INTRODUCTION TO SHIMURA VARIETIES AND THEIR COHOMOLOGY

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Plan:

I) Siegel modular varieties
II) General Shimura varieties
III) (Étale) Cohomology: Kottwitz conjecture

(The cohomology gives a realization of the global Langlands correspondence for certain number fields.)

The main reference are Sophie Morel's notes from the IHES '22 summer school.

§1. Siegel Modular Varieties

Analytically: we want to parametrize complex tori. This is fairly simple. Let V be a $\mathbf{C}$-vector space of dimension $m \geq 1$, $\Lambda \subset V$ a lattice (a discrete subgroup such that V/Λ is compact), then $X = V/\Lambda$ is a complex Lie group, which is a complex torus.

Exercise. — A morphism $f : X = V/\Lambda \to X' = V'/\Lambda'$ of complex Lie groups is given by a $\mathbf{C}$-linear map $V \to V'$ mapping Λ to Λ'.

Question: Which complex tori are algebraizable, i.e. $X \hookrightarrow \mathbb{P}^n(\mathbf{C})$ (equivalent to $X \simeq \underline{X}^{\mathrm{an}}$ for some projective $\underline{X}$ by Chow). Can we find a parametrization?

Example 1.0.1. — Let $n = 1$ complex tori are always algebraic. There is the Weierstrass $\wp$-function

$$\begin{aligned} \wp : V/\Lambda &\longrightarrow \mathbb{P}^1(\mathbf{C}) \\ v &\longmapsto \wp(v) = \frac{1}{v^2} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(\lambda + v)^2} - \frac{1}{\lambda^2}. \end{aligned}$$

It embeds V/Λ in $\mathbb{P}^2(\mathbf{C})$ via $[\wp : \wp' : 1]$ with image $y^2 = P_\Lambda(x)$ where $P_\Lambda \in \mathbf{C}[X]$ has degree 3. The coefficients of P_Λ are Eisenstein series (modular forms).

For $n > 1$, X is "almost never" algebraic.

Recall that X is algebraizable if and only if there exists $\mathscr{L} \in \operatorname{Pic}(X)$ which is ample (see Mumford's Abelian Varieties). Recall that $\operatorname{Pic}(X) \simeq H^1(X, \mathscr{O}_X^\times)$. There is a short exact sequence

(1)
$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp(2\pi i -)} \mathscr{O}_X^\times \longrightarrow 0$$

so we get a long exact sequence. The map

$$H^0(X, \mathscr{O}_X) \simeq \mathbf{C} \longrightarrow \mathbf{C}^\times \simeq H^0(X, \mathscr{O}_X^\times)$$

is surjective so we get

(2)
$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r, hook] \arrow[d, "\simeq"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d, "\simeq"', "Dolbeaut"] & H^1(X,\mathscr{O}_X) \arrow[r, "\delta"] \arrow[d, "\simeq"] & \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)) \arrow[d, "\simeq"] \\
H^1(\Lambda,\mathbb{Z}) \arrow[d, equal] & \overline{T} & H^1(\Lambda, \mathscr{O}(X)^\times) & \operatorname{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right) \\
\operatorname{Hom}(\Lambda,\mathbb{Z}) \arrow[dr, hook] & T \oplus \overline{T} \arrow[u, "\mathrm{pr}_2"] & & \\
 & \operatorname{Hom}_{\mathbf{R}}(V,\mathbf{C}) \arrow[u, "\simeq"] & &
\end{tikzcd}$$

We have $H^i(V,\mathbb{Z}) = 0$ for all $i > 0$ and $H^i(V,\mathscr{O}_V) = 0$ for all $i > 0$ so $\mathrm{Pic}(V) = 0$. $\overline{T}$ are the antilinear maps $V \to \mathbf{C}$ and $T = \mathrm{Hom}_{\mathbf{C}}(V,\mathbf{C})$. Observe that

$$\mathrm{Pic}^0(X) = \ker\delta \simeq \frac{\overline{T}}{\mathrm{pr}_2(\mathrm{Hom}(\Lambda,\mathbb{Z}))}$$

is also a complex torus. The last term is the Neron-Severi group

$$\begin{aligned} NS(X) &\simeq \{E : \Lambda\times\Lambda \longrightarrow \mathbb{Z} \text{ alt.} : E(iv_1, iv_2) = E(v_1,v_2), \forall v_1,v_2 \in V\} \\ &= \{\Im H : H : V\times V \longrightarrow \mathbf{C} \text{ Hermitian such that } (\Im H)(\Lambda\times\Lambda)\subset\mathbb{Z}\}. \end{aligned}$$

The Appel-Humbert theorem completely describes $\mathrm{Pic}(X)$ as $\{L(H,\alpha)\}$ with H as above and α an extra datum.

Theorem 1.0.1 (Lefschetz). — *The following are equivalent:*
1) *H is positive definite.*
2) *$L(H,\alpha)$ is ample (in fact, $L(H,\alpha)^{\otimes 3}$ is enough to embed X).*

Let $L \in \mathrm{Pic}(X)$ then

$$\begin{aligned} \phi_L : X &\longrightarrow \mathrm{Pic}^0(X) = \widehat{X} \\ x &\longmapsto T_x^* L \otimes L^{-1} \end{aligned}$$

is a morphism of Lie groups (here T_x is translation by x).

Theorem 1.0.2. — *The following are equivalent:*
- *L is ample.*
- $\ker\phi_L$ *is finite.*
- *ϕ_L is surjective (i.e. an isogeny).*

Exercise. — Check that phi_L is an isomorphism if and only if $E(\cdot,\cdot)$ is perfect ($\Lambda \simeq \mathrm{Hom}(\Lambda,\mathbb{Z})$).

Definition. — Say that such ϕ_L is a **polarization**. If ϕ_L is an isomorphism, then it is called a **principal polarization**.

Remark. — Not every algebraic X admits a principal polarization, but is isogenous to one that does.

We can define the moduli space

$$\mathscr{A}_n(\mathbf{C}) = \left\{(X,\phi) : X = V/\Lambda \text{ of dimension } n, \phi : X \longrightarrow \widehat{X} \text{ a principal polarization}\right\}.$$

Let (V,Λ,H) be a principally polarized complex torus. Choose a symplectic basis $(e_1,\dots,e_{2n})$ of Λ, i.e.

$$(E(e_i,e_j))_{i,j} = J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Exercise. — $L = L(H,\alpha)$ is ample if and only if $e_{n+1},\dots,e_{2n}$ is a basis of V over $\mathbf{C}$ such that

$$\tau = \mathrm{Mat}_{e_{n+1},\dots,e_{2n}}(e_1,\dots,e_n)$$

satisfies $\tau = {}^t\tau$ and $\Im(\tau)$ is positive definite.

