高级算法
Advanced Topics in Algorithms

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Chapter 6
Games with Continuous Strategy

Outline

Introduction

- So far, we have considered players choose action from a discrete set.
- it is possible for that the pure strategy set is from subsets of the real line, or infinite dimension.
- For example, the pure strategy (action) sets are a subset of real number $[a, b]$.
- A pure strategy is a choice $x \in [a, b]$.
- A mixed strategy is defined by giving a function $p(x)$ such that the probability that the choice lies between x and $x + dx$ is $p(x)dx$.
- The existence of NE for games with continuous pure-strategy sets was proved independently by Debreu, Glicksburg and Fan.
- Let us study some classical games with continuous strategy sets.

Cournot Duopoly

- Consider two firms competing for a market by producing some infinitely divisible product (e.g., petroleum).
- We allow firms to choose how much they produce, e.g., firm i decides on q_i, the quantity to produce, in which $q_i \in [0, \infty)$.
- Each unit production cost is c.
- Let $Q = q_1 + q_2$, which is the total quantity produced by both firms.
- The market price depends on Q, which is

$$P(Q) = \begin{cases} P_0(1 - \frac{Q}{Q_0}) & \text{if } Q < Q_0, \\ 0 & \text{if } Q \geq Q_0. \end{cases}$$

- Payoff for firm i is

$$\pi_i(q_1, q_2) = q_i P(Q) - c q_i \quad \text{for } i = 1, 2.$$

- Obviously, $q_i \in [0, Q_0]$.

Solution for firm 1

- Consider firm 1 against every possible choice of firm 2, the best response is to find $\hat{q}_1$ that maximizes $\pi_1(q_1, q_2)$, or $\frac{\partial \pi_1}{\partial q_1}(\hat{q}_1, q_2) = 0$.
- Solving $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{q_2}{Q_0} - \frac{c}{P_0}\right)$
- We need to check it is the "best", not "worst" response by

$$\frac{\partial^2 \pi_1}{\partial q_1^2}(\hat{q}_1, q_2) = -2\left(\frac{P_0}{Q_0}\right) < 0.$$

- Need to check $\hat{q}_1 + q_2 \leq Q_0$, or

$$\begin{aligned}
\hat{q}_1 + q_2 &= \frac{Q_0}{2}\left(1 - \frac{q_2}{Q_0} - \frac{c}{P_0}\right) + q_2 = \frac{Q_0}{2} + \frac{q_2}{2} - \frac{cQ_0}{2P_0} \\
&< \frac{Q_0}{2} + \frac{Q_0}{2} - \frac{cQ_0}{2P_0} = Q_0\left(1 - \frac{c}{2P_0}\right) < Q_0.
\end{aligned}$$

Overall solution

- Similarly, $\hat{q}_2 = \frac{Q_0}{2}\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right)$.
- A pure strategy NE is (q_1^*, q_2^*), each is a best response to the other. So we need to solve:

$$q_1^* = \frac{Q_0}{2}\left(1 - \frac{q_2^*}{Q_0} - \frac{c}{P_0}\right); \quad q_2^* = \frac{Q_0}{2}\left(1 - \frac{q_1^*}{Q_0} - \frac{c}{P_0}\right);$$

- The solution is: $q_1^* = q_2^* = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right) \equiv q_c^*$.
- Payoff of each firm:

$$\pi_1(q_c^*, q_c^*) = \pi_2(q_c^*, q_c^*) = q_c^* P(2q_c^*) - cq_c^* = \frac{Q_0 P_0}{9}\left(1 - \frac{c}{P_0}\right)^2.$$

Comparison with monopoly

- Under monopoly, the payoff is

$$\pi_m(q) = qP(q) - cq.$$

- Solving, we have

$$q_m^* = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right).$$

- Since $q_m < 2q_c^*$, the price for unit good is higher in the monopoly market than the competitive market. This implies competition can benefit consumer.

