INTRODUCTION TO SHIMURA VARIETIES AND THEIR COHOMOLOGY

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Plan:

I) Siegel modular varieties
II) General Shimura varieties
III) (Étale) Cohomology: Kottwitz conjecture

(The cohomology gives a realization of the global Langlands correspondence for certain number fields.)

The main reference are Sophie Morel's notes from the IHES '22 summer school.

§1. Siegel Modular Varieties

Analytically: we want to parametrize complex tori. This is fairly simple. Let V be a $\mathbf{C}$-vector space of dimension $m \geq 1$, $\Lambda \subset V$ a lattice (a discrete subgroup such that V/Λ is compact), then $X = V/\Lambda$ is a complex Lie group, which is a complex torus.

Exercise. — A morphism $f : X = V/\Lambda \to X' = V'/\Lambda'$ of complex Lie groups is given by a $\mathbf{C}$-linear map $V \to V'$ mapping Λ to Λ'.

Question: Which complex tori are algebraizable, i.e. $X \hookrightarrow \mathbb{P}^n(\mathbf{C})$ (equivalent to $X \simeq \underline{X}^{\mathrm{an}}$ for some projective $\underline{X}$ by Chow). Can we find a parametrization?

Example 1.0.1. — Let $n = 1$ complex tori are always algebraic. There is the Weierstrass $\wp$-function

$$\begin{aligned} \wp : V/\Lambda &\longrightarrow \mathbb{P}^1(\mathbf{C}) \\ v &\longmapsto \wp(v) = \frac{1}{v^2} + \sum_{\lambda \in \Lambda - 0} \frac{1}{(\lambda + v)^2} - \frac{1}{\lambda^2}. \end{aligned}$$

It embeds V/Λ in $\mathbb{P}^2(\mathbb{C})$ via $[\wp : \wp' : 1]$ with image $y^2 = P_\Lambda(x)$ where $P_\Lambda \in \mathbf{C}[X]$ has degree 3. The coefficients of P_Λ are Eisenstein series (modular forms).

For $n > 1$, X is "almost never" algebraic.

Recall that X is algebraizable if and only if there exists $\mathscr{L} \in \mathrm{Pic}(X)$ which is ample (see Mumford's Abelian Varieties). Recall that $\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X^\times)$. There is a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp(2\pi i -)} \mathscr{O}_X^\times \longrightarrow 0 \tag{1}$$

so we get a long exact sequence. The map

$$H^0(X, \mathscr{O}_X) \simeq \mathbf{C} \longrightarrow \mathbf{C}^\times \simeq H^0(X, \mathscr{O}_X^\times)$$

is surjective so we get

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r, hook] \arrow[d, "\simeq"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d, "\simeq"', "Dolbeaut"] & H^1(X,\mathscr{O}_X) \arrow[r, "\delta"] \arrow[d, "\simeq"] & \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)) \arrow[d, "\simeq"] \\
H^1(\Lambda,\mathbb{Z}) \arrow[d, equal] & \overline{T} & H^1(\Lambda, \mathscr{O}(X)^\times) & \mathrm{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right) \\
\mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[dr, hook] & T \oplus \overline{T} \arrow[u, "\mathrm{pr}_2"] & & \\
& \mathrm{Hom}_{\mathbf{R}}(V,\mathbf{C}) \arrow[u, "\simeq"] & &
\end{tikzcd} \tag{2}$$

We have $H^i(V,\mathbb{Z}) = 0$ for all $i > 0$ and $H^i(V,\mathscr{O}_V) = 0$ for all $i > 0$ so $\mathrm{Pic}(V) = 0$. $\overline{T}$ are the antilinear maps $V \to \mathbf{C}$ and $T = \mathrm{Hom}_{\mathbf{C}}(V,\mathbf{C})$. Observe that

$$\mathrm{Pic}^0(X) = \ker\delta \simeq \frac{\overline{T}}{\mathrm{pr}_2(\mathrm{Hom}(\Lambda,\mathbb{Z}))}$$

is also a complex torus. The last term is the Neron-Severi group

$$\begin{aligned} NS(X) &\simeq \{E : \Lambda\times\Lambda \longrightarrow \mathbb{Z} \text{ alt.} : E(iv_1,iv_2) = E(v_1,v_2), \forall v_1,v_2 \in V\} \\ &= \{\Im H : \ H : V\times V \longrightarrow \mathbf{C} \text{ Hermitian such that } (\Im H)(\Lambda\times\Lambda)\subset\mathbb{Z}\}\,. \end{aligned}$$

The Appel-Humbert theorem completely describes $\mathrm{Pic}(X)$ as $\{L(H,\alpha)\}$ with H as above and α an extra datum.

Theorem 1.0.1 (Lefschetz). — *The following are equivalent:*

1) H is positive definite.

2) $L(H,\alpha)$ is ample (in fact, $L(H,\alpha)^{\otimes 3}$ is enough to embed X).

