MOTIVIC SHEAVES

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Plan:
L1: Motivation, construction, examples.
L2: Six-functor formalism.
L3: Motivic t-structures and weight structures.
L4: ∞-categorical methods.

§1. Motivation from GRT and cohomology

1.1. Cohomology and sheaves for representation theory

Lecture 1

Question: How do you construct interesting representations?
Answer:

1) Find interesting actions.
2) Linearlize them.

Example 1.1.1. — Let K be a compact Lie group. The action of K on itself gives us an action of K on $L^2(K)$ with respect to a Haar measure. The Peter-Weyl theorem says that

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \text{ unitary}}} \pi^{\oplus dim(\pi)}.$$

"Lie theory $\subset$ algebraic geometry". Reductive groups are algebraic groups with many associated varieties with group actions: flag varieties...

The linearizations we consider in this course are the many types of cohomology theories.

Example 1.1.2 (Borel-Weil-Bott). — Let $T \subset B \subset G$ be a reductive group over $\mathbf{C}$. Let $\lambda \in X^\vee(T)$ such that there exists $w \in W$ with $w * \lambda = w(\lambda + \rho) - \rho > 0$ (where $\rho = (1/2)\sum_{\alpha\in\Phi^+}\alpha$). Then

$$\mathrm{R}\Gamma(G/B, L_\lambda) \simeq \pi_{w*\lambda}[-\ell(w)]$$

where $\ell(w)$ is the length of w.

Cohomology fits in the wider context of sheaf theory. If T is a locally contractible topological space, then

$$\mathrm{H}^n_{\mathrm{sing}}(T, \mathbb{Z}) \simeq \mathrm{H}^n(T, \underline{\mathbb{Z}}_T) \simeq \mathrm{R}^n(\pi_T)_*(\underline{\mathbb{Z}}_T)$$

where π_T is the morphism $\pi_T : T \to \mathrm{pt}$ with

$$\mathrm{R}\pi_{T*} : D(T, \mathbb{Z}) \longrightarrow D(\mathbb{Z}) \simeq D(*, \mathbb{Z}).$$

Cohomology (singular with $\mathbb{Q}$-coefficients) of algebraic varieties over $\mathbf{C}$ is ***very*** special.

- There is a weight filtration.
- There is a mixed hodge structure.

Sheaves on complex algebraic varieties are also very special:

- Perverse sheaves;
- Decomposition theorem;
- Mixed Hodge modules.

This leads to great success stories in GRT:

- Springer theory;
- Kazdhan-Lustig theory;
- geometric Satake...

1.2. From sheaves to motivic sheaves There are situations which can't be directly studied using these tools:

- Representation theory of reductive groups over other fields/rings/schemes.
- Modular/integral representation theory.
- q-deformations, quantum groups, canonical bases.

These can be attacked using:

- l-adic sheaves,
- sheaves cohomology with $\mathbb{Z}$-coefficients,
- K-theory.

Motivic sheaves will give us a unified perspective.

Motivic dream: There should exist universal cohomology/sheaf theories such that

1) they unify and "explain" the special structure in cohomology/sheaves;
2) they are of algebro-geometric nature;
3) they "explain" the realization of algebraic cycles and algebraic K-theory.

§2. Construction of $\mathsf{DA}^{\text{ét}}$ and SH (Morel-Voevodsky)

2.1. Triangulated categories and localization

Definition. — A **triangulated category** is the data:

- an additive category C,
- an automorphism $\Sigma = (-)[1] : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$,
- a collection of distinguished triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

The data satisfies the following conditions:

- shifted distinguished triangles are distinguished up to isomorphism,
- for all $f : A \to B$ there exists

  $$A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \xrightarrow{+}$$

  unique up to isomorphism and functorial,
-
  $$\begin{tikzcd} A \arrow[r] \arrow[d, "f", "\simeq"'] & B \arrow[r] \arrow[d, "g", "\simeq"'] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "{f[1]}", "\simeq"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$
- (??)