Comparison with cartel

- Suppose both firms form a cartel and agree to produce at $q_1 = q_2 = q_m^*/2$, and the payoff is

$$\pi_1(q_m^*/2, q_m^*/2) = \pi_2(q_m^*/2, q_m^*/2) = \frac{1}{2}q_m^* P(q_m^*) - \frac{1}{2}cq_m^* = \frac{Q_0 P_0}{8}\left(1 - \frac{c}{P_0}\right)^2,$$

  which is higher than the Cournot payoff and the price for customer is the same as the monopoly market.
- This conclusion is **unstable** because the best response to cartel:

$$\hat{q} = \frac{Q_0}{2}\left(1 - \frac{q_m^*}{2Q_0} - \frac{c}{P_0}\right) = \frac{3}{4}q_m^* > \frac{1}{2}q_m^*.$$

- We are not saying cartel is not possible, this only says cartel will not occur in the situations described by the Cournot model.

Bertrand Model of Duopoly

- Consider a case of *differentiated* products with two firms 1 and 2 choose prices p_1 and p_2 respectively.
- The quantity that consumers demand from firm i is

$$q_i(p_i, p_j) = a - p_i + bp_j, \;\; b > 0.$$

- Assume no fixed costs of production and marginal costs are constant at c, where $c < a$.
- Both firms act simultaneously.
- Each firm's strategy space is $S_i = [0, \infty)$.
- A typical strategy s_i is now a price choice, $p_i \geq 0$.

Bertrand Model of Duopoly

- Profit function of firm i:

$$\pi_i(p_i, p_j) = q_i(p_i, p_j)[p_i - c] = [a - p_i + bp_j][p_i - c]$$

- (p_1^*, p_2^*) is a NE if for each firm i, p_i^* solves:

$$\max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*) = \max_{0 \leq p_i < \infty} [a - p_i + bp_j^*][p_i - c]$$

- The solution to firm i's optimization is

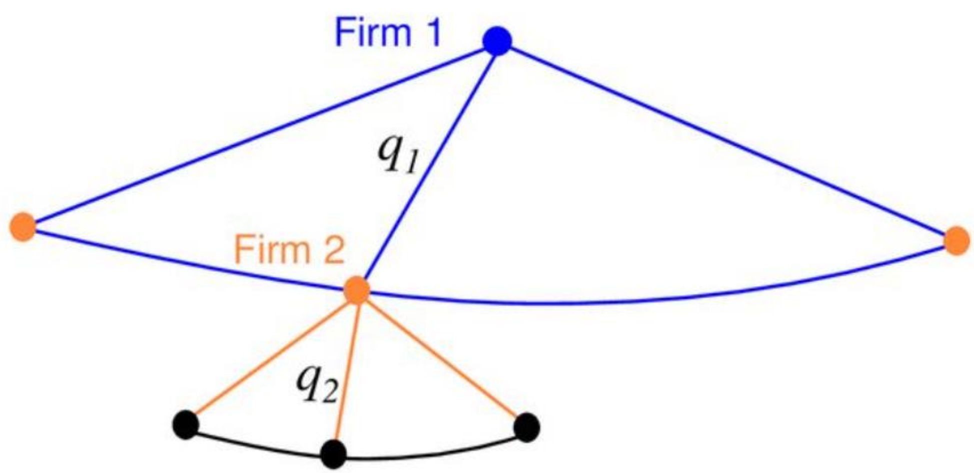
$$p_i^* = \frac{1}{2}(a + bp_j^* + c), \quad i = 1, 2.$$

