NATIONAL INSTITUTE OF TECHNOLOGY, JAMSHEDPUR

JHARKHAND-831014

Department of Mathematics

Session: 2023-2024

Course Handout (SEMESTER-I)

Course No : MA-1101 Date: 21/08/2023

Course title : Engineering Mathematics - I Credit-4(4-0-0)

Instructor In-charge : Dr. HARI SHANKAR PRASAD

Course Description

Review of Mean Value Theorem (Rolle's Theorem, Cauchy Mean Value Theorem), Taylor's and Maclaurin's Theorems with remainders, Asymptotes and Curvature.

Sequence and Series: Sequence, Convergence of Sequence using Comparison Test, Root Test, Ratio Test. Series and its convergence, Power Series, Radius of Convergence.

Limit, Continuity and Differentiability of function of several variables, partial derivatives and their geometrical interpretation, total derivatives of composite and implicit functions, derivatives of higher order and their commutativity, Euler's theorem on homogeneous functions, harmonic function, Taylor's expansion of function of several variables, Maxima and Minima of Function of several variables- Lagrange's method of multipliers.

Review of first order ordinary differential equations, Second order differential equations with constant coefficients, Euler's equation, system of differential equations. Series solution of ordinary differential equations: near ordinary and singular points.

Scope

- ❖ To study advanced calculus of one variable and several variables.
- ❖ To study different type of differential equations and methods to find their solutions.

Objective

- ❖ To develop skills so that student might be able to handle problems arising in engineering disciplines.
- ❖ To strengthen fields like Differential Equations and Calculus to solve different problems arising in engineering disciplines.

Text Books

- ➢ Higher Engineering Mathematics by B. S. Grewal
- ➢ Differential and Integral Calculus by B. C. Das and B. N. Mukherjee

Reference Books

- Advanced Engineering Mathematics by Erwin Kreyszig (Wiley)
- Advanced Engineering Mathematics by R. K. Jain and S. R. K. Iyengar

Course Plan:

Lecture No	Topics to be covered
1-6	Review of Mean Value Theorem: Rolle's Theorem, Cauchy Mean Vluae Theorem
7-11	Taylor's and Maclaurin's Theorems with remainders, Asymptotes and Curvature.
12-17	Sequence and Series: Sequence, Convergence of Sequence using Comparison Test, Root Test, Ratio Test. Series and its convergence, Power Series, Radius of Convergence.
18-23	Limit, Continuity and Differentiability of function of several variables, Partial derivatives and their geometrical interpretation
24-28	Total derivatives of composite and implicit functions, derivatives of higher order and their commutativity, Euler's theorem on homogeneous functions, harmonic function
29-32	Taylor's expansion of function of several variables, Maxima and Minima of Function of several variables-Lagrange's method of multipliers.
33-34	Review of first order ordinary differential equations
35-38	Second order differential equations with constant coefficients, Euler's equation
39-44	System of differential equations. Series solution of ordinary differential equations: near ordinary and singular points.

Evaluation Scheme:

EC No	Evaluation Component	Duration	Marks	Date and Time
1	Mid Sem. Exam	2 Hours	30	As per Academic Calendar
2	End Sem. Exam	3 Hours	50	As per Academic Calendar
3	Surprise Quizes/Tests	10 Minutes	10	During Theory Classes
4	Assignment		10	Take Home

Chamber Consultation Hour: Tuesday: 11.00 A.M. to 1.00 P.M.

Notices: All notices regarding the course will be displayed on the Notice Board of Department of Mathematics.

Instructor In-charge

Dr. H. S. PRASAD

NATIONAL INSTITUTE OF TECHNOLOGY, JAMSHEDPUR-14

Department of Mathematics

First Semester, Session: 2023-2024

Course Home Assignment

ELECTRICAL ENGG., 1st semester, 2023 Batch)

Date: 21-08-2023

Course No.: MA1101 **Credit: 4**

Course Title: Engg. Mathematics: I

Instructor-in-charge: Dr. Hari Shankar Prasad

1. Establish the following Standard results where D is the differential operator and a, b, c, n, m are constants**:**

(i) $D^n (ax+b)^m = m(m-1)(m-2).......(m-n+1)\, a^n (ax+b)^{m-n}$ (ii) $D^n (a^{mx}) = m^n (\log n)^n a^{mx}$

