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12. Use row operations to verify that the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & (b-a)(b+a) \\ 0 & (c-a) & (c-a)(c+a) \end{bmatrix} \xrightarrow[M_3\left(\frac{1}{c-a}\right)]{M_2\left(\frac{1}{b-a}\right)} (b-a)(c-a)\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 1 & (c+a) \end{bmatrix} \xrightarrow{A_{23}(-1)} \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 0 & (c+a)-(b+a) \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 0 & (c-b) \end{bmatrix}$$

$$(b-a)(c-a)\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 0 & (c-b) \end{bmatrix} \xrightarrow{M_3\left(\frac{1}{c-b}\right)} \begin{bmatrix} 1 & a & a^2 \\ & 1 & (b+a) \\ & & 1 \end{bmatrix} \Rightarrow (b-a)(c-b)(c-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & (b+a) \\ 0 & 0 & 1 \end{bmatrix} = (b-a)(c-a)(c-b)$$

$$\det\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = 1$$

$$(b-a)(c-a)(c-b)(1) = (b-a)(c-a)(c-b)$$

$$(b-a)(c-a)(c-b) \overset{\checkmark}{=} (b-a)(c-a)(c-b)$$

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16. Find these 4 by 4 determinants by Gaussian elimination:

$$\det\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix}.$$

$$\det\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \xrightarrow[A_{14}(-41/11)]{A_{12}(-21/11),\ A_{13}(-31/11)} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 0 & \frac{-10}{11} & \frac{-20}{11} & \frac{-30}{11} \\ 0 & \frac{-20}{11} & \frac{-40}{11} & \frac{-60}{11} \\ 0 & \frac{-30}{11} & \frac{-60}{11} & \frac{-90}{11} \end{bmatrix} \xrightarrow{A_{23}(-2)} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 0 & \frac{-10}{11} & \frac{-20}{11} & \frac{-30}{11} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-30}{11} & \frac{-60}{11} & \frac{-90}{11} \end{bmatrix} \Rightarrow \text{ROW3} = 0 \therefore \det|A| = 0$$

$$\det\begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix} \xrightarrow[A_{14}(-t^3)]{A_{12}(t),\ A_{13}(-t^2)} \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & (1-t^2) & (t-t^3) & (t^2-t^4) \\ 0 & (t-t^3) & (1-t^4) & (t-t^5) \\ 0 & (t^2-t^4) & (t-t^5) & (1-t^6) \end{bmatrix} \xrightarrow{A_{23}\left(-\frac{t-t^3}{1-t^2}\right)} \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & (1-t^2) & (t-t^3) & (t^2-t^4) \\ 0 & 0 & (1-t^4)-\frac{(t-t^3)^2}{1-t^2} & (t-t^5)-\frac{(t-t^3)(t^2-t^4)}{1-t^2} \\ 0 & (t^2-t^4) & (t-t^5) & (1-t^6) \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & (1-t^2) & (t-t^3) & (t^2-t^4) \\ 0 & 0 & (1-t^2) & (t-t^3) \\ 0 & (t^2-t^4) & (t-t^5) & (1-t^6) \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & (1-t^2) & (t-t^3) & (t^2-t^4) \\ 0 & 0 & (1-t^2) & (t-t^3) \\ 0 & 0 & 0 & (1-t^2) \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix} = 1(1-t^2)(1-t^2)(1-t^2) = (1-t^2)^3$$

$$(1-t^4)-\frac{(t-t^3)(t-t^3)}{1-t^2} = (1-t^4)-\frac{(t^2-2t^4+t^6)}{(1-t^2)} = (1-t^4)-\frac{(t^2-2t^4+t^6)}{(1-t)(1+t)}$$

$$(1-t^4)-\frac{t^2(1-2t^2+t^4)}{(1-t)(1+t)} = (1-t^4)-\frac{t^2(1-t^2)^2}{(1-t)(1+t)}$$

$$(1-t^2)(1+t^2)-\frac{t^2(1-t^2)^2}{(1-t)(1+t)} = \frac{(1-t^2)(1+t^2)(1-t)(1+t)-t^2(1-t^2)^2}{(1-t)(1+t)}$$

