MOTIVIC SHEAVES

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Plan:
L1: Motivation, construction, examples.
L2: Six-functor formalism.
L3: Motivic *t*-structures and weight structures.
L4: ∞-categorical methods.

§1. Motivation from GRT and cohomology

1.1. Cohomology and sheaves for representation theory

Question: How do you construct interesting representations?
Answer:

1) Find interesting actions.
2) Linearlize them.

Example 1.1.1. — Let K be a compact Lie group. The action of K on itself gives us an action of K on $L^2(K)$ with respect to a Haar measure. The Peter-Weyl theorem says that

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \text{ unitary}}} \pi^{\oplus dim(\pi)}.$$

"Lie theory $\subset$ algebraic geometry". Reductive groups are algebraic groups with many associated varieties with group actions: flag varieties...

The linearizations we consider in this course are the many types of cohomology theories.

Example 1.1.2 (Borel-Weil-Bott). — Let $T \subset B \subset G$ be a reductive group over $\mathbf{C}$. Let $\lambda \in X^\vee(T)$ such that there exists $w \in W$ with $w * \lambda = w(\lambda + \rho) - \rho > 0$ (where $\rho = (1/2)\sum_{\alpha \in \Phi^+} \alpha$). Then

$$\mathrm{R}\Gamma(G/B, L_\lambda) \simeq \pi_{w*\lambda}[-\ell(w)]$$

where $\ell(w)$ is the length of w.

Cohomology fits in the wider context of sheaf theory. If T is a locally contractible topological space, then

$$\mathrm{H}^n_{\mathrm{sing}}(T, \mathbb{Z}) \simeq \mathrm{H}^n(T, \underline{\mathbb{Z}}_T) \simeq \mathrm{R}^n(\pi_T)_*(\underline{\mathbb{Z}}_T)$$

where π_T is the morphism $\pi_T : T \to \mathrm{pt}$ with

$$\mathrm{R}\pi_{T*} : D(T, \mathbb{Z}) \longrightarrow D(\mathbb{Z}) \simeq D(*, \mathbb{Z}).$$

Cohomology (singular with $\mathbb{Q}$-coefficients) of algebraic varieties over $\mathbf{C}$ is ***very*** special.

- There is a weight filtration.
- There is a mixed hodge structure.

Sheaves on complex algebraic varieties are also very special:

- Perverse sheaves;
- Decomposition theorem;
- Mixed Hodge modules.

This leads to great success stories in GRT:

- Springer theory;
- Kazdhan-Lustig theory;
- geometric Satake...

1.2. From sheaves to motivic sheaves There are situations which can't be directly studied using these tools:

- Representation theory of reductive groups over other fields/rings/schemes.
- Modular/integral representation theory.
- q-deformations, quantum groups, canonical bases.

These can be attacked using:

- l-adic sheaves,
- sheaves cohomology with $\mathbb{Z}$-coefficients,
- K-theory.

Motivic sheaves will give us a unified perspective.

Motivic dream: There should exist universal cohomology/sheaf theories such that

1) they unify and "explain" the special structure in cohomology/sheaves;
2) they are of algebro-geometric nature;
3) they "explain" the realization of algebraic cycles and algebraic K-theory.

§2. Construction of $\mathsf{DA}^{\text{ét}}$ and SH (Morel-Voevodsky)

2.1. Triangulated categories and localization

Definition. — A **triangulated category** is the data:

- an additive category C,
- an automorphism $\Sigma = (-)[1] : \mathsf{C} \xrightarrow{\sim} \mathsf{C}$,
- a collection of distinguished triangles
$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

The data satisfies the following conditions:

- shifted distinguished triangles are distinguished up to isomorphism,
- for all $f : A \to B$ there exists
$$A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \xrightarrow{+}$$
unique up to isomorphism and functorial,
-
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\simeq"', "f"] & B \arrow[r] \arrow[d, "\simeq"', "g"] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\simeq"', "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$
- (??)

