D as a linear transform

Patrik Jansson

$V = \mathbb{R} \to \mathbb{R}$ LinTran(D)?

$D : V \to V$

const 0

$H_0(D, 0_F, 0_F) \equiv D\, 0_F = 0_F$ → Yes!

$\& \; H_2(D, +_F, +_F) \equiv \forall f, g : V.\; D(f +_F g) = Df +_F Dg$

$\& \; H_1(D, neg, neg) \equiv \forall f : V.\; D(neg\, f) = neg\,(Df)$

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$V = \mathbb{R} \to \mathbb{R}$ LinTran(D) !

$D : V \to V$

$H_0(D, zero_F, zero_F) \equiv D(const\ 0) = const\ 0$

$H_2(D, (+_F), (+_F)) \equiv \forall f, g : V.\ D(f +_F g) = Df +_F Dg$

$H_1(D, neg, neg) \equiv \forall f : V.\ D(neg\ f) = neg\ (D f)$

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$$D : V \to V$$

Examples: $\exp : \mathbb{R} \to \mathbb{R}$

$$D \exp = \exp$$

$$scale_F : \mathbb{R} \to V \to V$$

$$scale_F\ c\ f = \backslash t \to c \cdot f\ t$$

$s : \mathbb{R}$ $\qquad g_s : \mathbb{R} \to \mathbb{R}$

$$g_s\ t = \underbrace{\exp(-s \cdot t)}$$

$$D\ g_s\ t = \underbrace{\exp(-s \cdot t)} \cdot (-s)$$

$$= -s \cdot g_s\ t$$

$$D\ g_s = \backslash t \to -s \cdot g_s\ t$$

$$= scale_F\ (-s)\ g_s$$

D as a linear transform

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$D : V \to V$

Examples: $v_1 = \exp : \mathbb{R} \to \mathbb{R}$ $\qquad D \exp = \exp$

let $g_s(t) = \exp(-s \cdot t)$

then $g_s : V$ for every $s : \mathbb{R}$

$D\, g_s = \backslash t \to -s \cdot \exp(-s \cdot t) = \text{scale}\ (-s)\ g_s$

D as a linear transform

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$D : V \to V$

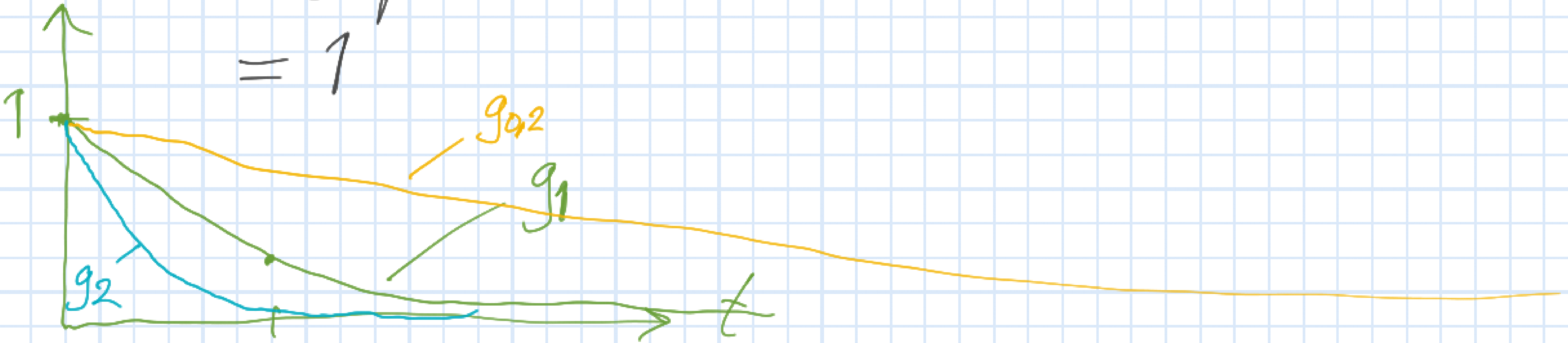
Examples: $v_1 = exp : \mathbb{R} \to \mathbb{R}$