Let $L \in \mathrm{Pic}(X)$ then

$$\begin{aligned} \phi_L : X &\longrightarrow \mathrm{Pic}^0(X) = \widehat{X} \\ x &\longmapsto T_x^* L \otimes L^{-1} \end{aligned}$$

is a morphism of Lie groups (here T_x is translation by x).

Theorem 1.0.2. — *The following are equivalent:*

- *L is ample.*
- $\ker\phi_L$ *is finite.*
- *ϕ_L is surjective (i.e. an isogeny).*

Exercise. — Check that phi_L is an isomorphism if and only if $E(\cdot,\cdot)$ is perfect ($\Lambda \simeq \mathrm{Hom}(\Lambda,\mathbb{Z})$).

Definition. — Say that such ϕ_L is a **polarization**. If ϕ_L is an isomorphism, then it is called a **principal polarization**.

Remark. — Not every algebraic X admits a principal polarization, but is isogenous to one that does.

We can define the moduli space

$$\mathscr{A}_n(\mathbf{C}) = \Big\{(X,\phi) : X = V/\Lambda \text{ of dimension } n, \phi : X \longrightarrow \widehat{X} \text{ a principal polarization}\Big\}\,.$$

Let (V,Λ,H) be a principally polarized complex torus. Choose a symplectic basis $(e_1,\dots,e_{2n})$ of Λ, i.e.

$$(E(e_i,e_j))_{i,j} = J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Exercise. — $L = L(H,\alpha)$ is ample if and only if $e_{n+1},\dots,e_{2n}$ is a basis of V over $\mathbf{C}$ such that

$$\tau = \mathrm{Mat}_{e_{n+1},\dots,e_{2n}}(e_1,\dots,e_n)$$

satisfies $\tau =^t \tau$ and $\Im(\tau)$ is positive definite.

Definition. — $\mathscr{H}_n^+$ is the set of such $\tau \in M_n(\mathbf{C})$. There is an algebraic group

$$\mathbf{Sp}_{2n,\mathbb{Z}} : R \longmapsto \left\{g \in M_{2n}(R) :^t gJ_ng = J_n\right\}.$$

There is an action of $\mathbf{Sp}_{2n}(\mathbb{Z})$ on $\mathscr{H}_n^+$ such that given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\tau\gamma = (\tau b + d)^{-1}(\tau a + c)$$

(this corresponds to replacing $\underline{e} = (e_1,\dots,e_{2n})$ by $\underline{e}\gamma$).

We prefer left actions: let ${}^t\gamma$ act so that $\gamma\tau = \tau *^t \gamma$, i.e.

$$(\tau^t c + d)^{-1}(\tau^t a +^t b) = (a\tau + b)(c\tau + d).$$

This extends to an action of $\mathbf{Sp}_{2n}(\mathbf{R})$ on $\mathscr{H}_n^+$. This action is transitive and

$$\mathrm{Stab}_{\mathbf{Sp}_{2n}(\mathbf{R})}(iI_n) \longrightarrow U(n)$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a + ib$$

is an isomorphism (this is a maximal compact subgroup).

So $\mathscr{A}_n(\mathbf{C}) \simeq \Gamma_n \setminus \mathscr{H}_n^+$ where $\Gamma_n \simeq \mathbf{Sp}_{2n}(\mathbb{Z})$.

Remark. — There exists $\gamma \in \Gamma_n \setminus \{\pm 1\}$and $\tau \in \mathscr{H}_n^+$ such that $\gamma\tau = \tau$.

There is a universal object

$$\mathscr{X}_n(\mathbf{C}) = \mathbb{Z}^{2n} \rtimes \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

where

$$\gamma(v,\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(v,\tau) = ((\tau^t c +^t d)^{-1} v, \gamma\tau)$$

and

$$(\lambda_1, \lambda_2)(v,\tau) = (v + \lambda_1 + \tau\lambda_2, \tau)$$

for $\lambda_i \in \mathbb{Z}^n$.

There is a morphism $\pi : \mathscr{X}(\mathbf{C}) \to \mathscr{A}_n(\mathbf{C})$ which admits a section e. The fiber of τ is $[\tau] \simeq \mathbf{C}^n / \Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z}^n \oplus \tau\mathbb{Z}^n$. We get the **Hodge bundle**: take $\Omega^1(V/\Lambda)$ which are translaton invariant 1-forms, which is isomorphic to V^* via e^*, then the Hodge bundle is

$$\mathscr{E}_n = e^* \Omega^1_{\mathscr{X}(\mathbf{C})/\mathscr{A}_n(\mathbf{C})} \simeq \Gamma_n \setminus \mathbf{C}^n \times \mathscr{H}_n^+$$

for action (??).

