

INFINITY CATEGORIES

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§1. SIMPLICIAL SETS

Definition. — The **simplex category** Δ is the category of

- finite non-empty totally ordered sets
- order-preserving maps.

Notation. — $[n] = \{0 < 1 < 2 < \dots < n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

Every object in Δ is (uniquely) isomorphic to some $[n]$.

Definition. — A **simplicial set** is a functor

$$\mathcal{S} : \Delta^{\text{op}} \longrightarrow \text{Sets}$$

Notation. — $\mathcal{S}_n := \mathcal{S}([n])$, call this the **set of n -simplices** of \mathcal{S} . 0-simplices are called **vertices**, 1-simplices are called **edges**.

Example 1.0.1. — Let C be a set. Let $\underline{C} : \Delta^{\text{op}} \rightarrow \text{Sets}$ be the constant functor:

$$\begin{aligned} \underline{C}_n &= C \quad \forall n, \\ \underline{C}(\alpha) &= \text{id} \quad \forall \alpha : [m] \longrightarrow [n] \text{ in } \Delta. \end{aligned}$$

This is called a **discrete simplicial set**.

Definition. — Let \mathcal{S} be a simplicial set. Given $\alpha : [n] \rightarrow [n-1]$ we get $\mathcal{S}(\alpha) : \mathcal{S}_{n-1} \rightarrow \mathcal{S}_n$. The n -simplices in the image are called **degenerate** simplices, i.e. σ is degenerate if there is an α such that $\sigma \in \text{im}(\mathcal{S}(\alpha))$.

Lemma 1.0.1. — A simplicial set is discrete if and only if for all $n \geq 1$ all n -simplices are degenerate.

Exercise. — Prove it.

Example 1.0.2. — Let (P, \geq) be a poset. Define a simplicial set $N(P, \leq)$ called the **nerve** of (P, \leq) by

$$N(P, \leq)_k = \{\text{chains } p_0 \leq p_1 \leq \dots \leq p_k : p_i \in P\}$$

where a chain is a totally ordered subset.

Exercise. — Finish the definition. Which simplices are degenerate?

Example 1.0.3 (“Standard n -simplex”). — The **standard n -simplex** is

$$\Delta^n := N([n]).$$

(Pictures)

Note. — For $j \in [n]$, we get a subsimplicial set

$$N([n] \setminus \{j\}) \subset \Delta^n$$

isomorphic to Δ^{n-1} called the j^{th} **face** of Δ^n . (Picture)

Example 1.0.4 (Horns). — Let $n \geq 0$ and $0 \leq j \leq n$, define the **horn**

$$\begin{aligned} \Lambda_j^n &:= \text{subsimplicial set of } \Delta^n = N([n]) \\ &\text{consisting of chains } p_0 \leq p_1 \leq \dots \leq p_k \text{ (Pictures)} \\ &\text{such that } \{p_0, \dots, p_k\} \not\supset [n] \setminus \{j\}. \end{aligned}$$

Example 1.0.5 (($n - 1$)-sphere $\partial\Delta^n$). — We define the $(n - 1)$ -**sphere**

$$\partial\Delta^n := \begin{array}{c} \text{subsimplicial set of } \Delta^n \\ \text{chains } p_0 \leq \cdots \leq p_k \text{ such that } \{p_0 \leq \cdots \leq p_k\} \neq [n] \end{array}$$

Example 1.0.6 (Products). — Let \mathcal{S}, \mathcal{T} be simplicial sets. We define their **product** $\mathcal{S} \times \mathcal{T}$ as

$$(\mathcal{S} \times \mathcal{T})_k = \mathcal{S}_k \times \mathcal{T}_k.$$

(Picture)

Example 1.0.7. — Let \mathbf{C} be an ordinary category. We define its **nerve** $N(\mathbf{C})$ as

$$N(\mathbf{C})_k := \left\{ \begin{array}{c} \text{composable sequences of morphisms} \\ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \rightarrow \cdots \xrightarrow{f_k} X_k \end{array} \right\}.$$

Example 1.0.8. — Let X be a topological space. The **singular simplicial set** of X is defined as

$$\text{Sing}(X)_k = \{\text{continuous maps } |\Delta^k| \rightarrow X\},$$

where $|\Delta^k|$ is the standard k -simplex

$$|\Delta^k| = \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1} \mid x_i \geq 0, \sum x_i = 1 \right\}.$$

Exercise. — What does this do to the morphisms in Δ ?