$$\frac{(1-t^2)^2(1+t^2)-t^2(1-t^2)^2}{(1-t)(1+t)} = (1-t^2)^2((1+t^2)-t^2)$$

$$= \frac{(1-t^2)^2}{(1-t^2)} = (1-t^2)$$

$$t-t^5-\frac{(t-t^3)(t^2-4)}{(1-t^2)}$$

$$(t-t^5)-\frac{(t-t^3)(t^2-t^4)}{(1-t^2)} = (t-t^5)-\frac{(t^3-t^5-t^5+t^7)}{(1-t^2)} =$$

$$\frac{(t-t^5)(1-t^2)-(t^3-2t^5+t^7)}{(1-t^2)} = \frac{t-t^3-t^5+t^7-t^3+2t^5-t^7}{(1-t^2)} =$$

$$\frac{t-2t^3+t^5}{(1-t^2)} = \frac{t(1-2t^2+t^4)}{(1-t^2)} = \frac{t(1-t^2)^2}{(1-t^2)} = (t-t^3)$$

4.2.p210.#35

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35. Suppose the 4 by 4 matrix M has four equal rows all containing a, b, c, d. We know that $\det(M) = 0$. The problem is to find $\det(I + M)$ by any method:

$$\det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

Partial credit if you find this determinant when $a = b = c = d = 1$. Sudden death if you say that $\det(I + M) = \det I + \det M$.

$$A = \begin{bmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{bmatrix} \xrightarrow[A_{43}(-1)]{A_{21}(-1),\ A_{32}(-1)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ a & b & c & 1+d \end{bmatrix} \xrightarrow{A_{14}(-a)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & b+a & c & 1+d \end{bmatrix} \xrightarrow{A_{24}^{-}(b+a)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & a+b+c & 1+d \end{bmatrix} \xrightarrow{A_{34}^{-}(a+b+c)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1+a+b+c+d \end{bmatrix}$$

$\Rightarrow \det(I+M) = 1+a+b+c$

IF $a=b=c=d=1$

$\det(I+M) = 5$

4.3.p216.#07

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7. (a) Evaluate this determinant by cofactors of row 1:

$$\begin{vmatrix}4&4&4&4\\1&2&0&1\\2&0&1&2\\1&1&0&2\end{vmatrix}.$$

(b) Check by subtracting column 1 from the other columns and recomputing.

a)

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$C_{11} = (-1)^{1+1} = (-1)^2 = 1$$

$$a_{11} = \begin{vmatrix}2&0&1\\0&1&2\\1&0&2\end{vmatrix} = (2(1\cdot 2-0\cdot 2) - 0(0\cdot 2-1\cdot 2) + 1(0\cdot 0-1\cdot 1))$$
$$= 2(2-0) - 0(0-1) + 1(0-1)$$
$$= 2(2) - 0(-1) + 1(-1)$$
$$= 4 - 0 - 1$$
$$= 3$$

$$a_{11}C_{11} = 3\cdot 1 = 3$$

$$C_{12} = (-1)^{1+2} = (-1)^3 = -1$$

$$a_{12} = \begin{vmatrix}1&0&1\\2&1&2\\1&0&2\end{vmatrix} = 1(1\cdot 2-0\cdot 2) - 0(2\cdot 2-1\cdot 2) + 1(2\cdot 0-1\cdot 1)$$
$$= 1(2) - 0(2) + 1(-1)$$
$$= 2 - 1$$
$$= 1$$

$$a_{12}C_{12} = 1\cdot -1 = -1$$

$$C_{13} = (-1)^{1+3} = (-1)^4 = 1$$

$$a_{13} = \begin{vmatrix}1&2&1\\2&0&2\\1&1&2\end{vmatrix} = 1(0\cdot 2-1\cdot 2) - 2(2\cdot 2-1\cdot 2) + 1(2\cdot 1-1\cdot 0)$$
$$= 1(-2) - 2(2) + 1(2)$$
$$= -2 - 4 + 2$$
$$= -4$$

$$a_{13}C_{13} = -4\cdot 1 = -4$$

$$C_{14} = (-1)^{1+4} = (-1)^5 = -1$$

$$a_{14} = \begin{vmatrix}1&2&0\\2&0&1\\1&1&0\end{vmatrix} = 1(0\cdot 0-1\cdot 1) - 2(2\cdot 0-1\cdot 1) + 0(2\cdot 1-1\cdot 0)$$
$$= 1(-1) - 2(-1) + 0(2)$$
$$= -1 + 2 + 0$$
$$= 1$$

$$a_{14}C_{14} = 1\cdot -1 = -1$$

$$\det(A) = 4(3 - 1 - 4 - 1)$$
$$= 4(-3)$$
$$\det(A) = -12$$

b)