We can apply Schur functors (use the action of $\mathfrak{S}_k$ on on $\mathscr{E}_n^{\otimes k}$ to act on subbundles, e.g. $\bigwedge^k \mathscr{E}_n$ for $0 \leq k \leq n$). (Equivalently see $\mathscr{E}_n$ as a $\mathbf{GL}_n(\mathbf{C})$-bundle on $\mathscr{A}_n(\mathbf{C})$ and apply a holomorphic representation $\rho : (\mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$.) Sections of such vector bundles on $\mathscr{A}_k(\mathbf{C})$ are (level Γ_n, weight ρ) Siegel modular forms on $\mathscr{A}_n(\mathbf{C})$.

Notation: Write

$$M_\rho(\Gamma_n) = \{f \in \Gamma(A_n(\mathbf{C}), \rho(\mathscr{E}_n) : f \text{ is holomorphic at } \infty\}$$

(the last condition is automatic if $n > 1$). We write

$$S_\rho(\Gamma_n) = \{f : \text{vanish at } \infty\} \subset M_\rho(\Gamma_n)$$

for the set of **cusp forms**.

We want a group theoretic description of the complex structure on $\mathscr{A}_n(\mathbf{C})$ and these vector bundles on $\mathscr{A}_m(\mathbf{C})$.

We have $Z(U(n)) \simeq U(1)$ and its centralizer in $\mathbf{Sp}_{2n}(\mathbf{R})$ is $U(n) = K(\mathbf{R})$ where $K \hookrightarrow \mathbf{Sp}_{2n,\mathbf{R}}$ is an algebraic subgroup.

Over $\mathbf{C}$ we have

$$Z(K_\mathbf{C}) \simeq \mathbf{GL}_{1,\mathbf{C}} \hookrightarrow \mathbf{Sp}_{2n,\mathbf{C}}.$$

This determines two opposite parabolic subgroups $Q_+ = K_\mathbf{C} N_+$, $Q_- K_\mathbf{C} N_-$.

1.1. Siegel modular forms as automorphic forms Let $\rho : \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$ be a holomorphic (equivalently algebraic) representation. **Siegel modular forms** are

$$M_\rho(\Gamma_n) = \left\{ \begin{array}{c} f : \mathscr{H}_n^+ \to W \\ \text{holomorphic} \end{array} \middle| \begin{array}{c} \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_n, \forall \tau \in \mathscr{H}_n^+, f(\gamma\tau) = \rho(c\tau + d) f(\tau) \\ \text{and } f \text{ holomorphic at } \infty \end{array} \right\}$$
$$\subset H^0(\mathscr{A}_n(\mathbf{C}), {}^\rho\mathscr{E}_n).$$

${}^\rho\mathscr{E}_m$ comes from a $\mathbf{Sp}_{2n}(\mathbf{R})$-equivariant vector bundle on

$$\begin{tikzcd} & \mathscr{H}_n^+ \arrow[r, hook] & \mathbf{Sp}_{2n}(\mathbf{C})/Q_-(\mathbf{C}) \\ \mathbf{Sp}_{2n}(\mathbf{R}) \arrow[r] & \mathbf{Sp}_{2n}(\mathbf{R})/U(n) \arrow[u, "\simeq"] & \end{tikzcd}$$

Define

$$j : \mathbf{Sp}_{2n}(\mathbf{R}) \times \mathscr{H}_n^+ \longrightarrow \mathbf{GL}_n(\mathbf{C})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \longmapsto c\tau + d.$$

This is a cocycle

$$j(gg', \tau) = j(g, g'\tau) j(g', \tau)$$

(so $j(-, i)|_{U(n)} : U(n) \to \mathbf{GL}_n(\mathbf{C})$ is a morphism). To $f \in M_\rho(\Gamma_n)$ associate

$$\phi_f : \Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}) \longrightarrow W$$
$$g \longmapsto \phi_f(g) = \rho(j(g, i))^{-1} f(gi)$$

a smooth function. Let $g \in \mathbf{Sp}_2 n(\mathbf{R})$ and $k \in U(n)$, then

$$\phi_f(gk) = \rho(j(k, i))^{-1} f(gi).$$

Assume $W = \mathbf{C}$ for simplicity, e.g. ${}^\rho\mathscr{E}_n = (\bigwedge^n \mathscr{E}_n)^{\otimes k}$. Then

$$\phi_f \in \mathscr{A}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \subset C^\infty(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}, \mathbf{C})$$

(? details). This space has actions by $\mathfrak{g}$ and $U(n)$. By the Cauchy-Riemann equations f is holomorphic if and only if ϕ_f is killed by Lie $N_- \subset \mathfrak{g} = \mathbf{C} \otimes_{\mathbf{R}} \operatorname{Lie} \mathbf{Sp}_{2n}(\mathbf{R})$. Note that $\operatorname{Lie}(\mathbf{Sp}_{2n}(\mathbf{R}))$ acts on $C^\infty(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$ by

$$(X \cdot \phi)(g) = \left.\frac{d}{dt}\right|_{t=0} \phi(ge^{tX}).$$

ϕ_f lies in some generalized Verma module in $\mathscr{A}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$.

