1 Question One

Note: Assume S_t = S in Black-Scholes-Merton PDE.

1.1 Verify Black-Scholes Call Option Price is a solution to Black-Scholes-Merton PDE

The Black-Scholes-Merton partial differentiation equation is:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + r\frac{\partial f}{\partial S}S - rf = 0$$

The partial derivatives corresponding to pricing a call option using the Black-Scholes Options Pricing Model are known as the Greeks. The greeks are displayed below with the price the call options denoted by f in this instance:

$$\begin{aligned}
\Delta_f &= \frac{\partial f}{\partial S} = N(d_1) \\
\Gamma_f &= \frac{\partial^2 f}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T}} \\
\Theta_f &= \frac{\partial f}{\partial t} = \frac{-SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \\
V_f &= \frac{\partial f}{\partial \sigma} = S\sqrt{T}N'(d_1) \\
\rho_f &= \frac{\partial f}{\partial r} = KTe^{-rT}N(d_2) \\
f &= SN(d_1) - Ke^{-rT}N(d_2)
\end{aligned}$$

Substitute the 'Greeks' into Black-Scholes-Merton PDE, rearrange equations, and cancel out terms.

$$\begin{aligned}
\frac{-SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) + \frac{1}{2}\frac{N'(d_1)}{S\sigma\sqrt{T}}\sigma^2 S^2 + rN(d_1)S - rSN(d_1) + rKe^{-rT}N(d_2) = 0 \\
(\frac{-SN'(d_1)\sigma}{2\sqrt{T}} + \frac{1}{2}\frac{N'(d_1)}{\sqrt{T}}\sigma S) + (rN(d_1)S - rSN(d_1)) + (rKe^{-rT}N(d_2) - rKe^{-rT}N(d_2)) = 0 \\
0 + 0 + 0 = 0
\end{aligned}$$

Therefore, Black-Scholes Call Option Price is a solution to the Black-Scholes-Merton PDE.

1.2 Replication of Titman (1985)

Titman (1985) analysis Land Prices under Uncertainty using an application of the Binominal Option Pricing Model. Sections I, II, III, V and VI from Titman (1985) inform the recreation of the figure. The theory explained in the comparative statistics is not necessary in this instance.

1.2.1 Mathematical Derivation

- q: Number of units,
- C: Cost of constructing a building on a given piece of land
- $\frac{\partial C}{\partial q} > 0$ and $\frac{\partial^2 C}{\partial q^2} > 0$ imply C is an increasing and convex function in the number of units.
- p_0 is the current market price per building unit.
- p_h and p_l ($p_h > p_l$) are the two prices vacant land can take in time, t when $t > 0$.
- Assumed there are only two possible values in the next time period $\pi(p_h)$ and $\pi(p_l)$.

- R_F is the risk-free rate of return on the Riskless assets.
- R_t is the per unit rental rate.
- s_h and s_l are state prices.

The following maximisation problem expresses the desires of a landowner to maximise wealth who wishes to construct a building at present time.

$$\max_{\{q\}} \Pi(P_0) = p_0 q - C(q) \tag{1}$$

The solution to the building maximisation problem is expressed below with q^* satisfying the optimal solution. Jensen's equality would show higher level of uncertainty increases the value of vacant land.

$$\frac{\partial C}{\partial q} = p_0 \tag{2}$$

The vacant land can be viewed in the same manner as options/contingent securities, valued by forming a hedge portfolio, consisting of the risk-free rate asset and the exongenously priced primitive asset, that is perfectly correlated with the contingent security.

$$p_0 = s_h p_h + s_l p_l + R_t(s_h + s_i) \tag{3}$$

$$\frac{1}{(1+R_f)} = s_l + s_h \tag{4}$$

Rearranging the above equations leads to:

$$s_h = \frac{p_0 - (\frac{p_l + R_t}{1+R_f})}{p_h - p_l} \tag{5}$$

$$s_l = \frac{(\frac{p_h + R_t}{1+R_f} - p_0)}{p_h - p_l} \tag{6}$$

Given state prices,following that no opportunities for risk-less arbitrage exist, the price of vacant land at is:

$$V = \Pi(p_h)s_h + \Pi(p_l)s_l \tag{7}$$

Subsequently, we apply this valuation equation to each stage to replicate figure one. If $\Pi(p_t) > V(p_t)$, the land will be developed at time t where $\frac{\partial C}{\partial q} = p_t$. Otherwise, the land will be left vacant.

1.2.2 Implementation

π_{2hh} = \$600,000, π_{2hl} = \$300,000, π_{21h} = \$400,000, π_{2ll} = \$200,000,
p_{2hh} = \$110,000, p_{2hl} = \$90,000, p_{2lh} = \$95,000, p_{2ll} = \$75,000.
π_{1h} = \$400,000, π_{1l} = \$300,000.
p_{1h} = \$100,000, p_{1l} = \$80,000.
R_{th1} = \$10,000, R_{tl1} = \$5,000.
π_0 = \$300,000, p_0 = \$90,000, R_0 = \$10,000, R_f = 10%.

