INFINITY CATEGORIES

LECTURER: PRAMOD ACHAR, TYPESETTER: MICHAŁ MRUGAŁA

§1. SIMPLICIAL SETS

Definition. — The **simplex category** Δ is the category of

- finite non-empty totally ordered sets
- order-preserving maps.

Notation. —
$$[n] = \{0 < 1 < 2 < \cdots < n\} \text{ for } n \in \mathbb{Z}_{>0}.$$

Every object in Δ is (uniquely) isomorphic to some [n].

Definition. — A **simplicial set** is a functor

$$\mathscr{S}:\Delta^{\mathsf{op}}\longrightarrow\mathsf{Sets}$$

Notation. — $\mathcal{S}_n := \mathcal{S}([n])$, call this the **set of** *n***-simplices** of \mathcal{S} . 0-simplices are called **vertices**, 1-simplices are called edges.

Example 1.0.1. — Let *C* be a set. Let $\underline{C} : \Delta^{op} \to \mathsf{Sets}$ be the constant functor:

$$\underline{C}_n = C \quad \forall n,$$

$$\underline{C}(\alpha) = \mathrm{id} \quad \forall \alpha : [m] \longrightarrow [n] \text{ in } \Delta.$$

This is called a **discrete simplicial set**.

Definition. — Let $\mathscr S$ be a simplicial set. Given $\alpha : [n] \to [n-1]$ we get $\mathscr S(\alpha) : \mathscr S_{n-1} \to \mathscr S_n$. The *n*-simplices in the image are called **degenerate** simplices, i.e. σ is degenerate if there is an α such that $\sigma \in \operatorname{im}(\mathscr{S}(\alpha))$.

Lemma 1.0.1. — A simplicial set is discrete if and only if for all n > 1 all n-simplices are degenerate.

Exercise. — Prove it.

Example 1.0.2. — Let (P, \geq) be a poset. Define a simplicial set $N(P, \leq)$ called the **nerve** of (P, \leq) by

$$N(P, \leq)_k = \{ \text{chains } p_0 \leq p_1 \leq \cdots \leq p_k : p_i \in P \}$$

where a chain is a totally ordered subset.

Exercise. — Finish the definition. Which simplices are degenerate?

Example 1.0.3 ("Standard n-simplex"). — The standard n-simplex is

$$\Delta^n := N([n]).$$

(Pictures)

Note. — For $j \in [n]$, we get a subsimplicial set

$$N([n] \setminus \{j\}) \subset \Delta^n$$

isomorphic to Δ^{n-1} called the j^{th} **face** of Δ^n . (Picture)

Example 1.0.4 (Horns). — Let $n \ge 0$ and $0 \le j \le n$, define the **horn**

subsimplicial set of
$$\Delta^n = N([n])$$

 $\Lambda^n_j := \text{consisting of chains } p_0 \leq p_1 \leq \cdots \leq p_k \text{ (Pictures)}$
such that $\{p_0, \dots, p_k\} \not\supset [n] \setminus \{j\}.$

Example 1.0.5 ((n-1)**-sphere** $\partial \Delta^n$ **).** — We define the (n-1)**-sphere**

$$\partial \Delta^n := \begin{array}{c} ext{subsimplicial set of } \Delta^n \\ ext{chains } p_0 \leq \cdots \leq p_k \text{ such that } \{p_0 \leq \cdots \leq p_k\} \neq [n] \end{array}$$

Example 1.0.6 (Products). — Let \mathscr{S} , \mathscr{T} be simplicial sets. We define their **product** $\mathscr{S} \times \mathscr{T}$ as

$$(\mathscr{S} \times \mathscr{T})_k = \mathscr{S}_k \times \mathscr{T}_k.$$

(Picture)

Example 1.0.7. — Let C be an ordinary category. We define its **nerve** N(C) as

$$N(\mathsf{C})_k := \left\{ \begin{array}{c} \text{composable sequences of morphisms} \\ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \dots \xrightarrow{f_k} X_k \end{array} \right\}.$$

Example 1.0.8. — Let *X* be a topological space. The **singular simplicial set** of *X* is defined as

$$\operatorname{Sing}(X)_k = \{ \operatorname{continuous maps} |\Delta^k| \longrightarrow X \},$$

where $|\Delta^k|$ is the standard k-simplex

$$|\Delta^k| = \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1} \middle| x_i \ge 0, \sum x_i = 1 \right\}.$$

Exercise. — What does this do to the morphisms in Δ ?

