1 MA 511 Final exam topics

- Compute the determinant of a matrix (2x2, 3x3, etc)
  - For a 2x2 matrix with elements a, b, d, c (CW from top left), the determinant is $ad - bc$.
  - For a 3x3 matrix, right-append the first two columns of the matrix to the right side of the initial matrix. Then, take multiply diagonally starting in the top left and add the results. Then multiply diagonally starting in the top right, subtracting the results from the sum of the first three multiplies.
  - For a larger matrix, it is often easiest to use cofactor expansion.
  - Summary of ways to compute determinants:
    1. Direct methods for 2x2 and 3x3 matrices
    2. Cofactor expansion
    3. Reduction to triangular form
    4. Through properties by reduction to I, whose determinant is 1.
- Know the properties of determinants
  - If B is obtained from A by swapping two rows, then $det(A) = -det(B)$
  - The determinant is linear on each row (see notes)
  - If A contains two identical rows, then $det(A) = 0$
  - If B is obtained from A by adding a multiple of one row of A to another, $det(A) = det(B)$ (determinant does not change).
  - If A contains a zero row, then $det(A) = 0$.
  - If B is obtained from A by multiplying a row by a constant c, then $det(B) = c * det(A)$.
  - If A is upper triangular, lower triangular, or diagonal, then $det(A)$ is the product of the elements on the main diagonal.
  - A is nonsingular if and only if $det(A) \neq 0$.
  - $det(AB) = det(A) * det(B)$
  - If A is nonsingular (invertible), then $det(A^{-1}) = \frac{1}{det(A)}$
  - $det(A) = det(A^T)$
- Compute the cofactors of a matrix
  1. Compute the submatrix $M_{i,j}$ by removing row i and column j from the initial matrix.
  2. Then, the cofactor $A_{i,j}$ for row i and column j of a matrix is: $A_{i,j} = (-1)^{i+j} * det(M_{i,j})$
- Compute the determinant of a matrix using cofactor expansion
  1. We expand the cofactors along either a row i or column j, with elements aIi, j in the row or column.
  2. The determinant can be computed as the sum of each cofactor times its corresponding element, i.e. $det(X) = \sum_{i,j} a_{i,j} A_{i,j}$
- Determine the number of inversions in a permutation
  1. Determine all possible pairs of elements in the sequence (i.e. the permutations of 2 elements)
  2. The number of inversions is given by the number of pairs for which the first value in the pair is larger than the second value.
- Compute the adjoint of a matrix

1. The adjoint of a matrix A, $adj(A)$ is the **transpose** of the cofactor matrix of A.

- Know some applications of determinants
  1. Computation of the inverse of a matrix: $A^{-1} = \frac{1}{det(A)} * adj(A)$
     - For a 2x2 matrix, this can be simplified to $A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  2. Determining whether a matrix has LDU decomposition: If all k-by-k upper left corner submatrices are non-singular (i.e. have determinant = 0), the matrix has LDU decomposition.
  3. Computing the volume of a parallelpiped: The volume of a parallelpiped given by a set of vectors organized as the columns of a matrix A is given by $V = |det(A)|$.
- Apply Cramer's Rule
  - If A is a square matrix, x is a column vector of unknowns, and b is a solution column vector, then the solution to $Ax = b$ contains elements $x_i = \frac{det(B_i)}{det(A)}$, where B_i is obtained by replacing the ith column of A with b.
- Compute the eigenvalues of a square matrix
  - Eigenvalues are values that, if subtracted from the elements along the main diagonal of a square matrix, cause the matrix to become singular.
  - Eigvenvalues λ associated with eigenvectors v satisfy $Av = \lambda v$ and $(A - \lambda I)v = 0$