Example 2.1.1. — Let A be an abelian category, $\mathsf{Ch}(\mathsf{A})$ be the abelian category of chain complexes in A. We define $(A[1])_n = A_{n-1}$. Given $f : A_\bullet \to B_\bullet$ the maping cone is given by

$$\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n, \qquad d_n = \begin{pmatrix} -d^A_{n-1} & 0 \\ f & d^B_n \end{pmatrix}.$$

Definition. — $f : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if for all $n \in \mathbb{Z}$, the map $H_n(A_\bullet) \simeq H_n(B_\bullet)$ is an isomorphism.

Definition. — $D(\mathsf{A})$ is defined as the localization of $\mathsf{Ch}(\mathsf{A})$ by quasi-isomorphisms.

Now we consider reflexive localizations (1-categorical ones lead to triangulated and ∞-categorical ones).

Definition. — Let C be a 1-category.

1) $\mathsf{C}' \subset \mathsf{C}$ is **reflexive** if $\iota : \mathsf{C}' \to \mathsf{C}$ has a left adjoint.
2) $L_W : \mathsf{C} \to \mathsf{C}[W^{-1}]$ is **reflexive** if L_W has a right adjoint.

Lemma 2.1.1 (Reflexive subcategories are the same thing as reflexive localizations). —

a) *Let $\mathsf{C}' \subset \mathsf{C}$ be reflexive, $L : \mathsf{C} \to \mathsf{C}'$ be the left adjoint to ι. Define*
$$W_L = \{f : L(f) \text{ is an isomorphism}\}.$$
Then $\mathsf{C}' \simeq \mathsf{C}[W_L^{-1}]$ and $L \simeq L_{W_L}$.

b) *If L is a reflexive localization, then its right adjoint ι is fully faithful and $\iota : \mathsf{C}[W^{-1}] \xrightarrow{\sim} \mathrm{EssIm}(\iota) \subset \mathsf{C}$.*

Definition. — Let $S \subset \mathsf{C}$ be a collection of morphisms.

a) $A \in \mathsf{C}$ is S**-local** if for all $f : B \to C$ in S
$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$
b) $f : B \to C$ is an S**-equivalence** if for all S-local A
$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

Lemma 2.1.2. — *If $L : \mathsf{C} \rightleftarrows \mathsf{C}' : \iota$ is a reflexive localizaton, W_L as before, then*

- *ι gives an isomorphism between C' and W_L-local objects.*
- *W_L are the W_L-equivalences.*

Definition. — Let D be a triangulated category with all small products.

- Let κ be a regular cardinal (for example $\kappa = \aleph_0$). Then $A \in \mathsf{D}$ is κ**-small**/κ**-compact** if and only if
$$\operatorname*{colim}_{\substack{I' \subset I \\ |I'| < \kappa}} \mathrm{Hom}\left(A, \bigoplus_{I'} B_i\right) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_{I} B_i\right).$$
- **Compact** means $\aleph_0$-small. A is compact if and only if
$$\bigoplus_I \mathrm{Hom}(A, B_i) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_I B_i\right).$$
- D is **presentable**/**well-generated** if and only if there exist κ and a set $S \subset \mathsf{D}$ of κ-small objects which generate D:
$$\forall B \in \mathsf{D}, (\forall A \in S, \mathrm{Hom}(A, B) = 0) \implies B = 0.$$
- D is **compactly generated** if it is $\aleph_0$-presentable.

Definition. — $\mathsf{E} \subset \mathsf{D}$ is **localizing** if it is

- triangulated,
- stable under coproducts,
- thick (stable under subobjects and subquotients).