Definition. — $\mathscr{H}_n^+$ is the set of such $\tau \in M_n(\mathbf{C})$. There is an algebraic group

$$\mathbf{Sp}_{2n,\mathbb{Z}} : R \longmapsto \left\{g \in M_{2n}(R) : {}^t g J_n g = J_n\right\}.$$

There is an action of $\mathbf{Sp}_{2n}(\mathbb{Z})$ on $\mathscr{H}_n^+$ such that given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\tau\gamma = (\tau b + d)^{-1}(\tau a + c)$$

(this corresponds to replacing $\underline{e} = (e_1,\dots,e_{2n})$ by $\underline{e}\gamma$).

We prefer left actions: let ${}^t\gamma$ act so that $\gamma\tau = \tau * {}^t\gamma$, i.e.

$$(\tau\, {}^t c + d)^{-1}(\tau\, {}^t a + {}^t b) = (a\tau + b)(c\tau + d).$$

This extends to an action of $\mathbf{Sp}_{2n}(\mathbf{R})$ on $\mathscr{H}_n^+$. This action is transitive and

$$\mathrm{Stab}_{\mathbf{Sp}_{2n}(\mathbf{R})}(iI_n) \longrightarrow U(n)$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a + ib$$

is an isomorphism (this is a maximal compact subgroup).

So $\mathscr{A}_n(\mathbf{C}) \simeq \Gamma_n \setminus \mathscr{H}_n^+$ where $\Gamma_n \simeq \mathbf{Sp}_{2n}(\mathbb{Z})$.

Remark. — There exists $\gamma \in \Gamma_n \setminus \{\pm 1\}$and $\tau \in \mathscr{H}_n^+$ such that $\gamma\tau = \tau$.

There is a universal object

$$\mathscr{X}_n(\mathbf{C}) = \mathbb{Z}^{2n} \rtimes \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

where

$$\gamma(v, \tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (v, \tau) = ((\tau^t c +^t d)^{-1} v, \gamma\tau)$$

and

$$(\lambda_1, \lambda_2)(v, \tau) = (v + \lambda_1 + \tau\lambda_2, \tau)$$

for $\lambda_i \in \mathbb{Z}^n$.

There is a morphism $\pi : \mathscr{X}(\mathbf{C}) \to \mathscr{A}_n(\mathbf{C})$ which admits a section e. The fiber of τ is $[\tau] \simeq \mathbf{C}^n/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z}^n \oplus \tau\mathbb{Z}^n$. We get the **Hodge bundle**: take $\Omega^1(V/\Lambda)$ which are translaton invariant 1-forms, which is isomorphic to V^* via e^*, then the Hodge bundle is

$$\mathscr{E}_n = e^*\Omega^1_{\mathscr{X}(\mathbf{C})/\mathscr{A}_n(\mathbf{C})} \simeq \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

for action (??).

We can apply Schur functors (use the action of $\mathfrak{S}_k$ on on $\mathscr{E}_n^{\otimes k}$ to act on subbundles, e.g. $\bigwedge^k \mathscr{E}_n$ for $0 \leq k \leq n$). (Equivalently see $\mathscr{E}_n$ as a $\mathbf{GL}_n(\mathbf{C})$-bundle on $\mathscr{A}_n(\mathbf{C})$ and apply a holomorphic representation $\rho : (\mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$.) Sections of such vector bundles on $\mathscr{A}_k(\mathbf{C})$ are (level Γ_n, weight ρ) Siegel modular forms on $\mathscr{A}_n(\mathbf{C})$.

Notation: Write

$$M_\rho(\Gamma_n) = \{f \in \Gamma(A_n(\mathbf{C}), \rho(\mathscr{E}_n) : f \text{ is holomorphic at } \infty\}$$

(the last condition is automatic if $n > 1$). We write

$$S_\rho(\Gamma_n) = \{f : \text{vanish at } \infty\} \subset M_\rho(\Gamma_n)$$

for the set of **cusp forms**.

We want a group theoretic description of the complex structure on $\mathscr{A}_n(\mathbf{C})$ and these vector bundles on $\mathscr{A}_m(\mathbf{C})$.

We have $Z(U(n)) \simeq U(1)$ and its centralizer in $\mathbf{Sp}_{2n}(\mathbf{R})$ is $U(n) = K(\mathbf{R})$ where $K \hookrightarrow \mathbf{Sp}_{2n,\mathbf{R}}$ is an algebraic subgroup.

Over $\mathbf{C}$ we have

$$Z(K_\mathbf{C}) \simeq \mathbf{GL}_{1,\mathbf{C}} \hookrightarrow \mathbf{Sp}_{2n,\mathbf{C}}.$$

This determines two opposite parabolic subgroups $Q_+ = K_\mathbf{C} N_+$, $Q_- K_\mathbf{C} N_-$.

1.1. Siegel modular forms as automorphic forms Let $\rho : \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$ be a holomorphic (equivalently algebraic) representation. **Siegel modular forms** are

$$M_\rho(\Gamma_n) = \left\{ \begin{array}{c} f : \mathscr{H}_n^+ \to W \\ \text{holomorphic} \end{array} \middle| \begin{array}{c} \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n, \forall \tau \in \mathscr{H}_n^+, f(\gamma\tau) = \rho(c\tau + d) f(\tau) \\ \text{and } f \text{ holomorphic at } \infty \end{array} \right\}$$
$$\subset H^0(\mathscr{A}_n(\mathbf{C}), {}^\rho\mathscr{E}_n).$$

${}^\rho\mathscr{E}_m$ comes from a $\mathbf{Sp}_{2n}(\mathbf{R})$-equivariant vector bundle on

$$\begin{tikzcd} & \mathscr{H}_n^+ \arrow[r, hook] & \mathbf{Sp}_{2n}(\mathbf{C})/Q_-(\mathbf{C}) \\ \mathbf{Sp}_{2n}(\mathbf{R}) \arrow[r] & \mathbf{Sp}_{2n}(\mathbf{R})/U(n) \arrow[u, "\simeq"] & \end{tikzcd}$$

Define

$$j : \mathbf{Sp}_{2n}(\mathbf{R}) \times \mathscr{H}_n^+ \longrightarrow \mathbf{GL}_n(\mathbf{C})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \longmapsto c\tau + d.$$

