REPRESENTATIONS OF REDUCTIVE ALGEBRAIC GROUPS

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Contents

§1. Generalities on affine group schemes and smooth representations

1.1. Affine group schemes Fix k a base field.

Recall that an **affine k-group scheme** is one of the following data:

(1) An affine scheme G over k endowed with morphisms of k-schemes
- $m : G \times G \to G$;
- $e : \mathrm{Spec}(k) \to G$;
- $\mathrm{inv} : G \to G$;

which satisfy the usual axioms of groups (with m multiplication, e the unit, inv the inverse).

(2) A functor $\mathsf{Alg}_k = \{k\text{-algebras}\} \xrightarrow{F} \mathsf{Gps}$ such that the composition $\mathsf{Alg}_k \xrightarrow{F} \mathsf{Gps} \to \mathsf{Sets}$ is representable.

(3) A commutative Hopf algebra over k, i.e. a commutative algebra A with morphisms of k-algebras
- $\Delta : A \to A \otimes A$;
- $\varepsilon : A \to k$;
- $S : A \to A$;

such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) : A \longrightarrow A \otimes A \otimes A$$
$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta : A \longrightarrow A$$
$$(\mathrm{id}, S) \circ \Delta = \Delta \circ \varepsilon = (S, \mathrm{id}) \circ \Delta : A \longrightarrow A$$

Notation. —

$$A \longrightarrow \mathrm{Spec}(A)$$
$$G \longrightarrow \mathscr{O}(G), \Delta_G.$$

Example 1.1.1. —

(1) *Diagonalizable groups:* If Λ is an abstract commutative group we have the affine k-group scheme $\mathrm{Diag}(\Lambda) := \mathrm{Spec}(k[\Lambda])$ with

$$\Delta(\lambda) = \lambda \otimes \lambda, \quad \varepsilon(\lambda) = 1, \quad S(\lambda) = \lambda^{-1} \quad (\forall \lambda \in \Lambda).$$

In particular for $\Lambda = \mathbb{Z}$, $k[\Lambda] = k[x, x^{-1}]$ and $\mathrm{Diag}(\Lambda) = \mathbb{G}_m$ (the **multiplicative group**).

A **(split) torus** is a group scheme of the form $\mathrm{Diag}(\Lambda)$ with Λ a finitely generated free abelian group.

(2) *Additive group:* $\mathbb{G}_a := \mathrm{Spec}(k[x])$ with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

More generally for V a k-vector space we have the functor $V_a : R \mapsto (R \otimes V, -)$ which is an affine k-group scheme if V is finite dimensional.

(3) If V is a k-vector space, $\mathbf{GL}(V)$ is the functor $R \mapsto \mathrm{Aut}_R(R \otimes V)$. If V is finite dimensional this is an affine k-group scheme.

In particular, if $V = k^n$ we get

$$\mathbf{GL}_n = \mathrm{Spec}(k[x_{ij}, 1 \leq i, j \leq n][\det{}^{-1}])$$

with

$$\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}, \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

Similarly we have $\mathbf{SL}(V), \mathbf{SL}_n$.

(4) For any abstract group Γ we have the functor (??)

1.2. Representations If G is an affine k-group scheme, a **representation** of G is the datum of a k-vector space V and a morphism of group valued functors $G \to \mathbf{GL}(V)$. [Equivalently, an action of G on V_a such that $G(R)$ acts R-linearly on $R \otimes V$.]

This datum is equivalent ot that of a **comodule** for $\mathscr{O}(G)$, i.e. a k-vector space V and a k-linear map $\Delta_V : V \to V \otimes \mathscr{O}(G)$ such that

$$(\Delta_V \otimes \mathrm{id}_{\mathscr{O}(G)} \circ \Delta_V = (\mathrm{id}_V \otimes \Delta_G) \circ \Delta_V : V \longrightarrow V \otimes \mathscr{O}(G) \otimes \mathscr{O}(G)$$
$$(\mathrm{id}_V \otimes \varepsilon) \otimes \Delta_V = \mathrm{id}_V : V \longrightarrow V.$$

[Δ_V corresponds to the image of $\mathrm{id}_{\mathscr{O}(G)} \in G(\mathscr{O}(G)) = \mathrm{End}_{k\text{-alg}}(\mathscr{O}(G))$ in $\mathrm{End}_{\mathscr{O}(G)}(\mathscr{O}(G) \otimes V)$.]

Example 1.2.1. —

(1) *(Right) Regular representation:* $V = \mathscr{O}(G)$ with $\Delta_V = \Delta_G$. More generally, given an action of G on an affine scheme X we get a representation with underlying vector space $\mathscr{O}(X)$.
(2) If V is a finite dimensional vector space, V is a representation of $\mathbf{GL}(V)$.
(3) For any G we have the trivial representation k.

Notation. — $\mathrm{Rep}(G)$ is the abelian category of representations of G. $\mathrm{Rep}^{\mathrm{fd}}(G)$ is the full subcategory of finite dimensional representations.

If $V \in \mathrm{Rep}(G)$ then V is the union of its finite dimensional subrepresentations.

Example 1.2.2 (Representations of diagonalizable group schemes). — Let Λ be a commutative group, $G = \mathrm{Diag}(\Lambda)$. If $V \in \mathrm{Rep}(G)$ we have

$$\Lambda_V : V \longrightarrow V \otimes \mathscr{O}(G) = \bigoplus_{\lambda \in \Lambda} V \otimes \lambda.$$

Hence there are morphisms $(\rho_\lambda : \lambda \in \Lambda)$ in $\mathrm{End}(V)$ such that

$$\Delta_V(v) = \sum_{\lambda \in \Lambda} \rho_\lambda(v) \otimes \lambda, \qquad \forall v \in V.$$

(Here $\rho_\lambda(v) = 0$ for all but finitely many λs.)

