Propositional Calculus

Patrik Jansson

$p_1 = a \wedge \neg a = F$

$p_2 = a \vee \neg a = T$

$p_3 = a \Rightarrow b$

$p_4 = (a \wedge b) \Rightarrow (b \wedge a) = T$

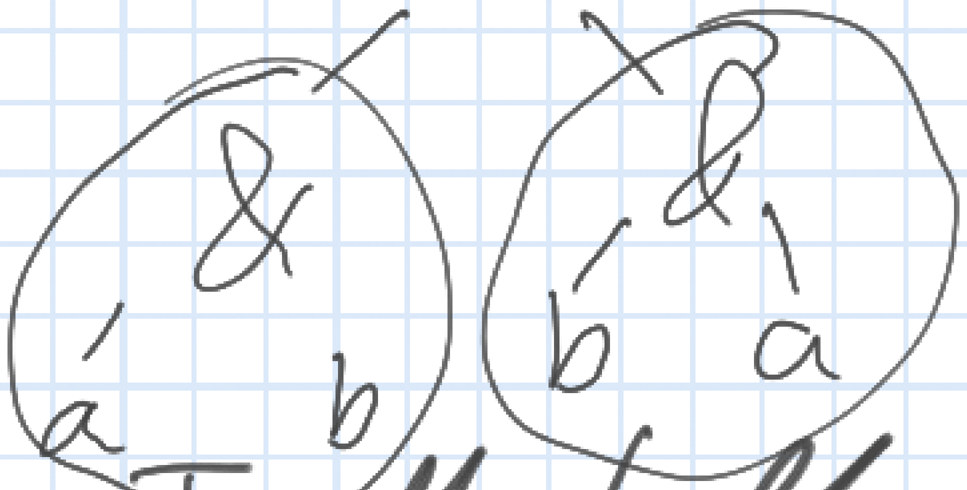
(a	∧	b)	⇒	(b	∧	a)
F	F	F	T	F	F	F
F	F	T	T	T	F	F
T	F	F	T	F	F	T
T	T	T	T	T	T	T
1	3	2		2	4	1

Syntax

False	$\bot$	F	nullary
True	$\top$	T	nullary
Not	$\neg$	~	unary
And	$\wedge$	&	binary
Or	$\vee$	|	binary
Implies	$\supset$	$\Rightarrow$	binary

a	⇒	b
T	T	T
T	F	F
F	T	T
F	T	F
1		2

$\Rightarrow$



Truth table

Tautology
p_4, p_2

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$p_1 = a \wedge \neg a$

$p_2 = a \vee \neg a$

$p_3 = a \Rightarrow b$

$p_4 = (a \wedge b) \Rightarrow (b \wedge a)$

```
p1, p2, p3, p4 :: Prop
p1 = And (Name "a") (Not (Name "a"))
p2 = Or  (Name "a") (Not (Name "a"))
p3 = Implies (Name "a") (Name "b")
p4 = Implies (And a b)  (And b a)
  where a = Name "a"; b = Name "b"
```

```
data Prop = Con     Bool
          | Not     Prop
          | And     Prop Prop
          | Or      Prop Prop
          | Implies Prop Prop
          | Name    Name
type Name = String
```

"assignment function"

syntax → semantics

type Tab = Name → 𝔹

type Tab' = [(Name, 𝔹)]

eval : Prop → $\boxed{\text{Tab} \to \text{Bool}}$

eval (Name n) t = t n

eval (Not p) t = not (eval p t)

⋮

: Prop → Tab' → Maybe Bool

Proofs

$a : A$ "a is.a.proof of A"

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B}\ \text{AndIntro}$$

and, &

```
data Either p q where
  Left  :: p → Either p q
  Right :: q → Either p q
```

$$\frac{a : A}{\text{Left } a : A \vee B}\ \text{OrIntroL} \qquad \frac{b : B}{\text{Right } b : A \vee B}\ \text{OrIntroR}$$

or, |

Proofs

$a : A$ "a is a proof of A"

$$\frac{a : A \qquad b : B}{(a,b) : A \wedge B} \text{AndIntro}$$

$$\frac{p : A \wedge B}{\mathrm{fst}\ p : A} \qquad \frac{p : A \wedge B}{\mathrm{snd}\ p : B}$$

