DUALIZABILITY

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Definition. — Let $(\mathsf{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. An object $c \in \mathsf{C}$ is *dualizable* if there exists $c^\vee \in C$ and maps

$$\text{coev} : \mathbb{1} \longrightarrow c \otimes c^\vee$$
$$\text{ev} : c^\vee \otimes c \longrightarrow \mathbb{1}$$

such that

$$\left[c \xrightarrow{\text{coev} \otimes \text{id}} c \otimes c^\vee \otimes c \xrightarrow{\text{id} \otimes \text{ev}} c\right] = \text{id}_c$$
$$\left[c^\vee \longrightarrow c^\vee \otimes c \otimes c^\vee \longrightarrow c^\vee\right] = \text{id}_{c^\vee}\,.$$

Example 0.0.1. — Let $C = \mathsf{Vect}_k \ni V$. V is dualizable if and only if $\dim V < \infty$.

If V is finite-dimensional define $V^\vee = \text{Hom}(V, k)$, pick a basis $v_1, \ldots, v_n$ and a dual basis $v^1, \ldots, v^n \in V^\vee$. We can define

$$\text{coev} : k \longrightarrow V \otimes V^\vee$$
$$1 \longmapsto \sum v_i \otimes v^i$$

and ev to be the usual evaluation map.

For the inverse direction we get that coev(1) is a finite sum. One shows that the v_i that show up form a basis.

Example 0.0.2. — In $(\mathsf{Set}, \times), (\mathsf{Top}, \times)$ and any $(\mathsf{C}, \times)$ the only dualizable object is $\{*\}$.

Recall that $\underline{\text{Hom}}(c, c')$ satisfies

$$\text{Hom}_C(t, \underline{\text{Hom}}(c, c')) = \text{Hom}_C(t \otimes c, c').$$

Corollary 0.0.1. — *An object $c \in C$ is dualizable if and only if*

a) $\underline{\text{Hom}}(c, \mathbb{1})$ *and* $\underline{\text{Hom}}(c, c)$ *exist,*
b) $c \otimes \underline{\text{Hom}}(c, \mathbb{1}) \to \underline{\text{Hom}}(c, c)$ *is an iso.*

(Hence $c^\vee = \underline{\text{Hom}}(c, \mathbb{1})$*).*

Example 0.0.3. — Take Mod_R for a commutative ring R. Then $M \in \mathsf{Mod}_R$ is dualizable if and only if M is projective.

Dualizable objects are closed under retracts. If M is a finite projective module, it is a retract of a finite free module, so it is dualizable.

Conversely, if M is dualizable there exists

$$\begin{array}{ccccc} M & \longrightarrow & R^? & \longrightarrow & M \\ m & \longmapsto & (f_i(m))_i & & \\ & & (r_i) & \longmapsto & \sum r_i m_i \end{array} \tag{1}$$

We define

$$R \longrightarrow M \otimes_R \text{Hom}(M, R)$$
$$1 \longmapsto \sum m_i \otimes f_i.$$

Example 0.0.4. — Consider $D(R), \otimes_R^{\mathbb{L}}$. An object is dualizable if and only if it is a perfect complex, i.e. quasi-isomorphic to a finite complex of finite projective modules.

Example 0.0.5. — Let X be a qcqs scheme, then $\mathscr{F} \in D(\mathsf{QCoh}(X))$ is dualizable if and only if $\mathscr{F}$ is perfect, i.e. $\mathscr{F}|_{\operatorname{Spec} R}$ is as above.

§1. Dualizability as a finiteness condition

Definition. — If C is a symmetric monoidal ∞-category, $c \in \mathsf{C}$ is dualizable if and only if $c \in H_0(\mathsf{C})$ is dualizable.

Lemma 1.0.1. — *Suppose C is a symmetric monoidal ∞-category which has filtered colimits, which are preserved under $\otimes$. If $\mathbb{1} \in \mathsf{C}$ is compact (i.e. $\mathrm{Map}_{\mathsf{C}}(\mathbb{1}, -)$ preserves filtered colimits), then any dualizable object is compact.*

Proof. $\mathrm{Map}_{\mathsf{C}}(c, -) = \mathrm{Map}_{\mathsf{C}}(\mathbb{1}, c^\vee \otimes -)$. □

Lemma 1.0.2. — *Suppose C is presentable and colimits are preserved under $\otimes$ (presentably symmetric monoidal). Then $c \in \mathsf{C}$ is dualizable if and only if $c \otimes -$ preserves limits.*

Proof. First suppose $\varphi = c \otimes -$ preserves limits. By the adjoint functor theorem φ admits a left adjoint φ^L, then $\varphi^L(\mathbb{1})$ is a dual for c. □

Lemma 1.0.3. — *Let X be a topological space, $\mathscr{F} \in D(\mathsf{Sh}(X, \mathsf{Ab}))$ is dualizable if and only if locally on X, $\mathscr{F}$ is constant and associated to a perfect complex of abelian groups.*

