INFINITY CATEGORIES

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§1. SIMPLICIAL SETS

Definition. — The **simplex category $\mathbf{\Delta}$** is the category of

- finite non-empty totally ordered sets
- order-preserving maps.

Notation. — $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n \in \mathbb{Z}_{\geq 0}$.

Every object in $\mathbf{\Delta}$ is (uniquely) isomorphic to some $[n]$.

Definition. — A **simplicial set** is a functor

$$\mathscr{S} : \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$$

Notation. — $\mathscr{S}_n := \mathscr{S}([n])$, call this the **set of n-simplices** of $\mathscr{S}$. 0-simplices are called **vertices**, 1-simplices are called **edges**.

Example 1.0.1. — Let C be a set. Let $\underline{C} : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Sets}$ be the constant functor:

$$\underline{C}_n = C \quad \forall n,$$
$$\underline{C}(\alpha) = \mathrm{id} \quad \forall \alpha : [m] \longrightarrow [n] \text{ in } \mathbf{\Delta}.$$

This is called a **discrete simplicial set**.

Definition. — Let $\mathscr{S}$ be a simplicial set. Given $\alpha : [n] \to [n-1]$ we get $\mathscr{S}(\alpha) : \mathscr{S}_{n-1} \to \mathscr{S}_n$. The n-simplices in the image are called **degenerate** simplices, i.e. σ is degenerate if there is an α such that $\sigma \in \mathrm{im}(\mathscr{S}(\alpha))$.

Lemma 1.0.1. — *A simplicial set is discrete if and only if for all $n \geq 1$ all n-simplices are degenerate.*

Exercise. — Prove it.

Example 1.0.2. — Let $(P, \geq)$ be a poset. Define a simplicial set $N(P, \leq)$ called the **nerve** of $(P, \leq)$ by

$$N(P, \leq)_k = \{\text{chains } p_0 \leq p_1 \leq \cdots \leq p_k : p_i \in P\}$$

where a chain is a totally ordered subset.

Exercise. — Finish the definition. Which simplices are degenerate?

Example 1.0.3 ("Standard n-simplex"). — The **standard n-simplex** is

$$\Delta^n := N([n]).$$

(Pictures)

Note. — For $j \in [n]$, we get a subsimplicial set

$$N([n] \setminus \{j\}) \subset \Delta^n$$

isomorphic to Δ^{n-1} called the j^{th} **face** of Δ^n. (Picture)

Example 1.0.4 (Horns). — Let $n \geq 0$ and $0 \leq j \leq n$, define the **horn**

$$\Lambda^n_j := \begin{array}{c} \text{subsimplicial set of } \Delta^n = N([n]) \\ \text{consisting of chains } p_0 \leq p_1 \leq \cdots \leq p_k \ \ (\textit{Pictures}) \\ \text{such that } \{p_0, \ldots, p_k\} \not\supset [n] \setminus \{j\}. \end{array}$$

Example 1.0.5 ($(n-1)$-sphere $\partial\Delta^n$). — We define the $(n-1)$-**sphere**

$$\partial\Delta^n := \begin{array}{c}\text{subsimplicial set of } \Delta^n \\ \text{chains } p_0 \leq \cdots \leq p_k \text{ such that } \{p_0 \leq \cdots \leq p_k\} \neq [n]\end{array}$$

Example 1.0.6 (Products). — Let $\mathscr{S}, \mathscr{T}$ be simplicial sets. We define their **product** $\mathscr{S} \times \mathscr{T}$ as

$$(\mathscr{S} \times \mathscr{T})_k = \mathscr{S}_k \times \mathscr{T}_k.$$

(Picture)

Example 1.0.7. — Let C be an ordinary category. We define its **nerve** $N(\mathsf{C})$ as

$$N(\mathsf{C})_k := \left\{ \begin{array}{c}\text{composable sequences of morphisms} \\ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \ldots \xrightarrow{f_k} X_k\end{array} \right\}.$$

Example 1.0.8. — Let X be a topological space. The **singular simplicial set** of X is defined as

$$\mathrm{Sing}(X)_k = \{\text{continuous maps } |\Delta^k| \longrightarrow X\},$$

where $|\Delta^k|$ is the standard k-simplex

$$|\Delta^k| = \left\{ (x_0, \ldots, x_k) \in \mathbf{R}^{k+1} \middle| x_i \geq 0, \sum x_i = 1 \right\}.$$

Exercise. — What does this do to the morphisms in $\mathbf{\Delta}$?

Definition. — A **Kan complex** is a simplicial set X such that for every diagram

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "\text{any map}"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

we can fill the dashed arrow. This is called an **extension problem**. If the arrow exists we say that the extension problem **admits a solution**.

