

REPRESENTATIONS OF REDUCTIVE ALGEBRAIC GROUPS

LECTURER: SIMON RICHE, TYPESETTER: MICHAŁ MRUGAŁA

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§1. GENERALITIES ON AFFINE GROUP SCHEMES AND SMOOTH REPRESENTATIONS

1.1. Affine group schemes

Fix k a base field.

Recall that an **affine k -group scheme** is one of the following data:

- (1) An affine scheme G over k endowed with morphisms of k -schemes

- $m : G \times G \rightarrow G$;
- $e : \text{Spec}(k) \rightarrow G$;
- $\text{inv} : G \rightarrow G$;

which satisfy the usual axioms of groups (with m multiplication, e the unit, inv the inverse).

- (2) A functor $\text{Alg}_k = \{k\text{-algebras}\} \xrightarrow{F} \text{Gps}$ such that the composition $\text{Alg}_k \xrightarrow{F} \text{Gps} \rightarrow \text{Sets}$ is representable.

- (3) A commutative Hopf algebra over k , i.e. a commutative algebra A with morphisms of k -algebras

- $\Delta : A \rightarrow A \otimes A$;
- $\varepsilon : A \rightarrow k$;
- $S : A \rightarrow A$;

such that

$$\begin{aligned} (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) : A \longrightarrow A \otimes A \otimes A \\ (\text{id} \otimes \varepsilon) \circ \Delta &= \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta : A \longrightarrow A \\ (\text{id}, S) \circ \Delta &= \Delta \circ \varepsilon = (S, \text{id}) \circ \Delta : A \longrightarrow A \end{aligned}$$

Notation. —

$$\begin{aligned} A &\longrightarrow \text{Spec}(A) \\ G &\longrightarrow \mathcal{O}(G), \Delta_G. \end{aligned}$$

Example 1.1.1. —

- (1) *Diagonalizable groups*: If Λ is an abstract commutative group we have the affine k -group scheme $\text{Diag}(\Lambda) := \text{Spec}(k[\Lambda])$ with

$$\Delta(\lambda) = \lambda \otimes \lambda, \quad \varepsilon(\lambda) = 1, \quad S(\lambda) = \lambda^{-1} \quad (\forall \lambda \in \Lambda).$$

In particular for $\Lambda = \mathbb{Z}$, $k[\Lambda] = k[x, x^{-1}]$ and $\text{Diag}(\Lambda) = \mathbf{G}_m$ (the **multiplicative group**).

A **(split) torus** is a group scheme of the form $\text{Diag}(\Lambda)$ with Λ a finitely generated free abelian group.

- (2) *Additive group*: $\mathbf{G}_a := \text{Spec}(k[x])$ with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

More generally for V a k -vector space we have the functor $V_a : R \mapsto (R \otimes V, -)$ which is an affine k -group scheme if V is finite dimensional.

- (3) If V is a k -vector space, $\mathbf{GL}(V)$ is the functor $R \mapsto \text{Aut}_R(R \otimes V)$. If V is finite dimensional this is an affine k -group scheme.

In particular, if $V = k^n$ we get

$$\mathbf{GL}_n = \text{Spec}(k[x_{ij}, 1 \leq i, j \leq n][\det^{-1}])$$

with

$$\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}, \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

Similarly we have $\mathbf{SL}(V), \mathbf{SL}_n$.

- (4) For any abstract group Γ we have the functor (??)

1.2. Representations If G is an affine k -group scheme, a **representation** of G is the datum of a k -vector space V and a morphism of group valued functors $G \rightarrow \mathbf{GL}(V)$. [Equivalently, an action of G on V_a such that $G(R)$ acts R -linearly on $R \otimes V$.]

This datum is equivalent to that of a **comodule** for $\mathcal{O}(G)$, i.e. a k -vector space V and a k -linear map $\Delta_V : V \rightarrow V \otimes \mathcal{O}(G)$ such that

$$\begin{aligned} (\Delta_V \otimes \text{id}_{\mathcal{O}(G)}) \circ \Delta_V &= (\text{id}_V \otimes \Delta_G) \circ \Delta_V : V \longrightarrow V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \\ (\text{id}_V \otimes \varepsilon) \otimes \Delta_V &= \text{id}_V : V \longrightarrow V. \end{aligned}$$

[Δ_V corresponds to the image of $\text{id}_{\mathcal{O}(G)} \in G(\mathcal{O}(G)) = \text{End}_{k\text{-alg}}(\mathcal{O}(G))$ in $\text{End}_{\mathcal{O}(G)}(\mathcal{O}(G) \otimes V)$.]

