

## Plane Strain Elastodynamics:

Now that we have finished our discussion of primal and mixed finite element analysis of the plane strain elasto-statics problem, we turn to our final topic of the class: plane strain elastodynamics. Unlike all of the other problems considered in this class, the plane strain elastodynamics problem requires discretization in both space and time. To discretize in space, we use a finite element method, while to discretize in time, we turn to standard time integration schemes. It is also possible to discretize in space-time using a finite element method, leading to a so-called space-time finite element method, but this is beyond the scope of the class.

The (primal) strong form of the plane strain elastodynamics problem is:

$$(S) \left\{ \begin{array}{l} \text{Find } u_1 : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R} \text{ and } u_2 : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R} \text{ such that:} \\ \\ \rho u_{1,tt} = \sigma_{11,1} + \sigma_{12,2} + f_1 \quad \text{in } \Omega \times (0, T) \\ \rho u_{2,tt} = \sigma_{21,1} + \sigma_{22,2} + f_2 \quad \text{in } \Omega \times (0, T) \\ u_1 = g_1 \quad \text{on } \Gamma_{D_1} \times (0, T) \\ u_2 = g_2 \quad \text{on } \Gamma_{D_2} \times (0, T) \\ t_1 = h_1 \quad \text{on } \Gamma_{N_1} \times (0, T) \\ t_2 = h_2 \quad \text{on } \Gamma_{N_2} \times (0, T) \\ \vec{u}(\cdot, 0) = \vec{u}_0 \quad \text{in } \Omega \\ \vec{u}_{,t}(\cdot, 0) = \vec{v}_0 \quad \text{in } \Omega \end{array} \right.$$

where:

$\Omega$  = The domain (cross-section) of interest

$T$  = The end time of interest

$\Gamma_{D_i}$  = The Dirichlet- $i$  boundary

$\Gamma_{N_i}$  = The Neumann- $i$  boundary

$u_i$  = The  $i^{\text{th}}$  component of displacement  $\vec{u}$

$\sigma_{ij}$  = The  $ij^{\text{th}}$  component of the Cauchy stress tensor  $\vec{\sigma}$

$t_i$  = The  $i^{\text{th}}$  component of the traction vector  $\vec{t}$

$f_i$  = The  $i^{\text{th}}$  component of body force per unit volume  $\vec{f}$

$\rho$  = The density of the body

$g_i$  = The prescribed value of the  $i^{\text{th}}$  component of displacement

$h_i$  = The prescribed value of the  $i^{\text{th}}$  component of traction

$\vec{u}_0$  = The prescribed initial displacement

$\vec{v}_0$  = The prescribed initial velocity

Just as in the setting of plane strain elastostatics:

$$\sigma_{11} = 2\mu \epsilon_{11} + \lambda (\epsilon_{11} + \epsilon_{22})$$

$$\sigma_{22} = 2\mu \epsilon_{22} + \lambda (\epsilon_{11} + \epsilon_{22})$$

$$\sigma_{12} = \sigma_{21} = 2\mu \epsilon_{12} = 2\mu \epsilon_{21}$$

where  $\mu$  and  $\lambda$  are the Lamé parameters and  $\epsilon_{11}$ ,  $\epsilon_{22}$ ,  $\epsilon_{12}$ , and  $\epsilon_{21}$  are the (infinitesimal) strain tensor components:

$$\varepsilon_{11} = u_{1,1}$$

$$\varepsilon_{22} = u_{2,2}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(u_{1,2} + u_{2,1})$$

It also holds that velocity and acceleration are the first and second time derivatives of displacement  $\vec{u}_{,t}$  and  $\vec{u}_{,tt}$  respectively. Note that in the plane strain elastodynamics problem we have displacement and velocity initial conditions:

$$\vec{u}(\cdot, t) = \vec{u}_0$$

$$\vec{u}_{,t}(\cdot, t) = \vec{v}_0$$

in addition to the displacement and traction boundary conditions we also had for the plane strain elastostatics problem. Moreover, the partial differential equations governing the plane strain elastodynamics problem are identical in form to those governing the plane strain elastostatics problem except the appearance of acceleration terms. These terms appear due to the fact that the governing partial differential equations for the plane elastodynamics problem correspond to Newton's second law:

$$\rho a_i = \underbrace{\sigma_{11,1} + \sigma_{12,2}}_{\text{Mass Times Acceleration}} + \underbrace{f_i}_{\text{Force}}$$

$$\rho \vec{a}_2 = \underbrace{\sigma_{21,1}}_{\substack{\text{Mass} \\ \text{Times}}} + \underbrace{\sigma_{22,2}}_{\substack{\text{Force}}} + f_2$$