Daniel Kan discovered Kan complexes in 1958. The *key fact* is that $\mathrm{Sing}(X)$ *is* a Kan complex. The theme from 1958 to today is that Kan complexes are a "combinatorial model" for algebraic topology which allows us to do homotopy theory.

Definition. — Let X be a Kan complex and $\mathscr{S}$ be any simplicial set. Two maps $f, g : \mathscr{S} \to X$ are said to be **homotopic** if there exists a map $H : \mathscr{S} \times \Delta^1 \to X$ such that

$$H|_{\mathscr{S}\times\{0\}} = f, \quad H|_{\mathscr{S}\times\{1\}} = g.$$

Lemma 1.0.2. — *This is an equivalence relation.*

Proof. Omitted, tricky for an exercise. This requires X to be a Kan complex. □

Definition. — Let X be a Kan complex and x_0 be a vertex of X. Let

$$\mathrm{Loops}_{x_0} = \{\text{maps } \gamma : \Delta^n \longrightarrow X \text{ such that } \gamma|_{\partial\Delta^n} \text{ is the constant map to } x_0\}.$$

We say $\gamma, \gamma' \in \mathrm{Loops}_{x_0}$ are **relatively homotopic** (**rel. homotopic**) if there exists $H : \Delta^n \times \Delta^1 \to X$ such that

$$H|_{\Delta^n\times\{0\}} = \gamma, \quad H|_{\Delta^n\times\{1\}} = \gamma', \quad H|_{\partial\Delta^n\times\Delta^1} = \text{const. map to } x_0.$$

Define

$$\pi_n(X, x_0) := \frac{\mathrm{Loops}_{x_0}}{\text{rel. homotopy}}.$$

Fact. — For $n \geq 1$, $\pi_n(X, x_0)$ is a group. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Definition. — An **∞-category** (or **quasi-category**) is a simplicial set $\mathscr{C}$ such that any extension problem

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

with $0 < j < n$ (**inner horns**) admits a solution. (Picture) An ∞-category is also called a **weak Kan complex**.

Lemma 1.0.3. — *Let* C *be an ordinary category, then* $N(\mathsf{C})$ *is an* ∞*-category.*

Digression: Let I^n be the simplicial set consisting of n consecutive 1-simplices (**n-spine**) (Picture). A naive alternative definition is: $\mathscr{C}$ is an infinity category if every

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & \mathscr{C} \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

has a solution. This is WRONG (but its wrongness is subtle), even though N(ord. cat.) satisfy this. There is a book by Markus Land "Introduction to ∞-categories" which explores this. The definition of ∞-categories was introduced by Boardman-Vogt in 1972. Joyal started generalizing results from ordinary category theory to ∞-categories in 2006. Lurie is largely responsible for how well this notion is developed in modern literature.

Remark. — Having a unique solution to the lifting problem characterizes nerves of ordinary categories.

Definition. — Let $\mathscr{C}$ be an ∞-category. An **object** is a vertex. A **morphism** is an edge. An **identity morphism** is a degenerate edge. Say that h is *a* **composition** of g and f if there exists a 2-simplex such that (Picture).

Remark. — Compositions are NOT unique in ∞-categories.

Example 1.0.9 (∞-categories). —

1) Topological spaces Top.
   - Objects are topological spaces.
   - Morphisms are continuous maps.
   - A 2-simplex is a (not necessarily commutative) diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"] & & X_2 \end{tikzcd}$$

   and a homotopy $H : X_0 \times [0,1] \to X_2$ from gf to h.
   - A 3-simplex is a diagram

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_1 \arrow[rr] \arrow[dr] & & X_3 \\ & X_2 \arrow[ur] & \end{tikzcd}$$

with continuous maps $f_{ij} : X_i \to X_j$ for $i < j$, homotopies $T_{ijk} X_i \times [0,1] \to X_k$ from $f_{jk} \circ f_{ij}$ to f_{ik}, and $H : X_0 \times [0,1]^2 \to X_3$ (**higher homotopy**) such that $H|_{\mathrm{bdry}}$ is

$$\begin{tikzcd} (0,0) \arrow[r, "T_{123} f_{01}"] \arrow[d, "f_{23} T_{012}"'] & (0,1) \arrow[d, "T_{013}"] \\ (1,0) \arrow[r, "T_{023}"'] & (1,0) \end{tikzcd}$$

2) The ∞-category of ordinary categories Cat_1.
- Objects are ordinary categories.
- Morphisms are functors.
- A 2-simplex is a (not necessarily commutative) diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"] & & X_2 \end{tikzcd}$$

  and a natural isomorphism $T : g \circ f \xrightarrow{\sim} h$.
- A 3-simplex is a diagram

$$\begin{tikzcd} & X_0 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_1 \arrow[rr] \arrow[dr] & & X_3 \\ & X_2 \arrow[ur] & \end{tikzcd}$$

  where f_{ij} are functors and T_{ijk} are natural isomorphism such that

$$\begin{tikzcd} \bullet \arrow[r, "T_{123} f_{01}"] \arrow[d, "f_{23} T_{012}"'] & \bullet \arrow[d, "T_{013}"] \\ \bullet \arrow[r, "T_{023}"'] & \bullet \end{tikzcd}$$

  commutes

A source of ∞-categories are

- ordinary categories with an equivalence relation on morphisms,
- ordinary categories with inverting some morphisms.

