MOTIVIC SHEAVES

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Plan:
L1: Motivation, construction, examples.
L2: Six-functor formalism.
L3: Motivic t-structures and weight structures.
L4: ∞-categorical methods.

§1. Motivation from GRT and cohomology

1.1. Cohomology and sheaves for representation theory
Lecture 1

Question: How do you construct interesting representations?
Answer:

1) Find interesting actions.
2) Linearize them.

Example 1.1.1. — Let K be a compact Lie group. The action of K on itself gives us an action of K on $L^2(K, \mathrm{Haar})$ with respect to a Haar measure. The Peter-Weyl theorem says that

$$L^2(K, \mathrm{Haar}) \simeq \widehat{\bigoplus_{\pi \text{ unitary}}} \pi^{\oplus dim(\pi)}.$$

"Lie theory $\subset$ algebraic geometry". Reductive groups are algebraic groups with many associated varieties with group actions: flag varieties...

The linearizations we consider in this course are the many types of cohomology theories.

Example 1.1.2 (Borel-Weil-Bott). — Let $T \subset B \subset G$ be a reductive group over $\mathbf{C}$. Let $\lambda \in X^\vee(T)$ such that there exists $w \in W$ with $w * \lambda = w(\lambda + \rho) - \rho > 0$ (where $\rho = (1/2)\sum_{\alpha \in \Phi^+} \alpha$). Then

$$\mathrm{R}\Gamma(G/B, L_\lambda) \simeq \pi_{w*\lambda}[-\ell(w)]$$

where $\ell(w)$ is the length of w.

Cohomology fits in the wider context of sheaf theory. If T is a locally contractible topological space, then

$$\mathrm{H}^n_{\mathrm{sing}}(T, \mathbb{Z}) \simeq \mathrm{H}^n(T, \underline{\mathbb{Z}}_T) \simeq \mathrm{R}^n(\pi_T)_*(\underline{\mathbb{Z}}_T)$$

where π_T is the morphism $\pi_T : T \to \mathrm{pt}$ with

$$\mathrm{R}\pi_{T*} : D(T, \mathbb{Z}) \longrightarrow D(\mathbb{Z}) \simeq D(*, \mathbb{Z}).$$

Cohomology (singular with $\mathbb{Q}$-coefficients) of algebraic varieties over $\mathbf{C}$ is ***very*** special.

- There is a weight filtration.
- There is a mixed hodge structure.

Sheaves on complex algebraic varieties are also very special:

- Perverse sheaves;
- Decomposition theorem;
- Mixed Hodge modules.

This leads to great success stories in GRT:

- Springer theory;
- Kazdhan-Lustig theory;
- geometric Satake...

1.2. From sheaves to motivic sheaves There are situations which can't be directly studied using these tools:

- Representation theory of reductive groups over other fields/rings/schemes.
- Modular/integral representation theory.
- q-deformations, quantum groups, canonical bases.

These can be attacked using:

- l-adic sheaves,
- sheaves cohomology with $\mathbb{Z}$-coefficients,
- K-theory.

Motivic sheaves will give us a unified perspective.

Motivic dream: There should exist universal cohomology/sheaf theories such that

1) they unify and "explain" the special structure in cohomology/sheaves;
2) they are of algebro-geometric nature;
3) they "explain" the realization of algebraic cycles and algebraic K-theory.

§2. Construction of $\mathsf{DA}^{\text{ÉT}}$ and SH (Morel-Voevodsky)

2.1. Triangulated categories and localization

Definition. — A **triangulated category** is the data of:

- an additive category C,
- an automorphism $\Sigma = (-)[1] : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$,
- a collection of distinguished triangles
$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

The data satisfies the following conditions:

- shifted distinguished triangles are distinguished up to isomorphism,
- for all $f : A \to B$ there exists
$$A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \xrightarrow{+}$$
unique up to isomorphism and functorial,
-
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f", "\simeq"'] & B \arrow[r] \arrow[d, "g", "\simeq"'] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "f[1]", "\simeq"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$
- (??)

Remark. — In modern language triangulated categories are replaced by stable ∞-categories. If C is a stable ∞-category, the homotopy category $h\mathsf{C}$ has a canonical structure of triangulated category. The reader who is familiar with this language can assume all triangulated categories to be stable ∞-categories with minimal changes.

Example 2.1.1. — Let A be an abelian category, $\mathsf{Ch}(\mathsf{A})$ be the abelian category of chain complexes in A. We define $(A[1])_n = A_{n-1}$. Given $f : A_\bullet \to B_\bullet$ the maping cone is given by
$$\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n, \qquad d_n = \begin{pmatrix} -d^A_{n-1} & 0 \\ f & d^B_n \end{pmatrix}.$$

Definition. — $f : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if for all $n \in \mathbb{Z}$, the map $H_n(A_\bullet) \simeq H_n(B_\bullet)$ is an isomorphism.

Definition. — $D(\mathsf{A})$ is defined as the localization of $\mathsf{Ch}(\mathsf{A})$ by quasi-isomorphisms.

Now we consider reflexive localizations[1] (1-categorical ones lead to triangulated and ∞-categorical ones).

[1] In [Lur09] these localizations are simply called **localizations**.

Definition. — Let C be a category (1 or ∞).

1) A full subcategory $\mathsf{C}' \subset \mathsf{C}$ is **reflexive** if $\iota : \mathsf{C}' \to \mathsf{C}$ has a left adjoint.
2) $L_W : \mathsf{C} \to \mathsf{C}[W^{-1}]$ is **reflexive** if L_W has a right adjoint.

Lemma 2.1.1 (Reflexive subcategories are the same thing as reflexive localizations). —

a) Let $\mathsf{C}' \subset \mathsf{C}$ be reflexive, $L : \mathsf{C} \to \mathsf{C}'$ be the left adjoint to ι. Define

$$W_L = \{f : L(f) \text{ is an isomorphism}\}.$$

Then $\mathsf{C}' \simeq \mathsf{C}[W_L^{-1}]$ and $L \simeq L_{W_L}$.

b) If L is a reflexive localization, then its right adjoint ι is fully faithful and $\iota : \mathsf{C}[W^{-1}] \xrightarrow{\sim} \mathrm{EssIm}(\iota) \subset \mathsf{C}$.

Definition. — Let $S \subset \mathsf{C}$ be a collection of morphisms.

a) $A \in \mathsf{C}$ is S-**local** if for all $f : B \to C$ in S

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

b) $f : B \to C$ is an S-**equivalence** if for all S-local A

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

Lemma 2.1.2. — *If $L : \mathsf{C} \rightleftarrows \mathsf{C}' : \iota$ is a reflexive localization, W_L as before, then*

- *ι gives an isomorphism between C' and W_L-local objects.*
- *W_L are the W_L-equivalences.*

Definition. — Let D be a triangulated category with all small coproducts.

- Let κ be a regular cardinal (for example $\kappa = \aleph_0$). Then $A \in \mathsf{D}$ is κ**-small**/κ**-compact** if and only if

$$\operatorname*{colim}_{\substack{I' \subset I \\ |I'| < \kappa}} \mathrm{Hom}\left(A, \bigoplus_{I'} B_i\right) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_I B_i\right).$$

- **Compact** means $\aleph_0$-small. A is compact if and only if

$$\bigoplus_I \mathrm{Hom}(A, B_i) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_I B_i\right).$$

- D is **presentable**/**well-generated** if and only if there exist κ and a set $S \subset \mathsf{D}$ of κ-small objects which generate D:

$$\forall B \in \mathsf{D}, (\forall A \in S, \mathrm{Hom}(A, B) = 0) \implies B = 0.$$

- D is **compactly generated** if it is $\aleph_0$-presentable.

