Laplace transform

Patrik Jansson

$$L\,f\,s = I\,(f \cdot g_s)\,\infty$$

$$= \int_0^\infty f(t) \cdot e^{-s \cdot t}\,dt$$

$$g_s\,t = e^{-st}$$

Assume $f\,x \cdot g_s\,x \to 0$ as $x \to \infty$

$$L\,(D f)\,s = -f\,0 + s \cdot L\,f\,s$$

$V \simeq \mathbb{R} \to \mathbb{R}$

$W \simeq \mathbb{C} \to \mathbb{C}$

$$L : V \to W$$

$$LinTrans\,(L, V, W)$$

Laplace transform

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$$L f s = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} \, dt \qquad L(Df)s = -f0 + s \cdot L f s$$

Compute L for some example "vectors": exp, sin, cos

We know $D \exp = \exp \wedge \exp 0 = 1$

$E : \mathbb{C} \to \mathbb{C}$

$E = L \exp$

$$L(D \exp) s = -\exp 0 + s \cdot L \exp s$$
$$L \exp s = -1 + s \cdot L \exp s$$
$$E s = -1 + s \cdot E s$$
$$(1-s) \cdot E s = -1$$
$$E s = -\frac{1}{1-s} = \frac{1}{s-1}$$

$L \exp s = \frac{1}{s-1}$

$\operatorname{Re} s > 1$

Laplace transform

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$$\cdot \ L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L (D f) s = - f 0 + s \cdot L f s$$

Compute L for some example "vectors": $g_\alpha = e^{-\alpha t}$

$g_{-1} = \exp$

$$D g_\alpha = \text{scale} \ (-\alpha) \ g_\alpha \quad , \quad g_\alpha 0 = 1$$

$$G_\alpha = L \, g_\alpha$$

$$L (D g_\alpha) s = - g_\alpha 0 + s \cdot L g_\alpha s$$

$$L (\text{scale} (-\alpha) g_\alpha) s = -1 + s \cdot L g_\alpha s$$

$$-\alpha \cdot L g_\alpha s = -1 + s \cdot L g_\alpha s$$

$$-\alpha \cdot G_\alpha s = -1 + s \cdot G_\alpha s$$

$$+(\alpha + s) \cdot G_\alpha s = +1 \quad \Rightarrow \quad G_\alpha s = \frac{1}{s + \alpha}$$

$$L \, g_\alpha s = \frac{1}{s + \alpha}$$

$$L (\backslash t \to e^{\beta t}) s = \frac{1}{s - \beta}$$

$$L (\backslash t \to e^{t}) s = \frac{1}{s - 1}$$

Laplace transform

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$L\,f\,s = \int_0^\infty f(t)\cdot e^{-s\cdot t}\,dt$

$L\,(D f)\,s = -f\,0 + s\cdot L\,f\,s$

Compute L for some example "vectors": sin, cos

What are their ODE's?

$D\,\sin = \cos \quad , \sin 0 = 0$

$D\,\cos = -\sin \quad , \cos 0 = 1$

Laplace sin & cos

$$\begin{cases} D\sin = \cos & , \sin 0 = 0 \\ D\cos = -\sin & , \cos 0 = 1 \end{cases}$$

$$L(\lambda t \to e^{\alpha t})s = \frac{1}{s-\alpha}$$

$$L f' s = -f0 + s \cdot L f s$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$\begin{cases} L \sin' s = -\sin 0 + s \cdot L \sin s \\ L \cos' s = -\cos 0 + s \cdot L \cos s \end{cases}$$

$$\begin{cases} L \cos s = s \cdot L \sin s \\ L(-\sin) s = -1 + s \cdot L \cos s \end{cases}$$

$$C = L \cos$$
$$S = L \sin$$

$$\begin{cases} C s = s \cdot S s \\ -S s = -1 + s \cdot C s = -1 + s \cdot s \cdot S s = -1 + s^2 \cdot S s \end{cases}$$

$$\begin{cases} C s = s \cdot S s \\ (s^2 + 1) \cdot S s = 1 \end{cases}$$

$$\begin{cases} S s = \frac{1}{1+s^2} \\ C s = \frac{s}{1+s^2} \end{cases}$$

$$L \sin s = \frac{1}{1+s^2}$$

$$L \cos s = \frac{s}{1+s^2}$$

Laplace transform

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$$L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L(Df)s = -f0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f 0 = 0$, $f' 0 = 1$

