高级算法

Advanced Topics in Algorithms

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Convex Optimization Basis

Convex set

line segment between x_1, x_2: all points

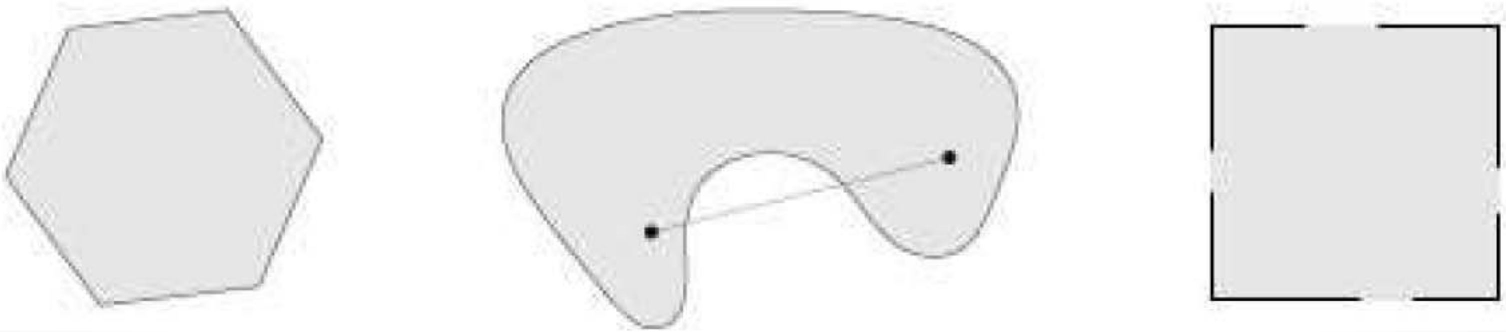
$$x = \theta x_1 + (1 - \theta) x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \Rightarrow \theta x_1 + (1 - \theta) x_2 \in C$$

examples(one convex, two nonconvex sets)



Definition

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if **dom** f is a convex set and

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\ f, 0 \le \theta \le 1$

- f is concave if $-f$ is convex
- f is strictly convex if **dom** f is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\ f, x \neq y, 0 < \theta < 1$

Examples on R

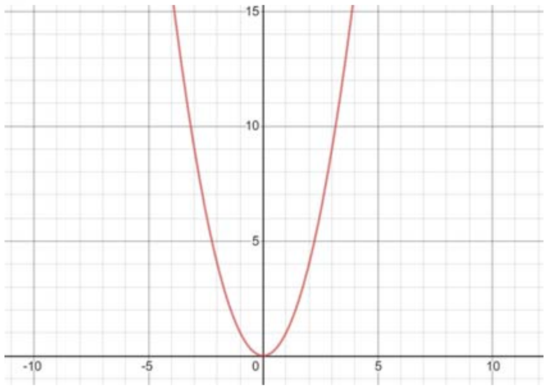
convex:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^a on $\mathbf{R}_{++}$, for $a \geq 1$ or $a \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

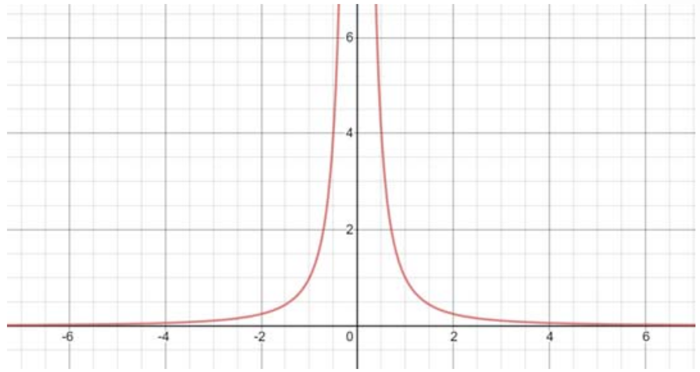
concave:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^a on $\mathbf{R}_{++}$, for $0 \leq a \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