Remark. — More generally, one defines compact objects as those whose covariant hom-functor commutes with filtered colimits. When C is triangulated or stable, it is equivalent to the definition given above.
The ∞-category of presentable ∞-categories, with colimit-preserving (and hence left-adjoint) functors, is denoted Pr^L; it admits a closed symmetric monoidal structure. See [Lur09, §5.5].

Definition. — $\mathsf{E} \subset \mathsf{D}$ is **localizing** if it is

- triangulated,
- stable under coproducts,
- thick (stable under subobjects and subquotients).

Theorem 2.1.1 (Adjoint Functor Theorem). — *Let D, D' be triangulated categories with all coproducts, $F : \mathsf{D} \to \mathsf{D}'$ be a triangulated functor and D be presentable. Then F admits a right adjoint if and only if F preserves all coproducts.*

Corollary 2.1.1 (Verdier Localization). — *Let D be a presentable category and E be a localizing subcategory. Define*

$$\mathsf{D}/\mathsf{E} = D[W_{\mathsf{E}}^{-1}], \quad W_{\mathsf{E}} = \{f : \mathrm{Cone}(f) \in \mathsf{E}\}.$$

Then $\mathsf{D} \to \mathsf{D}/\mathsf{E}$ is a reflexive localization.

Let $S \subset \mathsf{D}$ be a subset of objects, then $\langle\langle S \rangle\rangle$ is the smallest subcategory containing S such that $\mathsf{D}/\langle\langle S \rangle\rangle$ is a reflexive localization.

Let's get some inspiration from Betti homology, i.e. singular homology on complex algebraic varieties. Let $X \in \mathsf{Var}_{\mathbf{C}}^{\text{(f.t)}}$, then we get

$$C_*^{\text{sing}}(X(\mathbf{C}), \mathbb{Z}) \in D(\mathbb{Z}).$$

This comes with some data:

(a)
- $D(\mathbb{Z})$ has a symmetric monoidal structure: $\otimes^{\mathbb{Z}}$,
- (Künneth) $C_*(X \times Y) \simeq C_*(X) \otimes^{\mathbb{Z}} C_*(Y)$.

which satisfies sproperties:

(b) ($\mathbb{A}^1$-homotopy invariance) $C_*(X \times \mathbb{A}^1) \xrightarrow{\sim} C_*(X)$ ($(\mathbb{A}^1)^{\text{an}} = \mathbf{C}$ is contractible).

(c′) (Mayer-Vietoris sequence) Let $X = U \cup V$ be a Zariski cover, then

$$C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*(X) \xrightarrow{\partial} C_*(U \cap V)[1].$$

(c) (Étale descent) Let $U \to X$ be étale surjective. The **Čech nerve** $\check{C}_\bullet(U/X)$ of $U \to X$ is a simplicial scheme $\mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Sch}$ whose simplices are given by

$$\check{C}_n(U/X) = U^{\times_X n+1}$$

and whose morphisms are induced by the universal property of fibre product.
Composition with C_* yields a simplicial complex of abelian groups

$$C_*(\check{C}_\bullet(U/X))) : \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Sch} \xrightarrow{C_*} \mathsf{Ch}(\mathbb{Z}),$$

and we consider the **homotopy colimit**

$$\operatorname{hocolim} C_*(\check{C}(U/X)).$$

It can be explicitly constructed as follows: to the simplicial complex we can naturally associate a double complex of abelian groups $C(C_*(\check{C}_\bullet(U/X)))$, and then we have

$$\operatorname{hocolim} C_*(\check{C}(U/X)) \simeq \mathsf{Tot}^{\oplus} C(C_*(\check{C}_\bullet(U/X))).$$

Then the canonical map

$$\operatorname{hocolim} C_*(\check{C}(U/X)) \longrightarrow C_*(U/X)$$

is a quasi-isomorphism of chain complexes.
Concretely we have a descent spectral sequence which gives us ($U = U \cup V$) Mayer Vietoris.

(d) ($\mathbb{P}^1$-stabilization) We have

$$\begin{aligned} C_*(\mathbb{P}^1_{\mathbf{C}}) &\simeq C_*(\text{pt}) \oplus \widetilde{C}_*(\mathbb{P}^1_{\mathbf{C}}) \\ &\simeq \mathbb{Z}[0] \oplus \mathbb{Z}(1)[2], \end{aligned}$$

and $\mathbb{Z}(1) \simeq \mathbb{Z}$ is $\oplus$-invertible.

Smooth varieties play a special role:

- Poincaré duality
- Gysin sequences
- $C_*(-)$ also satisfies "h-descent", so $C_*(-)$ is "determined" by $C_*(-)_{|\mathsf{Sm}}$.

There is an associated sheaf theory:

$$D_B(-) : \mathsf{Var}_{\mathbf{C}} \longrightarrow \mathsf{TriCat}^{\otimes}, \quad X \longmapsto D_B(X) = D(\mathsf{Sh}(X^{\text{an}}, \mathbb{Z})).$$

Sketch of $\mathsf{DA}^{ét}$*:* Let S be a base scheme.

- Start with
$$\begin{cases} D(\mathsf{PSh}(\mathsf{Sm}_S, \mathbb{Z})) = D_{\mathsf{PSh}}(S) \\ \mathbb{Z}[-] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}.$$
- Impose $\mathbb{A}^1$-invariance, étale descent, and $\mathbb{P}^1$-stability. This will give us $\mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$ and $M_S(-) : \mathsf{Sm}_S \to \mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$.

The surprise is that the result satisfies many *other* properties of singular (co)homology and derived categories of sheaves on complex varieties which can largely be packaged into the six-functor formalism. The result is closely related to algebraic cycles/higher Chow groups and algebraic K-theory. Lecture 2

(Fill in H_* from the recall part)

Let S be a qcqs scheme, Λ be a coefficient ring. Define

$$\begin{cases} D_{\mathsf{PSh}}(S) := D(\mathsf{PSh}(\mathsf{Sm}_S, \Lambda)) \text{ a presentable, symmetric monoidal, triangulated category} \\ \Lambda[\cdot] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}$$

Étale descent:

$$\begin{aligned} D_{\text{ét}}(S) &:= D(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S, \Lambda)) \\ &= D_{\mathsf{PSh}}(S)[W_{\text{ét}}^{-1}] \end{aligned}$$

where $W_{\text{ét}}$ are étale-local weak equivalences, i.e. $(f : K_\bullet \to L_\bullet) \in W_{\text{ét}}$ if for all n we have

$$a_{\text{ét}}\mathrm{H}_n(K_\bullet) \xrightarrow{\sim} a_{\text{ét}}\mathrm{H}_n(L_\bullet),$$

and where $a_{\text{ét}}$ is the étale sheafification functor.

$\mathbb{A}^1$-*invariance* Let

$$I_{\mathbb{A}^1,(\text{ét})} = \left\{ \cdots \longrightarrow 0 \longrightarrow \Lambda_{(\text{ét})}[X \times \mathbb{A}^1] \longrightarrow \Lambda_{(\text{ét})}[X] \longrightarrow 0 \longrightarrow \cdots | X \in \mathsf{Sm}_S \right\}.$$

Define

$$D_{\mathbb{A}^1}(S) := D_{\mathsf{PSh}}(S) / \langle\langle I_{\mathbb{A}^1} \rangle\rangle = D_{\mathsf{PSh}}(S)[W_{\mathbb{A}^1}^{-1}].$$

We have

$$L_{\mathbb{A}^1} : D_{\mathsf{PSh}}(S) \longrightarrow D_{\mathsf{PSh}}(S)^{\mathbb{A}^1-\text{loc}} \hookrightarrow D_{\mathsf{PSh}}(S).$$

with the middle term isomorphic to $D_{\mathbb{A}^1}(S)$.

