2D Heat Conduction : Galerkin Approximations:

In the last lecture, we discussed strong and weak forms for the problem of steady two-dimensional heat conduction. In this brief note, we show how to discretize the weak form presented in the last lecture using a Galerkin approximation.

To begin, suppose $\mathcal{V}^h$ is a finite-dimensional subspace of $\mathcal{V}$ and that $g^h \in \mathcal{S}^h$. Then, a (Bubnov-) Galerkin approximation of the weak form presented in the last lecture using $\mathcal{V}^h$ and g^h simply reads:

$$
(G) \quad \left\{ \begin{array}{l}
\text{Find } T^h \in \mathcal{S}^h := \mathcal{V}^h + g^h \text{ such that:} \\[1em]
\qquad b(T^h, w^h) = \ell(w^h) \qquad \text{for all } w^h \in \mathcal{V}^h \\[1em]
\text{where:} \\[1em]
\qquad b(T^h, w^h) = \displaystyle\int_{\Omega} \kappa \vec{\nabla} T^h \cdot \vec{\nabla} w^h \, d\Omega + \int_{\Gamma_R} \beta T^h w^h \, d\Gamma \\[1.5em]
\qquad \ell(w^h) = \displaystyle\int_{\Omega} f w^h \, d\Omega + \int_{\Gamma_N} h w^h \, d\Gamma + \int_{\Gamma_R} \beta T_R w^h \, d\Gamma
\end{array} \right.
$$

The solution $T^h \in \mathcal{S}^h$ to the above problem is referred to as the Galerkin solution, just as we did for the 1D model problem studied earlier. Using Céa's lemma, we can prove the *a priori* error estimate:

$$\|T - T^h\|_{H^1(\Omega)} \leq C_{poin}^{-1} \frac{K_{max}}{K_{min}} \inf_{v^h \in \mathcal{S}^h} \|T - v^h\|_{H^1(\Omega)}$$

where $\|\cdot\|_{H^1(\Omega)}$ is the H^1- norm defined as:

$$\|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + L^2 \left(\|v_{,x}\|^2_{L^2(\Omega)} + \|v_{,y}\|^2_{L^2(\Omega)} \right)$$

where L is the size of the domain and $\|\cdot\|_{L^2(\Omega)}$ is the L^2-norm defined as:

$$\|v\|^2_{L^2(\Omega)} := \int_\Omega v^2 \, d\Omega$$

$C_{poin} > 0$ is a constant such that:

$$\|v_{,1}\|^2_{L^2(\Omega)} + \|v_{,2}\|^2_{L^2(\Omega)} \geq \frac{C_{poin}}{L^2} \|v\|^2_{H^1(\Omega)}$$

for all $v \in \mathcal{V}$, and:

$$K_{max} := \sup_{\vec{x} \in \Omega} K(\vec{x}) \quad \text{and} \quad K_{min} := \inf_{\vec{x} \in \Omega} K(\vec{x})$$

Thus, just as was the case for the 1D model problem studied earlier, the Galerkin solution is quasi-optimal in the H^1- norm.

Also as was done for the 1D model problem studied earlier, we will construct finite element Galerkin approximations by choosing V^h to be comprised of finite element functions, piecewise polynomial functions with respect to a finite element mesh. However, in the 2D setting considered here, the finite element mesh is typically an approximation of the domain rather than an exact representation of the domain. As a consequence, the Dirichlet, Neumann, and Robin boundaries must be approximated as well, as must the boundary conditions specified on these boundaries. This will be discussed in more detail in the next lecture.