MATH 511 HOMEWORK (ON GRADESCOPE)

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Homework # 5

Due on July 14, 2024, 11:59 PM

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Section 5.1: P241 #5, ~~P242 #14~~,
Section 5.2: P250 #3,
Section 5.3: P264 #11,
Section 5.4: P277 #15, P278 #21
Section 5.5: P291 #31,
Section 5.6: P303 #22

5.1.p241.#05

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5. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

$$\det(A - \lambda I) = 0$$

A)

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 0-\lambda \end{bmatrix} \to (3-\lambda)(1-\lambda)(0-\lambda) = 0$$

$\lambda = 3 \quad \lambda = 1 \quad \lambda = 0$

$$\lambda = 0 \to \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{aligned} 3x_1 + 4x_2 + 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ t &= x_3 \end{aligned}$$

$$\begin{aligned} x_2 + 2t &= 0 \\ x_2 &= -2t \end{aligned}$$

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 &= 0 \\ 3x_1 + 4(-2t) + 2t &= 0 \\ 3x_1 - 8t + 2t &= 0 \\ 3x_1 - 6t &= 0 \\ 3x_1 &= 6t \\ x_1 &= 2t \end{aligned}$$

$$t = 1 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{aligned} 2x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_3 &= 0 \\ -x &= 0 \end{aligned}$$

$$x_3 = 0 \to \begin{aligned} 2x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + 4x_2 + 2(0) &= 0 \\ 2x_1 + 4x_2 &= 0 \\ 2x_1 &= -4x_2 \\ x_1 &= -2x_2 \end{aligned} \to \begin{aligned} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned} \to \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 3 \to \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \to \begin{aligned} 4x_2 + 2x_3 &= 0 \\ -2x_2 + 2x_3 &= 0 \\ -3x_3 &= 0 \end{aligned}$$

$$x_3 = 0 \to \begin{aligned} -2x_2 + 2x_3 &= 0 \\ -2x_2 + 2(0) &= 0 \\ -2x_2 &= 0 \\ x_2 &= 0 \end{aligned} \to \begin{aligned} 4x_2 + 2x_3 &= 0 \\ 4(0) + 2(0) &= 0 \\ 0 &= 0 \end{aligned} \to x_1 = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{TRACE}(A) &= \lambda_1 + \lambda_2 + \lambda_3 = A_{11} + A_{22} + A_{33} = \\ &= 0 + 1 + 3 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{DET}(A) &= 0 \cdot 1 \cdot 3 \\ &= 0 \end{aligned}$$

B)

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 0-\lambda \end{bmatrix} \to \det = (0-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 2-\lambda \\ 2 & 0 \end{vmatrix}$$

$$(-\lambda)(2-\lambda)(-\lambda) + 2(-2(2-\lambda))$$

$$\lambda^2(2-\lambda) - 4(2-\lambda)$$

$$\lambda = 2, \quad \lambda = -2 \to \lambda_1 = \lambda_2 = 2, \lambda_3 = -2$$

$$\lambda = 2 \to \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \to \begin{aligned} -2x_1 + 2x_3 &= 0 \\ 2x_1 - 2x_3 &= 0 \end{aligned}$$

$$\begin{aligned} 2x_1 - 2x_3 &= 0 \\ 2x_1 &= 2x_3 \\ x_1 &= x_3 \\ x_2 &= t \end{aligned} \to t = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; \overset{t=0}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$\det(B) = -\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = -8$$

$$\begin{aligned} \text{TRACE}(B) &= \lambda_1 + \lambda_2 + \lambda_3 = A_{11} + A_{22} + A_{33} \\ &= 2 + 2 - 2 \\ &= 2 \end{aligned}$$

$$\lambda = -2 \to \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{aligned} 2x_1 + 2x_3 &= 0 \\ 4x_2 &= 0 \\ x_2 &= 0 \end{aligned} \to \begin{aligned} 2x_1 &= -2x_3 \\ x_1 &= -x_3 \end{aligned} \to \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \text{DET}(B) &= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = A_{11} \cdot A_{22} \cdot A_{33} \\ &= 2 \cdot 2 \cdot -2 \\ &= -8 \end{aligned}$$