  1. Compute the characteristic equation $P_A(\lambda) = det(A - \lambda I) = 0$
  2. Factor the polynomial to determine the possible value(s) of λ.
- Compute the eigenvectors of a square matrix
  1. Compute the eigenvalues of the matrix.
  2. For each eigenvalue λ, find its associated eigenvector by finding the null space solution to $A - \lambda I$ (i.e. by finding the null space vector of the matrix with λ subtracted from the elements on its main diagonal.
- Compute the trace of a matrix
  1. The trace of a matrix, $trace(A)$ is the sum of the elements along the main diagonal.
- Diagonalize a square matrix
  - First determine if the matrix is diagonalizable. An $n \times n$ matrix is diagonalizable if it has n distinct eigenvectors. Also, if the matrix has n distinct eigenvalues, the matrix is diagonalizable. The converse is not true.
  - Compute Λ, which is an eigenvalue matrix (matrix with eigenvalues along the main diagonal and zeros elsewhere).
  - Compute the eigenvectors associated with each λ and put them in the columns of a matrix S.
  - Compute S^{-1}.
  - The diagonalization of the matrix can be written $S\Lambda S^{-1}$
  - Note: If A and B are diagonalizable, then $AB = BA$ if A and B share the same BASIS of eigenvectors.
- Compute a matrix power, A^n
  - If A is diagonalizable...
    1. Compute the diagonalization of the matrix, $S\Lambda S^{-1}$

2. Then, $A^n = S\Lambda^n S^{-1}$ (i.e. just put each eigenvalue in Λ to the power n and recalculate the matrix product).
3. Note: If v is an eigenvector of A associated with λ, then v is also an eigenvector of A^m associated with λ^m.

- If A is not diagonalizable, we have to use the Jordan form J and transition matrix M:
  1. Compute the Jordan form J of A
  2. Compute J^n. This is a matrix with λ^n on the main diagonal, and the binomial expansion $(n\lambda^{n-1}, (nCk)\lambda^{n-2}, ...)$ on each diagonal above that. See 2020 sample final problem 9b.
  3. Compute the transition matrix M
  4. Then, $A^n = MJ^nM^{-1}$

- Solve a differential equation/Initial Value Problem (IVP) $x' = Ax$ involving matrices
  1. Compute the eigenvalues λ_i of A.
  2. Compute the eigenvectors v_i of A.
  3. The general solution to the differential equation (without initial conditions) is given by $x(t) = c_1e^{\lambda_1 t}v_1 + ... + c_ne^{\lambda_n t}v_n$, with arbitrary constants $c_1, ..., c_n$.
  4. (IVP): Solve for the constants $c_1, ..., c_n$ by plugging in $t = 0$ and solving the resulting matrix system.
  5. Alternatively, the particular solution to the IVP is given by $x(t) = e^{At}x_0$

- Solve a second order differential equation $X'' = AX$ given a matrix A, $X(0)$ and $X'(0)$.
  1. Compute the eigenvalues and eigenvectors of A. Then, for each eigenvalue λ...
     (a) If $\lambda < 0$, let $w^2 = -\lambda$. Then the solution will have a term $(cos(wt) + sin(wt)) * v$
     (b) If $\lambda > 0$, let $w^2 = \lambda$. Then the soltuion will have a term $(e^{wt} + e^{-wt})v$
  2. Append constants $c_1, ..., c_n$ in front of each term to obtain the general solution.
  3. Plug in the initial conditions and solve to get the particular solution.

- Compute e^{At}, where A is a matrix
  - If A is diagonalizable, then $e^{At} = Se^{\Lambda t}S^{-1}$. $e^{\Lambda t}$ is a diagonal matrix with $e^{\lambda_n t}$ values along the main diagonal, and S is a matrix containing the eigenvectors of A in its columns. Thus...
    1. Compute the eigenvalues of A and arrange them from smallest to largest.
    2. Compute $e^{\Lambda t}$, which has exponentials $e^{\lambda_n t}$ values along the main diagonal and zeros elsewhere.
    3. Compute the eigenvectors of A and put them in the columns of S, ordered smallest to largest according to their corresponding eigenvalue.
    4. Compute S^{-1}.
    5. Then, $e^{At} = Se^{\Lambda t}S^{-1}$
  - If A is NOT diagonalizable...
    1. Compute the Jordan form J of A
    2. Compute the transition matrix M
    3. Compute $e^{Jt} = e^{\lambda t}X$, where X is a matrix with 1s on the main diagonal, t on the diagonal above the main diagonal, $t^2/2$ on the diagonal above that, etc.
    4. Then, $e^{At} = Me^{Jt}M^{-1}$
  - Alternatively, if A is NOT diagonalizable, but $A^n = 0$ (i.e. A is nilpotent)...
    1. Recall that we can write e^{At} as $I + At + A^2t^2/2! + ... + A^nt^n/n!$
    2. At some point, the A^n values will be 0. So, by inspection, we only add the terms that are non-zero.
    3. e^{At} is then just the sum of the non-zero terms in the expansion.

