Exam 2019-08

Patrik Jansson

1. Algebraic structure
2. Typing maths
3. Proofs
4. Laplace

Each ~25p

including bonus

Pass grade: 5p on each & sum ≥ 48

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1. [25p] Algebraic structure: a DSL for "things"

Consider the following (made up[1]) mathematical definition:

$(X, \mathit{start}, \mathit{grow}, \mathit{merge})$ is a **thing** with zero element *start*:

$$\begin{array}{ll} \mathit{merge}\ \mathit{start}\ x == \mathit{start} == \mathit{merge}\ x\ \mathit{start} & \text{-- for all } x \text{ in } X \\ \mathit{merge}\ (\mathit{grow}\ x)\ (\mathit{grow}\ y) == \mathit{grow}\ (\mathit{merge}\ x\ y) & \text{-- for all } x,\ y \text{ in } X \end{array}$$

(a) Define a type class *Thing* that corresponds to the "thing" structure.

(b) Define a datatype $T\ v$ for the language of thing expressions (with variables of type v) and define a *Thing* instance for it. (These are expressions formed from applying the thing operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

(c) Find and implement two other instances of the *Thing* class. Hint: look for one number-based and one string-based instance. Make sure the laws are satisfied.

(d) Give a type signature for, and define, a general evaluator for $T\ v$ expressions on the basis of an assignment function.

(e) Specialise the evaluator to the two *Thing* instances defined in (1c). Take three thing expressions of type $T\ \mathit{String}$, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

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2. [25p] Typing maths: The Indefinite Integral

Consider the following quote from Adams & Essex, Ch. 2, p. 148:

> The general antiderivative of a function $f(x)$ on an interval I is $F(x) + C$, where $F(x)$ is any particular antiderivative of $f(x)$ on I and C is a constant. This general antiderivative is called the indefinite integral of $f(x)$ on I and is denoted $\int f(x)dx$.
>
> **DEFINITION 8**
>
> The **indefinite integral** of $f(x)$ on interval I is
>
> $$\int f(x)dx = F(x) + C \qquad \text{on } I,$$
>
> provided $F'(x) = f(x)$ for all x in I.

(a) [7p] Give the types of the following expressions: f, x, $f(x)$, I, $\int f(x)dx$, F, F'.

(b) [3p] Suggest other notation and restate Def. 8.

(c) [5p] Define a ternary logical predicate *IndefInt* which takes I, f, and F as parameters and captures the meaning of Def. 8 except for the C.

(d) [5p] How are F and G related if we know *IndefInt* $I\ f\ F$ and *IndefInt* $I\ f\ G$?

(e) [5p] What is f if we know *IndefInt* $I\ f\ f$ for $I = (-\infty, \infty)$ and $f(0) = -1$.

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3. [25p] Proofs: Unique limits

Consider the statement: "The limit of a convergent sequence is unique."

This can be formalised as $T = \forall\ a : \mathit{Seq}\ X.\ \ U\ a$ where $X \subseteq \mathbb{R}$, $\mathit{Seq}\ X = \mathbb{N} \to X$,

$$
\begin{array}{ll}
U : \mathit{Seq}\ X \to \mathit{Prop} & \text{-- ``}a\text{ has a unique limit''} \\
U\ \ a = \forall\ L_1 : \mathbb{R}.\ \ \forall\ L_2 : \mathbb{R}.\ \ Q\ a\ L_1 \wedge L_1 \neq L_2 \to \neg\, Q\ a\ L_2 & \\
Q : \mathit{Seq}\ X \to X \to \mathit{Prop} & \text{-- ``}a\text{ converges to }L\text{''} \\
Q\ a\ L = \forall\ \epsilon > 0.\ \ P\ a\ L\ \epsilon & \\
P : \mathit{Seq}\ X \to X \to \mathbb{R}_{>0} \to \mathit{Prop} & \text{-- ``some tail of }a\text{ is near }L\text{''} \\
P\ \ a\ L\ \epsilon = \exists\ N : \mathbb{N}.\ \ \forall\ n : \mathbb{N}.\ \ (n \geqslant N) \to (|a\ n - L| < \epsilon) &
\end{array}
$$

i.e., if a sequence converges to a limit (L_1), then it doesn't converge to anything else (L_2).

