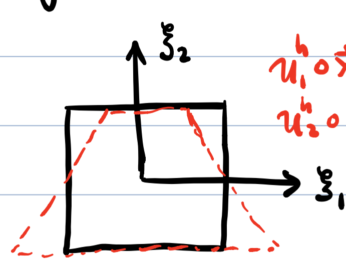
Plane Strain Elastostatics: Reduced and Selective Reduced Integration:

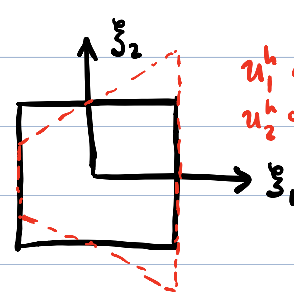
It was discovered early on in the history of structural finite element analysis that the effects of volumetric locking could be greatly reduced by underintegrating terms in the stiffness matrix for a primal displacement-only finite element approximation of the plane strain elastostatics problem. However, underintegration of all terms in the stiffness matrix yields a singular stiffness matrix approximation. This gives rise in turn to displacement vectors $\underline{v} \neq \underline{0}$ such that:

$$\underline{\underline{K}}\,\underline{v} = \underline{0}$$

Such displacement vectors are called spurious zero-energy modes to distinguish them from physical zero-energy modes associated with rigid body deformation. For instance, employing a one-point quadrature rule to integrate the stiffness matrix associated with a Q^1 finite element approximation yields spurious zero-energy modes that appear like an hourglass at the element level:



$$u_1^h \circ \vec{x}^e = c\,\xi\eta$$
$$u_2^h \circ \vec{x}^e = 0$$



$$u_1^h \circ \vec{x}^e = 0$$
$$u_2^h \circ \vec{x}^e = c\,\xi\eta$$

Such modes are typically called hourglass modes. One can overcome hourglassing by introducing hourglass stabilization terms to damp out spurious zero-energy modes. Alternatively, if one only underintegrates troublesome terms appearing in the stiffness matrix, then one arrives at a finite element approximation that is free of spurious zero-energy modes and also exhibits enhanced performance in the nearly incompressible setting. This is the basis of selective reduced integration.

For the plane strain elastostatics problem, the troublesome terms in the stiffness matrix are the λ-terms. Thus, when applied to the plane strain elastostatics problem, selective reduced integration yields an element stiffness matrix approximation of the form:

$$k^e_{pq} \approx \sum_{l=1}^{n_q} \left(\left(\underline{\varepsilon}\left(N^e_a \vec{e}_i\right)\right)\left(\vec{x}^e(\tilde{\xi}_l)\right)\right)^T \left(\bar{\underline{\underline{D}}}\left(\vec{x}^e(\tilde{\xi}_l)\right)\right)\left(\left(\underline{\varepsilon}\left(N^e_b \vec{e}_j\right)\right)\left(\vec{x}^e(\tilde{\xi}_l)\right)\right) j^e(\tilde{\xi}_l)\, \tilde{w}_l$$

$$+ \sum_{\tilde{l}=1}^{\tilde{n}_q} \lambda\left(\vec{x}^e(\tilde{\tilde{\xi}}_{\tilde{l}})\right)\left(\vec{\nabla}\cdot\left(N^e_a \vec{e}_i\right)\right)\left(\vec{x}^e(\tilde{\tilde{\xi}}_{\tilde{l}})\right)\left(\vec{\nabla}\cdot\left(N^e_b \vec{e}_j\right)\right)\left(\vec{x}^e(\tilde{\tilde{\xi}}_{\tilde{l}})\right) j^e(\tilde{\tilde{\xi}}_{\tilde{l}})\, \tilde{\tilde{w}}_{\tilde{l}}$$

where $\left\{\left(\tilde{\xi}_l, \tilde{w}_l\right)\right\}_{l=1}^{n_q}$ are the quadrature points and weights associated with full integration and $\left\{\left(\tilde{\tilde{\xi}}_{\tilde{l}}, \tilde{\tilde{w}}_{\tilde{l}}\right)\right\}_{\tilde{l}=1}^{\tilde{n}_q}$ are the quadrature points and weights associated with reduced integration. As $\lambda \to \infty$, we see this approach yields the constraints:

$$(\vec{\nabla} \cdot \vec{u}^h)(\vec{x}^e(\hat{\vec{\xi}}_{\hat{l}})) = 0 \quad \text{for } \hat{l} = 1, \ldots, \hat{n}_q \text{ and } e = 1, \ldots, n_{el}$$

at the quadrature points associated with reduced quadrature rather than the stronger constraint:

$$(\vec{\nabla} \cdot \vec{u}^h)(\vec{x}) = 0 \quad \text{for } \vec{x} \in \Omega^h$$

Thus selective reduced integration weakens the incompressibility constraint just like a mixed finite element approximation. In fact, in many cases, selective reduced integration yields the displacement solution associated with a particular mixed finite element approximation. To see this, suppose that Q^h is a finite element pressure space with discontinuous pressure approximations and that the following properties apply:

1. The number of nonzero pressure basis functions $\hat{n}_{en}$ over each element is equal to the number of quadrature points $\hat{n}_q$ associated with reduced quadrature.