Theorem 2.1.1 (Adjoint Functor Theorem). — *Let D, D' be triangulated categories with all coproducts, $F : \mathsf{D} \to \mathsf{D}'$ be a triangulated functor and D be presentable. Then F admits a right adjoint if and only if F preserves all coproducts.*

Corollary 2.1.1 (Verdier Localization). — *Let D be a presentable category and E be a localizing subcategory. Define*
$$\mathsf{D}/\mathsf{E} = D[W_{\mathsf{E}}^{-1}], \quad W_{\mathsf{E}} = \{f : \mathrm{Cone}(f) \in \mathsf{E}\}.$$
Then $\mathsf{D} \to \mathsf{D}/\mathsf{E}$ is a reflexive localization.

Let $S \subset \mathsf{D}$ be a subset of objects, then $\langle\langle S \rangle\rangle$ is the smallest subcategory containing S such that $\mathsf{D}/\langle\langle S \rangle\rangle$ is a reflexive localization.

Let's get some inspiration from Betti homology, i.e. singular homology on complex algebraic varieties. Let $X \in \mathsf{Var}_{\mathbf{C}}^{(\mathrm{f.t})}$, then we get
$$C_*^{\mathrm{sing}}(X(\mathbf{C}), \mathbb{Z}) \in D(\mathbb{Z}).$$
This comes with some data:

(a)
- $D(\mathbb{Z})$ has a symmetric monoidal structure: $\otimes^{\mathbb{Z}}$,
- (Künneth) $C_*(X \times Y) \simeq C_*(X) \otimes^{\mathbb{Z}} C_*(Y)$.

which satisfies sproperties:

(b) ($\mathbb{A}^1$-homotopy invariance) $C_*(X \times \mathbb{A}^1) \xrightarrow{\sim} C_*(X)$ ($(\mathbb{A}^1)^{\mathrm{an}} = \mathbf{C}$ is contractible).

(c′) (Mayer-Vietoris sequence) Let $X = U \cup V$ be a Zariski cover, then

$$C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*(X) \xrightarrow{\partial} C_*(U \cap V)[1].$$

(c) (Étale descent) Let $U \to X$ be étale surjective. Define

$$\check{C}_n(U/X) = U^{n+1}.$$

Then $\check{C}_\bullet(U/X)$ is a simplicial scheme, so $C_*(\check{C}_\bullet(U/X))$ is a simplicial complex of abelian groups and $C(C_*(\check{C}_\bullet(U/X)))$ is a double complex. (??)

Concretely we have a descent spectral sequence which gives us ($U = U \cup V$) Mayer Vietoris.

(d) ($\mathbb{P}^1$-stabilization)

$$\begin{aligned} C_*(\mathbb{P}^1_{\mathbf{C}}) &\simeq C_*(\mathrm{pt}) \oplus \widetilde{C}_*(\mathbb{P}^1_{\mathbf{C}}) \\ &\simeq \mathbb{Z}[0] \oplus \mathbb{Z}(1)[2]. \end{aligned}$$

$\mathbb{Z}(1)$ is $\oplus$-invertible.

Smooth varieties play a special role:

- Poincaré duality
- Gysin sequences
- $C_*(-)$ also satisfies "h-descent", so $C_*(-)$ is "determined" by $C_*(-)_{|(?)}$.

There is an associated sheaf theory:

$$D_B(-) : \mathsf{Var}_{\mathbf{C}} \longrightarrow \mathsf{TriCat}^{\otimes}, \quad X \longmapsto D_B(X) = D(\mathsf{Sh}(X^{\mathrm{an}}, \mathbb{Z})).$$

Sketch of $\mathsf{DA}^{\acute{e}t}$*:* Let S be a base scheme.

- Start with
$$\begin{cases} D(\mathsf{PSh}(\mathsf{Sm}_S, \mathbb{Z})) = D_{\mathsf{PSh}}(S) \\ \mathbb{Z}[-] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}.$$
- Impose $\mathbb{A}^1$-invariance, étale descent, and $\mathbb{P}^1$-stability. This will give us $\mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$ and $M_S(-) : \mathsf{Sm}_S \to \mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$.