Lecture 2

Definition. —

1. Let C be an ∞-category and $f : X \to Y$ be a morphism in C. f is called an **isomorphism** if there exists $g : Y \to X$ and two 2-simplices

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_X"] & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, "g"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd}$$

2. An ∞-category is called an **∞-groupoid** if *every* morphism is an isomorphism.

Theorem 1.0.1 (Joyal). — *An ∞-category is an ∞-groupoid if and only if it is a Kan complex.*

Proof. The forward direction is hard, the converse is an exercise. □

Definition. —

3. Say $f, g : \mathsf{C} \to \mathsf{D}$ are functors (morphisms of simiplicial sets) of ∞-categories. A **natural transformation** from f to g is a functor $T : \mathsf{C} \times \Delta^1 \to \mathsf{D}$ such that $T|_{\mathsf{C}\times\{0\}} = f$ and $T|_{\mathsf{C}\times\{1\}} = g$.

   A special case: the identity natural transformation $\mathrm{id}_f : f \to F$ is the map

$$\mathsf{C} \times \Delta^1 \xrightarrow{\text{proj}} \mathsf{C} \xrightarrow{f} \mathsf{D}.$$

   $T : f \to g$ is a **natural isomorphism** if there exists $T' : g \to f$ and two maps $H : \mathsf{C} \times \Delta^2 \to \mathsf{D}$, $H' : \mathsf{C} \times \Delta^2 \to \mathsf{D}$ such that

$$\begin{tikzcd} & g \arrow[dr, "T'"] & \\ f \arrow[ur, "T"] \arrow[rr, "\mathrm{id}_f"] & & f \end{tikzcd} \qquad \begin{tikzcd} & f \arrow[dr, "T"] & \\ g \arrow[ur, "T'"] \arrow[rr, "\mathrm{id}_g"] & & g \end{tikzcd}$$

In ordinary category theory a natural transformation assigns objects in C to morphisms in D and morphisms in C to commutative squares in D. For ∞-categories a natural transformation takes objects to morphisms, morphisms to diagrams of shape $\Delta^1 \times \Delta^1$ and generally an n-simplex to a diagram of shape $\Delta^n \times \Delta^1$.

Theorem 1.0.2 (Pointwise criterion for natural isomorphism). — *Let $f, g : \mathsf{C} \to \mathsf{D}$ be functors of ∞-categories and $T : f \to g$ be a natural transformation. T is a natural isomorphism if and only if for all objects x in C, $T(\{x\} \times \Delta^1)$ is an isomorphism in D.*

This is a consequence of Joyal's theorem.

Definition. — Define Cat_∞ as follows:

- Objects are ∞-categories.
- Morphisms are functors.
- 2-simplices are diagrams

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"] & & X_2 \end{tikzcd}$$

  and a natural isomorphism $T : g \circ f \xrightarrow{\sim} h$.
- 3-simplices and higher: copy the data of Top and replace $[0,1]^n$ by $(\Delta^1)^n$.

This is similar to Top and Cat_1.

Definition. — Define Spc same as above, except objects are ∞-groupoids.

In literature: ∞-groupoids, Kan complexes, spaces and anima are synonyms.

Definition. — A functor $f : \mathsf{C} \to \mathsf{D}$ is called a **categorical equivalence** if there exists $g : \mathsf{D} \to \mathsf{C}$ such that $f \circ g \simeq \mathrm{id}_\mathsf{D}$ and $g \circ f \simeq \mathrm{id}_\mathsf{C}$.

Theorem 1.0.3 (Fundamental Theorem of Category Theory). — *A functor $f : \mathsf{C} \to \mathsf{D}$ is a categorical equivalence if and only it it's essentially surjective and fully faithful.*

Note that we haven't defined essentially surjective or fully faithful. Let's pre-warm up first before we define them.

Lemma 1.0.4. — *Let X be a Kan complex. X is **contractible** (i.e., categorically equivalent to Δ^0) if and only if every lifting problem*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

admits a solution.