$$A = \begin{bmatrix}4&4&4&4\\1&2&0&1\\2&0&1&2\\1&1&0&2\end{bmatrix} \xrightarrow{C_{A_{12}}(-1),\ C_{A_{13}}(-1),\ C_{A_{14}}(-1)} \begin{bmatrix}4&0&0&0\\1&1&-1&0\\2&-2&-1&0\\1&0&-1&1\end{bmatrix}$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$C_{11} = (-1)^{1+1} = 1$$

$$a_{11} = \begin{vmatrix}1&-1&0\\-2&-1&0\\0&-1&1\end{vmatrix} = 1(-1\cdot 1+1\cdot 0) + 1(-2\cdot 1-0\cdot 0) + 0(-2\cdot -1-0\cdot -1)$$
$$= 1(-1) + 1(-2) + 0(2)$$
$$= -1 - 2 + 0$$
$$= -3$$

$$a_{11}C_{11} = -3\cdot 1 = -3$$

$$a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} = 0$$

$$\det(A) = 4(-3) + 0 + 0 + 0$$
$$\det(A) = -12$$

4.4.p226.#05

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5. (a) Draw the triangle with vertices $A = (2, 2)$, $B = (-1, 3)$, and $C = (0, 0)$. By regarding it as half of a parallelogram, explain why its area equals

$$\text{area}\,(ABC) = \frac{1}{2}\det\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}.$$

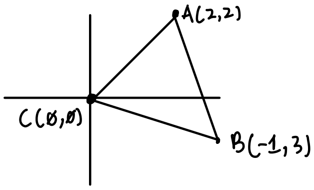
(b) Move the third vertex to $C = (1, -4)$ and justify the formula

$$\text{area}\,(ABC) = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix}.$$

Hint: Subtracting the last row from each of the others leaves

$$\det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 6 & 0 \\ -2 & 7 & 0 \\ 1 & -4 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 6 \\ -2 & 7 \end{bmatrix}.$$

Sketch $A' = (1, 6)$, $B' = (-2, 7)$, $C' = (0, 0)$ and their relation to A, B, C.



THE det|2X2| IS EQUIVALENT TO THE ARE OF A PARALLELOGRAM, WITH THIS INFORMATION, A TRIANGLE IS HALF THE AREA OF A PARALLELOGRAM.

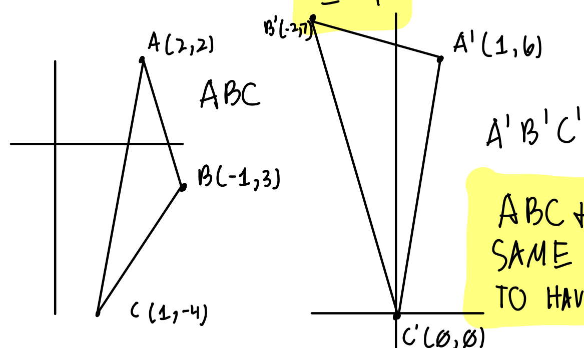
AREA OF A PARALLELOGRAM = BASE • HEIGHT$_{\text{BASE}}$

AREA OF A TRIANGLE = $\frac{1}{2}$ BASE • HEIGHT$_{\text{BASE}}$

$$\text{AREA} = \tfrac{1}{2}\,|2(3-\emptyset) + (-1)(\emptyset-2) + \emptyset(2-3)|$$
$$= \tfrac{1}{2}\,|2(3) - 1(-2) + \emptyset|$$
$$= \tfrac{1}{2}\,|6 + 2 + \emptyset|$$
$$= \tfrac{1}{2}\,|8|$$
$$= 4$$

$$\frac{1}{2}\det\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = \tfrac{1}{2}(2\cdot 3 - (-1)\cdot 2)$$
$$= \tfrac{1}{2}(6 + 2)$$
$$= \tfrac{1}{2}(8)$$
$$= 4$$

b)



ABC + A'B'C' CONSIST OF THE SAME det/AREA, A'B'C', WAS SHIFTED TO HAVE C AT THE ORIGIN.

$$\text{AREA} = \frac{1}{2}\det\begin{bmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{bmatrix} \rightarrow \frac{1}{2}\cdot 1\cdot\det\begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix} = \frac{1}{2}\cdot 1\cdot(1\cdot 7 - (-2)\cdot 6)$$
$$= \tfrac{1}{2}(7 + 12)$$
$$= \tfrac{1}{2}(19)$$
$$= 19/2$$

THE FORMULA IS POSSIBLE DUE TO THE NORMALIZATION OF C TO THE ORIGIN; REDUCING THE MATRIX TO A 2X2.