Example 1.2.1. —

- (1) (*Right*) *Regular representation*: $V = \mathcal{O}(G)$ with $\Delta_V = \Delta_G$. More generally, given an action of G on an affine scheme X we get a representation with underlying vector space $\mathcal{O}(X)$.
- (2) If V is a finite dimensional vector space, V is a representation of $\mathbf{GL}(V)$.
- (3) For any G we have the trivial representation k .

Notation. — $\text{Rep}(G)$ is the abelian category of representations of G . $\text{Rep}^{\text{fd}}(G)$ is the full subcategory of finite dimensional representations.

If $V \in \text{Rep}(G)$ then V is the union of its finite dimensional subrepresentations.

Example 1.2.2 (Representations of diagonalizable group schemes). — Let Λ be a commutative group, $G = \text{Diag}(\Lambda)$. If $V \in \text{Rep}(G)$ we have

$$\Delta_V : V \longrightarrow V \otimes \mathcal{O}(G) = \bigoplus_{\lambda \in \Lambda} V \otimes \lambda.$$

Hence there are morphisms $(\rho_\lambda : \lambda \in \Lambda)$ in $\text{End}(V)$ such that

$$\Delta_V(v) = \sum_{\lambda \in \Lambda} \rho_\lambda(v) \otimes \lambda, \quad \forall v \in V.$$

(Here $\rho_\lambda(v) = 0$ for all but finitely many λ s.)

It is easy to see that

$$\rho_\lambda \circ \rho_\mu = \begin{cases} \rho_\lambda & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

and $\text{id} = \sum_{\lambda \in \Lambda} \rho_\lambda$.

Hence $V = \bigoplus_{\lambda \in \Lambda} \rho_\lambda(V)$ with

$$\rho_\lambda(V) = \{v \in V : \Delta_V(v) = v \otimes \lambda\} = V_\lambda.$$

Hence $\text{Rep}(G)$ is isomorphic to the category of Λ -graded vector spaces (correct?).

1.3. Induction Let G be an affine k -group scheme.

A **subgroup** of G is a closed subscheme $H \subset G$ such that $e, \text{inv}|_H, m|_{H \times H}$ factor through H . Then H is an affine k -group scheme. In this setting we have the restriction functor $\text{Res}_H^G : \text{Rep}(G) \rightarrow \text{Rep}(H)$.

Proposition 1.3.1. — *The functor Res_H^G has a right adjoint $\text{Ind}_H^G : \text{Rep}(H) \rightarrow \text{Rep}(G)$.*

Explicitly, we have

$$\text{Ind}_H^G(V) = (V \otimes \mathcal{O}(G))^H$$

with H acting diagonally via the right-regular representation on $\mathcal{O}(G)$ and G acting on the fixed points via the left regular representation

$$\text{Ind}_H^G(V) = \left\{ \begin{array}{c} \text{morphisms of functors} \\ f : G \rightarrow V_a \end{array} \middle| \begin{array}{c} f(gh) = h^{-1}f(g) \\ \forall g \in G(R), h \in H(R), R \in \mathbf{Alg}_k \end{array} \right\}.$$

The canonical isomorphism

$$\text{Hom}_{\text{Rep}(G)}(V, \text{Ind}_H^G(V')) \simeq \text{Hom}_{\text{Rep}(H)}(V, V')$$

is called **Frobenius reciprocity**.

Properties. —

- *Transitivity:* Given subgroups $H_1 \subset H_2 \subset G$ we have

$$\mathrm{Ind}_{H_1}^G \simeq \mathrm{Ind}_{H_2}^G \circ \mathrm{Ind}_{H_1}^{H_2}.$$

- *Tensor identity:* For $V_1 \in \mathrm{Rep}(H), V_2 \in \mathrm{Rep}(G)$

$$\mathrm{Ind}_H^G(V_1 \otimes \mathrm{Res}_H^G(V_2)) \simeq \mathrm{Ind}_H^G(V_1) \otimes V_2.$$

- Ind_H^G sends injective objects of $\mathrm{Rep}(H)$ to injective objects of $\mathrm{Rep}(G)$. In particular,

$$\mathrm{Ind}_H^G(k) = \mathcal{O}(G)$$

is injective.