Now we're warm enough to warm up, so lets do that. Let $f : X \to Y$ be a map of Kan complexes. Suppose every lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution. Then f *is* a categorical equivalence (think: homotopy equivalence of topological spaces). But this condition is too strong for the converse. A simple counter-example is to take X contractible.

Definition. — Let $f : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories. Given

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "r"] \arrow[d] & \mathsf{C} \arrow[d, "f"] \\ \Delta^n \arrow[r, "s"] & \mathsf{D} \end{tikzcd} \tag{1}$$

we say it admits a **solution up to isomorphism** if

(i) there exists $u : \Delta^n \to \mathsf{C}$ such that

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "r"] \arrow[d] & \mathsf{C} \\ \Delta^n \arrow[ur, "u"] & \end{tikzcd}$$

(ii) $f \circ u : \Delta^n \to \mathsf{D}$ is naturally isomorphic to $s : \Delta^n \to \mathsf{D}$ *relative* (of relative homotopy) to $\partial\Delta^n$.

Definition. — Let $f : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories.

- It's **essentially surjective** if every diagram (1) with $n = 0$ admits a solution up to isomorphism.
- It's **full** if every diagram (1) with $n = 1$ admits a solution up to isomorphism.
- It's **fully faithful** if every diagram (1) with $n \geq 1$ admits a solution up to isomorphism.

So a functor of ∞-categories is fully faithfull and essentially surjective if all (1) admit a solution up to isomorphism.

Remark. — These definitions of fully and full faithful are *nonstandard*.

Now the Fundamental Theorem makes sense.

Proof idea. The forward direction is easy. Conversely, we factor through

$$\mathsf{C} \longrightarrow \mathsf{C}^{\text{enhanced}} \longrightarrow \mathsf{D}$$

where an n-simplex in $\mathsf{C}^{\text{enhanced}}$ is the data of

- n-simplex in C,
- a diagram of shape $\Delta^n \times \Delta^1$ in D satisfying some conditions.

The inverses of the intermediate maps are easy to construct. □

What is missing so far is *mapping spaces*. Given objects X, Y in an ∞-category C we expect to find a space (Kan complex) $\mathrm{map}_{\mathsf{C}}(X, Y)$ such that the objects of $\mathrm{map}_{\mathsf{C}}(X, Y)$ are morphisms $X \to Y$ *and* it should extend to a functor

$$\mathrm{map}_{\mathsf{C}} : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \longrightarrow \mathsf{Spc}.$$

For 1-categories this is usually called Hom or Mor. Lurie uses Hom for a non-functorial, but easier, version of map.

Here is one non-functorial approach to mapping spaces. Let C^{Δ^1} be the simplicial set such that $(\mathsf{C}^{\Delta^1})_k$ is the sest of maps $\Delta^1 \times \Delta^k \to \mathsf{C}$. By restricting to $\{0\} \times \Delta^k$ and $\{1\} \times \Delta^k$ we get a map

$\mathsf{C}^{\Delta^1} \to \mathsf{C} \times \mathsf{C}$. Define $\mathrm{map}_{\mathsf{C}}(X, Y)$ as the fiber product (of simplicial sets)

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(X,Y) \arrow[r] \arrow[d] & \mathsf{C}^{\Delta^1} \arrow[d] \\
\Delta^0 \arrow[r, "{(x,y)}"] & \mathsf{C} \times \mathsf{C}.
\end{tikzcd}$$

Theorem 1.0.4. — *Let $f : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories. Then f is fully faithful if and only if for all objects X, Y, the induced map*

$$\mathrm{map}_{\mathsf{C}}(X, Y) \longrightarrow \mathrm{map}_{\mathsf{D}}(f(x), f(y))$$

is a categorical equivalence of Kan complexes.

This theorem is actually the usual definition in the literature.

Somewhere along the way:

Theorem 1.0.5 (Whitehead's Theorem). — *A map $f : X \to Y$ of Kan complexes is a categorical equivalence if and only if*

$$\pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0)$$

are bijections for all n and all x_0.

Lecture 3

A monoidal category in ordinary category theory consists of:

- A category C.
- A functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$.
- An object $\mathbb{1} \in \mathsf{C}$.
- 3 natural transformations: the associator, left and right unitors.

We ask them to satisfy 3 axioms:

- Triangle axioms (they say $\mathbb{1} \otimes x = x = x \otimes \mathbb{1}$.
- Pentagon axiom (various ways to group 4 objects).

Mac Lane's Coherence Theorem tells us that every "reasonable" diagram made from the 3 natural transformations commutes.

