(x) \leq 1$$

Since $u(x)$is between the two functions, **by squeeze theorem**, $\lim_{x \to 0} (u(x)) = 1$

3 Example 2.3

For the limit to exist at $x = 0$,

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} (f(x))$$

$$\begin{aligned}\lim_{x\to0^-} (f(x)) &= \lim_{x\to0^-} (0) \\ &= 0\end{aligned}$$

$$\lim_{x\to0^+} (f(x)) = \lim_{x\to0^+} (1)$$

$\therefore$ Since $\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} (f(x))$,the limit $\lim_{x\to0} (f(x))$does not exist

4 Example 2.4

4.1 (a) $\lim_{x\to0} (f(x))$

$$\begin{aligned}\lim_{x\to0^-} (f(x)) &= \lim_{x\to0^-} (x+2) \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to0^+} (f(x)) &= \lim_{x\to0^+} (x+2) \\ &= 2\end{aligned}$$

$\therefore$ Since $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} (f(x))$,the limit $\lim_{x\to0} (f(x))$is 2

4.2 (b) $\lim_{x\to1} (f(x))$

$$\begin{aligned}\lim_{x\to1^-} (f(x)) &= \lim_{x\to1^-} (x+2) \\ &= 1+2 \\ &= 3\end{aligned}$$

$$\begin{aligned}\lim_{x\to1^+} (f(x)) &= \lim_{x\to1^+} \left(2x^2\right) \\ &= 2(1)^2 \\ &= 2\end{aligned}$$

$\therefore$ Since $\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} (f(x))$,the limit $\lim_{x\to0} (f(x))$does not exist

5 Example 2.5

To prove that the limit does not exist, $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} (f(x))$, where a is a number

1. Form the piecewise defined function (optional, just to make it easier)

$$f(x) = \begin{cases} \frac{-(x)}{x} = -1 & x < 0 \\ \frac{x}{x} = 1 & x \geq 0 \end{cases}$$

2. Prove that $\lim_{x\to 0} \left(\frac{|x|}{x}\right)$D.N.E.

$$\lim_{x\to 0^-} \left(\frac{|x|}{x}\right) = \lim_{x\to 0^-} (-1)$$
$$= -1$$

$$\lim_{x\to 0^+} \left(\frac{|x|}{x}\right) = \lim_{x\to 0^+} (1)$$
$$= 1$$

3. Draw inference
$\therefore$ Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} (f(x))$,the limit $\lim_{x\to 0} (f(x))$does not exist

6 Example 2.6

Note: $D.N.E$ means does not exist.

6.1 $\lim_{x\to 2^-} (g(x)) = 3$

6.2 $\lim_{x\to 2^+} (g(x)) = 1$

6.3 $\lim_{x\to 2} (g(x)) = D.N.E.$

6.4 $\lim_{x\to 5^-} (g(x)) = 2$

6.5 $\lim_{x\to 5^+} (g(x)) = 2$

6.6 $\lim_{x\to 5} (g(x)) = 2$

6.7 $g(5) = 1$

7 Example 2.7

Answer already given

8 Example 2.8

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ cos\, x & x > 0 \end{cases}$$

Determine continuity

1. $x = -1$

   (a) Condition: $f(-1)$ must exist

   $$\begin{aligned} f(-1) &= (-1)^2 + 1 \\ &= 2 \end{aligned}$$

   (b) Condition 2: $\lim_{x \to -1} (f(x))$must exist

   $$\begin{aligned} \lim_{x \to -1} \left(x^2 + 1\right) &= (-1)^2 + 1 \\ &= 2 \end{aligned}$$

   (c) Condition 3: $\lim_{x \to -1} (f(x)) = f(-1)$

   (d) Since all the three conditions are satisfied, the function is continuous at $x = -1$

2. $x = 0$

   (a) Condition: $f(0)$ must exist

   $$f(0) = undefined$$

   (b) Since the first condition test failed, the function is not continuous at $x = 0$

9 Example 2.9

1. $f(x) = \begin{cases} x^2 + 1 & x < 1 \\ x^3 - x + 2 & x \geq 1 \end{cases}$

   (a) Condition 1: Check if $f(1)$ exist

   $$\begin{aligned} f(1) &= (1)^3 - (1) + 2 \\ &= 1 - 1 + 2 \\ &= 2 \end{aligned}$$

(b) Condition 2: Check if $\lim_{x\to 1} f(x)$exist

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= (1) + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= 1 - 1 + 2 \\ &= 2\end{aligned}$$

i. $\therefore$ Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} (f(x))$,the limit $\lim_{x\to 1} (f(x)) = 2$.