If $f \in S_\rho(\Gamma_m)$ (vanishes at ∞) then

$$\phi_f \in \mathscr{A}_{\text{cusp}}(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \subset \mathscr{A}^2(-) \subset \mathscr{A}(-)$$

and $\mathscr{A}^2(-)$ decomposes inside $L^2(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R}))$ with the action of $\mathbf{Sp}_{2n}(\mathbf{R})$. This means that cusp forms have fast decay at cusps.

As a $(\mathfrak{g}, U(n))$-module,

$$\mathscr{A}_{cusp} \subset \mathscr{A}^2(\Gamma_n \setminus \mathbf{Sp}_{2n}(\mathbf{R})) \simeq \bigoplus_{\substack{\pi \text{ irr} \\ (\mathfrak{g}, U(n))\text{-mod}}} \pi^{\oplus m(\pi)}.$$

Siegel cusp forms correpond to special vectors in some of these πs ($U(n)$-equivariant and killed by Lie N).

1.2. Level structures Let $X = V/\Lambda$ be a complex torus with a principal polarization $phi : X \xrightarrow{\sim} \widehat{X}$. For $M \geq 1$

$$X[M] := \ker\left(X \xrightarrow{\times M} X\right) = \frac{1}{M}\Lambda/\Lambda \simeq (\mathbb{Z}/M)^{2n}.$$

The map $[M]_X : X \to X$ is an isogeny (i.e. surjective with finite kernel). For all isogenenies $f : X \to Y$ inducing $\widehat{f} = f^* : \widehat{Y} \to \widehat{X}$, also an isogeny. We get the Weil pairing

$$\begin{aligned} \ker f \times \ker \widehat{f} &\longrightarrow \mathbf{C}^\times \\ (x, [L]) &\longmapsto \langle x, [L] \rangle . \end{aligned}$$

Choose $t : f^*L \xrightarrow{\sim} \mathscr{O}_X$ we have

$$\begin{tikzcd} T_x^* f^* L \arrow[r, "T_x^*(t)"] \arrow[d, "\simeq"] & T_x^* \mathscr{O}_X \arrow[d, "\simeq"] \\ f^*L \arrow[r, "t \times \langle x, [L] \rangle"] & \mathscr{O}_X . \end{tikzcd}$$

$f = [M]_X$ is a special case, then we get $X[M] \times \widehat{X}[M] \to \mu_M(\mathbf{C})$ and usaing a polarization we get

$$\langle \cdot, \cdot \rangle_\phi : X[M] \times X[M] \longrightarrow \mu_M(\mathbf{C}).$$

Proposition 1.2.1. — $\langle \cdot, \cdot \rangle_\phi$ *is alternating and non-degenerate.*

Proof. Recall that ϕ is ϕ_L for some $L = L(H, \alpha)$, let $E = \Im H : \Lambda \times \Lambda \to \mathbb{Z}$. Then

$$\begin{tikzcd} X[M] \times X[M] \arrow[r, "\langle \cdot, \cdot \rangle_\phi"] \arrow[d, "\simeq"] & \mu_M(\mathbf{C}) \\ \left(\frac{1}{M}\Lambda/\Lambda\right)^2 \arrow[r, "ME(\cdot,\cdot)"] & \frac{1}{M}\mathbb{Z}/\mathbb{Z} \arrow[u, "\exp(2\pi i -)"'] \end{tikzcd}$$

$\square$

Definition. — Temporarily we define a level structure on (X, ϕ) to be

$$(\mathbb{Z}/M)^{2n} \underset{\eta}{\xrightarrow{\sim}} X[M]$$

such that $\eta^* \langle \cdot, \cdot \rangle_\phi$ is the standard pairing for metric J_n.

Fact. — By strong approximation $\mathbf{Sp}_{2n}(\mathbb{Z}) \twoheadrightarrow \mathbf{Sp}_{2n}(\mathbb{Z}/M\mathbb{Z})$. Define $\Gamma_m(M)$ to be the kernel.

Corollary 1.2.1. — *There is a bijection*

$$\{(X, \phi, \eta) | \textit{PPAV with a level } M \textit{ structure}\} / \sim \simeq \Gamma_n(M) \setminus \mathscr{H}_n^+ = \mathscr{A}_n'(M)(\mathbf{C}).$$

Exercise. — For $M \geq 3$, for all $\tau \in \mathscr{H}_n^+$ show that $\mathrm{Stab}_{\Gamma_n(M)}(\tau) = \{1\}$. (?)

We get a tower $(\mathscr{A}_n'(M)(\mathbf{C}))_{M \geq 1}$ ordered by divisibility. For $M \mid |M'$ we get $\mathscr{A}_n'(M')(\mathbf{C}) \to \mathscr{A}_n'(M)(\mathbf{C})$.