Definition. — Define

$$\mathbf{\Delta}^n_{\text{alg},S} := \mathrm{Spec}_S \left(\mathscr{O}_S[X_0, \ldots, X_n] / \left(\sum x_i - 1 \right) \right) \simeq \mathbb{A}^n_S$$

then $\mathbf{\Delta}^\bullet_{\text{alg},S}$ is a cosimplicial scheme over S.

Definition (Suslin-Voevodsky). — Let $K_\bullet \in \mathsf{Ch}(\mathsf{PSh}(\mathsf{Sm}_S, \Lambda))$, we define

$$\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) = \mathrm{hocolim}_{\Delta^{\mathrm{op}}} K_\bullet(\mathbf{\Delta}^\bullet_{\text{alg},S} \times_S X)$$

Example 2.1.2. — Let $F \in \mathsf{PSh}$ then

$$\mathrm{Sing}^{\mathbb{A}^1}(F)(U) = \left[\ldots \longrightarrow F(\mathbf{\Delta}^2 \times U) \longrightarrow F(\mathbb{A}^1 \times U) \xrightarrow{i_0^* - i_1^*} F(U) \longrightarrow 0 \right].$$

Proposition 2.1.1. — $L_{\mathbb{A}^1} \simeq \mathrm{Sing}^{\mathbb{A}^1}$.

Proof. The idea is to use

$$\begin{aligned} m : \mathbb{A}^1 \times \mathbb{A}^1 &\longrightarrow \mathbb{A}^1 \\ (x, y) &\longmapsto xy \end{aligned}$$

to prove

a) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet)$ is $\mathbb{A}^1$-local.

b) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) \to K_\bullet$ is $\mathbb{A}^1$-weak equivalence.

□

Definition. — The category of **effective étale motivic sheaves** on S is

$$\mathsf{DA}^{\text{ét,eff}}(S, \Lambda) := D_{\text{ét}}(S) / \left\langle\left\langle I_{\mathbb{A}^1,\text{ét}} \right\rangle\right\rangle.$$

Write $L^{\text{eff}}_{\text{mot}}$ for the associated localization functor.

Lemma 2.1.3. — *We have*

$$L_{mot}^{eff} = \underbrace{\cdots \mathrm{Sing}^{\mathbb{A}^1} L_{\acute{e}t} \mathrm{Sing}^{\mathbb{A}^1}}_{\textit{transfinite composition}}$$

Definition. — Let $X \in \mathsf{Sm}_S$. Define

$$M_S^{\text{ét,eff}}(X) := L_{\text{mot}}^{\text{eff}} \Lambda_{\text{ét}}[X] \in \mathsf{DA}^{\text{eff,ét}}(S, \Lambda)$$

(effective étale (relative homological) motive/motivic sheaf of X/S).
Here $\Lambda_{\text{ét}}$ is given by the composition

$$\mathsf{Sm}_S \xrightarrow{\Lambda[-]} \mathsf{PSh}(\mathsf{Sm}_S, \Lambda) \xrightarrow{a_{\text{ét}}} \mathsf{Sh}(\mathsf{Sm}_S, \Lambda).$$

We have

$$M_S^{\text{ét,eff}}(X \times_S Y) \simeq M_S^{\text{ét,eff}}(X) \otimes M_S^{\text{ét,eff}}(Y).$$

Proposition 2.1.2 (Artin-Shreier $+\Lambda\left[\frac{1}{p}\right]$). — *Let S be a $\mathbf{F}_p$-scheme, then*

$$\mathsf{DA}^{\acute{e}t,eff}(S, \Lambda) \xrightarrow{\sim} \mathsf{DA}^{\acute{e}t,eff}\left(S, \Lambda\left[\frac{1}{p}\right]\right).$$

Proof. We have the short-exact sequence

$$0 \longrightarrow \Lambda/p\Lambda \longrightarrow \mathbb{G}_a \otimes \Lambda \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbb{G}_a \longrightarrow 0$$

and hence an exact sequence

$$\Lambda_{\text{ét}}[\mathbb{G}_a] \otimes (\mathbb{G}_a \otimes \Lambda) \xrightarrow{a_{\mathbb{G}_a} \otimes \mathrm{id}} \mathbb{G}_a \otimes \mathbb{G}_a \otimes \Lambda \xrightarrow{m} \mathbb{G}_a \otimes \Lambda.$$

(Some remark??) Thus

$$L_{\mathbb{A}^1}(\mathbb{G}_a \otimes \Lambda) \simeq \Lambda(0)$$

and so

$$L_{\text{mot}}^{\text{eff}}(\Lambda/p\Lambda) = 0.$$

□

$\mathbb{P}^1$-*stabilization:* Let $x \in X(S)$, we have

$$M_S^{\text{eff}}(X) = \Lambda_S(0) \oplus M_S^{\text{eff}}(X, x).$$

Definition. — We define

$$T := M_S^{\text{eff}}(\mathbb{P}^1, 1)[-2]$$

which is equal to $\Lambda(1)$.

Exercise. — Any $x \in \mathbb{P}^1_S(S)$ gives the same decomposition.

We have a problem: T is not $\otimes$-invertible.

Definition. — The category of étale motivic sheaves over S is

$$\mathsf{DA}^{\text{ét}}(S, \Lambda) := \mathsf{DA}^{\text{ét,eff}}(S, \Lambda)[T^{\otimes -1}].$$

This is not a proper definition since we have not explained the construction. It's not clear what this means!

Spectra:

Definition. — Let C be a closed, symmetric monoidal 1-category and T be an object of C. A T**-prespectrum** A is given by the following datum:

$$A = \left\{(A_n, \sigma_n)_{n \in \mathbb{N}} | A_n \in \mathsf{C}, \sigma_n : T \otimes A_n \longrightarrow A_{n+1}\right\}.$$

By the adjunction $\otimes \dashv \underline{\mathrm{Hom}}$ and the fact that for each object $A \in \mathsf{C}$ we have $A \simeq \underline{\mathrm{Hom}}(\mathbb{1}, A)$, the datum of the maps σ_n is equivalent to the datum of maps

$$A_n \longrightarrow \underline{\mathrm{Hom}}(T, A_{n+1}).$$

A is a T**-spectrum** if for all $n \in N$ these maps are isomorphisms:

$$A_n \xrightarrow{\sim} \underline{\mathrm{Hom}}(T, A_{n+1}).$$

We write $\mathrm{PSp}_T(\mathsf{C})$ and $\mathrm{Sp}_T(\mathsf{C})$ for the T-prespectrum and T-spectrum respectively.

The evaluation map
$$\mathrm{Ev}_n(A) = A_n$$
has a left adjoint Σ_T^∞. To define it, put
$$\mathrm{Sus}^n(A)_m = \begin{cases} \varnothing & \text{if } m < n \\ T^{\otimes(m-n)} \otimes A & \text{if } m > n \end{cases}$$
and $\Sigma_T^\infty := \mathrm{Sus}^0$ is the ∞-suspension functor.