(iii) $D^n[1/(ax+b)] = [(-1)^n a^n (n!)]/(ax+b)^{1+n}$ (iv) $D^n (e^{mx}) = m^n e^{mx}$

(v) $D^n[\sin(ax+b)] = a^n \sin(ax+b+n\pi/2)$ (vi) $D^n[\sin(ax+b)] = a^n \cos(ax+b+n\pi/2)$

(vii) $D^n[e^{ax}\sin(bx+c)] = (a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n\tan^{-1} b/a)$

(viii) $D^n[e^{ax}\cos(bx+c)] = (a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n\tan^{-1} b/a)$,

2. Find the n^{th} derivative of the following**:**

(i) $\log[(ax+b)(cx+d)]$ (ii) $(\sin ax)(\cos bx)$ (iii) $e^{cx}(\sin ax)(\cos bx)$ (iv) $e^{cx}(\sin x)(\cos x)^2$

(v) $1/[1-6x+7x^2]$ (vi) $x^4/[(x+2)(2x+3)]$ (vii) $1/[(x-1)^3(x-2)]$ (viii) $\tan^{-1}[2x/(1-x^2)]$

3.(a) If $y = x(a^2+x^2)^{-1}$, prove that $y_n = (-1)^n n!\, a^{-n-1} \sin^{n+1}\phi \cos(n+1)\phi$, where $\phi = \tan^{-1}(x/a)$.

(b) If $y = \tan^{-1}[\{x(1+x^2)^{1/2} - 1\}/x]$, prove that $y_n = (0.5)(-1)^{n-1}(n-1)!\, a^{-n-1} \sin^n \phi \sin(n\phi)$,

where $\phi = \cot^{-1}(x)$.

4. (a) State and prove Labnitz's theorem for finding the n^{th} derivatives of product of two functions.

(b) Find the n^{th} derivative of the following:

(i) $x^3 \cos x$ (ii) $e^x\, x^3 \cos x$ (iii) $(\log x) x^2 \cos x$ (iv) $x^2 e^x \cos^3 x$ (v) $x^2 \tan^{-1} x$

5.(a) If $y = e^a \sin^{-1} x$, show that $(1-x^2) y_{n-2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0$.

(b) If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$.

(c) If $y^{1/m} + y^{-1/m} = 2x$, show that $(x^2-1) y_{n+2} + (2n+1) x y_{n+1} + (n^2-m^2) y_n = 0$.

(d) Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = [(-1)^n (n!)/x^{n+1}][\log x - 1 - 1/2 - 1/3 - .... - 1/n]$.

(e) If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, *prove that* $I_n = nI_{n-1} + (n-1)!$; hence show that

$I_n = n!(\log x + 1 + 1/2 + 1/3 + ... + 1/n)$.

(f) If $y = [x + (1+x^2)^{1/2}]^m$, *find* $(y_n)_0$. (g) If $y = \sin(m \sin^{-1} x)$, *find* $(y_n)_0$.

(h) If $y = \log[x + (1 + x^2)^{1/2}], find\ (y_n)_0$.

(i) If $y = (1/2)(\tan^{-1} x)^2, showthat\ (y_{n+2})_0 + 2n^2(y_n)_0 + n(n-1)^2(n-2)(y_{n-2})_0 = 0$.

6. (a) State and Prove (i) Rolle's, (ii) Lagrange's and (iii) Cauchy mean value theorems.

(b) Solve all the exercise problems (which are based on the mean value theorems) from the prescribed Text

Book of B. S. Grewal.

© State and Prove Taylors theorem with remainders for the function of single variable.

(d) Show that the Maclaurin's theorem is special case of Taylors theorem.

(e) Obtain the Taylor's polynomial approximation of degree n to the following function $f(x)$ about the point $x = a.$ Estimate the error in the given interval.

(i) $f(x) = \sqrt{x},\ n = 3, a = 1, 1 \le x \le 1.5.$ (ii) $f(x) = e^{-x^2}\ n = 3, a = 0, -1 \le x \le 1.$

(iii) $f(x) = x \sin x,\ n = 4, a = 0, -1 \le x \le 1.$ (iv) $f(x) = x^2 e^{-x}\ n = 4, a = 1, 0.5 \le x \le 1.5.$

(f) Obtain the Taylor's polynomial approximation of degree n to the following function $f(x)$ about the point $x = a.$ Find the error term and show that it tends to zero as $n \to \infty.$ Hence write its Taylor's series.