The surprise is that the result satisfies many *other* properties of singular (co)homology and derived categories of sheaves on complex varieties which can largely be packaged into the six-functor formalism. The result is closely related to algebraic cycles/higher Chow groups and algebraic K-theory. Lecture 2

(Fill in H_* frmo the recall part)

Let S be a qcqs scheme, Λ be a coefficient ring. Define

$$\begin{cases} D_{\mathsf{PSh}}(S) := D(\mathsf{PSh}(\mathsf{Sm}_S, \Lambda)) \text{a presentable, symmetric monoidal, triangulated category} \\ \Lambda[\cdot] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}$$

Étale descent:

$$\begin{aligned} D_{\text{ét}}(S) &:= D(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S, \Lambda)) \\ &= D_{\mathsf{PSh}}(S)[W_{\text{ét}}^{-1}] \end{aligned}$$

where $W_{\text{ét}}$ are étale-local weak equivalences, i.e. $(f : K_\bullet \to L_\bullet) \in W_{\text{ét}}$ if for all n we have

$$(?)_{\text{ét}}\mathrm{H}_n(K_\bullet) \xrightarrow{\sim} (?)_{\text{ét}}\mathrm{H}_n(L_\bullet).$$

$\mathbb{A}^1$-*invariance* Let

$$I_{\mathbb{A}^1,(\text{ét})} = \{\ldots \longrightarrow 0 \longrightarrow \Lambda_{(\text{ét})}[X \times \mathbb{A}^1] \longrightarrow \Lambda_{(\text{ét})}[X] \longrightarrow 0 \longrightarrow \ldots | X \in \mathsf{Sm}_S\}.$$

Define

$$D_{\mathbb{A}^1}(S) := D_{\mathsf{PSh}}(S) / \langle\langle I_{\mathbb{A}^1} \rangle\rangle = D_{\mathsf{PSh}}(S)[W_{\mathbb{A}^1}^{-1}].$$

We have

$$L_{\mathbb{A}^1} : D_{\mathsf{PSh}}(S) \longrightarrow D_{\mathsf{PSh}}(S)^{\mathbb{A}^1-\mathrm{loc}} \hookrightarrow D_{\mathsf{PSh}}(S).?$$

with the middle term isomorphic to $D_{\mathbb{A}^1}(S)$.

Definition. — Define

$$\boldsymbol{\Delta}^n_{\mathrm{alg},S} := \mathrm{Spec}_S\left(\mathscr{O}_S[X_0,\ldots,X_n]/\left(\sum x_i - 1\right)\right) \simeq \mathbb{A}^n_S$$

then $\boldsymbol{\Delta}^\bullet_{\mathrm{alg},S}$ is a cosimplicial scheme over S.

Definition (Suslin-Voevodsky). — Define

$$\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) = \mathrm{hocolim}_{\boldsymbol{\Delta}^{\mathrm{op}}} K_\bullet(\boldsymbol{\Delta}^\bullet_{\mathrm{alg},S} \times_S X)$$

Example 2.1.2. — Let $F \in \mathsf{PSh}$ then

$$\mathrm{Sing}^{\mathbb{A}^1}(F)(U) = \left[\ldots \longrightarrow F(\boldsymbol{\Delta}^2 \times U) \longrightarrow F(\mathbb{A}^{(?)} \times U) \xrightarrow{i_0^* - i_1^*} F(U) \longrightarrow 0\right].$$

Proposition 2.1.1. — $L_{\mathbb{A}^1} \simeq \mathrm{Sing}^{\mathbb{A}^1}$.

Proof. The idea is to use

$$\begin{aligned} m : \mathbb{A}^1 \times \mathbb{A}^1 &\longrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto xy \end{aligned}$$

to prove

a) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet)$ is $\mathbb{A}^1$-local.
b) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) \to K_\bullet$ is $\mathbb{A}^1$-weak equivalence.