5.2.p250.#03

[chapter].[section].[page].[problem]

3. Find *all* the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and write two different diagonalizing matrices S.

$$\det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \rightarrow (1-\lambda)((1-\lambda)^2-1) - 1((1-\lambda)-1) + 1(1-(1-\lambda))$$

$$= (1-\lambda)((1-\lambda)^2-1) - 1(-\lambda) + 1(\lambda)$$

$$(1-\lambda)(1-2\lambda+\lambda^2-1) + \lambda + \lambda$$

$$(1-\lambda)(\lambda^2-2\lambda) + 2\lambda$$

$$\lambda^2 \cancel{-2\lambda} - \lambda^3 + 2\lambda^2 \cancel{+2\lambda}$$

$$-\lambda^3 + 3\lambda^3$$

$$\lambda^2(\lambda-3)$$

$$\lambda_1 = \lambda_2 = 0, \lambda_3 = 3$$

$$\lambda = 0 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3 \rightarrow -1$$

$$x_2 = x_2 \rightarrow 1$$

$$x_3 = x_3 \rightarrow 0$$

$$\rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 3 \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{P_{13}} \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \xrightarrow[A_{13}(2)]{A_{12}(-1)} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{A_{23}(1)} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{M_2(-1/3)} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 + x_2 - 2x_3 = 0$$

$$\rightarrow x_1 + x_3 - 2x_3 = 0$$

$$x_1 - x_3 = 0 \; x$$

$$x_2 - x_3 = 0$$

$$x_3 = x_3$$

$$x_1 = x_3$$

$$x_2 = x_3$$

$$x_1 = x_2 = x_3$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{array}{ccc} & P & D \\ S_1(A) = & \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} & P & D \\ S_2(A) = & \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

5.3.p264.#11

[chapter].[section].[page].[problem]

11. (a) From the fact that column 1 + column 2 = 2(column 3), so the columns are linearly dependent, find one eigenvalue and one eigenvector of A:

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

(b) Find the other eigenvalues of A (it is Markov).

(c) If $u_0 = (0, 10, 0)$, find the limit of $A^k u_0$ as $k \to \infty$.

a) DUE TO LINEAR DEPENDENCE, $\det(A) = \emptyset$

$\lambda_1 = \emptyset$

$$A = \left[\begin{array}{ccc|c} .2 & .4 & .3 & \emptyset \\ .4 & .2 & .3 & \emptyset \\ .4 & .4 & .4 & \emptyset \end{array}\right] \to \begin{array}{l} .2x_1 + .4x_2 + .3x_3 = \emptyset \\ .4x_1 + .2x_2 + .3x_3 = \emptyset \\ .4x_1 + .4x_2 + .4x_3 = \emptyset \end{array} \to \begin{array}{l} -.2x_1 + .2x_2 = \emptyset \\ x_1 = x_2 \\ x_1 + x_2 + x_3 \end{array} \to \begin{array}{l} x_1 + x_1 + x_3 = \emptyset \\ 2x_1 + x_3 = \emptyset \\ x_3 = -2x_1 \end{array} \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

b) DUE TO MARKOV, $\lambda_2 = 1$

$$\text{TRACE} = \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$
$$\emptyset + 1 + \lambda_3 = .2 + .2 + .4$$
$$1 + \lambda_3 = .8$$
$$\lambda_3 = -.2$$

