July 23, 2019

1. Differentiate the following functions

   (a) $y = \frac{5}{x^2} - \frac{x^4}{3}$

$$\begin{aligned} y &= \frac{5}{x^2} - \frac{x^4}{3} \\ &= 5x^{-2} - \frac{1}{3}x^4 \\ \frac{dy}{dx} &= -10x^{-3} - \frac{4}{3}x^3 \end{aligned}$$

   (b) $y = \sqrt{x^3}$

$$\begin{aligned} y &= x^{\frac{3}{2}} \\ \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} \end{aligned}$$

   (c) $y = \frac{6}{\sqrt{x}}$

$$\begin{aligned} y &= 6 * \frac{1}{\sqrt{x}} \\ &= 6 * x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= 6 * -\frac{1}{2} * x^{-\frac{3}{2}} \\ &= -3x^{-\frac{3}{2}} \end{aligned}$$

   (d) $y = \frac{e^{3x}}{16}$

$$\begin{aligned} y &= \frac{1}{16}e^{3x} \\ \frac{dy}{dx} &= \frac{3}{16}e^{\frac{3}{x}} \end{aligned}$$

   (e) $y = 3\ln x$

$$\begin{aligned} \frac{dy}{dx} &= 3 * \frac{1}{x} \\ &= \frac{3}{x} \end{aligned}$$

(f) $y = \frac{x^2-1}{x}$

$$\begin{aligned} y &= x - \frac{1}{x} \\ &= x - x^{-1} \\ \frac{dy}{dx} &= 1 - (-1) * x^{-2} \\ &= 1 + \frac{1}{x^2} \end{aligned}$$

2.

(a) $y = x\sqrt{x+3}$

$$\begin{aligned} y &= x\left(x+3\right)^{\frac{1}{2}} \\ \frac{dy}{dx} &= 1 \cdot \left(x+3\right)^{\frac{1}{2}} + x \cdot \frac{1}{2}\left(x+3\right)^{-\frac{1}{2}} \\ &= \left(x+3\right)^{\frac{1}{2}} + \frac{x}{2}\left(x+3\right)^{-\frac{1}{2}} \\ &= \left(x+3\right)^{-\frac{1}{2}}\left(x+3+\frac{1}{2}x\right) \\ &= \left(x+3\right)^{-\frac{1}{2}}\left(\frac{3}{2}x+3\right) \\ &= \frac{\frac{3}{2}x+3}{\sqrt{x+3}} \\ &= \frac{\frac{3}{2}\left(x+2\right)}{\sqrt{x+3}} \\ &= \frac{3\left(x+2\right)}{2\sqrt{x+3}} \end{aligned}$$

(b) $y = \sqrt{x} \cdot e^x$

$$\begin{aligned} y &= x^{\frac{1}{2}}e^x \\ \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} * e^x + x^{\frac{1}{2}} * e^x \\ &= e^x x^{-\frac{1}{2}}\left(\frac{1}{2}+x\right) \\ &= \frac{e^x\left(2x+1\right)}{2\sqrt{x}} \end{aligned}$$

(c) $y = (1 + \sin x)\tan x$

$$\begin{aligned} y &= (\cos x)(\tan x) + (1+\sin x)\left(\sec^2 x\right) \\ &= (\cos x)\left(\frac{\sin x}{\cos x}\right) + (1+\sin x)\left(\sec^2 x\right) \\ &= \sin x + \sec^2 x + \tan x \sec x \\ &= \sin x + \sec^2 x + \frac{\tan x}{\cos x} \end{aligned}$$

(d) $y = \left(1 - x^3\right)\left(1 + x^3\right)$

i. $Ans : -6x^5$, note, the book answer is wrong

$$\begin{aligned} y &= -3x^2 * \left(1 + x^3\right) + \left(1 - x^3\right) * 3x^2 \\ &= -3x^2 - 3x^5 + 3x^2 - 3x^5 \\ &= -6x^5 \end{aligned}$$

3. **Quotient rule**

(a) $y = \frac{2x^2+1}{x-1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-1)\frac{d}{dx}\left(2x^2+1\right) - \left(2x^2+1\right)\frac{d}{dx}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(4x) - \left(2x^2+1\right)}{(x-1)^2} \\ &= \frac{4x^2 - 4x - 2x^2 - 1}{(x-1)^2} \\ &= \frac{2x^2 - 4x - 1}{(x-1)^2} \end{aligned}$$

(b) $y = \frac{e^x}{x^3+1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(x^3+1\right)e^x - e^x\left(3x^2\right)}{\left(x^3+1\right)^2} \\ &= \frac{\left(x^3+1\right)e^x - e^x\left(3x^2\right)}{\left(x^3+1\right)^2} \\ &= e^x\left(\frac{x^3 - 3x^2 + 1}{\left(x^3+1\right)^2}\right) \\ &= \frac{e^x\left(x^3 - 3x^2 + 1\right)}{\left(x^3+1\right)^2} \end{aligned}$$

(c) $y = \frac{\cos x}{\sin x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x \cdot (-\sin x) - \cos x \cdot (\cos x)}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \\ &= -\csc^2 x\end{aligned}$$