At t=1,

$$\begin{aligned}
V_{1h} &= \pi_{2hh} \times s_{h1h} + \pi_{2hl} \times s_{l1h} \\
&= \pi_{2hh} \times \frac{p_{1h} - (\frac{p_{2hl}+R_{t1h}}{1+R_f})}{p_{2hh} - p_{2hl}} + \pi_{2hl} \times \frac{(\frac{p_{2hh}+R_{t1h}}{1+R_f} - p_{1h})}{p_{2hh} - p_{2hl}} \\
&= \$600,000 \times \frac{\$100,000 - (\frac{\$90,000+\$10,000}{1.10})}{\$110,000 - \$90,000} + \$300,000 \times \frac{(\frac{\$110,000+\$10,000}{1.10} - \$100,000)}{\$110,000 - \$90,000} \\
&= \$409090.91 \\
V_{1l} &= \pi_{2lh} \times s_{l1h} + \pi_{2ll} \times s_{l1l} \\
&= \pi_{2lh} \times \frac{p_{1l} - (\frac{p_{2ll}+R_{t1l}}{1+R_f})}{p_{2lh} - p_{2ll}} + \pi_{2ll} \times \frac{(\frac{p_{2lh}+R_{t1l}}{1+R_f} - p_{1l})}{p_{2lh} - p_{2ll}} \\
&= \$400,000 \times \frac{\$80,000 - (\frac{\$75,000+\$5,000}{1.10})}{\$95,000 - \$75,000} + \$200,000 \times \frac{(\frac{\$95,000+\$5,000}{1.10} - \$80,000)}{\$95,000 - \$75,000} \\
&= \$254,545.45
\end{aligned}$$

$V_{1h} > \pi_{1h}$, i.e. \$409090.91 > \$400,000. Therefore, it more valuable to keep the land vacant and wait for future opportunities in this high state at t=1. $V_{1l} < \pi_{1l}$, i.e. \$254,545.45 < \$300,000. Therefore, it more valuable to develop the land than wait for future opportunities in this low state at t=1. At t=0,

$$\begin{aligned}
V_0 &= \max(\pi_{1h}, V_{1h}) \times s_{h0} + \max(\pi_{1l}, V_{1l}) \times s_{l0} \\
&= \max(\pi_{1h}, V_{1h}) \times \frac{p_0 - (\frac{p_{1l}+R_{t0l}}{1+R_f})}{p_{1h} - p_{1l}} + \max(\pi_{1l}, V_{1l}) \times \frac{(\frac{p_{1h}+R_{t1l}}{1+R_f} - p_0)}{p_{1h} - p_{1l}} \\
&= \max(\$400,000, \$409090.91) \times \frac{\$90,000 - (\frac{\$80,000+\$10,000}{1.10})}{\$100,000 - \$80,000} \\
&+ \max(\$300,000, \$254,545.45) \times \frac{(\frac{\$100,000+\$10,000}{1.10} - \$90,000)}{\$100,000 - \$80,000} \\
&= \$317,355.37
\end{aligned}$$

$V_0 > \pi_0$, i.e. \$317,355.37 > \$300,000. Therefore, it more valuable to keep the land vacant and wait for future opportunities in this initial state at t=0. A replication of figure 1 is shown in figure 1 below:

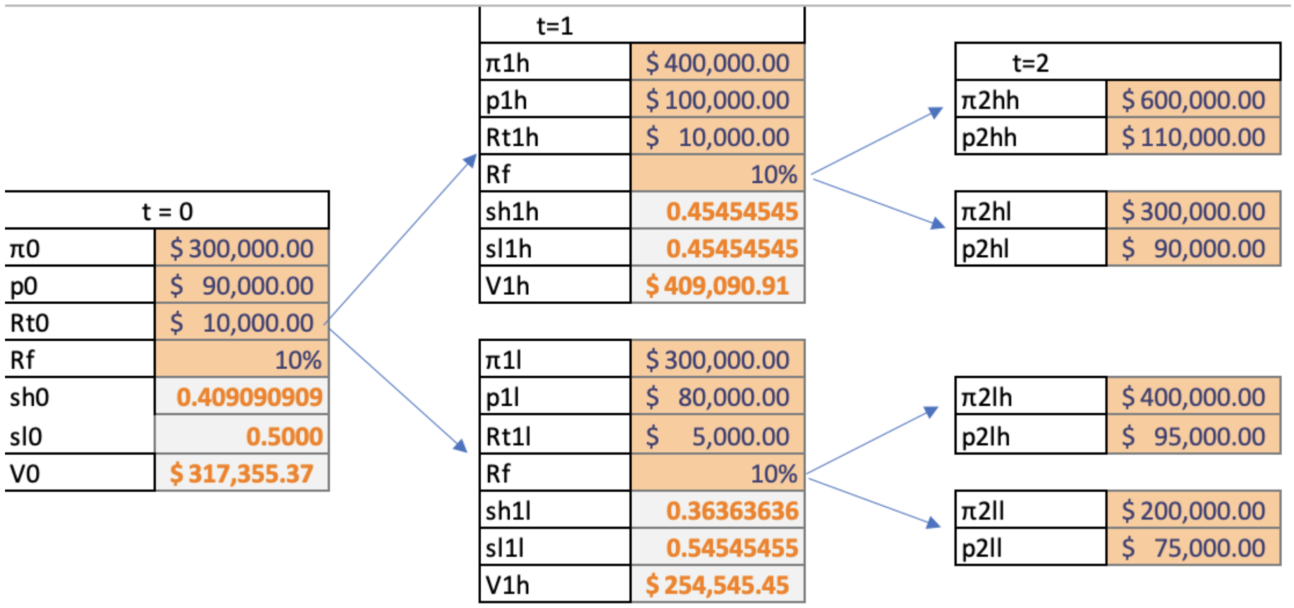

Figure 1: Replication of figure one (Titman, 1985)