Acceleration

A corresponding (primal) weak form can be constructed using weighted residuals, just as was done for the plane strain elasto-statics problem. This gives rise to the problem:

Find  $\vec{u} \in \mathcal{D}_T$  such that:

$$\int_{\Omega} \rho \vec{u}_{,tt}(\cdot, t) \cdot \vec{w} d\Omega + b(\vec{u}(\cdot, t), \vec{w}) = l(\vec{w}) \quad \text{for all } \vec{w} \in \mathcal{V} \text{ and } t \in (0, T)$$

$(\mathbf{w})$

$$\int_{\Omega} \rho \vec{u}(\cdot, 0) \cdot \vec{w} d\Omega = \int_{\Omega} \rho \vec{u}_0 \cdot \vec{w} d\Omega \quad \text{for all } \vec{w} \in \mathcal{V}$$

$$\int_{\Omega} \rho \vec{u}_{,t}(\cdot, 0) \cdot \vec{w} d\Omega = \int_{\Omega} \rho \vec{v}_0 \cdot \vec{w} d\Omega \quad \text{for all } \vec{w} \in \mathcal{V}$$

where:

$$\mathcal{D}_T := \left\{ \vec{v}: \Omega \times [0, T] \rightarrow \mathbb{R}^2 : \begin{array}{l} v_i(\cdot, t) \in H^1(\Omega), \\ v_{i,tt}(\cdot, t) \in L^2(\Omega), \text{ and} \\ v_i(\vec{x}, t) = g_i(\vec{x}, t) \text{ for } \vec{x} \in I_{D_i} \end{array} \right\}$$

is the set of (time-dependent) trial solutions,

$$\mathcal{V} := \left\{ \vec{v} \in (H^1(\Omega))^2 : v_i(\vec{x}) = 0 \text{ for } \vec{x} \in \Gamma_D \right\}$$

is the space of (time-independent) test functions,

$$b(\vec{u}(\cdot, t), \vec{w}) = \int_{\Omega} \underline{\Sigma}(\vec{w})^T \underline{D} \underline{\Sigma}(\vec{u}(\cdot, t)) d\Omega$$

$$l(\vec{w}) = \int_{\Omega} \vec{w} \cdot \vec{f} d\Omega + \int_{\Gamma_{N_1}} w_1 h_1 d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2 d\Gamma$$

are the bilinear and linear forms for the problem at hand,

$$\underline{\Sigma}(\vec{u}(\cdot, t)) = \begin{bmatrix} u_{1,1}(\cdot, t) \\ u_{2,2}(\cdot, t) \\ u_{1,2}(\cdot, t) + u_{2,1}(\cdot, t) \end{bmatrix} \quad \underline{\Sigma}(\vec{w}) = \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{bmatrix}$$

are the (time-instantaneous) strain and virtual strain vectors, and:

$$\underline{D} = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Now, suppose  $\mathcal{V}^h$  is a finite-dimensional subspace of  $\mathcal{V}$  and  $\vec{g}^h \in \mathcal{D}_T$ . Then a (Bubnov-) Galerkin semi-discrete approximation of the plane strain elastodynamics problem using  $\mathcal{V}^h$  and  $\vec{g}^h$  reads:

