Calc 2: Tutorial 5

November 15, 2019

1. Solve the following differential equations: (CHECK WITH LECTURER)

   (a) $x^2 \frac{dy}{dx} + xy = x + 1$

      i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned} x^2 \frac{dy}{dx} + xy &= x + 1 \\ \frac{1}{x^2}\left[x^2 \frac{dy}{dx} + xy\right] &= \frac{1}{x^2}\left[x+1\right] \\ \boldsymbol{\frac{dy}{dx} + \frac{y}{x}} &\boldsymbol{= \frac{1}{x} + \frac{1}{x^2}, p\left(x\right) = \frac{1}{x}} \end{aligned}$$

      ii. Find the integrating factor $\mu\left(x\right) = e^{\int p(x)dx}$.

$$\begin{aligned} \mu\left(x\right) &= e^{\int \frac{1}{x} dx} \\ &= e^{\ln x} \\ \boldsymbol{\mu\left(x\right)} &\boldsymbol{= x} \end{aligned}$$

      iii. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\begin{aligned} x\left[\frac{dy}{dx} + \frac{y}{x}\right] &= x\left[\frac{1}{x} + \frac{1}{x^2}\right] \\ x\frac{dy}{dx} + y &= 1 + \frac{1}{x} \\ \frac{d}{dx}\left[yx\right] &= 1 + \frac{1}{x} \\ \int \frac{d}{dx}\left[yx\right]dx &= \int 1 + \frac{1}{x}dx \\ yx &= x + \ln x + c \\ y &= 1 + \frac{\ln x + c}{x} \end{aligned}$$

   (b) $\frac{dy}{dx} + y \cot x = \csc x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

A. Already in correct standard form, $p(x) = \cot x$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\begin{aligned}\mu(x) &= e^{\int \cot x \, dx} \\ &= e^{\ln|\sin x|} \\ &= \sin x\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x) y]$ and write it as such.

$$\begin{aligned}\sin x \left[\frac{dy}{dx} + y \cot x\right] &= \sin x [\csc x] \\ \sin x \frac{dy}{dx} + y \sin x \cdot \frac{\cos x}{\sin x} &= \sin x \left[\frac{1}{\sin x}\right] \\ \sin x \frac{dy}{dx} + y \cos x &= 1 \\ \sin x \frac{dy}{dx} + y \cos x &= 1 \\ \frac{d}{dx}[y \sin x] &= 1 \\ \int \frac{d}{dx}[y \sin x] \, dx &= \int 1 dx \\ y \sin x + c &= x + c \\ y &= \frac{x + c}{\sin x}\end{aligned}$$

(c) $x\frac{dy}{dx} = y - x^2 e^{-x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} - x^2 e^{-x} \\ \frac{dy}{dx} - \frac{y}{x} &= -x^2 e^{-x}\end{aligned}$$

A. From the above, $p(x) = -\frac{1}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}\mu(x) &= e^{\int -\frac{1}{x} dx} \\ &= e^{-\int \frac{1}{x} dx} \\ &= e^{-\ln x} \\ \mu(x) &= e^{\ln x^{-1}} \\ &= x^{-1} \\ &= \frac{1}{x}\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\, y]$ and write it as such.

$$\begin{aligned}
\frac{1}{x}\left(\frac{dy}{dx}-\frac{y}{x}\right) &= \frac{1}{x}\left(-x^2e^{-x}\right)\\
-\left(-x^{-2}y+x^{-1}\frac{dy}{dx}\right) &= -xe^{-x}\\
-\frac{dy}{dx}\left(x^{-1}y\right) &= -xe^{-x}\\
-\int\frac{dy}{dx}\left(x^{-1}y\right)dx &= \int -xe^{-x}dx\\
-x^{-2}y &= -\int xe^{-x}dx
\end{aligned}$$

iv. Before we can proceed, again, integrate by parts, if you don't remember, GG.com...jk,