1.3 Hurdle Rates: A Reproduction

The replication draws on the analysis from Dixit (1992) that the optimal investment rule can be expressed as a constant hurdle rate rule.

$$R = \frac{C}{I} + \alpha \tag{8}$$

A hurdle rate calls for investing when a projects's IRR exceeds the hurdle rate (γ). A hurdle rate rule is equivalent to a cash-flow rule in which the cash flow trigger, C_γ, is given by 9, or in terms of project value $C_\gamma = V(\rho - \alpha)$, would invest when project value expressed in equation 10

$$C_\gamma = I(\gamma - \alpha) \tag{9}$$

$$V_\gamma = \frac{\gamma - \alpha}{\rho - \alpha} \times I \tag{10}$$

For an arbitrary trigger value (V_A), the corresponding hurdle rate rule would be to invest when the IRR (R) equals γ_A (equation 11)

$$\gamma_A = \alpha + (\rho - \alpha)\frac{V_A}{I} \tag{11}$$

Under the zero-NPV rule, investing when $V_A = I$ and $\gamma = \rho$. Given the optimal investment trigger under uncertainty is $V = V_H$, we define the optimal hurdle rate in equation 12.

$$\gamma_H = \alpha + (\rho - \alpha) \times \frac{b_1}{b_1 - 1} \tag{12}$$

$$b_1 = \left(\frac{1}{2} - \frac{r - (\rho - \alpha)}{\sigma^2}\right) + \sqrt{\left(\frac{r - (\rho - \alpha)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{13}$$

We can use equations 12 and 13 to replicate Table 2 in McDonald (2000). Specifically, the optimal hurdle rate with cash flow volatility (σ) = 30%, cash flow growth rate (α) = 3%, project discount rate

(ρ) = 12% and risk-free rate (r) = 8%.

$$b_1 = \left(\frac{1}{2} - \frac{r-(\rho-\alpha)}{\sigma^2}\right) + \sqrt{\left(\frac{r-(\rho-\alpha)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$b_1 = \left(\frac{1}{2} - \frac{0.08-(0.12-0.03)}{0.3^2}\right) + \sqrt{\left(\frac{0.08-(0.12-0.03)}{0.30^2} - \frac{1}{2}\right)^2 + \frac{2\times 0.08}{0.30^2}}$$

$$b_1 = 2.0778...$$

$$\gamma_H = \alpha + (\rho - \alpha) \times \frac{b_1}{b_1 - 1}$$

$$\gamma_H = 0.03 + (0.12 - 0.03) \times \frac{2.0778...}{2.0778... - 1}$$

$$\gamma_H = 20.3$$

Subsequently, using the above equations with different variables leads to a replication of the Table 2 in McDonald (2000) shown in figure 2.

Hurdle Rates

		b1			Hurdle Rate		
Risk-free Rate, r	0.08	Project Discount Rates (ρ)			Project Discount Rates (ρ)		
Cash Flow Volatility (σ)	Cash Flow Growth Rate (α)	8%	12%	16%	8%	12%	16%
20%	-3%	3.6085	5.2604	7.0661	12.2%	15.5%	19.1%
20%	0%	2.5616	4.0000	5.7016	13.1%	16.0%	19.4%
20%	3%	1.7656	2.8860	4.4075	14.5%	16.8%	19.8%
30%	-3%	2.4057	3.1245	3.9003	15.8%	19.1%	22.6%
30%	0%	1.9240	2.5784	3.3142	16.7%	19.6%	22.9%
30%	3%	1.5104	2.0778	2.7561	17.8%	20.4%	23.4%
40%	-3%	1.9010	2.3082	2.7400	20.2%	23.5%	26.9%
40%	0%	1.6180	2.0000	2.4142	20.9%	24.0%	27.3%
40%	3%	1.3602	1.7098	2.1010	21.9%	24.7%	27.8%

Figure 2: Replication of Table 2 (McDonald, 2000)

The adoption of a fixed hurdle rate i.e. 20%, may lead to errors in the timing of investments. The hurdle rate is most sensitive to changes σ and ρ.Therefore, we look at the extremes in these instances for worst case scenarios below:

- Low: σ, α and ρ of 20%, -3% and 8%, respectively has an optimal hurdle rate of 12.2%.
- High: σ, α and ρ of 40%, 3% and 16%, respectively has an optimal hurdle rate of 27.8%.

The former with low volatility and discount rate leads to suboptimal investment decisions as the investment option is worth the least, and optimal to invest at low project values. This combination of variables has the largest difference between fixed and optimal hurdle rates (20%-12.2%=7.8%). Subsequently, a 20% hurdle rate leads to excessive delay (underinvestment), waiting to realise value from the delaying investment under uncertainty from excess volatility not present. The opposite is true for the later with high volatility and high discount rate makes the investment option worth the most, and optimal to invest at high project values. This combination of variables equals the largest difference between fixed and optimal hurdle rates (27.8%-20%=7.8%). Subsequently, a 20% hurdle rate leads to premature investment (over investment), exercising value prematurely instead of maximising value by delaying investment under uncertainty from excess volatility. The high state is less value destructive than the low state. However, both inform an arbitrary rate of 20% may be inappropriate in these instances.