Given (X, ϕ)

$$\{\text{level } M \text{ structures on } (X, \phi)\}$$

is a right $\mathbf{Sp}_{2n}(\mathbb{Z}/M\mathbb{Z})$-torsor which gives us an action of

$$\mathbf{Sp}_{2n}(\widehat{\mathbb{Z}}) = \varprojlim_M \mathbf{Sp}_{2n}(\mathbb{Z}/M)$$

on this tower.

Also

$$\mathscr{A}_n'(M)(\mathbf{C}) \simeq \mathscr{A}_n'(M')(\mathbf{C}) / \left(K(M)/K(M')\right)$$

where

$$K(M) = \ker\left(\mathbf{Sp}_{2n}(\widehat{Z}) \longrightarrow \mathbf{Sp}_{2n}(\mathbb{Z}/M)\right).$$

The quotient $K(M)/K(M')$ is a finite group.

1.3. Hecke operators (adelically) The goal is to define more natural maps between $\mathscr{A}'_n(M)(\mathbf{C})$. The basic idea is that given (X,ϕ,η), we should also consider isogeneus complex tori (i.e. quotients of X by finite subgroups). But there are some problems: this is not strictly compatible with principal polarizations. Let $f : X \to Y$be an isogeny, ϕ be a principal polarization for Y, then $f^*\phi := \widehat{f}\circ\phi\circ f$ has degree $(\deg f)^2$, so it is not principal unless f is an isomorphism.

There are two solutions:

1) Rescale polarizations.
2) Consider quasi-isogenies

$$f \in \mathbb{Q}\otimes \mathrm{Hom}(X,Y) \text{ such that } \exists M \geq 1 \text{ with } Mf \in \mathrm{Hom}(X,Y) \text{ an isogeny.}$$

Let's do both.

Recall the ring of adeles $\mathbb{A} = \mathbf{R}\times\mathbb{A}_f$ where

$$\mathbb{A}_f = \prod_p{}'(\mathbb{Q}_p,\mathbb{Z}_p) = \left\{ (x_p)_{p \text{ prime}} \middle| \begin{array}{c} x_p\in\mathbb{Q}_p \\ \exists \text{ finite } S \text{ such that} \forall p\notin S, x_p\in\mathbb{Z}_p \end{array}\right\}.$$

Recall that

$$\begin{aligned}\text{lattices in } \mathbb{Q}\Lambda &\leftrightarrow \left\{ (\Lambda'_p)_p \middle| \begin{array}{c} \Lambda'_p \subset \mathbb{Q}_p\otimes_{\mathbb{Z}}\Lambda \text{ is a } \mathbb{Z}_p\text{-lattice} \\ \exists \text{ finite } S \text{ such that } \forall p\notin S, \Lambda'_p = \mathbb{Z}_p\Lambda \end{array}\right\} \\ &\leftrightarrow \mathbf{GL}(\mathbb{A}_f\otimes\Lambda)/\mathbf{GL}(\widehat{\mathbb{Z}}\otimes\Lambda).\end{aligned}$$

Proof. Reduce to the case where

$$M\Lambda \subset \Lambda' \subset \frac{1}{M}\Lambda$$

and use the chinese remainder theorem. □

Proposition 1.3.1. — *Let (X,ϕ) be a principally polarized abelian variety.*

(a) Let L be the set of principally polarized abelian varieties (X',ϕ') quasi-isogeneous to $(X,/\phi)$, i.e. there exists a quasi-isogeny $f : X' \dashrightarrow X$ such that $f^*\phi = c\phi'$, where $c\in\mathbb{Q}_{>0}$.

(b) Let R be the set of $(\Lambda'_p)_p$ such that

$$\Lambda'_p \subset V_pX := \mathbb{Q}_p\otimes_{\mathbb{Z}_p} T_pX$$

is a $\mathbb{Z}_p$-lattice such that there exists k_p making $p^{k_p}\langle\cdot,\cdot\rangle|_{\Lambda'_p\times\Lambda'_p}$ take values in $\mathbb{Z}_p(1) :== \varprojlim_k \mu_{p^k}(?)$ and is perfect, as well as there is a finite S such that for all $p\notin S$

$$\Lambda'_p = T_pX := \varprojlim_k X[p^k].$$

Then

$$L/\sim\simeq R.$$

This is also isomorphic to the set of $\mathbf{GSp}_{2n}(\widehat{\mathbb{Z}})$-orbits of symplectic trivializations

$$(\mathbb{A}_f^{2n}, \textit{standard } \langle\cdot,\cdot\rangle) \xrightarrow{\sim} \left(\mathbb{Q}\otimes\prod_p T_pX, \langle\cdot,\cdot\rangle_\phi\right).$$

Here $\mathbf{GSp}_{2n}$ is the $\mathbb{Z}$-group scheme

$$\mathbf{GSp}_{2n}(R) = \left\{(g,c)|g\in M_{2n}(R), c\in R^\times, {}^t gJ_ng = cJ_n\right\}.$$

Definition. — A **level structure** for (X,ϕ) is an isomorphism $(\mathbb{Z}/M)^{2n} \xrightarrow{\sim} X[M]$. $\mathbb{Z}/M \xrightarrow{\sim} \mu_M(\mathbf{C})$ such that the obvious diagram commutes.