    4. If A isn't nilpotent, try to write A in a different form such that one of the terms is nilpotent (see 2020 review final problem 8d).

- Determine the stability of a differential system $x' = Ax$
  - We can analyze the system based on its eigenvalues...
    1. Determine the eigenvalues λ of A.
    2. If $Re(\lambda) < 0$ for all eigenvalues, the system is **stable**.
    3. If $Re(\lambda) > 0$ for *at least one* eigenvalue, the system is **unstable**.
    4. If $Re(\lambda) \leq 0$ for all eigenvalues, AND $Re(\lambda) = 0$ for *at least one* eigenvalue, the system is **neutrally stable**.
  - We can analyze the system based on its determinant and trace...
    1. If $det(A) > 0$ and $trace(A) < 0$, the system is **stable**.
    2. If $det(A) < 0$ or $trace(A) > 0$, the system is **unstable**.
    3. If $det(A) = 0$ and $trace(A) = 0$, we cannot conclude the stability of the system.
    4. Otherwise, the system is **neutrally stable**.
  - Note: $trace(A) = \lambda_1 + ... + \lambda_n$
  - Note: $det(A) = \lambda_1 ... \lambda_n$

- Compute the inner product of complex vectors
  - By default, the inner product for complex numbers x and y is $< x, y >= \overline{x^T} y$, where $\overline{x^T}$ is the conjugate transpose of x.
  - Properties:
    1. $< x, y >= \overline{< y, x >}$
    2. $< x, y_1 + y_2 >=< x, y_1 > + < x, y_2 >$
    3. $< x, cy >= c < x, y >$
    4. $< x, x >$ is a non-negative real number

- Know the definition and properties of Hermitian matrices
  - The Hermitian transpose A^H of a matrix A is given by $A^H = \overline{A}^T$. With this, the inner product can be written as $< x, y >= x^H y$
  - Length of a complex vector $= \sqrt{< x, x >} = \sqrt{x^H x}$
  - Two complex vectors are orthogonal if $< x, y >= x^H y = 0$
  - A matrix is *Hermitian* if $A^H = A = \overline{A}^T$. This is the same as "symmetric" for a real valued matrix. Hermitian means *conjugate symmetric.*
  - Properties:
    1. If A is Hermitian, then $x^H A x$ is real valued for any $x \in$ the nth order complex numbers
    2. $(A + B)^H = A^H + B^H$
    3. $(AB)^H = B^H A^H$
    4. $(A^{-1})^H = (A^H)^{-1}$
    5. $(A^H)^H = A$
    6. If A is Hermitian, then all its eigenvalues are real valued
    7. The eigenvectors associated with distinct eigenvalues of a Hermitian matrix are orthogonal
    8. $U^{-1} A U = \Lambda$ can be written as $A = U \Lambda U^H$. Any matrix of this form is Hermitian.
  - A matrix is **normal** if $N^H N = N N^H$. A matrix can be normal without being Hermitian. Any normal matrix is unitarily diagonalizable.

