MOTION ANALYSIS

This study aims to characterize the motion of a system and the efforts (externals and internals) applied to it during a movement.

More precisely, the system studied is a single leg of the quadruped robot SASSA.
The movement studied is a **purely vertical jump**.

Hypothesis made :

- the movement can be considered as
- the mass of leg platic parts is neglectable compared to the mass of the two motors, which are placed coaxialy in the hips of the leg. Thus, the center of mass of the leg is superposed with the center of rotation of the hips joint.
- no friction in the joints
- no slipping between the foot and the ground

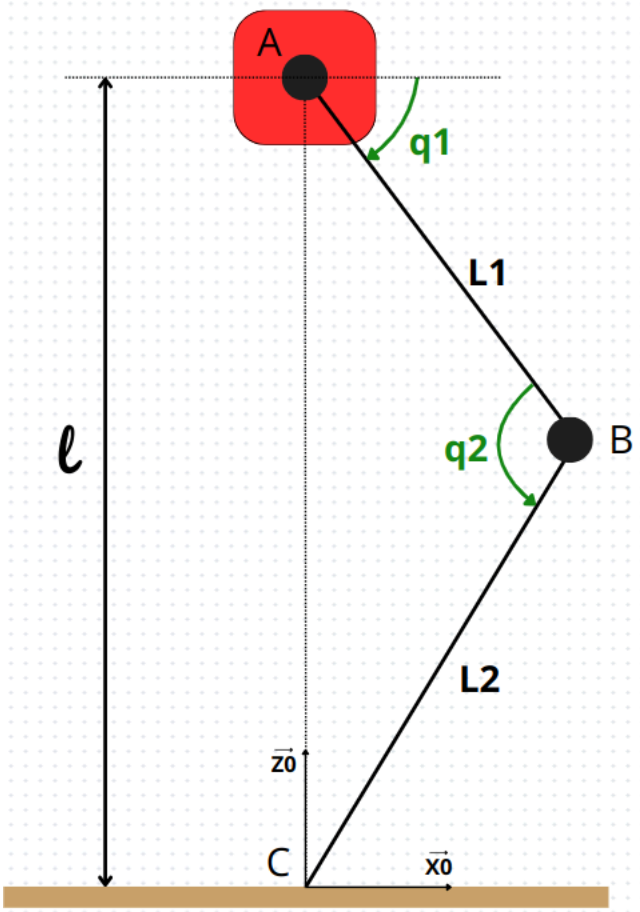
At first, we will write the equations of the **position of the center of mass of the leg** during the movement.
Next, we will use trigonometry to find the equation of the **leg joints angles** based on the position of the center of mass of the leg.
Then, we will calculate the **vertical pushing force** applied by the leg on the ground during the movement.
Finally, we will calculate the **torque on each joint** based on the pushing force of the leg.

Trajectory of the center of mass

The motion can be separated in 2 stapes :

- the **pushing movement** : as long as the leg keeps touching the ground.
- the **off ground movement** : the leg is no more touching the ground.

We call l the distance between the ground and the center of mass of the leg.



Pushing movement trajectory

We will study $l(t)$ during the pushing movement trajectory.
We choose to express $l(t)$ as a polynomial of degree 4.This equation depends on the initials conditions of the system.

$$l(t) = \frac{1}{24}c_0t^4 + \frac{1}{6}b_0t^3 + \frac{1}{2}a_0t^2 + v_0t + l_0$$

where $l_0 = l(t_0), v_0 = \dot{l}(t_0), a_0 = \ddot{l}(t_0), b_0 = \dddot{l}(t_0), c_0 = \ddddot{l}(t_0)$ are the initial conditions.