Definition. — A **Kan complex** is a simplicial set *X* such that for every diagram



we can fill the dashed arrow. This is called an **extension problem**. If the arrow exists we say that the extension problem **admits a solution**.

Daniel Kan discovered Kan complexes in 1958. The *key fact* is that Sing(X) is a Kan complex. The theme from 1958 to today is that Kan complexes are a "combinatorial model" for algebraic topology which allows us to do homotopy theory.

Definition. — Let X be a Kan complex and $\mathscr S$ be any simplicial set. Two maps $f,g:\mathscr S\to X$ are said to be **homotopic** if there exists a map $H:\mathscr S\times\Delta^1\to X$ such that

$$H|_{\mathcal{S}\times\{0\}}=f,\quad H|_{\mathcal{S}\times\{1\}}=g.$$

Lemma 1.0.2. — This is an equivalence relation.

Proof. Omitted, tricky for an exercise. This requires *X* to be a Kan complex.

Definition. — Let X be a Kan complex and x_0 be a vertex of X. Let

$$\text{Loops}_{x_0} = \{\text{maps } \gamma : \Delta^n \longrightarrow X \text{ such that } \gamma|_{\partial \Delta^n} \text{ is the constant map to } x_0\}.$$

We say $\gamma, \gamma' \in \text{Loops}_{x_0}$ are **relatively homotopic** (**rel. homotopic**) if there exists $H: \Delta^n \times \Delta^1 \to X$ such that

$$H|_{\Delta^n \times \{0\}} = \gamma$$
, $H|_{\Delta^n \times \{1\}} = \gamma'$, $H|_{\partial \Delta^n \times \Delta^1} = \text{const. map to } x_0$.

Define

$$\pi_n(X, x_0) := \frac{\text{Loops}_{x_0}}{\text{rel. homotopy}}.$$

Fact. — For $n \ge 1$, $\pi_n(X, x_0)$ is a group. For $n \ge 2$, $\pi_n(X, x_0)$ is abelian.

Definition. — An ∞ -category (or quasi-category) is a simplicial set $\mathscr C$ such that any extension problem



with 0 < j < n (inner horns) admits a solution. (Picture) An ∞ -category is also called a **weak Kan complex**.

Lemma 1.0.3. — Let C be an ordinary category, then N(C) is an ∞ -category.

Digression: Let I^n be the simplicial set consisting of n consecutive 1-simplices (n-spine) (Picture). A naive alternative definition is: \mathscr{C} is an infinity category if every



has a solution. This is WRONG (but its wrongness is subtle), even though N(ord. cat.) satisfy this. There is a book by Markus Land "Introduction to ∞ -categories" which explores this. The definition of ∞ -categories was introduced by Boardman-Vogt in 1972. Joyal started generalizing results from ordinary category theory to ∞ -categories in 2006. Lurie is largely responsible for how well this notion is developed in modern literature.

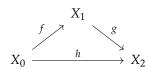
Remark. — Having a unique solution to the lifting problem characterizes nerves of ordinary categories.

Definition. — Let \mathscr{C} be an ∞ -category. An **object** is a vertex. A **morphism** is an edge. An **identity morphism** is a degenerate edge. Say that h is a **composition** of g and f if there exists a 2-simplex such that (Picture).

Remark. — Compositions are NOT unique in ∞-categories.

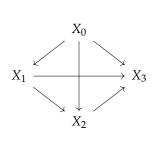
Example 1.0.9 (∞ -categories). —

- 1) Topological spaces Top.
 - Objects are topological spaces.
 - Morphisms are continuous maps.
 - A 2-simplex is a (not necessarily commutative) diagram



and a homotopy $H: X_0 \times [0,1] \to X_2$ from gf to h.

