## **INFINITY CATEGORIES**

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## §1. SIMPLICIAL SETS

**Definition.** — The **simplex category**  $\Delta$  is the category of

- finite non-empty totally ordered sets
- order-preserving maps.

**Notation.** — 
$$[n] = \{0 < 1 < 2 < \cdots < n\} \text{ for } n \in \mathbb{Z}_{>0}.$$

Every object in  $\Delta$  is (uniquely) isomorphic to some [n].

**Definition.** — A **simplicial set** is a functor

$$\mathscr{S}: \Delta^{\mathsf{op}} \longrightarrow \mathsf{Sets}$$

**Notation.** —  $\mathcal{S}_n := \mathcal{S}([n])$ , call this the **set of** *n***-simplices** of  $\mathcal{S}$ . 0-simplices are called **vertices**, 1-simplices are called edges.

**Example 1.0.1.** — Let *C* be a set. Let  $\underline{C} : \Delta^{op} \to \mathsf{Sets}$  be the constant functor:

$$\underline{C}_n = C \quad \forall n,$$

$$\underline{C}(\alpha) = \mathrm{id} \quad \forall \alpha : [m] \longrightarrow [n] \text{ in } \Delta.$$

This is called a **discrete simplicial set**.

**Definition.** — Let  $\mathscr S$  be a simplicial set. Given  $\alpha : [n] \to [n-1]$  we get  $\mathscr S(\alpha) : \mathscr S_{n-1} \to \mathscr S_n$ . The *n*-simplices in the eimage are called **degenerate** simplices, i.e.  $\sigma$  is degenerate if there is an  $\alpha$  such that  $\sigma \in \operatorname{im}(\mathscr{S}(\alpha))$ .

**Lemma 1.0.1.** — A simplicial set is discrete if and only if for all n > 1 all n-simplices are degenerate.

**Exercise.** — Prove it.

**Example 1.0.2.** — Let  $(P, \geq)$  be a poset. Define a simplicial set  $N(P, \leq)$  called the **nerve** of  $(P, \leq)$  by

$$N(P, \leq)_k = \{ \text{chains } p_0 \leq p_1 \leq \cdots \leq p_k : p_i \in P \}$$

where a chain is a totally ordered subset.

**Exercise.** — Finish the definition. Which simplices are degenerate?

Example 1.0.3 ("Standard n-simplex"). — The standard n-simplex is

$$\Delta^n := N([n]).$$

(Pictures)

**Note.** — For  $j \in [n]$ , we get a subsimplicial set

$$N([n] \setminus \{j\}) \subset \Delta^n$$

isomorphic to  $\Delta^{n-1}$  called the  $j^{th}$  **face** of  $\Delta^n$ . (Picture)

**Example 1.0.4 (Horns).** — Let  $n \ge 0$  and  $0 \le j \le n$ , define the **horn** 

subsimplicial set of 
$$\Delta^n = N([n])$$
  
 $\Lambda^n_j := \text{consisting of chains } p_0 \leq p_1 \leq \cdots \leq p_k \text{ (Pictures)}$   
such that  $\{p_0, \dots, p_k\} \not\supset [n] \setminus \{j\}.$ 

**Example 1.0.5 (**(n-1)**-sphere**  $\partial \Delta^n$ **).** — We define the (n-1)**-sphere** 

$$\partial \Delta^n := \begin{array}{c} ext{subsimplicial set of } \Delta^n \\ ext{chains } p_0 \leq \cdots \leq p_k \text{ such that } \{p_0 \leq \cdots \leq p_k\} \neq [n] \end{array}$$

**Example 1.0.6 (Products).** — Let  $\mathscr{S}$ ,  $\mathscr{T}$  be simplicial sets. We define their **product**  $\mathscr{S} \times \mathscr{T}$  as

$$(\mathscr{S} \times \mathscr{T})_k = \mathscr{S}_k \times \mathscr{T}_k.$$

(Picture)

**Example 1.0.7.** — Let C be an ordinary category. We define its **nerve** N(C) as

$$N(\mathsf{C})_k := \left\{ \begin{array}{c} \text{composable sequences of morphisms} \\ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \dots \xrightarrow{f_k} X_k \end{array} \right\}.$$

**Example 1.0.8.** — Let *X* be a topological space. The **singular simplicial set** of *X* is defined as

$$\operatorname{Sing}(X)_k = \{ \operatorname{continuous maps} |\Delta^k| \longrightarrow X \},$$

where  $|\Delta^k|$  is the standard k-simplex

$$|\Delta^k| = \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1} \middle| x_i \ge 0, \sum x_i = 1 \right\}.$$

**Exercise.** — What does this do to the morphisms in  $\Delta$ ?

**Definition.** — A **Kan complex** is a simplicial set *X* such that for every diagram



we can fill the dashed arrow. This is called an **extension problem**. If the arrow exists we say that the extension problem **admits a solution**.

Daniel Kan discovered Kan complexes in 1958. The *key fact* is that Sing(X) is a Kan complex. The theme from 1958 to today is that Kan complexes are a "combinatorial model" for algebraic topology which allows us to do homotopy theory.