$\lambda_1 = \emptyset, \lambda_2 = 1, \lambda_3 = -.2$

c) $x_1 + x_2 + x_3 = 1$

$$\begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix} \begin{array}{l} x_1 = x_2 \to v_1 = v_2 \\ \\ \to .4x_1 + .4x_2 + .4x_3 = v_3 \end{array} \quad \begin{array}{l} .4(x_1 + x_2 + x_3) = v_3 \\ .4(1) = v_3 \\ v_3 = .4 \end{array} \to \begin{array}{l} v_1 = v_2 \\ v_3 = .4 \\ v_1 + v_2 + v_3 = 1 \end{array} \to \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix}$$

$$u_\emptyset = (\emptyset, 1\emptyset, \emptyset) \quad \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} = \begin{bmatrix} 3\emptyset \\ 3\emptyset \\ 4\emptyset \end{bmatrix} = \text{LIMIT OF } A^k u_\emptyset \text{ as } k \to \infty$$

5.4.p277.#15

[chapter].[section].[page].[problem]

15. Solve the second-order equation

$$\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} -5-\lambda & -1 \\ -1 & -5-\lambda \end{bmatrix} \rightarrow (-5-\lambda)(-5-\lambda) - (-1)(-1)$$

$(\lambda+5)(\lambda+5)-1$

$\lambda^2+10\lambda+25-1$

$\lambda^2+\underset{\text{ADD}}{10}\lambda+\underset{\text{MUL}}{24}$

$(\lambda+6)(\lambda+4)$

$\lambda=-6,\ \lambda=-4$

$$\lambda_1 = -4 \rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} -x_1 - x_2 = 0 \\ x_2 = x_2 \end{matrix} \rightarrow \begin{matrix} -x_1 = x_2 \\ x_1 = -x_2 \end{matrix} \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -6 \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 - x_2 = 0 \\ x_2 = x_2 \end{matrix} \rightarrow x_1 = x_2 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u(0) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$C_1 + C_2 = 1$

$-C_1 + C_2 = 0$

$C_2 = C_1$

$C_1 + C_2 = 1 \rightarrow C_1 = 1/2,\ C_2 = 1/2$

$C_1 + C_1 = 1$

$2C_1 = 1$

$C_1 = 1/2$

$$u(t) = -1/2 \cos(\sqrt{4}t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1/2 \cos(\sqrt{6}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u(t) = -1/2 \cos(2t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1/2 \cos(\sqrt{6}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5.4.p278.#21

[chapter].[section].[page].[problem]

21. Find λ's and x's so that $u = e^{\lambda t}x$ solves

$$\frac{du}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} u.$$

What combination $u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ starts from $u(0) = (5, -2)$?

$$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix} \rightarrow (4-\lambda)(1-\lambda) \quad \lambda_1 = 4 \;\; \lambda_2 = 1$$

$$\lambda_1 = 4 \rightarrow \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = x_1 \\ 3x_2 = 0 \end{matrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow u_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \rightarrow \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_2 = x_2 \end{matrix} \rightarrow x_1 = -x_2 \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow u_2 = e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u(0) = \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{A_{21}(1)} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right] \rightarrow \begin{matrix} c_1 = 3 \\ c_2 = -2 \end{matrix}$$

$$u(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5.5.p291.#31

[chapter].[section].[page].[problem]

30. Which classes of matrices does P belong to: orthogonal, invertible, Hermitian, unitary, factorizable into LU, factorizable into QR?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

31. Compute P^2, P^3, and P^{100} in Problem 30. What are the eigenvalues of P?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^2 = P \cdot P = \overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \cdot \overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P^2$$

$$P^3 = P^2 \cdot P = \overset{P^2}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \cdot \overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = P^3$$

$$P^{100} = (P^3)^{33+1} = P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

5.6.p303.#22

[chapter].[section].[page].[problem]