- Know the definition and properties of Unitary matrices

- Recall: A matrix Q is orthogonal if $QQ^T = Q^TQ = I$, (i.e. $Q^T = Q^{-1}$)
- A complex matrix U is **unitary** if $UU^H = U^HU = I$, (i.e. $U^H = U^{-1}$)
- Properties:
  1. If U is unitary, then $||Ux|| = ||x||$, or $x^Hx = (Ux)^HUx = x^HU^HUx$
  2. If U is unitary, then all its eigenvalues have absolute value 1.
  3. If U is unitary, then the eigenvectors associated with distinct eigenvalues are orthogonal.
  4. If U_1 and U_2 are unitary then U_1U_2 is unitary
  5. Any unitary/normal/hermitian matrix A is also unitarily diagonalizable: There exists a unitary matrix U asuch that $U^HAU = \Lambda$ is diagonal. This implies $A = U\Lambda U^H$, where U is unitary/normal/hermitian.

- Know the Spectral Theorem:
  - Any real symmetric matrix A is orthogonally diagonalizable. That is, there exists an orthogonal matrix Q such that $Q^{-1}AQ = \Lambda$, where Λ is a diagonal matrix.
  - $Q^{-1}AQ = \Lambda$ can be written as $A = Q\Lambda Q^T$
  - Any matrix of the form Q^TAQ is symmetric.

- Find a unitary (or orthogonal) matrix Q such that Q^TAQ is upper triangular, given A:
  1. Compute the eigenvalues of A
  2. Compute the eigenvectors of A. For an $n \times n$ matrix, we will need n eigenvectors. We can start by finding only one using a single eigenvalue, and then finding others that are orthogonal to it by inspection.
  3. Normalize the eigenvectors
  4. Pick one of the normalized eigenvectors. Find other vectors that are orthogonal to it (use the fact that the inner product should be 0). Recall that we need n vectors.
  5. Normalize the new orthogonal vectors and put them in the columns of U_1
  6. Compute $U^{-1} = U^H = U^T$ (the latter only holds if U is real-valued)
  7. The first "upper triangular" matrix is given by $U_1^HAU_1$. Let this result be $\tilde{A}$. Note that it may not actually be upper triangular yet, but we are getting there.
  8. Repeat the above steps for the lower-right $n-1 \times n-1$ submatrix of $\tilde{A}$ until the submatrix is of size 1×1. However, now when you calculate U_2, since it is smaller size, you must "overlay" it with the identity matrix in the same place you took the submatrix from.
  9. The upper triangular form will be given by $U_n^T ... U_2^TU_1^TAU_1U_2...U_n$, so the product $U_1U_2...U_n$ is unitary.
  10. Note: If A is Hermitian, then there exists a unitary matrix U such that U^HAU is upper triangular.

- Know the definition of similar matrices
  - Two matrices A and B are **similar** if there exists an invertible matrix Q such that $A = Q^{-1}BQ$ or $B = QAQ^{-1}$.
  - Diagonal matrices are similar to all other diagonal matrices
  - Similar matrices have the same eigenvalues, but NOT NECESSARILY the same eigevectors.
  - If A and B are complex (or real), then A and B are similar if and only if they have the same Jordan Canonical Form (up to block permutations).

- Determine whether a function f is positive definite, negative definite, or indefinite
  1. f is positive definite if $f(x,y) > 0$ for all $(x,y) \neq (0,0)$
  2. f is negative definite if $f(x,y) < 0$ for all $(x,y) \neq (0,0)$

3. f is indefinite if $f(x,y) > 0$ for some $(x,y) \neq (0,0)$ and $f(x,y) < 0$ for some other $(x,y) \neq (0,0)$.

- Find and characterize the critical points of a function using matrices
  1. A quadratic polynomial can be expressed by a symmetric matrix: $ax^2+2bxy+cy^2 <-> \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
  2. A point p is a critical point of the function F if $F_X(p) = 0$ and $F_Y(p) = 0$ (i.e. partials wrt x and y are 0).
  3. If p is the critical point to characterize, calculate $F_{XX}(p), F_{XY}(p), F_{YX}(p), F_{YY}(p)$ (double partial derivatives)
  4. Arrange the above derivatives into a matrix $\begin{pmatrix} F_{XX}(p) & F_{XY}(p) \\ F_{YX}(p) & F_{YY}(p) \end{pmatrix}$
  5. If $F_{XX}(p) > 0$ and the determinant of the above matrix > 0, then p is a **local minimum**.
  6. If $F_{XX}(p) < 0$ and the determinant of the above matrix > 0, then p is a **local maximum**.
  7. If the determinant of the above matrix < 0, then p is a **saddle point**.