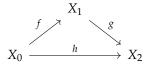
• A 3-simplex is a diagram



with continuous maps $f_{ij}: X_i \to X_j$ for i < j, homotopies $T_{ijk}X_i \times [0,1] \to X_k$ from $f_{jk} \circ f_{ij}$ to f_{ik} , and $H: X_0 \times [0,1]^2 \to X_3$ (**higher homotopy**) such that $H|_{\text{bdry}}$ is

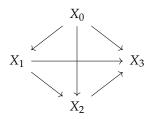
$$\begin{array}{ccc}
(0,0) & \xrightarrow{T_{123}f_{01}} & (0,1) \\
f_{23}T_{012} \downarrow & & \downarrow T_{013} \\
(1,0) & \xrightarrow{T_{023}} & (1,0)
\end{array}$$

- 2) The ∞-category of ordinary categories Cat_1 .
 - Objects are ordinary categories.
 - Morphisms are functors.
 - A 2-simplex is a (not necessarily commutative) diagram

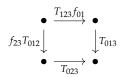


and a natural isomorphism $T: g \circ f \xrightarrow{\sim} h$.

• A 3-simplex is a diagram



where f_{ij} are functors and T_{ijk} are natural isomorphism such that



commutes

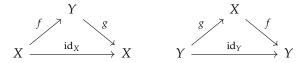
A source of ∞-categories are

- ordinary categories with an equivalence relation on morphisms,
- ordinary categories with inverting some morphisms.

Lecture 2

Definition. —

1. Let C be an ∞-category and $f: X \to Y$ be a morphism in C. f is called an **isomorphism** if there exists $g: Y \to X$ and two 2-simplices



2. An ∞ -category is called an ∞ -groupoid if *every* morphism is an isomorphism.

Theorem 1.0.1 (Joyal). — An ∞ -category is an ∞ -groupoid if and only if it is a Kan complex.

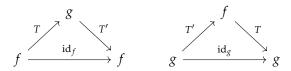
Proof. The forward direction is hard, the converse is an exercise.

Definition. —

3. Say $f,g: C \to D$ are functors (morphisms of simiplicial sets) of ∞-categories. A **natural transformation** from f to g is a functor $T: C \times \Delta^1 \to D$ such that $T|_{C \times \{0\}} = f$ and $T|_{C \times \{1\}} = g$. A special case: the identity natural transformation $\mathrm{id}_f: f \to F$ is the map

$$\mathsf{C} \times \Delta^1 \xrightarrow{\mathsf{proj}} \mathsf{C} \xrightarrow{f} \mathsf{D}.$$

 $T: f \to g$ is a **natural isomorphism** if there exists $T': g \to f$ and two maps $H: C \times \Delta^2 \to D, H': C \times \Delta^2 \to D$ such that



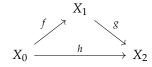
In ordinary category theory a natural transformation assigns objects in C to morphisms in D and morphisms in C to commutative squares in D. For ∞ -categories a natural transformation takes objects to morphisms, morphisms to diagrams of shape $\Delta^1 \times \Delta^1$ and generally an n-simplex to a diagram of shape $\Delta^n \times \Delta^1$.

Theorem 1.0.2 (Pointwise criterion for natural isomorphism). — Let $f,g: C \to D$ be functors of ∞ -categories and $T: f \to g$ be a natural transformation. T is a natural isomorphism if and only if for all objects x in C, $T(\{x\} \times \Delta^1)$ is an isomorphism in D.

This is a consequence of Joyal's theorem.

Definition. — Define Cat_{∞} as follows:

- Objects are ∞-categories.
- Morphisms are functors.
- 2-simplices are diagrams



and a natural isomorphism $T: g \circ f \xrightarrow{\sim} h$.

• 3-simplices and higher: copy the data of Top and replace $[0,1]^n$ by $(\Delta^1)^n$.

This is similar to Top and Cat_1 .

Definition. — Define Spc same as above, except objects are ∞ -groupoids.

In literature: ∞-groupoids, Kan complexes, spaces and anima are synonyms.

Definition. — A functor $f: C \to D$ is called a **categorical equivalence** if there exists $g: D \to C$ such that $f \circ g \simeq \mathrm{id}_D$ and $g \circ f \simeq \mathrm{id}_C$.

Theorem 1.0.3 (Fundamental Theorem of Category Theory). — A functor $f : C \to D$ is a categorical equivalence if and only it it's essentially surjective and fully faithful.

Note that we haven't defined essentially surjective or fully faithful. Let's pre-warm up first before we define them.