It is easy to see that

$$\rho_\lambda \circ \rho_\mu = \begin{cases} \rho_\lambda & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{id} = \sum_{\lambda_\Lambda} \rho_\lambda$.

Hence $V = \bigoplus_{\lambda \in \Lambda} \rho_\lambda(V)$ with

$$\rho_\lambda(V) = \{v \in V : \Delta_V(v) = v \otimes \lambda\} = V_\lambda.$$

Hence $\mathrm{Rep}(G)$ is isomorphic to the category of Λ-graded vector spaces (correct?).

1.3. Induction Let G be an affine k-group scheme.

A **subgroup** of G is a closed subscheme $H \subset G$ such that $e, \mathrm{inv}\,|_H, m|_{H \times H}$ factor through H. Then H is an affine k-group scheme. In this setting we have the restriction functor $\mathrm{Res}^G_H : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$.

Proposition 1.3.1. — *The functor* Res^G_H *has a right adjoint* $\mathrm{Ind}^G_H : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

Explicitly, we have

$$\mathrm{Ind}^G_H(V) = (V \otimes \mathscr{O}(G))^H$$

with H acting diagonally via the right-regular representation on $\mathscr{O}(G)$ and G acting on the fixed points via the left regular representation

$$\mathrm{Ind}^G_H(V) = \left\{ \begin{array}{c} \text{morphisms of functors} \\ f : G \to V_a \end{array} \;\middle|\; \begin{array}{c} f(gh) = h^{-1} f(g) \\ \forall g \in G(R), h \in H(R), R \in \mathsf{Alg}_k \end{array} \right\}.$$

The canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, \mathrm{Ind}^G_H(V')) \simeq \mathrm{Hom}_{\mathrm{Rep}(H)}(V, V')$$

is called **Frobenius reciprocity**.

Properties. —

- *Transitivity:* Given subgroups $H_1 \subset H_2 \subset G$ we have
$$\operatorname{Ind}_{H_1}^G \simeq \operatorname{Ind}_{H_2}^G \circ \operatorname{Ind}_{H_1}^{H_2}.$$
- *Tensor identity:* For $V_1 \in \operatorname{Rep}(H), V_2 \in \operatorname{Rep}(G)$
$$\operatorname{Ind}_H^G(V_1 \otimes \operatorname{Res}_H^G(V_2)) \simeq \operatorname{Ind}_H^G(V_1) \otimes V_2.$$
- Ind_H^G sends injective objects of $\operatorname{Rep}(H)$ to injective objects of $\operatorname{Rep}(G)$. In particular,
$$\operatorname{Ind}_H^G(k) = \mathscr{O}(G)$$
is injective.
- $\operatorname{Rep}(G)$ has enough injectives.

Geometric interpretation: We assume G is an algebraic group (over k), i.e. an affine k-group scheme such that $\mathscr{O}(G)$ is a finitely generated k-algebra. In this setting, for $H \subset G$ a subgroup we have a quotient scheme G/H of finite type over k with a faithfully flat quotient map $\pi : G \to G/H$. For $V \in \operatorname{Rep}(H)$, we have a quasicoherent sheaf $\mathscr{L}_{G/H}(V) \in \mathsf{QCoh}(G/H)$ with
$$\Gamma(V, \mathscr{L}_{G/H}(V)) = \left\{ \text{morphisms } f : \pi^{-1}(V) \longrightarrow V \,\middle|\, f(x,h) = h^{-1}f(x) \text{ for all } (?) \right\}.$$
We have $\operatorname{Ind}_H^G(V) = \Gamma(G/H, \mathscr{L}_{G/H}(V))$. If V is finite dimensional, then $\mathscr{L}_{G/H}(V)$ is coherent.

Consequences. —

- If G/H is affine then Ind_H^G is exact.
- If G/H is projective then Ind_H^G preserves finite dimensionality.

Since $\operatorname{Rep}(H)$ has enough injectives we can consider the derived functor
$$\mathrm{R}\operatorname{Ind}_H^G : D^b \operatorname{Rep}(H) \longrightarrow D^b \operatorname{Rep}(G).$$
The functor $\mathscr{L}_{G/H} : \operatorname{Rep}(H) \to \mathsf{QCoh}(G/H)$ is exact, hence we have
$$\mathscr{L}_{G/H} : D^b \operatorname{Rep}(H) \longrightarrow D^b\mathsf{QCoh}(G/H).$$
One can check that
$$\mathrm{R}\operatorname{Ind}_H^G(V) \simeq R\Gamma(G/H, \mathscr{L}_{G/H}(V)).$$
(??)

Consequences. —

- We have $\mathrm{R}^n \operatorname{Ind}_H^G(V) = 0$ for all $V \in \operatorname{Rep}(H)$ if $n > \dim(G/H)$.
- If G/H is projective, then $\mathrm{R}^n \operatorname{Ind}_H^G(V)$ is finite-dimensional for all $V \in \operatorname{Rep}^{\mathrm{fd}}(H), n \in \mathbb{Z}$.

§2. Reductive algebraic groups

From now on k is algebraically closed.

2.1. Definition A k-algebraic group G is called **unipotent** if every non-zero representation admits a non-zero fixed vector. [Equivalent condition: G is unipotent if and only if it is isomorphic to a subgroup of unipotent upper-triangular matrices in $\mathbf{GL}_n$ for some n.]