$$\left\{ \begin{array}{l} \text{Find } \vec{u}^h = \vec{v}^h + \vec{g}^h \text{ with } \vec{v}^h(t) \in \mathcal{V}^h \text{ for all } t \in [0, T] \text{ such that:} \\ \int_{\Omega} \rho \vec{u}_{,tt}^h(\cdot, t) \cdot \vec{w}^h d\Omega + b(\vec{u}^h(\cdot, t), \vec{w}^h) = l(\vec{w}^h) \\ \text{for all } \vec{w}^h \in \mathcal{V}^h \text{ and } t \in (0, T) \\ (w) \\ \int_{\Omega} \rho \vec{u}^h(\cdot, 0) \cdot \vec{w}^h d\Omega = \int_{\Omega} \rho \vec{u}_0 \cdot \vec{w}^h d\Omega \quad \text{for all } \vec{w}^h \in \mathcal{V}^h \\ \int_{\Omega} \rho \vec{u}_{,t}^h(\cdot, 0) \cdot \vec{w}^h d\Omega = \int_{\Omega} \rho \vec{v}_0 \cdot \vec{w}^h d\Omega \quad \text{for all } \vec{w}^h \in \mathcal{V}^h \end{array} \right.$$

A finite element semi-discretization results when  $\mathcal{V}^h$  is comprised of finite element functions. Now suppose that  $\mathcal{V}^h$  admits the basis:

$$\left\{ N_A \vec{e}_i \right\}_{A \in \gamma - \gamma_0, i=1,2}$$

and suppose that  $ID(A, i)$  maps each  $A \in \gamma - \gamma_0$ , for  $i=1, 2$  to a unique equation number between 1 and  $n_{eq}$  where:

$$n_{eq} := \dim(\mathcal{V}^h)$$

Then each  $\vec{w}^h \in V^h$  admits the form:

$$\vec{w}^h(\vec{x}) = \sum_{i=1}^2 \sum_{A \in \gamma - \gamma_D} (\vec{w}_A^h)_i \vec{e}_i N_A(\vec{x})$$

and the Galerkin displacement solution  $\vec{u}^h$  admits the form:

Time-Dependent Displacement Degrees of Freedom

$$\vec{u}^h(\vec{x}, t) = \sum_{j=1}^2 \sum_{B \in \gamma - \gamma_D} (\vec{u}_B^h)_j(t) \vec{e}_j N_B(\vec{x}) + \vec{g}^h(\vec{x}, t)$$

Using the above representations, we can follow the same steps as done for the plane strain elasto-statics problem to arrive at the following matrix problem:

$$(ODE) \left\{ \begin{array}{l} \text{Find } \underline{d} : [0, T] \rightarrow \mathbb{R}^{N_d} \text{ such that:} \\ \\ \underline{M} \ddot{\underline{d}}(t) + \underline{K} \underline{d}(t) = \underline{F}(t) \quad \text{for } t \in (0, T) \\ \underline{d}(0) = \underline{d}_0 \\ \dot{\underline{d}}(0) = \dot{\underline{d}}_0 \end{array} \right.$$

where :

$$M_{PQ} = \int_{\Omega} p N_A N_B \vec{e}_i \cdot \vec{e}_j d\Omega \quad \text{for } P = ID(A, i), Q = ID(B, j)$$

$$K_{PQ} = b(N_B \vec{e}_j, N_A \vec{e}_i) \quad \text{for } P = ID(A, i), Q = ID(B, j)$$

$$F_p(t) = l(N_A \dot{e}_i) - \int_{\Omega} p N_A \dot{\bar{g}}^h(t) \cdot \dot{e}_i d\Omega$$

$$- b(\dot{\bar{g}}^h(t), N_A \dot{e}_i) \quad \text{for } P = ID(A_j i)$$

$$d_Q(t) = (\bar{u}_B^h)_j(t) \quad \text{for } Q = ID(B_j j)$$

and:

$$\ddot{d}_0 = \underline{\underline{M}}^{-1} \ddot{u}_0$$

$$\dot{d}_0 = \underline{\underline{N}}^{-1} \dot{u}_0$$

where:

$$(\ddot{d}_0)_P = \int_{\Omega} p N_A (\ddot{u}_0 - \dot{\bar{g}}^h(\cdot, t)) \cdot \dot{e}_i d\Omega \quad \text{for } P = ID(A_j i)$$

$$(\dot{d}_0)_P = \int_{\Omega} p N_A (\dot{v}_0 - \dot{\bar{g}}^h(\cdot, t)) \cdot \dot{e}_i d\Omega \quad \text{for } P = ID(A_j i)$$

