Plane Strain Elastostatics: Stabilized Mixed Finite Element Approximations:

Consider a MINI mixed finite element approximation of the plane strain elastostatics problem. For this approximation:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^1_{+,\,\text{cont}}\left(\mathcal{M}^h\right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \quad \text{for } A \in \eta_{D_i},\ i=1,2 \right\}$$

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^1_{+,\,\text{cont}}\left(\mathcal{M}^h\right)\right)^2 : v_i^h(\vec{x}_A) = 0 \quad \text{for } A \in \eta_{D_i},\ i=1,2 \right\}$$

$$Q^h := \mathcal{P}^1_{\text{cont}}\left(\mathcal{M}^h\right)$$

We can decompose the displacement trial set and test space into a linear part and a bubble part:

$$\mathcal{S}^h = \mathcal{S}^h_L \oplus \mathcal{V}^h_b$$

$$\mathcal{V}^h = V^h_L \oplus \mathcal{V}^h_b$$

where:

$$\mathcal{S}^h_L := \left\{ \vec{v}^h \in \left(\mathcal{P}^1_{\text{cont}}\left(\mathcal{M}^h\right)\right)^2 : v_i^h(\vec{x}_A) = g_i(\vec{x}_A) \quad \text{for } A \in \eta_{D_i},\ i=1,2 \right\}$$

$$\mathcal{V}^h_L := \left\{ \vec{v}^h \in \left(\mathcal{P}^1_{cont}(\mathcal{M}^h)\right)^2 : v^h_i(\vec{x}_A) = 0 \text{ for } A \in \eta_{D_i}, i = 1,2 \right\}$$

$$\mathcal{V}^h_b := \left\{ \vec{v}^h \in C^0(\Omega^h) : \vec{v}^h \circ \vec{x}^e = \vec{\alpha} \hat{N}_{bub} \text{ where } \alpha \in \mathbb{R}^2 \right\}$$

meaning each $\vec{v}^h \in \mathcal{S}^h$ admits the unique decomposition $\vec{v}^h_L + \vec{v}^h_b$ with $\vec{v}^h_L \in \mathcal{S}^h_L$ and $\vec{v}^h_b \in \mathcal{V}^h_b$ and each $\vec{v}^h \in \mathcal{V}^h$ admits the unique decomposition $\vec{v}^h_L + \vec{v}^h_b$ with $\vec{v}^h_L \in \mathcal{V}^h_L$ and $\vec{v}^h_b \in \mathcal{V}^h_b$. Thus, the MINI mixed finite element approximation admits the alternative form:

$$(G_{MINI}) \begin{cases} \text{Find } \vec{u}^h_L \in \mathcal{S}^h_L, \vec{u}^h_b \in \mathcal{S}^h_b, \text{ and } p^h \in Q^h \text{ such that:} \\ \bar{b}(\vec{u}^h_L, \vec{w}^h_L) + \bar{b}(\vec{u}^h_b, \vec{w}^h_L) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h_L \, d\Omega \\ + \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h_L + \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h_b + \int_\Omega 1/\lambda \, p^h q^h \, d\Omega = \ell(\vec{w}^h_L) \\ \text{for all } \vec{w}^h_L \in \mathcal{V}^h_L \text{ and } q^h \in Q^h \text{ and:} \\ \bar{b}(\vec{u}^h_L, \vec{w}^h_b) + \bar{b}(\vec{u}^h_b, \vec{w}^h_b) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h_b \, d\Omega = \ell(\vec{w}^h_b) \\ \text{for all } \vec{w}^h_b \in \mathcal{V}^h_b. \end{cases}$$

We can analytically solve:

$$\bar{b}(\vec{u}^h_L, \vec{w}^h_b) + \bar{b}(\vec{u}^h_b, \vec{w}^h_b) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h_b \, d\Omega = \ell(\vec{w}^h_b)$$

for all $\vec{w}^h_b \in \mathcal{Y}^h_b$

for the bubble displacement solution $\vec{u}^h_b \in \mathcal{Y}^h_b$ in terms of the linear displacement solution $\vec{u}^h_L \in \mathcal{S}^h_b$ and pressure solution $p^h \in Q^h$ and insert this expression into:

$$\bar{b}(\vec{u}^h_L, \vec{w}^h_L) + \bar{b}(\vec{u}^h_b, \vec{w}^h_L) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h_L \, d\Omega$$

$$+ \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h_L + \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h_b + \int_\Omega \frac{1}{\lambda} p^h q^h \, d\Omega = \ell(\vec{w}^h_L)$$

for all $\vec{w}^h_L \in \mathcal{Y}^h_L$, $q^h \in Q^h$

to arrive at a reduced problem for the linear displacement solution $\vec{u}^h_L \in \mathcal{S}^h_L$ and $p^h \in Q^h$. When μ and $\vec{f}$ are constant over each element in the mesh, we arrive at the reduced problem:

Find $\vec{u}^h_L \in \mathcal{S}^h_L$ and $p^h \in Q^h$

$$\bar{b}(\vec{u}^h_L, \vec{w}^h_L) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h_L \, d\Omega + \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h_L + \int_\Omega \frac{1}{\lambda} p^h q^h \, d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \underline{\underline{\tau}}^e \vec{\nabla} q^h \cdot \vec{\nabla} p^h \, d\Omega = \ell(\vec{w}^h_L) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \underline{\underline{\tau}}^e \vec{\nabla} q^h \cdot \vec{f} \, d\Omega$$

for all $\vec{w}^h_L \in \mathcal{V}^h_L$ and $q^h \in Q^h$ where:

$$\underline{\underline{\tau}}^e := \begin{bmatrix} \dfrac{\int_{\Omega^e} \underline{\varepsilon}(N^e_{bub}\vec{e}_1) \underline{\underline{\bar{D}}}\, \underline{\varepsilon}(N^e_{bub}\vec{e}_1)\, d\Omega}{\int_{\Omega^e} N^e_{bub}\, d\Omega} & \dfrac{\int_{\Omega^e} \underline{\varepsilon}(N^e_{bub}\vec{e}_1) \underline{\underline{\bar{D}}}\, \underline{\varepsilon}(N^e_{bub}\vec{e}_2)\, d\Omega}{\int_{\Omega^e} N^e_{bub}\, d\Omega} \\ \dfrac{\int_{\Omega^e} \underline{\varepsilon}(N^e_{bub}\vec{e}_2) \underline{\underline{\bar{D}}}\, \underline{\varepsilon}(N^e_{bub}\vec{e}_1)\, d\Omega}{\int_{\Omega^e} N^e_{bub}\, d\Omega} & \dfrac{\int_{\Omega^e} \underline{\varepsilon}(N^e_{bub}\vec{e}_2) \underline{\underline{\bar{D}}}\, \underline{\varepsilon}(N^e_{bub}\vec{e}_2)\, d\Omega}{\int_{\Omega^e} N^e_{bub}\, d\Omega} \end{bmatrix}^{-1}$$

and $N^e_{bub} \circ \vec{x}^e = \hat{N}_{bub}$. The above problem is identical to that of a $\mathcal{P}^1/\mathcal{P}^1$ mixed finite element approximation except for the terms appearing in blue. We say the $\mathcal{P}^1/\mathcal{P}^1$ mixed finite element approximation is unstable as it does not satisfy the Babuška-Brezzi condition while the MINI mixed finite element approximation is stable as it does satisfy the Babuška-Brezzi condition. The blue terms above thus act to stabilize the pressure field associated with the unstable $\mathcal{P}^1/\mathcal{P}^1$ displacement-pressure pair, and we refer to the act of adding bubble degrees of freedom to the $\mathcal{P}^1/\mathcal{P}^1$ displacement-pressure pair as bubble stabilization.

For any $\mathcal{P}^k/\mathcal{P}^k$ or $\mathcal{Q}^k/\mathcal{Q}^k$ displacement-pressure pair, we can add bubble degrees of freedom of degree $k+2$ to stabilize the pair. However, it is simpler to directly employ a stabilized mixed finite element approximation similar in form to the reduced problem attained earlier for the MINI mixed finite element approximation. For instance, it is quite common to employ the so-called PSPG (Pressure Stabilizing Petrov-Galerkin) method for this purpose. The stabilized mixed finite element approximation associated with the PSPG method is:

Find $\vec{u}^h \in \mathcal{S}^h$ and $p^h \in Q^h$ such that:

$$\bar{b}(\vec{u}^h_L, \vec{w}^h_L) - \int_{\Omega} p^h \vec{\nabla} \cdot \vec{w}^h_L \, d\Omega + \int_{\Omega} q^h \vec{\nabla} \cdot \vec{u}^h_L + \int_{\Omega} 1/\lambda \, p^h q^h \, d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau^e \vec{\nabla} q^h \cdot \vec{R}_u(\vec{u}^h, p^h) \, d\Omega = \ell(\vec{w}^h_L)$$

where $\tau^e \sim (h^e)^2/\mu$ is a stabilization parameter and $\vec{R}_u(\vec{u}^h, p^h)$ is the residual vector:

$$\vec{R}_u(\vec{u}^h, p^h) := \begin{bmatrix} -(2\mu u^h_{1,1})_{,1} - (\mu(u^h_{1,2} + u^h_{2,1}))_{,2} + p^h_{,1} - f_1 \\ -(\mu(u^h_{1,2} + u^h_{2,1}))_{,1} - (2\mu u^h_{2,2})_{,2} + p^h_{,2} - f_2 \end{bmatrix}$$

Note that when μ is constant over each element and the P^1/P^1 displacement-pressure pair is employed:

$$R_u(\vec{u}^h, p) = \vec{\nabla} p^h - \vec{f}$$

so the PSPG method collapses to the reduced problem derived earlier for the MINI mixed finite element approximation with one minor difference - the matrix $\underset{\sim}{\tau}^e$ is replaced by a stabilization parameter τ^e.

As the additional term appearing in the PSPG method involves residuals associated with the mixed strong form of the plane strain elastostatics problem, it is in fact a weighted residual method. In fact, the PSPG method is commonly referred to as a residual-based stabilized method. As they are constructed from weighted residuals, residual-based stabilized methods are capable of returning higher-order convergence rates versus non-residual-based stabilized methods. Residual-based stabilized methods are particularly popular in the finite element analysis of incompressible and compressible fluid flows. For instance, the PSPG method is typically paired with the SUPG (Streamline Upwind Petrov-Galerkin) method and grad-div stabilization in the finite element analysis of incompressible fluid flow.