INFINITY CATEGORIES

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§1. SIMPLICIAL SETS

Definition. — The **simplex category $\mathbf{\Delta}$** is the category of

- finite non-empty totally ordered sets
- order-preserving maps.

Notation. — $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

Every object in $\mathbf{\Delta}$ is (uniquely) isomorphic to some $[n]$.

Definition. — A **simplicial set** is a functor

$$\mathscr{S} : \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$$

Notation. — $\mathscr{S}_n := \mathscr{S}([n])$, call this the **set of n-simplices** of $\mathscr{S}$. 0-simplices are called **vertices**, 1-simplices are called **edges**.

Example 1.0.1. — Let C be a set. Let $\underline{C} : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Sets}$ be the constant functor:

$$\underline{C}_n = C \quad \forall n,$$
$$\underline{C}(\alpha) = \mathrm{id} \quad \forall \alpha : [m] \longrightarrow [n] \text{ in } \mathbf{\Delta}.$$

This is called a **discrete simplicial set**.

Definition. — Let $\mathscr{S}$ be a simplicial set. Given $\alpha : [n] \to [n-1]$ we get $\mathscr{S}(\alpha) : \mathscr{S}_{n-1} \to \mathscr{S}_n$. The n-simplices in the eimage are called **degenerate** simplices, i.e. σ is degenerate if there is an α such that $\sigma \in \mathrm{im}(\mathscr{S}(\alpha))$.

Lemma 1.0.1. — *A simplicial set is discrete if and only if for all $n \geq 1$ all n-simplices are degenerate.*

Exercise. — Prove it.

Example 1.0.2. — Let $(P, \geq)$ be a poset. Define a simplicial set $N(P, \leq)$ called the **nerve** of $(P, \leq)$ by

$$N(P, \leq)_k = \{\text{chains } p_0 \leq p_1 \leq \cdots \leq p_k : p_i \in P\}$$

where a chain is a totally ordered subset.

Exercise. — Finish the definition. Which simplices are degenerate?

Example 1.0.3 ("Standard n-simplex"). — The **standard n-simplex** is

$$\Delta^n := N([n]).$$

(Pictures)

Note. — For $j \in [n]$, we get a subsimplicial set

$$N([n] \setminus \{j\}) \subset \Delta^n$$

isomorphic to Δ^{n-1} called the j^{th} **face** of Δ^n. (Picture)

Example 1.0.4 (Horns). — Let $n \geq 0$ and $0 \leq j \leq n$, define the **horn**

$$\Lambda^n_j := \begin{array}{c} \text{subsimplicial set of } \Delta^n = N([n]) \\ \text{consisting of chains } p_0 \leq p_1 \leq \cdots \leq p_k \ \ (\mathit{Pictures}) \\ \text{such that } \{p_0, \ldots, p_k\} \not\supset [n] \setminus \{j\}. \end{array}$$

Example 1.0.5 (($n-1$)-sphere $\partial\Delta^n$). — We define the $(n-1)$-**sphere**

$$\partial\Delta^n := \begin{array}{c} \text{subsimplicial set of } \Delta^n \\ \text{chains } p_0 \leq \cdots \leq p_k \text{ such that } \{p_0 \leq \cdots \leq p_k\} \neq [n] \end{array}$$

Example 1.0.6 (Products). — Let $\mathscr{S}, \mathscr{T}$ be simplicial sets. We define their **product** $\mathscr{S} \times \mathscr{T}$ as

$$(\mathscr{S} \times \mathscr{T})_k = \mathscr{S}_k \times \mathscr{T}_k.$$

(Picture)

Example 1.0.7. — Let C be an ordinary category. We define its **nerve** $N(\mathsf{C})$ as

$$N(\mathsf{C})_k := \left\{ \begin{array}{c} \text{composable sequences of morphisms} \\ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \rightarrow \ldots \xrightarrow{f_k} X_k \end{array} \right\}.$$

Example 1.0.8. — Let X be a topological space. The **singular simplicial set** of X is defined as

$$\mathrm{Sing}(X)_k = \{\text{continuous maps } |\Delta^k| \longrightarrow X\},$$

where $|\Delta^k|$ is the standard k-simplex

$$|\Delta^k| = \left\{ (x_0, \ldots, x_k) \in \mathbf{R}^{k+1} \middle| x_i \geq 0, \sum x_i = 1 \right\}.$$

Exercise. — What does this do to the morphisms in $\mathbf{\Delta}$?

Definition. — A **Kan complex** is a simplicial set X such that for every diagram

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "\text{any map}"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

we can fill the dashed arrow. This is called an **extension problem**. If the arrow exists we say that the extension problem **admits a solution**.

Daniel Kan discovered Kan complexes in 1958. The *key fact* is that $\mathrm{Sing}(X)$ *is* a Kan complex. The theme from 1958 to today is that Kan complexes are a "combinatorial model" for algebraic topology which allows us to do homotopy theory.