Proposition 2.1.3. — *Assume* C *is presentably symmetrical monoidal*[2]. *Then* $\mathrm{Sp}_T(\mathsf{C}) \subset \mathrm{PSp}_T(\mathsf{C})$ *is a reflexive subcategory.* W_{st} *is generated by*
$$\left\{\mathrm{Sus}^{n+1}(T \otimes A) \longrightarrow \mathrm{Sus}^n(A) : n \in \mathbb{N}, A \in \mathsf{C}\right\}.$$

Definition. — We define
$$\mathsf{DA}^{\text{ét}}(S,\Lambda) := \mathrm{Sp}_T\,\mathsf{DA}^{\text{eff,ét}}(S,\Lambda).$$

(This definition is correct "with ∞-categories".) We have
$$\begin{aligned} M_S : \mathsf{Sm}_S &\longrightarrow \mathsf{DA}^{\text{ét}}(S,\Lambda) \\ X &\longmapsto L_{(\mathbb{A}^1,\text{ét},?}\Sigma_T^\infty M_S^{\text{ét,eff}}(X). \end{aligned}$$

Remark. — $M \in \mathsf{DA}^{\text{ét}}(S,\Lambda)$ is isomorphic to a stable $(\mathbb{A}^1, \text{ét})$-local $\mathbb{P}^1$-prespectrum
$$\left\{K_n \in \mathsf{Ch}(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S,\Lambda))\ ;\ \sigma_n = \Lambda_{\text{ét}}[\mathbb{P}^1, 1] \otimes K_n \longrightarrow K_{n+1}\right\}$$
such that for all $X \in \mathsf{Sm}_S, i \in \mathbb{Z}$

- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X \times_S \mathbb{A}^1, K_n)$
- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^{i+2}_{\text{ét}}(X \times_S (\mathbb{P}^1, 1), K_{n+1})$.

2.2. Constructible motivic sheaves

Definition. — We define **constructible motivic sheaves**
$$\begin{aligned} \mathsf{DA}^{\text{ét}}_{\text{ct}} &= \langle M_S(X)(-n) | X \in \mathsf{Sm}_S, n \in \mathbb{N}\rangle^{\text{d.f.}} \\ &\subset \mathsf{DA}^{\text{ét}}(S,\Lambda). \end{aligned}$$
and **locally constructible motivic sheaves**
$$\mathsf{DA}^{\text{ét}}_{\text{lct}}(S,\Lambda) := \{M | \exists e : U \twoheadrightarrow S, e^*M \in \mathsf{DA}_{\text{ct}}\}.$$

There is a Betti realization for S finite type over $\mathbf{C}$
$$R_B : \mathsf{DA}^{\text{ét}}(S,\Lambda) \longrightarrow D(S^{\text{an}},\Lambda)$$
by the existence of relative homology and the universal property. If $X \in \mathsf{Sm}_S$
$$R_B(M_S(X)) \simeq \mathrm{H}^{\text{sing}}_*(X/S)$$
and
$$R_B(\mathsf{DA}^{\text{ét}}_{\text{lct}}(S,\Lambda) \subset D^b_{\text{ct}}(S^{\text{an}},\Lambda).$$

Another deep property is the *rigidity theorem*. Define
$$D_{\text{ét}}(S,\Lambda) = D(\mathsf{Sh}_{\text{ét}}(S,\Lambda))$$
and write
$$\iota : (\mathsf{Et}_S, \text{ét}) \hookrightarrow (\mathsf{Sm}_S, \text{ét})$$
for the inclusion, then we get
$$\iota_S^* : D_{\text{ét}}(S,\Lambda) \longrightarrow \mathsf{DA}^{\text{ét}}(S,\Lambda).$$

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[2]i.e., it is presentable symmetric monoidal and the tensor product bifunctor $-\otimes- : \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ preserves colimits separately in each variable.

Theorem 2.2.1 (Ayoub). — *Let S be an excellent, Noetherian, finite dimensional, Λ-finite scheme, with any prime invertible in Λ or $\mathscr{O}_S$. Then ι_S^* is an equivalence.*

This procedure is very flexible and admits many *variants*.

- We can change the input. Instead of complexes we can work with presheaves of simplicial sets or ∞-groupoids.
- We can change the topology. Instead of étale use Nisnevich or use presheaves with transfers.
- Change the geometric context. For example, to rigid analytic motivic sheaves.

Definition. — The **stable motivic homotopy category** over S is

$$\mathsf{SH}(S) := \mathrm{PSp}_T\left(\mathsf{PSh}(\mathsf{Sm}_S, \mathsf{sSet})\right)\left[W^{-1}_{(\mathbb{A}^1,\mathrm{Nis},\mathbb{P}^1)}\right].$$

Recall that the Nisnevich topology is between the Zariski and étale topologies. $\mathsf{DA}^{\text{ét}}(S)$ is the motivic version of $D(S^{\mathrm{an}}, \mathbb{Z})$ and $\mathsf{SH}(S)$ is the motivic version of sheaves of S^1-spectra on S^{an}. There is also $\mathsf{DM}(S, \Lambda)$ which are the Nisnevich-local motivic sheaves. Many important invariants of varieties only satisfy Nisnevich descent, but not étale descent; for example, K-theory or higher Chow groups.

§3. Motives over a field

Let $S = \mathrm{Spec}(k)$ and $\Lambda = \mathbb{Q}$. Define

$$\mathsf{DM}(k, \mathbb{Q}) := \mathsf{DA}^{\text{ét}}(k, \mathbb{Q}).$$

The analogies you should have in mind are

- $D(\mathrm{Ind}\,\mathsf{MHS}_{\mathbb{Q}})$,
- $D(\mathrm{Ind}\,\mathsf{Rep}^{\mathrm{f.d.}}_{\mathbb{Q}_l} G_k)$.

Even though the construction was very formal there are some surprises. Define

$$\mathbb{Q}\langle i\rangle := \mathbb{Q}(i)[2i]$$

be pure Tate twists and $M\langle i\rangle := M \otimes \mathbb{Q}\langle i\rangle$.

- *Projective bundle formula:* Let $E \to X$ be a vector bundle, then
$$M(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{\mathrm{rank}\,E-1} M(X)\langle i\rangle$$
$$M(\mathbb{P}^n_l) = \Lambda(0) \oplus \Lambda\langle 1\rangle \oplus \cdots \oplus \Lambda\langle n\rangle.$$
- *Gysin triangle:* Let $(c : Z \not\hookrightarrow X) \in \mathsf{Sm}_k$, then we have
$$M(X \setminus Z) \longrightarrow M(X) \longrightarrow M(Z)\langle c\rangle \xrightarrow{+}$$
- *Smooth blow-up formula:*
$$M(\mathrm{Bl}_Z(X)) \simeq M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)\langle i\rangle.$$
- *Poincaré duality 1*: Let X be smooth and projective over k, then $M(X)$ is *dualizable* with
$$M(X)^\vee \simeq M(X)\langle -\dim(X)\rangle.$$

We have $\mathsf{DM}(k, \mathbb{Q}) \simeq \mathrm{Ind}\,\mathsf{DM}_{\mathrm{ct}}$.

From here on out

$$\mathsf{DM}(k, \Lambda) = \begin{cases} \mathsf{DA}^{\text{ét}}(k, \Lambda) & \Lambda \text{ a } \mathbb{Q}\text{-algebra} \\ \mathsf{DM}(k, \Lambda) & \Lambda \text{ a } \mathbb{Z}\left[\frac{1}{p}\right]\text{-algebra.} \end{cases}$$

For singular varieties $X \in \mathsf{Sch}^{\mathrm{ft,sep}}_R$ we get $M(X) \in \mathsf{DM}(k, \Lambda)$. There are four theories

(i) $M(X)$,
(ii) Borel-Moore cohomoloy $M_{\mathrm{BM}}(X)$ (also denoted $M^c(X)$ in the literature),
(iii) $M^{\mathrm{coh}}(X)$,
(iv) $M^{\mathrm{coh}}_{\mathrm{c}}(X)$.