Definition. — A **Kan complex** is a simplicial set X such that for every diagram

$$\begin{array}{ccc} \Delta_j^n & \xrightarrow{\text{any map}} & X \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

we can fill the dashed arrow. This is called an **extension problem**. If the arrow exists we say that the extension problem **admits a solution**.

Daniel Kan discovered Kan complexes in 1958. The *key fact* is that $\text{Sing}(X)$ is a Kan complex. The theme from 1958 to today is that Kan complexes are a “combinatorial model” for algebraic topology which allows us to do homotopy theory.

Definition. — Let X be a Kan complex and \mathcal{S} be any simplicial set. Two maps $f, g : \mathcal{S} \rightarrow X$ are said to be **homotopic** if there exists a map $H : \mathcal{S} \times \Delta^1 \rightarrow X$ such that

$$H|_{\mathcal{S} \times \{0\}} = f, \quad H|_{\mathcal{S} \times \{1\}} = g.$$

Lemma 1.0.2. — *This is an equivalence relation.*

Proof. Omitted, tricky for an exercise. This requires X to be a Kan complex. □

Definition. — Let X be a Kan complex and x_0 be a vertex of X . Let

$$\text{Loops}_{x_0} = \{\text{maps } \gamma : \Delta^n \rightarrow X \text{ such that } \gamma|_{\partial\Delta^n} \text{ is the constant map to } x_0\}.$$

We say $\gamma, \gamma' \in \text{Loops}_{x_0}$ are **relatively homotopic (rel. homotopic)** if there exists $H : \Delta^n \times \Delta^1 \rightarrow X$ such that

$$H|_{\Delta^n \times \{0\}} = \gamma, \quad H|_{\Delta^n \times \{1\}} = \gamma', \quad H|_{\partial\Delta^n \times \Delta^1} = \text{const. map to } x_0.$$

Define

$$\pi_n(X, x_0) := \frac{\text{Loops}_{x_0}}{\text{rel. homotopy}}.$$

Fact. — For $n \geq 1$, $\pi_n(X, x_0)$ is a group. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Definition. — An ∞ -category (or **quasi-category**) is a simplicial set \mathcal{C} such that any extension problem

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

with $0 < j < n$ (**inner horns**) admits a solution. (Picture) An ∞ -category is also called a **weak Kan complex**.

Lemma 1.0.3. — Let C be an ordinary category, then $N(C)$ is an ∞ -category.

Digression: Let I^n be the simplicial set consisting of n consecutive 1-simplices (n -**spine**) (Picture). A naive alternative definition is: \mathcal{C} is an infinity category if every

$$\begin{array}{ccc} I^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a solution. This is WRONG (but its wrongness is subtle), even though $N(\text{ord. cat.})$ satisfy this. There is a book by Markus Land “Introduction to ∞ -categories” which explores this. The definition of ∞ -categories was introduced by Boardman-Vogt in 1972. Joyal started generalizing results from ordinary category theory to ∞ -categories in 2006. Lurie is largely responsible for how well this notion is developed in modern literature.

Remark. — Having a unique solution to the lifting problem characterizes nerves of ordinary categories.

Definition. — Let \mathcal{C} be an ∞ -category. An **object** is a vertex. A **morphism** is an edge. An **identity morphism** is a degenerate edge. Say that h is a **composition** of g and f if there exists a 2-simplex such that (Picture).

Remark. — Compositions are NOT unique in ∞ -categories.

Example 1.0.9 (∞ -categories). —

1) Topological spaces Top.