Example 2.1.1. — Let A be an abelian category, $\mathsf{Ch}(\mathsf{A})$ be the abelian category of chain complexes in A. We define $(A[1])_n = A_{n-1}$. Given $f : A_\bullet \to B_\bullet$ the maping cone is given by
$$\mathrm{Cone}(f)_n = A_{n-1} \oplus B_n, \qquad d_n = \begin{pmatrix} -d^A_{n-1} & 0 \\ f & d^B_n \end{pmatrix}.$$

Definition. — $f : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if for all $n \in \mathbb{Z}$, the map $H_n(A_\bullet) \simeq H_n(B_\bullet)$ is an isomorphism.

Definition. — $D(\mathsf{A})$ is defined as the localization of $\mathsf{Ch}(\mathsf{A})$ by quasi-isomorphisms.

Now we consider reflexive localizations (1-categorical ones lead to triangulated and ∞-categorical ones).

Definition. — Let C be a 1-category.

1) $\mathsf{C}' \subset \mathsf{C}$ is **reflexive** if $\iota : \mathsf{C}' \to \mathsf{C}$ has a left adjoint.
2) $L_W : \mathsf{C} \to \mathsf{C}[W^{-1}]$ is **reflexive** if L_W has a right adjoint.

Lemma 2.1.1 (Reflexive subcategories are the same thing as reflexive localizations). —

a) *Let $\mathsf{C}' \subset \mathsf{C}$ be reflexive, $L : \mathsf{C} \to \mathsf{C}'$ be the left adjoint to ι. Define*

$$W_L = \{f : L(f) \text{ is an isomorphism}\}.$$

Then $\mathsf{C}' \simeq \mathsf{C}[W_L^{-1}]$ and $L \simeq L_{W_L}$.

b) *If L is a reflexive localization, then its right adjoint ι is fully faithful and $\iota : \mathsf{C}[W^{-1}] \xrightarrow{\sim} \mathrm{EssIm}(\iota) \subset \mathsf{C}$.*

Definition. — Let $S \subset \mathsf{C}$ be a collection of morphisms.

a) $A \in \mathsf{C}$ is S**-local** if for all $f : B \to C$ in S

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

b) $f : B \to C$ is an S**-equivalence** if for all S-local A

$$\mathrm{Hom}_{\mathsf{C}}(C, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(B, A).$$

Lemma 2.1.2. — *If $L : \mathsf{C} \rightleftarrows \mathsf{C}' : \iota$ is a reflexive localizaton, W_L as before, then*

- *ι gives an isomorphism between C' and W_L-local objects.*
- *W_L are the W_L-equivalences.*

Definition. — Let D be a triangulated category with all small products.

- Let κ be a regular cardinal (for example $\kappa = \aleph_0$). Then $A \in \mathsf{D}$ is κ**-small**/κ**-compact** if and only if

$$\operatorname*{colim}_{\substack{I' \subset I \\ |I'| < \kappa}} \mathrm{Hom}\left(A, \bigoplus_{I'} B_i\right) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_{I} B_i\right).$$

- **Compact** means $\aleph_0$-small. A is compact if and only if

$$\bigoplus_I \mathrm{Hom}(A, B_i) \xrightarrow{\sim} \mathrm{Hom}\left(A, \bigoplus_I B_i\right).$$

- D is **presentable**/**well-generated** if and only if there exist κ and a set $S \subset \mathsf{D}$ of κ-small objects which generate D:

$$\forall B \in \mathsf{D}, (\forall A \in S, \mathrm{Hom}(A, B) = 0) \implies B = 0.$$

- D is **compactly generated** if it is $\aleph_0$-presentable.

Definition. — $\mathsf{E} \subset \mathsf{D}$ is **localizing** if it is

- triangulated,
- stable under coproducts,
- thick (stable under subobjects and subquotients).

Theorem 2.1.1 (Adjoint Functor Theorem). — *Let D, D' be triangulated categories with all coproducts, $F : \mathsf{D} \to \mathsf{D}'$ be a triangulated functor and D be presentable. Then F admits a right adjoint if and only if F preserves all coproducts.*

Corollary 2.1.1 (Verdier Localization). — *Let D be a presentable category and E be a localizing subcategory. Define*

$$\mathsf{D}/\mathsf{E} = D[W_{\mathsf{E}}^{-1}], \quad W_{\mathsf{E}} = \{f : \mathrm{Cone}(f) \in \mathsf{E}\}.$$

Then $\mathsf{D} \to \mathsf{D}/\mathsf{E}$ is a reflexive localization.