Localization: Consider a closed immersion $Z \hookrightarrow X$ and the open immersion $X \setminus Z \hookrightarrow X$. We have

$$M_{\mathrm{BM}}(Z) \longrightarrow M_{\mathrm{BM}}(X) \longrightarrow M_{\mathrm{BM}}(?) \xrightarrow{+}$$

$$M_{\mathrm{c}}^{\mathrm{coh}}(X \setminus Z) \longrightarrow M_{\mathrm{c}}^{\mathrm{coh}}(X) \longrightarrow M_{\mathrm{c}}^{\mathrm{coh}}(Z) \xrightarrow{+}$$

Poincaré duality 2: For $X \in \mathsf{Sm}_k$

$$\begin{cases} M(X)^\vee \simeq M_{\mathrm{BM}}(X)\langle -d\rangle \\ M^{\mathrm{coh}}(X)^\vee \simeq M^{\mathrm{coh}}(X)\langle d\rangle . \end{cases}$$

If $M \in \mathsf{DM}(k)$, then M is dualizable if and only if it is constructible, if and only if it is compact.

3.1. Motivic cohomology and algebraic cycles

Definition. — Let $X \in \mathsf{Sm}_k$, we define the **Motivic cohomology groups**

$$\begin{aligned} \mathrm{H}^{p,q}_{\mathrm{mot}}(X) = \mathrm{H}^{p}_{\mathrm{mot}}(X, \Lambda(q)) &:= \mathrm{Hom}_{\mathsf{DM}(k,\Lambda)}(M(X), \Lambda(q)[p]) \\ &\simeq \mathrm{Hom}_{\mathsf{DM}(X,\Lambda)}(\Lambda_X(0), \Lambda_X(q)[p]). \end{aligned}$$

For $X \in \mathsf{Sch}_k^{\mathrm{ft,sep}}$ define

$$\mathrm{H}^{\mathrm{BM}}_{p,q} := \mathrm{Hom}(\Lambda(q)[p], M_{\mathrm{BM}}(X)).$$

3.1.1. Weight 1 motivic cohomology

Lemma 3.1.1. — *We have*

$$M_S^{eff}(\mathbb{G}_m) \simeq \Lambda(0) \oplus \Lambda(1)[1].$$

Proof. $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$, so by Mayer-Vietoris we get

$$M(\mathbb{G}_m) \longrightarrow M(\mathbb{A})^{\oplus 2} \longrightarrow M(\mathbb{P}^1) \xrightarrow{+}$$

hence by $\mathbb{A}^1$-invariance

$$M(\mathbb{G}_m, 1) \simeq M(\mathbb{P}^1, 1)[-1].$$

□

The map $\alpha_{\mathbb{G}_m} : \Lambda_{\text{ét}}[\mathbb{G}_m] \to \mathbb{G}_m \otimes \Lambda$ induces

$$\Lambda(1)[1] \xrightarrow{(*)} \Sigma^\infty_T(\mathbb{G}_m \otimes \Lambda).$$

Theorem 3.1.1. —

1) (*) is an isomorphism, so

$$\mathrm{Pic}(s) \otimes \Lambda \xrightarrow{c_1} \mathrm{H}^{2,1}_{mot}(S)$$

2) For S regular

$$\mathrm{H}^{n,1}_{mot}(S) = \begin{cases} \mathscr{O}_S^\times \otimes \Lambda & n = 1 \\ \mathrm{Pic}(S) \otimes \Lambda & n = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

3.1.2. Higher Chow groups Let $\mathbf{\Delta}^\bullet_{\mathrm{alg},k} \in (\mathsf{Sm}_k)^{\mathbf{\Delta}}$.

Definition. — Let $X \in \mathsf{Sch}^{\mathrm{ft}}_k$ define

$$\mathfrak{z}_n(X,r) \subseteq Z_n(X \times \mathbf{\Delta}^r_{\mathrm{alg}}) \otimes \Lambda$$

generated by integral subvarieties of dimension n which intersect all faces properly.

(Picture) We get $d : \mathfrak{z}_n(X,r) \to \mathfrak{z}_{n-1}(X,r-1)$ so $\mathfrak{z}_n(X,\bullet)$ is a chain complex, called *Bloch cycle complex*. We define the higher Chow groups as the homology of this chain complex:

$$\mathrm{CH}_n(X,r) := \mathrm{H}_r(\mathfrak{z}_n(X,\bullet)).$$

Theorem 3.1.2 (Voevodsky+...). — *Let k be perfect, $X \in \mathsf{Sch}^{ft,sep}_k$ then*

$$\mathrm{H}^{BM}_{p,q}(X) \simeq \mathrm{CH}_q(X, p-2q, \Lambda).$$

If $X \in \mathsf{Sm}_k$ then

$$\mathrm{H}^{p,q}_{mot}(X) \simeq \mathrm{CH}^q(X, 2q-p, \Lambda)$$
$$\mathrm{H}^{2n,n}_{mot}(X) \simeq \mathrm{CH}^n(X, \Lambda).$$

Corollary 3.1.1. —

$$\mathrm{H}^{p,q}_{mot} = 0 \quad \textit{if} \begin{cases} p > 2q \\ q > \min\{p, \dim X\} \end{cases}$$

3.2. Examples (Tate)

Definition. — Define

$$\mathsf{DMT}(k,\Lambda) = \langle \Lambda(n) | n \in \mathbb{Z} \rangle^{\mathrm{df}}$$

the **mixed Tate motives**. It constains $\bigoplus \Lambda \langle i \rangle^{\oplus n_i}$ the **pure Tate motives**.

We have $M(\mathbb{A}^n) = \Lambda(0)$ and $M_{\mathrm{BM}}(\mathbb{A}^n) = \Lambda\langle n \rangle$.

Exercise. — Show that

$$M(\mathbb{A}^n \setminus \{0\}) \simeq \Lambda(0) \oplus \Lambda(n)[2n-1].$$

3.2.1. Cellular varieties

Definition. — $X \in \mathsf{Sch}^{\mathrm{ft}}_k$ is **cellular** if there exists a closed subscheme $Z \hookrightarrow X$ such that $X \setminus Z \xrightarrow{\sim} \mathbb{A}^i_k$ and Z is cellular.

Proposition 3.2.1. — *Suppose X is cellular:*

a) *We have*

$$M_{BM}(X) \simeq \bigoplus_{i=0}^{d} \Lambda \langle i \rangle^{n_i},$$

where n_i is the number of cells of dimension i.

b) *If X is also smooth*

$$M(X) \simeq \bigoplus_{j=0}^{d} \Lambda \langle j \rangle^{m_j},$$

where m_j is the number of cells of codimension j.

Example 3.2.1. —

1) Let G be split reductive, $B \subset G$ be a Borel, then G/B is cellular (Bruhat decomposition) and

$$M(G/B) \simeq \bigoplus_{i \geq 0} \Lambda \langle i \rangle^{n_i}$$

where n_i is the number of $w \in W$ of length i.

2) Let X be quasiprojective and smooth, with a (right?) action of G such that $\forall x \in X$, $\lim_{t\to 0} tx$ exists. Then (??)

3.2.2. Reductive groups

Theorem 3.2.1 (Biglami). — *If G is split reductive, then*
$$M(G) \simeq \mathrm{Sym}^* \left(\bigoplus_{i \geq 1} \Lambda(i)[2? - i]^{\oplus n_i} \right)$$
so by Chevalley
$$R[\mathfrak{g}]^G = k[q_1, \dots, q_{\mathrm{rk}\, G}$$
where $\deg q_j = d_j$ *and* n_i *is the number of* j *such that* $d_j = i$.