**Definition.** — Let X be a Kan complex and  $\mathscr S$  be any simplicial set. Two maps  $f,g:\mathscr S\to X$  are said to be **homotopic** if there exists a map  $H:\mathscr S\times\Delta^1\to X$  such that

$$H|_{\mathcal{S}\times\{0\}}=f,\quad H|_{\mathcal{S}\times\{1\}}=g.$$

**Lemma 1.0.2.** — This is an equivalence relation.

*Proof.* Omitted, tricky for an exercise. This requires *X* to be a Kan complex.

**Definition.** — Let X be a Kan complex and  $x_0$  be a vertex of X. Let

$$\text{Loops}_{x_0} = \{\text{maps } \gamma : \Delta^n \longrightarrow X \text{ such that } \gamma|_{\partial \Delta^n} \text{ is the constant map to } x_0\}.$$

We say  $\gamma, \gamma' \in \text{Loops}_{x_0}$  are **relatively homotopic** (**rel. homotopic**) if there exists  $H: \Delta^n \times \Delta^1 \to X$  such that

$$H|_{\Delta^n \times \{0\}} = \gamma$$
,  $H|_{\Delta^n \times \{1\}} = \gamma'$ ,  $H|_{\partial \Delta^n \times \Delta^1} = \text{const. map to } x_0$ .

Define

$$\pi_n(X, x_0) := \frac{\text{Loops}_{x_0}}{\text{rel. homotopy}}.$$

**Fact.** — For  $n \ge 1$ ,  $\pi_n(X, x_0)$  is a group. For  $n \ge 2$ ,  $\pi_n(X, x_0)$  is abelian.

**Definition.** — An  $\infty$ -category (or quasi-category) is a simplicial set  $\mathscr C$  such that any extension problem



with 0 < j < n (inner horns) admits a solution. (Picture) An  $\infty$ -category is also called a **weak Kan complex**.

**Lemma 1.0.3.** — Let C be an ordinary category, then N(C) is an  $\infty$ -category.

*Digression:* Let  $I^n$  be the simplicial set consisting of n consecutive 1-simplices (n-spine) (Picture). A naive alternative definition is:  $\mathscr{C}$  is an infinity category if every



has a solution. This is WRONG (but its wrongness is subtle), even though N(ord. cat.) satisfy this. There is a book by Markus Land "Introduction to  $\infty$ -categories" which explores this. The definition of  $\infty$ -categories was introduced by Boardman-Vogt in 1972. Joyal started generalizing results from ordinary category theory to  $\infty$ -categories in 2006. Lurie is largely responsible for how well this notion is developed in modern literature.

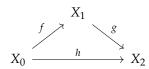
**Remark.** — Having a unique solution to the lifting problem characterizes nerves of ordinary categories.

**Definition.** — Let  $\mathscr{C}$  be an  $\infty$ -category. An **object** is a vertex. A **morphism** is an edge. An **identity morphism** is a degenerate edge. Say that h is a **composition** of g and f if there exists a 2-simplex such that (Picture).

**Remark.** — Compositions are NOT unique in ∞-categories.

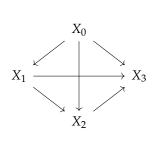
## Example 1.0.9 ( $\infty$ -categories). —

- 1) Topological spaces Top.
  - Objects are topological spaces.
  - Morphisms are continuous maps.
  - A 2-simplex is a (not necessarily commutative) diagram



and a homotopy  $H: X_0 \times [0,1] \to X_2$  from gf to h.

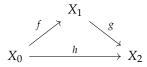
• A 3-simplex is a diagram



with continuous maps  $f_{ij}: X_i \to X_j$  for i < j, homotopies  $T_{ijk}X_i \times [0,1] \to X_k$  from  $f_{jk} \circ f_{ij}$  to  $f_{ik}$ , and  $H: X_0 \times [0,1]^2 \to X_3$  (**higher homotopy**) such that  $H|_{\text{bdry}}$  is

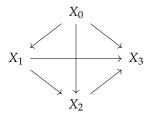
$$\begin{array}{ccc} (0,0) & \xrightarrow{T_{123}f_{01}} & (0,1) \\ f_{23}T_{012} \downarrow & & & \downarrow T_{013} \\ (1,0) & & & & \hline T_{023} & (1,0) \end{array}$$

- 2) The ∞-category of ordinary categories  $Cat_1$ .
  - Objects are ordinary categories.
  - Morphisms are functors.
  - A 2-simplex is a (not necessarily commutative) diagram



*and* a natural isomorphism  $T: g \circ f \xrightarrow{\sim} h$ .

• A 3-simplex is a diagram



where  $f_{ij}$  are functors and  $T_{ijk}$  are natural isomorphism such that

$$\begin{array}{ccc}
& \xrightarrow{T_{123}f_{01}} & \bullet \\
f_{23}T_{012} \downarrow & & \downarrow T_{013} \\
& \xrightarrow{T_{1023}} & \bullet
\end{array}$$

commutes

A source of ∞-categories are

- ordinary categories with an equivalence relation on morphisms,
- ordinary categories with inverting some morphisms.