- Solving these two equations, we have

$$p_1^* = p_2^* = \frac{a + c}{2 - b}.$$

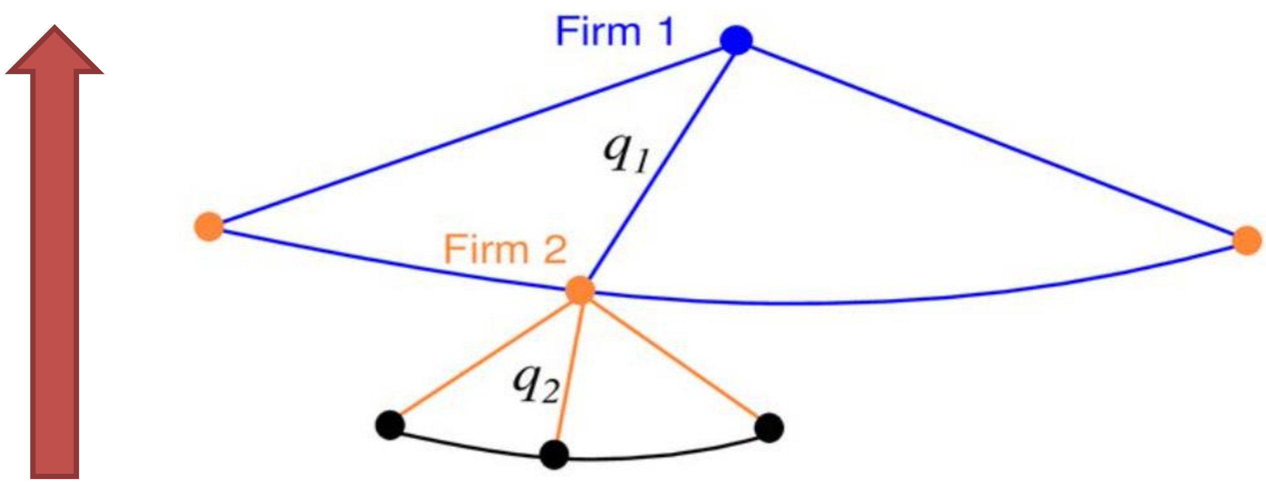
The Stackelberg Duopoly

- Similar to the Cournot model, we have two firms, each needs to determine the amount of production, and the same market price $P(Q) = P_0(1 - Q/Q_0)$ where $Q = q_1 + q_2$.
- However, we have sequential decision: Firm 1 (or market leader) decides first and then firm 2 decides. We assume each firm wants to maximize its profit, and $P_0 > c$.
- Determine q_1^*, q_2^*, payoffs $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$.



Solution to the Stackelberg Duopoly Model

- We first use backward induction to find the subgame perfect NE by finding the best response of firm 2, $\hat{q}_2(q_1)$, for every possible value of q_1.
- Given that firm 1 knows firm 2's best response, we find the best response of firm 1, $\hat{q}_1(\hat{q}_2)$, so as to find the NE for this game.



Solution to the Stackelberg Duopoly Model: continue

- Firm 2's profit: $\pi_2(q_1, q_2) = q_2[P(Q) - c]$ and the best response to a choice of q_1 is found by solving: $\frac{\partial \pi_2}{\partial q_2}(q_1, q_2) = 0$, which gives

$$\hat{q}_2(q_1) = \frac{Q_0}{2}\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).$$

- Firm 1 chooses q_1 based on the best response of $\hat{q}_2(q_1)$, firm 1's payoff:

$$\begin{aligned}
\pi_1(q_1, \hat{q}_2(q_1)) &= q_1\left[P_0\left(1 - \frac{q_1 + \hat{q}_2(q_1)}{Q_0}\right) - c\right] \\
&= q_1\left(\frac{P_0}{2}\right)\left(1 - \frac{q_1}{Q_0} - \frac{c}{P_0}\right).
\end{aligned}$$

- Firm 1 maximizes its profit at: $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)$.

Solution to the Stackelberg Duopoly Model: continue

- By evaluation $\frac{\partial \pi_1(q_1, \hat{q}_2)}{\partial q_1} = 0$, one can find that firm 1 maximizes its profit at: $\hat{q}_1 = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)$.
- The Nash equilibrium is:

$$q_1^* = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right); q_2^* = \hat{q}_2(q_1^*) = \frac{Q_0}{4}\left(1 - \frac{c}{P_0}\right).$$

- Some interesting note:
  1. Leader's advantage: since $q_1^* > q_2^*$, this implies $\pi_1(q_1^*, q_2^*) > \pi_2(q_1^*, q_2^*)$.
  2. The price of the good is cheaper under the Stackelberg duopoly than Cournot duopoly.