$g_s(t) = exp(-s \cdot t)$

$g_s(0) = exp(-s \cdot 0)$

$= exp\ 0$

$= 1$



$D\ exp = exp$

$D\ g_s = scale\ (-s)\ g_s$

$D(f \cdot g) = Df \cdot g + f \cdot Dg$

$\int$ as a linear transform

Patrik Jansson

$D : V \to V$

$D(f \cdot g) = Df \cdot g + f \cdot Dg$

$I : V \to V$

$I\, f\, x = \int_0^x f = \int_0^x f(t)\, dt$

$I\,(Df)\, x = \int_0^x f'(t)\, dt = \left[f(t) \right]_0^x = f\, x - f\, 0$

DSL → δσλ DSLsofMath

$\int$ as a linear transform

Patrik Jansson

$D : V \to V$ $\qquad D(f \cdot g) = Df \cdot g + f \cdot Dg$

$I : V \to V$

$I\,f\,x = \int_0^x f = \int_0^x f(t)\,dt$

$I\,(Df)\;x = f\,x - f\,0$

$$
\begin{aligned}
I\,(D\,(f \cdot g))\;x &= (f \cdot g)\;x - (f \cdot g)\;0 \\
&= f\,x \cdot g\,x - f\,0 \cdot g\,0 \\
&= f(x) \cdot g(x) - f(0) \cdot g(0)
\end{aligned}
$$

$\int$ as a linear transform

Patrik Jansson

$D : V \to V$ $\qquad D(f \cdot g) = Df \cdot g + f \cdot Dg$

$I : V \to V$

$I\,f\,x = \int_0^x f = \int_0^x f(t)\,dt$

$I(Df)\;x = f\,x - f\,0$

$I(D(f \cdot g))\,x = (f \cdot g)\,x - (f \cdot g)\,0 = I(Df \cdot g)\,x + I(f \cdot Dg)\,x$

Towards Laplace

Patrik Jansson

$$(f\cdot g)\,x - (f\cdot g)\,0 = I(Df\cdot g)\,x + I(f\cdot Dg)\,x$$

let $g = g_s = \backslash t \to e^{-st}$; $D g_s = scale\,(-s)\,g_s$

Assume $(f\cdot g_s)\,x \to 0$ as $x \to \infty$

$$0 - f\,0\cdot g_s\,0 = I\,(Df\cdot g_s)\,\infty + I\,(\overbrace{f\cdot scale\,(-s)\,g_s}^{scale\,(-s)\,(f\cdot g_s)})\,\infty$$

$$\Leftrightarrow\quad -f\,0 = I\,(Df\cdot g_s)\,\infty - s\cdot I\,(f\cdot g_s)\,\infty$$

Towards Laplace

Patrik Jansson

$$(f \cdot g)\, x - (f \cdot g)\, 0 = I\,(Df \cdot g)\, x + I\,(f \cdot Dg)\, x$$

let $g = g_s = \backslash t \to e^{-st}$; $D g_s = scale\,(-s)\, g_s$

Assume $(f \cdot g_s)\, x \to 0$ as $x \to \infty$

$$0 - f\,0 \cdot g_s\,0 = I\,(Df \cdot g_s)\,\infty + I\,(f \cdot scale\,(-s)\, g_s)\,\infty$$

$$-f\,0 = I\,(Df \cdot g_s)\,\infty - s \cdot I\,(f \cdot g_s)\,\infty$$

$$-f\,0 = L\,(Df)\, s - s \cdot L\, f\, s$$

$$L\,(Df)\, s = -f\,0 + s \cdot L\, f\, s$$

$$L\, f\, s = I\,(f \cdot g_s)\,\infty = \int_0^\infty f \cdot g_s$$

Towards Laplace

Patrik Jansson

$$(f \cdot g)\,x - (f \cdot g)\,0 = I\,(Df \cdot g)\,x + I\,(f \cdot Dg)\,x$$

let $g = g_s = \backslash t \to e^{-st}$; $D\,g_s = \text{scale}\,(-s)\,g_s$

Assume $(f \cdot g_s)\,x \to 0$ as $x \to \infty$

$$0 - f\,0 \cdot g_s\,0 = I\,(Df \cdot g_s)\,\infty + I\,(f \cdot \text{scale}\,(-s)\,g_s)\,\infty$$

$$-f\,0 = I\,(Df \cdot g_s)\,\infty - s \cdot I\,(f \cdot g_s)\,\infty$$

$$-f\,0 = L\,(Df)\,s - s \cdot L\,f\,s$$

$$L\,(Df)\,s = -f\,0 + s \cdot L\,f\,s$$

let $L\,f\,s = I\,(f \cdot g_s)\,\infty$; $\mathbb{C}$; $\mathbb{R} \to \mathbb{R}$; $\mathbb{C}$

$L : (\mathbb{R} \to \mathbb{R}) \to$
$(\mathbb{C} \to \mathbb{C})$

Laplace transform

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$L f s = I (f \cdot g_s) \infty$

$$= \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt$$

$g_s t = e^{-st}$

Assume $f x \cdot g_s x \to 0$ as $x \to \infty$

$L (D f) s = - f 0 + s \cdot L f s$

$V \approx \mathbb{R} \to \mathbb{R}$

$W \approx \mathbb{C} \to \mathbb{C}$

$L : V \to W$

$LinTrans (L, V, W)$ (exercise!)