MA 511 Solutions, Homework 1

1.3.17 We can subtract one copy of row one from row two to get

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right]$$

and subtract one copy of the resulting row two from row three to get

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{array}\right].$$

This is a singular matrix when $q = -4$. If $q = -4$, then the bottom row represents the equation $0 = t - 5$, and so there are infinitely many solutions when $t = 5$. When $q = -4$, $t = 5$, and $z = 1$, we get $y = 3$ from the second row and $x = -9$ from the first row, so the solution is $(-9, 3, 1)$.

1.3.31 The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{array}\right].$$

Subtracting row two from row three, and then subsequently subtracting row one from row two, we get

$$\left[\begin{array}{ccc|c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & 0 & a-4 & b_3 - b_2 \end{array}\right].$$

This matrix fails to have three pivots when $a = 0, 2$, or 4.

1.4.24 We have

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

which corresponds to the operations (1) subtract 4 copies of row 1 from row 2, (2) add 2 copies of row 1 to row 3, and (3) subtract two copies of (the new) row 2 from row 3. We have

$$M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 0 \end{bmatrix}.$$

1.5.29 We have the decomposition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

In order to get four pivots, we need $a \neq 0$, $a \neq b$, $b \neq c$, and $c \neq d$.

1.5.38 We can write, to represent subtracting 3 of row one from row two and subsequently subtracting $1/(c-6)$ times the resulting row two from row three, that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{c-6} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 0 & \frac{c-7}{c-6} \end{bmatrix},$$

whenever $c \neq 6$. This gives 3 pivots whenever $c \neq 6, 7$.

1.6.40 (a) True. A row of zeroes means there cannot be four pivots.

(b) False. Consider the 4x4 matrix of all 1's. Subtracting row three from row four results in a row of zeroes, which as in (a) means the matrix is not invertible.

(c) True. If A^{-1} exists, then $AA^{-1} = A^{-1}A = I$, so $A = (A^{-1})^{-1}$.

(d) True. We have that $A^{-1} = ((A^T)^T)^{-1} = ((A^T)^{-1})^T$, which exists since $(A^T)^{-1}$ exists.

1.6.41 If $c = 0$, we have a row of zeroes, so it is not invertible. Otherwise, subtract $8/c$ times the second row from the third row to get

$$\begin{bmatrix} 2 & c & c \\ c & c & c \\ 0 & -1 & c-8 \end{bmatrix}.$$

Next subtract $c/2$ times the first row from the second row to get

$$\begin{bmatrix} 2 & c & c \\ 0 & (2c-c^2)/2 & (2c-c^2)/2 \\ 0 & -1 & c-8 \end{bmatrix}.$$

If $c = 2$ or 0, then the middle row is all zeroes, so it is not invertible. Otherwise, add $2/(2c-c^2)$ times the second row to the third row to get

$$\begin{bmatrix} 2 & c & c \\ 0 & c/2 & c/2 \\ 0 & 0 & c-7 \end{bmatrix}.$$

If $c = 7$, this is not invertible. Thus $c = 0, 2, 7$ are the values of c for which the matrix is not invertible.

1.6.56 We have

$$(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A^T)^2 - (B^T)^2 = A^2 - B^2,$$

so (a) is symmetric. Next,

$$[(A + B)(A - B)]^T = (A - B)^T(A + B)^T = (A - B)(A + B),$$

which is not generally equal to $(A + B)(A - B)$, so (b) is not symmetric. Next,

$$(ABA)^T = A^T B^T A^T = ABA,$$

so (c) is symmetric. Finally, by the same method,

$$(ABAB)^T = BABA,$$

which is not generally equal to $ABAB$, so (d) is not symmetric.