2. Reduced quadrature fully integrates the gradient and mass matrices.

Then we can choose the pressure basis functions so they are interpolatory at the reduced quadrature points:

$$\hat{N}_{\hat{A}}\left(\vec{x}^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right)\right) = \begin{cases} 1 & \text{if } \hat{A} = \mathrm{IENP}(\check{\ell}, e) \\ 0 & \text{otherwise} \end{cases}$$

Recall the finite element solution $(\vec{u}^h, p^h)$ of the mixed finite element approximation associated with $(\mathcal{S}^h, \mathcal{V}^h, Q^h)$ satisfies:

$$\bar{b}(\vec{u}^h, \vec{w}^h) - \int_\Omega p^h \vec{\nabla}\cdot\vec{w}^h \, d\Omega = \ell(\vec{w}^h) \quad \forall \vec{w}^h \in \mathcal{V}^h$$

$$\int_\Omega q^h \vec{\nabla}\cdot\vec{u}^h \, d\Omega + \int_\Omega \tfrac{1}{\lambda} p^h q^h \, d\Omega = 0 \quad \forall q^h \in Q^h$$

Thus:

$$\int_\Omega \hat{N}_{\hat{A}} \vec{\nabla}\cdot\vec{u}^h \, d\Omega + \int_\Omega \tfrac{1}{\lambda} p^h \hat{N}_{\hat{A}} \, d\Omega = 0 \quad \forall \hat{A} \in \hat{\eta}$$

By assumption:

$$\int_\Omega \hat{N}_{\hat{A}} \vec{\nabla}\cdot\vec{u}^h \, d\Omega = \left(\vec{\nabla}\cdot\vec{u}^h\right)\left(\vec{x}^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right)\right) j^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right) \hat{\hat{w}}_{\hat{\ell}}$$

$$\int_\Omega \tfrac{1}{\lambda} p^h \hat{N}_{\hat{A}} \, d\Omega = \tfrac{1}{\lambda}\left(\vec{x}^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right)\right) p^h\left(\vec{x}^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right)\right) j^e\left(\hat{\hat{\xi}}_{\hat{\ell}}\right) \hat{\hat{w}}_{\hat{\ell}}$$

where $\hat{A} = \mathrm{IENP}(\check{\ell}, e)$, so:

$$p^h(\vec{x}^e(\tilde{\xi}_{\tilde{l}})) = -\lambda(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))(\vec{\nabla}\cdot\vec{u}^h)(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))$$

Also by assumption:

$$-\int_\Omega p^h \vec{\nabla}\cdot\vec{w}^h \, d\Omega = -\sum_{e=1}^{n_{el}} \sum_{\tilde{l}=1}^{\tilde{n}_q} p^h(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))(\vec{\nabla}\cdot\vec{w}^h)(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))j^e(\tilde{\xi}_{\tilde{l}})\tilde{w}_{\tilde{l}}$$

so it holds that:

$$\bar{b}(\vec{u}^h, \vec{w}^h) + \bar{\bar{b}}(\vec{u}^h, \vec{w}^h) = l(\vec{w}^h) \quad \forall \vec{w}^h \in \mathcal{V}^h$$

where:

$$\bar{\bar{b}}(\vec{u}^h, w^h) :=$$

$$\underbrace{\sum_{e=1}^{n_{el}} \sum_{\tilde{l}=1}^{\tilde{n}_q} \lambda(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))(\vec{\nabla}\cdot\vec{u}^h)(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))(\vec{\nabla}\cdot\vec{w}^h)(\vec{x}^e(\tilde{\xi}_{\tilde{l}}))j^e(\tilde{\xi}_{\tilde{l}})\tilde{w}_{\tilde{l}}}_{\text{Result of Applying Reduced Quadrature Rule to: } \int_\Omega \lambda \vec{\nabla}\cdot\vec{u}^h \, \vec{\nabla}\cdot\vec{w}^h \, d\Omega}$$

Thus the finite element displacement solution associated with the mixed finite element approximation with $(\mathcal{S}^h, \mathcal{V}^h, Q^h)$ is the same as that associated with a primal finite element approximation with $(\mathcal{U}^h, \mathcal{V}^h)$ if the λ-terms are under integrated using the reduced quadrature rule.

In the below table, a number of common selective reduced quadrature schemes are displayed alongside equivalent mixed finite element approximations:

Primal Finite Element	Full Integration Rule	Reduced Integration Rule	Equivalent Mixed Finite Element
$\mathcal{P}^1$	3 Pt. Rule	1 Pt. Rule	$\mathcal{P}^1/\mathcal{P}^0_d$
$\mathcal{Q}^1$	2x2 Pt. Rule	1x1 Pt. Rule	$\mathcal{Q}^1/\mathcal{Q}^0_d$
$\mathcal{P}^2$	6 Pt. Rule	3 Pt. Rule	$\mathcal{P}^2/\mathcal{P}^1_d$
$\mathcal{Q}^2$	3x3 Pt. Rule	2x2 Pt. Rule	$\mathcal{Q}^2/\mathcal{Q}^1_d$
$\mathcal{P}^3$	10 Pt. Rule	6 Pt. Rule	$\mathcal{P}^3/\mathcal{P}^2_d$
$\mathcal{Q}^3$	4x4 Pt. Rule	3x3 Pt. Rule	$\mathcal{Q}^3/\mathcal{Q}^2_d$

Selective reduced integration yields a smaller matrix system than an equivalent mixed finite element approximation, though the resulting stiffness matrix becomes increasingly ill-conditioned as $\lambda \to \infty$.