$f : A \Rightarrow B$

$f : A \to B$

data *Either p q* **where**
  Left :: $p \to$ *Either p q*
  Right :: $q \to$ *Either p q*

$$\frac{a : A}{\mathrm{Left}\ a : A \vee B} \text{OrIntroL} \qquad \frac{b : B}{\mathrm{Right}\ b : A \vee B} \text{OrIntroR}$$

$$\frac{e : A \vee B \qquad f : A \Rightarrow C \qquad g : B \Rightarrow C}{\mathrm{orElim}\ e\ f\ g : C}$$

$\mathrm{orElim}\ (\mathrm{Left}\ a)\ f\ g = f\ a$

$\mathrm{orElim}\ (\mathrm{Right}\ b)\ f\ g = g\ b$

Pure set theory

Abs. Syn. for sets

Empty : M

Sing : $M \to M$

Union : $M \to M \to M$

Intersect : $M \to M \to M$

$\emptyset = \{\ \} =$ empty set

$\{x\} =$ 1-element set of x

$A \cup B$

$A \cap B$

card : $M \to N$

Elem : $M \to M \to$ Prop

Elem x (Sing x) : Prop

$|\emptyset| = 0$

$x \in \{x\}$

$|\{x\}| = 1$

$|A \cup B| \geq |A|$

Sanity-checks

Pure set theory

Empty : M

Sing : $M \to M$

Union : $M \to M \to M$

Intersect : $M \to M \to M$

Elem : $M \to M \to \text{Prop}$

card : $M \to \mathbb{N}$

(for all x)

Union

$\boxed{x \in \{x\}}$

or

$\boxed{(x \in A \cup B) \Leftrightarrow (x \in A) \vee (x \in B)}$

$\boxed{(x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)}$

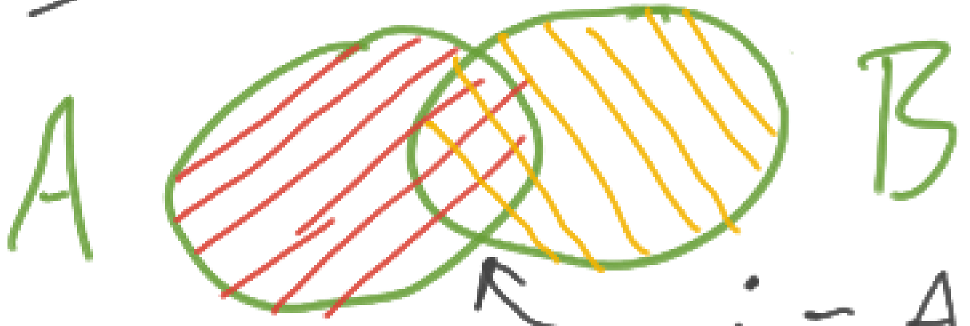
and

$A = \{1,2\}$

$a = A \setminus B = \{1\}$

$B = \{2,3,4\}$

$b = B \setminus A = \{3,4\}$



$i = A \cap B$

$A = a \cup i$

$B = b \cup i$

$$\underset{2}{|A|} + \underset{3}{|B|} = \underset{4}{|A \cup B|} + \underset{1}{|A \cap B|}$$

Pure set theory

"union - table"

$x \cup x = x$ idempotent

commutative

$m_0 = \text{Empty} = \{\}$

$x \cup m_0 = x$ $(x + 0 = x)$

$m_{i+1} = \{m_i\}$

$m_1 = \{m_0\} = \{\{\}\}$

$m_2 = \{m_1\}$

$m_1 \cup m_2 = \{m_0\} \cup \{m_1\} = \{m_0, m_1\} = t_{01}$

$t_{12}, t_{13}, t_{012}, \ldots \quad t_{ij} = \{m_i, m_j\}$

$\cup$	m_0	m_1	m_2	t_{01}	m_3
m_0	m_0				
m_1	m_1	m_1			
m_2	m_2	t_{01}	m_2		
t_{01}	t_{01}			t_{01}	
m_3	m_3				m_3
X	X				

$x \cup y = y \cup x$

Pure set theory

$m_0 = \emptyset$ $\quad |m_0| = 0$

$m_1 = \{m_0\}$ $\quad |m_1| = 1$

$m_2 = \{m_1\}$ $\quad |m_2| = 1$

$m_3 = \{m_2\}$ $\quad |m_3| = 1$

$$t_{01} = \{m_0, m_1\} = \{m_0\} \cup \{m_1\} = m_1 \cup m_2$$

$\cup$	m_0	m_1	m_2
m_0	m_0		
m_1		m_1	
m_2			m_2