Note above that  $(\cdot)$  denotes differentiation with respect to time. The matrix  $\underline{\underline{M}}$  is called the mass matrix (not to be confused with the mass matrix  $\underline{\underline{M}}$  that appeared earlier in the matrix representation of a mixed displacement-pressure (Galerkin approximation of the plane strain elastostatics problem)), while  $\underline{\underline{K}}$ ,  $\underline{d}(t)$ , and  $\underline{F}(t)$  are again referred to as the stiffness matrix, displacement vector, and force vector, respectively. We also refer to  $\dot{d}(t)$  as the velocity vector and  $\ddot{d}(t)$  as the acceleration vector.

As opposed to the matrix representation of a Galerkin approximation of the plane strain elasto-statics problem, the matrix representation of a Galerkin approximation of the plane strain elasto-dynamics problem consists of a system of ordinary differential equations rather than a system of linear algebraic equations. To solve this system for the unknown displacement vector  $\underline{d}(t)$ , we turn to numerical time integration to discretize in time. In particular, we turn to the Newmark method.

Suppose that we know the displacement, velocity, and acceleration vectors at time  $t_n$  and we seek the displacement, velocity, and acceleration vectors at time  $t_{n+1}$ . A Taylor series expansion of the displacement and velocity vectors gives:

$$\underline{d}(t_{n+1}) = \underline{d}(t_n) + \Delta t \dot{\underline{d}}(t_n) + \frac{(\Delta t)^2}{2} \ddot{\underline{d}}(t^*)$$

$$\dot{\underline{d}}(t_{n+1}) = \dot{\underline{d}}(t_n) + \Delta t \ddot{\underline{d}}(t^{**})$$

where  $\Delta t = t_{n+1} - t_n$  is the time-step size and:

$$t_n \leq t^*, t^{**} \leq t_{n+1}$$

are two unknown times. With the Newmark method, we approximate:

$$\ddot{\underline{d}}(t^*) \approx (1-2\beta) \ddot{\underline{d}}(t_n) + 2\beta \ddot{\underline{d}}(t_{n+1})$$

$$\ddot{\underline{d}}(t^{**}) \approx (1-\gamma) \ddot{\underline{d}}(t_n) + \gamma \ddot{\underline{d}}(t_{n+1})$$

where  $0 \leq \beta \leq \frac{1}{2}$  and  $0 \leq \gamma \leq 1$ . The above approximations together with the governing equation at time  $t_{n+1}$ :

$$\underline{M} \ddot{\underline{d}}(t_{n+1}) + \underline{K} \underline{d}(t_{n+1}) = \underline{F}(t_{n+1})$$

give rise to three matrix equations for the unknown displacement, velocity, and acceleration vectors at time  $t_{n+1}$ :

$$\left\{ \begin{array}{l} \text{Classical} \\ \text{Newmark} \\ \text{Method} \end{array} \right. \left\{ \begin{array}{l} \underline{M} \underline{a}_{n+1} + \underline{K} \underline{d}_{n+1} = \underline{F}_{n+1} \\ \underline{d}_{n+1} = \underline{d}_n + \Delta t \underline{v}_n + \frac{(\Delta t)^2}{2} ((1-2\beta) \underline{a}_n + 2\beta \underline{a}_{n+1}) \\ \underline{v}_{n+1} = \underline{v}_n + \Delta t ((1-\gamma) \underline{a}_n + \gamma \underline{a}_{n+1}) \end{array} \right.$$

Above, we have used  $\underline{d}_n$ ,  $\underline{v}_n$ , and  $\underline{a}_n$  and  $\underline{d}_{n+1}$ ,  $\underline{v}_{n+1}$ , and  $\underline{a}_{n+1}$  to denote the numerical approximations of  $\underline{d}(t_n)$ ,  $\dot{\underline{d}}(t_n)$ , and  $\ddot{\underline{d}}(t_n)$  and  $\underline{d}(t_{n+1})$ ,  $\dot{\underline{d}}(t_{n+1})$ , and  $\ddot{\underline{d}}(t_{n+1})$ , respectively. The update formulas:

$$\underline{d}_{n+1} = \underline{d}_n + \Delta t \underline{v}_n + \frac{(\Delta t)^2}{2} ((1-2\beta) \underline{a}_n + 2\beta \underline{a}_{n+1})$$

$$\underline{v}_{n+1} = \underline{v}_n + \Delta t ((1-\gamma) \underline{a}_n + \gamma \underline{a}_{n+1})$$

are commonly referred to as the Newmark formulas. Plugging the Newmark formulas into:

$$\underline{\underline{M}} \underline{\underline{a}}_{n+1} + \underline{\underline{K}} \underline{\underline{d}}_{n+1} = \underline{\underline{F}}_{n+1}$$

gives rise to a single matrix equation for  $\underline{\underline{a}}_{n+1}$ :

$$(\underline{\underline{M}} + \beta (\Delta t)^2 \underline{\underline{K}}) \underline{\underline{a}}_{n+1} = \underline{\underline{F}}_{n+1} - \underline{\underline{K}} (\underline{\underline{d}}_n + \Delta t \underline{\underline{y}}_n + \left( \frac{1-2\gamma}{2} \right) (\Delta t^2) \underline{\underline{a}}_n)$$

and once we solve the above for  $\underline{\underline{a}}_{n+1}$ , we can use the Newmark formulas to find  $\underline{\underline{d}}_{n+1}$  and  $\underline{\underline{y}}_{n+1}$ . Once we know  $\underline{\underline{d}}_{n+1}$ ,  $\underline{\underline{y}}_{n+1}$ , and  $\underline{\underline{a}}_{n+1}$ , we can proceed to the next time step.

The particular choices of the parameters  $\beta$  and  $\gamma$  determine the accuracy and stability properties of the Newmark method. For example, the choice  $2\beta \geq \gamma \geq \frac{1}{2}$  achieves unconditional stability while the choice  $\gamma = \frac{1}{2}$  achieves second-order accuracy. Since  $\beta$  and  $\gamma$  are free, the Newmark method is not just one method but an entirely family of methods.

Well-known members of this family include:

Method	Type	$\beta$	$\gamma$	Stability	Accuracy
Trapezoidal Rule	Implicit	$\frac{1}{4}$	$\frac{1}{2}$	Unconditional	2
Linear Acceleration	Implicit	$\frac{1}{6}$	$\frac{1}{2}$	Conditional	2

Fox-Goodwin	Implicit	$\frac{1}{12}$	$\frac{1}{2}$	Conditional	2
Central Difference	Explicit	0	$\frac{1}{2}$	Conditional	2

In practice, the Newmark equations are not solved exactly but rather a predictor - multicorrector algorithm is employed. At the beginning of each time step, an iteration counter  $i$  is set to zero and a predictor phase is entered where the desired quantities of interest are set to predicted values:

$$\hat{d}_{n+1}^i = \hat{d}_{n+1}, \quad \hat{v}_{n+1}^i = \hat{v}_{n+1}, \quad \hat{a}_{n+1}^i = \hat{a}_{n+1}$$

The predictors may be chosen in any way such that they satisfy the Newmark formulas:

$$\begin{aligned}\hat{d}_{n+1} &= d_n + \Delta t v_n + \frac{(\Delta t)^2}{2} ((1-2\beta) a_n + 2\beta a_{n+1}) \\ \hat{v}_{n+1} &= v_n + \Delta t ((1-\gamma) a_n + \gamma a_{n+1})\end{aligned}$$

This means one of  $\hat{d}_{n+1}$ ,  $\hat{v}_{n+1}$ , and  $\hat{a}_{n+1}$  may be tuned while the other two are determined by the Newmark formulas. Typical predictor choices are the constant displacement predictor (often preferred in nonlinear structural mechanics), the constant velocity predictor (often preferred in fluid mechanics and fluid-structure interaction), and the

zero acceleration predictor (often used for linear problems such as the one considered here):

Constant Displacement Predictor:

$$\begin{aligned}\hat{d}_{n+1} &= d_n \\ \hat{a}_{n+1} &= -\frac{1}{\beta \Delta t} \hat{v}_n - \frac{(1-2\beta)}{2\beta} a_n\end{aligned}$$

