Homework 4 Solution

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Let us consider a 3×3 Vandermonde determinant.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b & b^2 \\ 1 & c-b & c^2-b^2 \end{vmatrix} \quad (\mathbf{R_3} \to \mathbf{R_3} - \mathbf{R_2})$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-b & c^2-b^2 \end{vmatrix} \quad (\mathbf{R_2} \to \mathbf{R_2} - \mathbf{R_1})$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-b)(c-a) \end{vmatrix} \quad \left(\mathbf{R_3} \to \mathbf{R_3} - \frac{\mathbf{c-b}}{\mathbf{b-a}}\mathbf{R_2}\right)$$

We observe this is a triangular matrix, so we have the following:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

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Let us consider the following matrix:

$$\begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{pmatrix}$$

We need to find the determinant of this matrix using Gaussian Elimination: By Rule 5 from the text, a matrix has the same determinant even after subtracting a multiple of one row from another row. Thus,

$$\begin{vmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{vmatrix} = \begin{vmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{vmatrix} \quad (\mathbf{R_2 \to R_2 - R_1})$$

$$= \begin{vmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 20 & 20 & 20 & 20 \\ 41 & 42 & 43 & 44 \end{vmatrix} \quad (\mathbf{R_3 \to R_3 - 2R_1})$$

$$= \begin{vmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{vmatrix} \quad (\mathbf{R_4 \to R_4 - 3R_1})$$

$$= - \begin{vmatrix} 11 & 12 & 13 & 14 \\ 20 & 20 & 20 & 20 \\ 10 & 10 & 10 & 10 \\ 30 & 30 & 30 & 30 \end{vmatrix} \quad (\text{Exchange } R_3 \text{ to } R_2)$$

$$= \begin{vmatrix} 20 & 20 & 20 & 20 \\ 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 30 & 30 & 30 & 30 \end{vmatrix} \quad (\text{Exchange } R_2 \text{ to } R_1)$$

By Rule 3 from the text, the determinant of a matrix depends on the first row of the matrix linearly and then we exchange the first row 2 times. The negative sign got erased by applying Rule 2 twice. Thus, we have

$$\begin{vmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 30 & 30 & 30 & 30 \end{vmatrix}$$

Now, since there is a row of zeros, thus by rule 6 determinant of this matrix is zero. Thus,

$$\det \begin{vmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{vmatrix} = 0$$

Let us consider the following matrix,

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

We need to find the determinant of this matrix using Gaussian Elimination:

By Rule 5, a matrix has the same determinant even after subtracting a multiple of one row from another row.

Thus,

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{pmatrix}$$

$$\xrightarrow{R_4 \to R_4 - tR_3} \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ 0 & 0 & 0 & 1-t^2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - tR_2} \begin{pmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ 0 & 0 & 1-t^2 & t(1-t^2) \\ 0 & 0 & 0 & 1-t^2 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - tR_1} \begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1-t^2 & t(1-t^2) & t^2(1-t^2) \\ 0 & 0 & 1-t^2 & t(1-t^2) \\ 0 & 0 & 0 & 1-t^2 \end{pmatrix}$$

Hence, the determinant is $(1-t^2)^3$

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$$\det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$

$$= \begin{vmatrix} 1+a & a & a & a \\ b & 1+b & b & b \\ c & c & 1+c & c \\ d & d & d & 1+d \end{vmatrix} \quad \text{(Transpose Rule)}$$

$$= \begin{vmatrix} 1+a+b+c+d & 1+a+b+c+d & 1+a+b+c+d & 1+a+b+c+d \\ b & 1+b & b & b \\ c & c & 1+c & c \\ d & d & d & 1+d \end{vmatrix} \quad (\mathbf{R_1 \to R_1 + R_2 + R_3 + R_4})$$

$$= (1 + a + b + c + d)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & 1+b & b & b \\ c & c & 1+c & c \\ d & d & d & 1+d \end{vmatrix} \quad \text{(factor out first row)}$$

$$= (1 + a + b + c + d)\begin{vmatrix} 1 & b & c & d \\ 1 & 1+b & c & d \\ 1 & b & 1+c & d \\ 1 & b & c & 1+d \end{vmatrix} \quad \text{(Transpose Rule)}$$

Now add negative 1 times the first row to the second, third, and fourth rows and we get the following:

$$= (1 + a + b + c + d)\begin{vmatrix} 1 & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 + a + b + c + d \text{ (Upper triangular matrix)}$$