□

Definition. — The category of **effective étale motivic sheaves** on S is

$$\mathsf{DA}^{\text{ét,eff}}(S,\Lambda) := D_{\text{ét}}(S)/\left\langle\!\left\langle I_{\mathbb{A}^1,\text{ét}}\right\rangle\!\right\rangle.$$

Write $L^{\mathrm{eff}}_{\mathrm{mot}}$ for the associated localization functor.

Lemma 2.1.3. — *We have*

$$L^{\mathit{eff}}_{\mathit{mot}} = \underbrace{\ldots \mathrm{Sing}^{\mathbb{A}^1} L_{\acute{e}t}\, \mathrm{Sing}^{\mathbb{A}^1}}_{\textit{transfinie composition}\ldots}$$

Definition. — Let $X \in \mathsf{Sm}_S$. Define

$$M^{\text{ét,eff}}_S(X) := L^{\mathrm{eff}}_{\mathrm{mot}}\Lambda_{\text{ét}}[X] \in \mathsf{DA}^{\text{eff,ét}}(S,\Lambda)$$

(effecive étale (? homological) motive/motivic sheaf of ?).

We have

$$M^{\text{ét,eff}}_S(X \times_S Y) \simeq M^{\text{ét,eff}}_S(X) \otimes M^{\text{ét,eff}}_S(Y).$$

Proposition 2.1.2 (Artin-Shreier $+\Lambda\left[\frac{1}{p}\right]$). — *Let S be a $\mathbf{F}_p$-scheme, then*

$$\mathsf{DA}^{\acute{e}t,\mathit{eff}}(S,\Lambda) \xrightarrow{\sim} \mathsf{DA}^{\acute{e}t,\mathit{eff}}\left(S,\Lambda\left[\frac{1}{p}\right]\right).$$

Proof. We have the short-exact sequence

$$0 \longrightarrow \Lambda/p\Lambda \longrightarrow \mathbb{G}_a \otimes \Lambda \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbb{G}_a \longrightarrow 0$$

and hence an exact sequence

$$\Lambda_{\text{ét}}[\mathbb{G}_a] \otimes (\mathbb{G}_a \otimes \Lambda) \xrightarrow{a_{\mathbb{G}_a} \otimes \mathrm{id}} \mathbb{G}_a \otimes \mathbb{G}_a \otimes \Lambda \xrightarrow{m} \mathbb{G}_a \otimes \Lambda.$$

(Some remark??) Thus

$$L_{\mathbb{A}^1}(\mathbb{G}_a \otimes \Lambda) \simeq \Lambda(0)$$

and so

$$L^{\mathrm{eff}}_{\mathrm{mot}}(\Gamma/p\Gamma) = 0.$$

□

$\mathbb{P}^1$-*stabilization:* Let $x \in X(S)$, we have

$$M_S^{\text{eff}}(X) = \Lambda_S(0) \oplus M_S^{\text{eff}}(X, x).$$

Definition. — We define

$$T := M_S^{\text{eff}}(\mathbb{P}^1, 1)[-2]$$

which is equal to $\Lambda(?)$.

Exercise. — Any $x \in \mathbb{P}^1_S(S)$ gives the same decomposition.

We have a problem: T is not $\oplus$-invertible.

Definition. — The category of étale motivic sheaves over S is

$$\mathsf{DA}^{\text{ét}}(S, \Lambda) := \mathsf{DA}^{\text{ét,eff}}(S, \Lambda)[T^{\otimes -1}].$$

This is not a proper definition since we have not explained the construction. It's not clear what this means!

