Plane Strain Elastostatics : Mathematical Analysis of Volumetric Locking:

We can also use Céa's lemma to see the detoriation in quality of a finite element approximation in the limit as Poisson ratio ν goes to 0.5. For constant μ and λ, one can show that:

$$b(\vec{v}, \vec{v}) \geq \frac{C_{korn}\, 2\mu}{L^2} \|\vec{v}\|^2_{(H^1(\Omega))^2} \qquad \forall \vec{v} \in \mathcal{V}$$

$$|b(\vec{v}, \vec{w})| \leq \frac{(2\mu + \lambda)}{L^2} \|\vec{v}\|_{(H^1(\Omega))^2} \|\vec{w}\|_{(H^1(\Omega))^2} \qquad \forall \vec{w}, \vec{v} \in (H^1(\Omega))^2$$

where:

$$\|\vec{v}\|^2_{(H^1(\Omega))^2} := \|v_1^2\|^2_{H^1(\Omega)} + \|v_2^2\|^2_{H^1(\Omega)} \qquad \forall \vec{v} \in (H^1(\Omega))^2$$

and $C_{korn} > 0$ is the constant associated with Korn's inequality:

$$\int_\Omega \left(v_{1,1}^2 + v_{2,2}^2 + \frac{1}{2}(v_{1,2} + v_{2,1})^2\right) d\Omega \geq \frac{C_{korn}}{L^2} \|\vec{v}\|^2_{(H^1(\Omega))^2} \qquad \forall \vec{v} \in \mathcal{V}$$

Thus, by Céa's lemma, a Galerkin approximation of the plane strain elastostatics problem admits the error estimate:

$$\|\vec{u} - \vec{u}^h\|_{(H^1(\Omega))^2} \leq C^{-1}_{korn} \left(\frac{2\mu + \lambda}{2\mu}\right) \inf_{\vec{v}^h \in \mathcal{V}^h} \|\vec{u} - \vec{v}^h\|_{(H^1(\Omega))^2}$$

Since $\lambda \to \infty$ as $\nu \to 0.5$, we see the right hand side of the above error estimate becomes unbounded in the limit as Poisson ratio ν goes to 0.5. Even when $\nu \approx 0.5$, the factor $\left(\frac{2\mu+\lambda}{2\mu}\right)$ can be quite large, indicating that the solution error might be much larger than the best approximation error. Indeed, this is what we see with volumetric locking. The best approximation error must be made quite small in order to overcome the factor $\left(\frac{2\mu+\lambda}{2\mu}\right)$. With a finite element approximation, the mesh size must be made sufficiently small or the polynomial degree must be made sufficiently high.