Lemma 1.0.4. — Let X be a Kan complex. X is **contractible** (i.e., categorically equivalent to Δ^0) if and only if every lifting problem

$$\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \Delta^0
\end{array}$$

admits a solution.

Now we're warm enough to warm up, so lets do that. Let $f: X \to Y$ be a map of Kan complexes. Suppose every lifting problem

$$\frac{\partial \Delta^n}{\longrightarrow} X$$

$$\downarrow f$$

$$\Delta^n \longrightarrow Y$$

has a solution. Then *f* is a categorical equivalence (think: homotopy equivalence of topological spaces). But this condition is too strong for the converse. A simple counter-example is to take *X* contractible.

Definition. — Let $f: C \to D$ be a functor of ∞ -categories. Given

(1)
$$\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{r} & \mathsf{C} \\
\downarrow & & \downarrow_f \\
\Delta^n & \xrightarrow{s} & \mathsf{D}
\end{array}$$

we say it admits a **solution up to isomorphism** if

(i) there exists $u : \Delta^n \to C$ such that

$$\frac{\partial \Delta^n}{\downarrow} \xrightarrow{u} C$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Delta^n$$

(ii) $f \circ u : \Delta^n \to D$ is naturally isomorphic to $s : \Delta^n \to D$ relative (of relative homotopy) to $\partial \Delta^n$.

Definition. — Let $f : C \to D$ be a functor of ∞-categories.

- It's **essentially surjective** if every diagram (1) with n = 0 admits a solution up to isomorphism.
- It's **full** if every diagram (1) with n = 1 admits a solution up to isomorphism.
- It's fully faithful if every diagram (1) with $n \ge 1$ admits a solution up to isomorphism.

So a functor of ∞ -categories is fully faithfull and essentially surjective if all (1) admit a solution up to isomorphism.

Remark. — These definitions of fully and full faithful are *nonstandard*.

Now the Fundamental Theorem makes sense.

Proof idea. The forward direction is easy. Conversely, we factor through

$$C \longrightarrow C^{enhanced} \longrightarrow D$$

where an *n*-simplex in C^{enhanced} is the data of

- *n*-simplex in C,
- a diagram of shape $\Delta^n \times \Delta^1$ in D satisfying some conditions.

The inverses of the intermediate maps are easy to construct.

What is missing so far is *mapping spaces*. Given objects X, Y in an ∞ -category C we expect to find a space (Kan complex) $\operatorname{map}_{C}(X, Y)$ such that the objects of $\operatorname{map}_{C}(X, Y)$ are morphisms $X \to Y$ and it should extend to a functor

$$map_C : C^{op} \times C \longrightarrow Spc.$$

For 1-categories this is usually called Hom or Mor. Lurie uses Hom for a non-functorial, but easier, version of map.

Here is one non-functorial approach to mapping spaces. Let C^{Δ^1} be the simplicial set such that $(C^{\Delta^1})_k$ is the sest of maps $\Delta^1 \times \Delta^k \to C$. By restricting to $\{0\} \times \Delta^k$ and $\{1\} \times \Delta^k$ we get a map

 $C^{\Delta^1} \to C \times C$. Define $map_C(X,Y)$ as the fiber product (of simplicial sets)

$$\begin{array}{ccc}
\operatorname{map}_{\mathsf{C}}(X,Y) & \longrightarrow & \mathsf{C}^{\Delta^{1}} \\
\downarrow & & \downarrow \\
\Delta^{0} & \xrightarrow{(x,y)} & \mathsf{C} \times \mathsf{C}.
\end{array}$$

Theorem 1.0.4. — Let $f: C \to D$ be a functor of ∞ -categories. Then f is fully faithful if and only if for all objects X, Y, the induced map

$$\operatorname{map}_{\mathsf{C}}(X,Y) \longrightarrow \operatorname{map}_{\mathsf{D}}(f(x),f(y))$$

is a categorical equivalence of Kan complexes.

This theorem is actually the usual definition in the literature.

Somewhere along the way:

Theorem 1.0.5 (Whitehead's Theorem). — A map $f: X \to Y$ of Kan complexes is a categorical equivalence if and only if

$$\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

are bijections for all n and all x_0 .

Lecture 3

A monoidal category in ordinary category theory consists of:

- A category C.
- A functor $\otimes : C \times C \to C$.
- An object $\mathbb{1} \in C$.
- 3 natural transformations: the associator, left and right unitors.