- Objects are topological spaces.
- Morphisms are continuous maps.
- A 2-simplex is a (not necessarily commutative) diagram

$$\begin{array}{ccc} & X_1 & \\ f \nearrow & & \searrow g \\ X_0 & \xrightarrow{h} & X_2 \end{array}$$

and a homotopy $H : X_0 \times [0, 1] \rightarrow X_2$ from gf to h .

- A 3-simplex is a diagram

$$\begin{array}{ccccc} & & X_0 & & \\ & \swarrow & \downarrow & \searrow & \\ X_1 & & & & X_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X_2 & & \end{array}$$

with continuous maps $f_{ij} : X_i \rightarrow X_j$ for $i < j$, homotopies $T_{ijk} : X_i \times [0, 1] \rightarrow X_k$ from $f_{jk} \circ f_{ij}$ to f_{ik} , and $H : X_0 \times [0, 1]^2 \rightarrow X_3$ (**higher homotopy**) such that $H|_{\text{bdry}}$ is

$$\begin{array}{ccc} (0,0) & \xrightarrow{T_{123}f_{01}} & (0,1) \\ f_{23}T_{012} \downarrow & & \downarrow T_{013} \\ (1,0) & \xrightarrow{T_{023}} & (1,0) \end{array}$$

2) The ∞ -category of ordinary categories Cat_1 .

- Objects are ordinary categories.
- Morphisms are functors.
- A 2-simplex is a (not necessarily commutative) diagram

$$\begin{array}{ccc} & X_1 & \\ f \nearrow & & \searrow g \\ X_0 & \xrightarrow{h} & X_2 \end{array}$$

and a natural isomorphism $T : g \circ f \xrightarrow{\sim} h$.

- A 3-simplex is a diagram

$$\begin{array}{ccccc} & & X_0 & & \\ & \swarrow & & \searrow & \\ X_1 & & & & X_3 \\ & \nwarrow & & \nearrow & \\ & & X_2 & & \end{array}$$

where f_{ij} are functors and T_{ijk} are natural isomorphism such that

$$\begin{array}{ccc} \bullet & \xrightarrow{T_{123}f_{01}} & \bullet \\ f_{23}T_{012} \downarrow & & \downarrow T_{013} \\ \bullet & \xrightarrow{T_{023}} & \bullet \end{array}$$

commutes

A source of ∞ -categories are

- ordinary categories with an equivalence relation on morphisms,
- ordinary categories with inverting some morphisms.

Lecture 2

Definition. —

1. Let C be an ∞ -category and $f : X \rightarrow Y$ be a morphism in C . f is called an **isomorphism** if there exists $g : Y \rightarrow X$ and two 2-simplices

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{\text{id}_X} & X \end{array} \quad \begin{array}{ccc} & X & \\ g \nearrow & & \searrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$

2. An ∞ -category is called an **∞ -groupoid** if every morphism is an isomorphism.

Theorem 1.0.1 (Joyal). — *An ∞ -category is an ∞ -groupoid if and only if it is a Kan complex.*

Proof. The forward direction is hard, the converse is an exercise. □

Definition. —

3. Say $f, g : C \rightarrow D$ are functors (morphisms of simplicial sets) of ∞ -categories. A **natural transformation** from f to g is a functor $T : C \times \Delta^1 \rightarrow D$ such that $T|_{C \times \{0\}} = f$ and $T|_{C \times \{1\}} = g$.

A special case: the identity natural transformation $\text{id}_f : f \rightarrow f$ is the map

$$C \times \Delta^1 \xrightarrow{\text{proj}} C \xrightarrow{f} D.$$

$T : f \rightarrow g$ is a **natural isomorphism** if there exists $T' : g \rightarrow f$ and two maps $H : C \times \Delta^2 \rightarrow D, H' : C \times \Delta^2 \rightarrow D$ such that

$$\begin{array}{ccc} & g & \\ T \nearrow & & \searrow T' \\ f & \xrightarrow{\text{id}_f} & f \end{array} \quad \begin{array}{ccc} & f & \\ T' \nearrow & & \searrow T \\ g & \xrightarrow{\text{id}_g} & g \end{array}$$

In ordinary category theory a natural transformation assigns objects in C to morphisms in D and morphisms in C to commutative squares in D . For ∞ -categories a natural transformation takes objects to morphisms, morphisms to diagrams of shape $\Delta^1 \times \Delta^1$ and generally an n -simplex to a diagram of shape $\Delta^n \times \Delta^1$.