(i) $f(x) = \sin 3x,\ a = 0.$ (ii) $f(x) = \sin^2 x, a = 0.$ (iii) $f(x) = x^2 \ln x, a = 1.$ (iv) $f(x) = 2^x, a = 1.$

(g) Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x = 0$ for the function $\sin x\ \cos x$ in the interval [0,1] such that the $|error| < 0.005.$

(h) The function $\tan^{-1} x$ is approximated by the first two non zero terms in the Taylor's polynomial expansion about the point $x = 0$. Find c such that $|error| < 0.005.$ when $0 < x < c..$

(I) Expand $y(x) = \tan^{-1} x$ in powers of $x - \pi/4.$

7.(a) Define the (i) Sequence, (ii) Limit and Convergence of a sequence with examples, (iii) Power series and its

convergence, and (iv) Radius of Convergence for a power series.

(b) Solve all the exercise problems of the Chapter 'Infinite series of the prescribed text book of Grewal, B.S.

8.(a) State and prove the various type of comparison tests, root tests, and ratio tests.

(b) Solve at least four different problems using each tests stated in the above problem 8(a).

9.(a) Define vertical, horizontal, and inclined/oblique asymptotes to the curve with at least one example.

(b) Find all the asymptotes to the following curves:

(i) $(y-2)(x^2-1) = 5$ (ii) $y^2x^2 = a^2(y^2 - x^2)$ **(iii)** $y = (x-1)^3/[x^2(x+1)]$ (iv) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(v) $(y+x)^2(x+y+2) = x + 9y - 2.$ (vi) $r = a(\sec\theta + \tan\theta).$ (vii) $(x^3 - 4y^3 + 3x^2y) - x + y + 3 = 0.$

(viii) $x = \dfrac{3at}{1+t^3}, y = \dfrac{3at^2}{1+t^3}, a > 0.$ (ix) $y = \dfrac{t^2}{\sqrt{t^2-4}}.$

10. Define the Curvature, Circle of curvature, Centre of curvature and Radius of curvature of a curve.

11. Derive the formula for finding the Radius of curvature of Cartesian curve, Polar curve, and parametric curve respectively.

12. Derive the formula for finding the Centre and Chord of curvature of a Cartesian curve respectively.

13. Find the curvature $|\kappa|$, the radius of curvature ρ and the centre (α, β) of curvature of the circle of curvature of the following curve at the given point. The constant a is positive.

(i) $y = x^2 - 6x + 10 \ \ at \ \ (3,1)$ (ii) $a\,y^2 = x^3 \ \ at \ \ (a,a)$ **(iii)** $y^3 + x^3 - 2axy = 0, (a,a)$

(iv) $x^2 - 4ay = 0\, at\ (2a, a)$ (v) $x^2 + \log(x + \sqrt{(1+x^2)}\, at\ \ (0,0)$ (vi) $y = 4/(x^2+2)\ \ at\ \ (0,2)$

(vii) $y = \sin x\ at\ (\pi/2,\ 1)$ (viii) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t, at\ \ t = \pi/4.$

(ix) $x = a\ln(\sec t + \tan t), y = a\sec t\ \ at\ \ t = 0.$

14. Find the radius of curvature ρ at the origin $(0,0)$ using Newton's Method for the following curves:

(i) $3y^2 + 4x^3 - 2x^2 + 6y = 0$ (ii) $2x^4 - 3y^4 + x^3y + xy - y^2 + 2x = 0.$

15. Define the following:

(i) Evolute of a curve, (ii) Involute of a curve and (iii) Envelope of a family of a curve.