Proof. Given an open subset U of X write $u : U \hookrightarrow X$ for the open embedding. We claim that

$$\operatorname{colim}_{x \in U} \operatorname{Hom}(u^*\mathscr{F}, u^*\mathscr{G}) \longrightarrow \operatorname{Hom}(\mathscr{F}_x, \mathscr{G}_x)$$

is an isomorphism. (For a proof look at Cisinski, Déglise Étale motives.) □

Let X be a smooth, affine variety over k we can associate to it the de Rham complex $\Omega^*_{X/k}$ and de Rham cohomology $H^n_{\mathrm{dR}}(X) = H^n(X, \Omega^*_{X/k})$. If X^{an} is compact, then $\Omega^*_{X^{\mathrm{an}}}$ are holomorphic differential forms and we define $H^n_{\mathrm{dR}^{\mathrm{an}}}(X) = H^n(X^{\mathrm{an}}, \Omega^*_{X^{\mathrm{an}}})$.

Theorem 1.0.1 (Grothendieck, C-D "Weil…"). — *Fix $k \subset \mathbf{C}$, then there is an isomorphism*

$$H^n_{dR}(X) \otimes_k \mathbf{C} \xrightarrow{\sim} H^n_{dR^{an}}(X).$$

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathsf{Sm}_k^{\mathsf{op}} \arrow[r, shift left, "X \mapsto \Omega_{X/k} \otimes \mathbf{C}"] \arrow[r, shift right, "X \mapsto \Omega^{\mathrm{an}}_{X^{\mathrm{an}}}"'] \arrow[d, "X \mapsto M(X)"'] & D(\mathsf{Vec}_{\mathbf{C}}) \\ \mathrm{DA}^{\text{ét}}(k)^{\mathsf{op}} \text{ or } \mathrm{SH}(k)^{\mathsf{op}} & \end{tikzcd} \tag{2}$$

(there is a natural transformation from top arrow to bottom, add this). The functors at the top are symmetric monoidal.

Lemma 1.0.4. — *Suppose we have*

$$(\mathsf{C}, \otimes) \overset{F}{\underset{G}{\rightrightarrows}} (\mathsf{D}, \otimes) \tag{3}$$

where F, G are monoidal funcgtor and $\alpha : F \to G$ compatible with $\otimes$. There exists $c \in \mathsf{C}$ dualizable. Then

$$\alpha(c) : F(c) \xrightarrow{\sim} G(c)$$

is an isomorphism.

Fact: $\mathrm{DA}^{\text{ét}}(k)$ or $\mathrm{SH}(k)$ is generated by $M(X)(?)$ for X/k smooth and proper. (6 functor formalism and resolution of singularities)

$M(X)$ is dualizable for X smooth and proper (by the 6 functor formalism).

Look at Robalo's thesis to get a F, G factoring through $\mathrm{SH}(k)^{\mathsf{op}}$ as symmetric monoidal functors. (??) □

Recall (or wait until Friday) $(\mathsf{Pr}^L, \otimes)$ the category of presentable ∞-categories and a colimit preserving functors

- $P(\mathsf{C}_0) \otimes P(\mathsf{C}_1) = P(\mathsf{C}_0 \times \mathsf{C}_1)$
- If X, Y are (qcqs) schemes over k

$$D(\mathsf{QCoh}(X)) \otimes_{D(k)} D(\mathsf{QCoh}(Y)) = D(\mathsf{QCoh}(X \times_k Y)).$$

Example 1.0.1. — In Pr^L, $P(\mathsf{C}_0)$ is dualizable with dual $P(\mathsf{C}_0^{\mathsf{op}})$.

Now consider Pr^L_ω the category of presentable, compactly generated categories and functors that preserve compact objects. Define $\mathsf{Pr}^L_{\omega,k} = \mathsf{Mod}_{D(k)}\mathsf{Pr}^L_\omega$. Any $\mathsf{C} = \mathsf{Ind}(\mathsf{C}_0)$ is dual in Pr^L with dual $\mathsf{C}^\vee = \mathsf{Ind}(\mathsf{C}_0^{\mathsf{op}})$.

Theorem 1.0.2 (Kontsevich). — *Let X/k be an algebraic variety. Define $\mathsf{C} = D(\mathsf{QCoh}(X)) \in \mathsf{Pr}^L_{\omega,k}$.*

1) X is smooth if and only if (in Pr^L_k)

$$\begin{aligned} \mathrm{coev} : D(k) &\longrightarrow \mathsf{C} \otimes \mathsf{C}^\vee = D(\mathsf{QCoh}(X \times X)) \\ k &\longmapsto \Delta_* \mathscr{O}_X \end{aligned}$$

preserves compact objects.

2) X is proper if and only if $p_* \Delta^* = \mathrm{ev} : D(\mathsf{QCoh}(X \times X)) \to D(k)$ preserves compact objects.

Hence X is smooth and proper if and only if $D(\mathsf{QCoh}(X))$ is dual in $\mathsf{Pr}^L_{\omega,k}$.

(Kadyrev, Prikodko proved Atiyah-Bott which implies Borel-Weil-Bott)