Let's try to mimic this for ∞-categories. The naive approach is to start with:

- an ∞-category C;
- a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$;
- an object $\mathbb{1} : \Delta^0 \to \mathsf{C}$;
- an associator

$$\begin{tikzcd}
\mathsf{C}\times\mathsf{C}\times\mathsf{C} \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, "\otimes\times\mathrm{id}"] \arrow[dr] & \mathsf{C}\times\mathsf{C} \arrow[d, "\otimes"] \\
\mathsf{C}\times\mathsf{C} \arrow[r, "\otimes"] & \mathsf{C}
\end{tikzcd}$$

a diagram of shape $\Delta^1 \times \Delta^1$ in Cat_∞;

- a left unitor

$$\begin{tikzcd}
\Delta^0\times\mathsf{C} \arrow[rr, "\mathbb{1}\times\mathrm{id}"] \arrow[dr, "\sim"] & & \mathsf{C}\times\mathsf{C} \arrow[dl, "\otimes"] \\
& \mathsf{C} &
\end{tikzcd}$$

a 2-simplex in Cat_∞ and similarly a right unitor;

and ask it to satisfy

- the triangle identity

$$\begin{tikzcd}
\mathsf{C}\times\Delta^0\times\mathsf{C} \arrow[d, "\mathrm{id}\times\mathbb{1}\times\mathrm{id}"'] \arrow[drr] \arrow[ddr] \arrow[ddrr] & & \\
\mathsf{C}\times\mathsf{C}\times\mathsf{C} \arrow[rr] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[drr] & & \mathsf{C}\times\mathsf{C} \arrow[d] \\
& \mathsf{C}\times\mathsf{C} \arrow[r] & \mathsf{C}
\end{tikzcd}$$

two 3-simplices attached along a face.

We model these diagrams with totally ordered finite sets:

- Associator

$$\begin{tikzcd} 0123 & 013 \arrow[l] \\ 023 \arrow[u] & 03 \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd}$$

- Left unitor

$$\begin{tikzcd} \frac{1}{2}2 & & 012 \arrow[ll, "(01{,}2)\mapsto(\frac{1}{2}{,}2)"'] \\ & 02 \arrow[ul, "\sim"] \arrow[ur, hook] & \end{tikzcd}$$

- Triangle identity

$$\begin{tikzcd} 0\frac{3}{2}3 & & \\ 0123 \arrow[u] & & 013 \arrow[ull, "\sim"'] \arrow[ll, "\sim"] \\ & 023 \arrow[ul] \arrow[uul] & 013 \arrow[l] \arrow[u] \arrow[uull] \end{tikzcd}$$

The data for the naive approach is modeled by

$$N\left(\left(\begin{matrix}\text{nonempty, finite, totally ordered}\\ \text{sets of size} \leq 4\end{matrix}\right)^{\mathsf{op}}\right) \longrightarrow \mathsf{Cat}_\infty$$

We are still missing the pentagon axiom and Mac Lane's Coherence Theorem.

Definition. — Let C be an ∞-category. A **monoidal structure** on C is a functor $F : N(\mathbf{\Delta}^{\mathsf{op}}) \to \mathsf{Cat}_\infty$ such that

1) $F([1]) = \mathsf{C}$,
2) for all n

$$[n] \leftarrow \{0,1\}, \{1,2\}, \ldots, \{n-1, n\}$$

induces

$$F([n]) \longrightarrow \mathsf{C} \times \mathsf{C} \times \cdots \times \mathsf{C} = \mathsf{C}^n$$

which we require to be an equivalence of categories.

Note that $F([0]) \xrightarrow{\sim} \Delta^0$. *The idea* is that the pentagon axiom and all "higher coherences" are encoded in $\mathbf{\Delta}^{\mathsf{op}}$.

The problem with this definition is unusable. Actually writing down a functor $N(\mathbf{\Delta}^{\mathsf{op}}) \to \mathsf{Cat}_\infty$ is too complicated. What do we do? Lurie will rescue us.

Let's warm up. Suppose you have a Λ^2_0 horn

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\exists?"] \\ 0 \arrow[ur, "f"] \arrow[r, "h"'] & 2 \end{tikzcd}$$

so we are asking: given

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & \mathsf{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

when does a solution exist? It exists if f has a right inverse.

Definition. — Let $p : \mathsf{C} \to \mathsf{D}$ be a functor of ∞-categories, $f : x \to y$ be a morphism in C. We say f is p-**cocartesian** if

$$\begin{tikzcd}
\{0,1\} = \Delta^1 \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathsf{C} \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \mathsf{D}
\end{tikzcd}$$

has a solution.

