1D Model Problem: Convergence of Finite Element Approximations:

In the last lecture, we discussed how to construct a Bubnov-Galerkin finite element approximation of our 1D model problem. Today, we discuss the quality of such an approximation. To do so, we need to invoke powerful results from functional analysis.

A linear form $\ell : V \to \mathbb{R}$ is a linear map from a vector space V to the real numbers $\mathbb{R}$:

$$\ell(c_1 v_1 + c_2 v_2) = c_1 \ell(v_1) + c_2 \ell(v_2)$$

for all $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2 \in V$. A bilinear form $b : V \times V \to \mathbb{R}$ is a bilinear map from a vector space V to the real numbers $\mathbb{R}$:

$$\begin{aligned} b(c_1 v_1 + c_2 v_2, d_1 w_1 + d_2 w_2) = {} & c_1 d_1 b(v_1, w_1) + c_1 d_2 b(v_1, w_2) \\ & + c_2 d_1 b(v_2, w_1) + c_2 d_2 b(v_2, w_2) \end{aligned}$$

for all $c_1, c_2, d_1, d_2 \in \mathbb{R}$ and $v_1, v_2, w_1, w_2 \in V$. An inner-product is a special bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ satisfying:

Symmetry: $(v_1, v_2) = (v_2, v_1)$ for all $v_1, v_2 \in V$

Positive-Definiteness: $(v, v) > 0$ for all $0 \neq v \in V$

A vector space V equipped with an inner-product $(\cdot,\cdot)_V$ is called an inner-product space. Such a space is equipped with an induced norm $\|\cdot\|: V \to \mathbb{R}$ defined as:

$$\|v\|_V = (v,v)_V^{1/2} \quad \text{for all } v \in V$$

An induced norm satisfies all the usual properties of a norm:

Triangle Inequality: $\|v_1 + v_2\|_V \leq \|v_1\|_V + \|v_2\|_V$ for all $v_1, v_2 \in V$

Scalability: $\|c\,v\|_V = |c|\,\|v\|_V$ for all $c \in \mathbb{R}, v \in V$

Positive Definiteness: $\|v\|_V > 0$ for all $0 \neq v \in V$

Moreover, members of an inner-product space satisfy the Cauchy-Schwarz inequality:

$$|(v_1, v_2)_V| \leq \|v_1\|_V \|v_2\|_V \quad \text{for all } v_1, v_2 \in V$$

An inner-product space is said to be complete if all so-called Cauchy sequences, sequences whose elements become arbitrarily close to each other with respect to the induced norm as the sequence progresses, converge to a member in the inner-product space with respect to the induced norm. Complete inner-product spaces are referred to as Hilbert spaces.

Hilbert spaces play a pivotal role in the mathematical theory of Finite element analysis and the mathematical theory of partial differential equations more generally. The simplest Hilbert space is $\mathbb{R}^n$, which is equipped with the dot product as an inner-product:

$$\left(\underline{x}_1, \underline{x}_2 \right)_{\mathbb{R}^n} = \underline{x}_1 \cdot \underline{x}_2 = \sum_{i=1}^{n} (\underline{x}_1)_i (\underline{x}_2)_i$$

for all $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$. All of the function spaces discussed in lecture so far are also Hilbert spaces. For instance, the space $L^2((0,L))$ is a Hilbert space when equipped with the inner-product:

$$\left(f_1, f_2 \right)_{L^2((0,L))} = \int_0^L f_1 \cdot f_2 \, dx$$

for all $f_1, f_2 \in L^2((0,L))$. The space $H^1((0,L))$ is a Hilbert space when equipped with the inner-product:

$$\left(f_1, f_2 \right)_{H^1((0,L))} = \int_0^L f_1 \cdot f_2 \, dx + L^2 \int_0^L f_{1,x} \cdot f_{2,x} \, dx$$

for all $f_1, f_2 \in H^1((0,L))$. The space:

$$H^k((0,L)) := \left\{ v \in L^2((0,L)) : \frac{d^i v}{dx^i} \in L^2((0,L)) \text{ for all } i = 1, \ldots, k \right\}$$

is a Hilbert space when equipped with the inner-product:

$$\left(f_1, f_2 \right)_{H^k((0,L))} = \sum_{i=0}^{k} L^{2i} \int_0^L \frac{d^i f_1}{dx^i} \cdot \frac{d^i f_2}{dx_i} \, dx$$

for all $f_1, f_2 \in H^k((0,L))$. All complete subspaces of a Hilbert space are also Hilbert spaces when equipped with their parent's inner-product. Therefore, the space of test functions for the variational form of our 1D model problem:

$$\mathcal{V} := \left\{ v \in H^1((0,L)) : v(0) = v(L) = 0 \right\}$$

is a Hilbert space when equipped with the H^1-inner-product, and all finite-dimensional subspaces $\mathcal{V}^h \subset \mathcal{V}$ are also Hilbert spaces when equipped with the H^1-inner-product. The last result is due to the fact that all finite-dimensional inner-product spaces are complete. By contrast, not all infinite-dimensional inner-product spaces are complete. An examination of this curious result is beyond the scope of this class.