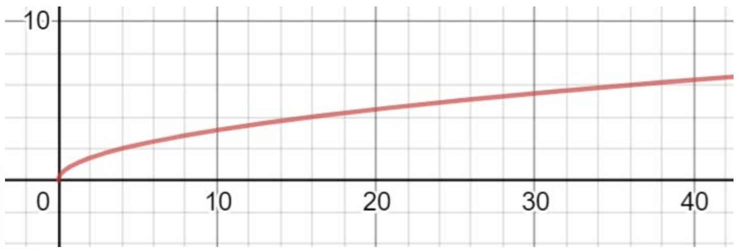
$Y=X^2$



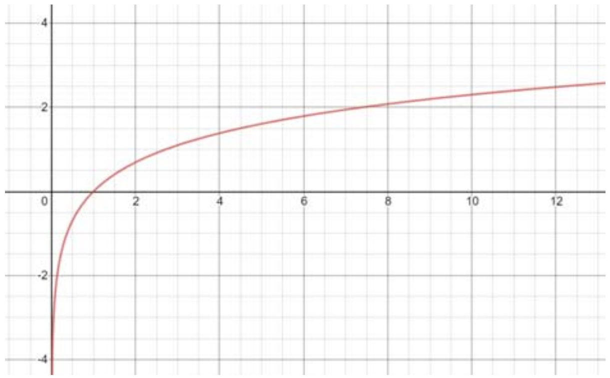
$Y=X^{-2}$



$Y=X^{-0.5}$



$Y=\ln X$



First-order condition

f is differentiable if **dom** f is open and the gradient:

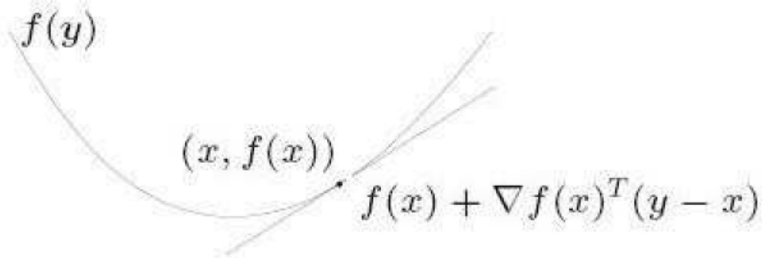
$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n}\right)$$

exists at each $x \in$ **dom** f

1st-order condition:

differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \textbf{for all } x, y \in \textbf{dom } f$$



first-order approximation of f is global underestimator

Second-order conditions

f is twice differentiable if **dom** f is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$:

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial_{x_i} \partial_{x_j}}, \; i, j = 1, \ldots, n$$

exists at each $x \in$ **dom** f

2nd-order condition:for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \textbf{ for all } \; x \in \textbf{dom } f$$

- if $\nabla^2 f(x) \succ 0$ **for all** $x \in$ **dom** f, then f is strictly convex

Examples

1. quadratic function: $f(x) = (1/2)x^T Px + q^T x + r$ (with $P \in$**S**)

$$\nabla f(x) = Px + q, \; \nabla^2 f(x) = P$$

convex if $P \succeq 0$

2. least-squares objective: $f(x) = \|Ax - b\|_2^2$

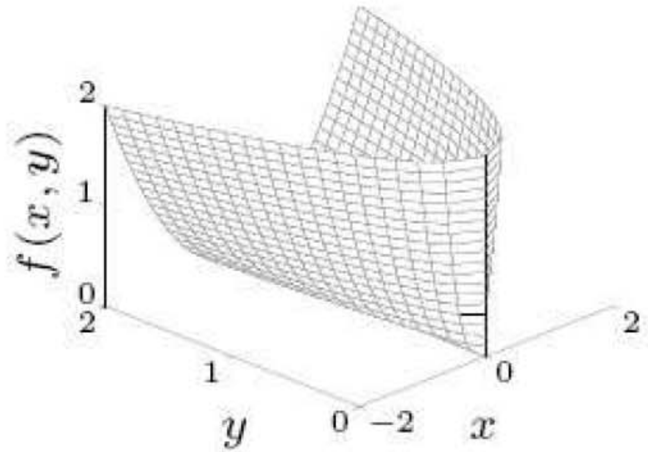
$$\nabla f(x) = 2A^T(Ax - b), \; \nabla^2 f(x) = 2A^T A$$

convex (for any A)

3. quadratic-over-linear: $f(x, y) = x^2/y$

$$\nabla^2 f(x,y) = \frac{2}{y^3}\begin{bmatrix} y \\ -x \end{bmatrix}\begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$



Lagrangian

standard form problem (not necessarily convex)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

variable $x \in \mathbf{R}^n$, **domain** D, optimal value P^*

Lagrangian: $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\operatorname{dom} L = D \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- λ_i is Lagrange multiplier associated with $f_i(x) \le 0$
- ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual function

Lagrange dual function: $g = \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$

$$\begin{aligned} g(\lambda,\nu) &= inf_{x\in D} L(x,\lambda,\nu) \\ &= inf_{x\in D}\left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)\right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

lower bound property: if $\lambda \geq 0$, then $g(\lambda,\nu) \leq p^*$

Proof

if $\tilde{x}$ is feasible and $\lambda \geq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x},\lambda,\nu) \geq inf_{x\in D} L(x,\lambda,\nu) = g(\lambda,\nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^* \geq g(\lambda,\nu)$