16. (a) Find the Evolute of the following curve:

(i) $x^2 = 4ay$ (ii) $xy = 1$ (iii) $(x^2/a^2) - (y^2/b^2) = 1, a > b.$ (iv) $x = a(\theta - \sin\theta), y = a(1-\cos\theta).$

(b) Find the involute of the following curve:

(i) $x = a\cos t, y = a\sin t.\ \ where\ \ a > 0.$

(ii) $x = a\cos t(\cos^2 t + 3\sin^2 t), y = a\sin t(\sin^2 t + 3\cos^2 t).\ \ where\ \ a > 0.$

(c) Find the equation of the envelope of the following given family of curves for positive parameter ($p.$):

(i) $py + p^2x - 10 = 0$ (ii) $x\tan p + y\sec p = 5$ (iii) $(x-p)^2 + (y-p)^2 = p^2$ (iv) $y = 3px - p^3$

17. Define the Limit, Continuity and Differentiability of a function of two and three variables respectively.

18. (a) Using $\delta - \varepsilon$ approach, establish the following limits:

(i) $\lim\limits_{(x,y)\to(1,1)} (x^2 + y^2 - 1) = 1.$ (ii) $\lim\limits_{(x,y)\to(2,1)} (x^2 + 2x - y^2) = 7.$ (iii) $\lim\limits_{(x,y)\to(0,0)} (x+y)/(x^2+y^2+1) = 0.$

(iv) $\lim\limits_{(x,y)\to(0,0)} (x^3 + y^3)/(x^2 + y^2) = 0.$

(b) Determine the following limits if they exists:

(i) $\lim\limits_{(x,y)\to(1,0)} (x-1)\sin y/(y\ln x) = 1.$ (ii) $\lim\limits_{(x,y)\to(2,0)} (1 + \dfrac{x}{y})^y = 7.$ (iii) $\lim\limits_{(x,y)\to(0,0)} x/\sqrt{x^2+y^2}$

(iv) $\lim\limits_{(x,y)\to(1,-1)} (x^3 - y^3)/(x-y)$ (v) $\lim\limits_{(x,y,z)\to(0,0,0)} (xy+z)/(x+y+z^2)$

(vi) $\lim\limits_{(x,y,z)\to(0,0,0)} x(x+y+z)/(x^2+y^2+z^2)$

19. Discuss the continuity of the following functions at the given points(0,0):

(i) $f(x,y)=\begin{cases}(x-y)^2/(x^2+y^2), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$ (ii) $f(x,y)=\begin{cases}(x-y)xy/(x^2+y^2), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$

(iii) $f(x,y)=\begin{cases}(x-y)^2/(x^2+y^2), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$ (iv) $f(x,y)=\begin{cases}e^{xy}/(x^2+1), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$

(v) $f(x,y)=\begin{cases}(x^2+y^2-2xy)/(x-y), & (x,y)\neq(1,-1)\\ 0 & ,\ (x,y)=(1,-1)\end{cases}$

(vi) $f(x,y)=\begin{cases}(2x^2+y^2)/(3+\sin x), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$

20.(a) Show that the function $f(x,y)=\begin{cases}(x^2+y^2)/(x-y), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$ possesses partial derivatives at (0,0), though it is not continuous at (0, 0)

(b) Show that the function $f(x,y)=\begin{cases}(xy)/\sqrt{x^2+y^2} & ,\ (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$ has the partial derivatives $f_x(0,0), f_y(0,0),$ but the partial derivatives are not continuous at (0, 0).

(c) Show that the function $f(x,y)=\sqrt{x^2+y^2}$ is not differentiable at (0, 0).

21.(a) Find all the partial derivatives of the specified order for the following functions at the given points:

(i) $f(x,y)=(x-y)/(x+y),$ second order at $(1,1)$. (ii) $f(x,y)=e^x\log y+(\cos y)\log x,$ third order at $(1,\pi/2)$. (iii) $f(x,y,z)=e^{(x^2+y^2+z^2)},$ second order at $(-1,1,-1)$. (iv) $f(x,y)=e^{\sin(x/y)},$ second order at $(\pi/2,1)$. (iv) $f(x,y,z)=\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x},$ second order at $(1,2,3)$.

(v) $f(x,y,z)=\sin xy+\sin yz+\sin zx,$ second order at $(1,\pi/2,\pi/2)$. (vi) $f(x,y,z)=x^x y^y z^z,$ second order at any finite points.

22.(a) For the function $f(x,y)=\begin{cases}(x^2y)(x-y)/(x^2+y^2), & (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$, show that $f_{xy}(0,0)\neq f_{yx}(0,0)$.

(b) Show that $f_{xyyz}=f_{yyxz}$ for all (x,y,z) when $f(x,y,z)=z^2e^{x+y^2}$.