We have

$$\begin{aligned}\mathscr{A}_n(M)(\mathbf{C}) &\simeq \{(X,\phi,\eta)|\text{PPAV with level } M \text{ structure}\}/\sim \\ &\simeq \left\{(X',\phi')\middle| K(M)\text{-orbit of trivalization of } \mathbb{Q}\otimes\prod_p T_pX'\right\}/\text{quasi-isogeny} \\ &\simeq \mathbf{GSp}_{2n}(\mathbb{Q})\backslash\left(\mathscr{H}_n^\pm\times\mathbf{GSp}_{2n}(\mathbb{A}_f)/K(M)\right)\end{aligned}$$

where
$$\mathscr{H}_n^{\pm} = \mathscr{H}_n^{+} \coprod \mathscr{H}_n^{-}$$
has an action of $\mathbf{GSp}_{2n}(\mathbf{R})$ and
$$K(M) := \ker\left(\mathbf{GSp}_{2n}(\widehat{\mathbb{Z}}) \longrightarrow \mathbf{GSp}_{2n}(\mathbb{Z}/M)\right).$$

From now on we write G for $\mathbf{GSp}_{2n}$. We have a tower $(\mathscr{A}_m(M)(\mathbf{C}))_{M\geq 1}$ (a $\mathbf{GSp}_2(\widehat{\mathbb{Z}})$-torsor over $\mathscr{A}_m(\mathbf{C})$ with a right action of $G(\mathbb{A}_f)$ and Lecture 3
$$\mathscr{A}_m(M)(\mathbf{C}) \simeq G(\mathbb{Q}) \setminus \left(\mathscr{H}_n^{\pm} \times G(\mathbb{A}_f)/K(M)\right).$$
For $g \in G(\mathbb{A}_f)$ and M, M' satisfying $K(M') \subset gK(M)g^{-1}$ define
$$\begin{aligned} T_g : \mathscr{A}_m(M')(\mathbf{C}) &\longrightarrow \mathscr{A}_m(M)(\mathbf{C}) \\ [\tau, h] &\longmapsto [\tau, hg]. \end{aligned}$$

There is also an action on Siegel modular forms. Note that $T_g^*\mathscr{E}_m \simeq \mathscr{E}_m$ and on $\mathscr{A}_m(M)(\mathbf{C})$, $\mathscr{E}_m = T_1^*\mathscr{E}_m$. Hence for $\rho : \mathbf{GL}_m(\mathbf{C}) \to \mathbf{GL}(W)$ there is an acton of $G(\mathbb{A}_f)$ on
$$M_\rho := \varinjlim_M M_\rho(K(M))$$
where
$$M_\rho(K(M)) := H^0(\mathscr{A}(M)(\mathbf{C}),^\rho \mathscr{E}_m)) + \text{holomorphy at } \infty \text{ if } m = 1.$$
M_ρ contains cusp forms S_ρ and by unitarity
$$S_\rho \simeq \bigoplus_{\pi_f \text{ irrep of } G(\mathbb{A}_f)} \pi_f^{\oplus m(\pi_f)}$$
(note that π_f are infinite-dimensional).

We recover
$$\mathrm{H}_B^k(\mathscr{A}_m(M)(\mathbf{C}), \mathbb{Q}) \simeq (\mathrm{H}_B^k)^{K(M)}$$
where the right hand side admits an action of Hecke operators $H(G(\mathbb{A}_f), K(M))$. (Trace map?)

Theorem 1.3.1 (Franke, Generalization of Matsushima's formula). — *We have*
$$\begin{aligned} \mathbf{C} \otimes_{\mathbb{Q}} \mathrm{H}_B^\bullet \xrightarrow[G(\mathbb{A}_f)\text{-equiv.}]{\sim} &\mathrm{H}^\bullet(\mathfrak{sp}_{2m}, U(m); \overbrace{\mathscr{A}(G(\mathbb{Q}) \setminus G(\mathbb{A})/\mathbf{R}_{>0})}^{\mathscr{A}(G)}) \\ &:= \mathrm{H}^\bullet\left(\mathrm{Hom}_{U(m)}\left(\bigwedge^{\bullet} \mathfrak{sp}_{2n}/\mathfrak{gl}_m\right), \mathscr{A}(G)\right). \end{aligned}$$

