高级算法
Advanced Topics in Algorithms

陈 旭

数据科学与计算机学院

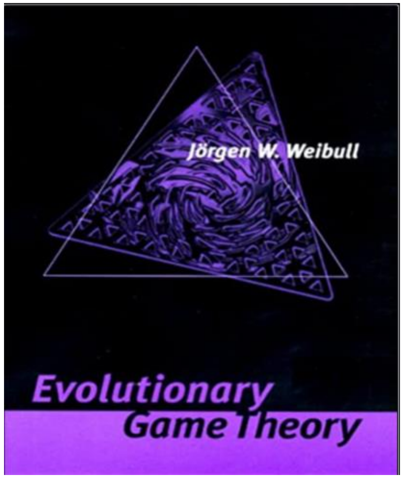
中山大學
SUN YAT-SEN UNIVERSITY

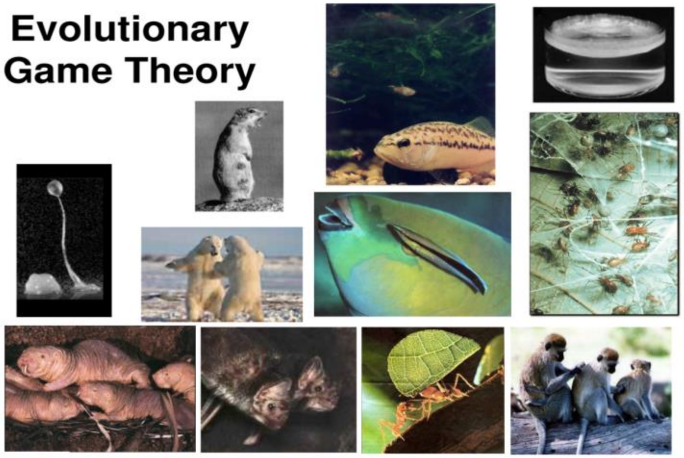
Chapter 8
Evolutionary Game Theory: Population Games

