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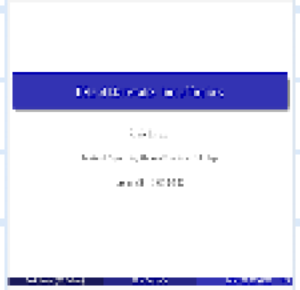

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DSLs of Mathematics: limit of functions

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Course goal and focus

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Goal

Encourage students to approach mathematical domains from a functional programming perspective.

Course focus

- Make functions and types explicit
- Explicit distinction between syntax and semantics
- Types as carriers of semantic information
- Organize the types and functions in DSLs

Now Make variable binding and scope explicit

Lecture notes and more available at: `https://github.com/DSLsofMath/DSLsofMath`

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Example: The limit of a function

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*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\varepsilon > 0$ there exists a number $\delta > 0$, possibly depending on ε, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \varepsilon$$

- Adams & Essex, Calculus - A Complete Course

$DSL \to \delta\sigma\lambda$

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Limit of a function – continued

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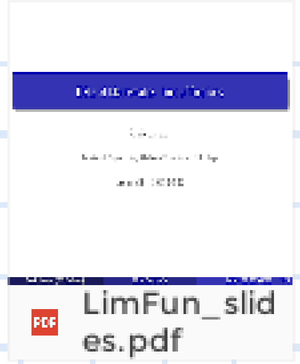

$$\lim_{x \to a} f(x) = L,$$

if

$$\forall \varepsilon > 0$$

$$\exists \delta > 0$$

such that if

$$0 < |x - a| < \delta,$$

then

$$x \in \mathit{Dom}\, f \wedge |f(x) - L| < \varepsilon$$

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Scope check:

a, f, L

First attempt at translation:

$$lim\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta$$

$$\text{where } P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow$$
$$(x \in Dom\ f \wedge |f\ x - L| < \epsilon)$$

ε

δ

x

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Limit of a function – continued

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$lim : X \to (X \to Y) \to Y \to$ FOL or Pred

$f : X \to Y$, $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$

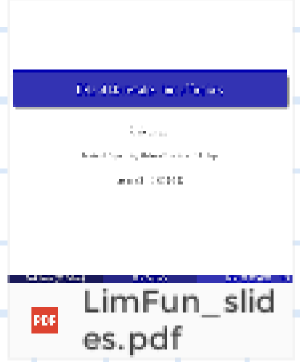
Finally (after adding a binding for x):

$$
\begin{aligned}
&lim\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = \ \ \forall\ x.\ \ Q\ \epsilon\ \delta\ x \\
&\qquad\qquad Q\ \epsilon\ \delta\ x = (0 < |x - a| < \delta) \Rightarrow \\
&\qquad\qquad\qquad\qquad (x \in Dom\ f \wedge |f\ x - L| < \epsilon)
\end{aligned}
$$

$\delta, \varepsilon : \mathbb{R}^+$

$P : \mathbb{R}^+ \to \mathbb{R}^+ \to FOL$

$x : X$



Limit of a function – continued

$$\boxed{\lim_{x \to a} f(x) = L}$$

Finally (after adding a binding for x):

$$
\begin{aligned}
&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\
&\quad \text{where } P\ \epsilon\ \delta = \quad \forall\ x.\ \ Q\ \epsilon\ \delta\ x \\
&\qquad\qquad Q\ \epsilon\ \delta\ x = (0 < |x - a| < \delta) \Rightarrow \\
&\qquad\qquad\qquad\qquad (x \in \mathit{Dom}\ f \wedge |f\ x - L| < \epsilon)
\end{aligned}
$$

Lesson learned: be careful with scope and binding (of x in this case).

lim properties

$\text{lim}\ a\ f\ L_1$

$\wedge\ \text{lim}\ a\ f\ L_2$

$\Rightarrow\ L_1 = L_2$

Thus lim can be used as a partial function from a and f.

lim typing

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$\text{lim} : X \to (X \to Y) \to Y \to \text{Prop}$

or

$\text{lim} : X \to (X \to Y) \to \text{Maybe}\ Y$

or

$\text{lim} : X \to (X \to Y) \to Y \quad \boxed{X \subseteq \mathbb{R},\ Y \subseteq \mathbb{R}}$

$\boxed{\text{lim}\ a}$ is linear:

$\text{lim}\ a\ (f \oplus g) = \text{lim}\ a\ f + \text{lim}\ a\ g$

$\text{lim}\ a\ (c \triangleleft f) = c \cdot (\text{lim}\ a\ f)$

$\oplus : (X \to Y) \to (X \to Y) \to (X \to Y)$ lifted addition

$f \oplus g = \backslash x \to (f\ x) + (g\ x)$

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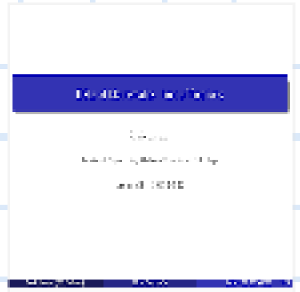
*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*



type of D?