2 Question Two

Note: The cumulative normal distribution is calculated using NORM.DIST('Value', 0,1,TRUE) for N(d1) and N(d2). The standard normal density is calculated using NORM.DIST('Value', 0,1,FALSE) for N'(d1) and N'(d2).

2.1 Value of the firm's Equity

The assumptions imply the value of equity for an organisation is synonymous with the value of a call option, with face-value of debt functioning as the strike price. The value of the company's assets follow a geometric brownian motion. The non-dividend paying nature, in combination with the one time issuance of a single zero-coupon bond, informs the life of an equity as a call option corresponds to the life of the bonds.

$$\begin{aligned}
\frac{\partial S_t}{S_t} &= \mu dt + \sigma dW \\
V_e &= SN(d_1) - Ke^{-rT}N(d_2) \\
d_1 &= \frac{(ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T)}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

$S = 100$, $K = 300$, $T = 10$, $\sigma = 30\%$, $\beta_A = 1$, $r_f = 5\%$ p.a, c.c, $(r - r_f) = 10\% p.a.$
Therefore, we can calculate the value of equity:

$$\begin{aligned}
d_1 &= \frac{(ln(\frac{100}{300}) + (0.05 + \frac{0.30^2}{2})10)}{0.30\sqrt{10}} \\
&= -0.156651107 \\
d_2 &= -0.156651107 - 0.30\sqrt{10} \\
&= -1.105334405 \\
V_E &= 100 \times N(-0.156651107) - 300e^{-0.05\times 10}N(-1.105334405) \\
&= 19.30114021
\end{aligned}$$

2.2 Present Value of Debt

The present value of debt is the present value of the one, zero coupon bond. We must use equation 14 as the risk-free rate is continuously compounding.

$$\begin{aligned}
PV_{cc} &= Ke^{-r_fT} \\
&= 300e^{-0.05\times 10} \\
&= 181.9591979
\end{aligned}$$

The current value of the debt would be the current value of assets minus the current value of equity (previously calculated as a call option)

$$\begin{aligned}
V_D &= S - V_E \\
&= 100 - 19.30114021 \\
&= 80.69885979
\end{aligned}$$

Therefore, the difference between the present and current value of debt is a Put option on the equity of the firm defined by Put-Call Parity.

$$\begin{aligned}
P_E + S &= V_E + Ke^{-r_f T} \\
P_E &= V_E + Ke^{-r_f T} - S \\
&= 19.30114021 + 181.9591979 - 10 \\
&= 101.2603381
\end{aligned}$$

2.3 F_V, F_{σ^2}, F_r

We use the following equations from [15] and above 'Greeks' to inform changes in the value of debt.

$$\begin{aligned}
F_V &= 1 - f_V \geq 0 \\
F_{\sigma^2} &= -f_{\sigma^2} \\
F_r &= -f_r < 0 \\
f_V &= \Delta_f = \frac{\partial f}{\partial S} = N(d_1) \\
f_{\sigma^2} &= V_f = \frac{\partial f}{\partial \sigma} = S\sqrt{T}N'(d_1) \\
f_r &= \rho_f = \frac{\partial f}{\partial r} = KTe^{-rT}N(d_2) \\
f &= SN(d_1) - Ke^{-rT}N(d_2)
\end{aligned}$$

For F_V:

$$\begin{aligned}
f_V &= \Delta_f = \frac{\partial f}{\partial S} = N(d_1) \\
&= N(-0.156651107) \\
&= 0.437759911 \\
F_V &= 1 - 0.437759911 \geq 0 \\
F_V &= 0.562240089 \geq 0
\end{aligned}$$

The positive sign makes sense intuitively as the value of the firm increases, there is an increase in the collateral available for debt holders. Therefore, the value of their options increase as there is greater insurance against debt. For F_{σ^2} ($\sigma = \sigma^2$ in this instance)

$$\begin{aligned}
f_{\sigma^2} &= V_f = \frac{\partial f}{\partial \sigma} = S\sqrt{T}N'(d_1) \\
&= 100\sqrt{10} \times 0.394077252 \\
&= 124.618169 \\
F_{\sigma^2} &= -f_{\sigma^2} < 0 \\
&= -124.618169 < 0
\end{aligned}$$

A negative co-efficient is intuitive as an increase in volatility raises the uncertainty of the underlying asset (value of the firm) at the expiration date. Equity holders receive the residual claim. Bondholders entitlements are up to a specific amount e.g B dollars. The increase in volatility increases the probability the value of the firm will be less than the entitlement owed to bondholders at expiration, decreasing the demand in bonds from investors. Therefore, an increase in volatility leads to a decrease in the value of corporate debt. For F_r:

$$\begin{aligned} f_r = \rho_f = \frac{\partial f}{\partial r} &= KTe^{-rT}N(d_2) \\ &= 300 \times 100e^{-0.05\times 10} \times N(-1.105334405) \\ &= 244.748509 \\ F_r &= -f_r < 0 \\ F_r &= -244.748509 < 0 \end{aligned}$$

A negative co-efficient is intuitive as a fall in interest rates lowers the cost of borrowing. Most bonds pay a fixed (usually lower) interest rate. Therefore a fall in interest rates will increase the competitiveness of bonds, reducing the opportunity cost for investors, increasing bonds demand and subsequent price of bonds and corporate liabilities.

