MOTIVIC SHEAVES

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Plan:

L1: Motivation, construction, examples.
L2: Six-functor formalism.
L3: Motivic t-structures and weight structures.
L4: ∞-categorical methods.

§1. Motivation from GRT and cohomology

1.1. Cohomology and sheaves for representation theory

Lecture 1

Question: How do you construct interesting representations?
Answer:

1) Find interesting actions.
2) Linearlize them.

Example 1.1.1. — Let K be a compact Lie group. The action of K on itself gives us an action of K on $L^2(K)$ with respect to a Haar measure. The Peter-Weyl theorem says that

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \text{ unitary}}} \pi^{\oplus dim(\pi)}.$$

"Lie theory $\subset$ algebraic geometry". Reductive groups are algebraic groups with many associated varieties with group actions: flag varieties…

The linearizations we consider in this course are the many types of cohomology theories.

Example 1.1.2 (Borel-Weil-Bott). — Let $T \subset B \subset G$ be a reductive group over $\mathbf{C}$. Let $\lambda \in X^\vee(T)$ such that there exists $w \in W$ with $w * \lambda = w(\lambda + \rho) - \rho > 0$ (where $\rho = (1/2)\sum_{\alpha \in \Phi^+} \alpha$). Then

$$\mathrm{R}\Gamma(G/B, L_\lambda) \simeq \pi_{w*\lambda}[-\ell(w)]$$

where $\ell(w)$ is the length of w.

Cohomology fits in the wider context of sheaf theory. If T is a locally contractible topological space, then

$$\mathrm{H}^n_{\mathrm{sing}}(T, \mathbb{Z}) \simeq \mathrm{H}^n(T, \underline{\mathbb{Z}}_T) \simeq \mathrm{R}^n(\pi_T)_*(\underline{\mathbb{Z}}_T)$$

where π_T is the morphism $\pi_T : T \to \mathrm{pt}$ with

$$\mathrm{R}\pi_{T*} : D(T, \mathbb{Z}) \longrightarrow D(\mathbb{Z}) \simeq D(*, \mathbb{Z}).$$

Cohomology (singular with $\mathbb{Q}$-coefficients) of algebraic varieties over $\mathbf{C}$ is ***very*** special.

- There is a weight filtration.
- There is a mixed hodge structure.

Sheaves on complex algebraic varieties are also very special:

- Perverse sheaves;
- Decomposition theorem;
- Mixed Hodge modules.

This leads to great success stories in GRT:

- Springer theory;
- Kazdhan-Lustig theory;
- geometric Satake…

1.2. From sheaves to motivic sheaves There are situations which can't be directly studied using these tools:

- Representation theory of reductive groups over other fields/rings/schemes.
- Modular/integral representation theory.
- q-deformations, quantum groups, canonical bases.

These can be attacked using:

- l-adic sheaves,
- sheaves cohomology with $\mathbb{Z}$-coefficients,
- K-theory.

Motivic sheaves will give us a unified perspective.

Motivic dream: There should exist universal cohomology/sheaf theories such that

1) they unify and "explain" the special structure in cohomology/sheaves;
2) they are of algebro-geometric nature;
3) they "explain" the realization of algebraic cycles and algebraic K-theory.

§2. Construction of $\mathsf{DA}^{\text{ét}}$ and SH (Morel-Voevodsky)

2.1. Triangulated categories and localization

Definition. — A **triangulated category** is the data:

- an additive category C,
- an automorphism $\Sigma = (-)[1] : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$,
- a collection of distinguished triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

The data satisfies the following conditions:

- shifted distinguished triangles are distinguished up to isomorphism,
- for all $f : A \to B$ there exists

$$A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \xrightarrow{+}$$

  unique up to isomorphism and functorial,
-

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f", "\simeq"'] & B \arrow[r] \arrow[d, "g", "\simeq"'] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "{f[1]}", "\simeq"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

- (??)

Example 2.1.1. — Let A be an abelian category, $\mathsf{Ch}(\mathsf{A})$ be the abelian category of chain complexes in A. We define $(A[1])_n = A_{n-1}$. Given $f : A_\bullet \to B_\bullet$ the maping cone is given by

$$\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n, \qquad d_n = \begin{pmatrix} -d^A_{n-1} & 0 \\ f & d^B_n \end{pmatrix}.$$

Definition. — $f : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if for all $n \in \mathbb{Z}$, the map $H_n(A_\bullet) \simeq H_n(B_\bullet)$ is an isomorphism.

Definition. — $D(\mathsf{A})$ is defined as the localization of $\mathsf{Ch}(\mathsf{A})$ by quasi-isomorphisms.

Now we consider reflexive localizations (1-categorical ones lead to triangulated and ∞-categorical ones).

Definition. — Let C be a 1-category.

1) $\mathsf{C}' \subset \mathsf{C}$ is **reflexive** if $\iota : \mathsf{C}' \to \mathsf{C}$ has a left adjoint.
2) $L_W : \mathsf{C} \to \mathsf{C}[W^{-1}]$ is **reflexive** if L_W has a right adjoint.