Theorem 1.0.2 (Pointwise criterion for natural isomorphism). — *Let $f, g : C \rightarrow D$ be functors of ∞ -categories and $T : f \rightarrow g$ be a natural transformation. T is a natural isomorphism if and only if for all objects x in C , $T(\{x\} \times \Delta^1)$ is an isomorphism in D .*

This is a consequence of Joyal's theorem.

Definition. — Define Cat_∞ as follows:

- Objects are ∞ -categories.
- Morphisms are functors.
- 2-simplices are diagrams

$$\begin{array}{ccc} & X_1 & \\ f \nearrow & & \searrow g \\ X_0 & \xrightarrow{h} & X_2 \end{array}$$

and a natural isomorphism $T : g \circ f \xrightarrow{\sim} h$.

- 3-simplices and higher: copy the data of Top and replace $[0, 1]^n$ by $(\Delta^1)^n$.

This is similar to Top and Cat_1 .

Definition. — Define Spc same as above, except objects are ∞ -groupoids.

In literature: ∞ -groupoids, Kan complexes, spaces and anima are synonyms.

Definition. — A functor $f : C \rightarrow D$ is called a **categorical equivalence** if there exists $g : D \rightarrow C$ such that $f \circ g \simeq \text{id}_D$ and $g \circ f \simeq \text{id}_C$.

Theorem 1.0.3 (Fundamental Theorem of Category Theory). — *A functor $f : C \rightarrow D$ is a categorical equivalence if and only if it's essentially surjective and fully faithful.*

Note that we haven't defined essentially surjective or fully faithful. Let's pre-warm up first before we define them.

Lemma 1.0.4. — *Let X be a Kan complex. X is **contractible** (i.e., categorically equivalent to Δ^0) if and only if every lifting problem*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

admits a solution.

Now we're warm enough to warm up, so let's do that. Let $f : X \rightarrow Y$ be a map of Kan complexes. Suppose every lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ & \nearrow \text{dashed} & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

has a solution. Then f is a categorical equivalence (think: homotopy equivalence of topological spaces). But this condition is too strong for the converse. A simple counter-example is to take X contractible.

Definition. — Let $f : C \rightarrow D$ be a functor of ∞ -categories. Given

$$(1) \quad \begin{array}{ccc} \partial\Delta^n & \xrightarrow{r} & C \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{s} & D \end{array}$$

we say it admits a **solution up to isomorphism** if

(i) there exists $u : \Delta^n \rightarrow C$ such that

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{r} & C \\ \downarrow & \nearrow u & \\ \Delta^n & & \end{array}$$

(ii) $f \circ u : \Delta^n \rightarrow D$ is naturally isomorphic to $s : \Delta^n \rightarrow D$ *relative* (of relative homotopy) to $\partial\Delta^n$.

Definition. — Let $f : C \rightarrow D$ be a functor of ∞ -categories.

- It's **essentially surjective** if every diagram (1) with $n = 0$ admits a solution up to isomorphism.
- It's **full** if every diagram (1) with $n = 1$ admits a solution up to isomorphism.
- It's **fully faithful** if every diagram (1) with $n \geq 1$ admits a solution up to isomorphism.

So a functor of ∞ -categories is fully faithful and essentially surjective if all (1) admit a solution up to isomorphism.

Remark. — These definitions of fully and full faithful are *nonstandard*.

Now the Fundamental Theorem makes sense.

Proof idea. The forward direction is easy. Conversely, we factor through

$$C \longrightarrow C^{\text{enhanced}} \longrightarrow D$$

where an n -simplex in C^{enhanced} is the data of

- n -simplex in C ,
- a diagram of shape $\Delta^n \times \Delta^1$ in D satisfying some conditions.