2.4 Firm Equity Beta

Galai and Masulis (1976) combine the Capital Asset Pricing Model (CAPM) and Option Pricing Model (OPM) to assess the risk of equity.

$$\beta_S = N(d_1)\frac{V}{S}\beta_V \equiv \eta_S\beta_V$$

However, the notation in Galai and Masulis, 1976 is different from the assignment. Therefore, adjustments to ensure consistency.

$$\beta_{V_E} = N(d_1)\frac{S}{V_E}\beta_S \equiv \eta_{V_E}\beta_S$$

The vectorization of a range of face values for debt leads to changes in the value of equity as a call option. The static firm β implies no change in systematic risk regardless of leverage. We use the Black Scholes Formula to calculate V_E. K = 0 is invalid as lim $ln(x) = \infty$ when calculate d_1. Therefore, we can only use of K from 10 to 300. Furthermore:

1. The Black Scholes Model in subsection 2.1 informs calculating values d_1, d_2, and V_E.
2. $PV = Ke^{-rfT}$ informs the present value of risk-less debt.
3. $V_D = S - V_E$ informs the market value of debt.
4. Leverage $= \frac{V_D}{S}$ informs leverage.

Figure 5 shows the implementation of these calculations. Subsequently, figure 3 shows the firms equity beta versus leverage.

2.5 Premium vs Leverage

The premium on a zero coupon bond is the yield to maturity minus the risk-free, calculated as follows:

$$V_D = Ke^{-YTM\times T} \tag{14}$$

$$YTM = -\frac{ln(\frac{V_D}{K})}{T} \tag{15}$$

$$Premium = YTM - r_f \tag{16}$$

(17)

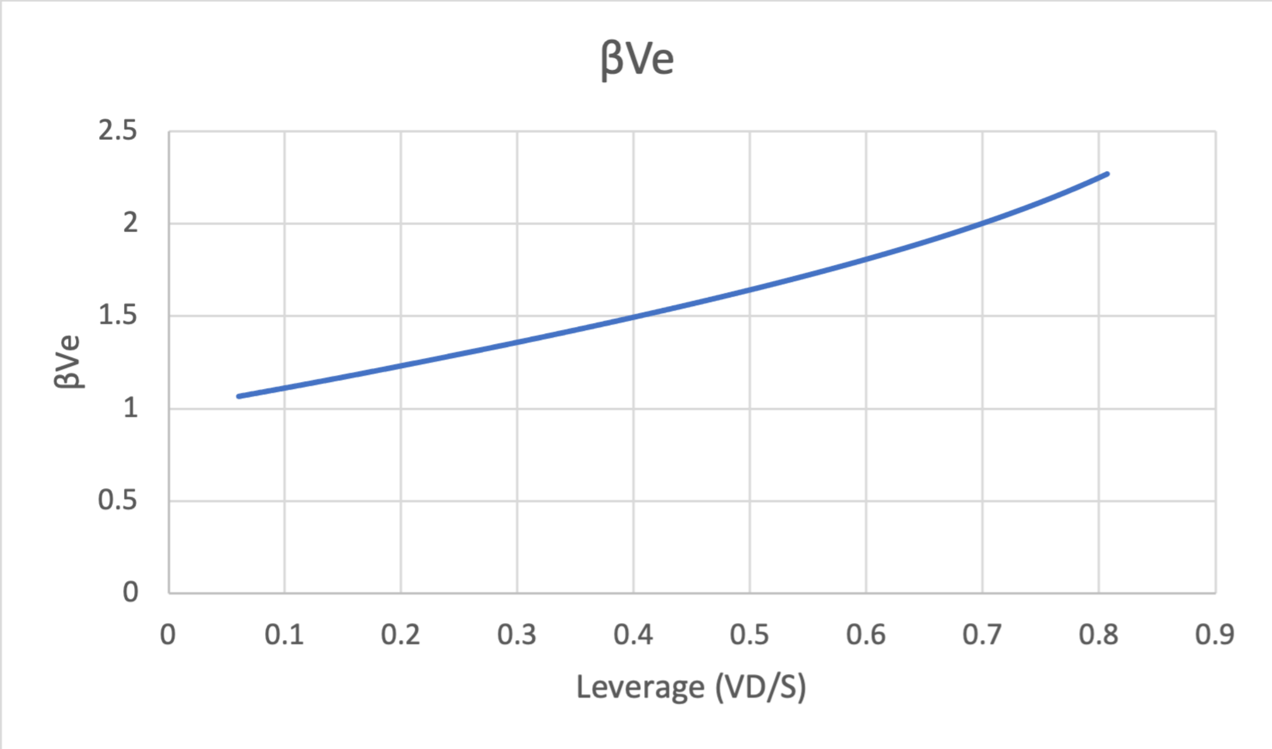

Figure 3: Free Equity Beta β_{V_E}

Subsequently, figure 4 the risk premium versus the 'quasi' market debt firm value ratio. The results replicate the figure in Merton (1974) from zero to points prior to plateau at subsequently exercise prices in excess to 300, assumably.