- Determine whether a matrix is positive definite or negative definite
  - A symmetric matrix is **positive definite** if all upper left corner submatrices have determinant > 0. Alternatively, we can say that $x^T Ax > 0$ for $x \in R^n$.
  - If A is an $n \times n$ symmetric matrix, then the following are equivalent:
    1. A is positive definite.
    2. A has positive eigenvalues.
    3. $det(A_k) > 0$ where A_k are $k \times k$ upper left corner submatrices
    4. The pivots in $REF(A)$ without row exchanges are positive.
    5. A can be written as $A = R^T R$ where R is $m \times n$

- Evaluate an ellipse using matrices
  1. If given a polynomial, write it in matrix form.
  2. Compute the eigenvalues and eigenvectors
  3. Normalize the eigenvectors. These give the directions of the semi-axes
  4. Solve $\lambda_1 x^2 + \lambda_2 y^2 = C$ for x and y by setting y and x to 0, respectively. The C parameter comes from the solution to an initial polynomial.

- Use Singular Value Decomposition (SVD) to decompose a matrix
  - An $m \times n$ matrix A can be decomposed with SVD to the form $A = U\Sigma V^T$
  - U is an $m \times m$ orthogonal matrix containing the normalized eigenvectors of AA^T
  - Σ is an $m \times n$ (same size as A) matrix containing the square roots of the nonzero eigenvalues of $A^T A$ along the main diagonal, and zeros on the right half/bottom. The values in Σ are arranged in decreasing order, starting from the top left. There are r singular values in Σ, where r is the rank of A.
  - V is an $n \times n$ matrix containing normalized eigenvectors of $A^T A$.

- The following method works best when A is a square matrix.
  1. First find AA^T and $A^T A$
  2. Find the eigenvalues of $A^T A$, arrange them in descending order, and take their square roots.
  3. Find the eigenvectors of AA^T and $A^T A$ and normalize them
  4. Populate the matrices U, Σ, and V