This is a cocycle

$$j(gg', \tau) = j(g, g'\tau) j(g', \tau)$$

(so $j(-, i)|_{U(n)} : U(n) \to \mathbf{GL}_n(\mathbf{C})$ is a morphism). To $f \in M_\rho(\Gamma_n)$ associate

$$\phi_f : \Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}) \longrightarrow W$$
$$g \longmapsto \phi_f(g) = \rho(j(g, i))^{-1} f(gi)$$

a smooth function. Let $g \in \mathbf{Sp}_2 n(\mathbf{R})$ and $k \in U(n)$, then

$$\phi_f(gk) = \rho(j(k, i))^{-1} f(gi).$$

Assume $W = \mathbf{C}$ for simplicity, e.g. ${}^\rho\mathscr{E}_n = (\bigwedge^n \mathscr{E}_n)^{\otimes k}$. Then

$$\phi_f \in \mathscr{A}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \subset C^\infty(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}, \mathbf{C})$$

(? details). This space has actions by $\mathfrak{g}$ and $U(n)$. By the Cauchy-Riemann equations f is holomorphic if and only if ϕ_f is killed by $\operatorname{Lie} N_- \subset \mathfrak{g} = \mathbf{C} \otimes_\mathbf{R} \operatorname{Lie} \mathbf{Sp}_{2n}(\mathbf{R})$. Note that $\operatorname{Lie}(\mathbf{Sp}_{2n}(\mathbf{R}))$ acts on $C^\infty(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$ by

$$(X \cdot \phi)(g) = \left.\frac{d}{dt}\right|_{t=0} \phi(ge^{tX}).$$

ϕ_f lies in some generalized Verma module in $\mathscr{A}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$.

If $f \in S_\rho(\Gamma_m)$ (vanishes at ∞) then

$$\phi_f \in \mathscr{A}_{\text{cusp}}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \subset \mathscr{A}^2(-) \subset \mathscr{A}(-)$$

and $\mathscr{A}^2(-)$ decomposes inside $L^2(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$ with the action of $\mathbf{Sp}_{2n}(\mathbf{R})$. This means that cusp forms have fast decay at cusps.

As a $(\mathfrak{g}, U(n))$-module,

$$\mathscr{A}_{cusp} \subset \mathscr{A}^2(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \simeq \bigoplus_{\substack{\pi \text{ irr} \\ (\mathfrak{g}, U(n))\text{-mod}}} \pi^{\oplus m(\pi)}.$$

Siegel cusp forms correpond to special vectors in some of these πs ($U(n)$-equivariant and killed by $\operatorname{Lie} N$).

1.2. Level structures Let $X = V/\Lambda$ be a complex torus with a principal polarization $phi : X \xrightarrow{\sim} \widehat{X}$. For $M \geq 1$

$$X[M] := \ker\left(X \xrightarrow{\times M} X\right) = \frac{1}{M}\Lambda/\Lambda \simeq (\mathbb{Z}/M)^{2n}.$$

The map $[M]_X : X \to X$ is an isogeny (i.e. surjective with finite kernel). For all isogenenies $f : X \to Y$ inducing $\widehat{f} = f^* : \widehat{Y} \to \widehat{X}$, also an isogeny. We get the Weil pairing

$$\begin{aligned} \ker f \times \ker \widehat{f} &\longrightarrow \mathbf{C}^\times \\ (x, [L]) &\longmapsto \langle x, [L]\rangle . \end{aligned}$$

Choose $t : f^*L \xrightarrow{\sim} \mathscr{O}_X$ we have

$$\begin{tikzcd} T_x^* f^* L \arrow[r, "T_x^*(t)"] \arrow[d, "\simeq"] & T_x^* \mathscr{O}_X \arrow[d, "\simeq"] \\ f^*L \arrow[r, "t \times \langle x, [L]\rangle"] & \mathscr{O}_X. \end{tikzcd}$$

$f = [M]_X$ is a special case, then we get $X[M] \times \widehat{X}[M] \to \mu_M(\mathbf{C})$ and usaing a polarization we get

$$\langle \cdot, \cdot \rangle_\phi : X[M] \times X[M] \longrightarrow \mu_M(\mathbf{C}).$$

Proposition 1.2.1. — *$\langle \cdot, \cdot \rangle_\phi$ is alternating and non-degenerate.*

Proof. Recall that ϕ is ϕ_L for some $L = L(H, \alpha)$, let $E = \Im H : \Lambda \times \Lambda \to \mathbb{Z}$. Then

$$\begin{tikzcd} X[M] \times X[M] \arrow[r, "\langle \cdot,\cdot \rangle_\phi"] \arrow[d, "\simeq"] & \mu_M(\mathbf{C}) \\ \left(\frac{1}{M}\Lambda/\Lambda\right)^2 \arrow[r, "ME(\cdot,\cdot)"] & \frac{1}{M}\mathbb{Z}/\mathbb{Z} \arrow[u, "\exp(2\pi i -)"'] \end{tikzcd}$$

□

Definition. — Temporarily we define a level structure on (X, ϕ) to be

$$(\mathbb{Z}/M)^{2n} \underset{\eta}{\xrightarrow{\sim}} X[M]$$

such that $\eta^* \langle \cdot, \cdot \rangle_\phi$ is the standard pairing for metric J_n.

Fact. — By strong approximation $\mathbf{Sp}_{2n}(\mathbb{Z}) \twoheadrightarrow \mathbf{Sp}_{2n}(\mathbb{Z}/M\mathbb{Z})$. Define $\Gamma_m(M)$ to be the kernel.

Corollary 1.2.1. — *There is a bijection*

$$\{(X, \phi, \eta) | \textit{PPAV with a level M structure}\} / \sim \simeq \Gamma_n(M) \setminus \mathscr{H}_n^+ = \mathscr{A}_n'(M)(\mathbf{C}).$$

Exercise. — For $M \geq 3$, for all $\tau \in \mathscr{H}_n^+$ show that $\mathrm{Stab}_{\Gamma_n(M)}(\tau) = \{1\}$. (?)

We get a tower $(\mathscr{A}_n'(M)(\mathbf{C}))_{M \geq 1}$ ordered by divisibility. For $M \mid |M'$ we get $\mathscr{A}_n'(M')(\mathbf{C}) \to \mathscr{A}_n'(M)(\mathbf{C})$.

Given (X, ϕ)

$$\{\text{level } M \text{ structures on } (X, \phi)\}$$

is a right $\mathbf{Sp}_{2n}(\mathbb{Z}/M\mathbb{Z})$-torsor which gives us an action of

$$\mathbf{Sp}_{2n}(\widehat{\mathbb{Z}}) = \varprojlim_M \mathbf{Sp}_{2n}(\mathbb{Z}/M)$$

on this tower.