Primal & Dual problem

Primal problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

variable $x \in \mathbf{R}^n$, **domain** D, optimal value P^*

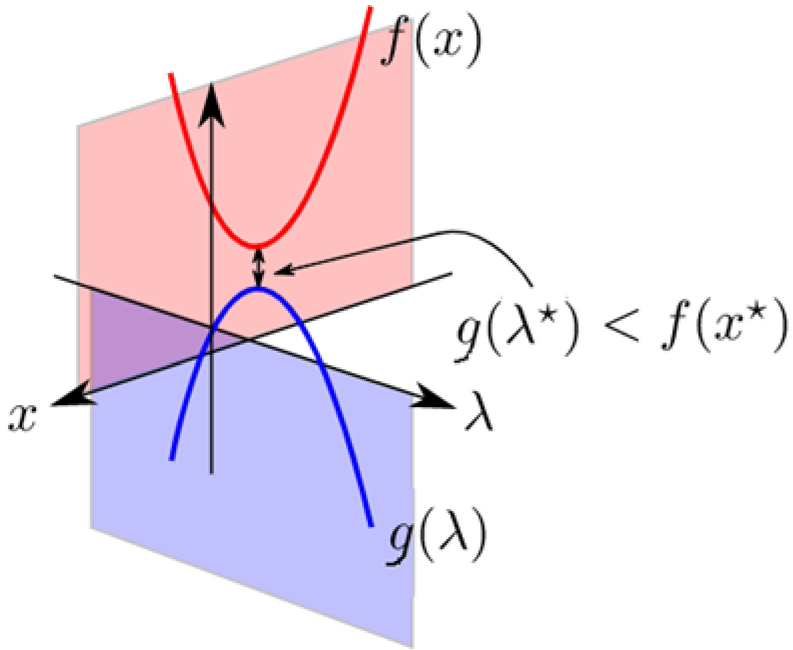
Dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

$$g(\lambda, \nu) \;=\; inf_{x \in D}\left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)\right)$$

Weak duality theorem

Let x' be a feasible solution to the primal problem, and λ' be a feasible solution to its dual problem, then $f(\mathbf{x}') \geq g(\lambda')$.



weak duality

Karush-Kuhn-Tucker (KKT) conditions

The following **four** conditions are called KKT conditions
(for a problem with differentiable f_i, h_i)

1. primal constraints: $f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p$

2. dual constraints: $\lambda \succeq 0$

3. complementary slackness: $\lambda_i f_i(x) = 0, i = 1, \ldots, m$

4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

KKT conditions for convex problem

if $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

  hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

Strong Duality Theorem:
Primal Optimum = Dual Optimum

Zero Duality Gap!

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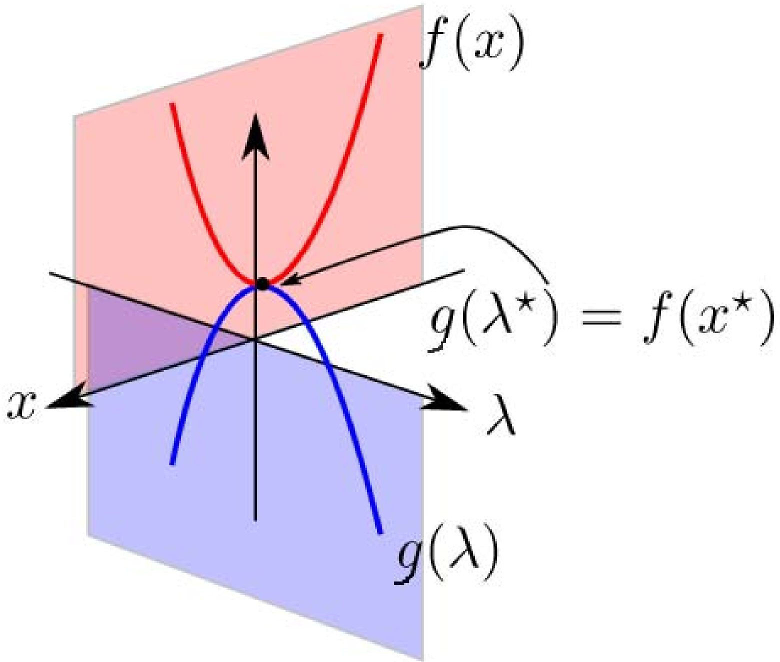
Zero Duality Gap!



strong duality

Example

water-filling (assume $\alpha_i > 0$)

$$\begin{aligned} \min_x \quad & -\sum_{i=1}^{n} \log(x_i + \alpha_i) \\ s.t. \quad & x \succeq 0, \quad \mathbf{1}^T x = 1 \end{aligned}$$

KKT Conditions:

x is optimal iff $x \succeq 0$, $\mathbf{1}^T x = 1$, and there exist $\lambda \in \mathbf{R}^n, \nu \in \mathbf{R}$ such that

$$\lambda \succeq 0, \quad \lambda_i x_i = 0, \quad \frac{1}{x_i + \alpha_i} + \lambda_i = \nu$$