$$\int u\,dv = uv-\int vdu$$

A. Let $u=x$, $dv=e^{-x}$

$$\begin{aligned}
&\frac{du}{dx}=1\\
&du=dx
\end{aligned}$$

$$\begin{aligned}
\int dv &= \int e^{-x}\\
v &= -e^{-x}
\end{aligned}$$

B. Lets find the answer

$$\int u\,dv = -x\cdot e^{-x}-\int e^{-x}dx$$

C. Again, we need to deal with integration, now use u substitution
Let $u=-x$, $du=-dx$

$$\begin{aligned}
-\int e^{-x}dx &= -\int e^{u}du\\
&= -e^{u}\\
-\int e^{-x}dx &= -e^{-x}+c
\end{aligned}$$

v. Substitute back in

$$\begin{aligned}
-x^{-1}y &= \int xe^{-x}dx \\
&= -e^{-x} + c \\
y &= -e^{-x} \cdot -x - cx \\
&= xe^{-x} - cx \\
y &= xe^{-x} + cx \text{ Note: c is constant, sign doesn't matter for now}
\end{aligned}$$

(d) $x\frac{dy}{dx} + 2y = \frac{\sin x}{x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int \frac{2}{x}dx} \\
&= e^{2\ln x} \\
\mu(x) &= x^2
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
x^2\left[\frac{dy}{dx} + \frac{2y}{x}\right] &= x^2\left[\frac{\sin x}{x^2}\right] \\
x^2\frac{dy}{dx} + 2xy &= \sin x \\
\frac{d}{dx}\left[x^2y\right] &= \sin x \\
\int \frac{d}{dx}\left[x^2y\right]dx &= \int \sin x\, dx + c \\
x^2y &= -\cos x + c \\
y &= \frac{c - \cos x}{x^2}
\end{aligned}$$

(e) $x\frac{dy}{dx} + 3y = 4x + 3$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
\frac{1}{x}\left[x\frac{dy}{dx} + 3y\right] &= \frac{1}{x}[4x + 3] \\
\frac{dy}{dx} + \frac{3y}{x} &= 4 + \frac{3}{x}
\end{aligned}$$

A. From above $p(x) = \frac{3}{x}$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int \frac{3}{x} dx} \\
&= e^{3\int \frac{1}{x} dx} \\
&= e^{3\ln x} \\
&= e^{\ln x^3} \\
\mu(x) &= x^3
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)y]$ and write it as such.

$$\begin{aligned}
x^3\left[\frac{dy}{dx} + \frac{3y}{x}\right] &= x^3\left[4 + \frac{3}{x}\right] \\
x^3\frac{dy}{dx} + 3x^2y &= 4x^3 + 3x^2 \\
\frac{d}{dx}\left[x^3y\right] &= 4x^3 + 3x^2 \\
\int \frac{d}{dx}\left[x^3y\right] dx &= \int \left(4x^3 + 3x^2\right) dx \\
x^3y &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\
x^3y &= x^4 + x^3 + c \\
y &= \frac{x^4}{x^3} + \frac{x^3}{x^3} + \frac{c}{x^3} \\
y &= x + 1 + \frac{c}{x^3}
\end{aligned}$$

(f) $\frac{dy}{dx} = 2y + e^{3x}$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
\frac{dy}{dx} &= 2y + e^{3x} \\
\frac{dy}{dx} - 2y &= e^{3x}, p(x) = -2
\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int -2dx} \\
&= e^{\int -2dx} \\
&= e^{-\int 2dx} \\
\mu(x) &= e^{-2x}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}[\mu(x)\,y]$ and write it as such.

$$\begin{aligned}
e^{-2x}\left[\frac{dy}{dx} - 2y\right] &= e^{-2x}\left[e^{3x}\right] \\
e^{-2x}\frac{dy}{dx} - 2e^{-2x}y &= e^{x} \\
\frac{dy}{dx}\left[e^{-2x}y\right] &= e^{x} \\
\int \frac{dy}{dx}\left[e^{-2x}y\right]dx &= \int e^{x}dx \\
e^{-2x}y &= e^{x} + c \\
y &= \frac{e^{x}+c}{e^{-2x}} \\
&= e^{3x} + ce^{-(-2x)} \\
y &= e^{3x} + ce^{2x}
\end{aligned}$$

