Plane Strain Elastostatics: Volumetric Locking:

While finite element approximation is an attractive approach for discretization of the plane strain elastostatics problem, the finite element approximation presented in the last two lectures suffers from a critical flaw: it exhibits a phenomenon called volumetric locking as the Poisson ratio ν approaches 0.5.

To understand the phenomenon of volumetric locking, it helps to write our weak form for the plane strain elastostatics problem in a slightly different way. Let:

$$\bar{\underline{\underline{D}}} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Then the plane strain elastostatics problem admits the alternative form:

$$(W) \left\{ \begin{array}{l} \text{Find } \vec{u} \in \mathcal{S} \text{ such that:} \\ \quad \bar{b}(\vec{u},\vec{w}) + \displaystyle\int_{\Omega} \lambda (\vec{\nabla}\cdot\vec{u})(\vec{\nabla}\cdot\vec{w})\, d\Omega = \ell(\vec{w}) \quad \text{for all } \vec{w} \in \mathcal{V} \\ \text{where:} \\ \quad \bar{b}(\vec{u},\vec{w}) = \displaystyle\int_{\Omega} \underline{\varepsilon}(\vec{w})^T \bar{\underline{\underline{D}}}\, \underline{\varepsilon}(\vec{u})\, d\Omega \\ \quad \ell(\vec{w}) = \displaystyle\int_{\Omega} \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Gamma_{N_1}} w_1 h_1\, d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2\, d\Gamma \end{array} \right.$$

Now suppose that λ is constant. Then the weak solution $\vec{u} \in \mathcal{S}$ of the above problem satisfies:

$$\int_{\Omega_h} (\vec{\nabla} \cdot \vec{u})(\vec{\nabla} \cdot \vec{w}) \, d\Omega_h = \frac{1}{\lambda} \left(\bar{b}(\vec{u}, \vec{w}) - \ell(\vec{w}) \right)$$

for all $\vec{w} \in \mathcal{V}$. Thus as $\lambda \to \infty$:

$$\int_{\Omega_h} (\vec{\nabla} \cdot \vec{u})(\vec{\nabla} \cdot \vec{w}) \, d\Omega_h \longrightarrow 0$$

for all $\vec{w} \in \mathcal{V}$, so:

$$\vec{\nabla} \cdot \vec{u} \longrightarrow 0$$

That is, the displacement field becomes divergence-free. Deformations characterized by divergence-free displacement fields are called incompressible deformations since each small portion of the body has the same volume before and after such a deformation. Note that λ approaches ∞ when ν approaches 0.5 since:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

Thus materials with a Poisson ratio of $\nu \approx 0.5$ are said to be nearly incompressible. Rubber is an example of a nearly incompressible material.

As $\lambda \to \infty$, the weak solution $\vec{u} \in \mathcal{S}$ of the plane strain elastostatics problem converges to the solution of:

$$(\mathring{W})\begin{cases} \text{Find } \mathring{u} \in \mathring{\mathcal{S}} \text{ such that:} \\ \quad \bar{b}(\mathring{u}, \mathring{w}) = \ell(\mathring{w}) \qquad \text{for all } \mathring{w} \in \mathring{\mathcal{V}} \\ \text{where:} \\ \quad \mathring{\mathcal{S}} := \{ \mathring{v} \in \mathcal{S} : \vec{\nabla} \cdot \mathring{v} \equiv 0 \} \\ \quad \mathring{\mathcal{V}} := \{ \mathring{v} \in \mathcal{V} : \vec{\nabla} \cdot \mathring{v} \equiv 0 \} \end{cases}$$

Likewise, finite element solutions $\vec{u}^h \in \mathcal{S}^h$ of finite element approximations of the plane strain elastostatics problem converge to the solution of:

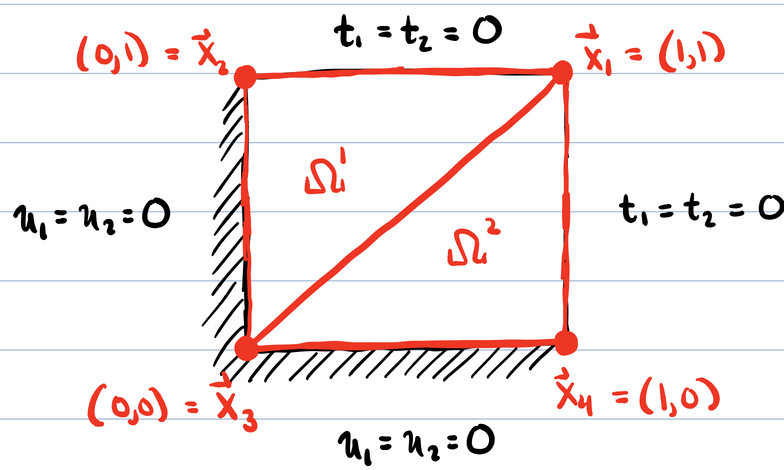
$$(\mathring{G})\begin{cases} \text{Find } \mathring{u}^h \in \mathring{\mathcal{S}}^h \text{ such that:} \\ \quad \bar{b}^h(\mathring{u}^h, \mathring{w}^h) = \ell^h(\mathring{w}^h) \qquad \text{for all } \mathring{w}^h \in \mathring{\mathcal{V}}^h \\ \text{where:} \\ \quad \bar{b}^h(\vec{u}^h, \vec{w}^h) = \int_{\Omega^h} \underset{\sim}{\varepsilon}(\vec{w}^h)^T \, \bar{\underset{\approx}{D}} \, \underset{\sim}{\varepsilon}(\vec{u}^h) \, d\Omega \\ \quad \ell^h(\vec{w}^h) = \int_{\Omega^h} \vec{w}^h \cdot \vec{f} \, d\Omega + \int_{\Gamma^h_{N_1}} w_1^h h_1 \, d\Gamma + \int_{\Gamma^h_{N_2}} w_2^h h_2 \, d\Gamma \\ \text{and:} \end{cases}$$

$$\left\{\begin{array}{l} \mathring{\mathcal{S}}^h := \left\{ \mathring{v}^h \in \mathcal{S}^h : \vec{\nabla} \cdot \mathring{v} \equiv 0 \right\} \\ \mathring{\mathcal{V}}^h := \left\{ \mathring{v}^h \in \mathcal{V}^h : \vec{\nabla} \cdot \mathring{v} \equiv 0 \right\} \end{array}\right.$$

Thus, the performance of a finite element approximation of the plane strain elastostatics problem in the incompressible limit $\lambda \to \infty$ is tied to how well divergence-free displacement fields in the space $\mathring{\mathcal{S}}$ are approximated by divergence-free finite element displacement fields in the space $\mathring{\mathcal{S}}^h$. Unfortunately, for many mesh and problem configurations, $\mathring{\mathcal{S}}^h$ consists of a single displacement field: zero displacement. For such cases, finite element approximations yield the nonphysical solution of zero displacement in the limit of $\lambda \to \infty$. In fact, finite element approximations yield unrealistically small displacement fields even for nearly incompressible materials unless the mesh size is sufficiently small. This is referred to as locking as the displacement is "locked" until the mesh size is sufficiently small. In particular, this is referred to as volumetric locking since locking is due to the incompressibility (constant volume) constraint being enforced on the displacement field. There are other types of locking that can be exhibited by finite element approximations. For instance, finite element Reissner-Mindlin shell approximations may suffer from membrane and shear locking. Strategies for alleviating membrane and shear locking are similar in spirit to strategies for alleviating volumetric locking.

To illustrate a case where volumetric locking arises, consider the following mesh and problem configuration:

$\boxed{k=1}$



For the above configuration, a finite element trial solution $\vec{v}^h \in \mathcal{S}^h$ takes the form:

$$\vec{v}^h(\vec{x}) = \vec{v}^h(\vec{x}_1)\, N_1(\vec{x})$$

Over Ω_1:

$$N_1(\vec{x}) = x_1$$

and over Ω_2:

$$N_1(\vec{x}) = x_2$$

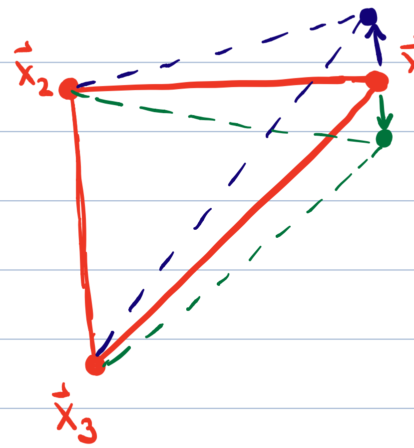
Thus, if $\vec{v}^h \in \mathring{\mathcal{S}}^h$, then it must hold that:

$$0 = \vec{\nabla}\cdot\vec{v}^h = v_1^h(\vec{x}_1)\underbrace{\frac{\partial N_1}{\partial x_1}}_{1}{}' + v_2^h(\vec{x}_1)\underbrace{\frac{\partial N_1}{\partial x_2}}_{0}{}' = v_1^h(\vec{x}_1)$$

over Ω^1 and:

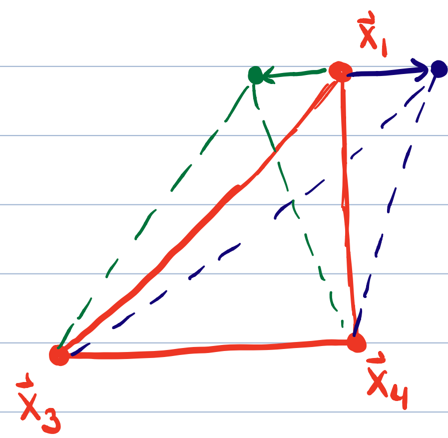
$$0 = \vec{\nabla}\cdot\vec{v}^h = v_1^h(\vec{x}_1)\underbrace{\frac{\partial N_1^1}{\partial x_1}}_{0} + v_2^h(\vec{x}_1)\underbrace{\frac{\partial N_1^1}{\partial x_2}}_{1} = v_2^h(\vec{x}_1)$$

over Ω^2, so $\vec{v}^h$ must identically be zero. Physically, $\vec{\nabla}\cdot\vec{v}^h = 0$ over Ω^1 if and only if the displacement at node $\vec{x}_1$ is completely in the vertical direction:



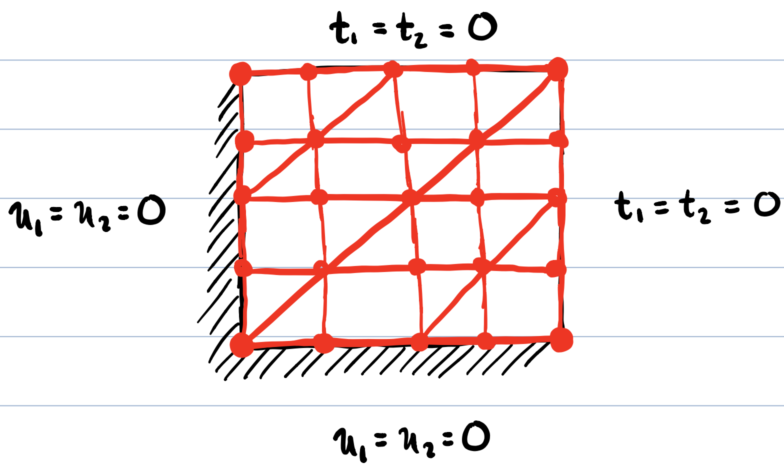
Possible Incompressible Deformations of Ω^1

and $\vec{\nabla}\cdot\vec{v}^h = 0$ over Ω^2 if and only if the displacement at node $\vec{x}_1$ is completely in the horizontal direction:



Possible Incompressible Deformations of Ω^2

So $\vec{\nabla}\cdot\vec{v}^h = 0$ if and only if there is no displacement at node $\vec{x}_1$. Volumetric locking also occurs on uniform refinements of the above mesh, for example:



With the above in mind, a question naturally arises:

Q: How do we alleviate volumetric locking?

The answer is quite simple to state but much more difficult to implement:

A: We weaken the incompressibility constraint.

To weaken the incompressibility constraint, we turn to <u>mixed finite element approximations</u> in the next few lectures.

As a final note, finite element approximations that exhibit volumetric locking are said to be <u>overly stiff</u>. That is because such approximations are overly stiff for deformations that should cause no volume change. This does not mean, however, that such approximations are overly stiff for arbitrary deformations.