Off ground trajectory

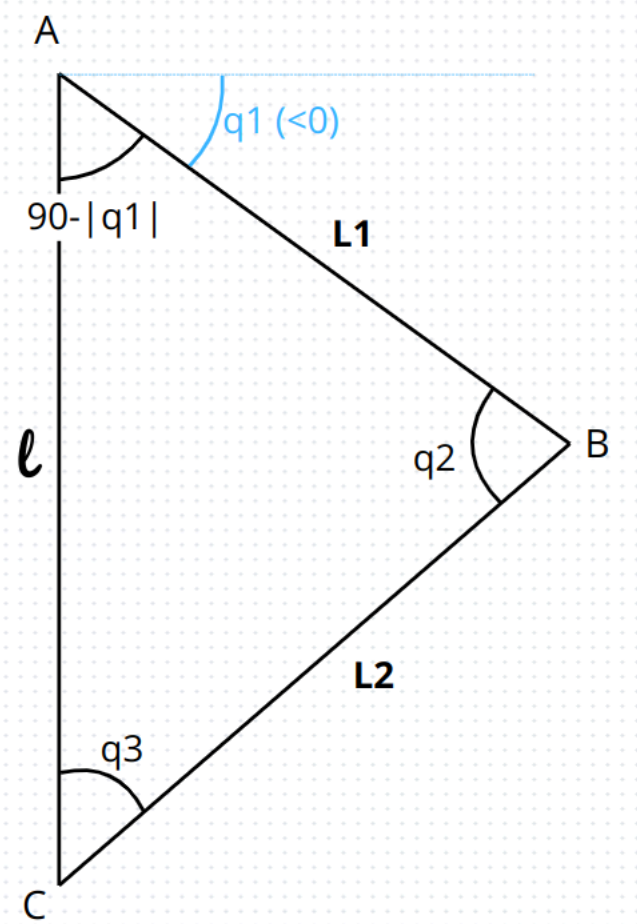
We will study $l(t)$ during the off ground trajectory.
At the moment where the leg leaves the ground, the only acceleration applied to it is the gravity.
The initial speed of the leg is the one at the moment it goes off the ground ; same for the initial height l_0.
We will call l_{to}, v_{to} the initials conditions at the takeoff. The time t is set to 0 such as we consider it is a different movement.

$$l(t) = l_{to} + v_{to}t - \frac{1}{2}gt^2$$

As you can see, two factors are implied on the maximal high l_{max} of the leg : the height and the speed at the takeoff.
Thus, to maximise the height of the jump, we must find the good compromise between these two values (we'll see next that it is not that easy).
Also note that the condition for the leg to go off the ground is that $v_{to}t > \frac{1}{2}gt^2$ at $t \to 0^+$.

Leg joints angles variation

We also can express the angles of the joints depending on $l(t)$, the height of the center of mass during the pushing phase, that we calculated above.



Using the law of cosines, we can write :

$$q_1 = -\frac{\pi}{2} + arccos(\frac{L_2{}^2 - L_1{}^2 - l^2}{-2L_1 l})$$
$$q_2 = arccos(\frac{l^2 - L_1{}^2 - L_2{}^2}{-2L_1 L_2})$$

We can derivate this formula to get the angular speed of each joints :

reminder : $arccos(u)' = \frac{-u'}{\sqrt{1-u^2}}$

Calculating $\dot{q}_1$: $u_1 = \frac{L_2{}^2 - L_1{}^2}{-2L_1}\frac{1}{l} + \frac{1}{2L_1}l$ and $u_1' = \frac{L_2{}^2 - L_1{}^2}{-2L_1}\frac{-l'}{l^2} + \frac{1}{2L_1}l'$

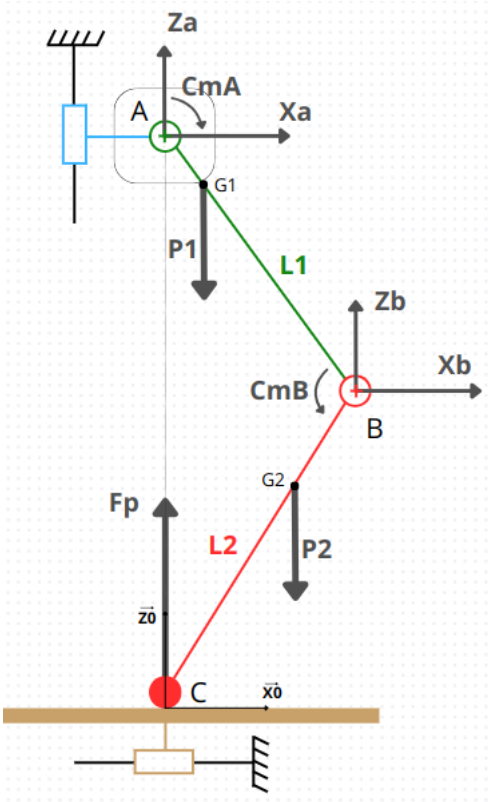
Calculating $\dot{q}_2$: $u_2 = \frac{l^2}{-2L_2 L_1} + \frac{-L_2{}^2 - L_1{}^2}{-2L_2 L_1}$ and $u_2' = \frac{-1}{L_2 L_1}l'l$

