Homomorphisms

DSLsofMath book Chapter 4

Patrik Jansson

Example:

$$
\begin{aligned}
exp &: \mathbb{R} && \to \mathbb{R}_{>0} \\
exp &\ \ 0 && = 1 && \text{-- } e^0 = 1 \\
exp &\ \ (a+b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

From: $(\mathbb{R}, (+), 0) \xrightarrow{exp}$ To: $(\mathbb{R}_{>0}, (*), 1)$

$(A, \oplus, z) \xrightarrow{h} (B, \otimes, u)$

$\forall x, y : A.\ h\,(x \oplus y) = (h\,x) \otimes (h\,y)$

$h\ \ z = u$

Homomorphisms

Types

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$$(A, \oplus, z) \xrightarrow{h} (B, \otimes, u)$$

$$\forall x,y:A.\ h(x \oplus y) = (h\,x) \otimes (h\,y)$$

$$h\ z = u$$

$$\boxed{\begin{matrix} x:A,\ y:A \\ x \oplus y : A \\ z : A \end{matrix}} \overset{h}{\longmapsto} \boxed{\begin{matrix} h\,x : B,\ h\,y : B & (h\,x) \otimes (h\,y) : B \\ h(x \oplus y) : B & \\ h\,z : B & u : B \end{matrix}}$$

Homomorphisms

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Example:

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\begin{aligned}
&exp : \mathbb{R} && \to \mathbb{R}_{>0} && \\
&exp \;\; 0 && = 1 && \text{-- } e^0 = 1 \\
&exp \;\; (a + b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

$$(A, \oplus, z) \xrightarrow{h} (B, \otimes, u)$$

$$H_2 : (A \to B) \times (A \to A \to A) \times (B \to B \to B) \longrightarrow Prop$$

$$H_2(h, \oplus, \otimes) \stackrel{def.}{=} \forall x, y : A.\ h(x \oplus y) == (h\,x) \otimes (h\,y)$$

We know $H_2(exp, +, *)$ & $H_0(exp, 0, 1)$

$$H_0(h, z, u) = h\,z == u$$

Homomorphisms

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Example:

$$
\begin{aligned}
&exp : \mathbb{R} && \to \mathbb{R}_{>0} && \\
&exp\ \ 0 && = 1 && \text{-- } e^0 = 1 \\
&exp\ \ (a+b) && = exp\ a * exp\ b && \text{-- } e^{a+b} = e^a e^b
\end{aligned}
$$

From

$$
(\mathbb{R}, (+), 0) \underset{log}{\overset{exp}{\rightleftarrows}} (\mathbb{R}_{>0}, (*), 1)
$$

$$
\begin{aligned}
&log : \mathbb{R}_{>0} && \to \mathbb{R} && \\
&log\ \ 1 && = 0 && \text{-- } \log 1 = 0 \\
&log\ \ (a * b) && = log\ a + log\ b && \text{-- } \log(ab) = \log a + \log b
\end{aligned}
$$

Homomorphisms

Types

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$$H_2 : (A \to B) \times (A \to A \to A) \times (B \to B \to B) \to Prop$$

$$H_2(h, \oplus, \otimes) \stackrel{\text{def.}}{=} \forall x, y : A.\ h\,(x \oplus y) == (h\,x) \otimes (h\,y)$$

$$H_1 : (A \to B) \times (A \to A) \times (B \to B) \to Prop$$

$$H_1(h, fa, fb) = \forall x : A.\ h\,(fa\ x) == fb\,(h\,x)$$

$$H_0 : (A \to B) \times A \times B \to Prop$$

$$H_0(h, a, b) = h\ a == b$$

Book section 4.2.1

DSL → δσλ

DSLsofMath

Homomorphisms

Predicates

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$H_2(h, opa, opb) = \forall x : A.\ \forall y : A.\ h\ (opa\ x\ y) == opb\ (h\ x)\ (h\ y)$

$H_1(h, fa, fb) = \forall x : A.\ h\ (fa\ x) == fb\ (h\ x)$

$H_0(h, a, b) = h\ a == b$

$h : A \to B$

Homomorphisms

Book section 4.3.1

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$$\exists \otimes.\ H_2(odd, +, \otimes) \qquad ?$$

$$odd : \mathbb{N} \to \mathbb{B}$$

$$\begin{tikzcd}
(+) \arrow[d, bend right] & : & \mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d] \\
(\otimes) & : & \mathbb{B} \arrow[r] & \mathbb{B} \arrow[r] & \mathbb{B}
\end{tikzcd}$$

Homomorphisms

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$\exists \otimes .\ H_2(odd, +, \otimes)$?

