Sample problems for FINAL EXAM MA 511

Part I- TRUE FALSE

1. If A and B are unitary matrices with the same eigenvalues then there is unitary U such that $UBU^H = A$.
2. If A is a square matrix such that the linear transformation $f(x) = Ax$, preserves angles and lengths then $A^T A$ is an orthogonal matrix.
3. If A is Hermitian and real then A is symmetric .
4. If U is a unitary matrix then all its eigenvalues are real.
5. If U_1 and U_2 are unitary matrices of the same size then $U_1 U_2^H$ is unitary.
6. If A is Hermitian and U is unitary of the same size then $U^{-1}AU$ is Hermitian.
7. If $x^T A = b^T$ has exactly one solution then rows of A are linerly independent.
8. If the characteristic polynomials of 2×2 matrices A and B are equal then A and B are similar. .
9. If u is a unit vector then $I - uu^T$ is a projection matrix.
10. If V and W are subspaces of R^{11} and $dim(V) = 5$ and $dim(W) = 8$ then $dim(V^\perp \cap W) \geq 4$.
11. All Fourier matrices F_n are unitarily diagonalizable
12. If A is $m \times n$ and $rank(A) = m$ then A has linearly independent rows
13. If A is symmetric postive definite then $A = B^4$ for some B which is symmetic positive definite.
14. If A has infinitely many inverses then it has no right inverse

15. All Fourier matrices are Hermitian.

16. The differential equation $du/dt = Au$ is stable if $Re(\lambda_i) > 0$ for all egenvalues λ_i,

17. A real $n \times n$ matrix is invertible if and only if A has n positive singular values.

18. A is diagonalizable if and only if A has distinct eigenvalues.

19. Let A be a unitary matrix. Then the eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

20. If A is similar to B, that is, there exists an invertible matrix S so that $A = SBS^{-1}$, then A and B share the same eigenvalues and eigenvectors.

21. Let A be the incidence matrix of a graph G. The summation of each all columns is zero vector.

Part II

1. Find the Singular Value Decomposition of the matrix

   (a) $A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$

   (b) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$

   (c) $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

2. Determine a type of the critical point $(0, 0)$ for the function

   (a) Let $F(x, y) = x^2 + 4xy + 3y^2$.

   (b) Let $G(x, y) = (x^2 - 2)(y^2 + 5)$

3. For the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ find a unitary matrix U such that $U^H AU$ is upper triangular.

4. Verify that the matrix is positive definite

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

5. Given the ellipsoid $x^T Ax = 4$, where

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

find the direction vectors and the length of its four principal axes.

6. Given the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ find its norm and the condition number.

7. Let $f : R^5 \to R^5$ be a linear transformation such that $f(e_i) = e_{i+1} + 2e_i$ for $i = 1, \ldots, 4$ and $f(e_5) = 2e_5$. Find the matrix A of f with respect to the standard basis. Find the Jordan form J of A and the matrix M such that $J = M^{-1}AM$.

8. Find the matrix exponential e^{At}, where

   (a) $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

   Use e^{At} to find explicitly the solution of the initial value problem $X' = AX$, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. Find A^n for the matrices

(a) $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

10. Decide about the stability of $dx/dt = Ax$, where

(a) $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 5 & -1 \\ 1 & -3 \end{bmatrix}$

11. Find eigenvalues of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

12. Which of the following matrices are similar and wchich are not. (Find their Jordan canonical form) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$,

$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$,

$D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$D = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

13. Find the projection matrix P onto the column space of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix},$$

Find the matrix Q with othonormal columns such that $P = QQ^T$. Find the eigenvalues and eigenvectors of P^3.

14. (a) Find the matrix of the projection $P : R^3 \to R^3$ onto the the plane $x + y + 3z = 0$.

    (b) Find the projection of $\mathbf{R}^3$ onto the line $span\{[1, 1, 3]$.

15. Find the closest function $c + dx^2$ to x^4 on the interval [-0,1], and the distance of x^4 to span$(1, x^4)$.
    (We consider the inner product $(f, g) = \int_0^1 fg dx$.)

16. Find the orthogonal basis of the vector space of quadratic polynomials $a + bx + cx^2$ over the interval $[0, 1]$.

17. Find QR factorization for

    (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

18. Find the least squares solution to

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

19. Solve the initial value problem $X'' = AX$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,
    $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $X'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,