We ask them to satisfy 3 axioms:

- Triangle axioms (they say $\mathbb{1} \otimes x = x = x \otimes \mathbb{1}$.
- Pentagon axiom (various ways to group 4 objects).

Mac Lane's Coherence Theorem tells us that every "reasonable" diagram made from the 3 natural transformations commutes.

Let's try to mimic this for ∞-categories. The naive approach is to start with:

- an ∞-category C;
- a functor \otimes : $C \times C \rightarrow C$;
- an object $1:\Delta^0\to C$;
- an associator

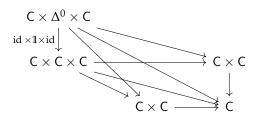
a diagram of shape $\Delta^1 \times \Delta^1$ in Cat_{∞} ;

• a left unitor

$$\Delta^0 \times C \xrightarrow{\quad \mathbb{1} \times id \quad} C \times C$$

a 2-simplex in Cat_∞ and similarly a right unitor; and ask it to satisfy

• the triangle identity



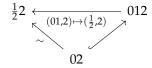
two 3-simplices attached along a face.

We model these diagrams with totally ordered finite sets:

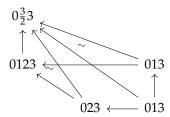
Associator

$$\begin{array}{ccc}
0123 &\longleftarrow & 013 \\
\uparrow & & \uparrow \\
023 &\longleftarrow & 03
\end{array}$$

• Left unitor



Triangle identity



The data for the naive approach is modeled by

$$N\left(\left(\begin{array}{c} \text{nonempty, finite, totally ordered} \\ \text{sets of size} \leq 4 \end{array}\right)^{\mathsf{op}}\right) \longrightarrow \mathsf{Cat}_{\infty}$$

We are still missing the pentagon axiom and Mac Lane's Coherence Theorem.

Definition. — Let C be an ∞ -category. A **monoidal structure** on C is a functor $F: N(\Delta^{op}) \to \mathsf{Cat}_{\infty}$ such that

- 1) F([1]) = C,
- 2) for all n

$$[n] \leftarrow \{0,1\}, \{1,2\}, \ldots, \{n-1,n\}$$

induces

$$F([n]) \longrightarrow \mathsf{C} \times \mathsf{C} \times \cdots \times \mathsf{C} = \mathsf{C}^n$$

which we require to be an equivalence of categories.

Note that $F([0]) \xrightarrow{\sim} \Delta^0$. The idea is that the pentagon axiom and all "higher coherences" are encoded in Δ^{op} .

The problem with this definition is unusable. Actually writing down a functor $N(\Delta^{op}) \to \mathsf{Cat}_{\infty}$ is too complicated. What do we do? Lurie will rescue us.

Let's warm up. Suppose you have a Λ_0^2 horn

$$0 \xrightarrow{f} 1 \\ \downarrow \exists ? \\ 2$$

so we are asking: given

$$\Lambda_0^2 \longrightarrow C$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\Lambda^2$$

when does a solution exist? It exists if f has a right inverse.

Definition. — Let $p : C \to D$ be a functor of ∞-categories, $f : x \to y$ be a morphism in C. We say f is p-cocartesian if

$$\{0,1\} = \Delta^1 \xrightarrow{f} C$$

$$\downarrow p$$

$$\Delta^n \longrightarrow D$$

has a solution.

Definition. — $p: C \to D$ is called a **cocartesian fibration** if lifting problems

$$\Lambda_j^n \xrightarrow{r} C
\downarrow \qquad \qquad \downarrow p
\Delta^n \longrightarrow D$$

have a solution when

- (1) 0 < j < n,
- (2) j = 0 and $n \ge 2$ if r sends $\{0,1\}$ to a p-cocartesian edge (this is actually automatic),
- (3) j = 0, n = 1; in this case we also *require* the solution $u : \Delta^1 \to C$ to be a *p*-cocartesian edge.