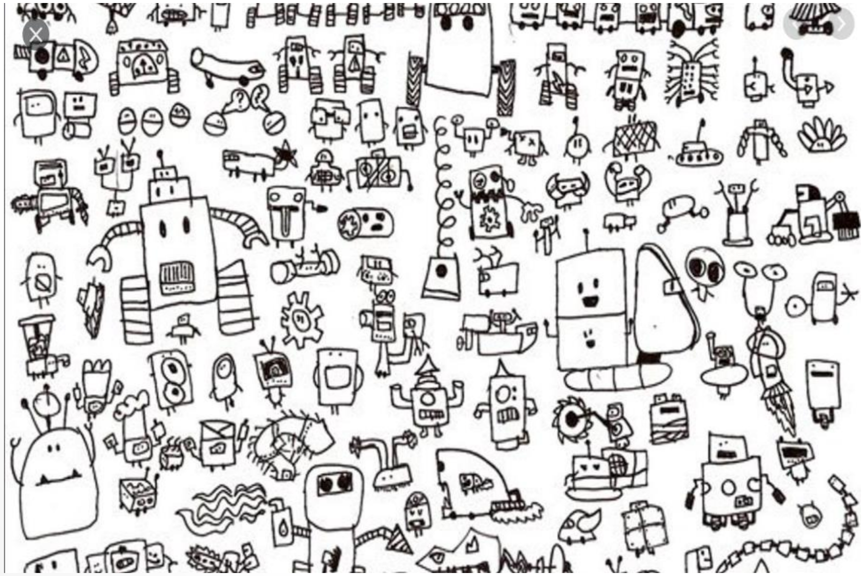
Introduction

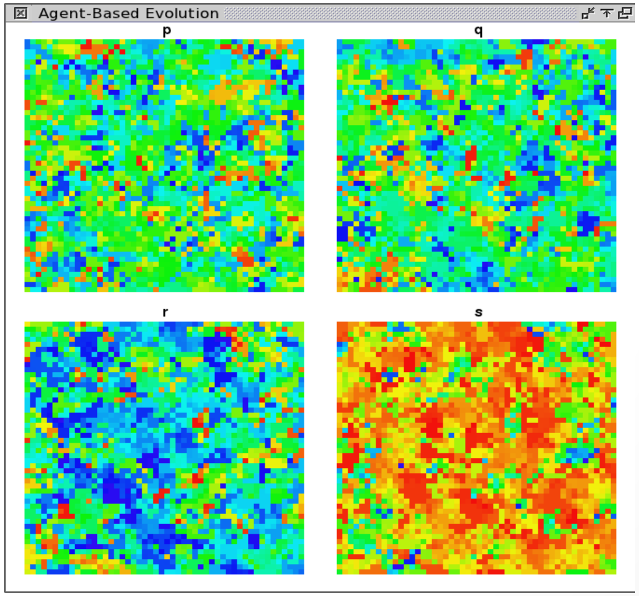
- So far we have considered "classical game theory", where outcome depends on the choice of rational individuals, and each individual uses a strategy that is the "best response" to other players' choice.
- If we map the same concept to "*population*", symmetric Nash equilibria $(\sigma_i^*, \sigma_{i-1}^*)$, has an alternative interpretation. In a population where everyone uses σ^*, the best thing to do is to follow the crowd. So if everyone using σ^*, it will remain that way.
- Some interesting (and important) questions:
  - What happens if the population is close to, but not at, the NE?
  - Will the population evolve toward the equilibrium?
  - Will the population move away from the equilibrium?
- Evolutionary Game Theory considers a population decision makers wherein the *frequency* with which a particular decision is made can be time varying. It is a theory started from Biology.

Evolutionary Game Theory









Under the evolutionary game, one type of end point (if any) is called an evolutionary stable strategy (ESS).

Definition

Consider an infinite population of individuals that can use a set of pure strategies, $\boldsymbol{S}$. A population profile is a vector $\boldsymbol{x}$ that gives a probability $x(s)$ with which each strategy $s \in \boldsymbol{S}$ is played in the population.

Note that the population profile needs not correspond to a strategy adopted by any members of the population !!

Example

- A population can use $\boldsymbol{S} = \{s_1, s_2\}$.
- If every member of the population randomizes by playing each of the pure strategies with probability $\frac{1}{2}$, then $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2})$. In this case, the population profile $\boldsymbol{x}$ is identical to the mixed strategy adopted by all members.
- If half of the population adopt the strategy s_1 and other half adopt strategy s_2. We have $\boldsymbol{x} = (\frac{1}{2}, \frac{1}{2})$, and this is NOT the same as the strategy adopted by **any** member of the population.

Definition

Consider a population where initially all the individuals adopt some strategy σ^*. Suppose a mutation occurs and a small proportion ϵ of individuals use some other strategy σ. The new population is called the post-entry population and will be denoted as $\boldsymbol{x}_\epsilon$.

Example

Consider a population with $\boldsymbol{S} = \{s_1, s_2\}$ and $\sigma^* = (\frac{1}{2}, \frac{1}{2})$. Suppose the mutant strategy is $\sigma = (\frac{3}{4}, \frac{1}{4})$, then

$$\boldsymbol{x}_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma = (1-\epsilon)\left(\frac{1}{2}, \frac{1}{2}\right) + \epsilon\left(\frac{3}{4}, \frac{1}{4}\right) = \left(\frac{1}{2} + \frac{\epsilon}{4}, \frac{1}{2} - \frac{\epsilon}{4}\right).$$

Stability of ESS

A mixed strategy σ^* is an evolutionary stable strategy (ESS) if there exists an $\bar{\epsilon}$ such that for every $0 < \epsilon < \bar{\epsilon}$ and every $\sigma \neq \sigma^*$

$$\pi(\sigma^*, \boldsymbol{x}_\epsilon) > \pi(\sigma, \boldsymbol{x}_\epsilon).$$

Physical meaning: a strategy σ^* is an ESS if mutants that adopt any other strategy σ leave fewer offspring in the post-entry population, provided that the proportion of mutants is sufficiently small.

Types of Population Games

In general, there are two types of population game: (1) *games against the field* and (2) *games with pairwise contests*.

Definition

A game against the field is one in which there is no specific “opponent” for a given individual - their payoff depends on what everyone in the population is doing.

Definition

A pairwise contest game describes a situation in which a given individual plays against an opponent that has been randomly selected (by nature) from the population and the payoff depends just on what both individual do.