The inverses of the intermediate maps are easy to construct. □

What is missing so far is *mapping spaces*. Given objects X, Y in an ∞ -category C we expect to find a space (Kan complex) $\text{map}_C(X, Y)$ such that the objects of $\text{map}_C(X, Y)$ are morphisms $X \rightarrow Y$ and it should extend to a functor

$$\text{map}_C : C^{\text{op}} \times C \longrightarrow \text{Spc}.$$

For 1-categories this is usually called Hom or Mor . Lurie uses Hom for a non-functorial, but easier, version of map .

Here is one non-functorial approach to mapping spaces. Let C^{Δ^1} be the simplicial set such that $(C^{\Delta^1})_k$ is the set of maps $\Delta^1 \times \Delta^k \rightarrow C$. By restricting to $\{0\} \times \Delta^k$ and $\{1\} \times \Delta^k$ we get a map

$C^{\Delta^1} \rightarrow C \times C$. Define $\text{map}_C(X, Y)$ as the fiber product (of simplicial sets)

$$\begin{array}{ccc} \text{map}_C(X, Y) & \longrightarrow & C^{\Delta^1} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(x, y)} & C \times C. \end{array}$$

Theorem 1.0.4. — *Let $f : C \rightarrow D$ be a functor of ∞ -categories. Then f is fully faithful if and only if for all objects X, Y , the induced map*

$$\text{map}_C(X, Y) \longrightarrow \text{map}_D(f(x), f(y))$$

is a categorical equivalence of Kan complexes.

This theorem is actually the usual definition in the literature.

Somewhere along the way:

Theorem 1.0.5 (Whitehead's Theorem). — *A map $f : X \rightarrow Y$ of Kan complexes is a categorical equivalence if and only if*

$$\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

are bijections for all n and all x_0 .

Lecture 3

A monoidal category in ordinary category theory consists of:

- A category C .
- A functor $\otimes : C \times C \rightarrow C$.
- An object $\mathbb{1} \in C$.
- 3 natural transformations: the associator, left and right unitors.

We ask them to satisfy 3 axioms:

- Triangle axioms (they say $\mathbb{1} \otimes x = x = x \otimes \mathbb{1}$).
- Pentagon axiom (various ways to group 4 objects).

Mac Lane's Coherence Theorem tells us that every "reasonable" diagram made from the 3 natural transformations commutes.

Let's try to mimic this for ∞ -categories. The naive approach is to start with:

- an ∞ -category C ;
- a functor $\otimes : C \times C \rightarrow C$;
- an object $\mathbb{1} : \Delta^0 \rightarrow C$;
- an associator

$$\begin{array}{ccc} C \times C \times C & \xrightarrow{\text{id} \times \otimes} & C \times C \\ \downarrow \otimes \times \text{id} & \searrow \otimes & \downarrow \otimes \\ C \times C & \xrightarrow{\otimes} & C \end{array}$$

a diagram of shape $\Delta^1 \times \Delta^1$ in Cat_∞ ;

- a left unitor

$$\begin{array}{ccc} \Delta^0 \times C & \xrightarrow{\mathbb{1} \times \text{id}} & C \times C \\ \searrow \sim & & \swarrow \otimes \\ & C & \end{array}$$

a 2-simplex in Cat_∞ and similarly a right unitor;

and ask it to satisfy

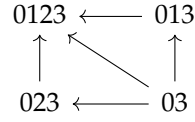
- the triangle identity

$$\begin{array}{ccccc} C \times \Delta^0 \times C & & & & \\ \text{id} \times \mathbb{1} \times \text{id} \downarrow & \searrow & \searrow & \searrow & \\ C \times C \times C & \xrightarrow{\quad} & C \times C & \xrightarrow{\quad} & C \\ & \searrow & \searrow & \searrow & \\ & C \times C & \xrightarrow{\quad} & C & \end{array}$$

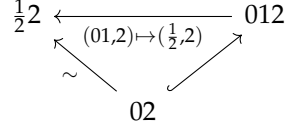
two 3-simplices attached along a face.