This offers an counter-example for det(M+I)=det(I)+det(M) when a=b=c=d=1

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a)

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix} = 4\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 4\left[2(2) + 1(-1)\right] - 4\left[1(2) + (-1)\right] + 4\left[(-2) - 2(2) + 2\right] - 4\left[0 - 1 - 2(-1)\right]$$

$$= 4(3) - 4(1) + 4(-4) - 4(1)$$

$$= 4(3 - 1 - 4 - 1)$$

$$= 4(-3)$$

$$= -12$$

b)

On subtracting 1st column from remaining columns,

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -2 & -1 & 0 \\ 1 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & -2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{vmatrix}$$

And, expanding by last columns

$$= 4 \begin{vmatrix} 1 & -1 \\ -2 & -1 \end{vmatrix} = 4(-1 - (-2)) = -12$$

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We are going to derive the general form first,

Let's say that $N(x_3, y_3), P(x_1, y_1), Q(x_2, y_2)$ are three vertices. Then

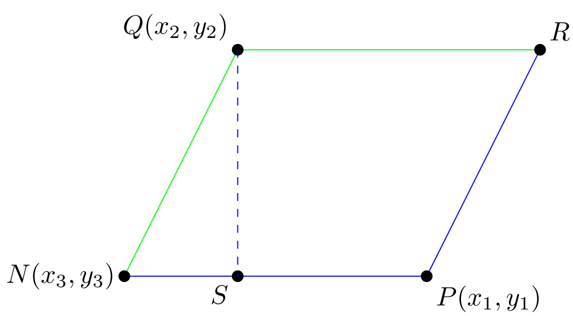
$$NP = (x_1 - x_3, y_1 - y_3),$$

$$NQ = (x_2 - x_3, y_2 - y_3)$$

are the adjoining sides of the parallelogram. We get NS is the projection of NQ upon NP given by

$$NS = \frac{NQ \cdot NP}{|NP|}$$

$$= \frac{(x_1 - x_3, y_1 - y_3) \cdot (x_2 - x_3, y_2 - y_3)}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}$$



The height of the right triangle NQS is QS

$$= \sqrt{\{(x_1 - x_3)^2 + (y_1 - y_3)^2\} - \left\{ \frac{(x_1 - x_3, y_1 - y_3) \cdot (x_2 - x_3, y_2 - y_3)}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \right\}^2}$$

$$= \sqrt{\frac{(x_2 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_3)}{|NP|}}$$

The area of the rectangle constructed is:

$$= |NP| \times \frac{(x_2 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_3)}{|NP|}$$

$$= (x_2 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_3)$$

The area of the parallelogram $NPQR$ is

$$= (x_2 - x_3)(y_1 - y_3) - (x_1 - x_3)(y_2 - y_3)$$

$$= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area of the triangle NPQ is

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Now we can substitute the value from part accordingly and we get the result from part a), that is,

$$= \frac{1}{2} \begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix}$$

Similarly, part b) is achieved by the same fashion as well.

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a) Solving the given system by Cramer's rule-

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now by replacing the first and second columns of A with b, we get the matrices B_1 and B_2

$$B_1 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

And,

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} \\ &= 3 \\ \det(B_1) &= \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 6 \\ \det(B_2) &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 3 \end{aligned}$$

Now we can compute x_1 and x_2

$$x_1 = \frac{det(B_1)}{det(A)} = -2$$

$$x_2 = \frac{det(B_2)}{det(A)} = 1$$

b) Solving the given system by Cramer's rule-

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similar to a),

$$\begin{aligned} \det(B_1) &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \\ &= 3 \end{aligned}$$

$$\det(B_2) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$
$$= -2$$
$$\det(B_3) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$
$$= 1$$

Thus, by Cramer's rule we have

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{3}{4}$$
$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{-2}{4} = -\frac{1}{2}$$
$$x_3 = \frac{\det(B_3)}{\det(A)} = \frac{1}{4}$$

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Since we know that the rows are orthogonal and we can compute HH^T

$$HH^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Recall this following rule:

Rightangle case $\ell_1^2 \ell_2^2 \cdots \ell_n^2 = \det(AA^{\mathrm{T}}) = (\det A)(\det A^{\mathrm{T}}) = (\det A)^2.$

Hence we have, det(H)=16 by the upper triangular matrix rule