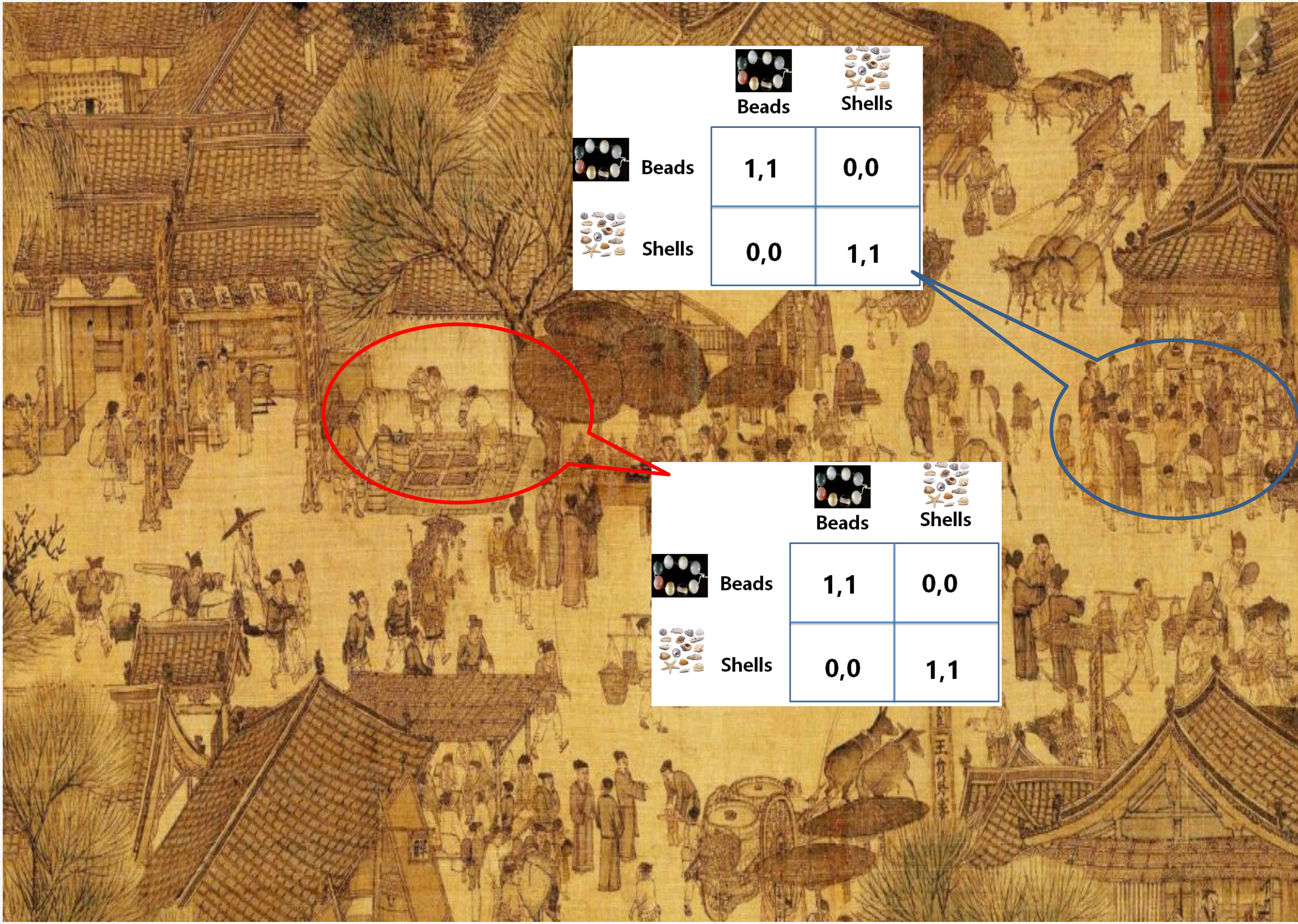
Example: The evolution of money

- In an remote island, inhabitants have to decide to use either "beads" or "shells" as tokens of money in commerce.
- A transaction is only successful if both parties use the same form of token.
- Assume that a trader gets a utility increment of 1 if the transaction is successful and 0 if it fails.

