Plane Strain Elastostatics : Mixed Finite Element Approximations:

During the last lecture, we first discussed the construction of mixed displacement-pressure Galerkin approximations for the plane strain elastostatics problem before discussing the behavior of such approximations in the incompressibility limit as $\nu \rightarrow 0.5$. We now turn to a discussion of mixed displacement-pressure finite element approximations for the plane strain elastostatics problem.

The simplest mixed displacement-pressure finite element approximations utilize the same finite element approximation space to represent the displacement and pressure fields. Such approximations are referred to as equal-order approximations. Equal-order triangular finite element approximations utilize the following trial solutions and test functions:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D,i}, i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right)\right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D,i}, i=1,2 \right\}$$

$$Q^h := \mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right)$$

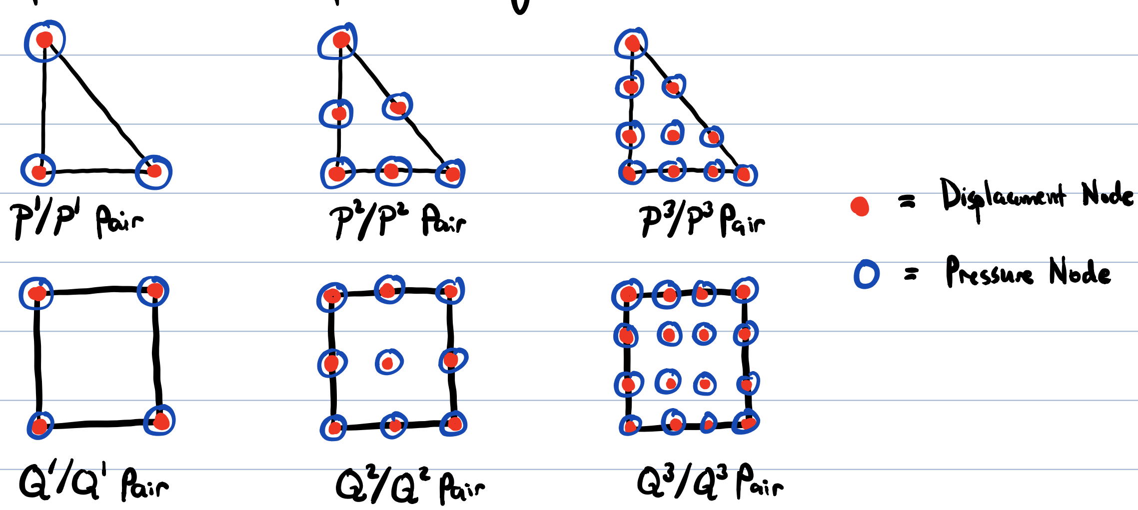
while equal-order tensor-product quadrilateral finite element approximations utilize the following trial solutions and test functions:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right) \right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right) \right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$Q^h := Q^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right)$$

where $k \geq 1$ is the desired polynomial degree. Equal-order triangular finite element approximations of degree k are commonly referred to as P^k/P^k approximations as they employ a P^k/P^k displacement-pressure finite element pair, while equal-order tensor-product quadrilateral finite element approximations of degree k are commonly referred to as Q^k/Q^k approximations as they employ a Q^k/Q^k displacement-pressure finite element pair. Visually:



Since the number of nodes is the same for the displacement and pressure fields, the ratio:

$$r := n_{eq} / \hat{n}_{eq}$$

is simply:

$$r = \frac{2\, n_{nod} - n_{nod_{D_1}} - n_{nod_{D_2}}}{n_{nod}}$$

for equal-order approximations where:

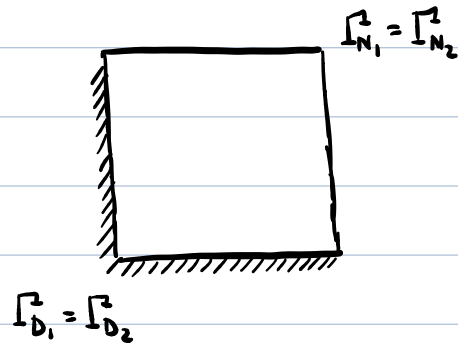
$$n_{nod} := \text{Number of Nodes } (|\eta|)$$
$$n_{nod_{D_1}} := \text{Number of Dirichlet-1 Nodes } (|\eta_{D_1}|)$$
$$n_{nod_{D_2}} := \text{Number of Dirichlet-2 Nodes } (|\eta_{D_2}|)$$

