Math 511 Solution Manual – Assignment 3

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Text: Strang's Linear Algebra and It's Applications 4th e.d.

Assignment 3

3.1 – Orthogonal Vectors and Subspaces

7. Find a vector $\mathbf{x}$ orthogonal to the row space of A, and a vector $\mathbf{y}$ orthogonal to the column space of A, and a vector $\mathbf{z}$ orthogonal to the nullspace of A,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

Solution: First, we find the orthogonal complement of the row space of A, which is the nullspace of A. For $Ax = 0$, we set $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solve this, we find that the null space is $span\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\}$. Choose $x = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, for example.

We then find the orthogonal complement of the column space of A, which is the nullspace of A^T. For $A^T y = 0$, we set $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solve this, we find that the null space is $span\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}$. Choose $y = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, for example.

The row space is the orthogonal complement of the nullspace, so, we choose any vector in the row space. Let $z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, for example.

3.2 – Cosines and Projections onto Lines

14. What matrix P projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x + y + t = 0$ and $x - t = 0$?

Solution: To be on the planes $x+y+t=0$ and $x-t=0$, an vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ must satisfy $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, since the vector is orthogonal to the normal vectors of planes. Solving this equation, we find the solutions to the matrix equation are any vectors in *span* $< \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} >$. Then, we use the formula of the projection matrix $P = \frac{aa^T}{a^Ta}$, and choose $a = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Substitute into the formula, we get $P = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$.

3.3 – Projections and Least Squares

3. Solve $Ax=b$ by least squares, and find $p = A\hat{x}$ if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Verify that the error $b-p$ is perpendicular to the columns of A.

Solution:

$$\begin{aligned} \hat{x} &= (A^TA)^{-1}A^Tb \\ &= (\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix})^{-1}(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Then $b-p = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$ Let us denote the columns $A_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, A_2 := \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. To verify $b-p$ is perpendicular to the columns of A, we need to show that $(b-p)^TA_1 = \frac{2}{3} - \frac{2}{3} = 0$ and $(b-p)^TA_2 = \frac{2}{3} - \frac{2}{3} = 0$. This is indeed the case.

6. Find the projection of b onto the column space of A:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p+q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

Solution:

$$\begin{aligned}
p &= A\vec{x} \\
&= A(A^TA)^{-1}A^Tb \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \\
&= \frac{1}{44} \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix}
\end{aligned}$$

We can find q by direct subtraction of $b - p = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} - \frac{1}{44} \begin{bmatrix} 92 \\ -56 \\ 260 \end{bmatrix} = \frac{1}{44} \begin{bmatrix} -48 \\ 144 \\ 48 \end{bmatrix}$. Check that p and q are indeed perpendicular. Since p is in the column space, and the left-nullspace is the orthogonal complement of the column space, q is contained in the left-nullspace.

12. If V is the subspace spanned by $(1,1,0,1)$ and $(0,0,1,0)$, find

(a) a basis for the orthogonal complement $V^{\perp}$.

(b) The projection matrix P onto V

(c) The vector in V closest to the vector $b = (0,1,0,-1)$ in $V^{\perp}$.

Solution:

(a) Begin by letting an arbitrary $w \in V^{\perp}$, then $(1,1,0,1).w = 0 = (0,0,1,0).w$. Hence if $w = (w_1, w_2, w_3, w_4)$, we obtain the system of equations

$$w_3 = 0, \qquad w_1 + w_2 + w_4 = 0$$

We can construct a basis for $V^\perp$ by letting $w_1 = 1$ and $w_2 = 0$ or $w_1 = 0$, $w_2 = 1$, respectively, since these are our linearly independent variables. Thus,

$$V^\perp = \text{span}\{(1, 0, 0, -1), (0, 1, 0, -1)\}$$

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More generally, we know that $C(A)^\perp = N(A^T)$. This we construct a matrix A so that the column space of A is V. This can be obtained by just letting the columns of A be basis vectors for V:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A basis for the orthogonal complement is then given by the special solutions to $A^T y = 0$.

(b) Since A has full column rank, the projection matrix P onto $V = C(A)$ is given by the formula

$$P = A(A^T A)^{-1} A^T$$

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Since $A^T A$ is diagonal, we can easily find its inverse as

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

So,

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{aligned}$$

(c) To obtain the vector p closest to b in $V^{\perp}$, we just apply P:

$$p = Pb = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note, since the two vectors $q_1 = (1,1,0,1)$ and $q_2 = (0,0,1,0)$ are orthogonal $q_1^T q_2 = 0$ the formula for the projector is

$$P = \frac{q_1 q_1^T}{q_1^T q_1} + \frac{q_2 q_2^T}{q_2^T q_2}$$

3.4 – Orthogonal Bases and Gram-Schmidt

15. Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Which fundamental subspace contains q_3? What is the least-squares solution of $Ax = b$ if $b = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T$?

Solution:

Let a, b denote the columns of the matrix A. We apply Gram-Schmidt process: $q_1 = \frac{a}{\|a\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix}$, and plug in for q_2 to get $q_2 = \frac{b-(q_1^T b)q_1}{\|b-(q_1^T b)q_1\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. And for q_3, pick an arbitrary vector c that is not in the column space of A, for example, $c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then, $q_3 = \frac{c-(q_1^T c)q_1-(q_2^T c)q_2}{\|c-(q_1^T c)q_1-(q_2^T c)q_2\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{-1}{3} \end{bmatrix}$. So we have an orthonormal set. Since q_3 is orthogonal to the column space of A, q_3 is contained in $Nul(A^T)$, the left-nullspace.

We now find the least-squares solution of $Ax = b$, where $b = [1,2,7]^T$ by solving $A^T Ax = A^T b$. From this equation, we get $\begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix} x = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$, so $x = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3.5 – The Fast Fourier Transform

11. Compute $y = F_4 c$ by the three steps of the Fast Fourier Transform if $c = (1,0,1,0)$.

Solution: The first step is to split c into two vectors c' and c'' by putting the odd indices in c' and the even indices in c''. We can do this with the even-odd permutation matrix (note that c is 0-indexed):

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}c'\\c''\end{bmatrix}$$

Then, we multiply the result by the 4×4 matrix with F_2 in the upper left and lower right blocks:

$$\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\0\\0\\0\end{bmatrix}$$

Finally, we recall from 3W in the book that we can reconstruct the result y by the following system of equations, where $m=\frac{1}{2}n$:

$$y_j = y'_j + w_n^j y''_j,\quad j=0,\ldots,m-1 \tag{1}$$

$$y_{j+m} = y'_j - w_n^j y''_j,\quad j=0,\ldots,m-1 \tag{2}$$

The matrix for this system when $n=4$ is

$$\begin{bmatrix}1&0&w_n^0&0\\0&1&0&w_n^1\\1&0&-w_n^0&0\\0&1&0&-w_n^1\end{bmatrix}=\begin{bmatrix}1&0&1&0\\0&1&0&i\\1&0&-1&0\\0&1&0&-i\end{bmatrix}$$

We multiply $\begin{bmatrix}y'_j\\y''_j\end{bmatrix}$ by the above to get our final answer:

$$\begin{bmatrix}1&0&1&0\\0&1&0&i\\1&0&-1&0\\0&1&0&-i\end{bmatrix}\begin{bmatrix}2\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\0\\2\\0\end{bmatrix}$$

We can check your work by multiplying the original vector c by the F_4 matrix. We should get the same answer:

$$\begin{bmatrix}1&1&1&1\\1&i&i^2&i^3\\1&i^2&i^4&i^6\\1&i^3&i^6&i^9\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\2\\0\end{bmatrix}$$