Lemma 2.1.1 (Reflexive subcategories are the same thing as reflexive localizations). —

a) *Let $\mathsf{C}' \subset \mathsf{C}$ be reflexive, $L : \mathsf{C} \to \mathsf{C}'$ be the left adjoint to ι. Define*

$$W_L = \{f : L(f) \text{ is an isomorphism}\}.$$

Then $\mathsf{C}' \simeq \mathsf{C}[W_L^{-1}]$ and $L \simeq L_{W_L}$.

b) *If L is a reflexive localization, then its right adjoint ι is fully faithful and $\iota : \mathsf{C}[W^{-1}] \xrightarrow{\sim} \mathrm{EssIm}(\iota) \subset \mathsf{C}$.*

Definition. — Let $S \subset \mathsf{C}$ be a collection of morphisms.

a) $A \in \mathsf{C}$ is S**-local** if for all $f : B \to C$ in S

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

b) $f : B \to C$ is an S**-equivalence** if for all S-local A

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

Lemma 2.1.2. — *If $L : \mathsf{C} \rightleftarrows \mathsf{C}' : \iota$ is a reflexive localizaton, W_L as before, then*

- *ι gives an isomorphism between C' and W_L-local objects.*
- *W_L are the W_L-equivalences.*

Definition. — Let D be a triangulated category with all small products.

- Let κ be a regular cardinal (for example $\kappa = \aleph_0$). Then $A \in \mathsf{D}$ is κ**-small**/κ**-compact** if and only if

$$\operatorname*{colim}_{\substack{I' \subset I \\ |I'| < \kappa}} \mathrm{Hom}\left(A, \bigoplus_{I'} B_i\right) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_{I} B_i\right).$$

- **Compact** means $\aleph_0$-small. A is compact if and only if

$$\bigoplus_I \mathrm{Hom}(A, B_i) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_I B_i\right).$$

- D is **presentable**/**well-generated** if and only if there exist κ and a set $S \subset \mathsf{D}$ of κ-small objects which generate D:

$$\forall B \in \mathsf{D}, (\forall A \in S, \mathrm{Hom}(A, B) = 0) \implies B = 0.$$

- D is **compactly generated** if it is $\aleph_0$-presentable.

Definition. — $\mathsf{E} \subset \mathsf{D}$ is **localizing** if it is

- triangulated,
- stable under coproducts,
- thick (stable under subobjects and subquotients).

Theorem 2.1.1 (Adjoint Functor Theorem). — *Let D, D' be triangulated categories with all coproducts, $F : \mathsf{D} \to \mathsf{D}'$ be a triangulated functor and D be presentable. Then F admits a right adjoint if and only if F preserves all coproducts.*

Corollary 2.1.1 (Verdier Localization). — *Let D be a presentable category and E be a localizing subcategory. Define*

$$\mathsf{D}/\mathsf{E} = D[W_{\mathsf{E}}^{-1}], \quad W_{\mathsf{E}} = \{f : \mathrm{Cone}(f) \in \mathsf{E}\}.$$

Then $\mathsf{D} \to \mathsf{D}/\mathsf{E}$ is a reflexive localization.

Let $S \subset \mathsf{D}$ be a subset of objects, then $\langle\langle S \rangle\rangle$ is the smallest subcategory containing S such that $\mathsf{D}/\langle\langle S \rangle\rangle$ is a reflexive localization.

Let's get some inspiration from Betti homology, i.e. singular homology on complex algebraic varieties. Let $X \in \mathsf{Var}_{\mathbf{C}}^{(\mathrm{f.t})}$, then we get

$$C_*^{\mathrm{sing}}(X(\mathbf{C}), \mathbb{Z}) \in D(\mathbb{Z}).$$

This comes with some data:

(a)
- $D(\mathbb{Z})$ has a symmetric monoidal structure: $\otimes^{\mathbb{Z}}$,
- (Künneth) $C_*(X \times Y) \simeq C_*(X) \otimes^{\mathbb{Z}} C_*(Y)$.

which satisfies sproperties:

(b) ($\mathbb{A}^1$-homotopy invariance) $C_*(X \times \mathbb{A}^1) \xrightarrow{\sim} C_*(X)$ ($(\mathbb{A}^1)^{\mathrm{an}} = \mathbf{C}$ is contractible).

(c′) (Mayer-Vietoris sequence) Let $X = U \cup V$ be a Zariski cover, then

$$C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*(X) \xrightarrow{\partial} C_*(U \cap V)[1].$$

(c) (Étale descent) Let $U \to X$ be étale surjective. Define

$$\check{C}_n(U/X) = U^{n+1}.$$

Then $\check{C}_\bullet(U/X)$ is a simplicial scheme, so $C_*(\check{C}_\bullet(U/X))$ is a simplicial complex of abelian groups and $C(C_*(\check{C}_\bullet(U/X)))$ is a double complex. (??)

Concretely we have a descent spectral sequence which gives us ($U = U \cup V$) Mayer Vietoris.

(d) ($\mathbb{P}^1$-stabilization)

$$\begin{aligned} C_*(\mathbb{P}^1_{\mathbf{C}}) &\simeq C_*(\mathrm{pt}) \oplus \widetilde{C}_*(\mathbb{P}^1_{\mathbf{C}}) \\ &\simeq \mathbb{Z}[0] \oplus \mathbb{Z}(1)[2]. \end{aligned}$$

$\mathbb{Z}(1)$ is $\oplus$-invertible.

Smooth varieties play a special role:

- Poincaré duality
- Gysin sequences
- $C_*(-)$ also satisfies "h-descent", so $C_*(-)$ is "determined" by $C_*(-)_{|(?)}$.