Mechanical efforts variation

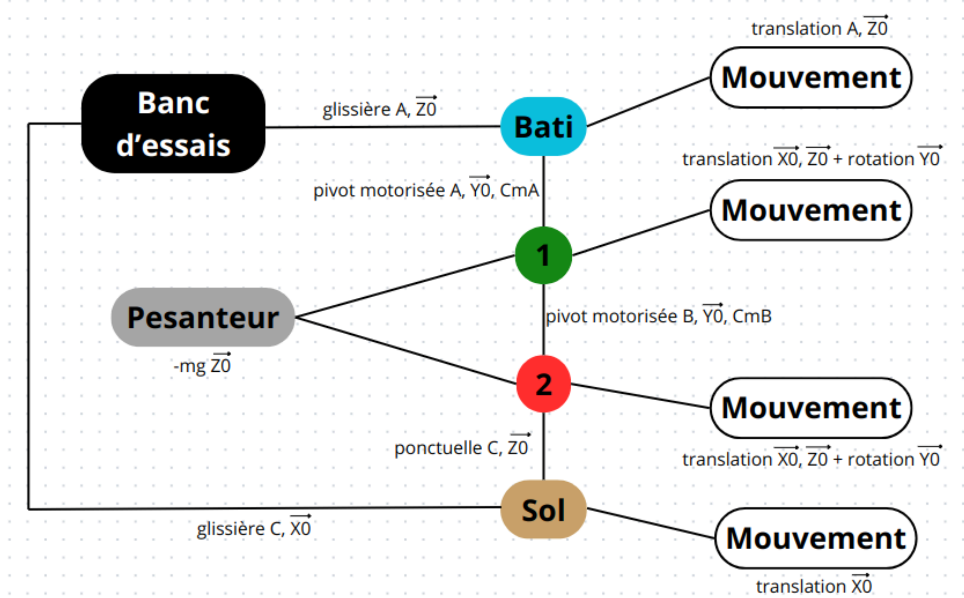
In order to calculate the pushing force and the torque on each joint at any moment of the pushing stape, we will Newton’s 3rd law.

In order to do that, we will have to solve the kinematic, kinetic and dynamic torsos for each of the to parts of the leg. We call the upper leg 1 and the lower leg 2, and their respective centers of mass are called G_1 and G_2.