Definition. — $p : \mathsf{C} \to \mathsf{D}$ is called a **cocartesian fibration** if lifting problems

$$\begin{tikzcd}
\Lambda^n_j \arrow[r, "r"] \arrow[d, hook] & \mathsf{C} \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathsf{D}
\end{tikzcd}$$

have a solution when

(1) $0 < j < n$,
(2) $j = 0$ and $n \geq 2$ *if* r sends $\{0,1\}$ to a p-cocartesian edge (this is actually automatic),
(3) $j = 0, n = 1$; in this case we also *require* the solution $u : \Delta^1 \to \mathsf{C}$ to be a p-cocartesian edge.

The idea is that a cocartesian fibration $p : \mathsf{C} \to \mathsf{D}$ should be thought of as a "functorial family of ∞-categories indexed by D". More precisely:

- For each object x in D let $\mathsf{C}_x = \{x\} \times_{\mathsf{D}} \mathsf{C}$. This is an ∞-category.
- For each edge $\gamma : x \to y$ in D and each n-simplex $\sigma : \Delta^n \to \mathsf{C}_x$ we can construct a map that under
$$\widetilde{\sigma} : \Delta^n \times \Delta^1 \longrightarrow \mathsf{C}$$
sends $\{j\} \times \Delta^1$ to a p-cocartesian edge.
- Moreover, we get a *functor*
$$\begin{aligned} \mathsf{C}_x &\longrightarrow \mathsf{C}_y \\ \sigma &\longmapsto \widetilde{\sigma}_{\Delta^n \times \{1\}} \end{aligned}$$
(this is slightly sloppy).

The definition is precisely set up so you can carry this out. Let's keep going:

- A 2-simplex in D gives a 2-simplex in Cat_∞

$$\begin{tikzcd}
& C_y \arrow[dd] \\
C_x \arrow[ur] \arrow[dr] & \\
& C_z
\end{tikzcd}$$

Theorem 1.0.6 (Straightening-Unstraightening Theorem). — *There is an equivalence of ∞-categories*

$$\mathsf{Cocart}(\mathsf{D}) \xrightarrow{\sim} \mathsf{Fun}(\mathsf{D}, \mathsf{Cat}_\infty)$$

between the ∞-category of cocartesian fibrations over D and the ∞-category of functors $\mathsf{D} \to \mathsf{CaT}_\infty$. The forward map is called ***straightening*** *and the inverse is called* ***unstraightening****.*

The left hand side is easier for humans:

- Writing down a functor $\mathsf{D} \to \mathsf{Cat}_\infty$ involves making millions of choices and checking that they're compatible.
- Writing down a cocartesian fibration is *easier*: you write down *all* possible choices and don't bother with compatibility.

Definition. — Let C be an ∞-category. A **monoidal structure** on C is a cocartesian fibration

$$\mathsf{C}^{\otimes} \longrightarrow N(\mathbf{\Delta}^{\mathsf{op}})$$

such that

$$\mathsf{C}^{\otimes}_{[1]} = \mathsf{C}$$
$$\mathsf{C}^{\otimes}_{[n]} \xrightarrow{\sim} \mathsf{C}^{\otimes}_{\{0,1\}} \times \mathsf{C}^{\otimes}_{\{1,2\}} \times \cdots \times \mathsf{C}^{\otimes}_{\{n-1,n\}}.$$

Let's go back to mapping spaces. We want a functor

$$\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{Spc} \subset \mathsf{Cat}_{\infty}$$
$$(x, y) \longmapsto \mathrm{map}_{\mathsf{C}}(x, y).$$

This is impossible for humans.

Let's introduce the **twisted arrow category** $\mathsf{Tw}(\mathsf{C})$ for an ∞-category C. We define

$$\mathsf{Tw}(\mathsf{C})_k = \left\{\text{maps } N(k' < (k-1)' < \cdots < 1' < 0' < 0 < 1 < \cdots < k) \longrightarrow \mathsf{C}\right\}.$$

Of course that poset is isomorphic to Δ^{2k+1} but we want it to have this notation. We make the unprimed indices correspond to C and the primed indices to correspond to C^{op}, so we get a map

$$\mathsf{Tw}(\mathsf{C}) \longrightarrow \mathsf{C}^{^{\mathsf{op}}} \times \mathsf{C}.$$

Lemma 1.0.5. — *This is a cocartesian fibration whose fibers are Kan complexes.*

So by Straightening-Unstraightening we get a functor

$$\mathrm{map}_{\mathsf{C}} : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{Spc} \subset \mathsf{Cat}_{\infty}.$$

By adjunction (1-categorical), we get

$$h : \mathsf{C} \longrightarrow \mathsf{Fun}(\mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Spc}).$$

Theorem 1.0.7 (∞-categorical Yoneda's Lemma). — *h is fully faithful.*

Remark. — Straightening-Unstraightening gives you a framework for generalizing 1-categorical notions to ∞-categories. Some themes for going between ordinary and ∞-categories:

1-categories	∞-categories
Sets	Spaces
existence and uniqueness	existence for $\partial\Delta^n \to \Delta^n$ and $\forall n$ uniqueness up to a contractible Kan complex

1.1. Pullbacks In an ordinary category

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & L \arrow[r] \arrow[d] & A \arrow[d] \\ & A' \arrow[r] & B \end{tikzcd}$$

L is the pullback of

$$\begin{tikzcd} & A \arrow[d] \\ A' \arrow[r] & B \end{tikzcd}$$

if given solid arrows there exists a unique dashed arrow.