There is an associated sheaf theory:

$$D_B(-) : \mathsf{Var}_{\mathbf{C}} \longrightarrow \mathsf{TriCat}^{\otimes}, \quad X \longmapsto D_B(X) = D(\mathsf{Sh}(X^{\mathrm{an}}, \mathbb{Z})).$$

Sketch of $\mathsf{DA}^{\acute{e}t}$*:* Let S be a base scheme.

- Start with
$$\begin{cases} D(\mathsf{PSh}(\mathsf{Sm}_S, \mathbb{Z})) = D_{\mathsf{PSh}}(S) \\ \mathbb{Z}[-] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}.$$
- Impose $\mathbb{A}^1$-invariance, étale descent, and $\mathbb{P}^1$-stability. This will give us $\mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$ and $M_S(-) : \mathsf{Sm}_S \to \mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$.

The surprise is that the result satisfies many *other* properties of singular (co)homology and derived categories of sheaves on complex varieties which can largely be packaged into the six-functor formalism. The result is closely related to algebraic cycles/higher Chow groups and algebraic K-theory. Lecture 2

(Fill in H_* frmo the recall part)

Let S be a qcqs scheme, Λ be a coefficient ring. Define

$$\begin{cases} D_{\mathsf{PSh}}(S) := D(\mathsf{PSh}(\mathsf{Sm}_S, \Lambda)) \text{a presentable, symmetric monoidal, triangulated category} \\ \Lambda[\cdot] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}$$

Étale descent:

$$\begin{aligned} D_{\text{ét}}(S) &:= D(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S, \Lambda)) \\ &= D_{\mathsf{PSh}}(S)[W_{\text{ét}}^{-1}] \end{aligned}$$

where $W_{\text{ét}}$ are étale-local weak equivalences, i.e. $(f : K_\bullet \to L_\bullet) \in W_{\text{ét}}$ if for all n we have

$$(?)_{\text{ét}}\mathrm{H}_n(K_\bullet) \xrightarrow{\sim} (?)_{\text{ét}}\mathrm{H}_n(L_\bullet).$$

$\mathbb{A}^1$-*invariance* Let

$$I_{\mathbb{A}^1,(\text{ét})} = \{\ldots \longrightarrow 0 \longrightarrow \Lambda_{(\text{ét})}[X \times \mathbb{A}^1] \longrightarrow \Lambda_{(\text{ét})}[X] \longrightarrow 0 \longrightarrow \ldots | X \in \mathsf{Sm}_S\}.$$

Define

$$D_{\mathbb{A}^1}(S) := D_{\mathsf{PSh}}(S) / \langle\langle I_{\mathbb{A}^1} \rangle\rangle = D_{\mathsf{PSh}}(S)[W_{\mathbb{A}^1}^{-1}].$$

We have

$$L_{\mathbb{A}^1} : D_{\mathsf{PSh}}(S) \longrightarrow D_{\mathsf{PSh}}(S)^{\mathbb{A}^1-\mathrm{loc}} \hookrightarrow D_{\mathsf{PSh}}(S).?$$

with the middle term isomorphic to $D_{\mathbb{A}^1}(S)$.

Definition. — Define

$$\Delta^n_{\mathrm{alg},S} := \mathrm{Spec}_S\left(\mathscr{O}_S[X_0,\dots,X_n]/\left(\sum x_i - 1\right)\right) \simeq \mathbb{A}^n_S$$

then $\Delta^\bullet_{\mathrm{alg},S}$ is a cosimplicial scheme over S.

Definition (Suslin-Voevodsky). — Define

$$\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) = \mathrm{hocolim}_{\Delta^{\mathrm{op}}} K_\bullet(\Delta^\bullet_{\mathrm{alg},S} \times_S X)$$

Example 2.1.2. — Let $F \in \mathsf{PSh}$ then

$$\mathrm{Sing}^{\mathbb{A}^1}(F)(U) = \left[\dots \longrightarrow F(\Delta^2 \times U) \longrightarrow F(\mathbb{A}^{(?)} \times U) \xrightarrow{i_0^* - i_1^*} F(U) \longrightarrow 0\right].$$

Proposition 2.1.1. — $L_{\mathbb{A}^1} \simeq \mathrm{Sing}^{\mathbb{A}^1}$.

Proof. The idea is to use

$$\begin{aligned} m : \mathbb{A}^1 \times \mathbb{A}^1 &\longrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto xy \end{aligned}$$

to prove

a) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet)$ is $\mathbb{A}^1$-local.
b) $\mathrm{Sing}^{\mathbb{A}^1}(K_\bullet) \to K_\bullet$ is $\mathbb{A}^1$-weak equivalence.

□

Definition. — The category of **effective étale motivic sheaves** on S is

$$\mathsf{DA}^{\text{ét,eff}}(S,\Lambda) := D_{\text{ét}}(S)/\left\langle\!\left\langle I_{\mathbb{A}^1,\text{ét}}\right\rangle\!\right\rangle.$$

Write $L^{\mathrm{eff}}_{\mathrm{mot}}$ for the associated localization functor.

Lemma 2.1.3. — *We have*

$$L^{eff}_{mot} = \underbrace{\dots \mathrm{Sing}^{\mathbb{A}^1} L_{\acute{e}t} \mathrm{Sing}^{\mathbb{A}^1}}_{transfinie\ composition\dots}$$

Definition. — Let $X \in \mathsf{Sm}_S$. Define

$$M^{\text{ét,eff}}_S(X) := L^{\mathrm{eff}}_{\mathrm{mot}}\Lambda_{\text{ét}}[X] \in \mathsf{DA}^{\text{eff,ét}}(S,\Lambda)$$

(effecive étale (? homological) motive/motivic sheaf of ?).