Remark. —

0) It's "easy" if we replace $\mathscr{A}(G)$ by C^∞ and use de Rham cohomology for the LHS.
1) If $\Gamma_m \setminus \mathscr{H}_m^+$ was compact, this would be obtained from the Hodge decomposition for Riemannian manifolds.
2) $\mathscr{A}(G)$ is not semi-simple at all.
3) If $m = 1$ we can use this to recover the Eichler-Shimura isomorphism. Let $\mathrm{H}^1_{B,\mathrm{cusp}}$ be the subspace of H^1_B defined by "vanishing at cusps". Then
$$\mathbf{C} \otimes_{\mathbb{Q}} \mathrm{H}^1_{B,\mathrm{cusp}} \xrightarrow[\mathbf{GL}_2(\mathbb{A}_f)]{\sim} S_2 \oplus \overline{S}_2.$$

   If $\Gamma_1 \setminus \mathscr{H}_1^+$ was compact (thus a projective curve over $\mathbf{C}$) this would follow from the Hodge decomposition because $\mathscr{E}_1^{\otimes 2} \simeq \Omega^1$ on $\mathscr{A}_1(\mathbf{C})$.

1.4. Siegel modular varieties, algebraically

Definition. — Let S be a scheme. An **abelian scheme** over S is an S-group scheme $X \to S$ which is smooth, proper with connected geometric fibers. If $S = \operatorname{Spec} k$ we call abelian schemes **abelian varieties**.

Proposition 1.4.1. — *Automatically commutative.*

Definition. — Let $X \to S$ be an abelian scheme and $e : S \to X$ be the identity section we define a functor

$$\begin{aligned} \operatorname{Pic}_{X/S,e} : \mathsf{Sch}_S &\longrightarrow \mathsf{Ab} \\ T &\longmapsto \{(L, \alpha) : L \in \operatorname{Pic}(X \times_S T) \text{ and } \alpha \text{ trivializes } e^*L\}. \end{aligned}$$

There is a subfunctor $\operatorname{Pic}^0_{X/S,e}$ defined by the data such that for all $t \in T$ and all smooth projective curves C over $K(t)$, for all $f : C \to X \times_S K(t)$, f^*L has degree 0.

Theorem 1.4.1 (Artin, Raynaud). — $\operatorname{Pic}^0_{X/S,e}$ *is represented by an abelian scheme over S.*

We write $\widehat{X}$ for this scheme.

Definition. — For $L \in \operatorname{Pic}(X)$, we have

$$\begin{aligned} \phi_L : X &\longrightarrow \widehat{X} \\ x \in X(T) &\longmapsto T_x^* L \otimes L^{-1}. \end{aligned}$$

A **polarization** is an isogeny (i.e. finite, faithfully flat) $\phi : X \to \widehat{X}$ such that for all geometric points $p : \operatorname{Spec} k \to S$, $\phi_p = \phi_L$ for some ample L. A polarization is **principal** if it is an isomorphism. A **principally polarized abelian variety** (PPAV) is the data (X, ϕ) of an abelian variety X and a principal polarization ϕ.

Proposition 1.4.2. — *If $M \geq 1$ is invertible on S then $X[M]$ (defined to be the kernel of $[M]_X$) is étale locally isomorphic to $\underline{(\mathbb{Z}/M)^{2m}}$.*

Definition. — Let $M \geq 1$, we define a functor

$$\begin{aligned} \mathscr{A}_m(M) : \mathsf{Sch}_{\mathbb{Z}[\frac{1}{M}]} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \{\text{PPAV } (X, \phi) \text{ with a level } M \text{ structure}\}/\sim \end{aligned}$$

(Groupoid when $M \leq 2$?)

Theorem 1.4.2 (Mumford). — *For $M \geq 3$, $\mathscr{A}_m(M)$ is represented by a smooth quasiprojective scheme over $\mathbb{Z}\left[\frac{1}{M}\right]$ of relative dimension $\frac{m(m+1)}{2}$.*

By the previous proposition, for all $M \mid M'$ with $M \geq 3$ there is a map

$$\mathscr{A}_m(M') \longrightarrow \mathscr{A}_m(M) \times_{\mathbb{Z}[\frac{1}{M}]} \mathbb{Z}\left[\frac{1}{M'}\right]$$

which is finite étale and a $\ker(G(\mathbb{Z}/M') \to G(\mathbb{Z}/M))$-torsor.

We stil have an action of $G(\mathbb{A}_f)$ on the tower $(\mathscr{A}_m(M) \times \mathbb{Q})_{M \geq 1}$ using the same interpretation of the moduli problem as in the analytic case (quasi-isogenies).

Variant: Let p be a prime and consider the tower $(\mathscr{A}_m(M) \times \mathbb{Z}_{(p)})_{(M,p)=1}$. It admits an actioon of $G(\mathbb{A}_f^{(p)})$, where $\mathbb{A}_f^{(p)}$ are finite adeles with $\mathbb{Q}_p$ omitted.

