Ice and Climate Project 1

Bennet Weiß, Lasse Vulto

January 6, 2024

1 Introduction

In this project, we use a simple ice sheet model to analyze response time and mass changes of a glacier to changes in equilibrium line altitude (ELA). In our changing climate, the ELA of various glaciers could increase due to a warmer average temperature. In addition to this analysis, we investigate the effect of model resolution on the model results. This is important as an increased resolution uses significantly more computational resources and thus a good balance of model performance and cost is required.

2 Underlying Equations of the Ice Sheet Model

In this project we use a simple set of equations originating from the shallow ice approximation and Weertman sliding. In the following, we will discretize these equations to make them ready for their use in numerical simulations.

2.1 Discretization of the Ice Flux Equation

To discretize the equation for the ice flux F we utilize the following identities for the average velocity

$$U = U_d + U_0 = \frac{2}{5} f_d H \tau_d^3 + f_s H^{-1} \tau_d^3, \tag{1}$$

and the stress τ_d resulting from gravitational forcing on the slope

$$\tau_d = \rho g H \frac{\partial h}{\partial x}. \tag{2}$$

Plugging these identities into the simple equation for the flux, which is simply the product of average velocity times glacier height, yields the following relationship

$$\begin{aligned} F &= HU \\ &= H\left(U_d + U_0\right) \\ &= H\left(\frac{2}{5} f_d H \tau_d^3 + f_s H^{-1} \tau_d^3\right) \\ &= H\left(\rho g H \frac{\partial h}{\partial x}\right)^3 \left(\frac{2}{5} f_d H + f_s H^{-1}\right) \\ &= c_d H^5 \left(\frac{\partial h}{\partial x}\right)^3 + c_s H^3 \left(\frac{\partial h}{\partial x}\right)^3. \end{aligned} \tag{3}$$

In the last step, we introduced the following constants

$$c_d = \frac{2}{5} \rho^3 g^3 f_d \tag{4}$$

$$c_s = \rho^3 g^3 f_s, \tag{5}$$

which we will use in the simulation.

Lastly, we discretize Equation (3) using the leap-frog scheme, and end up with

$$F_{i+1/2}^n = c_d \left(H_{i+1/2}^n\right)^5 \left(\frac{h_{i+1}^n - h_i^n}{\Delta x}\right)^3 + c_s \left(H_{i+1/2}^n\right)^3 \left(\frac{h_{i+1}^n - h_i^n}{\Delta x}\right)^3. \tag{6}$$

Here, we chose the Flux to be defined on the half grid points, which is an arbitrary decision. Consequently, h is defined on the full grid points. It is important to realize though, that, other that the equation suggests, H does not actually live on the same grid as F. Consequently, we don't a priori know $H^n_{i+1/2}$, but only know H^n_i and H^n_{i+1}. To get an estimate of H at $i + 1/2$ we are exploiting the idea of averaging the two heights at the edges to get an estimate for the height in the middle of each grid cell. Mathematically, there are two valid ways to do that. First, we could get an average of H and then raise it to respective power. Or, second, we could average it only after raising the values of H at the boundaries of the cell to the respective power. In this project we chose to follow the latter procedure as this preserves the strongly non-linear behaviour of the powers of 3 and 5 better. If we were average before raising the power, extreme values might be smoothed out and underestimated. With that, we arrive at the final discretized equation for the ice flux

$$F^n_{i+1/2} = c_d \frac{\left(H^n_i\right)^5 + \left(H^n_{i+1}\right)^5}{2} \left(\frac{h^n_{i+1} - h^n_i}{\Delta x}\right)^3 + c_s \frac{\left(H^n_i\right)^3 + \left(H^n_{i+1}\right)^3}{2} \left(\frac{h^n_{i+1} - h^n_i}{\Delta x}\right)^3 . \tag{7}$$

2.2 Discretization of the Equation for Glacier Height Evolution

Next, we want to discretize

$$\frac{\partial H}{\partial t} = -\frac{\partial F}{\partial x} + \dot{b} \tag{8}$$

with $\dot{b}$ representing the surface mass balance increasing linearly with height.

$$\dot{b} = \beta \left(h - h_{\mathrm{ELA}}\right) . \tag{9}$$

Here, we use the Euler forward scheme in time (left side of the equation) and leap frog scheme in space (right side of the equation) and get to the following expression.

$$\frac{H^{n+1}_i - H^n_i}{\Delta t} = \frac{F^n_{i-1/2} - F^n_{i+1/2}}{\Delta x} + \beta \left(h^n_i - (h_{\mathrm{ELA}})^n_i\right) \tag{10}$$

Fortunately, this way, H and h live on the same grid (full grid points) and F lives on half grid points as required. By rearranging equation (10) the ice height H ($hice$ in the model) can be calculated for the subsequent time step.