Example 2.1.1. — $\mathbb{G}_a$ is unipotent as

$$\mathbb{G}_a \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

If G is a smooth, connected algebraic group, the smooth, connected, unipotent, normal subgroups of G there is a largest element called the **unipotent radical** of G, denoted $R_u(G)$. An algebraic group G is called **reductive** if it is smooth, connected and $R_u(G)$ is trivial.

One possible motivation for studying representations of reductive algebraic groups is that any simple representation of a smooth connected algebraic group G factors through a simple representation of $G/R_u(G)$, which is a reductive algebraic group.

Example 2.1.2. —

(1) *Tori:* If Λ is a finitely generated, free abelian group, then $\mathrm{Diag}(\Lambda)$ is a reductive algebraic group.
(2) For any finite-dimensional k-vector space V, $\mathbf{GL}(V)$ and $\mathbf{SL}(V)$ are reductive algebraic groups.
(3) Symplectic groups, special orthogonal groups.

2.2. Structure From now on G is a redutive algebraic group.

We denote by B a **Borel subgroup** (a maximal, connected, smooth, solvable subgroup). Note that:

- a Borel subgroup is unique up to conjugation;
- the quotient G/B is a smooth, projective variety.

Example 2.2.1 (Main Example). — For $G = \mathbf{GL}_{n,k}$ one can take

$$B = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}.$$

In this case G/B parametrizes flags in k^n, i.e. data

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset k^n$$

with V_i a subspace of dimension i.

Let T be a maximal torus contained in B.

Example 2.2.2 (Main Example Continued). — We take

$$T = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \right\}.$$

Note that

$$T \simeq \mathrm{Diag}(\mathbb{X}), \quad \mathbb{X} = \{\text{morphisms } T \longrightarrow \mathbb{G}_m\}$$

we call elements of $\mathbb{X}$ **weights**.

Example 2.2.3 (Main Example Continued). — We have $\mathbb{X} \simeq \mathbb{Z}^n$ via

$$(\lambda_1, \dots, \lambda_n) \leftrightarrow \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \longmapsto \prod_{i=1}^{n} t_i^{\lambda_i}.$$

The **roots** $R \subset \mathbb{X}$ are the non-zero weights appearing in the action of T on $\mathfrak{g} = \mathrm{Lie}(G)$.

Example 2.2.4 (Main Example Continued). — We have

$$R = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}.$$

We define the **positive roots** $R_+ \subset R$: the weights appearing in the action of T on $\mathfrak{g}/\operatorname{Lie}(B)$, and the **simple roots** $R_s \subset R_+$: positive roots that cannot be written as a sum of two positive roots.

Note. — $R = R_+ \coprod -R_+$. Any element of R_+ can be uniquely written as a sum of simple roots.

Example 2.2.5 (Main Example Continued). — In our case

$$\begin{aligned} R_+ &= \{\varepsilon_i - \varepsilon_j : 1 \leq i < j < n\} \\ R_s &= \{\varepsilon_i - \varepsilon_{i+1}\}. \end{aligned}$$

The **cocharacters** of G are

$$\begin{aligned} \mathbb{X}^\vee &= \operatorname{Hom}_{\mathbb{Z}}(\mathbb{X}, \mathbb{Z}) \\ &= \{\text{morphisms } \mathbb{G}_m \longrightarrow T\}. \end{aligned}$$

We have **coroots** $R^\vee \subset \mathbb{X}^\vee$ and a bijection

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee. \end{aligned}$$

Then $(\mathbb{X}, R, \mathbb{X}^\vee, R^\vee)$ together with the identification $\mathbb{X}^\vee = \operatorname{Hom}(\mathbb{X}, \mathbb{Z})$ and the bijection $R \to R^\vee$ is the **root datum** of G. It determines G up to isomorphism.

There is an opposite Borel subgroup $B^+ \subset G$ containing T such that the non-zero weights of T acting on $\operatorname{Lie}(B^+)$ are R_+.

$W = N_G(T)/T$ is a the **Weyl group**, it is a constant group scheme, associated with a finite group also denoted W. W acts faithfully on $\mathbb{X}$. For $\alpha \in R$ there is an element $s_\alpha \in W$ which acts on $\mathbb{X}$ via

$$\lambda \longmapsto \lambda - \left\langle \lambda, \alpha^\vee \right\rangle \alpha.$$

If we set

$$S = \{s_\alpha : \alpha \in R_s\} \subset W,$$

then (W, S) is a Coxeter system. In particular, we have the length function

$$\begin{aligned} \ell : W &\longrightarrow \mathbb{Z}_{\geq 0} \\ w &\longmapsto \min\{r \geq 0 | \text{there exist } s_1, \ldots, s_r \in S \text{ such that } w = s_1 \cdots s_r\}. \end{aligned}$$

Example 2.2.6 (Main Example Continued). — For example, $W = \mathfrak{S}_n$ is the symmetric group via permutation matrices. The action on $\mathbb{X} = \mathbb{Z}^n$ is by permuting entries

$$S = \{(i, i+1) : 1 \leq i < n\}$$

The length function counts inversions of permutations.

Example 2.2.7. — Let $G = \mathbf{SL}_2$,

$$\begin{aligned} B &= \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \subset \mathbf{SL}_{2,k} \\ T = T &= \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in k^\times \right\} \simeq \mathbb{G}_m. \end{aligned}$$

We have $\mathbb{X} \simeq \mathbb{Z}$ via

$$\lambda \leftrightarrow \left[\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto t^\lambda \right].$$

We have $R = \{2, -2\}$, $R_+ = \{2\} = R_s$ and $W = \mathfrak{S}_2 = \mathbb{Z}/2\mathbb{Z}$.