Now for the ∞-categorical verson we need auxiliary simplicial sets

$$P_n = N(??)$$

(I need to think about how to write this, it's a poset $0 < \cdots < n+1$ with a square made at the and with a, a', b.) The standard notation for this simplicial set is $\Delta^{n+1} * \Lambda^2_2$, where $*$ stands for the join operation, which we won't define. Also define P_n^0 to be the subsimplicial set of P_n where we take chains that omit one of $0, 1, \ldots, n+1$. The standard notation is

$$\Delta^n * \Lambda^2_2 \coprod_{\partial\Delta^n * \Lambda^2_2} \partial\Delta^n * (\Delta^1 \times \Delta^1)$$

it also might be $\partial\Delta^{n+1} * \Lambda^2_2$ but Pramod wasn't sure.

(Pictures of P_i and P_i^0.)

Definition. — Let C be an ∞-category. A diagram of shape $\Delta^1 \times \Delta^1$, say (Picture) is called a **pullback** if every extension problem (Diagram) admits a solution.

For $n = 0$ this should remind you of 1-categories. The moral is to think of filling in a sphere.

WARNING: Suppose

$$\begin{tikzcd} & A \arrow[d] \\ A' \arrow[r] & B \end{tikzcd}$$

is a diagram of topological spaces, or ∞-categories, or Kan complexes. The 1-categorical and ∞-categorical pullbacks exist, but they ***don't agree*** in general. There is a map from the 1-limit to the ∞-limit.

In Cat_∞ the ∞-categorical pullback

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr] & A \arrow[d] \\ A' \arrow[r] & B \end{tikzcd}$$

is given by

$$L_n = \left\{ (\sigma, \sigma', J) \;\middle|\; \begin{array}{c} \sigma \in A_n, \sigma' \in A'_n, J : \Delta^n \times \Delta^1 \to B \\ J \text{ a natural isomorphism from } f \circ \sigma \text{ to } f' \circ \sigma' \end{array} \right\}.$$

In Top consider the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[d, "x_0"] \\ \mathrm{pt} \arrow[r, "x_0"] & X \end{tikzcd}$$

The 1-categorical pullback is a point. The ∞-categorical pullback is $\Omega(X, x_0)$, the space of loops in X based at x_0.

1-categorical limits are unique up to unique isomorphism, but ∞-categorical limits it's unique up to a contractible Kan complex of (∞-categorical) isomorphisms.

Example 1.1.1. — The following diagram is an ∞-categorical pullback

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & S^1 \end{tikzcd}$$

where $\mathbb{Z}$ is the space of continuous maps $\gamma : [0, 1] \to \mathbf{R}$ such that $\gamma(0) = 0$ and $\gamma(1) \in \mathbb{Z}$.

Recall that an **abelian category** is a category such that:

- There is a zero (initial and final) object.
- All pullbacks and pushouts exist.
- If $f : x \to y$ is a monomorphism then the pushout

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

is also a pullback.

- If $g : y \to z$ is an epimorphism then the pullback

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d, "g"] \\ 0 \arrow[r] & z \end{tikzcd}$$

  is also a pushout.

Remark. — You can actually recover the abelian group structure on hom sets from only this!

Definition. — An ∞-category is called **stable** if:

- There is a zero object (defining this is an exercise).
- All pullbacks and pushouts exist.
- Any diagram of shape $\Delta^1 \times \Delta^1$

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

  is a pullback if an donly if it's a pushout.

Remark. — 0 is the zero object and is actually an optional entry, we can ask this for all squares, the conditions are equivalent.

Example 1.1.2. — Start with an ordinary additive category A. Defne an ∞-category $\mathsf{Ch}(A)$ as follows:

- objects are chain complexes of objects in A,
- morphisms are chain maps,
- 2-simplices are diagrams

$$\begin{tikzcd} & B^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[ur, "f"] \arrow[r, "g"'] & C^\bullet \end{tikzcd}$$

  and a chain homotopy $s : A^\bullet \to C^{\bullet-1}$ such that
$$ds + sd = gf - h.$$

- 3 and higher and higher chain homotopies.

This category is explicitly written out by Lurie. Look up the "dg nerve".