We have

$$M^{\text{ét,eff}}_S(X \times_S Y) \simeq M^{\text{ét,eff}}_S(X) \otimes M^{\text{ét,eff}}_S(Y).$$

Proposition 2.1.2 (Artin-Shreier $+\Lambda\left[\frac{1}{p}\right]$). — *Let S be a $\mathbf{F}_p$-scheme, then*

$$\mathsf{DA}^{\acute{e}t,eff}(S,\Lambda) \xrightarrow{\sim} \mathsf{DA}^{\acute{e}t,eff}\left(S,\Lambda\left[\frac{1}{p}\right]\right).$$

Proof. We have the short-exact sequence

$$0 \longrightarrow \Lambda/p\Lambda \longrightarrow \mathbb{G}_a \otimes \Lambda \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbb{G}_a \longrightarrow 0$$

and hence an exact sequence

$$\Lambda_{\text{ét}}[\mathbb{G}_a] \otimes (\mathbb{G}_a \otimes \Lambda) \xrightarrow{a_{\mathbb{G}_a}\otimes \mathrm{id}} \mathbb{G}_a \otimes \mathbb{G}_a \otimes \Lambda \xrightarrow{m} \mathbb{G}_a \otimes \Lambda.$$

(Some remark??) Thus

$$L_{\mathbb{A}^1}(\mathbb{G}_a \otimes \Lambda) \simeq \Lambda(0)$$

and so

$$L^{\mathrm{eff}}_{\mathrm{mot}}(\Gamma/p\Gamma) = 0.$$

□

$\mathbb{P}^1$-*stabilization:* Let $x \in X(S)$, we have

$$M_S^{\mathrm{eff}}(X) = \Lambda_S(0) \oplus M_S^{\mathrm{eff}}(X, x).$$

Definition. — We define

$$T := M_S^{\mathrm{eff}}(\mathbb{P}^1, 1)[-2]$$

which is equal to $\Lambda(?)$.

Exercise. — Any $x \in \mathbb{P}^1_S(S)$ gives the same decomposition.

We have a problem: T is not $\oplus$-invertible.

Definition. — The category of étale motivic sheaves over S is

$$\mathsf{DA}^{\text{ét}}(S, \Lambda) := \mathsf{DA}^{\text{ét,eff}}(S, \Lambda)[T^{\otimes -1}].$$

This is not a proper definition since we have not explained the construction. It's not clear what this means!

Spectra:

Definition. — Let C be a closed, symmetric monoidal 1-category and T be an object of C. A T**-prespectrum** is

$$A = \left\{(A_n, \sigma_n)_{n \in \mathbb{N}} | A_n \in \mathsf{C}, \sigma_n : T \otimes A_n \longrightarrow A_{n+1}\right\}.$$

A is a T**-spectrum** if for all $n \in N$

$$A_n \xrightarrow{\sim} \underline{\mathrm{Hom}}(T, A_{n+1}).$$

We write $\mathrm{PSp}_T(\mathsf{C})$ and $\mathrm{Sp}_T(\mathsf{C})$ for the T-prespectrum and T-spectrum respectively.

The evaluation map

$$\mathrm{Ev}_n(A) = A_n$$

has a left adjoint. We define

$$\mathrm{Sus}^n(A)_m = \begin{cases} \varnothing & \text{if } m < n \\ T^{\otimes(m-n)} \otimes A & \text{if } m > n \end{cases}$$

and $\Sigma_T^\infty := \mathrm{Sus}^0$ is the ∞-suspension functor.

Proposition 2.1.3. — *Assume C is presentably, symmetrical monoidal. Then $\mathrm{Sp}_T(\mathsf{C}) \subset \mathrm{PSp}_T(\mathsf{C})$ is a reflexive subcategory. W_{st} is generated by*

$$\left\{\mathrm{Sus}^{n+1}(T \otimes A) \longrightarrow \mathrm{Sus}^n(A) : n \in \mathbb{N}, A \in \mathsf{C}\right\}.$$

Definition. — We define

$$\mathsf{DA}^{\text{ét}}(S, \Lambda) := \mathrm{Sp}_T \, \mathsf{DA}^{\text{eff,ét}}(S, \Lambda).$$

(This definition is correct "with ∞-categories".) We have

$$\begin{aligned} M_S : \mathsf{Sm}_S &\longrightarrow \mathsf{DA}^{\text{ét}}(S, \Lambda) \\ X &\longmapsto L_{(\mathbb{A}^1, \text{ét}, ?} \Sigma_T^\infty M_S^{\text{ét,eff}}(X). \end{aligned}$$

Remark. — $M \in \mathsf{DA}^{\text{ét}}(S, \Lambda)$ is isomotphic to a stable $(\mathbb{A}^1, \text{ét})$-local (??)

$$K_n \in \mathsf{Ch}(\mathsf{Sh}_{\text{ét}}(\mathsf{Sm}_S, \Lambda)) + \sigma_n = \Lambda_{\text{ét}}[\mathbb{P}^1, 1] \otimes K_n \longrightarrow K_{n+1}$$

such that for all $X \in \mathsf{Sm}_S, i \in \mathbb{Z}$

- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X \times_S \mathbb{A}^1, K_n)$
- $\mathrm{H}^i_{\text{ét}}(X, K_n) \xrightarrow{\sim} \mathrm{H}^{i+2}_{\text{ét}}(X \times_S (\mathbb{P}^1, 1), K_{n+1})$.