We model these diagrams with totally ordered finite sets:

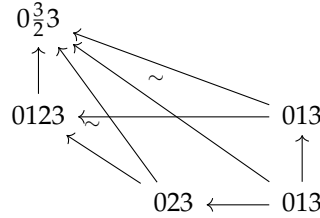
- Associator



- Left unitor



- Triangle identity



The data for the naive approach is modeled by

$$N \left(\left(\begin{array}{c} \text{nonempty, finite, totally ordered} \\ \text{sets of size } \leq 4 \end{array} \right)^{\text{op}} \right) \longrightarrow \text{Cat}_{\infty}$$

We are still missing the pentagon axiom and Mac Lane's Coherence Theorem.

Definition. — Let \mathcal{C} be an ∞ -category. A **monoidal structure** on \mathcal{C} is a functor $F : N(\Delta^{\text{op}}) \rightarrow \text{Cat}_{\infty}$ such that

- 1) $F([1]) = \mathcal{C}$,
- 2) for all n

$$[n] \leftarrow \{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}$$

induces

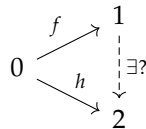
$$F([n]) \longrightarrow \mathcal{C} \times \mathcal{C} \times \dots \times \mathcal{C} = \mathcal{C}^n$$

which we require to be an equivalence of categories.

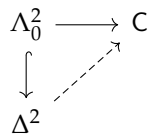
Note that $F([0]) \xrightarrow{\sim} \Delta^0$. The idea is that the pentagon axiom and all “higher coherences” are encoded in Δ^{op} .

The problem with this definition is unusable. Actually writing down a functor $N(\Delta^{\text{op}}) \rightarrow \text{Cat}_{\infty}$ is too complicated. What do we do? Lurie will rescue us.

Let's warm up. Suppose you have a Λ_0^2 horn



so we are asking: given



when does a solution exist? It exists if f has a right inverse.

Definition. — Let $p : C \rightarrow D$ be a functor of ∞ -categories, $f : x \rightarrow y$ be a morphism in C . We say f is *p -cocartesian* if

$$\begin{array}{ccccc} \{0,1\} = \Delta^1 & \hookrightarrow & \Lambda_0^n & \xrightarrow{\quad} & C \\ & & \downarrow & \nearrow & \downarrow p \\ & & \Delta^n & \longrightarrow & D \end{array}$$

f (curved arrow from $\{0,1\}$ to C)

has a solution.

Definition. — $p : C \rightarrow D$ is called a **cocartesian fibration** if lifting problems

$$\begin{array}{ccc} \Lambda_j^n & \xrightarrow{r} & C \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & D \end{array}$$

have a solution when

- (1) $0 < j < n$,
- (2) $j = 0$ and $n \geq 2$ if r sends $\{0,1\}$ to a p -cocartesian edge,
- (3) $j = 0, n = 1$; in this case we also *require* the solution $u : \Delta^1 \rightarrow C$ to be a p -cocartesian edge.

The idea is that a cocartesian fibration $p : C \rightarrow D$ should be thought of as a “functorial family of ∞ -categories indexed by D ”. More precisely:

- For each object x in D let $C_x = \{x\} \times_D C$. This is an ∞ -category.
- For each edge $\gamma : x \rightarrow y$ in D and each n -simplex $\sigma : \Delta^n \rightarrow C_x$ we can construct a map that under

$$\tilde{\sigma} : \Delta^n \times \Delta^1 \longrightarrow C$$

sends $\{j\} \times \Delta^1$ to a p -cocartesian edge.

- Moreover, we get a *functor*

$$\begin{array}{ccc} C_x & \longrightarrow & C_y \\ \sigma & \longmapsto & \tilde{\sigma}_{\Delta^n \times \{1\}} \end{array}$$

(this is slightly sloppy).

The definition is precisely set up so you can carry this out. Let’s keep going:

- A 2-simplex in D gives a 2-simplex in Cat_∞

$$\begin{array}{ccc} & & C_y \\ & \nearrow & \downarrow \\ C_x & & C_z \\ & \searrow & \end{array}$$