2.3. Classification of simple representations The Borel is a semidirect product $B = T \ltimes U$ with $U = R_u(B)$. Similarly $B^+ = T \ltimes U^+$ with $U^+ = R_u(B^+)$. In particular, $T \xrightarrow{\sim} B/U$, so any $\lambda \in \mathbb{X}$ provides a morphism $B \to \mathbb{G}_m$, which is a one-dimensional representation $k_B(\lambda)$. Define

$$\nabla(\lambda) = \mathrm{Ind}_B^G(k_B(\lambda)).$$

It's easy to see that:

- $\dim(\nabla(\lambda)) < \infty$ for all $\lambda \in \mathbb{X}$ (because G/B is projective).
- The action of T on $\nabla(\lambda)$ determines an $\mathbb{X}$-grading

$$\nabla(\lambda) = \bigoplus_{\mu \in \mathbb{X}} \nabla(\lambda)_\mu.$$

  Here if $\nabla(\lambda) \neq 0$, we have
  - $\nabla(\lambda)_\lambda = \nabla(\lambda)^{U^+}$ and this is one-dimensional,
  - if $\nabla(\lambda)_\mu \neq 0$ then $\lambda - \mu \in \mathbb{Z}_{\geq 0} R_s$.

  This follows from the open embedding

$$U^+ \times B \hookrightarrow G$$

  induced by multiplication.

Corollary 2.3.1. — *We have a bijection*

$$\begin{aligned}\{\lambda \in \mathbb{X} | \nabla(\lambda) \neq 0\} &\xrightarrow{\sim} \{\textit{simple objects in } \mathrm{Rep}(G)\} / \simeq \\ \lambda &\longmapsto L(\lambda) = \textit{ unique simple subrepresentation in } \nabla(\lambda).\end{aligned}$$

It's less easy to show:

Proposition 2.3.1. — *For $\lambda \in \mathbb{X}$, we have*

$$\nabla(\lambda) \neq 0 \quad \iff \quad \forall \alpha \in R_s, \langle \lambda, \alpha^\vee \rangle \geq 0.$$

Idea of the proof. The forward direction is easy using the fact that W permutes

$$\left\{\mu \in \mathbb{X} \middle| \nabla(\lambda)_\mu \neq 0\right\}.$$

Conversely, one can construct a function

$$\bigcup_{\alpha \in R_s} s_\alpha U^+ B \longrightarrow k$$

and then use the fact that the LHS has complement of codimension 2 in G, cf. Bruhat decomposition. □

We set

$$\mathbb{X}_+ = \left\{\lambda \in \mathbb{X} \middle| \forall \alpha \in R_s, \langle \lambda, \alpha^\vee \rangle \geq 0\right\}$$

the **dominant weights**.

Example 2.3.1. — (1) $\nabla(0) = k$ is the trivial representation (because G/B is connected and projective).

(2) Let $G = \mathbf{GL}_{n,k}$

$$\mathbb{X}_+ = \{(\lambda_1, \ldots, \lambda_n) | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

For $r \geq 0$

$$\nabla(r, 0, \ldots, 0) \simeq S^r(V)$$

with $V = k^n$ the natural representation, and

$$\nabla(0, \ldots, 0, -r) \simeq S^r(V^*)$$

(cf. sections of line bundles on $\mathbb{P}^n$).

For $s \in \{1, \dots, n\}$,

$$\nabla(\underbrace{1, \dots, 1}_{s}, 0, \dots, 0) = \bigwedge^{s}(V)$$
$$= L(\underbrace{1, \dots, 1}_{s}, 0, \dots, 0).$$

For $r \in \mathbb{Z}$

$$\nabla(r, \dots, r) = k_{\det^r} = L(r, \dots, r).$$

(3) Let $G = \mathbf{SL}_2$, then $\mathbb{X}_+ = \mathbb{Z}_{\geq 0}$. For $r \geq 0$

$$\nabla(r) = S^r(k^2)$$

(cf. sections of line bundles on $G/B = \mathbb{P}^1$). If $\mathrm{char}(k) = 0$ then $\nabla(r)$ is simple for all $r \geq 0$. If $\mathrm{char}(k) = p > 0$ this is not always true:

$$\nabla(p) = kx^p \oplus kx^{p-1}y \oplus \cdots \oplus kxy^{p-1} \oplus ky^p$$

with x, y a canonical basis of k^2. Then $kx^p \oplus ky^p$ is a non-trivial G-stable subspace. In fact, $L(p) = kx^p \oplus ky^p$.

More generally, $\nabla(r)$ is simple if and only if $r \leq p - 1$.

(4) For all $\lambda \in \mathbb{X}_+$, $L(\lambda)^* \simeq L(-w_0\lambda)$ where $w_0 \in W$ is the longest element.

2.4. Characters If $V \in \mathrm{Rep}^{\mathrm{fd}}(G)$ then the action of T determines a grading $V = \bigoplus_{\lambda \in \mathbb{X}} V_\lambda$ with

$$V_\lambda = \{v \in V | \forall t \in T, tv = \lambda(t)v\}.$$

We set

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathbb{X}} \dim(V_\lambda) e^\lambda \in \mathbb{Z}[\mathbb{X}].$$

It's easy to check that:

- ch factors through $K^0(\mathrm{Rep}^{\mathrm{fd}}(G)) \to \mathbb{Z}[\mathbb{X}]$.
- $\mathrm{ch}(V \otimes V') = \mathrm{ch}(V)\,\mathrm{ch}(V')$, so the map above is a *ring morphism*.
- ch takes values in $\mathbb{Z}[\mathbb{X}]^W$.