- Another method to compute the SVD of a matrix A is as follows:
  1. Compute A^TA. (I.e. first begin by computing parameters associated with V)
  2. Compute the eigenvalues and eigenvectors of A^TA and normalize them
  3. Compute the singular values σ_n of A^TA by taking the square roots of the nonzero eigenvalues.
  4. Then, the vectors in U are $u_n = Av_n/\sigma_n$
  5. Note the sizes that the final matrices should be. If you do not have enough vectors, you'll have to create an orthonormal set such that you do have enough.
- Sometimes it can be annoying to compute the eigenvalues/vectors of a 3×3 matrix. This might be the case if A is a horizontal rectangular matrix, where there are more columns than rows. In this case...
  1. Compute AA^T. (I.e. first begin by computing parameters associated with U)
  2. Compute the eigenvalues and eigenvectors of AA^T and normalize them
  3. Compute the singular values σ_n of AA^T by taking the square roots of the nonzero eigenvalues
  4. Then, the vectors in V are $v_n = A^Tu_n/\sigma_n$
  5. Note the sizes that the final matrices should be. If you do not have enough vectors, you'll have to create an orthonormal set such that you do have enough.
- Determine the Jordan Canonical Form of a matrix
  1. Compute the eigenvalues
  2. Determine the number of Jordan blocks associated with each unique eigenvalue λ. This is given by $dim(N(A - \lambda I)) = n - rank(A - \lambda I)$.
  3. Create the Jordan blocks. Each block's size is determined by the multiplicity of the eigenvalue it is associated with. The block is then a square matrix with λ on the main diagonal, and 1s on the diagonal directly above the main diagonal. The total number of Jordan blocks (# of unique eigenvalues) also tells us the number of "strings".
- Determine the transition matrix M for a given matrix such that $J = M^{-1}AM$
  1. Determine the number of "strings". This will be the total number of Jordan blocks (and also the total number of unique eigenvalues).
  2. For each string, we need to find the starting vector v for the string such that $(A - \lambda I)^n v = 0$ and $(A - \lambda I)^{n-1} v \neq 0$
  3. Note that a quick way to compute matrix powers, i.e. $(A - \lambda I)^n$ is that if there are only 1s on the diagonal directly *below* the main diagonal, each power n shifts the columns *left* by $n - 1$. If there are only 1s on the diagonal directly *above* the main diagonal, each power n shifts the columns *right* by $n - 1$.
  4. Then, find the remaining vectors $v_{n-1}, ..., v_1$ by using $v_{n-1} = (A - \lambda I)v_n$, starting with the v_n just found (the starting vector in the string).
  5. M is then $\begin{pmatrix} v_1 & v_2 & ... & v_n \end{pmatrix}$ (A matrix with $v_1, ..., v_n$ as its columns).
- Compute the norm of a matrix A, $||A||$
  1. In general, $||A|| = \sqrt{\lambda_{A^TA,max}}$
  2. If A is positive definite, then $||A|| = \lambda_{A,max}$
  3. If A is symmetric, then $||A|| = |\lambda_{A,max}|$ (the maximum of the absolute value of all eigenvalues)
  4. If A is invertible, then $||A^{-1}|| = \frac{1}{\sqrt{\lambda_{A^TA,min}}}$

  – Properties:

1. $||A|| \leq ||A|| * ||x||$, where x is a vector
2. $||A + B|| \leq ||A|| + ||B||$
3. $||AB|| \leq ||A|| * ||B||$

- Compute the condition number $c(A)$ associated with the norm
  1. The condition number is a measure of the sensitivity of A to perturbations.
  2. If A is symmetric/positive definite, then $c(A) = \frac{\lambda_{A,max}}{\lambda_{A,min}}$
  3. In general, $c(A) = ||A^{-1}|| * ||A|| = \sqrt{\frac{\lambda_{A^T A,max}}{\lambda_{A^T A,min}}}$
- Determine the sensitivity of a matrix system to small changes
  1. Find the eigenvalues of the matrix
  2. Find the eigenvectors associated with the eigenvalues
  3. The worst error occurs in the direction of the eigenvector associated with the lowest eigenvalue
  4. The perturbation δb occurs in the direction of the highest eigenvalue, i.e. $\delta b = \epsilon v_n$.
- Know properties associated with orthogonal matrices. The following are equivalent for an $n \times n$ real matrix A.
  1. A is orthogonal
  2. $A^T A = I$ (which is also orthogonal because I is orthogonal)
  3. $A^T = A^{-1}$
  4. A^T is orthogonal
  5. The columns of A form an orthonormal basis of R^n
  6. $< Ax, Ay > = < x, y >$, so A preserves the standard inner product on R^n
  7. $||Ax|| = ||x||$ so A preserves angles and vector lengths

2 Other topics for final exam

The following topics were covered on the midterm, but seem to show up fairly often in past finals.

- Compute the projection onto a span of vectors
- Find a matrix Q with orthonormal columns such that the projection matrix $P = QQ^T$
  1. Find the projection matrix P given A.
  2. Use Gram Schmidt to orthonormalize the columns of A.
  3. Then, since $P = Q(Q^T Q)^{-1} Q^T$, but $Q^T Q = QQ^T = I$ for orthogonal matrices, then $P = QQ^T$.
- Compute the least squares solution to a matrix system
- Compute the QR factorization of a matrix
- Compute the least squares solution to a system given functions (use the inner product integral)
- Topics requiring ORTHONORMAL vectors
  - QR factorization
  - SVD
  - Ellipse axis directions
  - Decomposing A so $U^H AU$ is upper triangular