Example: The evolution of money




	Beads	Shells
Beads	1,1	0,0
Shells	0,0	1,1



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- A transaction is only successful if both parties use the same form of token.
- Assume that a trader gets a utility increment of 1 if the transaction is successful and 0 if it fails.
- The general strategy to an individual is to use beads with p, i.e., $\sigma = (p, 1 - p)$. The population profile $\boldsymbol{x} = (x, 1 - x)$.
- What is an ESS ?

Solution

- An individual attempts to trade with a randomly selected member of the population, his payoff

$$\pi(\sigma, \mathbf{x}) = px + (1-p)(1-x) = (1-x) + p(2x-1).$$

We see that

$$x > \frac{1}{2} \longrightarrow \hat{p} = 1 \quad \text{and} \quad p = 1 \longrightarrow x = 1.$$

So $\sigma_b^* = (1,0)$ is a potential ESS with $\mathbf{x} = (1,0)$.

- The post-entry population is:

$$\mathbf{x}_\epsilon = (1-\epsilon)(1,0) + \epsilon(p, 1-p) = (1-\epsilon(1-p), \epsilon(1-p)).$$

Solution: continue

- In this population, the payoff for an arbitrary strategy is

$$\pi(\sigma, \boldsymbol{x}_\epsilon) = \epsilon(1-p) + p(1 - 2\epsilon(1-p)).$$

- The payoff for the candidate ESS is $\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) = 1 - \epsilon(1-p)$, so

$$\begin{aligned} \pi(\sigma_b^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) &> 0, \\ \Longleftrightarrow (1-p)(1-2\epsilon(1-p)) &> 0. \end{aligned}$$

- Now, $\forall p \neq p^*$, we have $(1-p) > 0$, so σ_b^* is an ESS if and only iff $\epsilon(1-p) < \frac{1}{2}$. That is $\bar{\epsilon} = \frac{1}{2}$.

Solution: continue

- The strategy $\sigma_s^* = (0, 1)$ is another ESS because the post-entry population,

$$\boldsymbol{x}_\epsilon = (\epsilon p, 1 - \epsilon p),$$

the payoff for an arbitrary strategy is

$$\pi(\sigma, \boldsymbol{x}_\epsilon) = (1 - \epsilon p) - p(1 - 2\epsilon p),$$

and the payoff for the candidate ESS is

$$\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) = 1 - \epsilon p.$$

- We have:

$$\pi(\sigma_b^*, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) > 0 \iff p(1 - 2\epsilon p) > 0.$$

- Now, $\forall p \neq p^*$, we have $p > 0$, so σ_s^* is an ESS if and only if $\epsilon p < \frac{1}{2}$, i.e., $\bar{\epsilon} = \frac{1}{2}$.

ESSs and Nash Equilibria

- In this section, we show that ESSs in a pairwise contest population game correspond to a (possibly empty) subset of the set of Nash equilibria for an associated two-player game.
- In a pairwise contest population game, the payoff to a focal individual using σ in a population with profile $\boldsymbol{x}$ is

$$\pi(\sigma, \boldsymbol{x}) = \sum_{s \in \boldsymbol{S}} \sum_{s' \in \boldsymbol{S}} p(s) x(s') \pi(s, s'). \tag{1}$$

- Note that the above payoff is the same as a two-player game against an opponent using a strategy $\sigma^{'}$ that assigns $p^{'}(s) = x(s) \forall s \in \boldsymbol{S}$. So there is an association between a two-player game with a population game involving pairwise contests.

Definition

In a pairwise contest population game has payoffs given by Eq. (1), then the associated two-player game is the game with the payoffs given by the numbers $\pi_1(s, s') = \pi(s, s') = \pi_2(s', s)$.

Theorem

Let σ^* be an ESS in a pairwise contest, then $\forall \sigma \neq \sigma^*$, either

1. $\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$, *or*
2. $\pi(\sigma^*, \sigma^*) = \pi(\sigma, \sigma^*)$ *and* $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.

Conversely, if either (1) or (2) holds for each $\sigma \neq \sigma^*$ in a two-player game, then σ^* is an ESS in the corresponding population game.