2.3 Implementation of Equations in Code

Lastly, the given code is updated using equations (7) and (10) and the model is ready for simulation.

3 Investigation of Model Behaviour and Spatial Resolution Influence

In this section, we will use the simple ice model set up earlier to study the response of a glacier to change of ELA in the model. We chose to dive deeper into the following research direction

Investigate the effect of the chosen dx on the glacier mass and response time to changes in the ELA.

To investigate, we will first touch the topic on response time as a function of ELA, as well as the total glacier mass as a function of ELA. Thereafter, the influence of model resolution will be analyzed by running simulations with varying dx.

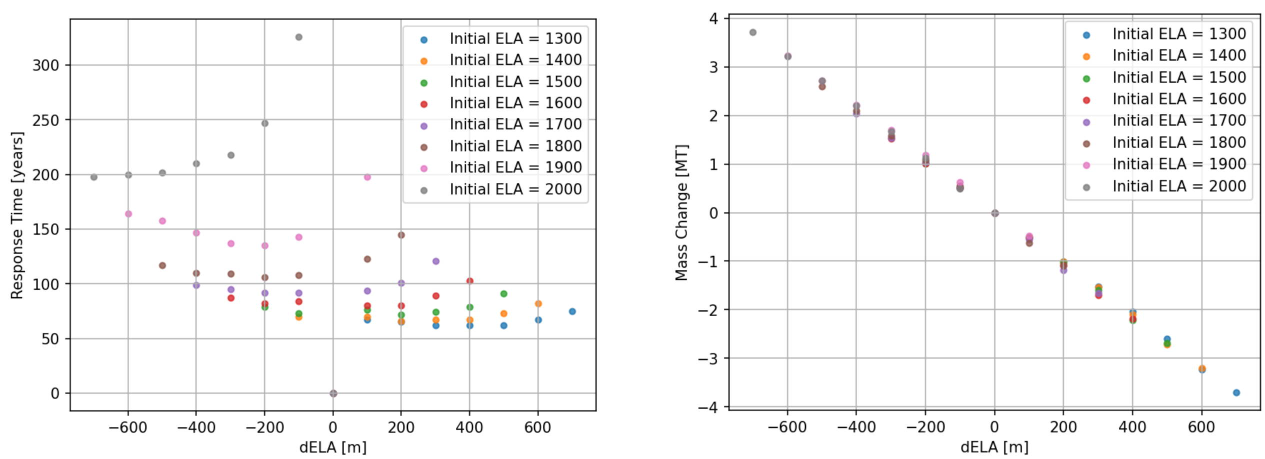
Response Time After Sudden ELA Changes

After a sudden change of ELA, which could be induced by a climate shift for example, the glacier takes a certain time before it reaches a new equilibrium, i.e. the glacier does not grow or shirink anymore. A way to quantify this time is the response time. It is defined as the time it takes the glacier to undergo $(1 - 1/e) \cdot 100\,\%$ ($\sim 63\,\%$) of the total change. While there are other possible variables along which this change can be measured, we chose glacier length as a proxy for change.

Using these definitions, first we will describe how response time changes with changes in ELA. Our approach to answer this is to make a spin-up glacier state for with different ELA between 1300 m and 2000 m in steps of 100 m. We chose that range as a lower ELA would result in the glacier flowing over our spatial bounds and a higher ELA would be above the maximum bedrock height. This spin-up phase is run for 1000 yr and we assume the glacier to have grown to its initial state and be in equilibrium after this time. This is especially true for glaciers with low ELA, which tend to equilibriate quite quick, while high ELA glaciers are just barely in equilibrium after that time due to the low smb above ELA (not shown). The average surface mass balance (SMB) over the whole glacier does, however, not reach exactly zero and a perfect equilibrium will consequently never be reached. To make sure that potential model drift does not deteriorate our results, for every simulation run in this project, its initial state is also run as a control simulation. All results shown are then taken with respect to this control simulation.

From each initial glacier state with different ELA, we then introduced a sudden ELA pertubation, such that the new ELA is once again in the range of 1300 m and 2000 m. Each of those simulations is once again run for 1000 years, and, assuming an equilibrium after that time, the equilibrium time is calculated and plotted in figure 1(a). A general feature observable in the plot is an approximately constant response time of around 70 years for medium to low initial ELA values. Thus, in this regime, the response time seems to be independent of the change in ELA; a glacier would take similar time to adjust to a temperature change, independent on the magnitude in temperature change.