22. Find a unitary U and triangular T so that $U^{-1}AU = T$, for

$$A = \begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

$$A = \begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix} \to \det(A) = \begin{bmatrix}5-\lambda & -3\\ 4 & -2-\lambda\end{bmatrix} \to (5-\lambda)(-2-\lambda)-(4)(-3)$$

$$-10-5\lambda+2\lambda+\lambda^2+12$$

$$\lambda^2-3\lambda+2$$

$$(\lambda-1)(\lambda-2)$$

$\lambda = 1$

$$\begin{bmatrix}4 & -3\\ 4 & -3\end{bmatrix} \xrightarrow{A_{12}(-1)} \left[\begin{array}{cc|c}4 & -3 & 0\\ 0 & 0 & 0\end{array}\right] \to \begin{aligned}4x_1 - 3x_2 &= 0\\ 4x_1 &= 3x_2\\ x_1 &= \tfrac{3}{4}x_2\end{aligned} \xrightarrow{x_2 = 4} \begin{bmatrix}3\\4\end{bmatrix} \to \frac{1}{\sqrt{3^2+4^2}} \cdot \begin{bmatrix}3\\4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3/5\\4/5\end{bmatrix}$$

$$9+16$$

$$25$$

$$\lambda = 2 \begin{bmatrix}3 & -3\\ 4 & -4\end{bmatrix} \xrightarrow{M_1(1/3)} \begin{bmatrix}1 & -1\\ 4 & -4\end{bmatrix} \xrightarrow{A_{12}(-4)} \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} \to \begin{aligned}x_1 - x_2 &= 0\\ x_1 &= x_2\end{aligned} \to \begin{bmatrix}1\\1\end{bmatrix} \to \frac{1}{\sqrt{1^2+1^2}} \cdot \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$U = \begin{bmatrix}1/\sqrt{2} & 3/5\\ 1/\sqrt{2} & 4/5\end{bmatrix} \to \underset{\text{UNITARY + REAL MATRIX}}{U^{-1} = U^T} \to \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 3/5 & 4/5\end{bmatrix} = U^{-1}$$

$$T = U^{-1} A U$$

$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 3/5 & 4/5\end{bmatrix} \begin{bmatrix}5 & -3\\ 4 & -2\end{bmatrix} \begin{bmatrix}1/\sqrt{2} & 3/4\\ 1/\sqrt{2} & 4/5\end{bmatrix}$$

$$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 3/5 & 4/5\end{bmatrix} \begin{bmatrix}\sqrt{2} & 3/5\\ \sqrt{2} & 4/5\end{bmatrix}$$

$$T = \begin{bmatrix}2 & \frac{7}{5\sqrt{2}}\\ \frac{7\sqrt{2}}{5} & 1\end{bmatrix} \to \begin{bmatrix}2 & \frac{7}{5\sqrt{2}}\\ 0 & 1\end{bmatrix} = T$$

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} \to \det\left(\begin{bmatrix}0-\lambda & 1 & 0\\ 0 & 0-\lambda & 0\\ 1 & 0 & 0-\lambda\end{bmatrix}\right) = -\lambda \det\begin{vmatrix}-\lambda & 0\\ 0 & -\lambda\end{vmatrix} - 1 \det\begin{vmatrix}0 & 0\\ 1 & -\lambda\end{vmatrix} + 0$$

$$= -\lambda(\lambda^2) - 0 = -\lambda^3 = 0 \to \lambda_{1,2,3} = 0$$

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} \to \begin{aligned}x_1 &= 0\\ x_2 &= 0\\ x_3 &= x_3\end{aligned} \to \begin{bmatrix}0\\0\\x_3\end{bmatrix} \to \begin{bmatrix}0\\0\\1\end{bmatrix} \to \left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{array}\right] \to \begin{aligned}x_1 &= 1\\ x_2 &= 0\\ x_3 &= x_3\end{aligned} \to \begin{bmatrix}1\\0\\0\end{bmatrix} \to \left[\begin{array}{ccc|c}0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right] \to \begin{aligned}x_1 &= 0\\ x_2 &= 1\\ x_3 &= x_3\end{aligned} \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$U = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix} \to U^{-1} = U^T = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \to T = U^{-1} A U = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = T$$