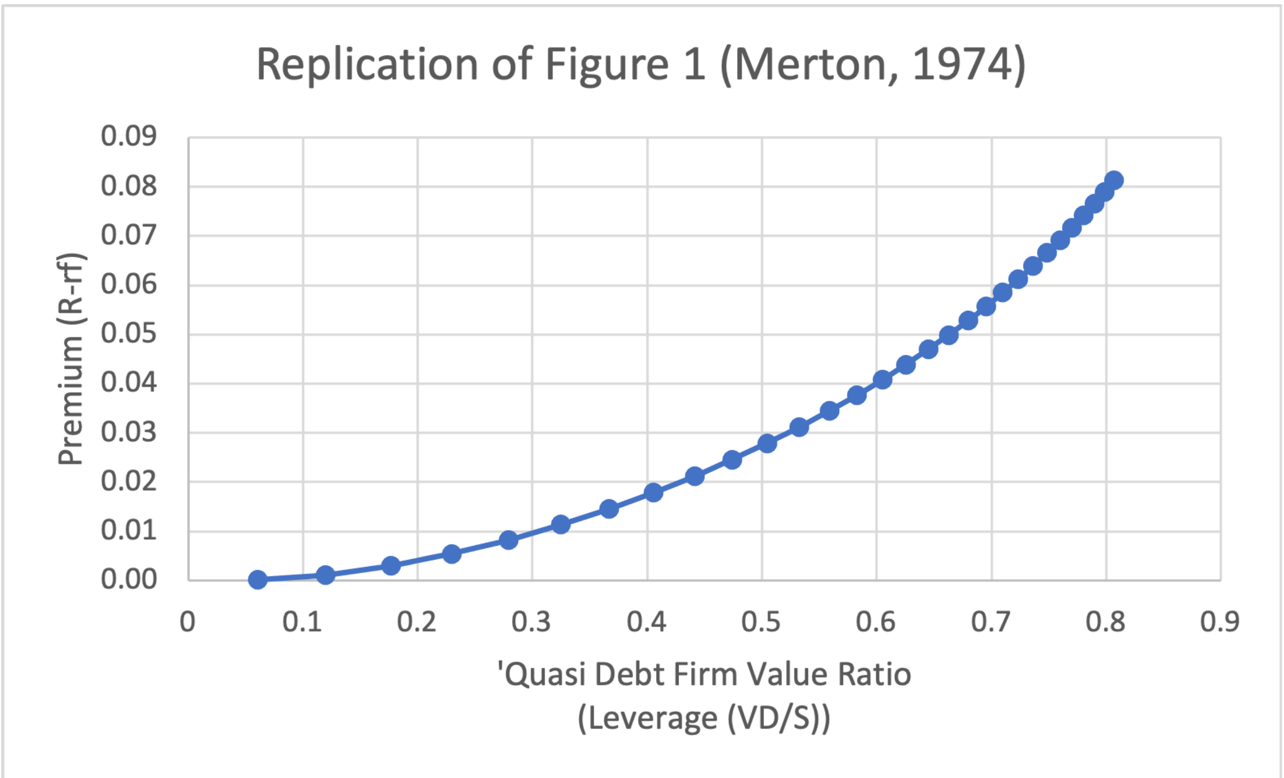

Figure 4: Replication of figure one Merton, 1974

Firm Equity Beta

βS	1

K (FV of Debt)	d1	d2	N(d1)	N(d2)	VE	PV (Debt)	VD	Leverage	βVe	Premium
10	3.428525726	2.479842428	0.999696565	0.993427977	93.944211	6.0653066	6.055788715	0.060557887	1.0641385	0.00016
20	2.697884444	1.749201146	0.996510917	0.959871872	88.007257	12.130613	11.99274269	0.119927427	1.1323054	0.00114
30	2.270486693	1.321803395	0.988410965	0.906883186	82.339523	18.19592	17.66047721	0.176604772	1.2004089	0.00299
40	1.967243163	1.018559865	0.975422408	0.845794017	77.022241	24.261226	22.97775932	0.229777593	1.2664166	0.00544
50	1.732029207	0.783345909	0.958365819	0.783287995	72.082173	30.326533	27.91782732	0.279178273	1.3295462	0.00828
60	1.539845412	0.591162114	0.93820098	0.722794098	67.516291	36.39184	32.48370883	0.324837088	1.389592	0.01136
70	1.377356328	0.42867303	0.915798946	0.665919405	63.306857	42.457146	36.69314291	0.366931429	1.4466031	0.01459
80	1.236601881	0.287918583	0.891882539	0.613295469	59.429654	48.522453	40.5703465	0.405703465	1.5007366	0.01790
90	1.112447661	0.163764363	0.867027139	0.565041676	55.858355	54.587759	44.14164517	0.441416452	1.5521888	0.02124
100	1.001387926	0.052704628	0.84168035	0.521016374	52.566795	60.653066	47.43320547	0.474332055	1.6011635	0.02458
110	0.900922175	-0.04776112	0.81618515	0.48095331	49.530093	66.718373	50.46990714	0.504699071	1.6478571	0.02791
120	0.80920413	-0.13947917	0.79080113	0.444535758	46.725165	72.783679	53.27483499	0.53274835	1.6924523	0.03120
130	0.724831708	-0.22385159	0.76572236	0.411436393	44.130894	78.848986	55.86910631	0.558691063	1.7351164	0.03445
140	0.646715047	-0.30196825	0.741091808	0.381338133	41.728123	84.914292	58.27187685	0.582718768	1.7760008	0.03765
150	0.573990175	-0.37469312	0.717012775	0.353944354	39.499562	90.979599	60.50043784	0.605004378	1.8152423	0.04080
160	0.5059606	-0.4427227	0.693557845	0.328983161	37.429645	97.044906	62.57035527	0.625703553	1.852964	0.04389
170	0.442056638	-0.50662666	0.670775889	0.306208399	35.504376	103.11021	64.49562408	0.644956241	1.8892767	0.04692
180	0.381806379	-0.56687692	0.648697507	0.285398903	33.711177	109.17552	66.28882266	0.662888227	1.9242802	0.04989
190	0.324814524	-0.62386877	0.627339274	0.266356884	32.03874	115.24083	67.96125977	0.679612598	1.9580647	0.05281