The idea is that a cocartesian fibration p : C → D should be thought of as a "functorial family of ∞ -categories indexed by D". More precisely:

- For each object x in D let $C_x = \{x\} \times_D C$. This is an ∞-category.
- For each edge $\gamma: x \to y$ in D and each n-simplex $\sigma: \Delta^n \to C_x$ we can construct a map that under

$$\widetilde{\sigma}: \Delta^n \times \Delta^1 \longrightarrow \mathsf{C}$$

sends $\{j\} \times \Delta^1$ to a *p*-cocartesian edge.

• Moreover, we get a functor

$$C_x \longrightarrow C_y$$

$$\sigma \longmapsto \widetilde{\sigma}_{\Delta^n \times \{1\}}$$

(this is slightly sloppy).

The definition is precisely set up so you can carry this out. Let's keep going:

 $\bullet\,$ A 2-simplex in D gives a 2-simplex in Cat_∞



Theorem 1.0.6 (Straightening-Unstraightening Theorem). — *There is an equivalence of* ∞ *-categories*

$$Cocart(D) \xrightarrow{\sim} Fun(D, Cat_{\infty})$$

between the ∞ -category of cocartesian fibrations over D and the ∞ -category of functors D \to CaT $_\infty$. The forward map is called **straightening** and the inverse is called **unstraightening**.

The left hand side is easier for humans:

- Writing down a functor $D \to \mathsf{Cat}_\infty$ involves making millions of choices and checking that they're compatible.
- Writing down a cocartesian fibration is *easier*: you write down *all* possible choices and don't bother with compatibility.

Definition. — Let C be an ∞-category. A **monoidal structure** on C is a cocartesian fibration

$$C^{\otimes} \longrightarrow N(\Delta^{op})$$

such that

$$\begin{split} \mathsf{C}_{[1]}^\otimes &= \mathsf{C} \\ \mathsf{C}_{[n]}^\otimes &\xrightarrow{\sim} \mathsf{C}_{\{0,1\}}^\otimes \times \mathsf{C}_{\{1,2\}}^\otimes \times \dots \times \mathsf{C}_{\{n-1,n\}}^\otimes. \end{split}$$

Let's go back to mapping spaces. We want a functor

$$C^{op} \times C \longrightarrow Spc \subset Cat_{\infty}$$

 $(x,y) \longmapsto map_{C}(x,y).$

This is impossible for humans.

Let's introduce the **twisted arrow category** Tw(C) for an ∞-category C. We define

$$\mathsf{Tw}(\mathsf{C})_k = \{ \mathsf{maps} \ N(k' < (k-1)' < \dots < 1' < 0' < 0 < 1 < \dots < k) \longrightarrow \mathsf{C} \}.$$

Of course that poset is isomorphic to Δ^{2k+1} but we want it to have this notation. We make the unprimed indices correspond to C and the primed indices to correspond to C^{op}, so we get a map

$$\mathsf{Tw}(\mathsf{C}) \longrightarrow \mathsf{C}^{\mathsf{op}} \times \mathsf{C}.$$

Lemma 1.0.5. — This is a cocartesian fibration whose fibers are Kan complexes.

So by Straightening-Unstraightening we get a functor

$$\mathsf{map}_\mathsf{C} : \mathsf{C}^\mathsf{op} \times \mathsf{C} \longrightarrow \mathsf{Spc} \subset \mathsf{Cat}_\infty.$$

By adjunction (1-categorical), we get

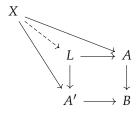
$$h: \mathsf{C} \longrightarrow \mathsf{Fun}(\mathsf{C}^\mathsf{op} \longrightarrow \mathsf{Spc}).$$

Theorem 1.0.7 (∞-categorical Yoneda's Lemma). — h is fully faithful.

Remark. — Straightening-Unstraightening gives you a framework for generalizing 1-categorical notions to ∞-categories. Some themes for going between ordinary and ∞-categories:

1-categories	∞-categories
Sets	Spaces
existence and uniqueness	existence for $\partial \Delta^n \to \Delta^n$ and $\forall n$ uniqueness
	up to a contractible Kan complex

1.1. Pullbacks In an ordinary category



L is the pullback of

$$A' \longrightarrow B$$

if given solid arrows there exists a unique dashed arrow.