Definition. — Let X be a Kan complex and $\mathscr{S}$ be any simplicial set. Two maps $f, g : \mathscr{S} \to X$ are said to be **homotopic** if there exists a map $H : \mathscr{S} \times \Delta^1 \to X$ such that

$$H|_{\mathscr{S} \times \{0\}} = f, \quad H|_{\mathscr{S} \times \{1\}} = g.$$

Lemma 1.0.2. — *This is an equivalence relation.*

Proof. Omitted, tricky for an exercise. This requires X to be a Kan complex. $\square$

Definition. — Let X be a Kan complex and x_0 be a vertex of X. Let

$$\mathrm{Loops}_{x_0} = \{\text{maps } \gamma : \Delta^n \longrightarrow X \text{ such that } \gamma|_{\partial\Delta^n} \text{ is the constant map to } x_0\}.$$

We say $\gamma, \gamma' \in \mathrm{Loops}_{x_0}$ are **relatively homotopic** (**rel. homotopic**) if there exists $H : \Delta^n \times \Delta^1 \to X$ such that

$$H|_{\Delta^n \times \{0\}} = \gamma, \quad H|_{\Delta^n \times \{1\}} = \gamma', \quad H|_{\partial\Delta^n \times \Delta^1} = \text{const. map to } x_0.$$

Define

$$\pi_n(X, x_0) := \frac{\mathrm{Loops}_{x_0}}{\text{rel. homotopy}}.$$

Fact. — For $n \geq 1$, $\pi_n(X, x_0)$ is a group. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Definition. — An **∞-category** (or **quasi-category**) is a simplicial set $\mathscr{C}$ such that any extension problem

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

with $0 < j < n$ (**inner horns**) admits a solution. (Picture) An ∞-category is also called a **weak Kan complex**.

Lemma 1.0.3. — *Let C be an ordinary category, then $N(\mathsf{C})$ is an ∞-category.*

Digression: Let I^n be the simplicial set consisting of n consecutive 1-simplices (**n-spine**) (Picture). A naive alternative definition is: $\mathscr{C}$ is an infinity category if every

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

has a solution. This is WRONG (but its wrongness is subtle), even though $N(\text{ord. cat.})$ satisfy this. There is a book by Markus Land "Introduction to ∞-categories" which explores this. The definition of ∞-categories was introduced by Boardman-Vogt in 1972. Joyal started generalizing results from ordinary category theory to ∞-categories in 2006. Lurie is largely responsible for how well this notion is developed in modern literature.

Remark. — Having a unique solution to the lifting problem characterizes nerves of ordinary categories.

Definition. — Let $\mathscr{C}$ be an ∞-category. An **object** is a vertex. A **morphism** is an edge. An **identity morphism** is a degenerate edge. Say that h is *a* **composition** of g and f if there exists a 2-simplex such that (Picture).

Remark. — Compositions are NOT unique in ∞-categories.

Example 1.0.9 (∞-categories). —

1) Topological spaces Top.
   - Objects are topological spaces.
   - Morphisms are continuous maps.
   - A 2-simplex is a (not necessarily commutative) diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"] & & X_2 \end{tikzcd}$$

   and a homotopy $H : X_0 \times [0,1] \to X_2$ from gf to h.
   - A 3-simplex is a diagram

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_1 \arrow[rr] \arrow[dr] & & X_3 \\ & X_2 \arrow[ur] & \end{tikzcd}$$

with continuous maps $f_{ij} : X_i \to X_j$ for $i < j$, homotopies $T_{ijk} X_i \times [0,1] \to X_k$ from $f_{jk} \circ f_{ij}$ to f_{ik}, and $H : X_0 \times [0,1]^2 \to X_3$ (**higher homotopy**) such that $H|_{\mathrm{bdry}}$ is

$$\begin{tikzcd} (0,0) \arrow[r, "T_{123} f_{01}"] \arrow[d, "f_{23} T_{012}"'] & (0,1) \arrow[d, "T_{013}"] \\ (1,0) \arrow[r, "T_{023}"'] & (1,0) \end{tikzcd}$$

2) The ∞-category of ordinary categories Cat_1.
  - Objects are ordinary categories.
  - Morphisms are functors.
  - A 2-simplex is a (not necessarily commutative) diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"] & & X_2 \end{tikzcd}$$

    and a natural isomorphism $T : g \circ f \xrightarrow{\sim} h$.
  - A 3-simplex is a diagram

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_1 \arrow[rr] \arrow[dr] & & X_3 \\ & X_2 \arrow[ur] & \end{tikzcd}$$

    where f_{ij} are functors and T_{ijk} are natural isomorphism such that

$$\begin{tikzcd} \bullet \arrow[r, "T_{123} f_{01}"] \arrow[d, "f_{23} T_{012}"'] & \bullet \arrow[d, "T_{013}"] \\ \bullet \arrow[r, "T_{023}"'] & \bullet \end{tikzcd}$$

    commutes

A source of ∞-categories are
  - ordinary categories with an equivalence relation on morphisms,
  - ordinary categories with inverting some morphisms.