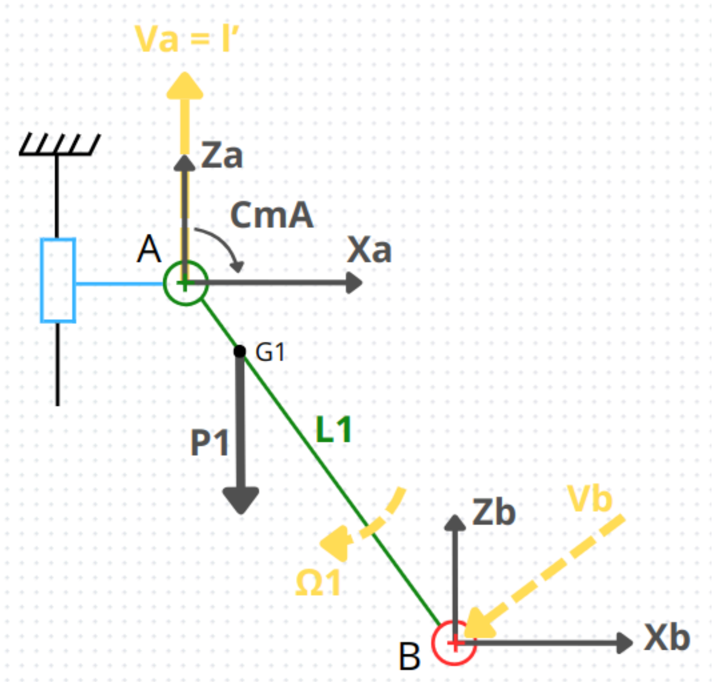
Kinematic diagramm



Connection Graph

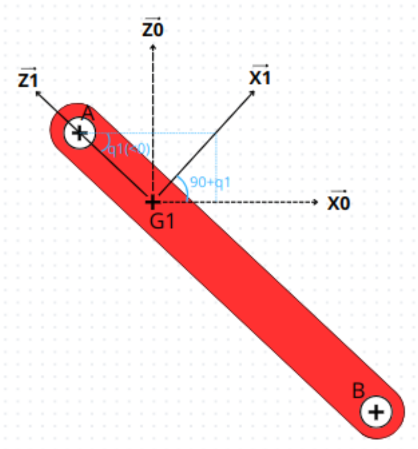


ISOLATING THE UPPER LEG 1



All the torsos will be expressed at A point because it is the center of rotation of 1.

A. Inertia matrix at A, B_0



The first stape is to get the matrix of inertia expressed in the base B_1 (base of the upper leg 1) at the point G_1.
That can be easily done by using the CAD software *Onshape* ; however, keep in mind that this matrix is an approximation, insofar as the filling of the part is considered as 100%, as well as the aluminium sliding skate and the adaptator are not taken into account.
This matrix is noted $I_{G_1}(1)_{B_1}$.

We must then transport this matrix to the point where we express all the torsors: the point A.

$$I_A(1)_{B_1} = I_{G_1}(1)_{B_1} + m_1 \begin{bmatrix} b^2+c^2 & -bc & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -ab & a^2+b^2 \end{bmatrix} \text{ where } \vec{AG_1} = \begin{bmatrix} a = X(G_1)_{B_1} - X(A)_{B_1} \\ b = 0 \\ c = Z(G_1)_{B_1} - Z(A)_{B_1} \end{bmatrix}$$

Finally, we have to switch the bases to express the matrix of inertia in the Galilean base B_0.
To remind, in the base B_0, the upper leg 1 is rotating around the point A.

Relation between B_0 and B_1 :

$$\begin{cases} \overrightarrow{x_0} = cos(\frac{\pi}{2}+q_1)\overrightarrow{x_1} - sin(\frac{\pi}{2}+q_1)\overrightarrow{z_1} \\ \overrightarrow{y_0} = \overrightarrow{y_1} \\ \overrightarrow{z_0} = sin(\frac{\pi}{2}+q_1)\overrightarrow{x_1} + cos(\frac{\pi}{2}+q_1)\overrightarrow{z_1} \end{cases}$$

Pass matrix : $[P_{B_1 \to B_0}] = \begin{bmatrix} cos(\frac{\pi}{2}+q_1) & 0 & -sin(\frac{\pi}{2}+q_1) \\ 0 & 1 & 0 \\ sin(\frac{\pi}{2}+q_1) & 0 & cos(\frac{\pi}{2}+q_1) \end{bmatrix}$

Now we get $I_A(1)_{B_0} = [P_{B_1 \to B_0}]^T . I_A(1)_{B_1} . [P_{B_1 \to B_0}]$

So we can associate $I_A(1)_{B_0} \text{to the matrix} \begin{bmatrix} A & F & D \\ E & B & F \\ D & E & C \end{bmatrix}$.

When calculating $I_A(1)_{B_0}$, we realise that $F = E = 0$.

Thus, we can simplify the matrix as following : $I_A(1)_{B_0} = \begin{bmatrix} A & 0 & D \\ 0 & B & 0 \\ D & 0 & C \end{bmatrix}$.

Keep in mind that the matrix of inertia $I_A(1)_{B_0}$ changes depending on the orientation of the leg in the base B_0, but it inertia products F and E will always be zero.

B. Kinematic torso at A, B_0

$$\{K_A(1/B_0)\} = \begin{Bmatrix} \vec{\Omega}(1/B_0) \\ \vec{V}(A \in 1/B_0) \end{Bmatrix}$$

with $\vec{\Omega}(1/B_0) = \begin{bmatrix} 0 \\ \dot{q}_1 \\ 0 \end{bmatrix}$ and $\vec{V}(A \in 1/B_0) = \begin{bmatrix} 0 \\ 0 \\ \dot{l} \end{bmatrix}$.