$odd : \mathbb{N} \to \mathbb{B}$

$+ \ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

$\otimes \ : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

$\exists \otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}.\ \forall x, y : \mathbb{N}.$

$$\underbrace{odd\,(x+y)}_{\mathbb{B}} == \underbrace{\underbrace{odd\,x}_{} \otimes \underbrace{odd\,y}_{}}_{\mathbb{B}}$$

Homomorphisms

Patrik Jansson

$$\exists \otimes.\ H_2(odd, +, \otimes) \ ?$$

$$odd : \mathbb{N} \to \mathbb{B}$$
$$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$$

2:nd arg.

1st. arg

$\otimes$	F	T
F	F	T
T	T	F

$$\exists \otimes.\ \forall x,y:\mathbb{N}.\ odd(\underbrace{x+y}_{0}) == \underbrace{odd\ x}_{F} \otimes \underbrace{odd\ y}_{F}$$

$x=0, y=0$ $\quad$ F

$x=y=1$ $\quad odd(1+1) = F$

$x=0, y=1$ $\quad$ T $== odd\ 0 \otimes odd\ 1$

$== F \otimes T$

Homomorphisms

Patrik Jansson

$$\exists \otimes.\ H_2(odd, +, \otimes)$$

$odd : \mathbb{N} \to \mathbb{B}$

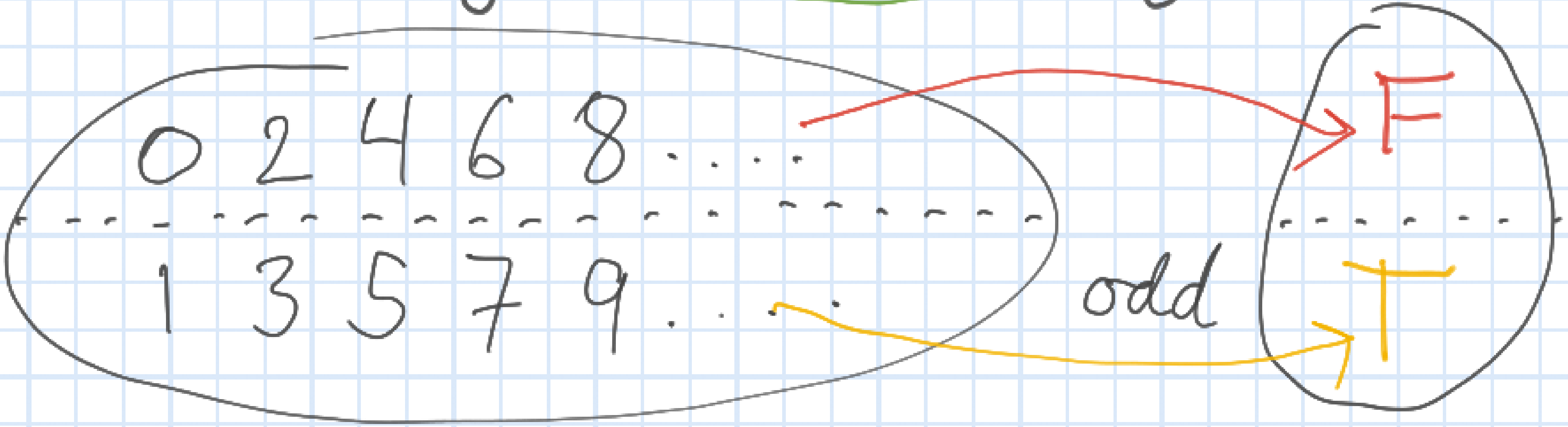
$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

$$\exists \otimes.\ \forall x, y : \mathbb{N}.\ \boxed{odd(x+y)} == odd\,x \otimes odd\,y$$