2.2. Constructible motivic sheaves

Definition. — We define **constructible motivic sheaves**

$$\begin{aligned}\mathsf{DA}^{\text{ét}}_{\text{ct}} &= \langle M_S(X)(-n) | X \in \mathsf{Sm}_S, n \in \mathbb{N} \rangle^{\text{d.f.}} \\ &\subset \mathsf{DA}^{\text{ét}}(S, \Lambda).\end{aligned}$$

and **locally constructible motivic sheaves**

$$\mathsf{DA}^{\text{ét}}_{\text{lct}}(S, \Lambda) := \{M | \exists e : U \twoheadrightarrow S, e^* M \in \mathsf{DA}_{\text{ct}}\}.$$

There is a Betti realization for S finite type over $\mathbf{C}$

$$R_B : \mathsf{DA}^{\text{ét}}(S, \Lambda) \longrightarrow D(S^{\text{an}}, \Lambda)$$

by the existence of relative homology and the universal property. If $X \in \mathsf{Sm}_S$

$$R_B(M_S(X)) \simeq \mathrm{H}^{\text{sing}}_*(X/S)$$

and

$$R_B(\mathsf{DA}^{\text{ét}}_{\text{lct}}(S, \Lambda) \subset D^b_{\text{ct}}(S^{\text{an}}, \Lambda).$$

Another deep property is the *rigidity theorem*. Define

$$D_{\text{ét}}(S, \Lambda) = D(\mathsf{Sh}_{\text{ét}}(S, \Lambda))$$

and write

$$\iota : (\mathsf{Et}_S, \text{ét}) \hookrightarrow (\mathsf{Sm}_S, \text{ét})$$

for the inclusion, then we get

$$\iota_S^* : D_{\text{ét}}(S, \Lambda) \longrightarrow \mathsf{DA}^{\text{ét}}(S, \Lambda).$$

Theorem 2.2.1 (Ayoub). — *Let S be an excellent, Noetherian, finite dimensional, Λ-finite, with any prime invertible in Λ or $\mathscr{O}_S$. Then ι_S^* is an equivalence.*

This procedure is very flexible and admits many *variants*.

- We can change the input. Instead of complexes we can work with presheaves of simplicial sets or ∞-groupoids.
- We can change the topology. Instead of étale use Nisnevich or use presheaves with transfers.
- Change the geometric context. For example, to rigid analytic motivic sheaves.

Definition. — The **stable motivic homotopy category** over S is

$$\mathsf{SH}(S) := \mathrm{PSp}_T(\mathsf{PSh}(\mathsf{Sm}_S, \mathsf{sSet}))[W^{-1}_{(\mathbb{A}^1, \text{Nis}, \mathbb{P}^1)}].$$

Recall that the Nisnevich topology is between the Zariski and étale topologies. $\mathsf{DA}^{\text{ét}}(S)$ is the motivic version of $D(S^{\text{an}}, \mathbb{Z})$ and $\mathsf{SH}(S)$ is the motivic version of sheaves of $S^?$-spectra on S^{an}. There is also $\mathsf{DM}(S, \Lambda)$ which are the Nisnevich-local motivic sheaves. Many important invariants of varieties only satisfy Nisnevich descent, but not étale descent; for example, K-theory or higher chow groups.

§3. Motives over a field

Let $S = \mathrm{Spec}(k)$ and $\Lambda = \mathbb{Q}$. Define

$$\mathsf{DM}(k, \mathbb{Q}) := \mathsf{DA}^{\text{ét}}(k, \mathbb{Q}).$$

The analogies you should have in mind are

- $D(\mathrm{Ind}\, \mathsf{MHS}_{\mathbb{Q}})$,
- $D(\mathrm{Ind}\, \mathsf{Rep}^{\text{f.d.}}_{\mathbb{Q}_l} G_k)$.

Even though the construction was very formal there are some surprises. Define

$$\mathbb{Q}\langle i \rangle := \mathbb{Q}(i)[2i]$$

be pure Tate twists and $M\langle i \rangle := M \otimes \mathbb{Q}\langle i \rangle$.

- *Projective bundle formula:* Let $E \to X$ be a vector bundle, then
$$M(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{\operatorname{rank} E-1} M(X)\langle i\rangle$$
$$M(\mathbb{P}^n_l) = \Lambda(0) \oplus \Lambda\langle 1\rangle \oplus \cdots \oplus \Lambda\langle n\rangle .$$
- *Gysin triangle:* Let $(c : Z \not\hookrightarrow X) \in \mathsf{Sm}_k$, then we have
$$M(X \setminus Z) \longrightarrow M(X) \longrightarrow M(Z)\langle c\rangle \xrightarrow{+}$$
- *Smooth blow-up formula:*
$$M(\mathrm{Bl}_Z(X)) \simeq M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)\langle i\rangle .$$
- *Poincaré duality 1*: Let X be smooth and projective over k, then $M(X)$ is *dualizable* with
$$M(X)^\vee \simeq M(X)\langle -\dim(X)\rangle .$$

We have $\mathsf{DM}(k,\mathbb{Q}) \simeq \operatorname{Ind}\mathsf{DM}_{\mathrm{ct}}$.