- $\mathrm{Rep}(G)$ has enough injectives.

Geometric interpretation: We assume G is an algebraic group (over k), i.e. an affine k -group scheme such that $\mathcal{O}(G)$ is a finitely generated k -algebra. In this setting, for $H \subset G$ a subgroup we have a quotient scheme G/H of finite type over k with a faithfully flat quotient map $\pi : G \rightarrow G/H$. For $V \in \mathrm{Rep}(H)$, we have a quasicoherent sheaf $\mathcal{L}_{G/H}(V) \in \mathrm{QCoh}(G/H)$ with

$$\Gamma(V, \mathcal{L}_{G/H}(V)) = \left\{ \text{morphisms } f : \pi^{-1}(V) \longrightarrow V \mid f(x, h) = h^{-1}f(x) \text{ for all } (?) \right\}.$$

We have $\mathrm{Ind}_H^G(V) = \Gamma(G/H, \mathcal{L}_{G/H}(V))$. If V is finite dimensional, then $\mathcal{L}_{G/H}(V)$ is coherent.

Consequences. —

- If G/H is affine then Ind_H^G is exact.
- If G/H is projective then Ind_H^G preserves finite dimensionality.

Since $\mathrm{Rep}(H)$ has enough injectives we can consider the derived functor

$$\mathrm{RInd}_H^G : D^b \mathrm{Rep}(H) \longrightarrow D^b \mathrm{Rep}(G).$$

The functor $\mathcal{L}_{G/H} : \mathrm{Rep}(H) \rightarrow \mathrm{QCoh}(G/H)$ is exact, hence we have

$$\mathcal{L}_{G/H} : D^b \mathrm{Rep}(H) \longrightarrow D^b \mathrm{QCoh}(G/H).$$

One can check that

$$\mathrm{RInd}_H^G(V) \simeq R\Gamma(G/H, \mathcal{L}_{G/H}(V)).$$

(??)

Consequences. —

- We have $\mathrm{R}^n \mathrm{Ind}_H^G(V) = 0$ for all $V \in \mathrm{Rep}(H)$ if $n > \dim(G/H)$.
- If G/H is projective, then $\mathrm{R}^n \mathrm{Ind}_H^G(V)$ is finite-dimensional for all $V \in \mathrm{Rep}^{\mathrm{fd}}(H), n \in \mathbb{Z}$.

§2. REDUCTIVE ALGEBRAIC GROUPS

From now on k is algebraically closed.

2.1. Definition A k -algebraic group G is called **unipotent** if every non-zero representation admits a non-zero fixed vector. [Equivalent condition: G is unipotent if and only if it is isomorphic to a subgroup of unipotent upper-triangular matrices in \mathbf{GL}_n for some n .]

Example 2.1.1. — G_a is unipotent as

$$G_a \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

If G is a smooth, connected algebraic group, the smooth, connected, unipotent, normal subgroups of G there is a largest element called the **unipotent radical** of G , denoted $R_u(G)$. An algebraic group G is called **reductive** if it is smooth, connected and $R_u(G)$ is trivial.

One possible motivation for studying representations of reductive algebraic groups is that any simple representation of a smooth connected algebraic group G factors through a simple representation of $G/R_u(G)$, which is a reductive algebraic group.

Example 2.1.2. —

- (1) *Tori*: If Λ is a finitely generated, free abelian group, then $\text{Diag}(\Lambda)$ is a reductive algebraic group.
- (2) For any finite-dimensional k -vector space V , $\mathbf{GL}(V)$ and $\mathbf{SL}(V)$ are reductive algebraic groups.
- (3) Symplectic groups, special orthogonal groups.

2.2. Structure From now on G is a reductive algebraic group.

We denote by B a **Borel subgroup** (a maximal, connected, smooth, solvable subgroup). Note that:

- a Borel subgroup is unique up to conjugation;
- the quotient G/B is a smooth, projective variety.

Example 2.2.1. — For $G = \mathbf{GL}_{n,k}$ one can take

$$B = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}.$$

In this case G/B parametrizes flags in k^n , i.e. data

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset k^n$$

with V_i a subspace of dimension i .