C. Kinetic torso at A, B_0

$$\{C_A(1/B_0)\} = \begin{Bmatrix} \overrightarrow{R_c}(1/B_0) = m\vec{V}(G_1 \in 1/B_0) \\ \overrightarrow{\sigma_A}(1/B_0) = I_A(1)_{B_0} . \vec{\Omega}(1/B_0) + \overrightarrow{AG}_{1B_0} \wedge \vec{V}(A \in 1/B_0) \end{Bmatrix}$$

with $\overrightarrow{R_c}(1/B_0) = \begin{bmatrix} m(\frac{d}{dt}[L_{AG_1}cos(q_1)] + \dot{l}) = -mL_{AG_1}\dot{q}_1 sin(q_1) + m\dot{l} \\ 0 \\ m(\frac{d}{dt}[L_{AG_1}sin(q_1)] + \dot{l}) = mL_{AG_1}\dot{q}_1 cos(q_1) + m\dot{l} \end{bmatrix}$.

and

$\overrightarrow{\sigma_A}(1/B_0) = \begin{bmatrix} A & 0 & D \\ 0 & B & 0 \\ D & 0 & C \end{bmatrix} . \begin{bmatrix} 0 \\ \dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} L_{AG_1}cos(q_1) \\ 0 \\ L_{AG_1}sin(q_1) \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 0 \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 \\ B\dot{q}_1 - \dot{l}\,L_{AG_1}cos(q_1) \\ 0 \end{bmatrix}$

.

D. Dynamic torso at A, B_0

$$\{D_A(1/B_0)\} = \begin{Bmatrix} \overrightarrow{R_d}(1/B_0) = m\vec{\Gamma}(G_1 \in 1/B_0) \\ \overrightarrow{\delta_A}(1/B_0) = \frac{d}{dt}[\sigma_A] \end{Bmatrix}$$

with $\overrightarrow{R_d}(1/B_0) = \frac{d}{dt}[\overrightarrow{R_c}(1/B_0)] = mL_{AG_1} \begin{bmatrix} -\ddot{q}_1 sin(q_1) + \dot{q}_1{}^2 cos(q_1) + m\ddot{l} \\ 0 \\ \ddot{q}_1 cos(q_1) - \dot{q}_1{}^2 sin(q_1) + m\ddot{l} \end{bmatrix}$.

and $\overrightarrow{\delta_A}(1/B_0) = \frac{d}{dt}[\overrightarrow{\sigma_A}(1/B_0)] = \begin{bmatrix} 0 \\ \dot{B}\dot{q}_1 + B\ddot{q}_1 - L_{AG_1}(\ddot{l}cos(q_1) - \dot{l}\dot{q}_1 sin(q_1)) \\ 0 \end{bmatrix}$.

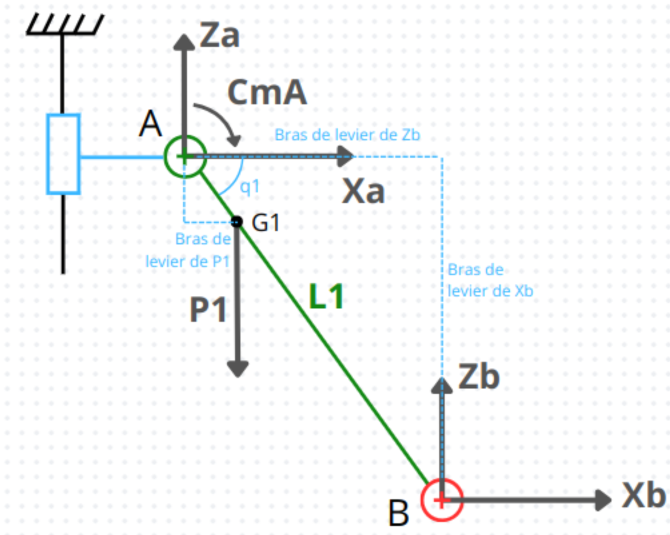
E. PFD at A, B_0

Calculate all the forces $\sum F_{ext \to 1}$ applied to 1 :

$$\sum F_{ext \to 1} = \overrightarrow{R_d}(1/B_0) \Leftrightarrow \begin{cases} \overrightarrow{x_0} : X_A + X_B = -\ddot{q}_1 sin(q_1) + \dot{q}_1{}^2 cos(q_1) + m\ddot{l} \\ \overrightarrow{z_0} : Z_A + Z_B - P_1 = \ddot{q}_1 cos(q_1) - \dot{q}_1{}^2 sin(q_1) + m\ddot{l} \\ \_ \Leftrightarrow Z_B = \ddot{q}_1 cos(q_1) - \dot{q}_1{}^2 sin(q_1) + m\ddot{l} + P_1 - Z_A \end{cases}$$

Isolating the Frame we find that $Z_A = 0$.