Also

$$\mathscr{A}_n'(M)(\mathbf{C}) \simeq \mathscr{A}_n'(M')(\mathbf{C}) / \left(K(M)/K(M')\right)$$

where

$$K(M) = \ker\left(\mathbf{Sp}_{2n}(\widehat{Z}) \longrightarrow \mathbf{Sp}_{2n}(\mathbb{Z}/M)\right).$$

The quotient $K(M)/K(M')$ is a finite group.

1.3. Hecke operators (adelically) The goal is to define more natural maps between $\mathscr{A}'_n(M)(\mathbf{C})$. The basic idea is that given (X, ϕ, η), we should also consider isogeneus complex tori (i.e. quotients of X by finite subgroups). But there are some problems: this is not strictly compatible with principal polarizations. Let $f : X \to Y$be an isogeny, ϕ be a principal polarization for Y, then $f^*\phi := \widehat{f} \circ \phi \circ f$ has degree $(\deg f)^2$, so it is not principal unless f is an isomorphism.

There are two solutions:

1) Rescale polarizations.
2) Consider quasi-isogenies

$$f \in \mathbb{Q} \otimes \mathrm{Hom}(X, Y) \text{ such that } \exists M \geq 1 \text{ with } Mf \in \mathrm{Hom}(X, Y) \text{ an isogeny.}$$

Let's do both.

Recall the ring of adeles $\mathbb{A} = \mathbf{R} \times \mathbb{A}_f$ where

$$\mathbb{A}_f = \prod_p{}' (\mathbb{Q}_p, \mathbb{Z}_p) = \left\{ (x_p)_{p \text{ prime}} \middle| \begin{array}{c} x_p \in \mathbb{Q}_p \\ \exists \text{ finite } S \text{ such that} \forall p \notin S, x_p \in \mathbb{Z}_p \end{array} \right\}.$$

Recall that

$$\begin{aligned} \text{lattices in } \mathbb{Q}\Lambda &\leftrightarrow \left\{ (\Lambda'_p)_p \middle| \begin{array}{c} \Lambda'_p \subset \mathbb{Q}_p \otimes_{\mathbb{Z}} \Lambda \text{ is a } \mathbb{Z}_p\text{-lattice} \\ \exists \text{ finite } S \text{ such that } \forall p \notin S, \Lambda'_p = \mathbb{Z}_p \Lambda \end{array} \right\} \\ &\leftrightarrow \mathbf{GL}(\mathbb{A}_f \otimes \Lambda)/\mathbf{GL}(\widehat{\mathbb{Z}} \otimes \Lambda). \end{aligned}$$

Proof. Reduce to the case where

$$M\Lambda \subset \Lambda' \subset \frac{1}{M}\Lambda$$

and use the chinese remainder theorem. □

Proposition 1.3.1. — *Let (X, ϕ) be a principally polarized abelian variety.*

(a) Let L be the set of principally polarized abelian varieties (X', ϕ') quasi-isogeneous to $(X, /\phi)$, i.e. there exists a quasi-isogeny $f : X' \dashrightarrow X$ such that $f^*\phi = c\phi'$, where $c \in \mathbb{Q}_{>0}$.

(b) Let R be the set of $(\Lambda'_p)_p$ such that

$$\Lambda'_p \subset V_p X := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p X$$

is a $\mathbb{Z}_p$-lattice such that there exists k_p making $p^{k_p} \langle \cdot, \cdot \rangle |_{\Lambda'_p \times \Lambda'_p}$ take values in $\mathbb{Z}_p(1) :== \varprojlim_k \mu_{p^k}(?)$ and is perfect, as well as there is a finite S such that for all $p \notin S$

$$\Lambda'_p = T_p X := \varprojlim_k X[p^k].$$

Then

$$L/\sim \simeq R.$$

This is also isomorphic to the set of $\mathbf{GSp}_{2n}(\widehat{\mathbb{Z}})$-orbits of symplectic trivializations

$$(\mathbb{A}_f^{2n}, \textit{standard } \langle \cdot, \cdot \rangle) \xrightarrow{\sim} \left(\mathbb{Q} \otimes \prod_p T_p X, \langle \cdot, \cdot \rangle_\phi \right).$$

Here $\mathbf{GSp}_{2n}$ is the $\mathbb{Z}$-group scheme

$$\mathbf{GSp}_{2n}(R) = \left\{ (g, c) | g \in M_{2n}(R), c \in R^\times, {}^t g J_n g = c J_n \right\}.$$

Definition. — A **level structure** for (X, ϕ) is an isomorphism $(\mathbb{Z}/M)^{2n} \xrightarrow{\sim} X[M]$. $\mathbb{Z}/M \xrightarrow{\sim} \mu_M(\mathbf{C})$ such that the obvious diagram commutes.

We have

$$\begin{aligned} \mathscr{A}_n(M)(\mathbf{C}) &\simeq \{(X, \phi, \eta) | \text{PPAV with level } M \text{ structure}\} / \sim \\ &\simeq \left\{ (X', \phi') \middle| K(M)\text{-orbit of trivalization of } \mathbb{Q} \otimes \prod_p T_p X' \right\} / \text{quasi-isogeny} \\ &\simeq \mathbf{GSp}_{2n}(\mathbb{Q}) \backslash \left(\mathscr{H}_n^\pm \times \mathbf{GSp}_{2n}(\mathbb{A}_f)/K(M) \right) \end{aligned}$$

where

$$\mathscr{H}_n^{\pm} = \mathscr{H}_n^{+} \coprod \mathscr{H}_n^{-}$$

has an action of $\mathbf{GSp}_{2n}(\mathbf{R})$ and

$$K(M) := \ker\left(\mathbf{GSp}_{2n}(\widehat{\mathbb{Z}}) \longrightarrow \mathbf{GSp}_{2n}(\mathbb{Z}/M)\right).$$

From now on we write G for $\mathbf{GSp}_{2n}$. We have a tower $(\mathscr{A}_m(M)(\mathbf{C}))_{M\geq 1}$ (a $\mathbf{GSp}_2(\widehat{\mathbb{Z}})$-torsor over $\mathscr{A}_m(\mathbf{C})$ with a right action of $G(\mathbb{A}_f)$ and Lecture 3

$$\mathscr{A}_m(M)(\mathbf{C}) \simeq G(\mathbb{Q}) \setminus \left(\mathscr{H}_n^{\pm} \times G(\mathbb{A}_f)/K(M)\right).$$

For $g \in G(\mathbb{A}_f)$ and M, M' satisfying $K(M') \subset gK(M)g^{-1}$ define

$$\begin{aligned} T_g : \mathscr{A}_m(M')(\mathbf{C}) &\longrightarrow \mathscr{A}_m(M)(\mathbf{C}) \\ [\tau, h] &\longmapsto [\tau, hg]. \end{aligned}$$