Applications:

1) We have a $\mathbb{Q}$-structure on modular forms.
2) Étale cohomology: the comparison theorem tells us that

$$\mathrm{H}^\bullet_{\text{ét}}(\mathscr{A}_m(M)_{\overline{\mathbb{Q}}}, \mathbb{Q}_l) \simeq \mathbb{Q}_l \otimes_{\mathbb{Q}} \mathrm{H}^\bullet_B(\mathscr{A}_m(M)(\mathbf{C}), \mathbb{Q}).$$

The LHS has an action of $G(\mathbb{A}) \times \operatorname{Gal}_{\mathbb{Q}}$ where

$$\operatorname{Gal}_{\mathbb{Q}} := \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Example 1.4.1. — Take $m = 1$. Eichler-Shimura and Deligne associated Galois representations to eigenforms of weight ≥ 2. There eigenforms correspond to automorphic representations

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p \hookrightarrow S_k$$

(such that $\pi_\infty \simeq D_k$, an irreducible $(\mathfrak{gl}_2, U(1))$-module). For almost all p, π_p is unramified:

$$\underbrace{\pi^{\mathbf{GL}_2(\mathbb{Z}_p)}}_{\dim 1} \neq 0$$

with an action of $\mathscr{H}(\mathbf{GL}_2(\mathbb{Q}_p), \mathbf{GL}_2(\mathbb{Z}_p))$ which is commutative. There correspond to $c(\pi_p)$ which are semi-simple conjugacy classes in $\mathbf{GL}_2(\mathbf{C})$.

Suitably normalized, there exists a number field $F \subset \mathbf{C}$ such that for almost any p,

$$\operatorname{tr} c(\pi_p) \in F, \quad \det c(\pi_p) \in F.$$

Theorem 1.4.3. — *For all $\iota : F \to \overline{\mathbb{Q}_l}$ there is a continuous irreducible representation $\rho_{\pi,\iota} : \mathrm{Gal}_{\mathbb{Q}} \to \mathbf{GL}_2(\overline{\mathbb{Q}_l})$ such that for almost any p, $\rho_{\pi,\iota}$ is unramified at p and*

$$\operatorname{tr} \rho_{\pi,\iota}(\mathrm{Frob}_p) = \iota(\operatorname{tr}(c(\pi_p)))$$

where the Frobenius on the right is geometric. (??)

§2. General Shimura varieties

Definition. — Let $\mathbb{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{GL}_{1,\mathbf{C}})$, so

$$\mathbb{S}(A) = (A \otimes_{\mathbf{R}} \mathbf{C})^\times$$

for an $\mathbf{R}$-algebra A.

$\mathsf{Rep}(\mathbb{S})$ correspond to real Hodge structures, i.e. finite dimensional real vector spaces V with a decomposition

$$V_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} V = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$ and there is an action of $\mathfrak{z} \in \mathbb{S}(\mathbf{R}) = \mathbf{C}^\times$ via $\mathfrak{z}^{-p}\bar{\mathfrak{z}}^{-q}$.

Definition. — A **Shimura datum** is a pair (G, X) of a commutative reductive group G over $\mathbb{Q}$ and a $G(\mathbf{R})$-orbit X of morphisms $h : \mathbb{S} \to G_{\mathbf{R}}$ such that

1) $\mathbb{S}$ acs via $\mathrm{Ad}(h)$ on

$$\mathfrak{g} := \mathbf{C} \otimes_{\mathbb{Q}} \mathrm{Lie}\, G = \bigoplus_{p,q} \mathfrak{g}^{p,q}$$

has kernel of type $\{(-1,1),(0,0),(1,-1)\}$, i.e. $\mathfrak{g}^{p,q} = 0$ unless (p,q) lies in this set. (This implies that $\mathbf{GL}_{1,\mathbf{R}} \hookrightarrow \mathbb{S}$ maps to $Z(G)$.)

2) $\mathrm{Ad}(h(i))$ is a Cartan involution of $G_{\mathrm{ad},\mathbf{R}} = (G/Z(G))_{\mathbf{R}}$, i.e. an ivolution θ of $H = G_{\mathrm{ad},\mathbf{R}}$ such that

$$\{g \in H(\mathbf{C}) | \theta(g) = g\}$$

is compact.

3) $G_{\mathrm{ad}} \simeq \prod_i G_{\mathrm{ad},i}$ (with $G_{\mathrm{ad},i}$ simple over $\mathbb{Q}$) has no factor $G_{\mathrm{ad},i}$ such that $G_{\mathrm{ad},i}(\mathbf{R})$ is compact.

Example 2.0.1. — Let $G = \mathbf{GSp}_{2n}$ and

$$h(a+ib) = \begin{pmatrix} aI_n & bI_n \\ -bI_n & aI_n \end{pmatrix}.$$