Proposition 2.4.1. — ch *induces an isomorphism*

$$K^0(\mathrm{Rep}^{fd}(G)) \xrightarrow{\sim} \mathbb{Z}[\mathbb{X}]^W.$$

Proof idea. Show that

$$\{\mathrm{ch}(L(\lambda)) | \lambda \in \mathbb{X}_+\}$$

is a basis of $\mathbb{Z}[\mathbb{X}]^W$. □

§3. Some general results about ? of reductive algebraic groups

3.1. Kempf's vanishing theorem

Theorem 3.1.1. — *If $\lambda \in \mathbb{X}_+$ then*

$$\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda)) = 0 \quad \forall n > 0.$$

Note. — We have

$$\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda)) = \mathrm{H}^n(G/B, \mathscr{L}_{G/B}(k_B(\lambda))),$$

where $\mathscr{L}_{G/B}(k_B(\lambda))$ is a line bundle equal to $\mathscr{O}_{G/B}(\lambda)$.

So we are in fact computing cohomology of some line bundles on G/B.

Closely related fact: (??)

In fact, in case $p = 0$, we get Kempf's vanishing theorem from this proposition using the *Kodaira vanishing theorem*.

Example 3.1.1. —

(1) $\mathrm{H}^n(G/B, \mathscr{O}_{G/B}) = 0$ for $n > 0$, this is the $\lambda = 0$ case.

(2) For $G = \mathbf{SL}_{2,k}, \mathbb{X} = \mathbb{Z}, G/B = \mathbb{P}^1$ and $\mathscr{O}_{G/B}(\lambda) = \mathscr{O}_{\mathbb{P}^1}(\lambda)$. So we recover the fact that
$$\mathrm{H}^n(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m)) = 0$$
if $n > 0$ and $m \geq 0$.

Remark. — Serre duality for G/B: $\omega_{G/B} \simeq \mathscr{O}_{G/B}(-\rho)$ where
$$\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$$
implies that for $\lambda \in \mathbb{X}$ we have
$$\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda)) \simeq \left(\mathrm{R}^{|R^+|-n} \operatorname{Ind}_B^G(-(\lambda+2\rho))\right)^*$$
(note that $|R_+| = \dim(G/B)$). So if $\lambda \in -2\rho - \mathbb{X}_+$ then $\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda)) = 0$ if $n \neq |R_+|$.

For $\mathbf{SL}_2$, this says
$$\mathrm{H}^n(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m)) = 0$$
for $n \neq 1$ if $m \leq -2$.

Here is an interesting application. For $\lambda \in \mathbb{X}_+$ we set
$$\Delta(\lambda) = (\nabla(-w_0\lambda))^* = \mathrm{R}^{|R|_+} \operatorname{Ind}_B^G(k_B(w_0\lambda - 2\rho))$$
(w_0 is longest length in W). These modules are called **Weyl modules**. We have
$$\Delta(\lambda) \twoheadrightarrow L(\lambda).$$

Proposition 3.1.1. — *For $\lambda, \mu \in \mathbb{X}_+$ we have*
$$\operatorname{Ext}^n_{\mathrm{Rep}(G)}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if } \lambda = \mu, n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The unique (up to scalar) non-zero morphism for $\lambda = \mu$ and $n = 0$ is the composition
$$\Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda).$$
This statement says that $\mathrm{Rep}(G)$ is a "highest weight category".

3.2. Borel-Bott-Weil theorem We consider the action of W on X given by
$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$
More precisely, this defines an action on $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}$, which stabilizes $\mathbb{X}$ since $w\rho - \rho \in \mathbb{Z}R$ for all $w \in W$.

Set
$$\overline{C} = \begin{cases} \{\lambda \in \mathbb{X} | \forall \beta \in R_+, \langle \lambda + \rho, \beta^\vee \rangle \geq 0\} & \text{if } p = 0 \\ \{\lambda \in \mathbb{X} | \forall \beta \in R_+, 0 \leq \langle \lambda + \rho, \beta^\vee \rangle\} & \text{if } p > 0. \end{cases}$$

Example 3.2.1. —

(1) Let $G = \mathbf{SL}_2, \mathbb{X} = \mathbb{Z}$, then
$$\overline{C} = \begin{cases} \{-1, 0, 1, \dots\} & \text{if } p = 0 \\ \{-1, 0, \dots, p-1\} & \text{if } p > 0. \end{cases}$$
(2) Let $G = \mathbf{SL}_3$, then (picture).

Theorem 3.2.1 (Borel-Bott-Weil). —

(1) If $\lambda \in \overline{C} \setminus \mathbb{X}_+$ then
$$\mathrm{R}^n \operatorname{Ind}_B^G(k_B(w \cdot \lambda)) = 0$$
for all $w \in W, n \in \mathbb{Z}$.

(2) If $\lambda \in \overline{C} \cap \mathbb{X}_+$, then for $n \in \mathbb{Z}, w \in W$, we have
$$\mathrm{R}^n \operatorname{Ind}_B^G(k_B(w \cdot \lambda)) = \begin{cases} \nabla(\lambda) & \text{if } n = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

The proof is by induction on $\ell(w)$, the case $\ell(w) = 0$ follows from Kempf's vanishing theorem. This uses a decomposition of $\mathrm{R}^n \operatorname{Ind}_B^{P(\alpha)}(k_B(\lambda))$ for $\alpha \in R_+$, which is a $\mathbf{SL}_2$ computation (?)

Remark. — If $p = 0$ then $W \cdot C = \mathbb{X}$, so we understand all $\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda))$. For $p > 0$, this then only describes a small number of such spaces. In particular, in this case there can exist $\lambda \in \mathbb{X}$ such that $\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda)) \neq 0$ for several n's.

