Domain-Specific Languages of Mathematics
Course codes: DAT326 / DIT982

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2022-03-15

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Results	Announced within 15 workdays
Exam check	2022-03-25, 12.15-13.00 in EDIT 6452
Aids	One textbook of your choice (Domain-Specific Languages of Mathematics, or Beta - Mathematics Handbook, or Rudin, or Adams and Essex, or ...). No printouts, no lecture notes, no notebooks, etc.
Grades	To pass you need **a minimum of 5p on each question (1 to 4)** and also reach these grade limits: 3: >=48p, 4: >=65p, 5: >=83p, max: 100p

Remember to write legibly. Good luck!

For reference: the learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain-Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [25p] **Algebraic structure**: Group (lightly edited from the Wikipedia entry)

   A group is a set, G, together with an operation $\cdot$ (called the group law of G) that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms:

   - Closure: For all a, b in G, the result of the operation, $a \cdot b$, is also in G.
   - Associativity: For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - Identity element: There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds.
   - Inverse element: For each a in G, there exists an element b in G, commonly denoted a^{-1}, such that $a \cdot b = b \cdot a = e$, where e is the identity element.

   (a) Define a type class *Group* that corresponds to the group structure.

   (b) Define a datatype $G\ v$ for the language of group expressions (with variables of type v) and define a *Group* instance for it. (These are expressions formed from applying the group operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find and implement two other instances of the *Group* class.

   (d) Give a type signature for, and define, a general evaluator for $G\ v$ expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *Group* instances defined in (1c). Take three group expressions of type G *String*, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 5pts.

2. [25p] **Laplace**

   Consider the following differential equation:

   $$f'' + 2f' + 10f = 0, \quad f(0) = 1, \quad f'(0) = -1$$

   (a) [10p] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs*$'$, *fs*$''$. What are the first three coefficients of *fs*? Explain how you compute them.

   (b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be a complex number) and the rules for linearity + derivative:

   $$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\; s = 1/(s - \alpha)$$

   Show that your solution does indeed satisfy the three requirements.

3. [20p] **Equational reasoning and endomorphisms**

   Note: The learning outcome tested here is mainly "perform calculational proofs", thus you are expected to use equational reasoning and motivate your steps.

   An endohomomorphism on an additive group $(G, (+), 0)$ is a function from G to G that distributes over addition. In other words: $\mathit{Endo}(f) = H_2(f, (+), (+)) \wedge H_0(f, 0, 0)$ where $H_2(h, op_1, op_2) = \forall\ x.\ \ \forall\ y.\ \ h\ (op_1\ x\ y) == op_2\ (h\ x)\ (h\ y)$ and $H_0(h, e_1, e_2) = h\ e_1 == e_2$.

   If the group is abelian, endohomomorphisms form a ring with function composition for multiplication and pointwise addition:

   $$\begin{array}{ll} \mathit{mulE}\ f_1\ f_2 = f_1 \circ f_2 & \text{-- Def. } \mathit{mulE} \\ \mathit{addE}\ f_1\ f_2 = \mathit{lift2}\ (+)\ f_1\ f_2 & \text{-- Def. } \mathit{addE} \end{array}$$

   $$\begin{array}{ll} f_1 \circ f_2 = \lambda x \to f_1\ (f_2\ x) & \text{-- Def. } (\circ) \\ \mathit{lift2}\ op\ f_1\ f_2 = \lambda x \to op\ (f_1\ x)\ (f_2\ x) & \text{-- Def. } \mathit{lift2} \end{array}$$

   (a) [5p] Define the additive unit zeroE of this ring and prove that it is the additive unit.

   (b) [5p] Define the multiplicative unit oneE of this ring and prove that it is the multiplicative unit.

   (c) [10p] Prove $\mathit{Endo}(f) \Rightarrow \mathit{mulE}\ f\ (\mathit{addE}\ g\ h) == \mathit{addE}\ (\mathit{mulE}\ f\ g)\ (\mathit{mulE}\ f\ h)$ for all f, g, h of appropriate type. (This proves the distributivity law for the ring instance.)

4. [30p] **Typing maths: limits**

   Consider the following two quotes from Adams and Essex: the first from page 90, Def. 10:

   > **Limit at infinity**
   > We say that $f(x)$ **approaches the limit** L **as** x **approaches infinity**, and we write
   >
   > $$\lim_{x \to \infty} f(x) = L,$$
   >
   > if the following condition is satisfied:
   > for every number $\epsilon > 0$ there exists a number R, possibly depending on ϵ, such that if $x > R$, then x belongs to the domain of f and $|f(x) - L| < \epsilon$.

   which we can call $\mathit{LimAtInf}(f, L)$ and the second from page 498, Def. 2:

   > We say that sequence a_n converges to the limit L, and we write $\lim_{n \to \infty} a_n = L$, if for every positive real number ϵ there exists an integer N (which may depend on ϵ) such that if $n > N$, then $|a_n - L| < \epsilon$.

   which we can call $\mathit{LimSeq}(a, L)$.

   (a) [7p] Define the first-order logic predicates $\mathit{LimAtInf}(f, L)$ and $\mathit{LimSeq}(a, L)$ encoding the quotes. Explain the differences.

   (b) [8p] Give the types of f, x, L, ϵ, R, $\mathit{LimAtInf}$ and give the types of a, L, n, N, ϵ, LimSeq. Explain your reasoning.

   (c) [5p] In both cases the idea of "divergence" is introduced in the same manner: let $\mathit{DivF}(f) = \neg\ (\exists\ L.\ \ \mathit{LimAtInf}(f, L))$ and $\mathit{DivS}(a) = \neg\ (\exists\ L.\ \ \mathit{LimSeq}(a, L))$. Simplify the predicate $\mathit{DivF}(f)$ by pushing the negation through all the way.

   (d) [10p] Prove $\forall\ a.\ \ \mathit{DivS}(a) \Rightarrow \mathit{DivF}((/2) \circ a \circ \mathit{round})$.