From here on out
$$\mathsf{DM}(k,\Lambda) = \begin{cases} \mathsf{DA}^{\text{ét}}(k,\Lambda) & \Lambda \text{ a } \mathbb{Q}\text{-algebra} \\ \mathsf{DM}(k,\Lambda) & \Lambda \text{ a } \mathbb{Z}\left[\frac{1}{p}\right]\text{-algebra.} \end{cases}$$

For singular varieties $X \in \mathsf{Sch}_R^{\mathrm{ft,sep}}$ we get $M(X) \in \mathsf{DM}(k,\Lambda)$. There are four theories

(i) $M(X)$,

(ii) Borel-Moore cohomoloy $M_{\mathrm{BM}}(X)$ (also denoted $M^c(X)$ in the literature),

(iii) $M^{\mathrm{coh}}(X)$,

(iv) $M^{\mathrm{coh}}_{\mathrm{c}}(X)$.

Localization: Consider a closed immersion $Z \hookrightarrow X$ and the open immersion $X \setminus Z \hookrightarrow X$. We have
$$M_{\mathrm{BM}}(Z) \longrightarrow M_{\mathrm{BM}}(X) \longrightarrow M_{\mathrm{BM}}(?) \xrightarrow{+}$$

$$M^{\mathrm{coh}}_{\mathrm{c}}(X \setminus Z) \longrightarrow M^{\mathrm{coh}}_{\mathrm{c}}(X) \longrightarrow M^{\mathrm{coh}}_{\mathrm{c}}(Z) \xrightarrow{+}$$

Poincaré duality 2: For $X \in \mathsf{Sm}_k$
$$\begin{cases} M(X)^\vee \simeq M_{\mathrm{BM}}(X)\langle -d\rangle \\ M^{\mathrm{coh}}(X)^\vee \simeq M^{\mathrm{coh}}(X)\langle d\rangle . \end{cases}$$

(??)

3.1. Motivic cohomology and algebraic cycles

Definition. — Let $X \in \mathsf{Sm}_k$, we define the **Motivic cohomology groups**
$$\begin{aligned} \mathrm{H}^{p,q}_{\mathrm{mot}}(X) = \mathrm{H}^p_{\mathrm{mot}}(X,\Lambda(q)) &:= \operatorname{Hom}_{\mathsf{DM}(k,\Lambda)}(M(X),\Lambda(q)[p]) \\ &\simeq \operatorname{Hom}_{\mathsf{DM}(X,\Lambda)}(\Lambda_X(0),\Lambda_X(q)[p]). \end{aligned}$$

For $X \in \mathsf{Sch}_k^{\mathrm{ft,sep}}$ define
$$\mathrm{H}^{\mathrm{BM}}_{p,q} := \operatorname{Hom}(\Lambda(q)[p], M_{\mathrm{BM}}(X)).$$

3.1.1. Weight 1 motivic cohomology

Lemma 3.1.1. — *We have*

$$M_S^{\mathit{eff}}(\mathbb{G}_m) \simeq \Lambda(0) \oplus \Lambda(1)[1].?$$

Proof. $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$, so by Mayer-Vietoris we get

$$M(\mathbb{G}_m) \longrightarrow M(\mathbb{A})^{\oplus 2} \longrightarrow M(\mathbb{P}^1) \xrightarrow{+}$$

hence by $\mathbb{A}^1$-invariance

$$M(\mathbb{G}_m, 1) \simeq M(\mathbb{P}^1, 1)[-1].$$

□

The map $\alpha_{\mathbb{G}_m} : \Lambda_{\text{ét}}[\mathbb{G}_m] \to \mathbb{G}_m \otimes \Lambda$ induces

$$\Lambda(1)[1] \xrightarrow{(*)} \Sigma_T^\infty(\mathbb{G}_m \otimes \Lambda).$$

Theorem 3.1.1. —
1) (*) is an isomorphism, so

$$\mathrm{Pic}(s) \otimes \Lambda \xrightarrow{c_1} \mathrm{H}^{2,1}_{mot}(S)$$

2) For S regular

$$\mathrm{H}^{n,1}_{mot}(S) = \begin{cases} \mathscr{O}_S^\times \otimes \Lambda & n = 1 \\ \mathrm{Pic}(S) \otimes \Lambda & n = 2 \\ 0 & \textit{otherwise}. \end{cases}$$

3.1.2. Higher Chow groups Let $\mathbf{\Delta}^\bullet_{\mathrm{alg},k} \in (\mathsf{Sm}_k)^{\mathbf{\Delta}}$.

Definition. — Let $X \in \mathsf{Sch}_k^{\mathrm{ft}}$ define

$$\mathfrak{z}_n(X, r) \subseteq Z_n(X \times \mathbf{\Delta}^r_{\mathrm{alg}}) \otimes \Lambda$$

generated by integral subvarieties of dimension n which intersect all faces properly.

(Picture) We get $d : \mathfrak{z}_n(X, r) \to \mathfrak{z}_{n-1}(X, r-1)$ so $\mathfrak{z}_n(X, \bullet)$ is a *Bloch cycle complex*. (??)