4. $G(r) = \sqrt{r} + \sqrt[3]{r}$

$$\begin{aligned}G(r) &= r^{\frac{1}{2}} + r^{\frac{1}{3}} \\ G'(r) &= \frac{1}{2} r^{-\frac{1}{2}} + \frac{1}{3} r^{-\frac{2}{3}} \\ G''(r) &= -\frac{1}{4} r^{-\frac{3}{2}} - \frac{2}{9} r^{-\frac{5}{3}}\end{aligned}$$

5. If $F(x) = f(x)\,g(x)$

$$\begin{aligned}F(x) &= f(x)\,g(x) \\ F'(x) &= f'(x)\,g(x) + f(x)\,g'(x) \\ F''(x) &= \frac{d}{dx}\left(f'(x)\,g(x)\right) + \frac{d}{dx}\left(f(x)\,g'(x)\right) \\ &= f''(x)\,g(x) + f'(x)\,g'(x) + f'(x)\,g'(x) + f(x)\,g''(x) \\ &= f''g(x) + 2f'g'(x) + fg''(x) \\ &= f''g + 2f'g' + fg''\end{aligned}$$

6.

(a) $f(x) = x \sin x$

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x) * \sin x + x * \frac{d}{dx}(\sin x) \\ &= \sin x + x * (\cos x) \\ &= \sin x + x \cos x\end{aligned}$$

(b) $g(t) = 4 \sec t + \tan t$

$$\frac{dg}{dt} = 4(\sec x \tan x) + \sec^2 x$$

(c) $y = e^u (\cos u + cu)$

$$\begin{aligned}y &= 1 * e^u * (\cos u + cu) + (e^u) * (-\sin u + c) \\ &= e^u (\cos u - \sin u + cu + c)\end{aligned}$$

(d) $y = \frac{\sin x}{1+\cos x}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} \\
&= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\
&= \frac{\cos x + 1}{(1+\cos x)^2}, \text{note:} \sin^2 x + \cos^2 x = 1 \\
&= \frac{1}{1+\cos x}
\end{aligned}$$

(e) $y = \frac{\tan x - 1}{\sec x}$

$$\begin{aligned}
y &= \frac{\tan x}{\sec x} - \frac{1}{\sec x} \\
&= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} - \cos x \\
&= \frac{\sin x}{\cos x} * \cos x - \cos x \\
&= \sin x - \cos x \\
\frac{dy}{dx} &= \cos x - (-\sin x) \\
&= \sin x + \cos x
\end{aligned}$$

i. Alternatively

$$\begin{aligned}
y &= \frac{\sec x \left(\sec^2 x\right) - (\tan x - 1)(\sec x \tan x)}{(\sec x)^2} \\
&= \frac{\sec^3 x + (1 - \tan x)(\sec x \tan x)}{(\sec x)^2} \\
&= \frac{\sec^3 x + \left(\sec x \tan x - \sec x \tan^2 x\right)}{\sec^2 x} \\
&= \frac{\sec^2 x + \tan x - \tan^2 x}{\sec x} \\
&= \frac{1 + \tan^2 (x) + \tan x - \tan^2 x}{\sec x}; \text{Note:} \sec^2 x = 1 + \tan^2 (x) \\
&= \frac{1 + \tan x}{\sec x} \\
&= \frac{\tan x + 1}{\sec x} \\
&= \frac{\tan x}{\sec x} + \frac{1}{\sec x} \\
&= \sin x + \cos x
\end{aligned}$$

7. Find the equation of the tangent line to the curve at the given point. $y = e^x \cos x, (0, 1)$

   (a) The equation of a tangent line is $y = mx + c$

   (b) First, find m, the gradient, or $\frac{dy}{dx}$, at point $(0, 1)$

$$\begin{aligned}\frac{dy}{dx} &= e^x \cos x + e^x (-\sin x)\\ &= e^0 \cos 0 + e^0 (-\sin 0)\\ &= 1\end{aligned}$$

   (c) Second, find the c, or the $y - intercept$.

$$\begin{aligned}y &= x + c\\ 1 &= 0 + c\\ c &= 1\end{aligned}$$

   (d) Construct the complete equation

$$y = x + 1$$

8. Find the equation of the tangent line to the curve $y = \sec x - 2\cos x$ at the point $(\frac{\pi}{3}, 1)$.

   (a) The equation of a tangent line is $y = mx + c$

   (b) First, find m, the gradient, or $\frac{dy}{dx}$, at point $\left(\frac{\pi}{3}, 1\right)$

$$\begin{aligned}\frac{dy}{dx} &= \sec x \tan x + 2\sin x\\ &= \sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3}\\ &= 2\left(\sqrt{3}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\\ &= 2\sqrt{3} + \sqrt{3}\\ &= 3\sqrt{3}\end{aligned}$$

   (c) Second, find the c, or the $y - intercept$.

$$\begin{aligned}y &= \left(3\sqrt{3}\right)x + c\\ 1 &= \frac{\pi}{3}\left(3\sqrt{3}\right) + c\\ c &= 1 - \pi\sqrt{3}\\ &=\end{aligned}$$

   (d) Construct the complete equation

$$y = 3\sqrt{3}x + 1 - \pi\sqrt{3}$$