Since $(n_{nod_{D_1}} + n_{nod_{D_2}}) / n_{nod} \to 0$ as $n_{nod} \to \infty$:

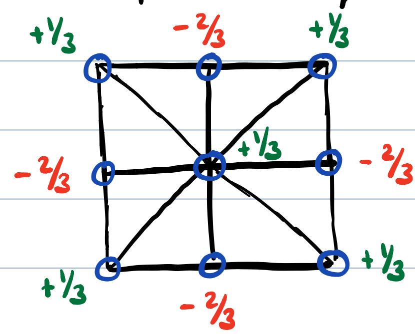
$$r \to 2 \quad \text{as} \quad n_{nod} \to \infty$$

However, for finite n_{nod}, $r < 2$, suggesting that equal-order approximations may not satisfy a Babuška-Brezzi condition. Indeed, equal-order approximations suffer from having spurious pressure modes. Consider, for instance, plane strain elastostatics within a square domain subject to Dirichlet displacement boundary conditions in the first and

Second directions along the left and bottom sides:



Then the following P^1/P^1 pressure field q^h with degrees of freedom $\underline{q}$:



is a spurious pressure mode. To see this, suppose the nodes are locally numbered such that $q^h(\vec{x}^e_1) = -2/3$ and $q^h(\vec{x}^e_2) = q^h(\vec{x}^e_3)$. Then:

$$\int_{\Omega^e} q^h \, \vec{\nabla} \cdot \vec{v}^h \, d\Omega = \vec{\nabla} \cdot \vec{v}^h \Big|_{\Omega^e} j^e \int_{\hat{\Omega}} \left(-2/3 \, \hat{N}_1 + 1/3 \left(\hat{N}_2 + \hat{N}_3 \right) \right) d\hat{\Omega}$$

for all $\vec{v}^h \in \mathcal{V}^h$ for the e^{th} element since $\vec{\nabla} \cdot \vec{v}^h$ and j^e are constant over the element. A direct computation shows:

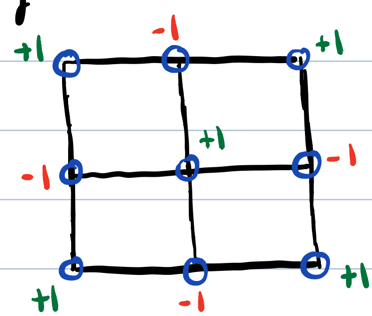
$$\int_{\hat{\Omega}} \left(-2/3 \, \hat{N}_1 + 1/3 \left(\hat{N}_2 + \hat{N}_3 \right) \right) d\hat{\Omega} = \int_0^1 \int_0^{1-\xi_2} \left(-2/3 + \xi_1 + \xi_2 \right) d\hat{\Omega}$$

$$= 0$$

so indeed:

$$\int_{\Omega^e} q^h \vec{\nabla} \cdot \vec{v}^h \, d\Omega = 0 \quad \text{for all } \vec{v}^h \in \mathcal{V}^h$$

and q^h is a spurious pressure mode. Since q^h alternates between two values at the nodes, $-2/3$ and $+1/3$, it is referred to as a checkerboard mode. It also turns out the following Q^1/Q^1 pressure field q^h with degrees of freedom q:



is also a spurious pressure mode of checkerboard type. Showing this is left as an exercise to the reader. While equal-order approximations suffer from having spurious pressure modes, they do outperform primal finite element approximations for the nearly incompressible case $\nu \approx 0.5$. In the incompressible case $\nu = 0.5$, the big matrix associated with an equal-order approximation:

$$\begin{bmatrix} \underline{\underline{K}} & \underline{\underline{G}} \\ -\underline{\underline{G}}^T & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{d} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{F} \\ \underline{H} \end{bmatrix}$$