Let $S \subset \mathsf{D}$ be a subset of objects, then $\langle\langle S \rangle\rangle$ is the smallest subcategory containing S such that $\mathsf{D}/\langle\langle S \rangle\rangle$ is a reflexive localization.

Let's get some inspiration from Betti homology, i.e. singular homology on complex algebraic varieties. Let $X \in \mathsf{Var}_{\mathbf{C}}^{(\mathrm{f.t})}$, then we get

$$C_*^{\mathrm{sing}}(X(\mathbf{C}), \mathbb{Z}) \in D(\mathbb{Z}).$$

This comes with some data:

(a)
- $D(\mathbb{Z})$ has a symmetric monoidal structure: $\otimes^{\mathbb{Z}}$,
- (Künneth) $C_*(X \times Y) \simeq C_*(X) \otimes^{\mathbb{Z}} C_*(Y)$.

which satisfies sproperties:

(b) ($\mathbb{A}^1$-homotopy invariance) $C_*(X \times \mathbb{A}^1) \xrightarrow{\sim} C_*(X)$ ($(\mathbb{A}^1)^{\mathrm{an}} = \mathbf{C}$ is contractible).
(c) (Mayer-Vietoris sequence) Let $X = U \cup V$ be a Zariski cover, then

$$C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*(X) \xrightarrow{\partial} C_*(U \cap V)[1].$$

(d) (Étale descent) Let $U \to X$ be étale surjective. Define

$$\check{C}_n(U/X) = U^{n+1}.$$

Then $\check{C}_\bullet(U/X)$ is a simplicial scheme, so $C_*(\check{C}_\bullet(U/X))$ is a simplicial complex of abelian groups and $C(C_*(\check{C}_\bullet(U/X)))$ is a double complex. (??)

Concretely we have a descent spectral sequence which gives us ($U = U \cup V$) Mayer Vietoris.
(e) ($\mathbb{P}^1$-stabilization)

$$\begin{aligned} C_*(\mathbb{P}^1_{\mathbf{C}}) &\simeq C_*(\mathrm{pt}) \oplus \widetilde{C}_*(\mathbb{P}^1_{\mathbf{C}}) \\ &\simeq \mathbb{Z}[0] \oplus \mathbb{Z}(1)[2]. \end{aligned}$$

$\mathbb{Z}(1)$ is $\oplus$-invertible.

Smooth varieties play a special role:

- Poincaré duality
- Gysin sequences
- $C_*(-)$ also satisfies "h-descent", so $C_*(-)$ is "determined" by $C_*(-)_{|(?)}$.

There is an associated sheaf theory:

$$D_B(-) : \mathsf{Var}_{\mathbf{C}} \longrightarrow \mathsf{TriCat}^{\otimes}, \quad X \longmapsto D_B(X) = D(\mathsf{Sh}(X^{\mathrm{an}}, \mathbb{Z})).$$

Sketch of $\mathsf{DA}^{\acute{e}t}$*:* Let S be a base scheme.

- Start with
$$\begin{cases} D(\mathsf{PSh}(\mathsf{Sm}_S, \mathbb{Z})) = D_{\mathsf{PSh}}(S) \\ \mathbb{Z}[-] : \mathsf{Sm}_S \to D_{\mathsf{PSh}}(S) \end{cases}.$$
- Impose $\mathbb{A}^1$-invariance, étale descent, and $\mathbb{P}^1$-stability. This will give us $\mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$ and $M_S(-) : \mathsf{Sm}_S \to \mathsf{DA}^{\text{ét}}(S, \mathbb{Z})$.

The surprise is that the result satisfies many *other* properties of singular (co)homology and derived categories of sheaves on complex varieties which can largely be packaged into the six-functor formalism. The result is closely related to algebraic cycles/higher Chow groups and algebraic K-theory.