Corollary 3.2.1. —
(1) If $\lambda \in \overline{C} \cap \mathbb{X}_+$ then $\nabla(\lambda) = L(\lambda)$.
(2) If $\lambda, \mu \in \overline{C} \cap \mathbb{X}_+$ then
$$\operatorname{Ext}^1_{\operatorname{Rep}(G)}(L(\lambda), L(\mu)) = 0.$$
In particular, if $p = 0$ the category $\operatorname{Rep}(G)$ *is semisimple.*

Proof.
(1) By BBW and Serre duality
$$\nabla(\lambda)^* \simeq \nabla(-w_0\lambda).$$
Hence $\nabla(\lambda)$ has a unique simple quotient isomorphic to $L(\lambda)$. Since $L(\lambda)$ is also the unique simple submodule of $\nabla(\lambda)$ and has multiplicity 1 as a composition factor (because $\dim \nabla(\lambda)_\lambda = 1$) we must have $\nabla(\lambda) \simeq L(\lambda)$.
(2) We have
$$\operatorname{Ext}^1(L(\lambda), L(\mu)) \simeq \operatorname{Ext}^1(\Delta(\lambda), \nabla(\mu)) \simeq 0.$$
Then we use local finiteness of representations.

□

Remark. —
(1) In Milne's book (§22.C) there is a different proof of semisimplicity using the action of $\mathfrak{g}$ and Casimir operators.
(2) For a connected algebraic group H over k of characteristic 0 one proves that H is reductive if and only if every finite dimensional representation is semisimple. [Thoerem 22.42 in Milne's book.]

Example 3.2.2. — For $G = \mathbf{SL}_2$, write $V = k^2$ for the natural representation. One recovers that $S^r(V)$ is simple if $p = 0$ or $p > 0$ and $r < p$.

3.3. Weyl's character formula For $\lambda \in \mathbb{X}$ we set
$$\chi(\lambda) = \sum_{n \geq 0} (-1)^n \operatorname{ch}\left(\mathrm{R}^n \operatorname{Ind}_B^G(k_B(\lambda))\right)$$

Note. — If $\lambda \in \mathbb{X}_+$, by Kempf's vanishing theorem we have
$$\chi(\lambda) = \operatorname{ch}(\nabla(\lambda)).$$

Theorem 3.3.1. — *For $\lambda \in \mathbb{X}$ we have*
$$\chi(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}} \in \operatorname{Frac}(\mathbb{Z}[\mathrm{X}])$$

In particular, this gives a formula for $\operatorname{ch}(\nabla(\lambda))$ (this formula does not depend on p!). Along the way to prove this theorem, one proves that $\operatorname{ch}(\Lambda(\lambda)) = \operatorname{ch}(\nabla(\lambda))$.

Remark. — The proof in general follows from an analysis of $\operatorname{ch} \mathrm{R}^1 \operatorname{Ind}_B^{P(\alpha)}(k_B(\lambda))$ for $\alpha \in R_+$ already used for the BBW theorem.

When $p = 0$, there is an alternative proof using a Lefschetz-type fixed point formula [cf. reference in Milne's book or §6.1.16 in Chriss-Ginzburg *Representation theory and complex geometry*].

§4. THE CASE OF POSITIVE CHARACTERISTIC

We assume that $p > 0$.

4.1. Frobenius morphism and Frobenius kernel Set

$$G^{(1)} = \mathrm{Spec}(k) \times_{\mathrm{Spec}(k)} G$$

where $\mathrm{Spec}(k) \to \mathrm{Spec}(k)$ corresponds to

$$\begin{aligned} k &\longrightarrow k \\ x &\longmapsto x^p. \end{aligned}$$

In other words, $\mathscr{O}(G^{(1)} = \mathscr{O}(G)$ with k acting by $\lambda \cdot f = \lambda^{1/p} f$ for $\lambda \in k, f \in \mathscr{O}(G)$.

We have a Frobenius morphism $\mathrm{Fr} : G \to G^{(1)}$ associated with

$$\begin{aligned} \mathscr{O}(G^{(1)}) &\longrightarrow O(G) \\ f &\longmapsto f^p. \end{aligned}$$

Here $G^{(1)}$ is an affine k-group scheme and Fr_G is a morphism of k-group schemes.

We have

$$T^{(1)} \subset B^{(1)} \subset G^{(1)}$$

a maximal torus and Borel subgroup and $G^{(1)}$ is again reductive.

We have

$$\begin{aligned} \phi : X^*(T^{(1)}) &\longrightarrow \mathbb{X} \\ \lambda &\longmapsto \lambda \circ \mathrm{Fr}_T\,. \end{aligned}$$

This morphism is injective, with image $p\mathbb{X}$. The roots of $G^{(1)}$ with respect to $T^{(1)}$ are pR.

Remark. — $G \simeq G^{(1)}$ as k-group schemes. A choice of such isomorphism amounts to choosing a "lift" of G to an $\mathbb{F}_p$-group scheme.

The **Frobenius kernel** is $G_1 = \ker(\mathrm{Fr}_G)$ (scheme-theoretic kernel). Here $\mathscr{O}(G_1)$ is a finite-dimensional Hopf-algebra and

$$\mathscr{O}(G_1)^* \simeq U\mathfrak{g} / \left\langle x^p - x^{[p]} \right\rangle$$

where $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ is the restricted p-th power operation. The Frobenius morphism induces an isomorphism $G/G_1 \xrightarrow{\sim} G^{(1)}$.

4.2. Curtis' and Steinberg's theorems Let

$$\mathbb{X}_+^{\mathrm{res}} = \left\{ \lambda \in \mathbb{X} \middle| \forall \alpha \in R_s, 0 \leq \left\langle \lambda, \alpha^\vee \right\rangle \leq p-1 \right\}.$$

Theorem 4.2.1 (Curtis). —

(1) For $\lambda \in \mathbb{X}_+^{\mathrm{res}}$, teh representation $\mathrm{Res}_{G_1}^G(L(\lambda))$ is a simple G_1-representation.