Once we know all the forces in exerted on 1, we still have to calculate the levers of each force to write their torque at the point A.



Calculate levers of $\sum F_{ext \to 1}$ at the point A :

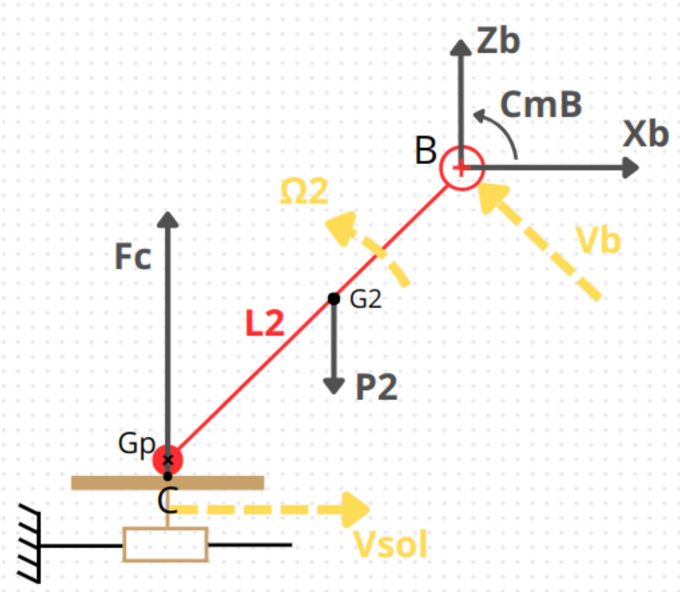
$Lever_{X_B} = -L_1 sin(q_1)$
$Lever_{Z_B} = L_1 cos(q_1)$
$Lever_{P_1} = -L_{AG_1} sin(\frac{-\pi}{2} - q_1)$
(remind that $q_1 < 0$)

Calculate all the torques $\sum M_{F_{ext} \to A \in 1}$ of the forces $\sum F_{ext \to 1}$ at the point A :

$$\sum M_{F_{ext} \to A \in 1} = \vec{\delta}_A(1/B_0) \Leftrightarrow \begin{cases} \overrightarrow{y_0} : -Cm_A - P_1.Lever_{P_1} + X_B.Lever_{X_B} + Z_B.Lever_{Z_B} = \dot{B}\dot{q}_1 + B\ddot{q}_1 - L_A \\ \_ \Leftrightarrow Cm_A = -\dot{B}\dot{q}_1 - B\ddot{q}_1 + L_{AG_1}(\ddot{l}cos(q_1) - \dot{l}\dot{q}_1 sin(q_1)) - P_1.Lever_{P_1} + X \end{cases}$$

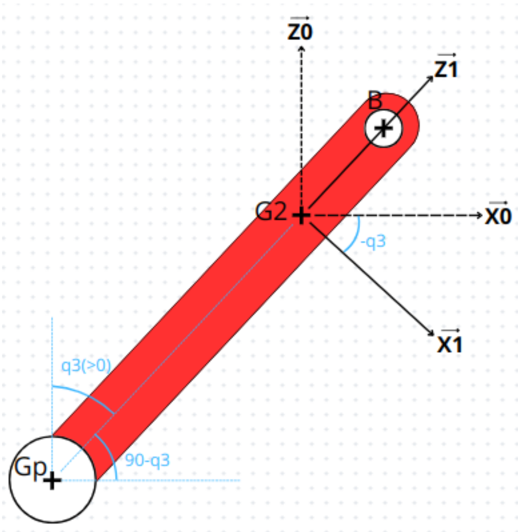
(warning : Cm_A must be expressed in $N.mm$ bbecause all the length are in mm)
For the moment, X_B is not known. We'll calculate them when isolating the lower leg 2.

ISOLATING THE LOWER LEG 2



Notice that the center of rotation of 2 is the center of the ball-foot, named G_p. That means G_p is fixed (rotation only).
For that reason, the torsos will be expressed at G_p point.