There is also an action on Siegel modular forms. Note that $T_g^*\mathscr{E}_m \simeq \mathscr{E}_m$ and on $\mathscr{A}_m(M)(\mathbf{C})$, $\mathscr{E}_m = T_1^*\mathscr{E}_m$. Hence for $\rho : \mathbf{GL}_m(\mathbf{C}) \to \mathbf{GL}(W)$ there is an acton of $G(\mathbb{A}_f)$ on

$$M_\rho := \varinjlim_M M_\rho(K(M))$$

where

$$M_\rho(K(M)) := H^0(\mathscr{A}(M)(\mathbf{C}),^\rho \mathscr{E}_m)) + \text{holomorphy at } \infty \text{ if } m = 1.$$

M_ρ contains cusp forms S_ρ and by unitarity

$$S_\rho \simeq \bigoplus_{\pi_f \text{ irrep of } G(\mathbb{A}_f)} \pi_f^{\oplus m(\pi_f)}$$

(note that π_f are infinite-dimensional).

We recover

$$\mathrm{H}_B^k(\mathscr{A}_m(M)(\mathbf{C}), \mathbb{Q}) \simeq (\mathrm{H}_B^k)^{K(M)}$$

where the right hand side admits an action of Hecke operators $H(G(\mathbb{A}_f), K(M))$. (Trace map?)

Theorem 1.3.1 (Franke, Generalization of Matsushima's formula). — *We have*

$$\begin{aligned} \mathbf{C} \otimes_{\mathbb{Q}} \mathrm{H}_B^\bullet \xrightarrow[G(\mathbb{A}_f)\text{-equiv.}]{\sim} &\ \mathrm{H}^\bullet(\mathfrak{sp}_{2m}, U(m); \overbrace{\mathscr{A}(G(\mathbb{Q}) \setminus G(\mathbb{A})/\mathbf{R}_{>0})}^{\mathscr{A}(G)}) \\ := &\ \mathrm{H}^\bullet\left(\mathrm{Hom}_{U(m)}\left(\bigwedge^\bullet \mathfrak{sp}_{2n}/\mathfrak{gl}_m\right), \mathscr{A}(G)\right). \end{aligned}$$

Remark. —

0) It's "easy" if we replace $\mathscr{A}(G)$ by C^∞ and use de Rham cohomology for the LHS.
1) If $\Gamma_m \setminus \mathscr{H}_m^+$ was compact, this would be obtained from the Hodge decomposition for Riemannian manifolds.
2) $\mathscr{A}(G)$ is not semi-simple at all.
3) If $m = 1$ we can use this to recover the Eichler-Shimura isomorphism. Let $\mathrm{H}^1_{B,\text{cusp}}$ be the subspace of H_B^1 defined by "vanishing at cusps". Then

$$\mathbf{C} \otimes_{\mathbb{Q}} \mathrm{H}^1_{B,\text{cusp}} \xrightarrow[\mathbf{GL}_2(\mathbb{A}_f)]{\sim} S_2 \oplus \overline{S}_2.$$

If $\Gamma_1 \setminus \mathscr{H}_1^+$ was compact (thus a projective curve over $\mathbf{C}$) this would follow from the Hodge decomposition because $\mathscr{E}_1^{\otimes 2} \simeq \Omega^1$ on $\mathscr{A}_1(\mathbf{C})$.

1.4. Siegel modular varieties, algebraically

Definition. — Let S be a scheme. An **abelian scheme** over S is an S-group scheme $X \to S$ which is smooth, proper with connected geometric fibers. If $S = \operatorname{Spec} k$ we call abelian schemes **abelian varieties**.

Proposition 1.4.1. — *Automatically commutative.*

Definition. — Let $X \to S$ be an abelian scheme and $e : S \to X$ be the identity section we define a functor

$$\begin{aligned} \operatorname{Pic}_{X/S,e} : \mathsf{Sch}_S &\longrightarrow \mathsf{Ab} \\ T &\longmapsto \{(L, \alpha) : L \in \operatorname{Pic}(X \times_S T) \text{ and } \alpha \text{ trivializes } e^*L\}. \end{aligned}$$

There is a subfunctor $\operatorname{Pic}^0_{X/S,e}$ defined by the data such that for all $t \in T$ and all smooth projective curves C over $K(t)$, for all $f : C \to X \times_S K(t)$, f^*L has degree 0.

Theorem 1.4.1 (Artin, Raynaud). — $\operatorname{Pic}^0_{X/S,e}$ *is represented by an abelian scheme over S.*

We write $\widehat{X}$ for this scheme.

Definition. — For $L \in \operatorname{Pic}(X)$, we have

$$\begin{aligned} \phi_L : X &\longrightarrow \widehat{X} \\ x \in X(T) &\longmapsto T_x^* L \otimes L^{-1}. \end{aligned}$$

A **polarization** is an isogeny (i.e. finite, faithfully flat) $\phi : X \to \widehat{X}$ such that for all geometric points $p : \operatorname{Spec} k \to S$, $\phi_p = \phi_L$ for some ample L. A polarization is **principal** if it is an isomorphism. A **principally polarized abelian variety** (PPAV) is the data (X, ϕ) of an abelian variety X and a principal polarization ϕ.

Proposition 1.4.2. — *If $M \geq 1$ is invertible on S then $X[M]$ (defined to be the kernel of $[M]_X$) is étale locally isomorphic to $\underline{(\mathbb{Z}/M)^{2m}}$.*

Definition. — Let $M \geq 1$, we define a functor

$$\begin{aligned} \mathscr{A}_m(M) : \mathsf{Sch}_{\mathbb{Z}[\frac{1}{M}]} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \{\text{PPAV } (X, \phi) \text{ with a level } M \text{ structure}\}/\sim \end{aligned}$$

(Groupoid when $M \leq 2$?)

Theorem 1.4.2 (Mumford). — *For $M \geq 3$, $\mathscr{A}_m(M)$ is represented by a smooth quasiprojective scheme over $\mathbb{Z}\left[\frac{1}{M}\right]$ of relative dimension $\frac{m(m+1)}{2}$.*

By the previous proposition, for all $M \mid M'$ with $M \geq 3$ there is a map

$$\mathscr{A}_m(M') \longrightarrow \mathscr{A}_m(M) \times_{\mathbb{Z}[\frac{1}{M}]} \mathbb{Z}\left[\frac{1}{M'}\right]$$

which is finite étale and a $\ker(G(\mathbb{Z}/M') \to G(\mathbb{Z}/M))$-torsor.