Now for the ∞-categorical verson we need auxiliary simplicial sets

$$P_n = N(??)$$

(I need to think about how to write this, it's a poset $0 < \cdots < n+1$ with a square made at the and with a, a', b.) The standard notation for this simplicial set is $\Delta^{n+1} * \Lambda_2^2$, where * stands for the join

operation, which we won't define. Also define P_n^0 to be the subsimplicial set of P_n where we take chains that omit one of 0, 1, ..., n + 1. The standard notation is

$$\Delta^n*\Lambda^2_2\coprod_{\partial\Delta^n*\Lambda^2_2}\partial\Delta^n*(\Delta^1\times\Delta^1)$$

it also might be $\partial \Delta^{n+1} * \Lambda_2^2$ but Pramod wasn't sure.

(Pictures of P_i and P_i^0 .)

Definition. — Let C be an ∞-category. A diagram of shape $\Delta^1 \times \Delta^1$, say (Picture) is called a **pullback** if every extension problem (Diagram) admits a solution.

For n = 0 this should remind you of 1-categories. The moral is to think of filling in a sphere. WARNING: Suppose

$$A' \longrightarrow B$$

is a diagram of topological spaces, or ∞ -categories, or Kan complexes. The 1-categorical and ∞ -categorical pullbacks exist, but they *don't agree* in general. There is a map from the 1-limit to the ∞ -limit.

In Cat_∞ the ∞-categorical pullback

$$\begin{array}{ccc}
L & \longrightarrow & A \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B
\end{array}$$

is given by

$$L_n = \left\{ (\sigma, \sigma', J) \middle| \begin{array}{c} \sigma \in A_n, \sigma' \in A'_n, J : \Delta^n \times \Delta^1 \to B \\ J \text{ a natural isomorphism from } f \circ \sigma \text{ to } f' \circ \sigma' \end{array} \right\}.$$

In Top consider the diagram

$$\begin{array}{c}
 \text{pt} \\
\downarrow x_0 \\
 \text{pt} \xrightarrow{x_0} X
\end{array}$$

The 1-categorical pullback is a point. The ∞ -categorical pullback is $\Omega(X, x_0)$, the space of loops in X based at x_0 .

1-categorical limits are unique up to unique isomorphism, but ∞ -categorical limits it's unique up to a contractible Kan complex of (∞ -categorical) isomorphisms.

Example 1.1.1. — The following diagram is an ∞-categorical pullback

$$\mathbb{Z} \longrightarrow \mathsf{pt}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathsf{pt} \longrightarrow S^1$$

where \mathbb{Z} is the space of continuous maps $\gamma:[0,1]\to \mathbf{R}$ such that $\gamma(0)=0$ and $\gamma(1)\in\mathbb{Z}$.

Recall that an **abelian category** is a category such that:

- There is a zero (initial and final) object.
- All pullbacks and pushouts exist.
- If $f: x \to y$ is a monomorphism then the pushout

$$\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
0 & \longrightarrow z
\end{array}$$

is also a pullback.

• If $g: y \to z$ is an epimorphism then the pullback

$$\begin{array}{ccc}
x & \longrightarrow & y \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & z
\end{array}$$

is also a pushout.

Remark. — You can actually recover the abelian group structure on hom sets from only this!

Definition. — An ∞-category is called **stable** if:

- There is a zero object (defining this is an exercise).
- All pullbacks and pushouts exist.
- Any diagram of shape $\Delta^1 \times \Delta^1$



is a pullback if an donly if it's a pushout.

Remark. — 0 is the zero object and is actually an optional entry, we can ask this for all squares, the conditions are equivalent.

Example 1.1.2. — Start with an ordinary additive category A. Define an ∞ -category Ch(A) as follows:

- objects are chain complexes of objects in *A*,
- morphisms are chain maps,
- 2-simplices are diagrams



and a chain homotopy $s: A^{\bullet} \to C^{\bullet-1}$ such that

$$ds + sd = gf - h$$
.

• 3 and higher and higher chain homotopies.

This category is explicitly written out by Lurie. Look up the "dg nerve".

Lemma 1.1.1. — Ch(A) *is stable.*

Proof idea. The main step is:

- Take a chain map $f: A^{\bullet} \to B^{\bullet}$.
- Prove by hand that

$$A^{\bullet} \xrightarrow{f} B^{\bullet}$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \operatorname{Cone}(f)$$

is a pullback and a pushout.