Tariffs and Imperfect Competition

- Consider two countries, denoted by $i = 1, 2$, each setting a tariff rate t_i per unit of product.

Tariffs and Imperfect Competition

- Consider two countries, denoted by $i = 1, 2$, each setting a tariff rate t_i per unit of product.
- A firm produces output, both for home consumption and export.
- Consumer can buy from a local firm or foreign firm.
- The market clearing price for country i is $P(Q_i) = a - Q_i$, where Q_i is the quantity on the market in country i.
- A firm in i produces $h_i(e_i)$ units for local (foreign) market, i.e., $Q_i = h_i + e_j$.
- The production cost of firm i is $C_i(h_i, e_i) = c(h_i + e_i)$ and it pays $t_j e_i$ to country j.

Tariffs and Imperfect Competition Game

- First, the government *simultaneously* choose tariff rates t_1 and t_2.
- Second, the firms observe the tariff rates, decide (h_1, e_1) and (h_2, e_2) *simultaneously*.
- Third, payoffs for both firms and governments:
  (1) Profit for firm *i*:

$$\begin{aligned}\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) &= [a-(h_i+e_j)]h_i + [a-(e_i+h_j)]e_i \\ &\quad -c(h_i+e_i) - t_j e_i\end{aligned}$$

  (2) Welfare for government *i*:

$$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j$$

Tariffs and Imperfect Competition Game: 2nd stage

- Suppose the governments have chosen t_1 and t_2.
- If $(h_1^*, e_1^*, h_2^*, e_2^*)$ is a NE for firm 1 and 2, firm i needs to solve $\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*)$. After re-arrangement, it becomes two separable optimizations:

$$\max_{h_i \geq 0} h_i[a - (h_i + e_j^*) - c]; \ \max_{e_i \geq 0} e_i[a - (e_i + h_j^*) - c] - t_j e_i.$$

- Assuming $e_j^* \leq a - c$ and $h_j^* \leq a - c - t_j$, we have

$$h_i^* = \frac{1}{2}\left(a - e_j^* - c\right) \ ; \ e_i^* = \frac{1}{2}\left(a - h_j^* - c - t_j\right), \ \ i = 1, 2.$$

- Solving, we have

$$h_i^* = \frac{a - c - t_i}{3} \ ; \ e_i^* = \frac{a - c - 2t_j}{3}, \ \ i = 1, 2.$$

Tariffs and Imperfect Competition Game: 1st stage

- In the first stage, government i payoff is:

$$W_i(t_i, t_j, h_1^*, e_1^*, h_2^*, e_2^*) = W_i(t_i, t_j)$$

since h_i^* (e_i^*) is a function of t_i (t_j).

- If (t_1^*, t_2^*) is a NE, each government solves:

$$\max_{t_i \geq 0} W_i(t_i, t_j^*).$$

- Solving the optimization, we have $t_i^* = \frac{a-c}{3}$, for $i = 1, 2.$ which is a *dominant strategy* for each government.
- Substitute t_i^*, we have

$$h_i^* = \frac{4(a-c)}{9} \; ; \; e_i^* = \frac{a-c}{9}, \text{ for } i = 1, 2.$$

Comment on Tariffs and Imperfect Competition Game

- In the subgame-perfect outcome, the aggregate quantity on each market is $5(a-c)/9$.
- But if two governments cooperate, they seek socially optimal point and they solve the following optimization problem :

$$\max_{t_1, t_2 \geq 0} W_1(t_1, t_2) + W_2(t_1, t_2)$$

  The solution is $t_1^* = t_2^* = 0$ (no tariff) and the aggregate quantity is $2(a-c)/3$.
- Therefore, for the above game, we have a unique NE, and it is socially inefficient.