We stil have an action of $G(\mathbb{A}_f)$ on the tower $(\mathscr{A}_m(M) \times \mathbb{Q})_{M \geq 1}$ using the same interpretation of the moduli problem as in the analytic case (quasi-isogenies).

Variant: Let p be a prime and consider the tower $(\mathscr{A}_m(M) \times \mathbb{Z}_{(p)})_{(M,p)=1}$. It admits an actioon of $G(\mathbb{A}_f^{(p)})$, where $\mathbb{A}_f^{(p)}$ are finite adeles with $\mathbb{Q}_p$ omitted.

Applications:

1) We have a $\mathbb{Q}$-structure on modular forms.
2) Étale cohomology: the comparison theorem tells us that

$$\mathrm{H}^\bullet_{\text{ét}}(\mathscr{A}_m(M)_{\overline{\mathbb{Q}}}, \mathbb{Q}_l) \simeq \mathbb{Q}_l \otimes_{\mathbb{Q}} \mathrm{H}^\bullet_B(\mathscr{A}_m(M)(\mathbf{C}), \mathbb{Q}).$$

The LHS has an action of $G(\mathbb{A}) \times \operatorname{Gal}_{\mathbb{Q}}$ where

$$\operatorname{Gal}_{\mathbb{Q}} := \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Example 1.4.1. — Take $m = 1$. Eichler-Shimura and Deligne associated Galois representations to eigenforms of weight ≥ 2. There eigenforms correspond to automorphic representations

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p \hookrightarrow S_k$$

(such that $\pi_\infty \simeq D_k$, an irreducible $(\mathfrak{gl}_2, U(1))$-module). For almost all p, π_p is unramified:

$$\underbrace{\pi^{\mathbf{GL}_2(\mathbb{Z}_p)}}_{\dim 1} \neq 0$$

with an action of $\mathscr{H}(\mathbf{GL}_2(\mathbb{Q}_p), \mathbf{GL}_2(\mathbb{Z}_p))$ which is commutative. There correspond to $c(\pi_p)$ which are semi-simple conjugacy classes in $\mathbf{GL}_2(\mathbf{C})$.

Suitably normalized, there exists a number field $F \subset \mathbf{C}$ such that for almost any p,

$$\operatorname{tr} c(\pi_p) \in F, \quad \det c(\pi_p) \in F.$$

Theorem 1.4.3. — *For all $\iota : F \to \overline{\mathbb{Q}_l}$ there is a continuous irreducible representation $\rho_{\pi,\iota} : \mathrm{Gal}_{\mathbb{Q}} \to \mathbf{GL}_2(\overline{\mathbb{Q}_l})$ such that for almost any p, $\rho_{\pi,\iota}$ is unramified at p and*

$$\operatorname{tr} \rho_{\pi,\iota}(\mathrm{Frob}_p) = \iota(\operatorname{tr}(c(\pi_p)))$$

where the Frobenius on the right is geometric. (??)

§2. General Shimura varieties

Definition. — Let $\mathbb{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{GL}_{1,\mathbf{C}})$, so

$$\mathbb{S}(A) = (A \otimes_{\mathbf{R}} \mathbf{C})^\times$$

for an $\mathbf{R}$-algebra A.

$\mathsf{Rep}(\mathbb{S})$ correspond to real Hodge structures, i.e. finite dimensional real vector spaces V with a decomposition

$$V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} V = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$ and there is an action of $\mathfrak{z} \in \mathbb{S}(\mathbf{R}) = \mathbf{C}^\times$ via $\mathfrak{z}^{-p}\bar{\mathfrak{z}}^{-q}$.

Definition. — A **Shimura datum** is a pair (G, X) of a commutative reductive group G over $\mathbb{Q}$ and a $G(\mathbf{R})$-orbit X of morphisms $h : \mathbb{S} \to G_{\mathbf{R}}$ such that

1) $\mathbb{S}$ acs via $\mathrm{Ad}(h)$ on
$$\mathfrak{g} := \mathbf{C} \otimes_{\mathbb{Q}} \mathrm{Lie}\, G = \bigoplus_{p,q} \mathfrak{g}^{p,q}$$
has kernel of type $\{(-1,1), (0,0), (1,-1)\}$, i.e. $\mathfrak{g}^{p,q} = 0$ unless (p,q) lies in this set. (This implies that $\mathbf{GL}_{1,\mathbf{R}} \hookrightarrow \mathbb{S}$ maps to $Z(G)$.)
2) $\mathrm{Ad}(h(i))$ is a Cartan involution of $G_{\mathrm{ad},\mathbf{R}} = (G/Z(G))_{\mathbf{R}}$, i.e. an ivolution θ of $H = G_{\mathrm{ad},\mathbf{R}}$ such that
$$\{g \in H(\mathbf{C}) | \theta(g) = g\}$$
is compact.
3) $G_{\mathrm{ad}} \simeq \prod_i G_{\mathrm{ad},i}$ (with $G_{\mathrm{ad},i}$ simple over $\mathbb{Q}$) has no factor $G_{\mathrm{ad},i}$ such that $G_{\mathrm{ad},i}(\mathbf{R})$ is compact.

Example 2.0.1. — Let $G = \mathbf{GSp}_{2n}$ and

$$h(a+ib) = \begin{pmatrix} aI_n & bI_n \\ -bI_n & aI_n \end{pmatrix}.$$

Proposition 2.0.1. — *X (a priori a real manifold) is a finite disjoint unon of Hermitian symmetric spaces (of non-compact type).*

A symmetric space is a quotient H/K_H where H is a commutative semisimple Lie group (real) and $K_H \subset H$ is a maximal compact subgroup, with an H-invariant Riemannian structure. The Hermitian part requires a complex structure. These were classified by Cartan (Harish-Chandra, Borel).

Complex structure as in the Siegel case:

Definition. — Let $h \in X$, define

$$\begin{aligned} h_{\mathbf{C}} : \mathbb{S}_{\mathbf{C}} &\longrightarrow G_{\mathbf{C}} \\ (\mathfrak{z}_1, \mathfrak{z}_2) &\longmapsto \mu_h(\mathfrak{z}_1)\mu'_h(\mathfrak{z}_2) \end{aligned}$$

From μ_h we get two opposite parabolic subgroups $Q^+_{\mu_h}, Q^-_{\mu_h}$. Embed $X \hookrightarrow G(\mathbf{C})/Q^-_{\mu_h}(\mathbf{C})$ which endows X with a complex structure.