(c) Show that $f_{xyz} = f_{yzx}$ for all (x, y, z) when $f(x, y, z) = e^{4y}\sin z$.

23. (a) State and prove Euler's theorem and establish the following results:

(i) If $u = \sin^{-1}\left[\dfrac{x^2+y^2}{x+y}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$. (ii) If $u = \log\left[\dfrac{(x^2+y^2)^{1/2}}{x}\right]$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

(iii) If $u = \tan\left[\dfrac{x^3+y^3}{x-y}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u.$$

24.(a) If $x(1-y^2)^{1/2} + y(1-x^2)^{1/2} = c, c$ any constant, $|x| < 1, |y| < 1,$ then find $\dfrac{dy}{dx}$ *and* $\dfrac{d^2y}{dx^2}$.

(b) If $z = f(u,v), u = x/(x^2+y^2), v = y/(x^2+y^2), (x,y) \neq (0,0),$ then show that

$$z_{uu} + z_{vv} = (x^2+y^2)^2(z_{xx} + z_{yy}).$$

(25) (a) State and Prove Taylors theorem with remainders for the function of two variable.

(b) Show that the Maclaurin's theorem is special case of Taylors theorem.

(c) Obtain the n^{th} order Taylor's series approximation to the following function about the given point. Also, estimate the maximum absolute error in the indicated region.

(i) $f(x,y) = xy^2 + y\cos(x-y)$ for $n = 2,$ about the point $(1,1)$ in the region $|x-1| < 0.05, |y-1| < 0.1.$

(ii) $f(x,y) = \sqrt{x+y}$ for $n = 2,$ about the point $(1,3)$ in the region $|x-1| < 0.2, |y-3| < 0.1.$

(iii) $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ for $n = 3,$ about the point $(-1,1)$ in the region $|x-1| < 0.1, |y-1| < 0.1.$

(iv) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ for $n = 1,$ about the point $(2,2,1)$ in the region $|x-2| < 0.1, |y-2| < 0.1, \ |z-1| < 0.1.$

(v) $f(x,y,z) = e^z\sin(x+y)$ for $n = 2,$ about the point $(0,0,0)$ in the region $|x| < 0.1, |y| < 0.1, \ |z| < 0.1.$

(d) Obtain the relative and absolute maximum and minimum values for the following functions in the given closed region R:

(i) $x^2 + y^2 - 2y, R: x^2 + y^2 \leq 1.$ (ii) $4x^2 + y^2 - 2x + 1, R: 2x^2 + y^2 \leq 1.$

(iii) $\cos x + \cos y + \cos(x+y), R: 0 \leq x \leq \pi, 0 \leq y \leq \pi.$

(iv) $x^3 + y^3 - xy, R: x = 1, y = 0, y = 2x.$

(e) Find the extreme value of $x^3 + 8y^3 + 4z^3$, *when* $xyz = 1.$

(f) Divide a number into three parts such that the product of the first, square of the second and cube of the third is maximum.

(g) Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to the coordinate axes that can be inscribed in the ellipsoids $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(h) Find the maximum value of xyz under the constraints $x^2+z^2=1$ *and* $x=y=0$.

(i) Find the extreme value of $x^2+2xy+z^2$ under the constraints $2x+y=0$ *and* $x+y+z=1$.

(j) Find the smallest and largest distance between the points P and Q such that P lies on the plane $x+y+z=2a$. and Q lies on the sphere $x^2+y^2+z^2=a^2$, where a is any constant.

(k) Find the shortest distance between the line $y=10-2x$ and the ellipse $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$.

26. (a) What is differential equation and its (i) general solution (ii) particular solution and (iii) singular solution?

(b) How a differential equation is formed? Illustrate with examples.

(c) Explain the methods of undetermined coefficients with examples for finding the particular integral of the given differential equation.

(d) Explain the method of variation of parameters for finding the particular integral of the given differential equation.

(e) Find the solution of general form of Euler's equation.

(f) What do you mean by a system of differential equations? Illustrate with examples.

(g) Describe at least one method for finding the solution of a system of linear differential equations.

27. Solve all the exercise problems of Chapter-11 and Chapter-13 related to differential equations of first and second order respectively of the Grewal, B.S. book.

28. Solve all the exercise problems of Chapter-16 related to the series solution of differential equations of the Grewal, B.S. book.

****