(2) If G is semisimple and simply connected, then

$$\lambda \longmapsto \mathrm{Res}_{G_1}^G(L(\lambda))$$

induces a bijection

$$\mathbb{X}_+^{\mathrm{res}} \xrightarrow{\sim} \{\textit{simple } G_1\textit{-modules}\}/\sim.$$

Remark. — If G is semisimple and simply connected, $\mathbb{X}_+^{\mathrm{res}} \xrightarrow{\sim} \mathbb{X}/p\mathbb{X}$. For general G, simple G_1-modules are parametrized by $\mathbb{X}/p\mathbb{X}$.

A closely related statement is:

Theorem 4.2.2 (Steinberg). — *If $\lambda \in \mathbb{X}_+^{\mathrm{res}}, \mu \in X^*(T^{(1)})_+$ we have*

$$L(\lambda) \otimes \mathrm{Fr}_G^*(L^{(1)}(\mu)) \simeq L(\lambda + \phi(\mu)).$$

If G, B, T are obtained from similar data over $\mathbb{F}_p$ we have

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "\simeq"] & B \arrow[r, hook] \arrow[d, "\simeq"] & G \arrow[d, "\simeq"] \\ T^{(1)} \arrow[r, hook] & B^{(1)} \arrow[r, hook] & G^{(1)} \end{tikzcd}$$

The formula becomes
$$L(\lambda)\otimes \mathrm{Fr}_G^*(L(\mu))\simeq L(\lambda+\rho\mu)$$
for $\lambda\in\mathbb{X}_+^{\mathrm{res}},\mu\in\mathbb{X}_+$.

More generally, for $\lambda_0,\dots,\lambda_r\in\mathbb{X}_+^{\mathrm{res}},\mu\in\mathbb{X}_+$.
$$L(\lambda_0+p\lambda_1+p^2\lambda_2+\cdots+p^r\lambda_r+p^{r+1}\mu)\simeq L(\lambda_0)\otimes \mathrm{Fr}_G^* L(\lambda_1)\otimes\cdots\otimes(\mathrm{Fr}_G^r)^*L(\lambda_r)\otimes(\mathrm{Fr}_G^{r+1})^*(\mu).$$
Note that $L(p\lambda)\simeq \mathrm{Fr}_G^* L(\lambda)$ for all $\lambda\in\mathbb{X}_+$.

Remark. — If G is semisimple and simply connected any $\lambda\in\mathbb{X}_+$ can be written uniquely as
$$\lambda_0+p\lambda_1+\cdots+p^r\lambda_r$$
with $\lambda_0,\dots,\lambda_r\in\mathbb{X}_+^{\mathrm{res}}$. This reduces the description of all simple representations to those corresponding to elements in $\mathbb{X}_+^{\mathrm{res}}$.

4.3. Linkage principle

4.3.1. Affine Weyl group Define
$$W_{\mathrm{aff}}:=W\ltimes\mathbb{Z}R$$
which acts (affinely, not linearly) on $\mathbf{R}\otimes_{\mathbb{Z}}\mathbb{X}$. A fundamental domain is
$$\overline{A_0}=\left\{v\in V\middle|\forall\alpha\in R_+,0\le\langle v,\alpha^\vee\rangle\le 1\right\}.$$
Set
$$S_{\mathrm{aff}}:=\left\{w\in W_{\mathrm{aff}}\middle|V^W\cap\overline{A_0}\text{ has codimension 1 in }V\right\}.$$

Fact. — The pair $(W_{\mathrm{aff}},S_{\mathrm{aff}})$ is a Coxeter system.

Example 4.3.1. —

(1) Take $G=\mathbf{SL}_2,V=\mathbf{R}$, then $S_{\mathrm{aff}}=\{s,s_0\},S=\{s\}$. (Picture) W_{aff} is the infinite dihedral group generated by s,s_0.
(2) Take $G=\mathbf{SL}_3$ and $\dim V=2$ (Picture).

What is relevant for the study of $\mathrm{Rep}(G)$ is the **dot-action** of W_{aff} on $\mathbb{X}$
$$(w,\lambda)\bullet\mu=w(\mu+p\lambda+\rho)-\rho.$$

Theorem 4.3.1 (Linkage principle). — *For $\lambda,\mu\in\mathbb{X}_+$ we have*
$$\mathrm{Ext}^1_{\mathrm{Rep}(G)}(L(\lambda),L(\mu))\neq 0\implies W_{\mathrm{aff}}\bullet\lambda=W_{\mathrm{aff}}\bullet\mu.$$

For $c\in\mathbb{X}(W_{\mathrm{aff}},\bullet)$ we set
$$\mathrm{Rep}_c(G)=\left\{M\in\mathrm{Rep}(G)\middle|\begin{array}{c}\text{all composition factors of }M\\ \text{are of the form }L(\lambda)\text{ with }\lambda\in\mathbb{X}_+\cap c\end{array}\right\}$$
a full subcategory of $\mathrm{Rep}(G)$.