Now comes a definition motivated by variations of Hodge structure, e.g. if $G \to \mathbf{GL}(V)$ is some representation (over $\mathbb{Q}$) we get a rational Hodge structure on V for all $h \in X$. There exists a classification (see Deligne in Corvallis), $G_{\mathbf{C}}$ can only have a Dynkin diagram of type A, B, C, D, E_6, E_7. In the A case $G_{\mathbf{R}}$ is isogeneous to $\prod \mathbf{SU}(p,q)$ (no $\mathbf{SL}_{n,\mathbf{R}}$!).

Definition. — Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup, define

$$\mathrm{Sh}(G, X, K)(\mathbf{C}) := G(\mathbb{Q}) \setminus \Big(X \times G(\mathbb{A}_f)/K \Big).$$

Choose a commutative compact subgroup X_0 of X and define

$$G(\mathbb{Q})^+ := \mathrm{Stab}_{G(\mathbb{Q})}(X_0).$$

Then

$$\mathrm{Sh}(G, X, K)(\mathbf{C}) \simeq \coprod_{[g] \in G(\mathbb{Q}) \setminus G(\mathbb{A}_f)/K} G(\mathbb{Q})^+ \cap gKg^{-1} \setminus X_0.$$

If K is small enough ("neat") then for all $h \in X_0$

$$\mathrm{Stab}_?(h) = \{1\}$$

so $\mathrm{Sh}(G, X, K)(\mathbf{C})$ is a complex manifold. We still have the maps T_g (right multiplication by g).

Theorem 2.0.1 (Baily-Borel, Borel). — *For K neat,* $\mathrm{Sh}(G, X, K)(\mathbf{C})$ *has a canonical structure of a smooth group scheme over* $\mathbf{C}$.

We still have finite étale maps between them.

Definition. — A **morphism of Shimura data** $(G_1, X_1) \to (G_2, X_2)$ is a morphism $f : G_1 \to G_2$ such that $f_{\mathbf{R}}$ maps X_1 to X_2.

In this case, Borel's uniqueness theorem implies that for all K_1, K_2 in $G_1(\mathbb{A}_f), G_2(\mathbb{A}_f)$ respectively such that $f(K_1) \subset K_2$ we get

$$\mathrm{Sh}(G_1, X_1, K_1)(\mathbf{C}) \longrightarrow \mathrm{Sh}(G_2, X_2, K_2)(\mathbf{C})$$

("obviously" ?) is algebraic.

2.1. Canonical model

Definition. — A **model** of $(\mathrm{Sh}(G, X, K))_{K \text{ neat}}$ over some number field E are $(S_K)_{K \text{ neat}}$ of smooth quasiprojective schemes and isomorphisms $S_K \times_E \mathbf{C} \simeq \mathrm{Sh}(G, X, K)$ such that all T_g are defined over E (so finite étale).

Definition. — Let (G, X) be a Shimura datum. The **reflex field** $E(G, X)$ is the smallest field $E \subset \mathbf{C}$ over which the conjugacy class of μ_h (any h) is defined.

This is always a number field.

In the case of tori: if G is a torus, each $\mathrm{Sh}(G, X, \mathbf{C})$ is finite. From global class field theory we get a model of $(\mathrm{Sh}(G, X, K))_K$ over $E(G, X)$.

Definition. — Let (G, X) be a Shimura datum. A **canonical model** of $(\mathrm{Sh}(G, X, K))_K$ is a model over $E(G, X)$ such that for all $T \subset G$ (over $\mathbb{Q}$) all $h \in X$ factor through $T_{\mathbf{R}}$.

So $(T, \{h\})$ is a Shimura datum $E(G, X) \subset E(T, \{h\})$. For all K

$$\mathrm{Sh}(T, \{h\}, K \cap T(\mathbb{A}_f))(\mathbf{C}) \longrightarrow \mathrm{Sh}(G, X, K)(\mathbf{C})$$

are defined over $E(T, \{h\})$.

Proposition 2.1.1 (Shimura, Deligne). — *Such models are unique.*

Theorem 2.1.1 (Shimura, Deligne, Tamigawa). — *In the Siegel case, the moduli space $(\mathscr{A}_m(M) \times \mathbb{Q})_{M\geq 1}$ is a canonical model.*

Note that $(K(M))_{M\geq 1}$ is cofial in all compact open subgroups. This is equivalent to the main theorem of complex multiplication.

Theorem 2.1.2 (Shimura, Deligne, ?, Milne, Shih, Moonen). — *For every Shimura datum (G, X) there exists a canonical model.*

In some cases, proved by reduction to the Siegel case:

- Hodge type Shimura varieties: $(G, X) \hookrightarrow (\mathbf{GSp}_{2n}, \mathscr{H}_n^{\pm})$.
- Abelian type Shimura varieties: "isogeneous" to Hodge type.
- PEL type Shimura varieties which are included in Hodge type Shimura varieties. P stands for polarizations, E stands for endomorphism, L stands for level. These are moduli spaces of polarized abelian varieties with an "endomorphism". Canonical models come from the moduli interpretation (follows from Mumford's representability theorem). There are also integral models over $\mathscr{O}_{E,S}$.

§3. The Kottwitz conjecture

Let (G, X) be a Shimura datum and $E = E(G, X)$ be the reflex field. Assume G_{der} is anisotropic (i.e. no non-trivial $\mathbf{GL}_{1,\mathbb{Q}} \to G_{\mathrm{der}}$), by reduction theory this is equivalent to $\mathrm{Sh}(G, X, K)$ being projective. Let l be a prime and define

$$\mathrm{H}^k_{\text{ét}} := \varinjlim_K \mathrm{H}^k_{\text{ét}}\left(\mathrm{Sh}(G, X, K)_{\overline{E}}, \overline{\mathbb{Q}_l}\right)$$

which admits an action of $G(\mathbb{A}_f) \times \mathrm{Gal}_E$ (where $\mathrm{Gal}_E := \mathrm{Gal}(\overline{E}/E)$). Recall the comparison theorem

$$\mathrm{H}^k_{\text{ét}} \simeq \overline{\mathbb{Q}_l} \otimes_{\mathbb{Q}} \mathrm{H}^k_B.$$

H^k_B is computed using de Rham cohomology (real) and Hodge decomposition. This gives us $(\mathfrak{g}, K_\infty)$-cohomology of automorphic forms ($\mathscr{A} = \mathscr{A}_{\mathrm{cusp}}$ is semisimple).

