Contrapositive Proof Presentation in LaTeX

CM-IV

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1 The Contrapositive of Conditionals

The equivalence between a conditional statement and its contrapostive is a fundamental law of logic and reason. Epp (2011) tells us that the contrapositive of a conditional statement of the form "*If p then q*" written symbolically is

$$\sim q \to \sim p. \tag{1}$$

In other words, equation 1 states that the contrapositive of "*If p then q*" is "*If NOT q then NOT p*". It is important to remember the fact that

A conditional statement is logically equivalent to its contrapositive.

2 Direct Proofs

In order to prove statements of the form "*If p then q*", we must show that since p is true, q must be true also. According to Hammack (2018), the setup for writing a direct proof is actually quite simple. You start with a **proposition** sentence, which is a true statement that is less significant than a theorem, and then write down your **proof** beginning with your supposition that finally ends with the conclusion section.

Direct Proof Outline

Proposition If p, then q.

Proof. Suppose p.

$\vdots$

Therefore q. $\square$

Logic, definitions, and mathematics are used in between the first and last lines of the proof to reach the conclusion.

3 Contrapositive Proofs

The contrapositive proof is often used to prove statements of the form "*If p then q*". Remembering that the conditional statement is logically equivalent to its contrapostive form, we use the direct proof to show that the contrapositive of the afforementioned conditional is true. The first line of the proof includes the sentence "*Suppose q is NOT true.*" The last line of the proof, following the logic and definitions, goes "*Therefore p is not true.*"

Contrapositive Proof Outline

Proposition If p, then q.

Proof. Suppose $\sim q$.

$\vdots$

Therefore $\sim p$. $\square$

3.1 Proof Exercises

Proposition Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Proof. (Contrapositive) Suppose if $x|y$ or $x|z$, then $x|yz$. In Definition 4.4, Hammack (2018) tells us that $x|y$ means $y = xc$, for some $c \in \mathbb{Z}$. Multiplying the previous equation by z will then give us $yz = xzc$. Fact 4.1 tells us that given any two integers, their products are also integers. Knowing this fact, we can take *c'* of zc and we end up with $yz = xc'$. Therefore, if $x|y$, then $x|yz$.

Once again using Definition 4.4, $x|z$ means $z = xc$, for some $c \in \mathbb{Z}$. We multiply the previous equation by y to end up with $yz = xyc$. Taking *c'* of yc and we have $yz = xc'$. So, if $x|z$, then $x|yz$. Therefore, if either $x|y$ or $x|z$, then $x|yz$.

$\square$

Proposition If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

Proof. (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Epp (2011) tells us in Theorem 8.4.1 Modular Equivalences that, given a, b, and $n \in \mathbb{Z}$ and that $n > 1$, we have the statement $n|(a-b)$. By definition of congruence modulo n, we immediately conclude that $a \equiv b \pmod{n}$. Theorem 8.4.1 also tells us that there exists some integers s and t such that $a = c + sn$ and $b = d + tn$. Then we arrive at $ac = (b + sn)(d + tn)$ through substitution. Next, we get $ac = bd + n(bt + sd + sdn)$ with some algebra. Let $k = bt + sd + stn$. Then k is an integer and $ac = bd + nk$. Therefore, by using Theorem 8.4.1, $ac \equiv bd$ (mod n). $\square$

References

Epp, Susanna S. (2011). *Discrete Mathematics with Applications, Fourth Edition*. Stratton, Richard.

Hammack, Richard (2018). *Book of Proof, Third Edition*. Hammack, Richard.