Lemma 1.1.2. — In any stable ∞ -category C the functors that take X to the pullback/pushout of the diagram

$$\begin{array}{ccc} 0 & & X \longrightarrow 0 \\ \downarrow & & \downarrow \\ 0 \longrightarrow X & & 0 \end{array}$$

respectively.

The first functor is denoted Ω or [-1] and called the **loop space** functor. The second functor is denoted Σ or [1] and called the **suspension** functor.

Remark. — This is pretty closed to C being a stable ∞ -cateogry implying that hC is triangulated. The octahedral axiom is similar to the third isomorphism theorem.

Also we get derived categories of abelian categories: take Ch(A) and invert quasi-isomorphisms. In ordinary category theory:

- There is a canonical example of an abelian category, which is the category of abelian groups.
- *Every* abelian category is enriched over abelian groups.

Let's go back to ∞-categories.

Definition. — The ∞-category of **spectra** Sptr is

$$\mathsf{Sptr} = \varprojlim \left(\dots \xrightarrow{\Omega} \mathsf{Spc}_* \xrightarrow{\Omega} \mathsf{Spc}_* \xrightarrow{\Omega} \mathsf{Spc}_* \right)$$

where Spc_∗ are pointed spaces and we take the ∞-categorical limit in Cat_∞.

An object of Sptr is (roughly) a sequence of pointed spaces ..., X_3 , X_3 , X_1 , X_0 and isomorphisms $\Omega X_3 \simeq X_2$, $\Omega X_2 \simeq X_2$, $\Omega X_1 \simeq X_0$, etc.

1-land	∞-land
Canonical example: Ab	Canonical example: Sptr
Most important abelian group: Z	Most important spectrum: S with $\pi_0(S) = \mathbb{Z}$
symmetric monoidal with \otimes	symmetric monoidal with smash product \wedge
Any: hom-sets are abelian groups	
Any: modules over Ab, can \otimes by fg ab grp	

We define the **sphere spectrum**: Start with

...,
$$Sing(S^2)$$
, $Sing(S^1)$, $Sing(S^0)$

with isomorphisms $S^2 = \Sigma S^1$, $S^1 = \Sigma S^0$ etc. This is not quite an object of Sptr. By adjunction we get maps $S_1 \to \Omega S^2$, $S^0 \to \Omega S^1$ etc. We fix the failure to be isomorphisms universally ("spectrification") which gives us the sphere spectrum S. It is also classically called the Ω -spectrum.

We have homotopy groups of spectra

$$\pi_n(X_{\bullet}) = \pi_{n+k}(X_k)$$

for $n \in \mathbb{Z}$ and $k \ge 0$. In particular

$$\pi_0(S) = \mathbb{Z}$$
, $\pi_n(S) = \text{stable homotopy groups of spheres.}$

Theorem 1.1.1. — Every stable ∞ -category is a module over Sptr^{fin} .

Theorem 1.1.2 (Lurie). — Let C be a stable ∞ -category. There is a functor

$$\operatorname{smap}_{\mathsf{C}}:\mathsf{C}^{\mathsf{op}}\times\mathsf{C}\longrightarrow\mathsf{Sptr}$$

such that for all spectra z and objects x, y in C

$$\operatorname{map}_{\mathsf{Sptr}}(z,\operatorname{smap}_(x,y)) \simeq \operatorname{map}_{\mathsf{C}}(z \wedge x,y).$$

Let C be a stable ∞ -category and hC be triangulated. Let x, y be objects in C then

$$\pi_n \operatorname{smap}(x, y) \simeq \operatorname{Hom}_{h\mathsf{C}}(x, y[-n]).$$

If $n \ge 0$, we have

$$\pi_n(\operatorname{smap}(x,y)) = \pi_n(\operatorname{map}(x,y)).$$

Think that smap corresponds to R Hom and map corresponds to $\tau^{\leq 0}$ R Hom.

Recall a construction of the tensor product of abelian groups A, B. Start with functions

$$F: A \times B \longrightarrow \mathbb{Z}$$

with finite support. Take a quotient to enforce functions to be linear in both variables.

The Lurie tensor product of stable ∞ -categories A, B is constructed as follows:

• Start with functors

$$A \times B \longrightarrow \mathsf{Sptr}.$$

• Take a quotient (or localization) to enforce that the functors preserve direct sums and pushouts in both variables.