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$\boxed{\begin{aligned} &D f \triangleq f' \\ &D : (X \to Y) \to (X \to Y) \end{aligned}}$$

$x : X$

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Example 2: derivative

*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$D\ f\ x = lim\ 0\ (\varphi\ x) \ \text{where} \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$g : \mathbb{R}^+ \to Y$

$g = \varphi\ x$

$\varphi : X \to \mathbb{R}^+ \to Y$

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Example 2: derivative

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*The **derivative** of a function f is another function f' defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = \mathit{lim}\ 0\ g \qquad \text{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$g = \varphi\ x$$

$$\varphi = \psi\ f$$

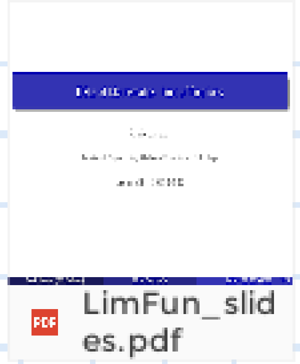
$$D\ f\ x = \mathit{lim}\ 0\ (\varphi\ x)\ \text{where}\quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$= \mathit{lim}\ 0\ (\psi\ f\ x) = (\mathit{lim}\ 0 \circ \psi\ f)\ x$$

$$D\ f \quad = \mathit{lim}\ 0 \circ \psi\ f\ \text{ where } \psi\ f\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

function composition

$$\begin{tikzcd} & \mathbb{R} \arrow[dl, "D\ f"'] \arrow[d, "\psi\ f"] & \ni & x \arrow[d, mapsto] \\ \mathbb{R} & \underbrace{(\mathbb{R}^{+} \to \mathbb{R})}_{} \arrow[l, "\mathit{lim}\ 0"] & \ni & \psi\ f\ x \end{tikzcd}$$



let $\mathbb{R}^{+} = \mathbb{R} \setminus \{0\}$

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Derivatives, cont.

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Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&sq\ x \quad\ \ = x\char`^2 \\
&double\ x = 2 * x \\
&c_2\ x \quad\ \ = 2 \\
&sq' \ == D\ sq \ == D\ (\lambda x \to x\char`^2) == D\ (\char`^2) == (2*) == double \\
&sq'' == D\ sq' == D\ double == c_2 == const\ 2
\end{aligned}
$$

pointfree

Note: we cannot *implement* D (of this type) in Haskell.
Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.
We need the "source code" of f to apply rules from calculus.

$$\boxed{D\, sq = \lim 0 \circ \psi\, sq} \qquad \psi : (\mathbb{R}\to\mathbb{R}) \to (\mathbb{R}\to\mathbb{R}^{+}\to\mathbb{R})$$

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$$\psi\, sq\; x\; h = \frac{1}{h}\left((x+h)^2 - x^2\right) = \frac{1}{h}\left(\cancel{x^2} + 2xh + h^2 - \cancel{x^2}\right) =$$

$$= \frac{1}{h}\left(2xh + h^2\right) = 2x + h$$

$$\psi\, sq\; x = (2\cdot x\, +) \qquad :: \mathbb{R}^{+}\to\mathbb{R}$$

$$D\, sq\; x = \lim 0\ (\psi\, sq\; x) = \lim 0\ (2\cdot x +) = 2\cdot x + 0 = 2\cdot x$$

$$\boxed{D\, sq = (2\cdot)}$$

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$$D\ sq = lim\ 0 \circ \Psi\ sq$$

$$\Psi\ sq\ x\ h = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} =$$

$$= 2x + h$$

$$\Psi\ sq\ x = (2 \cdot x +)$$

$$D\ sq\ x = lim\ 0\ (\Psi\ sq\ x) = lim\ 0\ (2 \cdot x +) = 2x + 0 = 2 \cdot x$$

$$D\ sq = (2 \cdot)$$

$$D\ ({}^\wedge 2) = (2 \cdot)$$

$$D(f \oplus g) = \lim 0 \circ \Psi\,(f \oplus g) \qquad D(\hat{}\,2) = (2\,\cdot)$$

$$\Psi\,(f \oplus g)\,x\,h = \frac{1}{h}\cdot\big(\,(f \oplus g)(x+h) - (f \oplus g)(x)\,\big)$$

$$= \frac{1}{h}\big((f(x+h) + g(x+h)) - (f\,x + g\,x)\big)$$

$$= \frac{1}{h}\big(f(x+h) - f(x)\big) + \frac{1}{h}\big(g(x+h) - g(x)\big)$$

$$= \Psi f\,x\,h + \Psi g\,x\,h = (\Psi f\,x \oplus \Psi g\,x)\,h$$

$$:: R^{+} \to R$$

$$\Psi\,(f \oplus g)\,x = \Psi f\,x \oplus \Psi g\,x$$

$$D(f \oplus g)\,x = \lim 0\,(\Psi f\,x \oplus \Psi g\,x) = \lim 0\,(\Psi f\,x) + \lim 0\,(\Psi g\,x)$$

$$= Df\,x + Dg\,x$$

$$= (Df \oplus Dg)\,x$$

$$\boxed{D(f \oplus g) = Df \oplus Dg}$$

$$D(f \oplus g) = \lim 0 \circ \Psi(f \oplus g)$$

$$D({}^{\wedge}2) = (2\cdot)$$

$$\Psi(f \oplus g)\, x\, h = \frac{(f \oplus g)(x+h) - (f \oplus g)\, x}{h} =$$

$$= \frac{1}{h}\Big(\big(f(x+h) + g(x+h)\big) - \big(f x + g x\big)\Big) =$$

$$= \frac{1}{h}\Big(\big(f(x+h) - f x\big) + \big(g(x+h) - g x\big)\Big) =$$

$$= \Psi f\, x\, h \quad + \quad \Psi g\, x\, h$$

$$\Psi(f \oplus g)\, x = \Psi f\, x \oplus \Psi g\, x$$

$$D(f \oplus g)\, x = \lim 0\, (\Psi f\, x \oplus \Psi g\, x) =$$

$$= \lim 0\, (\Psi f\, x) + \lim 0\, (\Psi g\, x) =$$

$$= D f\, x \quad + \quad D g\, x$$

$$D(f \oplus g) = Df \oplus Dg$$