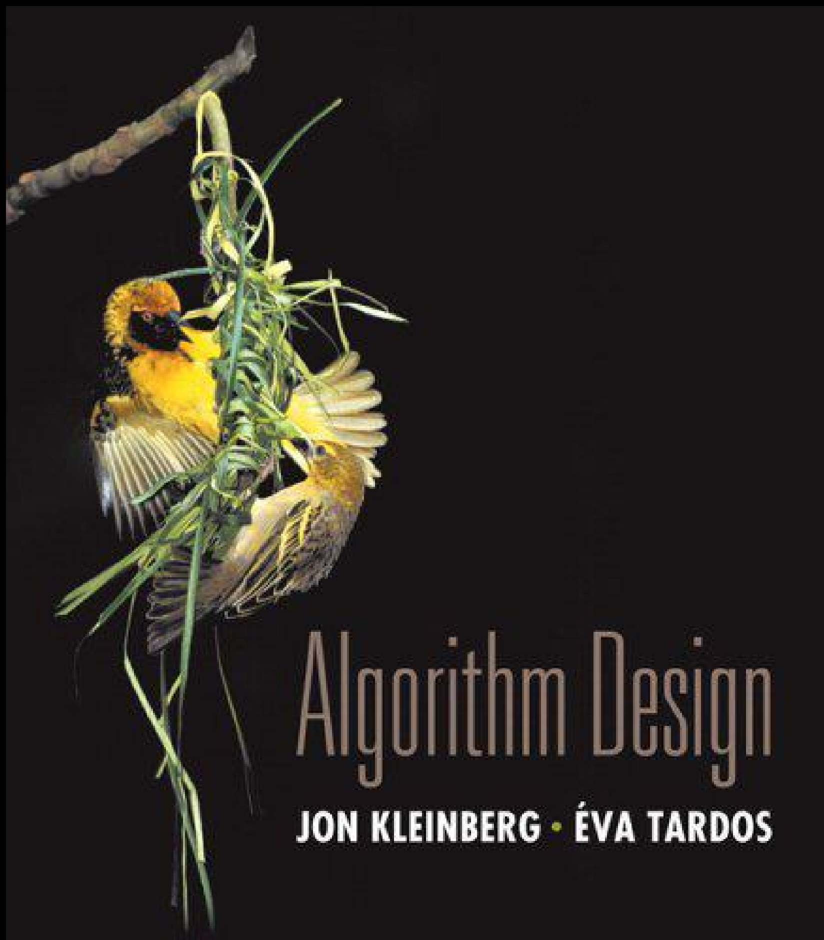


Chapter 13

Randomized Algorithms

Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, cryptography.

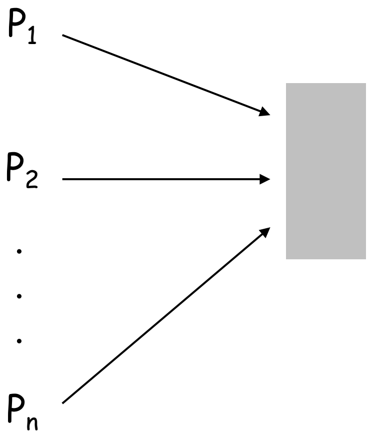
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability $p = 1/n$.

Claim. Let $S[i, t]$ = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access

none of remaining n-1 processes request access

- Setting $p = 1/n$, we have $Pr[S(i, t)] = 1/n\, \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪

value that maximizes Pr[S(i, t)]

Useful facts from calculus. As n increases from 2, the function:
- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in en rounds is at most 1/e. After $e \cdot n(c \ln n)$ rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have $\Pr[F(i, t)] \le (1 - 1/(en))^{t}$.

- Choose $t = \lceil e \cdot n \rceil$: $\Pr[F(i,t)] \le \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \le \left(1 - \frac{1}{en}\right)^{en} \le \frac{1}{e}$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $\Pr[F(i,t)] \le \left(\frac{1}{e}\right)^{c \ln n} = n^{-c}$

Contention Resolution: Randomized Protocol

Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is at least $1 - 1/n$.

Pf. Let $F[t]$ = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr[F[t]] \;=\; \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \;\le\; \sum_{i=1}^{n} \Pr[F[i,t]] \;\le\; n\left(1-\tfrac{1}{en}\right)^{t}$$

union bound   previous slide

- Choosing $t = 2\lceil en \rceil \lceil c \ln n \rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.3 Linearity of Expectation

Expectation

Expectation. Given a discrete random variables X, its expectation E[X] is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Waiting for a first success. Coin is heads with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j] = \sum_{j=0}^{\infty} j \, \underbrace{(1-p)^{j-1}}_{\text{j-1 tails}} \underbrace{p}_{\text{1 head}} = \frac{p}{1-p} \sum_{j=0}^{\infty} j \, (1-p)^{j} = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$

Expectation: Two Properties

Useful property. If X is a 0/1 random variable, E[X] = Pr[X = 1].

Pf.
$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X=j] = \sum_{j=0}^{1} j \cdot \Pr[X=j] = \Pr[X=1]$$

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Decouples a complex calculation into simpler pieces.

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf.

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1/n$.
- $E[X] \underset{\uparrow}{=} E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/n = 1.$ ▪

linearity of expectation

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - i - 1)$.
- $E[X] = E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/2 + 1/1 = \Theta(\log n)$. ▪

linearity of expectation

13.4 MAX 3-SAT

Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}
$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is 7k/8.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = number of clauses satisfied by random assignment.

$$
\begin{aligned}
E[Z] &= \sum_{j=1}^{k} E[Z_j] \\
&= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\
&= \tfrac{7}{8} k
\end{aligned}
$$

linearity of expectation

The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm?

Lemma. The probability that a random assignment satisfies $\geq 7k/8$ clauses is at least $1/(8k)$.

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq 7k/8$ clauses are satisfied.

$$
\begin{aligned}
\tfrac{7}{8}k \;=\; E[Z] \;&=\; \sum_{j \geq 0} j\, p_j \\
&=\; \sum_{j < 7k/8} j\, p_j \;+\; \sum_{j \geq 7k/8} j\, p_j \\
&\leq\; \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \geq 7k/8} p_j \\
&\leq\; \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\, p
\end{aligned}
$$

Rearranging terms yields $p \geq 1/(8k)$. ▪

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least $1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ▪

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

very unlikely to improve over simple randomized algorithm for MAX-3SAT

13.9 Chernoff Bounds

Chernoff Bounds

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \leq E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] < e^{-\delta^2 \mu / 2}$$

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Analysis.

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \dfrac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n / \log\log n)$ jobs.