$$L f' s = -0 + s \cdot L f s = s \cdot L f s = s \cdot F s$$

$$L f'' s = L(Df')s = -f'0 + s \cdot L f' s = -1 + s^2 \cdot L f s$$

$$F = L f \qquad = -1 + s^2 \cdot F s$$

Laplace transform

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$$L\,f\,s = \int_0^\infty f(t)\cdot e^{-s\cdot t}\,dt \qquad L\,(D f)\,s = -f\,0 + s\cdot L\,f\,s$$

Solve $f'' + 2\cdot f = 3\cdot f'$, $f\,0 = 0$, $f'\,0 = 1$

$$L\,(f'' + 2\cdot f)\,s = L\,(3\cdot f')\,s$$

$$\underline{L\,f''\,s} + 2\cdot L\,f\,s = 3\cdot \underline{L\,f'\,s}$$

$$(-1 + s^2\cdot F\,s) + 2\cdot F\,s = 3\cdot s\cdot F\,s$$

$$(s^2 - 3\cdot s + 2)\cdot F\,s = 1$$

$$F\,s = \frac{1}{(s^2 - 3\cdot s + 2)} = \frac{1}{(s-1)\cdot(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Laplace transform

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$$\mathcal{L} f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad \mathcal{L}(Df)s = -f0 + s \cdot \mathcal{L} f s$$

$$f'' = Df'$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f0 = 0$, $f'0 = 1$

$$\mathcal{L}(f'' + 2 \cdot f)s = \mathcal{L}(3 \cdot f')s$$

$$\boxed{\mathcal{L} f' s = -0 + s \cdot \mathcal{L} f s}$$

$$\mathcal{L} f'' s + 2 \cdot \mathcal{L} f s = 3 \cdot \mathcal{L} f' s$$

$$\| $$

$$\boxed{-f'0 + s \cdot \mathcal{L} f' s = -1 + s^2 \cdot \mathcal{L} f s}$$

Laplace transform

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$$L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L(Df) s = -f 0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f 0 = 0$, $f' 0 = 1$

$$L(f'' + 2 \cdot f) s = L(3 \cdot f') s$$

$$\boxed{L f' s = -0 + s \cdot L f s}$$

$$L f'' s + 2 \cdot L f s = 3 \cdot \underline{L f' s}$$

let $F = L f$

$$-1 + s^2 \cdot F s + 2 \cdot F s - 3 \cdot s \cdot F s = 0$$

$$(s^2 - 3 \cdot s + 2) \cdot F s = 1$$

$$F s = \frac{1}{(s-1)\cdot(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Ansatz

Laplace transform

Patrik Jansson

$$L f s = \int_0^{\infty} f(t) \cdot e^{-s \cdot t} \, dt \qquad L (D f) s = -f 0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f 0 = 0$, $f' 0 = 1$, $F = L f$

$$F s = \frac{1}{(s-1) \cdot (s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = -\frac{1}{s-1} + \frac{1}{s-2}$$

$1 = A \cdot (s-2) + B \cdot (s-1)$

$s=1$: $1 = A \cdot (1-2) + B \cdot (1-1) = -A \Leftrightarrow A = -1$

$s=2$: $1 = A \cdot (2-2) + B \cdot (2-1) = B \Leftrightarrow B = 1$

$$L \exp = \frac{1}{s-1} \qquad L (\lambda t \to e^{2t}) s = \frac{1}{s-2}$$

$$L(-\exp) = -\frac{1}{s-1}$$

$f t = -e^{t} + e^{2 \cdot t}$

$f' t = -e^{t} + 2 \cdot e^{2 \cdot t}$

$f'' t = -e^{t} + 4 \cdot e^{2 \cdot t}$

$f 0 = -e^{0} + e^{0} = -1 + 1 = 0$

$f' 0 = -e^{0} + 2 e^{0} = -1 + 2 = 1$

$LHS = (-1-2) \cdot e^{t} + (4+2) \cdot e^{2 \cdot t}$

$RHS = -3 \cdot e^{t} + 6 \cdot e^{2 \cdot t}$

Laplace transform

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$$\cdot\ L f s = \int_0^\infty f(t) \cdot e^{-s \cdot t} \, dt \qquad L (D f) s = -f 0 + s \cdot L f s$$