$\hat{v}_{n+1}$  defined by Newmark formula

Constant Velocity Predictor:

$$\begin{aligned}\hat{v}_{n+1} &= v_n \\ \hat{a}_{n+1} &= -\frac{(1-\gamma)}{\gamma} a_n\end{aligned}$$

$\hat{d}_{n+1}$  defined by Newmark formula

Zero Acceleration Predictor:

$$\hat{a}_{n+1} = 0$$

$\hat{v}_{n+1}$     } defined by Newmark formula  
 $\hat{d}_{n+1}$

The  $i^{\text{th}}$  values are used to compute a residual as:

$$\underline{\Delta F}_{n+1}^i = F_{n+1} - \underline{M} \hat{a}_{n+1}^i - \underline{K} \hat{d}_{n+1}^i$$

and this residual is then used to calculate a correction to the acceleration:

$$\underline{M}^* \underline{\Delta a} = \underline{\Delta F}_{n+1}^i$$

where  $\underline{M}^*$  is a chosen approximation of the true left hand side matrix:

$$\underline{\underline{M}}^* \approx \underline{\underline{M}} + \beta (\Delta t)^2 \underline{\underline{K}}$$

Once  $\Delta \underline{a}$  is known, a corrector phase is entered where the displacement, velocity, and acceleration vector approximations at time  $t_{n+1}$  are updated:

$$\begin{aligned}\underline{a}_{n+1}^{i+1} &= \underline{a}_{n+1}^i + \Delta \underline{a} \\ \underline{v}_{n+1}^{i+1} &= \underline{v}_{n+1}^i + \gamma \Delta t \Delta \underline{a} \\ \underline{d}_{n+1}^{i+1} &= \underline{d}_{n+1}^i + \beta (\Delta t)^2 \Delta \underline{a}\end{aligned}$$

Many different choices for  $\underline{\underline{M}}^*$  are commonly used. One choice is to use the true or consistent left hand side matrix:

$$\underline{\underline{M}}^* = \underline{\underline{M}} + \beta (\Delta t)^2 \underline{\underline{K}}$$

Then, the predictor-multicorrector algorithm converges after one step. However, the resulting matrix system for the acceleration term might be expensive to solve. An alternative choice is:

$$\underline{\underline{M}}^* = \underline{\underline{M}}_{\text{lumped}}$$

where  $\underline{\underline{M}}_{\text{lumped}}$  is the so-called lumped mass matrix whose diagonal components are row sums. The resulting matrix system is trivial to solve, but several corrector steps may be required for convergence. Other choices

include using the true or consistent mass matrix:

$$\overline{M}^* = \overline{M}$$

or to use the consistent left hand side matrix but solve the resulting matrix system inexactly using an iterative linear solver such as a preconditioned conjugate gradient method with a fixed number of iterations or loose solver criteria.

At the end of the Corrector phase, the residual is checked for convergence.

If  $\|\Delta_{\text{nti}}^i\| \leq \varepsilon \|\Delta_{\text{nti}}^o\|$  for some user-defined tolerance  $\varepsilon$ ,

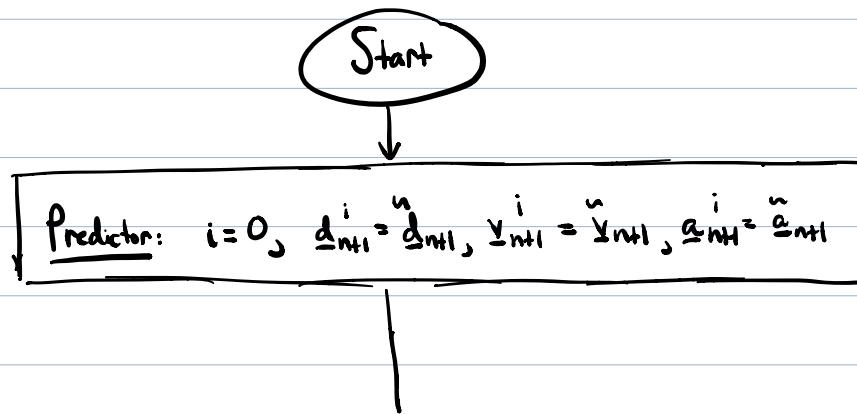
the time integration scheme proceeds to the next time step. Otherwise, the