The space $L^2((0,L))$ is an example of a Lebesgue space, hence why it is denoted with an "L". Lebesgue spaces play a pivotal role in real analysis and measure theory. The spaces $H^k((0,L))$ are examples of Sobolev spaces. In fact, they are the Hilbert Sobolev spaces, hence why they are denoted by "H". Sobolev spaces play a critical role in functional analysis and the theory of partial differential equations.

One of the most powerful results in functional analysis is the Lax Milgram theorem. This theorem may be utilized to establish well-posedness results for the variational form of our 1D model problem.

Lax-Milgram Theorem: Let X be a Hilbert space, H be a complete subspace of X, $b : X \times X \to \mathbb{R}$ be a bilinear form satisfying:

Coercivity: There exists a constant $\alpha > 0$ such that:

$$b(w, w) \geq \alpha \|w\|_X^2$$

for all $w \in H$.

Continuity: There exists a constant $M > 0$ such that:

$$|b(v, w)| \leq M \|v\|_X \|w\|_X$$

for all $v, w \in X$.

and $\ell : H \to \mathbb{R}$ be a linear form satisfying:

Continuity: There exists a constant $C > 0$ such that:

$$|\ell(w)| \leq C \|w\|_X$$

for all $w \in H$.

Then, for $x_0 \in X$, the problem:

Find $u \in H + x_0$ such that:

$$b(u,w) = \ell(w) \quad \text{for all } w \in H$$

has a unique solution satisfying:

$$\|u\|_X \leq \frac{c}{\alpha} + \left(\frac{M}{\alpha}+1\right)\|x_0\|_X$$

Now, recall the variational form of our 1D model problem:

$$(V)\begin{cases} \text{Find } u \in \mathcal{V} + g \text{ such that:} \\ \quad \int_0^L K u_{,x} w_{,x}\, dx = \int_0^L f w \, dx \\ \text{for all } w \in \mathcal{V}. \end{cases}$$

Problem (V) looks just like the problem appearing in the Lax-Milgram theorem with:

$$X = H^1((0,L)), \quad H = \mathcal{V}, \quad x_0 = g$$

$$b(v,w) = \int_0^L K v_{,x} w_{,x}\, dx$$

$$\ell(w) = \int_0^L f w \, dx$$

Now suppose:

i. $f \in L^2((0,L))$

ii. $K_{max} := \sup_{x \in (0,L)} K < \infty$

iii. $K_{min} := \inf_{x \in (0,L)} K > 0$

Then, for all $v \in \mathcal{V}$:

$$|\ell(w)| = \left| \int_0^L f w \, dx \right|$$

$$\leq \|f\|_{L^2((0,L))} \|w\|_{L^2((0,L))}$$

$$\leq \|f\|_{L^2((0,L))} \|w\|_{H^1((0,L))}$$

Thus, ℓ is a continuous linear form with continuity constant:

$$C = \|f\|_{L^2((0,L))}$$

For all $v, w \in H^1((0,L))$:

$$|b(v,w)| = \left| \int_0^L K v_{,x} w_{,x} \, dx \right|$$

$$\leq K_{max} \|v_{,x}\|_{L^2((0,L))} \|w_{,x}\|_{L^2((0,L))}$$

$$\leq \frac{K_{max}}{L^2} \|v\|_{H^1((0,L))} \|w\|_{H^1((0,L))}$$

So b is a continuous bilinear form with continuity constant:

$$M = \frac{K_{max}}{L^2}$$

For all $w \in \mathcal{V}$:

$$b(w,w) = \int_0^L K\, w_{,x}^2 \, dx \geq K_{min} \| w_{,x} \|^2_{L^2((0,L))}$$