(a) [10p] Let $nQ\ a\ L_2 = \neg Q\ a\ L_2$. Simplify this to eliminate the negation. (By pushing the negation $\neg$ inwards until it meets an ordering which it can negate). Explain the steps in your equational reasoning.

(b) [5p] Give nQ a functional interpretation – that is, explain what form a value *prf* of type $nQ\ a\ L_2$ would have.

(c) [10p] Sketch a proof of T using the functional interpretation – that is, provide pseudo code for $t : T$.

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4. [25p] Laplace: Consider the following differential equation:

$$\frac{f''(x) + 3f'(x)}{2} = 1 - f(x), \quad f(0) = -1, \quad f'(0) = 3$$

(a) [10p] Solve the equation assuming that f can be expressed by a power series fs, that is, use *integ* and the differential equation to express the relation between fs, fs', fs'', and the power series *rhs* for the right hand side. What are the first four coefficients of fs?

(b) [15p] Solve the equation using the Laplace transform. You should need this formula (note that α can be zero) and the rules for linearity + derivative:

$$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

Show that your solution does indeed satisfy the three requirements.

$f(t) = \ldots.$

3. Proofs

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$T = \forall a.\ U\,a$ $\qquad \neg(P \Rightarrow Q) = \neg(\neg P \vee Q) = \neg\neg P \wedge \neg Q = P \wedge \neg Q$

$U\,a = \forall L_1, L_2.\ Q\,a\,L_1 \wedge L_1 \neq L_2 \Rightarrow \neg Q\,a\,L_2$

$Q\,a\,L = \forall \varepsilon > 0.\ P\,a\,L\,\varepsilon$

$P\,a\,L\,\varepsilon = \exists N.\ \forall n.\ (n \geq N) \Rightarrow (|a_n - L| < \varepsilon)$

$nQ\,a\,L_2 = \neg Q\,a\,L_2 = \exists \varepsilon > 0.\ \neg P\,a\,L_2\,\varepsilon$

$= \exists_{\varepsilon>0}.\ \forall N.\ \neg(\forall n. \ldots) = \exists_{\varepsilon>0}.\ \forall N. \exists n. \neg(\quad \Rightarrow \quad)$

$= \exists \varepsilon > 0.\ \forall N. \exists n.\ (n \geq N) \wedge (|a_n - L_2| \geq \varepsilon)$

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$T = \forall a.\ U\, a$

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$= \exists \varepsilon > 0.\ \forall N.\ \exists n.\ \neg(\ (\ ) \Rightarrow (\ )\ )$

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$\neg(P \Rightarrow Q) = \neg(\neg P \vee Q) = P \wedge \neg Q$

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$P\,a\,L\,\varepsilon = \exists N.\ \forall n.\ (n \geq N) \Rightarrow (|a_n - L| < \varepsilon)$

$nQ\,a\,L_2 = \exists \varepsilon > 0.\ \forall N.\ \exists n.\ (n \geq N) \wedge |a_n - L| \geq \varepsilon$

$prf : nQ\,a\,L_2$

$prf = (\varepsilon, \backslash N \rightarrow (n, (p_1, p_2)))$

where $n = \ldots$

$p_1 = \ldots$

$p_2 = \ldots$

3. Proofs

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$$T = \forall a.\ U\,a$$

$$U\,a = \forall L_1, L_2.\ (Q\,a\,L_1 \wedge L_1 \neq L_2) \Rightarrow (\neg Q\,a\,L_2)$$

$$Q\,a\,L = \forall \varepsilon > 0.\ P\,a\,L\,\varepsilon$$

$$P\,a\,L\,\varepsilon = \exists N.\ \forall n.\ (n \geq N) \Rightarrow (|a_n - L| < \varepsilon)$$

$$t : T$$

$$t\ a\ L_1\ L_2\ (q, p) = prf$$

(not finished)

en funktion från $\varepsilon > 0$ ...

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$$\mathscr{L}\left(\lambda t.\, e^{\alpha * t}\right) s = 1/(s - \alpha)$$

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$f'' + 3 \cdot f' + 2 \cdot f = \underline{2}$, $f(0) = -1$, $f'(0) = 3$

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4. Laplace

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$$f'' + 3 \cdot f' + 2 \cdot f = \underline{2} \quad , f(0) = -1 \; , f'(0) = 3$$