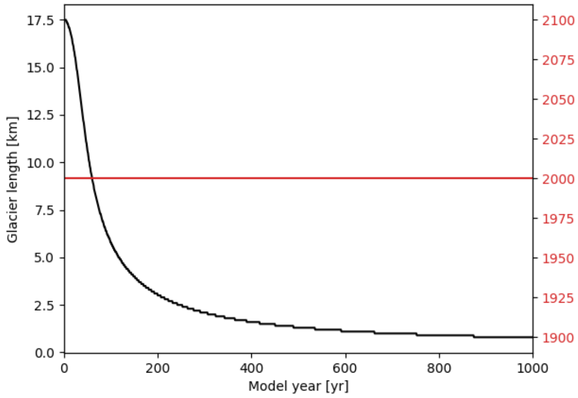
There are, however, other features observable in the figure. Generally, the response time is shorter across all $dELA$ for lower initial ELA, but seems to approach the limit discussed above. For a growing glacier, this is probably related to the fact that glaciers with higher ELA are very short to non-existent (for ELA 2000 m) and grow very slowly initially when the ELA reduces. In the other direction, for a shrinking glacier, the distribution of response time seems counter-intuitive at first. As the glacier disappears completely for ELA $\geq$ 2000 m, one would expect this to happen faster the shorter the glacier is at the start of the simulation. In the figure, however, we see the exact opposite behaviour. A possible explanation is two-fold. First, the models are not in complete equilibrium, even after a millennium, which can be seen by looking at the non-zero glacier-integrated SMB at the end of the simulation (not shown) or by verifying that the glacier indeed has not been completely removed yet. The latter is shown in Figure 2 that shows the glacier length time series for a glacier adjusting to a new ELA of 2000 m for initial ELA of 1400 m (left) and 1900 m (right). From this, it is evident that even after 1000 years, the glacier has not completely vanished. Second and probably more important, the difference of change in glacier length is much greater for lower ELA glaciers with a high ELA perturbation. This implies that especially the begin of the glacier melt happens incredibly fast such that 63 % of the total change are over quite quickly, resulting in a shorter response time. The higher the ELA perturbation, the faster the change in the beginning occurs, leading to the observed pattern. This is also visible in Figure 2. On the left, $dELA$ is significantly bigger and thus the response at the beginning of the simulation is quite strong. This leads to a shorter response time compared to the plot on the right where a more gradual change is recorded due to the lower change of ELA.



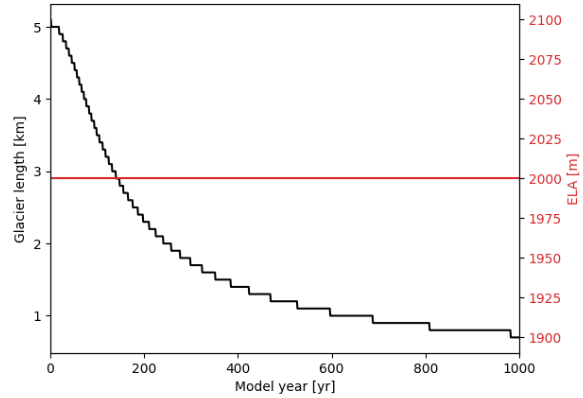
(a) Response time

(b) Mass change

Figure 1: Response time and mass change as function of ELA perturbation for various initial states with different ELA. Note that for a positive dEla the glacier is shrinking, and that for a negative dEla the glacier is growing.



(a) Initial ELA = 1400 m



(b) Initial ELA = 1900 m

Figure 2: Glacier length response (black line) to a new ELA of 2000 m for low (left) and high (right) initial ELA. The red line shows the new position of ELA.

Figure 1(b) shows the mass change the glacier undergoes as a function of ELA change. We defined the mass change simply as the difference of mass at the end of the simulation compared to the initial mass. Here, a clear linear downward trend is recognizible. This is logical, since increasing ELA leads to ablation that drives negative mass change. On the other hand, decreasing ELA drives positive mass change. The variance between different Initial ELA is relatively low and might be related to numerical residuals, influences of glacier shape or incomplete equilibrium, however, no clear pattern is detectable.