By the classical Poincaré inequality, there is a constant $C_{poin} > 0$ such that:

$$\| w_{,x} \|^2_{L^2((0,L))} \geq \frac{C_{poin}}{L^2} \| w \|^2_{H^1((0,L))}$$

for all $w \in \mathcal{V}$. Hence, for all $w \in \mathcal{V}$:

$$b(w,w) \geq \frac{C_{poin} K_{min}}{L^2} \| w \|^2_{H^1((0,L))}$$

So b is a coercive bilinear form with coercivity constant:

$$\alpha = \frac{C_{poin} K_{min}}{L^2}$$

Since b is coercive and continuous and ℓ is continuous, the Lax-Milgram theorem yields that Problem (V) has a unique solution satisfying:

$$\| u \|_{H^1((0,L))} \leq \frac{C}{\alpha} + \left(\frac{M}{\alpha} + 1 \right) \| g \|_{H^1((0,L))}$$

$$\leq \frac{L^2}{C_{poin} K_{min}} \| f \|_{L^2((0,L))} + \left(\frac{K_{max}}{C_{poin} K_{min}} + 1 \right) \| g \|_{H^1((0,L))}$$

Thus, as claimed in an earlier lecture, Problem (V) is well-posed under the given conditions on K and f.

It should be noted the Lax-Milgram theorem cannot be used to prove the strong and weak weak forms of our 1D model problem are well-posed. However, as also claimed in an earlier lecture, we can use a generalization of the Lax-Milgram theorem, the so-called Babuška-Lax-Milgram theorem, to do so. The Babuška-Lax-Milgram theorem is an advanced topic that will not be covered in this class.

The Lax-Milgram theorem can also be used to show a Bubnov-Galerkin approximation of the variational form of our 1D model problem is well-posed. In fact, coercivity, continuity, and a property called consistency can be used to derive *a priori* error estimates for such an approximation.

Theorem: Let X be a Hilbert space, $H \subset X$ be a complete subspace of X, and $H^h \subset H$ be a finite-dimensional subspace of H. Moreover, let $b: X \times X \to \mathbb{R}$ be a bilinear form and $\ell: H \to \mathbb{R}$ be a linear form satisfying the properties listed in the Lax-Milgram theorem.

Then, for $x_0 \in X$, the discrete problem:

Find $u^h \in H^h + x_0$ such that:

$$b(u^h, w^h) = \ell(w^h) \qquad \text{for all } w^h \in H^h$$

has a unique solution $u^h \in H^h + x_0$ satisfying:

$$\|u^h\|_X \leq \frac{C}{\alpha} + \left(\frac{M}{\alpha}+1\right)\|x_0\|_X$$

and the a priori error estimate:

$$\|u-u^h\|_X \leq \frac{M}{\alpha} \inf_{v^h \in H^h + x_0} \|u-v^h\|_X$$

where $u \in H + x_0$ is the solution of the continuous problem:

Find $u \in H + x_0$ such that:

$$b(u,w) = \ell(w) \qquad \text{for all } w \in H$$

Proof: Well-posedness is an immediate result of the Lax-Milgram theorem. To prove the error estimate, we exploit coercivity, continuity, and a third property, consistency. For all $w^h \in H^h$, note both:

$$b(u,w^h) = \ell(w^h) \quad \text{and} \quad b(u^h,w^h) = \ell(w^h)$$

Subtracting the second equation from the first gives:

$$b(u-u^h, w^h) = 0 \qquad \text{for all } w^h \in H^h$$

The above property, consistency, states the discrete problem is consistent with the continuous problem. The above property is also known as Galerkin orthogonality. Now, note $u-u^h \in H$. Then by the coercivity of b:

$$\|u-u^h\|_X^2 \leq \frac{1}{\alpha} b(u-u^h, u-u^h)$$

By bilinearity:

$$\begin{aligned} b(u-u^h, u-u^h) &= b(u-u^h, u) - b(u-u^h, u^h) \\ &= b(u-u^h, u) - b(u-u^h, u^h) \\ &\quad + b(u-u^h, v^h) - b(u-u^h, v^h) \\ &= b(u-u^h, u-v^h) + b(u-u^h, v^h-u^h) \end{aligned}$$

for arbitrary $v^h \in \mathcal{S}^h$. Note that $v^h - u^h \in H^h$, so by consistency:

$$b(u-u^h, v^h-u^h) = 0$$

and:

$$b(u-u^h, u-u^h) = b(u-u^h, u-v^h)$$

It follows that:

$$\|u-u^h\|_X^2 \leq \frac{1}{\alpha} b(u-u^h, u-v^h)$$

Finally, by continuity:

$$\|u-u^h\|_X^2 \leq \frac{M}{\alpha} \|u-u^h\|_X \|u-v^h\|_X$$

Dividing by $\|u-u^h\|_X$ yields:

$$\|u-u^h\|_X \leq \frac{M}{\alpha} \|u-v^h\|_X$$

The error estimate follows as $v^h \in \mathcal{S}^h$ is arbitrary. $\square$

The well-posedness result appearing in the above theorem is simply a corollary of the Lax-Milgram theorem. The a priori error estimate is referred to as Céa's lemma, first proven by Jean Céa in his dissertation in 1964. To demonstrate the power of the above theorem, recall a Bubnov-Galerkin approximation of the variational form of our 1D model problem takes the form:

$$(G) \begin{cases} \text{Find } u^h \in \mathcal{V}^h + g^h \text{ such that:} \\ \qquad \displaystyle\int_0^L K u^h_{,x} w^h_{,x} \, dx = \int_0^L f w^h \, dx \\ \text{for all } w^h \in \mathcal{V}^h. \end{cases}$$