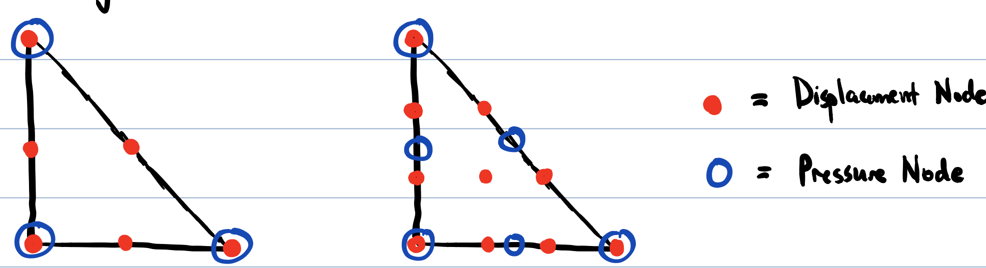
is singular, indicating equal-order approximations cannot be applied in the incompressible limit.

To recover the Babuška-Brezzi condition, we can do one of two things:

(i) We can decrease the number of pressure degrees of freedom, or

(ii) We can increase the number of displacement degrees of freedom.

One way to decrease the number of pressure degrees of freedom is to lower the polynomial degree of the pressure approximations. This yields the so-called Taylor-Hood approximations. The Taylor-Hood P^k/P^{k-1} approximations employ P^k/P^{k-1} displacement-pressure finite element pairs for degree $k \geq 2$:



P^2/P^1 Pair $\qquad$ P^3/P^2 Pair

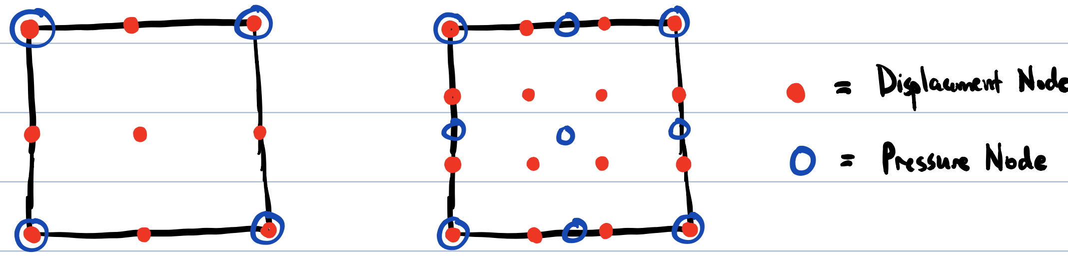
resulting in:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(P^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right) \right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(P^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}} \right) \right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$Q^h := \mathcal{P}^{k-1}_{cont}\left(\Omega^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)$$

Likewise, the Taylor-Hood Q^k/Q^{k-1} approximations employ Q^k/Q^{k-1} displacement-pressure finite element pairs for degree $k \geq 2$:



Q^2/Q^1 Pair Q^3/Q^2 Pair

resulting in:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\Omega^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i},\ i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\Omega^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i},\ i=1,2 \right\}$$

$$Q^h := Q^{k-1}_{cont}\left(\Omega^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)$$

Except for very particular coarse meshes, Taylor-Hood approximations do satisfy the Babuška-Brezzi condition, and sequences of Taylor-Hood approximations attained using mesh refinement satisfy a uniform Babuška-Brezzi condition. Thus, Taylor-Hood approximations are locking-free, and they can be employed in the incompressible case $\nu = 0.5$. Taylor-Hood approximations are among the most popular mixed displacement-pressure finite element approximations for the plane strain elastostatics problem, and they are also commonly employed in the finite element simulation of viscous incompressible flows. However, the lowest displacement polynomial degree possible with a Taylor-Hood approximation is $k=2$ rather than $k=1$.

One way to increase the dimension of the displacement space is to add additional degrees of freedom to each element associated with bubble basis functions - basis functions whose support lie within a single element. This is the basis of the so-called MINI element introduced by Brezzi, Douglas, and Marini. For an affine triangular mesh $\mathcal{M}^h$, a MINI finite element approximation employs the following trial solutions and test functions:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^1_{+, cont}\left(\mathcal{M}^h\right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}_{+, cont}\left(\mathcal{M}^h\right)\right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i}, i=1,2 \right\}$$

$$Q^h := \mathcal{P}^1_{cont}\left(\mathcal{M}^h\right)$$

where:

$$\mathcal{P}^1_{+,\text{cont.}}(\mathcal{M}^h) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{P}^1_+(\hat{\Omega}), e = 1, \ldots, nel \right\}$$

with:

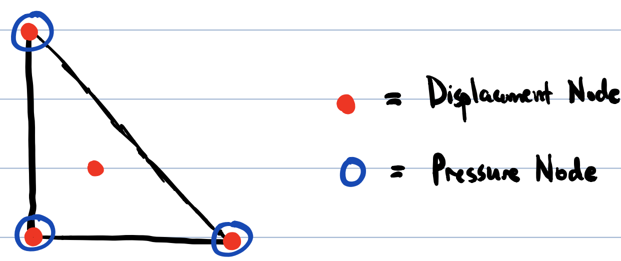
$$\mathcal{P}^1_+(\hat{\Omega}) := \left\{ \hat{v}^h + \alpha \hat{N}_{bub} : v^h \in \mathcal{P}^1(\hat{\Omega}) \text{ and } \alpha \in \mathbb{R} \right\}$$

where:

$$\hat{N}_{bub}(\vec{\xi}) := \hat{N}_1(\vec{\xi}) \hat{N}_2(\vec{\xi}) \hat{N}_3(\vec{\xi})$$

$$\hat{N}_1(\vec{\xi}) := 1 - \xi_1 - \xi_2$$
$$\hat{N}_2(\vec{\xi}) := \xi_1$$
$$\hat{N}_3(\vec{\xi}) := \xi_2$$

is a cubic bubble function defined on the parent element. The element degrees of freedom associated with the MINI element are:



MINI Element

MINI finite element approximations satisfy the Babuška-Brezzi condition, and sequences of MINI finite element approximations attained

attained using mesh refinement satisfy a uniform Babuška-Brezzi condition. MINI finite element approximations are perhaps the simplest and most computational efficient mixed finite element approximations satisfying the Babuška-Brezzi condition. We will later show MINI finite element approximations are closely related to so-called stabilized mixed finite element approximations for the plane strain elastostatics problem. Stabilized finite element methods are the most popular finite element methods for the simulation of fluid flow.

Mixed finite element approximations using discontinuous pressure approximations are also quite common. The simplest such approximations are the P^k/P_d^{k-1} and Q^k/Q_d^{k-1} approximations. P^k/P_d^{k-1} approximations employ the following trial solutions and test functions on a triangular mesh $\mathcal{M}^h$:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D,i}, i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D,i}, i=1,2 \right\}$$

$$Q^h := \mathcal{P}^{k-1}_{dis}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)$$

where:

$$\mathcal{P}^{k-1}_{dis}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right) := \left\{ v^h \in C^{-1}(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{P}^{k-1}(\hat{\Omega}), \ e = 1, \dots, n_{el} \right\}$$

and $C^{-1}(\bar{\Omega}^h)$ is the set of discontinuous functions over the finite element domain. Q^k/Q^{k-1}_d approximations employ the following trial solutions and test functions on a quadrilateral mesh $\mathcal{M}^h$:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v^h_i(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i}, \ i = 1,2 \right\}$$

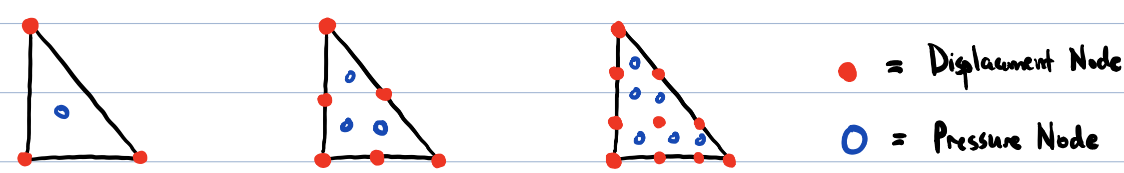
$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(Q^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)\right)^2 : v^h_i(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i}, \ i = 1,2 \right\}$$

$$Q^h := Q^{k-1}_{dis}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right)$$