200	0.270746644	-0.67793665	0.60670705	0.248905929	30.47689	121.30613	69.52311046	0.695231105	1.9907118	0.05567
210	0.219317296	-0.729366	0.586798555	0.232888904	29.016461	127.37144	70.98353924	0.709835392	2.0222954	0.05847
220	0.170280893	-0.7784024	0.567605382	0.218165911	27.649189	133.43675	72.35081087	0.723508109	2.0528826	0.06121
230	0.123424622	-0.82525868	0.54911457	0.204612374	26.367611	139.50205	73.63238897	0.73632389	2.0825344	0.06390
240	0.078562849	-0.87012045	0.531309831	0.192117292	25.164976	145.56736	74.83502357	0.748350236	2.1113067	0.06654
250	0.035532689	-0.91315061	0.514172509	0.180581668	24.035171	151.63266	75.96482862	0.759648286	2.1392504	0.06912
260	-0.00580957	-0.95449287	0.497682329	0.169917108	22.97265	157.69797	77.0273504	0.770273504	2.1664124	0.07165
270	-0.04559137	-0.99427467	0.481817973	0.160044582	21.972372	163.76328	78.02762808	0.780276281	2.1928355	0.07414
280	-0.08392623	-1.03260953	0.46655754	0.150893333	21.029753	169.82858	78.97024708	0.789702471	2.2185593	0.07657
290	-0.12091573	-1.06959903	0.45187889	0.142399916	20.140614	175.89389	79.85938628	0.798593863	2.2436203	0.07896
300	-0.15665111	-1.1053344	0.437759911	0.134507358	19.30114	181.9592	80.69885979	0.806988598	2.2680521	0.08131

Figure 5: Excel calculations for 2d/2e

3 Question Three

3.1 Implied Volatiles using Numerical Optimisation

The Black-Scholes Option Pricing model described in subsection 2.1 informs the calculation of d_1, d_2 and the value of the call option (V). Excel's goal seek function adjust the volatility to change the value of the call option to match the desired value at each exercise prices, deriving the implied volatilities. Figure 6 displays the calculations and figure 7 displays the plot.

Implied Volatility using Numerical Optimisation

S	100.00				
T	1.00				
rf	5%				
σ	30%				
K (Exercise)	50.00	75.00	100.00	125.00	150.00
V	54.483	31.298	14.231	6.592	2.558
σ (Implied)	0.597155608	0.351534196	0.3	0.324440946	0.3204788
d1	1.543056079	1.136362329	0.316666667	-0.37144691	-0.94893
d2	0.945900472	0.784828133	0.016666667	-0.69588786	-1.269408
N(d1)	0.938591415	0.872097523	0.624251728	0.355152345	0.1713282
N(d2)	0.827900324	0.783722828	0.50664873	0.243249531	0.1021477
V	54.48298408	31.29723613	14.23125479	6.591970578	2.5579324