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Problems 13–17 are about Cramer's Rule for $x = A^{-1}b$.

13. Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2. \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 \quad\quad &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \quad A_2(b) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$x_1 = \frac{\det A_1(b)}{\det(A)} = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}}{\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}} = \frac{(1\cdot4-2\cdot5)}{(2\cdot4-1\cdot5)} = \frac{4-1\varnothing}{8-5} = \frac{-6}{3} = -2 = x_1$$

$$x_2 = \frac{\det A_2(b)}{\det(A)} = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}{\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}} = \frac{(2\cdot2-1\cdot1)}{(2\cdot4-1\cdot5)} = \frac{4-1}{8-5} = \frac{3}{3} = 1 = x_2$$

$$A = \begin{bmatrix} 2 & 1 & \varnothing \\ 1 & 2 & 1 \\ \varnothing & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ \varnothing \\ \varnothing \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & \varnothing \\ 1 & 2 & 1 \\ \varnothing & 1 & 2 \end{bmatrix} \qquad \begin{aligned} \det(A) &= (-1)^2 2(2\cdot2-1\cdot1)+(-1)^3 1(1\cdot2-\varnothing\cdot1)+(-1)^4(\varnothing) \\ &= 2(4-1)+(-1)(2-\varnothing)+\varnothing \\ &= 2(3)-2 \\ &= 4 \end{aligned}$$

$$A_1(b) = \begin{bmatrix} 1 & 1 & \varnothing \\ \varnothing & 2 & 1 \\ \varnothing & 1 & 2 \end{bmatrix} \qquad \begin{aligned} \det(A_1) &= (-1)^2 1(2\cdot2-1\cdot1)+(-1)^3 1(\varnothing)+(-1)^4(\varnothing) \\ &= 1(4-1) \\ &= 3 \end{aligned}$$

$$A_2(b) = \begin{bmatrix} 2 & 1 & \varnothing \\ 1 & \varnothing & 1 \\ \varnothing & \varnothing & 2 \end{bmatrix} \qquad \begin{aligned} \det(A_2(b)) &= (-1)^2 2(\varnothing)+(-1)^3 1(1\cdot2-\varnothing\cdot1)+(-1)^4(\varnothing) \\ &= \varnothing+(-1)(2)+\varnothing \\ &= -2 \end{aligned}$$

$$A_3(b) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & \varnothing \\ \varnothing & 1 & \varnothing \end{bmatrix} \qquad \begin{aligned} \det(A_3(b)) &= (-1)^2 2(\varnothing)+(-1)^3 1(\varnothing)+(-1)^4 1(1) \\ &= \varnothing+\varnothing+1 \\ &= 1 \end{aligned}$$

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{3}{4}$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{-2}{4} = \frac{-1}{2}$$

$$x_3 = \frac{\det(A_3(b))}{\det(A)} = \frac{1}{4}$$

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Problems 27–36 are about area and volume by determinants.

31. The Hadamard matrix H has orthogonal rows. The box is a hypercube!

$$\text{What is } \det H = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } \mathbf{R}^4?$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad \det(H) = 1\begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\begin{array}{l|l|l|l}
+1(1 \overset{0}{-} 1) & +1(1 \overset{0}{-} 1) & +1(1 \overset{2}{-} -1) & +1(-1 \overset{-2}{-} 1) \\
-(-1)(1 \overset{2}{+} 1) & -(-1)(-1 \overset{-2}{-} 1) & -1(-1 \overset{-2}{-} 1) & -1(1 \overset{2}{-} -1) \\
+(-1)(-1 \overset{-2}{-} 1) & +(-1)(1 \overset{2}{-} -1) & +(-1)(-1 \overset{0}{-} -1) & +(-1)(-1 \overset{0}{-} -1) \\
= 2 + 2 & = -2 - 2 & = 2 + 2 & = -2 - 2 \\
= 4 & = -4 & = 4 & = -4
\end{array}$$

$$4 \quad - \quad -4 \quad + \quad 4 \quad - \quad -4$$

$$\det(H) = 4 + 4 + 4 + 4$$

$$\det(H) = 16$$