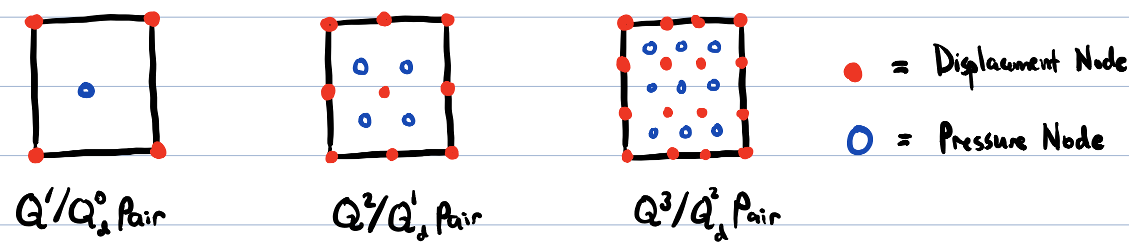
where:

$$Q^{k-1}_{dis}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right) := \left\{ v^h \in C^{-1}(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in Q^{k-1}(\hat{\Omega}), \ e = 1, \dots, n_{el} \right\}$$

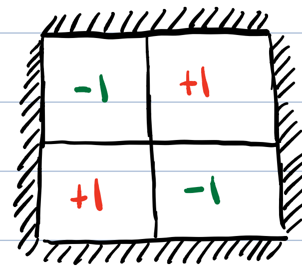
Visually:



P^1/P^0_d Pair $\qquad$ P^2/P^1_d Pair $\qquad$ P^3/P^2_d Pair



Above, we display the pressure nodes as being within the element since continuity is not enforced between adjacent elements. As it turns out that, except for special meshes, P^k/P^{k-1}_d and Q^k/Q^{k-1}_d approximations do not satisfy the Babuška-Brezzi condition. In fact, for affine meshes, P^k/P^{k-1}_d approximations yield the same displacement field as a primal finite element approximation using an affine triangular finite element approximation space. However, Q^k/Q^{k-1}_d approximations are often quite effective in practice, and they are much more effective in simulating the structural response of nearly incompressible materials than primal finite element approximations using a quadrilateral finite element approximation space. This is due to the fact that Q^k/Q^{k-1}_d approximations satisfy a weakened inf-sup condition. In practice, the influence of spurious pressure modes, such as the following checkerboard mode exhibited by Q^1/Q^0_d approximations:



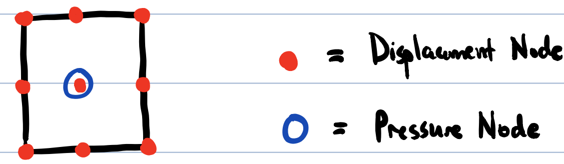
is alleviated by projecting the pressure solution into a different pressure space. This is referred to as pressure smoothing.

One convenient aspect of using discontinuous pressure approximations is one can solve for the pressure degrees of freedom element-by-element in terms of the displacement degrees of freedom, yielding a matrix system of the form:

$$\underbrace{\left(\underline{\underline{\bar{K}}} + \underline{\underline{G}}\,\underline{\underline{M}}^{-1}\underline{\underline{G}}^T\right)}_{\underline{\underline{K}}} \underline{d} = \underbrace{\left(\underline{\bar{F}} - \underline{\underline{G}}\,\underline{\underline{M}}^{-1}\underline{H}\right)}_{\underline{F}}$$

for the unknown displacement degrees of freedom. This is referred to as static condensation. For affine meshes, the attained stiffness matrix $\underline{\underline{K}}$ is the same as the stiffness matrix associated with a primal finite element approximation if one were to underintegrate the λ-contributions to the stiffness matrix. This was first noted by Malkus and Hughes. We will discuss selective reduced integration of the stiffness matrix later in this class.

While P^k/P^{k-1}_d and Q^k/Q^{k-1}_d approximations do not satisfy the Babuška-Brezzi condition, there are mixed finite element approximations with discontinuous pressure approximations that do. For instance, the Q^2/Q^0_d approximation does:



Q^2/Q^0_d Pair

However, Q^2/Q^0_d approximations suffer from suboptimal convergence of the displacement field. Other more exotic mixed finite element approximations with discontinuous pressure approximations also exist, but they will not be discussed in more detail here.

A large number of other mixed finite element approximations other than the ones presented here have been proposed and used on a near daily basis in academia, government, and industry. They each have their relative advantages and disadvantages, both in terms of accuracy, robustness, and computational cost. Sadly, there is only so much that can be presented here, so I have limited our discussion to the mixed finite element approximations presented here. The reader is encouraged to do a deep dive into the research literature if they are interested in learning more about this topic.