Homework 2 Solutions

1 Pg 76 #24

We may write the augmented matrix for the first given system as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

Subtracting row 2 from row 1 and then row 3 from row 2 we get the following reduced echelon matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_2 \\ 0 & 1 & 1 & b_2 - b_3 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

By back substitution, a solution for the above system of linear equation*s has the form $(b_1 - b_2, b_2 - b_3, b_3)$, which exists for every tuple (b_1, b_2, b_3).

For the second matrix, we subtract row 2 from row 1 to obtain the following row echelon matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right]$$

To make sure that the system of equation*s is consistent, we require $b_3 = 0$. Then, a general solution is of the form $(b_1 - b_2, b_2, 0)$. Thus, the set of tuples for which the above system has a solution is $\{(b_1, b_2, b_3) | b_3 = 0\}$.

2 Pg. 88 #33

For the first system of equation*s, we have the following augmented coefficient matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right]$$

Carry out the following row operations to obtain its row echelon matrix. Add R_1 to R_3 and subtract $2R_1$ from R_2. Finally, subtract $2R_2$ from R_1 and divide R_2 by 3. This gives us the following matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By back substitution, we get $z = 1$ and $x = -2 - 3y$. Thus, a general solution is of the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 - 3y \\ y \\ 1 \end{bmatrix}$$

and we may write the complete solution as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} y$$

For the second system, we obtain the following row echelon matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, we have $2z + 4t = 1$ and $x + 3y + z + 2t = 1$. The general solution is:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 3y \\ y \\ \frac{1}{2} - 2t \\ t \end{bmatrix}$$

We may write the complete solution as:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} t$$

3 Pg. 89 #43

The rank of a matrix is the number of pivots that the matrix has. To find this we find the row echelon form.

For the first matrix, add $2R_2$ to R_1 and add $3R_2$ to R_3. Rearranging rows, we get:

$$\begin{bmatrix} -3 & -2 & -1 \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, if $q \neq 3$, the matrix has rank 2, and if $q = 3$, the matrix has rank 1. There is no value of q where the matrix has rank 3.

For the second matrix, subtract $2R_1$ from R_2. This gives:

$$\begin{bmatrix} 3 & 1 & 3 \\ q-6 & 0 & q-6 \end{bmatrix}$$

If $q = 6$, then the matrix has rank 1. If $q \neq 6$, the matrix has rank 2. This matrix cannot have rank 3 since the maximum possible number of pivots is 2.

4 Pg. 91 #68

Part (a): Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A has a null space of dimension 1, while A^T has a null space of dimension 2.

Part (b): Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

This matrix has no free variables, while its transpose has 1 free variable.

Part (c): This is false in general since the transpose of a row reduced matrix need not be row reduced. Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

.

The transpose of the above matrix is not row reduced.

5 Pg 98 #2:

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Then rref(A) is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension of the null space of this matrix is 3, since there are 3 free variables, which means the maximum number of linearly independent vectors is 3.

6 Pg. 99 #13

For A, the reduced echelon matrix is:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix has 2 pivots, so the dimension of the column space is 2. For the row space we look at the column space of A^T.

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The echelon form is:

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This has 2 pivots so the dimension of the row space is 2.

The matrix U is the echelon form of A. Row equivalent matrices have the same row space, thus the dimension of the row space of U is 2. Since this matrix is already in echelon form, we can see there are 2 pivots, so again the dimension of the column space of U is 2.

7 Pg. 101 #29

For A:

First suppose $c = 0$. We get the matrix:

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

If $d = 2$, then we get 2 pivots and A has rank 2. If $d \neq 2$, then A has rank 3. If $c \neq 0$, then there is no way to eliminate c, so the rank is always 3.

For B:

Say $c = 0$. Then B is of the form

$$\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \sim \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$$

Above, the $\sim$ means row equivalent. This matrix has rank 2 only when $d \neq 0$. Now say $c \neq 0$. We can divide by c and obtain the following echelon form:

$$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & \frac{c}{d} - \frac{d}{c} \end{bmatrix}$$

This matrix has rank 2 when $\frac{c}{d} - \frac{d}{c} \neq 0$, that is, when $c^2 - d^2 \neq 0$. Combining both conditions, B has rank 2 when $c \neq \pm d$.

8 Pg. 113 #32

For A:
The elements of the row space of A are of the form:

$$a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$$

for scalars a, b.
The elements of the column space of A are of the form:

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

for scalars a, b. For the null space, the augmented matrix is:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is already in echelon form. We have $y = 0$ and $z = 0$, so the elements of the null space of A are of the form:

$$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

for scalar z.

The augmented matrix for A^T is:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

By rearranging rows, we get the echelon form. Regardless, we can see that the left null space is the same as the null space of A since the rows are just rearranged in this case, and back-substitution yields the same answer.

For $I + A$:

The echelon for this matrix is the identity matrix. This has 3 pivots so the column space must be the whole space, i.e. $\mathbb{R}^3$. Similary, one can see that the echelon form for $(I + A)^T$ is also the identity. Thus, the row space is also $\mathbb{R}^3$. Since row equivalent matrices have the same null space, the null space and left null space of $I + A$ are both trivial, that is they contain only the zero vector.

9 Pg. 113 #37

Part (a): This is true, because the number of pivots is the dimension of column space. We know the dimension of the column space of A is equal to the dimension of the row space of A, which in turn is the dimension of the column space of A^T.

Part (b): This is not true, consider the 2×3 matrix with first row all ones and second row all zeroes.

Part (c): This is false: Consider the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Part (d): This is true. Consider any general matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Using the condition given, we can simplify the above matrix to the form:

$$\begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}.$$

This matrix clearly has the same row space and column space.