Example 3.2.2. — We have
$$\begin{aligned} M(\mathbf{GL}_n) &= \mathrm{Sym}^* \left(\Lambda(1)[1] \oplus \Lambda(2)[3] \oplus \cdots \oplus \Lambda(n)[2n-1] \right) \\ M(\mathbf{SL}_n) &= \times(??) \end{aligned}$$

Exercise. — What is $M(\mathrm{Sp}_{2n})$?

3.3. Examples (non-Tate)

3.3.1. Curves

Proposition 3.3.1. — *Let C be a smooth projective curve with a 0-cycle (with Λ-coefficients) of degree 1 (or if Λ is a $\mathbb{Q}$-algebra)*
$$M(C) \simeq \Lambda(0) \oplus M_1(C) \oplus \Lambda \langle 1 \rangle .$$
If $g(C) > 0$ then $M_n(C) \notin \mathsf{DMT}(k, \Lambda)$.

3.3.2. Commutative algebraic groups

Theorem 3.3.1 (?). — *We take $\Lambda = \mathbb{Q}$ and G/k a smooth commutative group (e.g. a (semi-)abelian variety). Define*
$$M_1(G) := \Sigma^\infty_T (G \otimes \mathbb{Q}) \in \mathsf{DM}(k, G).$$
Then
$$M(G) \simeq \left(\bigoplus_{i=0}^{?} \mathrm{Sym}_i(M_1(G)) \right) \otimes M(\pi_0(G/R)]).$$

Example 3.3.1. —
$$M_1(C) = M_1(\mathrm{Jac}(C))$$

§4. Six functor formalism

4.1. Betti sheaves

Definition. — Define
$$\begin{aligned} D_B(-) : \mathsf{Var}_{\mathbf{C}}^{\mathsf{op}} &\longrightarrow \mathsf{TriCat}^{\otimes} \quad (\text{better } \mathsf{CAlg}(\mathrm{Pr}^L)) \\ X &\longmapsto D\left(\mathsf{Sh}(X^{\mathrm{an}}, \Lambda)\right) \\ f &\longmapsto f^* = \mathrm{L}f^* \quad \text{pullback} \end{aligned}$$

D_B is symmetric monoidal
$$f^*(F \otimes G) \simeq f^*(F) \otimes f^*(G)$$
(note that we write $\otimes = \otimes^{\mathbb{L}}$).

Proposition 4.1.1. — *$(f^*, f_* = \mathrm{R}f_*)$ is an adjoint pair. And $D_B(X)$ is closed, i.e. there exists $\underline{\mathrm{Hom}}(F, G)$.*

Definition. — A **sheaf theory** is a symmetric monoidal functor
$$D(-) : (\mathsf{Sch}_S^{\mathrm{ft}})^{\mathsf{op}} \longrightarrow \mathsf{TriCat}^{\otimes} / \mathsf{CAlg}(\mathrm{Pr}^L)$$

So we have four functors $(\otimes, \underline{\mathrm{Hom}})$ and (f^*, f_*) which form adjoint pairs.

Example 4.1.1. —

- Derived categories of étale/l-adic sheaves.
- Derived categories of (holomorphic) D-modules.

- Derived categories of mixed Hodge modules.
- $\mathsf{DA}^{\text{ét}}(-)$, $\mathsf{SH}(-)$, ...
- $D(\mathsf{QCoh}(-))$.

Let $f : Y \to X$ be separated of finite type, then we have two functors $f_! : D_B(Y) \leftrightarrows D_B(Y) : f^!$ and $f_!$ gives us relative cohomology with compact support.

These satisfy a bunch of natural transformations:

- *Base change:* Let
$$\begin{tikzcd} Y' \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$
be Cartesian, then we get a natural transformation
$$f^* g_*(-) \longrightarrow \widetilde{g}_* \widetilde{f}^*(-). \quad \text{(BC)}$$
- *Projection:* We have a natural transformation
$$f_*(-) \otimes - \longrightarrow f_*(- \otimes f^*(-)). \quad \text{(Proj)}$$
- *Künneth:* We have the natural transformation
$$f_*(-) \otimes g_*(-) \longrightarrow (f \times g)_*(- \boxtimes_X -) \quad \text{(Kü)}$$
where $\boxtimes_X := \mathrm{pr}_1^*(-) \otimes \mathrm{pr}_2^*(-)$.

In general none of these natural transformations is an isomorphism. One goal of the six functor formalism is to study how much they fail to be isomorphisms.

Theorem 4.1.1. — *Let $D = D_B$.*

- *If g is proper, then (BC) is an isomorphism.*
- *If f is proper, then (Proj) is an isomorphism.*
- *If f and g are both proper, then (Kü) is an isomorphism.*

Proposition 4.1.2 (Open base change). — *Assume f is an open immersion. Then (BC) is an isomorphism.*

Definition. — Let $f : Y \to X$ be separated of finite type and $F \in \mathsf{Sh}(X^{\text{an}}, \Lambda)$. Define
$$(f_! F)(U) := \left\{ s \in F(f^{-1}(U)) \,\middle|\, f|_{\mathrm{Supp}(s)} \text{ is proper} \right\} \subset (f_* F)(U)$$
is the **pushforward with compact support**. We also write
$$f_! := \mathrm{R} f_! : D(Y) \longrightarrow D(X).$$

$f_! \to f_*$ is an isomorphism for f proper.

Lemma 4.1.1. — *Suppose $j : U \hookrightarrow X$ is an open immersion.*

1) $j_! : \mathsf{Sh}(U^{an}) \to \mathsf{Sh}(X^{an})$ is "extension by zero"
$$(j_! F)_x = \begin{cases} F_x & x \in U \\ 0 & \textit{otherwise.} \end{cases}$$
2) $j_!$ is left adjoint to j^*.

3) We have open BC: $f^* j_! \simeq \widetilde{j}_! \widetilde{f}^*$ and open Proj
$$j_!(- \otimes j^*(-)) \simeq j_!(-) \otimes -.$$

Let $f : Y \to X$ be a separated morphism of finite type, then there exists a *Nagata compactification*: f factors as
$$Y \overset{j}{\hookrightarrow} \overline{Y} \overset{p}{\to} X$$
where j is an open immersion and p is proper. Then
$$f_! \simeq p_! j_! \simeq p_* j_!.$$

Theorem 4.1.2. — *(BC) We have* $g^* f_! \xrightarrow{\sim} \widetilde{f}_! \widetilde{g}^*$.

(Proj) $f_!(- \otimes f^*(-)) \xrightarrow{\sim} f_!(-) \otimes -$.

(Kü) $f_!(-) \otimes g_!(-) \xrightarrow{\sim} (f \times g)_!(- \boxtimes -)$.

Proposition 4.1.3. — *Let f be a separated morphism of finite type. The functor $f_! : D_B(Y) \to D_B(X)$ commutes with all coproducts. In particular, by the Adjoint Functor Theorem $f_!$ has a right adjoint $f^! : D_B(X) \to D_B(Y)$ called the* ***exceptional pullback****.*

Example 4.1.2. — If j is an open immersion (étale) then $j^! \simeq j^*$.

Proposition 4.1.4 (Formal local duality). — *There is an isomorphism*

$$\underline{\mathrm{Hom}}(f_! F, G) \xrightarrow{\sim} f_* \underline{\mathrm{Hom}}(F, f^! G).$$

Exercise. — Prove this!

Example 4.1.3. — Let $\pi : X \to \mathrm{Spec}(\mathbf{C})$, then

$$\mathrm{H}^*_c(X, \mathbb{Q})^\vee \simeq \mathrm{H}^*(X, \pi^! \mathbb{Q}).$$

To recover Poincaré duality, we need to compute $\pi^! \mathbb{Q}$ for X smooth.