3.1. Conjectural picture The Langlands group of E is a topological group L_E extending the Weil group W_E that fits into an sequence

$$\begin{tikzcd} L_E \arrow[r] \arrow[dr, "|\cdot|"] & W_E \arrow[r, two heads] \arrow[d, "|\cdot|"] & \mathrm{Gal}_E \\ & \mathbf{R}_{>0} & \end{tikzcd}$$

and $\ker(L_E \to \mathbf{R}_{>0})$ should be compact. For all places v of E we should have

$$\begin{tikzcd} L_{E_v} \arrow[r] \arrow[d] & L_E \arrow[d] \\ W_{E_v} \arrow[r] \arrow[d, hook] & W_E \arrow[d, two heads] \\ \mathrm{Gal}_{E_v} \arrow[r, hook] & \mathrm{Gal}_E \end{tikzcd}$$

Also E is functorial in E. If v is archimedean:

- If $E_v \simeq \mathbf{C}$ we have $L_{E_v} = E_v^\times$ $(= W_{E_v})$.
- If $E_v \simeq \mathbf{R}$ we have a short exact sequence

$$1 \longrightarrow \mathbf{C}^\times \longrightarrow W_{\mathbf{R}} \longrightarrow \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \longrightarrow 1$$

There is a relation to (pure) motives (over E).

Definition. — A continuous and semisimple representation $\rho : L_E \to \mathbf{GL}(W)$ is **algebraic** if for all archimedean places v of E

$$(\rho|_{L_{E_v}})|_{\mathbf{C}^\times} \simeq \left(\mathfrak{z} \longmapsto \begin{pmatrix} \mathfrak{z}^{a_1}\bar{\mathfrak{z}}^{b_1} & & \\ & \ddots & \\ & & \mathfrak{z}^{a_n}\bar{\mathfrak{z}}^{b_n} \end{pmatrix} \right)$$

for $a_i, b_i \in \mathbb{Z}$.

Conjecture 3.1.1. — *There exists $L_E \to \mathsf{GMot}_E(\mathbf{C})$ (GMot_E are pro-reductive groups over $\mathbb{Q}$ whose representations are pure motives over E) such that ρ is continuous and semisimple. ρ is algebraic if and only if it factors through $L_E \to \mathsf{GMot}_E(\mathbf{C})$. In particular, for all algebraic ρ and all $\iota : \mathbf{C} \simeq \overline{\mathbb{Q}_l}$ (only $\iota_{\overline{\mathbb{Q}}}$ matters) we have*

$$\mathrm{real}_L(\rho) : \mathrm{Gal}_E \longrightarrow \mathbf{GL}_n(\overline{\mathbb{Q}_l})$$

continuous, semisimple with $n = \dim_{\mathbf{C}} W$ which is "geometric" such that for almost all non-archimedeal places v of E, both ρ and $\mathrm{real}_L(\rho)$ are unramified at v and

$$\mathrm{real}_L(\rho)(\mathrm{Frob}_V) \sim \iota(\rho(\mathrm{Frob}_V))$$

(by Chebotarev this characterizes $\mathrm{real}_L(\rho)$).

3.2. Langlands dual group The isomorphism class of $G_{\overline{\mathbb{Q}}}$ is parametrized by a root datum $(X, R, X^\vee, R^\vee)$. Choose a maximal torus $T \subset G_{\overline{\mathbb{Q}}}$. The dual root datum $(X^\vee, R^\vee, X, R)$ corresponds to a dual group $\hat{G}_{\overline{\mathbb{Q}}}$ with a Galois action of $\mathrm{Gal}(E'/E)$ for some finite E'. Define

$${}^L G := \widehat{G}(\mathbf{C}) \rtimes \mathrm{Gal}_{\mathbb{Q}} .$$

For example

G	$\widehat{G}$
$\mathbf{GL}_m/U(n)$	$\mathbf{GL}_m$
$\mathbf{SL}_m$	$\mathbf{PGL}_m$
$\mathbf{Sp}_{2n}$	$\mathbf{SO}_{2n+1}$
$\mathbf{GSp}_{2n}$	$\mathbf{GSpin}_{2n+1}$

Definition. — A **Langlands parameter** is a continuous semisimple

$$\begin{tikzcd} L_{\mathbb{Q}} \arrow[rr, "\phi"] \arrow[dr] & & L_G \arrow[dl] \\ & \mathrm{Gal}_{\mathbb{Q}} . & \end{tikzcd}$$

ϕ is **discrete** if $\mathrm{Cont}(\phi, \widetilde{G}(\mathbf{C}))/Z(\widehat{G}(\mathbf{C}))^{\mathrm{Gal}_{\mathbb{Q}}}$ is finite.

If $G = \mathbf{GL}_n$, ϕ is irreducible.

3.3. Étale cohomology Let $d = \dim \mathrm{Sh}(G, X, K)$. Define

$$P^d_{\text{ét}} := \text{primitive part of } \mathrm{H}^d_{\text{ét}}$$

(hard Lefschetz, "not coming from cohomology of a lower dimensional variety"). Recall $\mu = \mu_h : \mathrm{Gal}_{1,\mathbf{C}} \to G_{\mathbf{C}}$ (up to conjugation). We can see μ as a character of a maximal torus $T \subset \widehat{G}$. This gives an irreducible (algebraic and finite dimensional) representation V_μ of $\widehat{G}(\mathbf{C})$ extends to

$${}^L(G_E) \xrightarrow{r_\mu} \mathbf{GL}(V_\mu).$$

A discrete Langlands parameter $\phi : L_{\mathbb{Q}} \to L_G$ induces an action of L_E on $V_\mu(\phi) := V_\mu$ by $|\cdot|^{-d/2}(r_\mu \circ \phi|_{L_E}$. The group

$$C_\phi := \mathrm{Cont}(\phi, \widehat{G}(\mathbf{C}))$$

also acts on $V_\mu(\phi)$

$$V_\mu(\phi) = \bigoplus_{\chi \in X^*(C_\phi)} V_\mu(\phi)_\chi$$

(in our case C_ϕ is abelian insider the maximal torus of $\widehat{G}$).

Now only consider ϕ such that

$$(\phi|_{W_{\mathbf{R}}})|_{\mathbf{C}^\times} \sim \mathfrak{z} \longmapsto (\mathfrak{z}/|\mathfrak{z}|)^{2\rho}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$.

Lemma 3.3.1. — *Each $V_\mu(\phi)_\chi$ is algebraic.*

Conjecture 3.3.1 (Kottwitz, simplified and vague). — *We have*

$$P^d_{\acute{e}t} \simeq \bigoplus_{\textit{these } \phi} \bigoplus_{\substack{\pi_f = \otimes'_p \pi_p \\ (\pi_f)_p \in \prod' \Pi(\phi|_{L_{\mathbf{Q}_p}})}} \bigoplus_{\chi} \mathrm{real}_L(V_\mu(\phi)_\chi)^{\oplus m(\phi, p_f, \chi)}$$

where $m(\phi, \pi_f, \chi)$ are "explicit" integers.