INTRODUCTION TO SHIMURA VARIETIES AND THEIR COHOMOLOGY

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Plan:

I) Siegel modular varieties
II) General Shimura varieties
III) (Étale) Cohomology: Kottwitz conjecture

(The cohomology gives a realization of the global Langlands correspondence for certain number fields.)

The main reference are Sophie Morel's notes from the IHES '22 summer school.

§1. Siegel Modular Varieties

Analytically: we want to parametrize complex tori. This is fairly simple. Let V be a $\mathbf{C}$-vector space of dimension $m \geq 1$, $\Lambda \subset V$ a lattice (a discrete subgroup such that V/Λ is compact), then $X = V/\Lambda$ is a complex Lie group, which is a complex torus.

Exercise. — A morphism $f : X = V/\Lambda \to X' = V'/\Lambda'$ of complex Lie groups is given by a $\mathbf{C}$-linear map $V \to V'$ mapping Λ to Λ'.

Question: Which complex tori are algebraizable, i.e. $X \hookrightarrow \mathbb{P}^n(\mathbf{C})$ (equivalent to $X \simeq \underline{X}^{\mathrm{an}}$ for some projective $\underline{X}$ by Chow). Can we find a parametrization?

Example 1.0.1. — Let $n = 1$ complex tori are always algebraic. There is the Weierstrass $\wp$-function

$$\begin{aligned}\wp : V/\Lambda &\longrightarrow \mathbb{P}^1(\mathbf{C})\\ v &\longmapsto \wp(v) = \frac{1}{v^2} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(\lambda+v)^2} - \frac{1}{\lambda^2}.\end{aligned}$$

It embeds V/Λ in $\mathbb{P}^2(\mathbb{C})$ via $[\wp : \wp' : 1]$ with image $y^2 = P_\Lambda(x)$ where $P_\Lambda \in \mathbf{C}[X]$ has degree 3. The coefficients of P_Λ are Eisenstein series (modular forms).

For $n > 1$, X is "almost never" algebraic.

Recall that X is algebraizable if and only if there exists $\mathscr{L} \in \mathrm{Pic}(X)$ which is ample (see Mumford's Abelian Varieties). Recall that $\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X^\times)$. There is a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp(2\pi i -)} \mathscr{O}_X^\times \longrightarrow 0 \tag{1}$$

so we get a long exact sequence. The map

$$H^0(X, \mathscr{O}_X) \simeq \mathbf{C} \longrightarrow \mathbf{C}^\times \simeq H^0(X, \mathscr{O}_X^\times)$$

is surjective so we get

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r, hook] \arrow[d, "\simeq"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d, "\simeq"', "Dolbeaut"] & H^1(X,\mathscr{O}_X) \arrow[r, "\delta"] \arrow[d, "\simeq"] & \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)) \arrow[d, "\simeq"] \\
H^1(\Lambda,\mathbb{Z}) \arrow[d, equal] & \overline{T} & H^1(\Lambda, \mathscr{O}(X)^\times) & \mathrm{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right) \\
\mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[dr, hook] & T \oplus \overline{T} \arrow[u, "\mathrm{pr}_2"'] & & \\
 & \mathrm{Hom}_{\mathbf{R}}(V,\mathbf{C}) \arrow[u, "\simeq"'] & &
\end{tikzcd} \tag{2}$$

We have $H^i(V,\mathbb{Z}) = 0$ for all $i > 0$ and $H^i(V,\mathscr{O}_V) = 0$ for all $i > 0$ so $\mathrm{Pic}(V) = 0$. $\overline{T}$ are the antilinear maps $V \to \mathbf{C}$ and $T = \mathrm{Hom}_{\mathbf{C}}(V,\mathbf{C})$. Observe that

$$\mathrm{Pic}^0(X) = \ker\delta \simeq \frac{\overline{T}}{\mathrm{pr}_2(\mathrm{Hom}(\Lambda,\mathbb{Z}))}$$

is also a complex torus. The last term is the Neron-Severi group

$$\begin{aligned} NS(X) &\simeq \{E : \Lambda\times\Lambda \longrightarrow \mathbb{Z} \text{ alt.} : E(iv_1,iv_2) = E(v_1,v_2), \forall v_1,v_2 \in V\} \\ &= \{\Im H : H : V\times V \longrightarrow \mathbf{C} \text{ Hermitian such that } (\Im H)(\Lambda\times\Lambda)\subset\mathbb{Z}\}. \end{aligned}$$

The Appel-Humbert theorem completely describes $\mathrm{Pic}(X)$ as $\{L(H,\alpha)\}$ with H as above and α an extra datum.

Theorem 1.0.1 (Lefschetz). — *The following are equivalent:*

1) H is positive definite.
2) $L(H,\alpha)$ is ample (in fact, $L(H,\alpha)^{\otimes 3}$ is enough to embed X).

Let $L \in \mathrm{Pic}(X)$ then

$$\begin{aligned} \phi_L : X &\longrightarrow \mathrm{Pic}^0(X) = \widehat{X} \\ x &\longmapsto T_x^* L \otimes L^{-1} \end{aligned}$$

is a morphism of Lie groups (here T_x is translation by x).

Theorem 1.0.2. — *The following are equivalent:*

- *L is ample.*
- $\ker\phi_L$ *is finite.*
- *ϕ_L is surjective (i.e. an isogeny).*

Exercise. — Check that phi_L is an isomorphism if and only if $E(\cdot,\cdot)$ is perfect ($\Lambda \simeq \mathrm{Hom}(\Lambda,\mathbb{Z})$).

Definition. — Say that such ϕ_L is a **polarization**. If ϕ_L is an isomorphism, then it is called a **principal polarization**.