Spectra:

Definition. — Let C be a closed, symmetric monoidal 1-category and T be an object of C. A T**-prespectrum** is

$$A = \{(A_n, \sigma_n)_{n \in \mathbb{N}} | A_n \in \mathsf{C}, \sigma_n : T \otimes A_n \longrightarrow A_{n+1}\}.$$

A is a T**-spectrum** if for all $n \in N$

$$A_n \xrightarrow{\sim} \underline{\mathrm{Hom}}(T, A_{n+1}).$$

We write $\mathrm{PSp}_T(\mathsf{C})$ and $\mathrm{Sp}_T(\mathsf{C})$ for the T-prespectrum and T-spectrum respectively.

The evaluation map

$$\mathrm{Ev}_n(A) = A_n$$

has a left adjoint. We define

$$\mathrm{Sus}^n(A)_m = \begin{cases} \varnothing & \text{if } m < n \\ T^{\otimes(m-n)} \otimes A & \text{if } m > n \end{cases}$$

and $\Sigma_T^\infty := \mathrm{Sus}^0$ is the ∞-suspension functor.

Proposition 2.1.3. — *Assume* C *is presentably, symmetrical monoidal. Then* $\mathrm{Sp}_T(\mathsf{C}) \subset \mathrm{PSp}_T(\mathsf{C})$ *is a reflexive subcategory.* W_{st} *is generated by*

$$\left\{\mathrm{Sus}^{n+1}(T \otimes A) \longrightarrow \mathrm{Sus}^n(A) : n \in \mathbb{N}, A \in \mathsf{C}\right\}.$$

Definition. — We define

$$\mathsf{DA}^{\text{ét}}(S, \Lambda) := \mathrm{Sp}_T \, \mathsf{DA}^{\text{eff,ét}}(S, \Lambda).$$

(This definition is correct "with ∞-categories".) We have

$$\begin{aligned} M_S : \mathsf{Sm}_S &\longrightarrow \mathsf{DA}^{\text{ét}}(S, \Lambda) \\ X &\longmapsto L_{(\mathbb{A}^1, \text{ét}, ?}\Sigma_T^\infty M_S^{\text{ét,eff}}(X). \end{aligned}$$

Remark. — $M \in \mathsf{DA}^{\text{ét}}(S, \Lambda)$ is isomotphic to a stable $(\mathbb{A}^1, \text{ét})$-local (??)

$$K_n \in \mathsf{Ch}(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S, \Lambda)) + \sigma_n = \Lambda_{\text{ét}}[\mathbb{P}^1, 1] \otimes K_n \longrightarrow K_{n+1}$$

such that for all $X \in \mathsf{Sm}_S, i \in \mathbb{Z}$

- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X \times_S \mathbb{A}^1, K_n)$
- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^{i+2}_{\text{ét}}(X \times_S (\mathbb{P}^1, 1), K_{n+1})$.

2.2. Constructible motivic sheaves

Definition. — We define **constructible motivic sheaves**

$$\begin{aligned}\mathsf{DA}^{\text{ét}}_{\text{ct}} &= \langle M_S(X)(-n) | X \in \mathsf{Sm}_S, n \in \mathbb{N}\rangle^{\text{d.f.}} \\ &\subset \mathsf{DA}^{\text{ét}}(S, \Lambda).\end{aligned}$$

and **locally constructible motivic sheaves**

$$\mathsf{DA}^{\text{ét}}_{\text{lct}}(S,\Lambda) := \{M | \exists e : U \twoheadrightarrow S, e^*M \in \mathsf{DA}_{\text{ct}}\}.$$

There is a Betti realization for S finite type over $\mathbf{C}$

$$R_B : \mathsf{DA}^{\text{ét}}(S,\Lambda) \longrightarrow D(S^{\text{an}}, \Lambda)$$

by the existence of relative homology and the universal property. If $X \in \mathsf{Sm}_S$

$$R_B(M_S(X)) \simeq \mathrm{H}^{\text{sing}}_*(X/S)$$

and

$$R_B(\mathsf{DA}^{\text{ét}}_{\text{lct}}(S,\Lambda) \subset D^b_{\text{ct}}(S^{\text{an}}, \Lambda).$$