Theorem 3.1.2 (Voevodsky+...). — *Let k be perfect,* $X \in \mathsf{Sch}_k^{\mathit{ft,sep}}$ *then*

$$\mathrm{H}^{BM}_{p,q}(X) \simeq \mathrm{CH}_q(X, p - 2q, \Lambda).$$

If $X \in \mathsf{Sm}_k$ *then*

$$\mathrm{H}^{p,q}_{mot}(X) \simeq \mathrm{CH}^q(X, 2q - p, \Lambda)$$
$$\mathrm{H}^{2n,n}_{mot}(X) \simeq \mathrm{CH}^n(X, \Lambda).$$

(??)

3.2. Examples (Tate)

Definition. — Define
$$\mathsf{DMT}(k,\Lambda) = \langle \Lambda(n) | n \in \mathbb{Z} \rangle^{\mathrm{df}}$$
the **mixed Tate motives**. It constains $\bigoplus \Lambda \langle i \rangle^{\oplus n_i}$ the **pure Tate motives**.

We have $M(\mathbb{A}^n) = \Lambda(0)$ and $M_{\mathrm{BM}}(\mathbb{A}^n) = \Lambda \langle n \rangle$.

Exercise. — Show that
$$M(\mathbb{A}^n \setminus \{0\}) \simeq \Lambda(0) \oplus \Lambda(n)[2n-1].$$

3.2.1. Cellular varieties

Definition. — $X \in \mathsf{Sch}_k^{\mathrm{ft}}$ is **cellular** if there exists a closed subscheme $Z \hookrightarrow X$ such that $X \setminus Z \xrightarrow{\sim} \mathbb{A}^i_k$ and Z is cellular.

Proposition 3.2.1. — *Suppose X is cellular:*

a) We have
$$M_{BM}(X) \simeq \bigoplus_{i=0}^{d} \Lambda \langle i \rangle^{n_i},$$
where n_i is the number of cells of dimension i.

b) If X is also smooth
$$M(X) \simeq \bigoplus_{j=0}^{d} \Lambda \langle j \rangle^{m_j},$$
where m_j is the number of cells of codimension j.

Example 3.2.1. —

1) Let G be split reductive, $B \subset G$ be a Borel, then G/B is cellular (Bruhat decomposition) and
$$M(G/B) \simeq \bigoplus_{i \geq 0} \Lambda \langle i \rangle^{n_i}$$
where n_i is the number of $w \in W$ of length i.

2) Let X be quasiprojective and smooth (??)

3.2.2. Reductive groups

Theorem 3.2.1 (Biglami). — *If G is split reductive, then*
$$M(G) \simeq \mathrm{Sym}^* \left(\bigoplus_{i \geq 1} \Lambda(i)[2?-i]^{\oplus n_i} \right)$$
so by Chevalley
$$R[\mathfrak{g}]^G = k[q_1, \ldots, q_?]$$
where $\deg q_j = d_j$ and n_i is the number of j such that $d_j = i$.

Example 3.2.2. — We have
$$M(\mathbf{GL}_n) = \mathrm{Sym}^* \left(\Lambda(1)[1] \oplus \Lambda(2)[3] \oplus \cdots \oplus \Lambda(n)[2n-1] \right)$$
$$M(\mathbf{SL}_n) = \times(??)$$

Exercise. — What is $M(\mathrm{Sp}_{2n})$?

3.3. Examples (non-Tate)

3.3.1. Curves

Proposition 3.3.1. — *Let C be a smooth projective curve with a 0-cycle (with Λ-coefficients) of degree 1 (or if Λ is a $\mathbb{Q}$-algebra)*
$$M(C) \simeq \Lambda(0) \oplus M_1(C) \oplus \Lambda\langle 1\rangle .$$
If $g(C) > 0$ then $M_n(C) \notin \mathsf{DMT}(k, \Lambda)$.

3.3.2. Commutative algebraic groups

Theorem 3.3.1 (?). — *We take $\Lambda = \mathbb{Q}$ and G/k a smooth commutative group (e.g. a (semi-)abelian variety). Define*
$$M_1(G) := \Sigma_T^\infty(G \otimes \mathbb{Q}) \in \mathsf{DM}(k, G).$$
Then
$$M(G) \simeq \left(\bigoplus_{i=0}^{?} \mathsf{Sym}_i(M_1(G))\right) \otimes M(?).$$

§4. Six functor formalism

4.1. Betti sheaves

Definition. — Define
$$\begin{aligned} D_B(-) : \mathsf{Var}_{\mathbf{C}}^{\mathsf{op}} &\longrightarrow \mathsf{TriCat}^{\otimes} \quad (\text{better } \mathsf{CAlg}(\mathrm{Pr}^L)) \\ X &\longmapsto D\left(\mathsf{Sh}(X^{\mathrm{an}}, \Lambda)\right) \\ f &\longmapsto f^* = \mathrm{L}f^* \quad \text{pullback} \end{aligned}$$

D_B is symmetric monoidal
$$f^*(F \otimes G) \simeq f^*(F) \otimes f^*(G) + \dots$$
(note that we write $\otimes = \otimes^{\mathbb{L}}$).

Proposition 4.1.1. — *$(f^*, f_* = \mathrm{R}f_*)$ is an adjoint pair. And $D_B(X)$ is closed, i.e. there exists $\underline{\mathrm{Hom}}(F, G)$.*

Definition. — A **sheaf theory** is a symmetric monoidal functor
$$D(-) : (\mathsf{Sch}_S^{\mathsf{ft}})^{\mathsf{op}} \longrightarrow \mathsf{TriCat}^{\otimes}/\mathsf{CAlg}(\mathrm{Pr}^L)$$

So we have four functors $(\otimes, \underline{\mathrm{Hom}})$ and (f^*, f_*) which form adjoint pairs.