(c) Condition 3: $\lim_{x\to 1} f(x) = f(1)$

i. PASS

(d) Inference

i. $f(x)$ is continuous at $x = 1$

2. $f(x) = \begin{cases} x^2 + 1 & x < 1 \\ 4 & x = 1 \\ x^3 - x + 2 & x > 1 \end{cases}$

(a) Condition 1: Check if $f(1)$ exist

$$f(1) = 4$$

(b) Condition 2: Check if $\lim_{x\to 1} f(x)$exist

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= (1)^2 + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= 1 - 1 + 2 \\ &= 2\end{aligned}$$

i. $\therefore$ Since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} (f(x))$,the limit $\lim_{x\to 1} (f(x)) = 2$.

(c) Condition 3: $\lim_{x\to 1} f(x) = f(1)$

i. FAIL.

$$\lim_{x\to 1} f(x) \neq f(1)$$

(d) Inference

i. $f(x)$ is NOT continuous at $x = 1$

10 Example 2.10

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 4 & x = 3 \end{cases}$$

Determine whether the $f(x)$ is continuous at $x = 3$.

1. Condition 1: $f(3)$ must exist

$$f(3) = 4$$

2. Condition 2: $\lim_{x\to 3}(f(x))$must exist

$$\begin{aligned} \lim_{x\to 3}(f(x)) &= \lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) \\ &= \frac{0}{0}\,(indeterminate) \\ &= \lim_{x\to 3}\left(\frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}\right) \\ &= \lim_{x\to 3}((x+3)) \end{aligned}$$

$$\boldsymbol{\lim_{x\to 3}(f(x)) = 6}$$

3. Condition 3: $\lim_{x\to 3}(f(x) = f(3))$

$$\lim_{x\to 3}(f(x) \neq f(3))$$

4. Inference

   (a) $f(x)$ is NOT continuous at $x = 3$

11 Example 2.11

$$g(x) = \begin{cases} x^2 + ax - 3 & x < 1 \\ 0 & x = 1 \\ 3x + b & x > 1 \end{cases}$$

continuous at x = 1.

1. Since the function is continuous at $x = 1$:

   (a) $f(1) = 0$

   (b) $\lim_{x\to 1} f(x) = f(1) = 0$

$$\begin{aligned}
\lim_{x\to 1^-}(f(x)) &= f(1) \\
(1)^2 + a(1) - 3 &= 0 \\
a - 2 &= 0 \\
\boldsymbol{a} &\boldsymbol{= 2} \\
\lim_{x\to 1^+}(f(x)) &= f(1) \\
3(1) + b &= 0 \\
\boldsymbol{b} &\boldsymbol{= -3}
\end{aligned}$$

12 Example 2.12

Find $\lim_{x\to 3^+}\left(\frac{2x}{x-3}\right)$ and $\lim_{x\to \mathbf{3}^-}\left(\frac{2x}{x-3}\right)$.

Note: There's an errata in the questions: the bolded part is the correction.

$$\begin{aligned}
\lim_{x\to 3^+}\left(\frac{2x}{x-3}\right) &= \frac{2(3)}{3^+ - 3}, 3^+ - 3 \to 0 \\
&= \infty \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 3^+}\left(\frac{2x}{x-3}\right) &= \frac{2(3)}{3^- - 3}, 3^- - 3 \to 0^- \\
&= -\infty
\end{aligned}$$

13 Example 2.13

Find $\lim_{x\to -3}\left(\frac{1}{(x+3)^2}\right)$

$$\begin{aligned}
\lim_{x\to -3^+}(x+3)^2 &\to \left(0^-\right)^2 \\
&\to 0^+
\end{aligned}$$

$$\lim_{x\to -3^-}(x+3)^2 \to \left(0^+\right)^2 \to 0^+$$

$$\begin{aligned}
\lim_{x\to -3^+}\left(\frac{1}{(x+3)^2}\right) &= \frac{1}{0^+} \\
&= \infty
\end{aligned}$$

$$\lim_{x \to -3^-} \left(\frac{1}{(x+3)^2} \right) = \frac{1}{0^+}$$
$$= \infty$$