K (Exercise)	σ (Implied)
50	59.72%
75	35.15%
100	30.00%
125	32.44%
150	32.05%

Figure 6: Implied Volatilities using Numerical Optimisation - Calculations

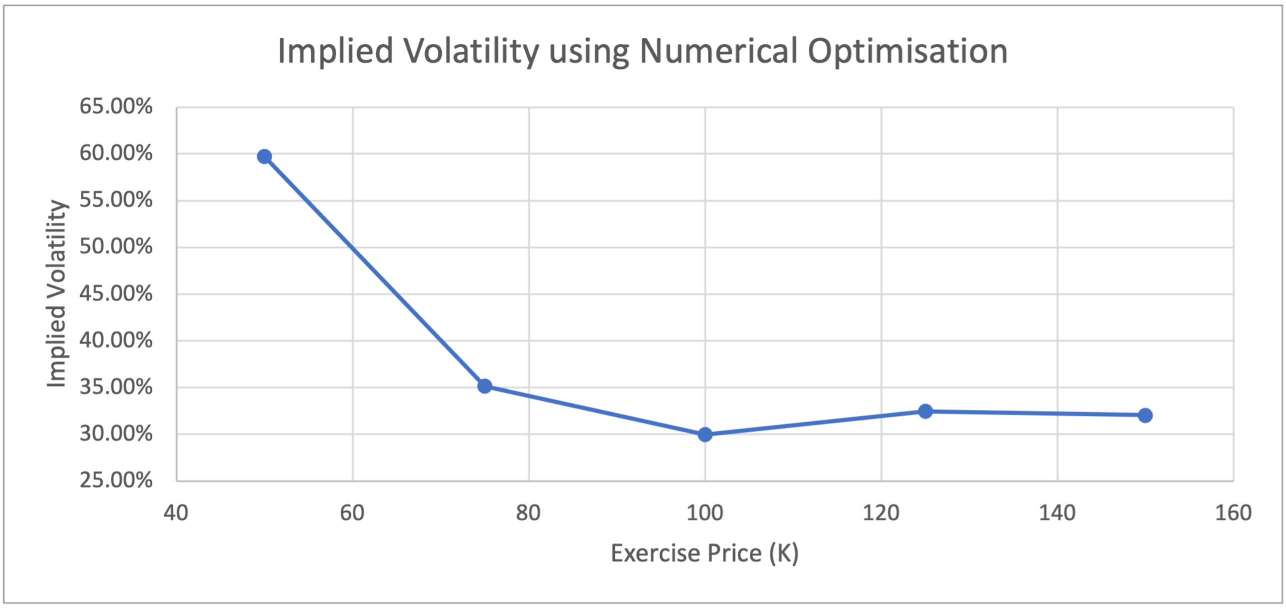

Figure 7: Implied Volatilities using Numerical Optimisation - Plot

3.2 Hypothetical Black Scholes Options Pricing

The Black-Scholes Option Pricing model described in subsection 2.1 informs the calculation of d_1, d_2 and the value of the call option (V) for all exercise prices. $\sigma = 30\%$ is the implied volatility from K = 100. Subsequently, the combination of the above points finds the following results in figure 8.

Implied Volatility (σ = 0.3)

σ (Implied)	0.3	0.3	0.3	0.3	0.3
d1	2.62715727	1.27560691	0.31666667	-0.42714517	-1.034884
d2	2.32715727	0.97560691	0.01666667	-0.72714517	-1.334884
N(d1)	0.99569492	0.89895274	0.62425173	0.3346368	0.1503616
N(d2)	0.99002155	0.83537035	0.50664873	0.23356851	0.0909572
V (Call Option)	52.4826108	30.2981102	14.2312548	5.69152419	2.0579857

Figure 8: Hypothetical Black Scholes Option Pricing

Implied volatility refers to the market's estimation of the constant volatility parameter. Notably, implied volatility refers to the average volatility over the remaining life of the option, if the underlying asset's volatility is allowed to vary deterministically over time. Both above rely on the strict assumptions of Black Scholes Options Pricing model. Options act as insurance against adverse price fluctuations on the underlying asset. Implied volatility is the measure of uncertainty for underlying asset price movements. A higher option price implies the options are more valuable, implying greater need to protect against increasingly underlying asset price movements. Option pricing models are reversed engineered to find implied volatilities from option prices. A bi-directional relationship exists. Additional $V = \frac{\partial S}{\partial \sigma}$ is always positive. Therefore, a higher options price would result in higher implied volatilities.

3.3 Risk-neutral Price Dynamics with Stochasticity

Risk-neutral price dynamics inform the probability each share price is exactly equal to the discounted expectation of the share price. Risk-neutral probabilities are probabilities of potential future outcomes adjusted for risk used in computations of asset prices. The Black Scholes model already assumes Geometric Brownian Motion with constant drift and volatility. However, the discounted component requires further thinking. Risk-neutral price dynamics relies on the discounted expectation of the share prices. Jump reflect the introduction of non-marginal information incorporated into the stock price. Hypothetically, a symmetric jump component (same magnitude, equally likely in either direction) should have an expectation of zero. Therefore, geometric brownian motion will make the only stochastic contribution, to implied volatilities, on average. The return dynamics will be equivalent to those in Black Scholes. Black Scholes leads to deterministic variation in the price of the underlying asset from constant drift and volatility parameters. Implied volatility is the average volatility over the remaining life of the option. Therefore, the observed pattern would be the flattening of the implied volatility curve from the geometric brownian motion contribution.

3.4 Positive Correlation to Price

Again, jumps reflect the introduction of non-marginal information incorporated into the stock price. A positive correlation creates contributions to the price as stated by Merton (1976). They will make contributions to the discounted expectation of the share price. Option prices will further depart from those predicted in the Black Scholes Model. Therefore, implied volatilities will increase in out-of-the-money/in-the-money options, leading a smiling of the the volatility curve.

3.5 Additional Explanations for Observations in Numerical Observations

An assessment of the underlying assumptions of Black-Scholes inform discrepancies in real options prices, and the hypothetical prices from the Black Scholes models, expose alternative explanations. In particular,

- The demand for in-the-money (ITM) and out-of-the-money (OTM) options is greater than at-the-money (ATM). ITM have a greater chance of being in-the-money at expiration and lessens the impact of time decay, as carry both intrinsic and time value. OTM has a lower cost than ITM, have no intrinsic value and can give a trader serious leverage. The greater demand leads to higher prices, increasing implied volatility, explaining the relationship observed above.
- One assumptions is 'The stock price follows a "geometric" Brownian motion through time which produces a log-normal distribution for stock price between any two points in time.' If the Stock distribution has fat tails, real option prices will be higher for deep ITM and OTM options will be higher than Black Scholes, leading to higher volatilities.

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