Example 4.1.1. —
- Derived categories of étale/l-adic sheaves.
- Dervied categories of (holonomic) D-modules.
- Derived categories of mixed Hodge modules.
- ??
- $D(\mathsf{QCoh}(-))$.

Let $f : Y \to X$ be separated of finite type, then we have two functors $f_! : D_B(Y) \leftrightarrows D_B(Y) : f^!$ and $f_!$ gives us relative cohomology with compact support.

These satisfy a bunch of natural transformations:
- *Base change:* Let
$$\begin{array}{ccc} Y' & \xrightarrow{\widetilde{f}} & X' \\ \downarrow{\scriptstyle \widetilde{g}} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$
be Cartesian, then we get a natural transformaton $f^* g_*(-) \to \widetilde{g}_* \widetilde{f}^*(-)$.
- *Projection:* We have a natural transformation
$$f_*(-) \otimes - \longrightarrow f_*(- \otimes f^*(-)).$$

- *Künneth:* We have the natural transformation
$$f_*(-) \otimes g_*(-) \longrightarrow (f \times g)_*(- \boxtimes_X -)$$
where $\boxtimes_X := \mathrm{pr}_1^*(-) \otimes \mathrm{pr}_2^*(-)$.

Theorem 4.1.1. — *Let $D = D_B$. Assume g is proper the (BC) and (Proj) are isomorphisms. If f is also proper then (Kü) is also an isomorphism.*

Proposition 4.1.2 (Open base change). — *Assume f is an open immersion. Then (BC) is an isomorphism.*

Definition. — Let $f : Y \to X$ be separated of finite type and $F \in \mathsf{Sh}(X^{\mathrm{an}}, \Lambda)$. Define
$$(f_!F)(U) := \left\{ s \in F(f^{-1}(U)) \,\middle|\, f|_{\mathrm{Supp}(s)} \text{ is proper} \right\} \subset (f_*F)(U)$$
is the **pushforward with compact support**. We also write
$$f_! := \mathrm{R}f_! : D(Y) \longrightarrow D(X).$$
$f_! \to f_*$ is an isomorphism for f proper (??).

Lemma 4.1.1. — *Suppose $j : U \hookrightarrow X$ is an open immersion.*
1) $j_! : \mathsf{Sh}(U^{an}) \to \mathsf{Sh}(X^{an})$ is "extension by zero"
$$(j_!F)_x = \begin{cases} F_x & x \in U \\ 0 & \textit{otherwise}. \end{cases}$$
2) $j_!$ is left adjoint to j^*.
3) We have open BC: $f^*j_! \simeq \widetilde{j}_!\widetilde{f}^*$ and open Proj
$$j_!(- \otimes j^*(-)) \simeq j_!(-) \otimes -.$$

Let $f : Y \to X$ be a separated morphism of finite type, then there exists a Nagata compactification where f factors as
$$Y \overset{j}{\hookrightarrow} \overline{Y} \overset{p}{\to} X$$
where j is an open immersion and p is proper. Then
$$j_! \simeq p_!j_! \simeq p_*j_!.$$

Theorem 4.1.2. — *(BC) We have $g^*f_! \xrightarrow{\sim} \widetilde{f}_!\widetilde{g}^*$.*
(Proj) $f_!(- \otimes f^*(-)) \xrightarrow{\sim} f_!(-) \otimes -$.
(Kü) $f_!(-) \otimes g_!(-) \xrightarrow{\sim} (f \times g)_!(- \boxtimes -)$.

Proposition 4.1.3. — *Let f be a separated morphism of finite type. The functor $f_! : D_B(Y) \to D_B(X)$ commutes with all coproducts. So by the Adjoint Functor Theorem, $f_!$ has a right adjoint $f^! : D_B(X) \to D_B(Y)$ called the **exceptional pullback**.*

Example 4.1.2. — If j is an open immersion (étale) then $j^! \simeq j^*$.

Proposition 4.1.4 (Formal local duality). — *There is an isomorphism*
$$\underline{\mathrm{Hom}}(f_!F, G_)) \xrightarrow{\sim} f_*\underline{\mathrm{Hom}}(F, f^!G).$$

Exercise. — Prove this!

Example 4.1.3. — Let $\pi : X \to \mathrm{Spec}(\mathbf{C})$, then
$$\mathrm{H}_c^*(X, \mathbb{Q})^\vee \simeq \mathrm{H}^*(X, \pi^!\mathbb{Q}).$$
To recover Poincaré duality, we need to compute $\pi^!\mathbb{Q}$ for X smooth.

Theorem 4.1.3 (Duality for smooth morphisms). — *Let $q : Y \to X$ be a separated morphism of finite type.*
1) There is a canonical natural transformation
$$\alpha_f : f^!\Lambda \otimes f^*(-) \longrightarrow f^!(-).$$
2) Let f be smooth separated of relative dimension d, then

- *α_f is an isomorphism,*
- *$f^!\Lambda \simeq \Lambda\langle d\rangle$.*

(Better $\Lambda(1) \simeq \Lambda$.)

3) If f is smooth then f^* has a left adjoint

$$f_\sharp = f_! \langle d\rangle .$$