A. Inertia matrix at G_p, B_0



In the same way we calculated $I_A(1)_{B_0}$, we will now calculate $I_{G_p}(2)_{B_0}$.

$$I_{G_p}(2)_{B_2} = I_{G_2}(2)_{B_2} + m_2 \begin{bmatrix} b^2+c^2 & -bc & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -ab & a^2+b^2 \end{bmatrix} \text{ where } \overrightarrow{G_pG_2} = \begin{bmatrix} a = X(G_2)_{B_2} - X(G_p)_{B_2} \\ b = 0 \\ c = Z(G_2)_{B_2} - Z(G_p)_{B_2} \end{bmatrix}$$

Relation between B_0 and B_2 (note that $q_3 > 0$)

$$\begin{cases} \overrightarrow{x_0} = cos(-q_3)\overrightarrow{x_2} - sin(-q_3)\overrightarrow{z_2} \\ \overrightarrow{y_0} = \overrightarrow{y_2} \\ \overrightarrow{z_0} = sin(-q_3)\overrightarrow{x_2} + cos(-q_3)\overrightarrow{z_2} \end{cases}$$

Pass matrix : $[P_{B_2 \to B_0}] = \begin{bmatrix} cos(-q_3) & 0 & -sin(-q_3) \\ 0 & 1 & 0 \\ sin(-q_3) & 0 & cos(-q_3) \end{bmatrix}$

Now we get $I_{G_p}(2)_{B_0} = [P_{B_2 \to B_0}]^T . I_{G_p}(2)_{B_2} . [P_{B_2 \to B_0}]$

We can associate $I_{G_p}(2)_{B_0} \text{to the matrix} \begin{bmatrix} A & F & D \\ E & B & F \\ D & E & C \end{bmatrix} = \begin{bmatrix} A & 0 & D \\ 0 & B & 0 \\ D & 0 & C \end{bmatrix}$.

B. Kinematic torso at G_p, B_0

$$\{K(2/B_0)\}_{G_p} = \begin{Bmatrix} \vec{\Omega}(2/B_0)) \\ \vec{V}(G_p \in 2/B_0) \end{Bmatrix} = \begin{Bmatrix} \dot{q}_3 \overrightarrow{y_0} \\ \vec{V}(G_p \in 2/B_0) = \vec{0} \end{Bmatrix}$$

C. Kinetic torso at G_p, B_0

$$\{C(2/B_0)\}_{G_p} = \begin{Bmatrix} \overrightarrow{Rc}(2/B_0)) \\ \vec{\sigma}_{G_p}(2/B_0) \end{Bmatrix} = \begin{Bmatrix} m\vec{V}(G_2 \in 2/B_0) \\ I_{G_p}(2)_{B_0} . \vec{\Omega}(2/B_0) + m(\overrightarrow{G_p G_2})_{B_0} \wedge \vec{V}(G_p \in 2/B_0) \end{Bmatrix}$$

with $\overrightarrow{Rc}(2/B_0) = m\frac{d}{dt}[\begin{bmatrix} L_{G_pG_2} sin(q_3) \\ 0 \\ L_{G_pG_2} cos(q_3) \end{bmatrix}] = mL_{G_pG_2} \begin{bmatrix} \dot{q}_3 cos(q_3) \\ 0 \\ -\dot{q}_3 sin(q_3) \end{bmatrix}$.

and $m(\overrightarrow{G_p G_2})_{B_0} \wedge \vec{V}(G_p \in 2/B_0) = m \begin{bmatrix} L_{G_pG_2} sin(q_3) \\ 0 \\ L_{G_pG_2} cos(q_3) \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ (Remind that $q_3 > 0$)

Also, due to the form of the matrix of inertia $I_{G_p}(2)_{B_0}$, we can deduce that $I_{G_p}(2)_{B_0} . \vec{\Omega}(2/B_0)$ is of the form $\begin{bmatrix} 0 \\ B\dot{q}_3 \\ 0 \end{bmatrix}$.

Thus, $\vec{\sigma}_{G_p}(2/B_0) = \begin{bmatrix} 0 \\ B\dot{q}_3 \\ 0 \end{bmatrix}$.