$$\lim_{x \to -3} \left(\frac{1}{(x+3)^2} = \infty \right)$$

14 Example 2.14

1. $x = 3$
2. $x = \frac{\pi}{2}$
3. $x = 0$

15 Example 2.15

1.
   (a) $\lim_{x \to \infty} \left(\frac{1}{x^8} \right)$

$$\lim_{x \to \infty} \left(\frac{1}{x^8} \right) = \frac{1}{\infty^8}$$
$$= 0$$

   (b) $\lim_{x \to -\infty} \left(\frac{500}{x^3} \right)$

$$\frac{500}{(-\infty)^3} = 0$$

   (c) $\lim_{x \to \infty} \left(\left(2x^2 + 3x - 5 \right) \right)$

$$2(\infty)^2 + 3(\infty) - 5 = \infty$$

   (d) $\lim_{x \to \infty} \left(\frac{3x+2}{4x+9} \right)$

$$\lim_{x \to \infty} \left(\frac{3x+2}{4x+9} \right) = \frac{\infty}{\infty} \, (undefined)$$
$$= \lim_{x \to \infty} \left(\frac{3 + \frac{2}{x}}{4 + \frac{9}{x}} \right) \text{ (divide all by } x\text{)}$$
$$= \frac{3}{4}$$

(e) $\lim_{x\to\infty}\left(\frac{x}{x^2-1}\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x}{x^2-1}\right) &= \lim_{x\to\infty}\left(\frac{1}{x-\frac{1}{x}}\right) \text{(divide all by } x\text{)}\\ &= \frac{1}{\infty}\\ &= 0\end{aligned}$$

(f) $\lim_{x\to\infty}\left(\frac{x^3-8x+1}{5x^3-1}\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x^3-8x+1}{5x^3-1}\right) &= \lim_{x\to\infty}\left(\frac{1-\frac{8}{x^2}+\frac{1}{x^3}}{5-\frac{1}{x^3}}\right) \text{(divide all by } x^3\text{)}\\ &= \frac{1}{5}\end{aligned}$$

(g) $\lim_{x\to\infty}\left(\frac{x^2-6}{x-3}\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x^2-6}{x-3}\right) &= \lim_{x\to\infty}\left(\frac{x-\frac{6}{x}}{1-\frac{3}{x}}\right) \text{(divide all by } x\text{)}\\ &= \infty\end{aligned}$$

(h) $\lim_{x\to\infty}\left(\frac{1+x^2}{1-x^2}\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{1+x^2}{1-x^2}\right) &= \lim_{x\to\infty}\left(\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}-1}\right) \text{(divide all by } x^2\text{)}\\ &= 1\end{aligned}$$

16 Notes & Exercise

16.1 Note

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.5937
100	2.704814
1000	2.7169
.	.
.	.
100,000,000	2.718282...
100,000,000,000	2.718282...
100,000,000,000,000	2.718282...

$$\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^x\right) = e\,(Euler's\,number)$$

To remember:

1. Count the number of characters in each word
   (a) To
   (b) express
   (c) e
   (d) remember
   (e) to
   (f) memorize
   (g) a
   (h) sentence
   (i) to
   (j) memorize
   (k) this
2. Or, you can remember that after 2.7, 1828 appears twice:
   (a) 2.7 1828 1828
   (b) After that are just the digits of the angles $45°, 90°, 45°$(no real reason)
      i. 2.7 1828 1828 45 90 45
3. e's slope is its value.
4. e is defined as the value at **Maximum**, **Continuous compounding**, **100% growth**, **one time period**($e \cdot 1$).

16.2 Notes

1. $$\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}=1\right)$$
   $y = 1$ is a horizontal asymtote.
2. $\lim_{x\to\infty}\left(\left(\frac{1}{x}-2\right)=-2\right)$
   (a) $y = -2$ is a horizontal asymtote.

17 Example

Since $f(x)$ is continous at $[1, 2]$

$$\begin{aligned} f(1) &= 4 - 6 + 3 - 2 \\ &= \mathbf{-1 < 0} \end{aligned}$$

$$\begin{aligned} f(2) &= 4(2)^3 - 6(2)^2 + 3(2) - 2 \\ &= \mathbf{12 > 0} \end{aligned}$$

According to the intermediate value theorem, since $f(x)$ is continuous between $[1, 2]$, and $f(1) < 0 < f(2)$. There exists a number c between 1 and 2 such that $f(c) = 0$