Theorem 4.1.3 (Duality for smooth morphisms). — *Let $q : Y \to X$ be a separated morphism of finite type.*

1) There is a canonical natural transformation

$$\alpha_f : f^! \Lambda \otimes f^*(-) \longrightarrow f^!(-).$$

2) Let f be smooth separated of relative dimension d, then

- *α_f is an isomorphism,*
- *$f^! \Lambda \simeq \Lambda \langle d \rangle$.*

(Better $\Lambda(1) \simeq \Lambda$.)

3) If f is smooth then f^* has a left adjoint

$$f_\sharp = f_! \langle d \rangle .$$

Exercise (Zariski/étale separation). — Let $\{j_i : U_i \to X\}$ be a Zariski/étale covering then $\{j_i^* = j_i^!\}$ is jointly conservative.

Proof sketch. Étale separation reduces 2) to $f : \mathbb{A}^n \times X \to X$ (?).
and $(?) \to \mathrm{H}^*_c(\mathbb{C}^n)$
3) is a corollary of 2). □

Proposition 4.1.5. — *Let $\pi : X \to \mathrm{Spec}(\mathbf{C})$ be separated, then*

$$\begin{aligned}
\mathrm{H}^*_{sing}(X^{an}, \Lambda) &\simeq \mathrm{H}^*(\pi_* \overbrace{\pi^* \Lambda}^{\Lambda}) \\
\mathrm{H}^*_c(X^{an}, \Lambda) &\simeq H^*(\pi_! \pi^* \Lambda) \\
\mathrm{H}_*(X^{an}, \Lambda) &\simeq \mathrm{H}_*(\pi_! \pi^! \Lambda) \\
\mathrm{H}^{BM}_*(X^{an}, \Lambda) &\simeq \mathrm{H}(\pi_* \pi^! \Lambda).
\end{aligned}$$

Remark. — Let q be smooth, then $q_\sharp \Lambda \simeq q_! q^! \Lambda$.

For a quasiprojective morphism f we get two factorizations

$$\begin{cases} f = pj & f_! = p_! j_! \quad p \text{ proper, } j \text{ open immersion} \\ f = qi & f^! = i^! q^! \quad q \text{ smooth, } i \text{ closed immersion} \end{cases}$$

Proposition 4.1.6 (Localization/gluing). — *Let $i : Z \hookrightarrow X$ be a closed immersion and $j : X \setminus Z = U \to X$*

$$\begin{cases} j_* j^* \longrightarrow \mathrm{id} \longrightarrow i_! i^! \xrightarrow{+} \\ \\ j_! j^! \longrightarrow \mathrm{id} \longrightarrow i_* i^* \xrightarrow{+} \end{cases}$$

(note that $i_! = i_*$).

Proposition 4.1.7 (Absolute purity). — *Let $i : Z \hookrightarrow X$ be a regular closed immersion of codimension c (for example, when both X and Z are smooth), then*

$$i^!(\Lambda_X) \simeq \Lambda_Z \langle -c \rangle .$$

So we get $i^!\Lambda_X$ for $i : D \hookrightarrow X$ a SNCD.

4.2. What are six functor formalisms? (Lurie, Gaitsgory-Rozenblyum, Liu-Zhang, Mann, . . .)

Definition (Fake). — Let C be an ∞-category with finite limits and E be a class of morphisms stable under composition and pullbacks. $\mathsf{Span}(\mathsf{C}, E)$ is the ∞-category of **spans**, which can be informally described as follows:

- Objects are the objects of C.
- 1-morphisms are diagrams

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f \in E"] & \\ X & & Y \end{tikzcd}$$

- 2-morphisms are diagrams

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr, "\in E"] & \\ X & & Y \\ & Z' \arrow[ul] \arrow[ur, "\in E"'] \arrow[uu, "\sim"] & \end{tikzcd}$$

- composition is given by pullbacks

$$\begin{tikzcd} & & Z'' \arrow[dl] \arrow[dr, "\in E"] \arrow[dll, bend right] \arrow[drr, bend left, "\in E"] & & \\ & Z \arrow[dl] \arrow[dr] & & Z' \arrow[dl] \arrow[dr, "\in E"] & \\ X & & X' & & X'' \end{tikzcd}$$

$\mathsf{Span}(\mathsf{C}, E)$ has a symmetric monoidal structure, and there is a symmetric monoidal functor

$$(\mathsf{C}^{\mathsf{op}}, \times) \longrightarrow (\mathsf{Span}(\mathsf{C}, E), \otimes)$$

which maps $g : Z \to X$ to the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, equal, "\mathrm{id}"] & \\ X & & Z. \end{tikzcd}$$

Definition (Mann). — A **3-functor formalism** is a lax symmetric monoidal[3] functor

$$\widetilde{D} : \mathsf{Span}(\mathsf{C}, E) \longrightarrow \mathsf{Cat}_\infty,$$

where Cat_∞ is equipped with the cartesian symmetric monoidal structure. A **6-functor formalism** is a 3-functor formalism where "right adjoints exist".

Fact. — $D_B(-)$ extends to a 3-functor formalism

$$\widetilde{D}_B : \mathsf{Span}(\mathsf{Sch}^?_{\mathsf{C}}, \mathsf{ft,sep}) \longrightarrow \mathsf{Cat}_\infty.$$

[3] A symmetric monoidal functor D is required to strongly preserve the symmetric monoidal structure: it is equipped with a natural isomorphism $D(- \otimes -) \simeq D(-) \otimes D(-)$. A lax symmetric monoidal functor relaxes this condition, merely requiring a non-necessarily invertible natural transformation.

$\widetilde{D}_B$ takes the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X \end{tikzcd}$$

to $f_!g^*$. It's lax symmetric monoidal, we have $\boxtimes_X$ and we can apply Δ_X^* to get $\otimes_X$. We have functoriality for composition of spans which gives us

$$BC : f_!g^* = \widetilde{g}^*\widetilde{f}_!.$$

Theorem 4.2.1 (Fake). — *Let $P, J \subseteq E$ such that $E = P \circ J$ and consider*

$$D : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{CAlg}(\mathsf{Cat}_\infty),$$

such that the following holds.

1) *For all $p \in P$ we have an adjoint pair (p^*, p_*), and PBC and PProj hold.*
2) *For all $j \in J$ we have an adjoint pair $(j_!, j^*)$, and OBC and OProj hold.*
3) *Let*

$$\begin{tikzcd} \bullet \arrow[r, "\widetilde{p}"] \arrow[d, "\widetilde{j}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "j"] \\ \bullet \arrow[r, "p"] & \bullet \end{tikzcd}$$

be cartesian, then

$$j!\widetilde{p}_* \xrightarrow{\sim} p_*\widetilde{j}_!$$

(Supp).

Then D extends to a 3-functor formalism.

4.3. Six functor formalism for motivic sheaves Let $f : T \to S$ be a morphism, we have the functor

$$\begin{aligned} f^{-1} : \mathsf{Sm}_S &\longrightarrow \mathsf{Sm}_T \\ X &\longmapsto X \times_S T \end{aligned}$$

which gives us $\mathsf{DA}^{\text{ét}}(-, \Lambda)$ and $\mathsf{SH}(-)$ sheaf theories. We already have $\otimes$, f^* and $\underline{\mathrm{Hom}}$, f_*.

Theorem 4.3.1. — $\mathsf{DA}^{\acute{e}t}(-, \Lambda), \mathsf{SH}(-)$ *extend to six-functor formalisms.*

This is a hard theorem, much harder than the Betti and étale cases. The main difficulty is that proper base change is hard!