Solve $f'' + 2 \cdot f = 3 \cdot f'$, $f\,0 = 0$, $f'0 = 1$, $F = L f$

$$F s = \frac{1}{(s-1) \cdot (s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \qquad = -\frac{1}{s-1} + \frac{1}{s-2}$$

$$f\,t = -e^{t} + e^{2 \cdot t}$$
$$f'\,t = -e^{t} + 2 \cdot e^{2 \cdot t}$$
$$f''\,t = -e^{t} + 4 \cdot e^{2 \cdot t}$$

$LHS =$

$RHS =$

$f\,0 =$

$f'0 =$

Laplace summary

$$L : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{C} \to \mathbb{C})$$

$$L\, f\, s = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

$$\mathrm{LinTran}\,(L)$$

$$L\,(\lambda t \to e^{\alpha t})\, s = \frac{1}{s-\alpha}$$

$$L\, f'\, s = -f\,0 + s \cdot L\, f\, s$$

$$L\,(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

Solve ODE by:

- transform
- solve rational expr.
- partial fraction decomposition
- transform back by "pattern matching"

$\boxed{\text{+ Check}}$

☆

Laplace foscale α

$$L\,(\backslash \underbrace{t \to f(\alpha \cdot t)}_{g})\, s = ?$$

$$D g\, t = \alpha \cdot D f (\alpha \cdot t), \quad g\, 0 = f\, 0$$

Assume $F = L\, f$

then $G\, s = \frac{1}{\alpha} \cdot F\left(\frac{s}{\alpha}\right)$

$$L\,(\backslash t \to e^{\alpha t})\, s = \frac{1}{s-\alpha}$$

$$L\, f'\, s = -f\, 0 + s \cdot L\, f\, s$$

$$L\,(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$L\, \sin\, s = 1/(s^2+1)$$

$$L\, \cos\, s = s/(s^2+1)$$

Laplace f∘scale α

$$L\,(\backslash t \to f(\alpha \cdot t))\,s = ?$$

$$\underbrace{\backslash t \to f(\alpha \cdot t)}_{g}$$

$$D\,g\,t = \alpha \cdot D\,f\,(\alpha \cdot t)$$

$$D\,g = \mathrm{scale}\,\alpha\,(D\,f \circ \mathrm{scale}\,\alpha)$$

$$g = f \circ \mathrm{scale}\,\alpha$$

$$\vdots$$

$$L\,g\,s = \frac{1}{\alpha} \cdot L\,f\left(\frac{s}{\alpha}\right)$$

$$L\,(\backslash t \to e^{\alpha t})\,s = \frac{1}{s-\alpha}$$

$$L\,f'\,s = -f\,0 + s \cdot L\,f\,s$$

$$L\,(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L\,f + \beta \cdot L\,g$$

$$L\,\sin\,s = 1/(s^2+1)$$

$$L\,\cos\,s = s/(s^2+1)$$

Laplace sin & cos

$D \sin = \cos \quad , \sin 0 = 0$

$D \cos = -\sin \quad , \cos 0 = 1$

let $S = L \sin$, $C = L \cos$

$$L(\lambda t \mapsto e^{\alpha t})s = \frac{1}{s-\alpha}$$

$$L f' s = -f0 + s \cdot L f s$$

$$L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot L f + \beta \cdot L g$$

$$\begin{cases} L \sin' s = -\sin 0 + s \cdot L \sin s \\ L \cos' s = -\cos 0 + s \cdot L \cos s \end{cases}$$

$$\begin{cases} C s = s \cdot S s \\ -S s = -1 + s \cdot C s = -1 + s^2 \cdot S s \end{cases}$$

$$S s = \frac{1}{s^2+1} \quad , \quad C s = \frac{s}{s^2+1} = \frac{s}{(s-i)\cdot(s+i)} = \frac{A}{s-i} + \frac{B}{s+i}$$

$$A \cdot L(\lambda t \mapsto e^{it})s \quad \| \quad B \cdot L(\lambda t \mapsto e^{-it})s \quad \|$$