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- if $\nu < 1/\alpha_i$: $\lambda_i = 0$ and $x_i = 1/\nu - \alpha_i$
- if $\nu \geq 1/\alpha_i$: $\lambda_i = \nu - 1/\alpha_i$ and $x_i = 0$
- determine ν from $\mathbf{1}^T x = \sum_{i=1}^n \max\{0, 1/\nu - \alpha_i\} = 1$

thus, the optimal point is given by

$$x_i^* = \max\{0, 1/\nu^* - \alpha_i\}$$

where ν^* satisfies $\sum_{i=1}^n \max\{0, 1/\nu^* - \alpha_i\} = 1$

interpretation:

- n patches; level of patch i is at height a_i
- flood area with unit amount of water
- resulting level is $1/\nu^*$
- depth of water above patch i is x_i^*

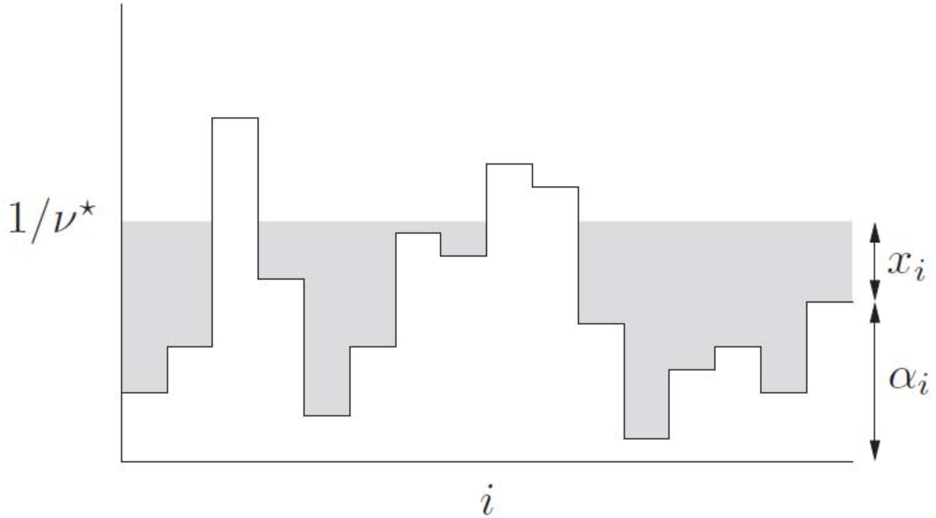


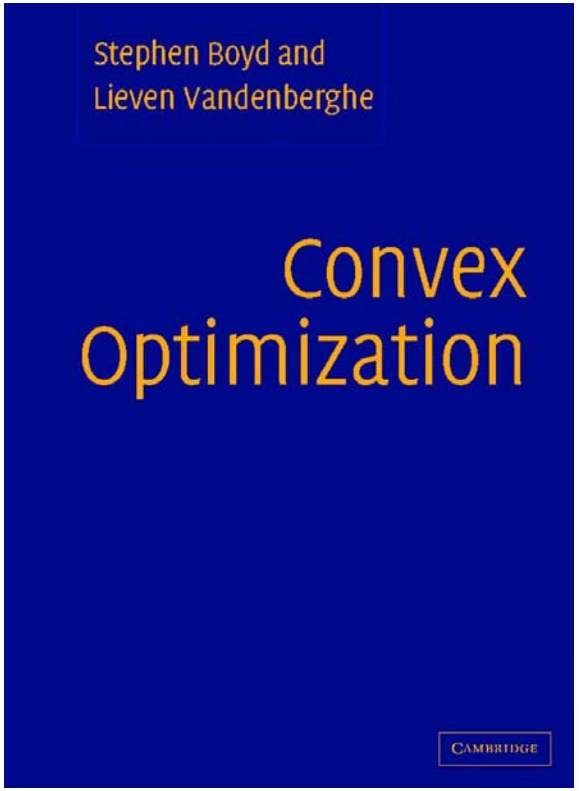
Illustration of water-filling algorithm. The height of each patch is given by α_i. The region is flooded to a level $1/\nu^\star$ which uses a total quantity of water equal to one. The height of the water (shown shaded) above each patch is the optimal value of $x_i^\star$.

Convex optimization problems can be solved by the following contemporary methods

- Primal-Dual methods
- Subgradient projection methods
- Interior-point methods
- ……

Numerical Optimization Solvers

- CVX – convex optimization solver
- CPLEX – integer, linear and quadratic programming
- MATLAB Optimization Toolbox – linear, integer, quadratic, and nonlinear problems
- MOSEK – linear, quadratic, conic and convex nonlinear, continuous and integer optimization
- ……



https://see.stanford.edu/Course/EE364A