Lemma 1.1.1. — $\mathsf{Ch}(A)$ *is stable.*

Proof idea. The main step is:

- Take a chain map $f : A^\bullet \to B^\bullet$.
- Prove by hand that

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d] \arrow[dr] & B^\bullet \arrow[d] \\ 0 \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

  is a pullback and a pushout.

□

Lemma 1.1.2. — *In any stable ∞-category C the functors that take X to the pullback/pushout of the diagram*

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

respectively.

The first functor is denoted Ω or $[-1]$ and called the **loop space** functor. The second functor is denoted Σ or $[1]$ and called the **suspension** functor.

Remark. — This is pretty closed to C being a stable ∞-cateogry implying that $h\mathsf{C}$ is triangulated. The octahedral axiom is similar to the third isomorphism theorem.

Also we get derived categories of abelian categories: take $\mathsf{Ch}(A)$ and invert quasi-isomorphisms. In ordinary category theory:

- There is a canonical example of an abelian category, which is the category of abelian groups.
- *Every* abelian category is enriched over abelian groups.

Let's go back to ∞-categories.

Definition. — The ∞-category of **spectra** Sptr is

$$\mathsf{Sptr} = \varprojlim \left(\dots \xrightarrow{\Omega} \mathsf{Spc}_* \xrightarrow{\Omega} \mathsf{Spc}_* \xrightarrow{\Omega} \mathsf{Spc}_* \right)$$

where Spc_* are pointed spaces and we take the ∞-categorical limit in Cat_∞.

An object of Sptr is (roughly) a sequence of pointed spaces $\dots, X_3, X_3, X_1, X_0$ and isomorphisms $\Omega X_3 \simeq X_2, \Omega X_2 \simeq X_2, \Omega X_1 \simeq X_0$, etc.

1-land	∞-land
Canonical example: Ab	Canonical example: Sptr
Most important abelian group: $\mathbb{Z}$	Most important spectrum: $\mathbb{S}$ with $\pi_0(\mathbb{S}) = \mathbb{Z}$
symmetric monoidal with $\otimes$	symmetric monoidal with smash product $\wedge$
Any: hom-sets are abelian groups	
Any: modules over Ab, can $\otimes$ by fg ab grp	

We define the **sphere spectrum**: Start with

$$\dots, \mathsf{Sing}(S^2), \mathsf{Sing}(S^1), \mathsf{Sing}(S^0)$$

with isomorphisms $S^2 = \Sigma S^1, S^1 = \Sigma S^0$ etc. This is not quite an object of Sptr. By adjunction we get maps $S_1 \to \Omega S^2, S^0 \to \Omega S^1$ etc. We fix the failure to be isomorphisms universally ("spectrification") which gives us the sphere spectrum $\mathbb{S}$. It is also classically called the Ω-spectrum.

We have homotopy groups of spectra

$$\pi_n(X_\bullet) = \pi_{n+k}(X_k)$$

for $n \in \mathbb{Z}$ and $k \geq 0$. In particular

$$\pi_0(\mathbb{S}) = \mathbb{Z}, \quad \pi_n(\mathbb{S}) = \text{stable homotopy groups of spheres.}$$

Theorem 1.1.1. — *Every stable ∞-category is a module over Sptr^{fin}.*

Theorem 1.1.2 (Lurie). — *Let C be a stable ∞-category. There is a functor*

$$\mathrm{smap}_{\mathsf{C}} : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{Sptr}$$

such that for all spectra z and objects x, y in C

$$\mathrm{map}_{\mathsf{Sptr}}(z, \mathrm{smap}_(x, y)) \simeq \mathrm{map}_{\mathsf{C}}(z \wedge x, y).$$

Let C be a stable ∞-category and $h\mathsf{C}$ be triangulated. Let x, y be objects in C then

$$\pi_n \,\mathrm{smap}(x, y) \simeq \mathrm{Hom}_{h\mathsf{C}}(x, y[-n]).$$

If $n \geq 0$, we have

$$\pi_n(\mathrm{smap}(x, y)) = \pi_n(\mathrm{map}(x, y)).$$

Think that smap corresponds to $\mathrm{R}\,\mathrm{Hom}$ and map corresponds to $\tau^{\leq 0}\mathrm{R}\,\mathrm{Hom}$.

Recall a construction of the tensor product of abelian groups A, B. Start with functions

$$F : A \times B \longrightarrow \mathbb{Z}$$

with finite support. Take a quotient to enforce functions to be linear in both variables.

The Lurie tensor product of stable ∞-categories A, B is constructed as follows:

- Start with functors
$$A \times B \longrightarrow \mathsf{Sptr}.$$
- Take a quotient (or localization) to enforce that the functors preserve direct sums and pushouts in both variables.