What is a Finite Element?

Now that we have discussed finite element approximation of steady two-dimensional heat conduction in detail, we turn to the question:

What exactly is a finite element?

A precise definition is given by Ciarlet in his famous text "The Finite Element Method for Elliptic Problems":

A finite element is a triple $(K, \mathcal{P}, \Sigma')$ with:

- $K \subset \mathbb{R}^d$ a connected Lipschitz domain with non-empty interior,
- $\mathcal{P}$ a finite dimensional vector space of functions $p: K \to \mathbb{R}$, and
- $\Sigma' = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ a set of linear forms on $\mathcal{P}$, called local degrees of freedom, such that the mapping $\Lambda_{\Sigma'}: \mathcal{P} \to \mathbb{R}^s$ defined by:

$$\Lambda_{\Sigma'}(p) = \left(\sigma_1(p), \sigma_2(p), \ldots, \sigma_s(p)\right)$$

is bijective.

There is a unique set of basis functions $\{p_a\}_{a=1}^{s}$, called local basis functions, associated with any $(K, \mathcal{P}, \Sigma')$ such that:

$$p = \sum_{a=1}^{s} \sigma_a(p)\, p_a$$

for all $p \in \mathcal{P}$. Given a collection of elements $(K_\alpha, P_\alpha, \Sigma_\alpha)$, a global approximation space and associated global degrees of freedom and basis functions can be constructed from the elements provided the elements meet certain compatibility conditions. Then, global basis functions can be constructed by gluing together local functions from neighboring elements. In fact, each of the finite element approximation spaces discussed previously can be constructed in such a fashion.

An isoparametric Lagrange triangular finite element of degree k is:

$$(\Omega^e, \mathcal{P}^k(\Omega^e), \Sigma^k(\Omega^e))$$

With:

$$\Omega^e := \{ \vec{x}^e(\vec{\xi}) : \vec{\xi} \in \hat{\Omega}_{tri} \}$$

$$\mathcal{P}^k(\Omega^e) := \{ \hat{p} \circ \vec{\xi}^e : \hat{p} \in \mathcal{P}^k(\hat{\Omega}_{tri}) \}$$

$$\Sigma^k(\Omega^e) := \{ \sigma_a^k \}_{a=1}^{n_{en}} \quad \text{with} \quad \sigma_a^k(p) = p(\vec{x}_a^e)$$

The local basis functions $\{ N_a^e \}_{a=1}^{n_{en}}$ are then simply:

$$N_a^e = \hat{N}_a \circ \vec{\xi}^e$$

We glue together these local basis functions using an element connectivity over each element:

$$N_A(\vec{x}) = \begin{cases} N_a^e(\vec{x}) & \text{if there is an } a \text{ such that } A = IEN(a,e) \\ 0 & \text{otherwise} \end{cases}$$

Provided the local nodes are related to the global nodes via:

$$\vec{x}_a^e = \vec{x}_A \qquad \text{for } A = IEN(a,e)$$

and the triangular mesh is conforming, the global basis functions are C^0-continuous.

Since we define isoparametric Lagrange triangular finite element using nodal degrees of freedom, we often refer to such elements as n_{en}-node elements. For instance, we call an isoparametric Lagrange triangular finite element of degree one a three-node triangular element, an isoparametric Lagrange triangular of degree two a six-node triangular element, and an isoparametric Lagrange triangular of degree three a ten-node triangular element. We often also call these elements linear, quadratic, and cubic triangular elements, though this terminology is not precise as it does not indicate what the degrees of freedom are taken to be. That is, there are other linear, quadratic, and cubic triangular elements besides Lagrange elements.

An isoparametric Lagrange tensor-product quadrilateral finite element of degree k is:

$$\left(\Omega^e, Q^k(\Omega^e), \Sigma^k(\Omega^e)\right)$$

with:

$$\Omega^e := \left\{ \vec{x}^e(\vec{\xi}) : \vec{\xi} \in \hat{\Omega}_{quad} \right\}$$

$$Q^k(\Omega^e) := \left\{ \hat{p} \circ \vec{\xi}^e : \hat{p} \in Q^k(\hat{\Omega}_{quad}) \right\}$$

$$\Sigma^k(\Omega^e) := \left\{ \sigma_a^k \right\}_{a=1}^{n_{en}} \quad \text{with} \quad \sigma_a^k(p) = p(\vec{x}_a^e)$$

while an isoparametric Lagrange serendipity quadrilateral finite element of degree k is:

$$\left(\Omega^e, \mathcal{S}^k(\Omega^e), \Sigma^k(\Omega^e)\right)$$

with:

$$\Omega^e := \left\{ \vec{x}^e(\vec{\xi}) : \vec{\xi} \in \hat{\Omega}_{quad} \right\}$$

$$\mathcal{S}^k(\Omega^e) := \left\{ \hat{p} \circ \vec{\xi}^e : \hat{p} \in \mathcal{S}^k(\hat{\Omega}_{quad}) \right\}$$

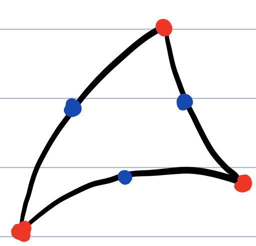
$$\Sigma^k(\Omega^e) := \left\{ \sigma_a^k \right\}_{a=1}^{n_{en}} \quad \text{with} \quad \sigma_a^k(p) = p(\vec{x}_a^e)$$

The local basis functions $\left\{ N_a^e \right\}_{a=1}^{n_{en}}$ are then simply:

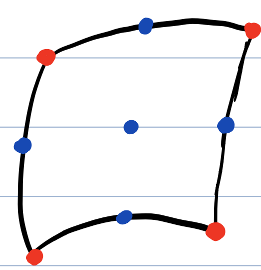
$$N_a^e = \hat{N}_a \circ \vec{\xi}^e$$

for both elements, and these can be glued together to form global basis functions using an element connectivity just as described earlier for isoparametric triangular finite elements.

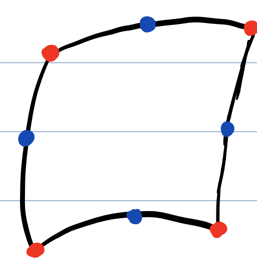
We also refer to isoparametric Lagrange tensor-product and serendipity quadrilateral finite elements as n_{en}-node elements. For instance, we call an isoparametric Lagrange tensor-product element of degree one a four-node quadrilateral element, an isoparametric Lagrange tensor-product of degree two a nine-node quadrilateral element, and an isoparametric Lagrange tensor-product element of degree three a sixteen-node quadrilateral element. Likewise, we call isoparametric Lagrange serendipity elements of degrees two and three eight- and twelve-node quadrilateral elements, respectively. Note we also often call isoparametric Lagrange tensor-product elements of degrees one, two, and three bilinear, biquadratic, and bicubic elements, though this terminology again is not precise.



Six-Node Triangular Element



Nine-Node Quadrilateral Element



Eight-Node Quadrilateral Element