Remark. —

- This also holds for other variants: $\mathsf{DM}(-, \Lambda)$, KGL-modules which are "KH-motives", MGL-modules,...
- At the end of the day there are still major differences:
  1) Let q be smooth of relative dimension d. In $\mathsf{DA}^{\text{ét}}(-, \Lambda), \mathsf{DM}(-, \Lambda), KGL, MGL$ we have $q^!\mathbb{1}_X \simeq \mathbb{1}_Y\langle d\rangle$ (the GL-oriented theories/complex oriented cohomology theories in SH^{top} with Chern classes for vector bundles). In $\mathsf{SH}(-)$, $q^!\mathbb{1}_X \simeq \mathrm{Th}_Y(\Omega_q)$ which is the Thom space/spectrum.
  2) $\mathsf{DA}^{\text{ét}}(-, \Lambda)$ has much stronger descent properties, it satisfies h-descent. The h-topology is defined by étale coverings and proper surjective morphisms.

If q is smooth, then q^{-1} has a left adjoint given by a very silly formula

$$\begin{aligned} q_\sharp : \mathsf{Sm}_T &\longrightarrow \mathsf{Sm}_S \\ X &\longmapsto X. \end{aligned}$$

This induces a left adjoint to $q^* : D(S) \to D(T)$ for $D = \mathsf{DA}^{\text{ét}}(-, \Lambda), \mathsf{SH}(-)$.

Theorem 4.3.2 (Voevodsky, Ayoub). — *A sheaf theory that satisfies:*

- *for q-smooth there is an adjoint pair $(q_\sharp, q^*)$ with base change and the projection formula,*

- *(Gluing) for all closed embeddings $i : Z \hookrightarrow X$ and open embeddings $j : X \setminus Z \hookrightarrow X$ the pair (i^*, j^*) is conservative and i_* is fully faithful,*
- *$\mathbb{A}^1$-invariance and $\mathbb{P}^1$-stability*

satisfies proper base change.

Note that the gluing axiom is a gluing theorem of Morel-Voevodsky, it uses smooth sites and at least Nisnevich descent. This type of sheaf theory is called a **motivic sheaf theory** or a **coefficient system**.

Theorem 4.3.3 (Drew-Gallauer). — $\mathsf{SH}(-)$ *is the initial motivic sheaf theory.* $\mathsf{DA}^{\acute{e}t}(-,\Lambda)$ *is initial among those satisfing étale descent and is Λ-linear.*

(Something about the Drew-Tubach mixed module realization?)

There are also good theories of:

- constructibility and Verdier duality,
- nearby and vanishing cycles.

4.4. Motivic t-structure conjecture and algebraic cycles Let $D = \mathsf{DA}^{\text{ét}}(-,\mathbb{Q}) = \mathsf{DM}(-,\mathbb{Q})$.

Definition. — Let D be a triangulated category. A **t-structure** is a pair $(D_{\geq 0}, D_{\leq 0}$ of full subcategories with

1) $D_{\geq 0}, D_{\leq 0}$ are replete (stable under isomorphisms),
2) $D_{\geq 0}[1] \subseteq D_{\geq 0}, D_{\leq 0}[-1] \subseteq D_{\leq 0}$,
3) $\mathrm{Hom}(D_{\geq 0}, D_{\leq 0}[-1]) = 0$,
4) for all $X \in D$, there exists a distinguished triangle

$$\tau_{\geq 0}X \longrightarrow X \longrightarrow \tau_{<0}X \xrightarrow{+}$$

where $\tau_{\geq 0}X \in D_{\geq 0}$ and $\tau_{<0}X \in D_{\leq 0}[-1]$.

Taking $D^{\heartsuit} = D_{=0} = D_{\geq 0} \cap D_{\leq 0}$ gives us the **heart** which is an abelian category.

Example 4.4.1. — Let

$$D(A)_{\geq 0} = \{K_\bullet | \forall n < 0, H_n(K_\bullet) = 0\}$$

and similarly for ≤ 0. The heart is A.

Example 4.4.2. — Let

$$\mathsf{hSptr}_{\geq 0} = \{K_\bullet | \forall n < 0, \pi_n(K_\bullet) = 0\}$$

similarly for ≤ 0 and the heart is Ab.

Conjecture 4.4.1 (T_k). — *Let k be a field. There exists a t-structure on $\mathsf{DM}(k,\mathbb{Q})$ such that*

1) for all $l \neq \mathrm{char}(k)$, $R_l : \mathsf{DM}(k,\mathbb{Q}) \to D(\mathbb{Q}_l)$ is t-exact.
2) The t-structure restricts to $\mathsf{DM}_{ct}(k,\mathbb{Q})$, define $\mathsf{MM}_{(d)}(k,\mathbb{Q})$ to be the heart of $\mathsf{DM}_{(d)}(k,\mathbb{Q})$.
3) $\mathsf{DM}_{ct}(k,\mathbb{Q}) \simeq D^b(\mathsf{MM}_{ct}(k,\mathbb{Q}))$.

Lemma 4.4.1. — *(T_k) implies that MM_{ct} is a Tannakian category.*

So $\mathsf{MM}_{\mathrm{ct}}(k,\mathbb{Q})$ is approximately isomorphic to $\mathsf{Rep}^{\mathrm{f.d.}}(G_{\mathrm{mot}}(k))$, were $G_{\mathrm{mot}}(k)$ is a pro-algebraic group over $\mathbb{Q}$, the "motivic Galois group".

Proposition 4.4.1. — *Let $\sigma : k \hookrightarrow \mathbf{C}$ be a field embedding. Then (T_k) is equivalent to the Nori realization functor*

$$R_{Nori} : \mathsf{DM}_{ct}(k,\mathbb{Q}) \longrightarrow D(\mathrm{NM}(k,\mathbb{Q}))$$

being an equivalence. Here $\mathrm{NM}(k,\mathbb{Q}))$ is the categoru of Nori motives.

Theorem 4.4.1. — *(T_k) implies*

a) (Conservativity): $R_l : \mathsf{DM}_{ct}(k,\mathbb{Q}) \to D(\mathbb{Q}_l)$ is conservative.
b) (char k) standard conjectures of Grothendieck on algebraic cycles up to homological equivalence on smooth projective varieties.
c) Bloch-Beilinson-Murre conjecture on filtrations of Chow groups of smooth projective varieties.

d) *Beilinson-Soulé conjecture: Fix* $X \in \mathsf{Sm}_k$, *then*

$$\mathrm{H}^q_{mot}(X, \mathbb{Q}(p)) = 0$$

for $q < 0$. *Call this statement* (BS_X).

Theorem 4.4.2 (Levine). — *If* $X \in \mathsf{Sm}_k$, *then* (BS_X) *implies the existence of a motivic t-structure on* $\mathsf{DMT}_{ct}(X, k)$ *(not satisfying property 3) in general.*

Theorem 4.4.3. — (BS_X) *is a known when*

1) *k is a number field, function field, finite field. The number field case is a difficult theorem of Borel, the function field case was proven by Harder, and the finite field case by Quillen.*
2) $M(X) \in \mathsf{DMT}(k)$ *for* $X = \mathbb{G}_m^m \times \mathbb{A}^n \times \mathbb{P}^n$.

Definition. — Let $i : Z \hookrightarrow X$ be a closed embedding and $j : X \setminus Z \hookrightarrow X$ be the complementary open embedding. We say it is **Whitney-Tate** if $i^* j_* \mathsf{DMT}(X \setminus Z) \subset \mathsf{DMT}(Z)$.

References

[Lur09] Jacob Lurie. *Higher Topos Theory*. Vol. 170. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009.