Problem (G) looks just like the discrete problem appearing in the above theorem with:

$$X = H^1((0,L)), \quad H = \mathcal{V}, \quad H^h = \mathcal{V}^h, \quad x_0 = g^h$$

$$b(v,w) = \int_0^L K v_{,x} w_{,x} \, dx$$

$$\ell(w) = \int_0^L f w \, dx$$

and Problem (V) looks just like the continuous problem appearing in the above theorem with the same selections and $g = g^h$. We already know that b and ℓ satisfy the properties listed in the Lax-Milgram theorem if:

i. $f \in L^2((0,L))$

ii. $K_{max} := \sup_{x \in (0,L)} K < \infty$

iii. $K_{min} := \inf_{x \in (0,L)} K > 0$

so in this case, the above theorem guarantees Problem (G) has a unique solution $u^h \in \mathcal{V}^h + g^h$ satisfying:

$$\| u^h \|_{H^1((0,L))} \leq \frac{C}{\alpha} + \left(\frac{M}{\alpha} + 1 \right) \| g^h \|_{H^1((0,L))}$$

$$\leq \frac{L^2}{C_{poin} K_{min}} \| f \|_{L^2((0,L))} + \left(\frac{K_{max}}{C_{poin} K_{min}} + 1 \right) \| g^h \|_{H^1((0,L))}$$

and the a priori error estimate:

$$\| u - u^h \|_{H^1((0,L))} \leq \frac{M}{\alpha} \inf_{v^h \in \mathcal{V}^h + g^h} \| u - v^h \|_{H^1((0,L))}$$

$$\leq \underbrace{\frac{K_{max}}{C_{poin} K_{min}}}_{\text{Proportionality Constant}} \underbrace{\inf_{v^h \in \mathcal{V}^h + g^h} \| u - v^h \|_{H^1((0,L))}}_{\text{Best Approximation Error}}$$

The term:

$$\inf_{v^h \in \mathcal{S}^h} \| u - v^h \|_{H^1((0,L))}$$

in the above estimate is called the best approximation error. It is the smallest error in the H^1-norm achievable by a member of the given finite-dimensional space of trial solutions. Thus the above estimate says the Galerkin solution $u^h \in \mathcal{V}^h + g^h$ to Problem (G) attains an error comparable to the best approximation error with a constant of proportionality:

$$\frac{K_{max}}{C_{poin} K_{min}}$$

We say then that the Galerkin solution is quasi-optimal in the H^1-norm.

When a finite element approximation is employed, we can say even more. In particular, if we employ finite element functions of degree k and the exact solution $u \in \mathcal{S}$ to Problem (V) lies in the Sobolev space $H^s((0,L))$, the best approximation error satisfies a bound of the form:

$$\inf_{v^h \in \mathcal{S}^h} \| u - v^h \|_{H^1((0,L))} \leq C_{interp} \left(\frac{h}{L} \right)^r \| u \|_{H^{r+1}((0,L))}$$

where $r = \min\{k, s-1\}$ and $C_{interp} > 0$ is a constant dependent on degree k but not on mesh size h or the solution u. The above is known as an interpolation estimate. With an interpolation estimate in hand, we have the following *a priori* error estimate for finite element approximations:

$$\boxed{\| u - u^h \|_{H^1((0,L))} \leq \frac{C_{interp}}{C_{poin}} \frac{K_{max}}{K_{min}} \left(\frac{h}{L} \right)^r \| u \|_{H^{r+1}((0,L))}}$$

The above estimate is quite powerful, as it states the error associated with a finite element approximation goes to zero as $h \to 0$. That is, finite element approximations converge under mesh refinement, also referred to as h-refinement. Moreover, the rate of convergence increases with increasing degree k, provided the solution u is sufficiently smooth. When the solution u is not smooth, on the other hand, the rate of convergence is arrested. In this case, the low cost of low-order finite element approximations renders them a more attractive candidate than high-order finite element approximations. Some state-of-the-art methods rely on local mesh refinement and degree elevation to arrive at a finite element approximation that exhibits optimal accuracy for a given number of degrees of freedom. These methods, often referred to as hp-methods, are just one example of many advanced finite element methods utilized in academia, government, and industry today.