?

$$\begin{array}{c|c|c|c}
 & & y=2 & y=3 \\
 & \otimes & F & T \\
\hline
x=2 & F & F & T \\
\hline
x=3 & T & T & F
\end{array}$$



Homomorphisms

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$\exists \otimes.\ H_2(\mathit{odd}, +, \otimes)$

$\mathit{odd} : \mathbb{N} \to \mathbb{B}$

$+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

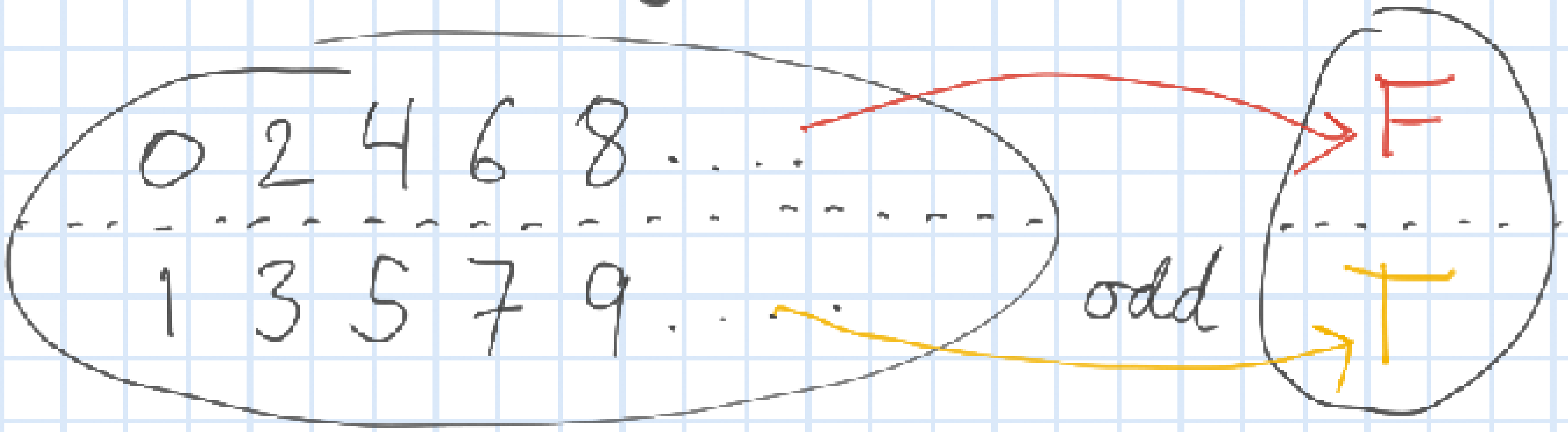
$\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$

xor = "addition mod 2"

?

$\otimes$	F	T
F	F	T
T	T	F

$\exists \otimes.\ \forall x,y:\mathbb{N}.\ \boxed{\mathit{odd}(x+y)} == \mathit{odd}\,x \otimes \mathit{odd}\,y$



Homomorphisms

Proof

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$$odd\,(x+y) = odd\,x \otimes odd\,y$$

$$\exists \otimes.\ H_2(odd, +, \otimes)$$

Proof: We know $\forall n:\mathbb{N}.\ \exists k:\mathbb{N}.\ \exists b:\mathbb{Z}_2.\ n = 2\cdot k + b$