Example: The evolution of money

	Beads	Shells
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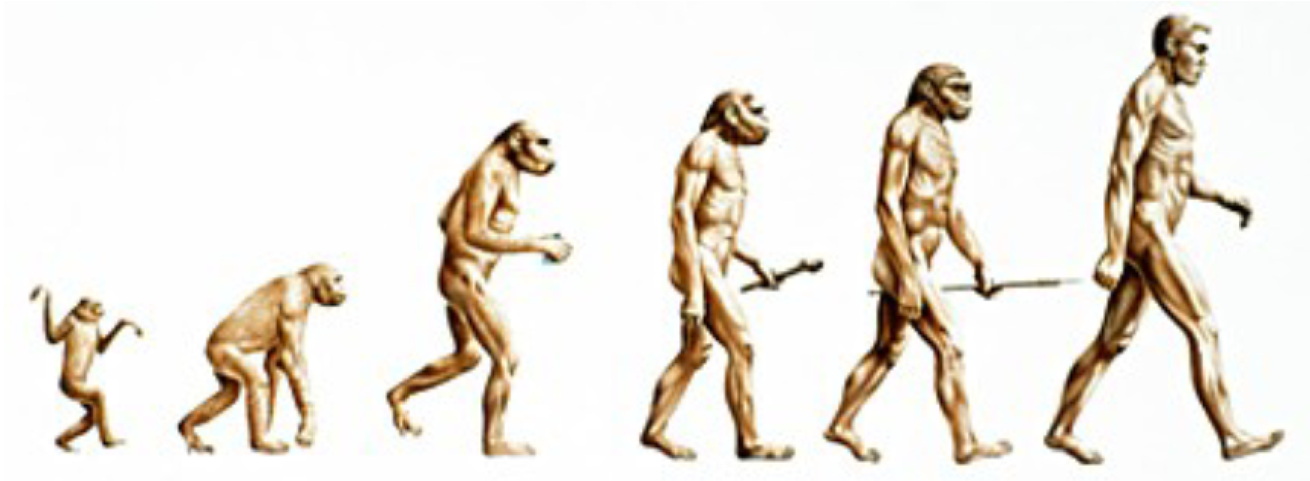
$$\pi(\sigma^*, \sigma^*) > \pi(\sigma, \sigma^*)$$

Evolutionary Simulations

- An evolutionary simulation is a stochastic game whose structure is intended to model certain aspects of evolutionary environments
  - At each **stage** (or **generation**) there is a large set (e.g., hundreds) of agents
- Different agents may use different strategies
  - A strategy *s* is represented by the set of all agents that use strategy *s*
  - Over time, the number of agents using *s* may grow or shrink depending on how well *s* performs
- *s*'s **reproductive success** is the fraction of agents using *s* at the end of the simulation,
  - i.e., (number of agents using *s*)/(total number of agents)



Replicator Dynamics

- **Replicator dynamics** works as follows:
  - $p_i^{new} = p_i^{curr}\, r_i / R$,

  where

  - p_i^{new} is the proportion of agents of type i in the next stage
  - p_i^{curr} is the proportion of agents of type i in the current stage
  - r_i = average payoff received by agents of type i in the current stage
  - R_i = average payoff received by all agents in the current stage

- Under the replicator dynamics, an agent's numbers grow (or shrink) proportionately to how much better it does than the average
- Probably the most popular reproduction dynamics
  - e.g., does well at reflecting growth of animal populations

Example: A Simple Lottery Game

- A repeated lottery game
- At each stage, agents make choices between two lotteries
  - "Safe" lottery: guaranteed reward of 4
  - "Risky" lottery: [0, 0.5; 8, 0.5],
    - i.e., probability ½ of 0, and probability ½ of 8
- Let's just look at stationary strategies
- Two pure strategies:
  - S: always choose the "safe" lottery
  - R: always choose "risky" lottery
- Many mixed strategies, one for every p in [0,1]
  - R_p: probability p of choosing the "risky" lottery, and probability $1-p$ of choosing the "safe" lottery

Lottery Game with Replicator Dynamics

- At each stage, each strategy's average payoff is 4
  - Thus on average, each strategy's population size should stay roughly constant
- Verified by simulation for S and R
- Would get similar behavior with any of the R_p strategies