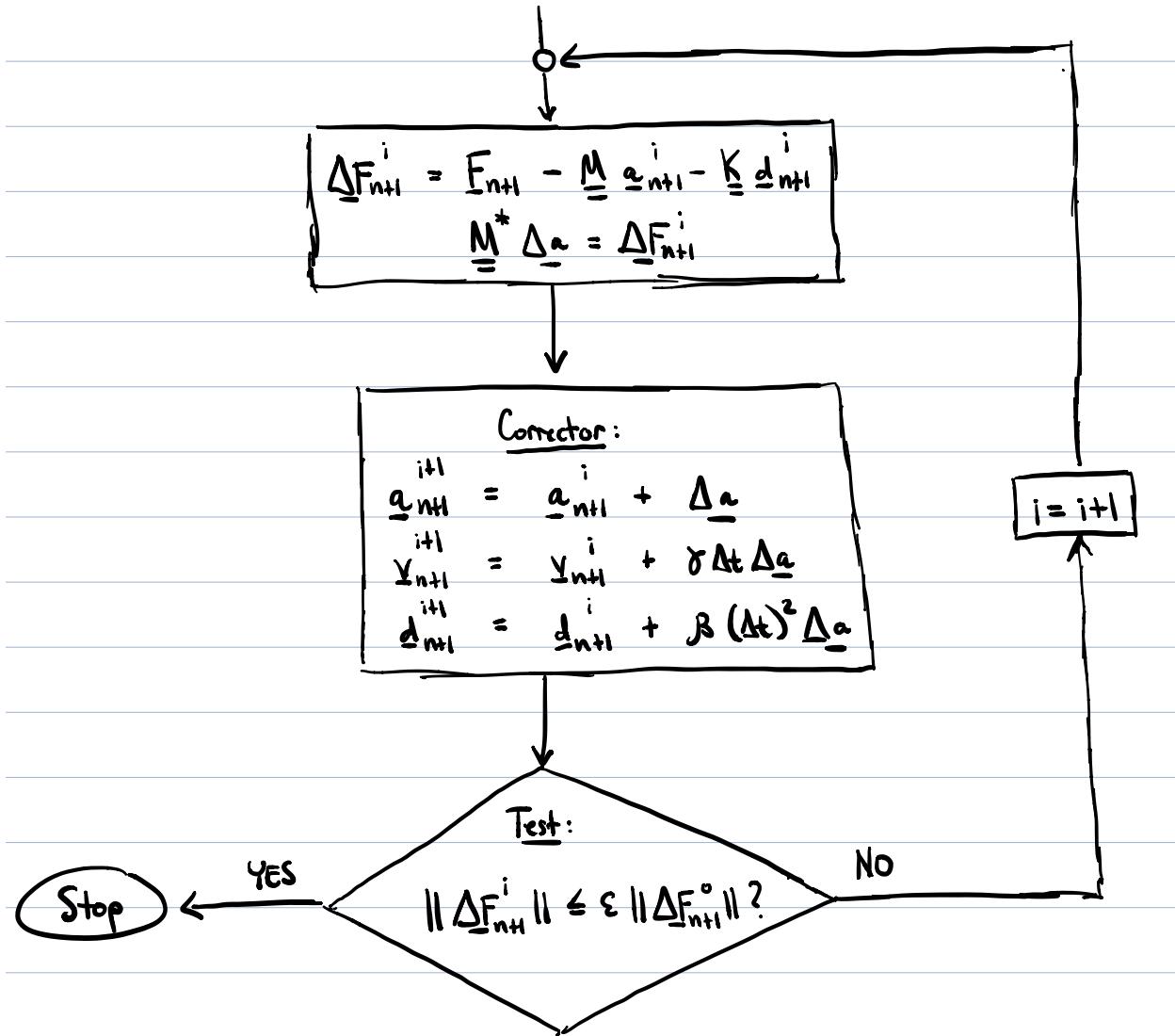
iteration counter  $i$  is incremented by 1 and the above corrector steps are

repeated. It should be noted that even if the exact  $\underline{M}^*$  is used, if an inexact linear solver is employed, additional corrector steps may be need to

be performed.