Thus $x = 2\cdot k_x + b_x \quad \wedge \quad y = 2\cdot k_y + b_y$

$odd\ x = odd\ b_x \quad \wedge \quad odd\ y = odd\ b_y$

$x+y = 2\cdot(k_x+k_y) + (b_x+b_y)$

$odd\,(x+y) = odd\,(b_x+b_y)$

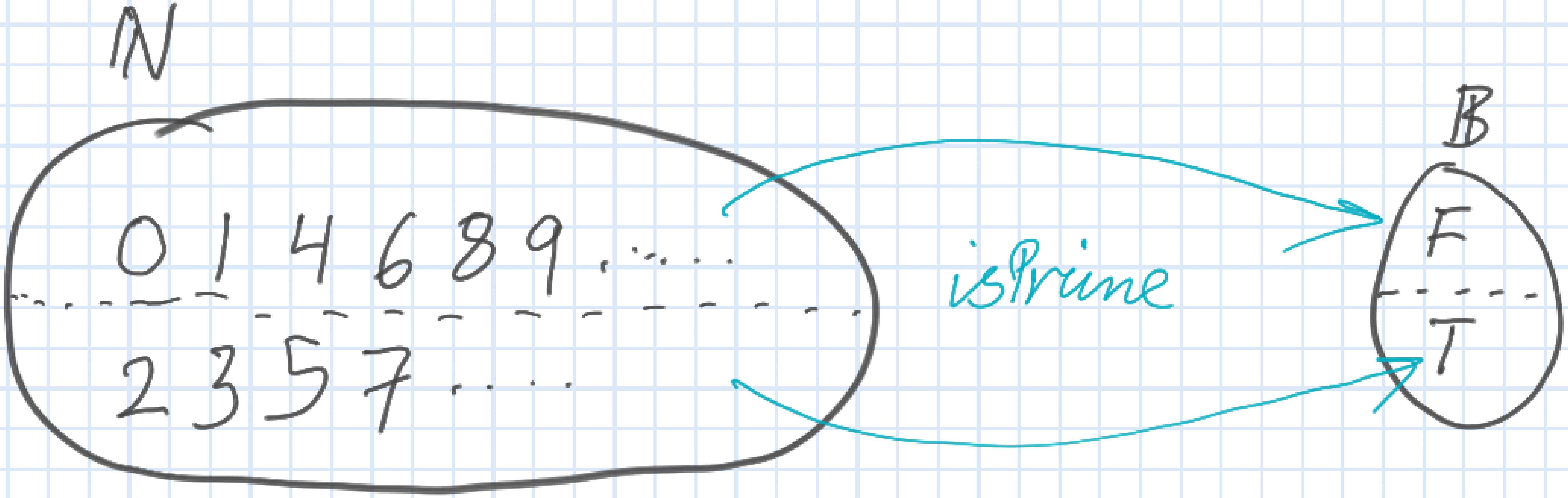
Check the four cases:

b_x	$b_y = 0$ or	$b_y = 1$
0	$F = F$	$T = T$
1	$T = T$	$F = F$

Book section 4.3.2

$$\exists \otimes.\ H_2(\text{isPrime}, +, \otimes) \quad ?$$

$$\text{isPrime} : \mathbb{N} \to \mathbb{B}$$
$$(+) : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$(\otimes) : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$$



$\exists \otimes.\ H_2(\text{isPrime}, +, \otimes)$? No. No $\otimes$ exists.

Proof by contradiction: Assume $\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ ~~exists~~ s.t.

$$\forall x, y : \mathbb{N}.\ isP(x+y) == isP\,x \otimes isP\,y$$

let $b = T \otimes T : \mathbb{B}$

$x = y = 2$

$$isP(2+2) == isP\,2 \otimes isP\,2$$

$$isP\,4 == T \otimes T == b$$

$$\overset{\shortparallel}{F} \implies b == F$$

$\bar{x} = 3, y = 2$: $isP(3+2) == isP\,3 \otimes isP\,2$

$$isP\,5 == T \otimes T == b$$

$$\overset{\shortparallel}{T} \implies b == T$$

$\exists \otimes.\ H_2(\mathrm{isPrime}, +, \otimes)$?

Proof by contradiction: Assume $\otimes : \mathbb{B} \to \mathbb{B} \to \mathbb{B}$ exists.

isPrime 2 = T

isPrime 3 = T

isPrime 4 = F

isPrime 5 = T

$2+2=4 \mapsto T \otimes T = F$

$2+3=5 \mapsto T \otimes T = T$

contradiction

Thus no such $\otimes$ can exist.