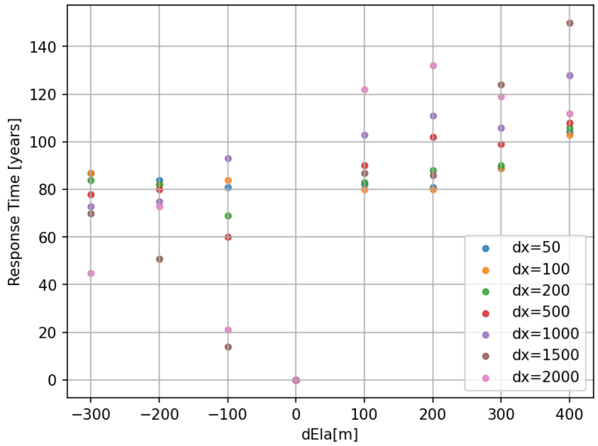
3.1 Investigation of Model Behavior with ELA and Resolution Change.

To answer the research question

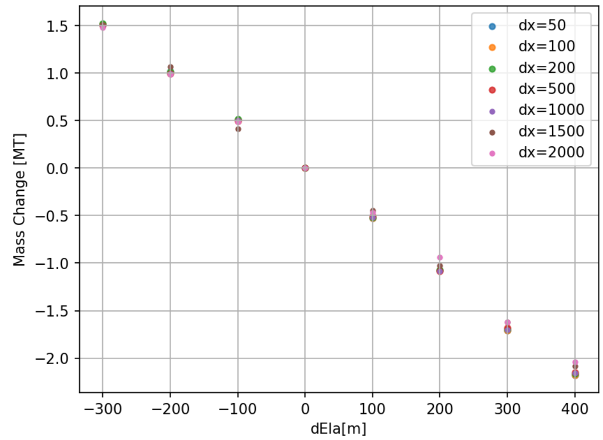
Investigate the effect of the chosen dx on the glacier mass and response time to changes in the ELA.

We can now conduct the same experiment as above, but for different dx. For simplicity sake, we only look at one initial ELA of 1600 meters. We looked at dx between dx=50m to dx=2000m. According to the CFL criterion, the temporal resolution needs to be at least doubled when the spatial resolution is doubled. This leads to overflow errors for $dx = 50$ m, potentially related to the nonlinearities of the functions underlying our simulation, and thus the number time steps was further increased by a factor of 8 for every doubling of resolution. For the other cases the number of time steps per year stayed constant since these simulations are numerically stable and changing the number of time steps per year do not change the result. While using more time steps than needed might heavily increase computational cost, we do not expect that to deterioate our results in any way.

This initial ELA was chosen since it is nicely in between 1300 m and 2000 m. Therefore, if we change the ELA between 1300 m and 2000 m we can see both the effects of a shrinking glacier as a growing one. The results are shown in Figure 3.



(a) Response time as a function of ELA for different dx.



(b) Mass change as a function of ELA for different dx.

Figure 3: Response time and mass change for different spatial resolutions.

The response time as a function of dEla shows quite a lot of variance for different dx. Especially for dx = 1500 m and dx = 2000 m the Response Time seems to vary a lot from the smaller dx. The reason for this is because the high value of dx only lets the glacier length change in steps of 1500 m or 2000 m, respectively. With these big steps, the response time is reached after only one step in glacier length for small dEla. The exact timing of these jumps then determine the exact time of the response time, leading to a seemingly random pattern of response time as function of ELA change.

Nevertheless, the mass change shows a very clear linear trend as observed in Figure 1(b) with a much lower variance between the resolutions. This is interesting, especially considering the aforementioned behaviour of quantized glacier lengths. Consequently, especially for low resolutions, the equilibrium glacier lengths of different resolutions can differ quite significantly (not shown). This could explain the variance of mass change, but it might be partly offset by a different glacier heights. This was not studied as part of this investigation, though. Additionally, the coarse resolution models might have problems reaching equilibrium for this reason deteriorating the accuracy of the response time.

4 Summary

In this project, a simple ice sheet model was investigated to characterize its behaviour with changing ELA change and different spatial resolutions. The response time of a simulated glacier was found to be a complex function of the change of ELA and the initial state of the glacier. However, for big glaciers and small changes in ELA, the response time was found to be approximately constant at around 70 yr. The mass change showed a strictly linear behaviour with ELA change across all initial states. Lower resolution model response time deviated significantly from their high resolution counterparts. A good compromise of model performance and cost could be a spatial resolution of $dx = 100$ m, as it is relatively close to the highest resolution run but saves significant computational cost. The mass change of only showed low variance between different resolutions and kept its linear shape with ELA change.