(g) $x\frac{dy}{dx} = 2y + x^3\ln x$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x} + x^2\ln x \\
\frac{dy}{dx} - \frac{2y}{x} &= x^2\ln x \\
\frac{dy}{dx} - \frac{2y}{x} &= x^2\ln x, p(x) = -\frac{2}{x}
\end{aligned}$$

ii. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.

$$\begin{aligned}
\mu(x) &= e^{\int p(x)dx} \\
\mu(x) &= e^{\int -2x^{-1}dx} \\
&= e^{-2\int \frac{1}{x}dx} \\
&= e^{\ln x^{-2}dx} \\
&= x^{-2} \\
\mu(x) &= \frac{1}{x^2}
\end{aligned}$$

iii. Multiply everything in the D.E. by $\mu(x)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

$$\frac{1}{x^2}\left[\frac{dy}{dx}-\frac{2y}{x}\right]=\frac{1}{x^2}\left[x^2\ln x\right]$$
$$x^{-2}\frac{dy}{dx}-2x^{-3}y=\ln x$$
$$\frac{dy}{dx}\left[x^{-2}y\right]=\ln x$$
$$\int\frac{dy}{dx}\left[x^{-2}y\right]dx=\int\ln x\,dx$$

A. Time to use Calc 1 skills, Integration by parts

$$\int u\,dv=uv-\int vdu$$

Let $u=\ln x$ and $v'=1$
$\frac{du}{dx}=\frac{1}{x}, v=x$

B. Finally we get

$$\begin{aligned}\int\ln x\,dx&=x\ln x-\int x\,\frac{1}{x}dx\\&=x\ln x-\int 1dx\\&=x\ln x-x+c\end{aligned}$$

C. Continue plugging in

$$\begin{aligned}\int\frac{dy}{dx}\left[x^{-2}y\right]dx&=x\ln x-x+c\\x^{-2}y&=x\ln x-x+c\\y&=x^3\ln x-x^3+cx^2\end{aligned}$$

(h) $x\frac{dy}{dx}-3y=x^4$

i. Put $D.E.$ in the correct standard form, coefficient of $\frac{dy}{dx}$ is 1.

ii. Find the integrating factor $\mu\left(x\right)=e^{\int p(x)dx}$.

iii. Multiply everything in the D.E. by $\mu\left(x\right)$, and verify that the left side becomes the product rule $\frac{d}{dx}\left[\mu\left(x\right)y\right]$ and write it as such.

2. Solve the following I.V.P.:

(a) $\frac{dy}{dx}=98-0.196y, y\left(0\right)=48$

(b) $x\frac{dy}{dx}+y=\frac{x}{x+1}, y\left(1\right)=1$

(c) $x\frac{dy}{dx}+3y=x^2-4x+3, y\left(1\right)=0$

3. **Separable Equations:**
   Solve the following differential equations:

   (a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

   (b) $\frac{dy}{dx} = \frac{xy}{2\ln y}$

   (c) $\frac{dx}{dt} + e^{t+x} = 0$

   (d) $2\sqrt{xy}\frac{dy}{dx} = 1, x, y > 0$

   (e) $\frac{dy}{dx} = \frac{xy}{x+2}$

   (f) $y\left(x^2 - 1\right)\frac{dy}{dx} = 1$

   (g) $\frac{dx}{dt} = \frac{4\sin t + 6\cos 2t}{x}$

   (h) $\frac{dy}{dx} = e^{-y}\left(2x - 4\right)$

   (i) $\sec x\frac{dy}{dx} = e^{y+\sin x}$

   (j) $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}$

4. Solve the following I.V.P.:

   (a) $y' = \frac{y\cos x}{1+y^2}, y\left(0\right) = 1$

   (b) $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0, y\left(0\right) = 1$

   (c) $\frac{dy}{dt} = te^y, y\left(1\right) = 0$

   (d) $t\left(t-1\right)\frac{dx}{dt} = x\left(x+1\right), x\left(2\right) = 2$

   (e) $\frac{dx}{dt} = e^{x+t}, x\left(0\right) = a$