Another deep property is the *rigidity theorem*. Define

$$D_{\text{ét}}(S,\Lambda) = D(\mathsf{Sh}_{\text{ét}}(S,\Lambda))$$

and write

$$\iota : (\mathsf{Et}_S, \text{ét}) \hookrightarrow (\mathsf{Sm}_S, \text{ét})$$

for the inclusion, then we get

$$\iota^*_S : D_{\text{ét}}(S,\Lambda) \longrightarrow \mathsf{DA}^{\text{ét}}(S,\Lambda).$$

Theorem 2.2.1 (Ayoub). — *Let S be an excellent, Noetherian, finite dimensional, Λ-finite, with any prime invertible in Λ or $\mathscr{O}_S$. Then ι^*_S is an equivalence.*

This procedure is very flexible and admits many *variants*.

- We can change the input. Instead of complexes we can work with presheaves of simplicial sets or ∞-groupoids.
- We can change the topology. Instead of étale use Nisnevich or use presheaves with transfers.
- Change the geometric context. For example, to rigid analytic motivic sheaves.

Definition. — The **stable motivic homotopy category** over S is

$$\mathsf{SH}(S) := \mathrm{PSp}_T(\mathsf{PSh}(\mathsf{Sm}_S, \mathsf{sSet}))[W^{-1}_{(\mathbb{A}^1, \text{Nis}, \mathbb{P}^1)}].$$

Recall that the Nisnevich topology is between the Zariski and étale topologies. $\mathsf{DA}^{\text{ét}}(S)$ is the motivic version of $D(S^{\text{an}}, \mathbb{Z})$ and $\mathsf{SH}(S)$ is the motivic version of sheaves of $S^?$-spectra on S^{an}. There is also $\mathsf{DM}(S,\Lambda)$ which are the Nisnevich-local motivic sheaves. Many important invariants of varieties only satisfy Nisnevich descent, but not étale descent; for example, K-theory or higher chow groups.

2.3. Motives over a field Let $S = \mathrm{Spec}(k)$ and $\Lambda = \mathbb{Q}$. Define

$$\mathsf{DM}(k, \mathbb{Q}) := \mathsf{DA}^{\text{ét}}(k, \mathbb{Q}).$$

The analogies you should have in mind are

- $D(\mathrm{Ind}\,\mathsf{MHS}_{\mathbb{Q}})$,
- $D(\mathrm{Ind}\,\mathsf{Rep}^{\text{f.d.}}_{\mathbb{Q}_l} G_k)$.

Even though the construction was very formal there are some surprises. Define

$$\mathbb{Q}\langle i\rangle := \mathbb{Q}(i)[2i]$$

be pure Tate twists and $M\langle i\rangle := M \otimes \mathbb{Q}\langle i\rangle$.

- *Projective bundle formula:* Let $E \to X$ be a vector bundle, then
$$M(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{\operatorname{rank} E-1} M(X)\langle i\rangle$$
$$M(\mathbb{P}^n_l) = \Lambda(0) \oplus \Lambda\langle 1\rangle \oplus \cdots \oplus \Lambda\langle n\rangle .$$
- *Gysin triangle:* Let $(c : Z \not\hookrightarrow X) \in \mathsf{Sm}_k$, then we have
$$M(X \setminus Z) \longrightarrow M(X) \longrightarrow M(Z)\langle c\rangle \xrightarrow{+}$$
- *Smooth blow-up formula:*
$$M(\mathrm{Bl}_Z(X)) \simeq M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)\langle i\rangle .$$
- *Poincaré duality 1*: Let X be smooth and projective over k, then $M(X)$ is *dualizable* with
$$M(\mathrm{X})^\vee \simeq M(\mathrm{X})\langle -\dim(X)\rangle .$$

We have $\mathsf{DM}(k, \mathbb{Q}) \simeq \mathrm{Ind}\,\mathsf{DM}_{\mathrm{ct}}$.