Remark. — Not every algebraic X admits a principal polarization, but is isogenous to one that does.

We can define the moduli space

$$\mathscr{A}_n(\mathbf{C}) = \left\{(X,\phi) : X = V/\Lambda \text{ of dimension } n, \phi : X \longrightarrow \widehat{X} \text{ a principal polarization}\right\}.$$

Let (V,Λ,H) be a principally polarized complex torus. Choose a symplectic basis $(e_1,\dots,e_{2n})$ of Λ, i.e.

$$(E(e_i,e_j))_{i,j} = J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Exercise. — $L = L(H,\alpha)$ is ample if and only if $e_{n+1},\dots,e_{2n}$ is a basis of V over $\mathbf{C}$ such that

$$\tau = \mathrm{Mat}_{e_{n+1},\dots,e_{2n}}(e_1,\dots,e_n)$$

satisfies $\tau =^t \tau$ and $\Im(\tau)$ is positive definite.

Definition. — $\mathscr{H}_n^+$ is the set of such $\tau \in M_n(\mathbf{C})$. There is an algebraic group

$$\mathbf{Sp}_{2n,\mathbb{Z}} : R \longmapsto \left\{g \in M_{2n}(R) :^t gJ_n g = J_n\right\}.$$

There is an action of $\mathbf{Sp}_{2n}(\mathbb{Z})$ on $\mathscr{H}_n^+$ such that given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\tau\gamma = (\tau b + d)^{-1}(\tau a + c)$$

(this corresponds to replacing $\underline{e} = (e_1,\dots,e_{2n})$ by $\underline{e}\gamma$).

We prefer left actions: let ${}^t\gamma$ act so that $\gamma\tau = \tau *^t \gamma$, i.e.

$$(\tau^t c + d)^{-1}(\tau^t a +^t b) = (a\tau + b)(c\tau + d).$$

This extends to an action of $\mathbf{Sp}_{2n}(\mathbf{R})$ on $\mathscr{H}_n^+$. This action is transitive and

$$\mathrm{Stab}_{\mathbf{Sp}_{2n}(\mathbf{R})}(iI_n) \longrightarrow U(n)$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a+ib$$

is an isomorphism (this is a maximal compact subgroup).

So $\mathscr{A}_n(\mathbf{C}) \simeq \Gamma_n \setminus \mathscr{H}_n^+$ where $\Gamma_n \simeq \mathbf{Sp}_{2n}(\mathbb{Z})$.

Remark. — There exists $\gamma \in \Gamma_n \setminus \{\pm 1\}$and $\tau \in \mathscr{H}_n^+$ such that $\gamma\tau = \tau$.

There is a universal object

$$\mathscr{X}_n(\mathbf{C}) = \mathbb{Z}^{2n} \rtimes \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

where

$$\gamma(v,\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(v,\tau) = ((\tau^t c +^t d)^{-1}v, \gamma\tau)$$

and

$$(\lambda_1, \lambda_2)(v,\tau) = (v + \lambda_1 + \tau\lambda_2, \tau)$$

for $\lambda_i \in \mathbb{Z}^n$.

There is a morphism $\pi : \mathscr{X}(\mathbf{C}) \to \mathscr{A}_n(\mathbf{C})$ which admits a section e. The fiber of τ is $[\tau] \simeq \mathbf{C}^n/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z}^n \oplus \tau\mathbb{Z}^n$. We get the **Hodge bundle**: take $\Omega^1(V/\Lambda)$ which are translaton invariant 1-forms, which is isomorphic to V^* via e^*, then the Hodge bundle is

$$\mathscr{E}_n = e^*\Omega^1_{\mathscr{X}(\mathbf{C})/\mathscr{A}_n(\mathbf{C})} \simeq \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

for action (??).

We can apply Schur functors (use the action of $\mathfrak{S}_k$ on on $\mathscr{E}_n^{\otimes k}$ to act on subbundles, e.g. $\bigwedge^k \mathscr{E}_n$ for $0 \leq k \leq n$). (Equivalently see $\mathscr{E}_n$ as a $\mathbf{GL}_n(\mathbf{C})$-bundle on $\mathscr{A}_n(\mathbf{C})$ and apply a holomorphic representation $\rho : (\mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$.) Sections of such vector bundles on $\mathscr{A}_k(\mathbf{C})$ are (level Γ_n, weight ρ) Siegel modular forms on $\mathscr{A}_n(\mathbf{C})$.

Notation: Write

$$M_\rho(\Gamma_n) = \{f \in \Gamma(A_n(\mathbf{C}), \rho(\mathscr{E}_n) : f \text{ is holomorphic at } \infty\}$$

(the last condition is automatic if $n > 1$). We write

$$S_\rho(\Gamma_n) = \{f : \text{vanish at } \infty\} \subset M_\rho(\Gamma_n)$$

for the set of **cusp forms**.

We want a group theoretic description of the complex structure on $\mathscr{A}_n(\mathbf{C})$ and these vector bundles on $\mathscr{A}_m(\mathbf{C})$.

We have $Z(U(n)) \simeq U(1)$ and its centralizer in $\mathbf{Sp}_{2n}(\mathbf{R})$ is $U(n) = K(\mathbf{R})$ where $K \hookrightarrow \mathbf{Sp}_{2n,\mathbf{R}}$ is an algebraic subgroup.

Over $\mathbf{C}$ we have

$$Z(K_\mathbf{C}) \simeq \mathbf{GL}_{1,\mathbf{C}} \hookrightarrow \mathbf{Sp}_{2n,\mathbf{C}}.$$

This determines two opposite parabolic subgroups $Q_+ = K_\mathbf{C} N_+$, $Q_- K_\mathbf{C} N_-$