A flowchart of the full predictor-multipointer algorithm is displayed below:





While not included above, the Newmark method and presented predictor-corrector algorithm can be easily modified to account for physical damping such as Rayleigh damping. They can also be easily extended to nonlinear problems of interest.

One advantageous property of the Newmark method is that  $\beta$  and  $\gamma$  may be tuned in order to damp unphysical high-frequency behavior due to spatial

discretization. Unfortunately,  $\gamma > \frac{1}{2}$  is necessary to introduce high-frequency dissipation in the Newmark method, limiting the accuracy of the method to first order. For this reason, Hilber, Hughes, and Taylor introduced the HHT- $\alpha$  method in:

H.M. Hilber, T.J.R. Hughes, and R.L. Taylor, "Improved numerical dissipation for time integration algorithms in structural mechanics," *Earthquake Engineering and Structural Dynamics*, 5: 283 - 292. (1977)

The basis of the HHT- $\alpha$  method is to replace the equation of motion in Newmark's method by:

$$\underline{M} \ddot{\underline{q}}_{n+1} + \underline{K} \underline{d}_{n+\alpha} = \underline{F}_{n+1}$$

where:

$$\underline{d}_{n+\alpha} = (1-\alpha) \underline{d}_n + \alpha \underline{d}_{n+1}$$

and  $\alpha$  is a tunable parameter. Hilber, Hughes, and Taylor showed the method is both second-order accurate and unconditionally stable when:

$$\frac{2}{3} \leq \alpha \leq 1$$

$$\beta = (2-\alpha)^2/4$$

$$\gamma = \frac{3}{2} - \alpha$$

and the method possesses high frequency dissipation when  $\alpha < 1$ . Moreover, the smaller the  $\alpha$ , the greater the dissipation. The HHT- $\alpha$  enjoys widespread use throughout the field of structural dynamics and other fields such as molecular dynamics. Chung and Hulbert introduced a further generalization, the generalized- $\alpha$  method, for more precise control of numerical dissipation in:

J. Chung and G.M. Hulbert, "A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized- $\alpha$  method," Journal of Applied Mechanics, 60: 371-375. (1993)

In the generalized- $\alpha$  method, the equation of motion is:

$$\underline{\underline{M}} \underline{\underline{q}}_{n+\alpha m} + \underline{\underline{K}} \underline{\underline{d}}_{n+\alpha f} = \underline{\underline{F}}_{n+1}$$

where both  $\alpha_f$  and  $\alpha_m$  are tunable parameters. Second-order accuracy is attained if:

$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m$$

$$\beta = \frac{1}{4} (1 - \alpha_f + \alpha_m)^2$$

while unconditional stability requires:

$$\alpha_m \geq \alpha_f \geq \frac{1}{2}$$

Chung and Hulbert further showed an optimal combination of high-frequency and low-frequency dissipation occurs if:

$$\begin{aligned}\alpha_m &= \frac{2 - \rho_\infty}{1 + \rho_\infty} \\ \alpha_f &= \frac{1}{1 + \rho_\infty}\end{aligned}$$

where  $\rho_\infty \in [0, 1]$ . Moreover, if  $\rho_\infty$  is chosen to be one, there is no damping and when  $\rho_\infty$  is chosen to be zero, the highest frequency in the system is completely damped out in one time step while the low frequencies are minimally damped. The generalized- $\alpha$  method was later extended to first-order systems encountered in fluid mechanics by Jansen, Whiting, and Hulbert.

The Newmark method, HHT- $\alpha$  method, and generalized- $\alpha$  methods are only three potential candidates for numerical time integration of the system of ordinary differential equations in Problem (ODE). Virtually any numerical integration scheme could be employed for this purpose. For instance, explicit or implicit Runge-Kutta methods could be utilized. Other popular time integration schemes for structural dynamics include the Wilson-θ and Houbolt schemes. Of course, each scheme has its relative advantages and disadvantages that should be accounted for in selecting a time integration scheme.