Corollary 4.3.1. — *We have*
$$\mathrm{Rep}(G)=\prod_{c\in\mathbb{X}/(W_{\mathrm{aff}},\bullet)}\mathrm{Rep}_c(G).$$

Remark. —

(1) The linkage principle was conjectured by Verma. Proved by Humphreys, Carter-Lusztig, Jatzen, Andersen.
(2) Under mild assumptions on p, the linkage principle follows from a "central character" argument and the description of the center of G and of $U\mathfrak{g}$.
(3) In fact, what Andersen proves is a "strong linkage principle": if $\lambda\in\mathbb{X}$ satisfies
$$\langle\lambda,\alpha^\vee\rangle\ge -1$$
for all $\alpha\in R_s,w\in W$, if $L(\mu)$ is a composition factor of $\mathrm{R}^i\,\mathrm{Ind}_B^G(k_B(w\bullet\lambda))$ then $\mu\in W_{\mathrm{aff}}\bullet\lambda$.

Example 4.3.2. — Let $G = \mathbf{SL}_2$, then

$$\begin{tikzcd} 0 \arrow[r] & L(p) \arrow[r] \arrow[d, equal] & \nabla(p) \arrow[r] \arrow[d, equal] & L(p-2) \arrow[r] & 0 \\ & k\langle x^p, y^p\rangle & k\langle x^p, x^{p-1}y, \dots, xy^{p-1}, y^p\rangle & & \end{tikzcd}$$

(Picture)

4.4. Translation functors Recall

$$\overline{C} = \left\{\lambda \in \mathbb{X} \middle| \forall \alpha \in R_+, 0 \le \left\langle \lambda + \rho, \beta^\vee \right\rangle \le p\right\}.$$

Then $\overline{C}$ is a fundamental domain for W_{aff} acting on $\mathbb{X}$. For $\lambda \in \overline{C}$ we consider the projection

$$\mathrm{pr}_\lambda : \mathrm{Rep}(G) = \prod_{\mu \in \overline{C}} \mathrm{Rep}_{W_{\text{aff}} \bullet \mu}(G) \longrightarrow \mathrm{Rep}_{W_{\text{aff}} \bullet \lambda}(G).$$

Given $\lambda, \mu \in \overline{C}$, denote by ν the unique element in $W(\mu - \lambda) \cap \mathbb{X}_+$ and set

$$\begin{aligned} T_\lambda^\mu : \mathrm{Rep}(G) &\longrightarrow \mathrm{Rep}(G) \\ M &\longmapsto \mathrm{pr}_\mu \left(L(\nu) \otimes \mathrm{pr}_\lambda M\right). \end{aligned}$$

Fact. —
(1) T_λ^μ is exact.
(2) T_λ^μ is left and right adjoint to T_μ^λ.

A subset $I \subset S_{\text{aff}}$ is called **finitary** if $\langle I \rangle \subset W_{\text{aff}}$ is finite. Then

$$\overline{C} = \coprod_{\substack{I \subset S_{\text{aff}} \\ \text{finitary}}} \overline{C}_I$$

where

$$\overline{C}_I = \left\{\lambda \in \overline{C} \middle| \mathrm{Stab}_{(W_{\text{aff}}\bullet)}(\lambda) = \langle I \rangle \right\}.$$

Example 4.4.1. — Let $G = \mathbf{SL}_3$ (Picture)

Theorem 4.4.1. — *If $\lambda, \mu \in \overline{C}_I$ with I finitary, then T_λ^μ and T_μ^λ restrict to quasi-inverse equivalences*

$$\mathrm{Rep}_{W_{\text{aff}} \bullet \lambda}(G) \xrightarrow{\sim} \mathrm{Rep}_{W_{\text{aff}} \bullet \mu}(G).$$

For $I \subset S_{\text{aff}}$ finitary we set

$$W_{\text{aff}}^I = \{w \in W_{\text{aff}} | \forall x \in W, y \in \langle I \rangle, \ell(xwy) = \ell(x) + \ell(w) + \ell(y)\}.$$

Fact. — For $\lambda \in \overline{C}_I$ we have an isomorphism

$$\begin{aligned} W_{\text{aff}}^I &\longrightarrow (W_{\text{aff}} \bullet \lambda) \cap \mathbb{X}_+ \\ w &\longmapsto w \bullet \lambda. \end{aligned}$$

So W_{aff}^I induces the simples/induced modules in $\mathrm{Rep}_{W_{\text{aff}} \bullet \lambda}(G)$. For λ, μ as in the theorem

$$\begin{aligned} T_\lambda^\mu L(w \bullet \lambda) &\simeq L(w \bullet \mu) \\ T_\lambda^\mu \nabla(w \bullet \lambda) &\simeq \nabla(w \bullet \mu). \end{aligned}$$

We have $C_\emptyset \neq \emptyset$ if and only if $p \ge h$ (Coxeter number of G). In this case, $0 \in \overline{C}_\emptyset$. For $\lambda \in \overline{C}_I$, consider

$$T_0^\lambda : \mathrm{Rep}_{W_{\text{aff}} \bullet 0}(G) \longrightarrow \mathrm{Rep}_{W_{\text{aff}} \bullet \lambda}(G).$$

Proposition 4.4.1. —
(1) For $W_{\text{aff}}^\emptyset$ we have

$$T_0^\lambda \nabla(w \bullet 0) = \begin{cases} \nabla(w \bullet \lambda) & \textit{if } w \bullet \lambda \in \mathbb{X}_+ \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *For $y \in W_{\mathrm{aff}}^I$, T_λ^0 has a filtration with subquotients*

$$\{\nabla(yx \bullet 0) | x \in ?\}.$$

(3) *For $w \in W_{\mathrm{aff}}^{\varnothing}$ we have*

$$T_0^\lambda L(w \bullet 0) = \begin{cases} L(w \bullet 0) & \textit{if } w \in W_{\mathrm{aff}}^I \\ 0 & \textit{otherwise}. \end{cases}$$

This reduces the study of $\mathrm{Rep}(G)$ to that of $\mathrm{Rep}_{W_{\mathrm{aff}} \bullet 0}(G)$.