D. Dynamic torso at G_p, B_0

$$\{D(2/B_0)\}_{G_p} = \begin{Bmatrix} \overrightarrow{R_d}(2/B_0) \\ \vec{\delta}_{G_p}(2/B_0) \end{Bmatrix} = \begin{Bmatrix} m\vec{\tau}(G_2 \in 2/B_0) = \frac{d}{dt}[\overrightarrow{R_c}] \\ \frac{d}{dt}[\vec{\sigma}_{G_p}(2/B_0)] \end{Bmatrix}$$

with $m\vec{\tau}(G_2 \in 2/B_0) = mL_{G_pG_2} \begin{bmatrix} \ddot{q}_3 cos(q_3) - \dot{q}_3^2 sin(q_3) \\ 0 \\ -\ddot{q}_3 sin(q_3) - \dot{q}_3^2 cos(q_3) \end{bmatrix}$.

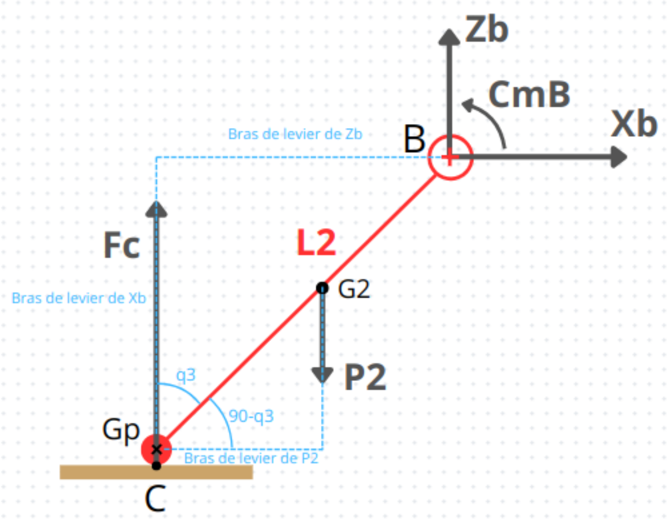
and $\frac{d}{dt}[\vec{\sigma}_{G_p}(2/B_0)] = \begin{bmatrix} 0 \\ \dot{B}\dot{q}_3 + B\ddot{q}_3 \\ 0 \end{bmatrix}$.

E. PFD at G_p, B_0

Calculate all the forces $\sum F_{ext \to 2}$ applied to 2 :

$$\sum F_{ext \to 2} = \overrightarrow{R_d}(2/B_0) \Leftrightarrow \begin{cases} \overrightarrow{x_0} : X_B = \ddot{q}_3 cos(q_3) - \dot{q}_3{}^2 sin(q_3) \\ \overrightarrow{z_0} : Z_B + F_C - P_2 = -\ddot{q}_3 sin(q_3) - \dot{q}_3{}^2 cos(q_3) \\ \_ \Leftrightarrow Z_B = -\ddot{q}_3 sin(q_3) - \dot{q}_3{}^2 cos(q_3) - F_C + P_2 \end{cases}$$

Once we know all the forces in exerted on 2, we still have to calculate the levers of each force to write their torque at the point G_p.



<u>Calculate levers of $\sum F_{ext \to 2}$ at the point G_p :</u>

$Lever_{X_B} = -L_2 cos(q_3)$
$Lever_{Z_B} = L_2 sin(q_3)$
$Lever_{P_2} = L_{G_p G_2} cos(\frac{\pi}{2} - q_3)$

<u>Calculate all the torques $\sum M_{F_{ext} \to G_p \in 2}$ of the forces $\sum F_{ext \to 2}$ at the point G_p :</u>

$$\sum \vec{M}_{F_{ext} \to G_p \in 2} = \vec{\delta}_{G_p}(2/B_0) \Leftrightarrow \begin{cases} \overrightarrow{y_0} : Cm_B . L_2 + Z_B . Lever_{Z_B} + X_B . Lever_{X_B} - P_2 . Lever_{P_2} = \dot{B}\dot{q}_3 + B\ddot{q}_3 \\ \_ \Leftrightarrow Cm_B = \frac{\dot{B}\dot{q}_3 + B\ddot{q}_3 - Z_B . Lever_{Z_B} - X_B . Lever_{X_B} + P_2 . Lever_{P_2}}{L2} \end{cases}$$

(warning : Cm_B must be expressed in $N.mm$ because all the length are in mm)

IN ADDITION

<u>PS :</u> due to the fact that F_c and P_1, P_2 are not exactly aligned, a torque is created, which could make the leg off balance. That torque created is countered by the sliding mate between the Frame and the Test Bench.