Plane Strain Elastostatics: Finite Element Approximations:

Now that we have presented strong and weak forms for the plane strain elastostatics problem, we next discuss finite element approximation. Finite element approximation of the plane strain elastostatics problem is predicated on the use of a Galerkin approximation with suitably chosen trial solutions and test functions. With this in mind, let:

$$\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{\vec{x}_A\}_{A=1}^{n_{nod}}\right) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{P}^k(\hat{\Omega}), e = 1, \ldots, n_{el} \right\}$$

be a given isoparametric triangular finite element approximation space. In order to be applied to the finite element analysis of plane strain elastostatics the mesh of curved elements must satisfy:

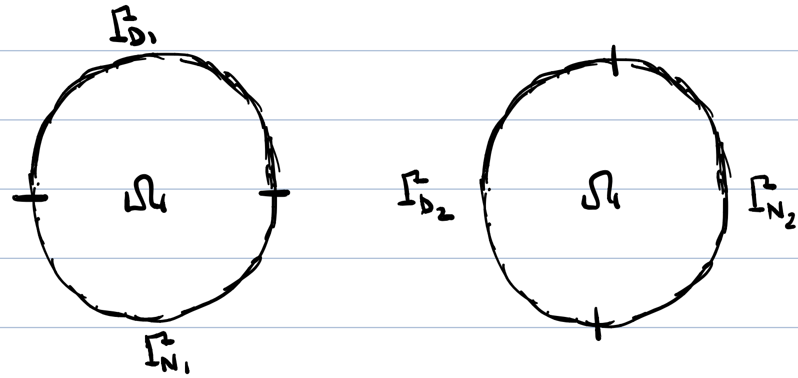
Criterion 1: Each node lie in the domain of interest or its boundary.

Criterion 2: All nodes within or on the ends of a boundary edge lie either in $\bar{\Gamma}_{D_1}$ or $\bar{\Gamma}_{N_1}$.

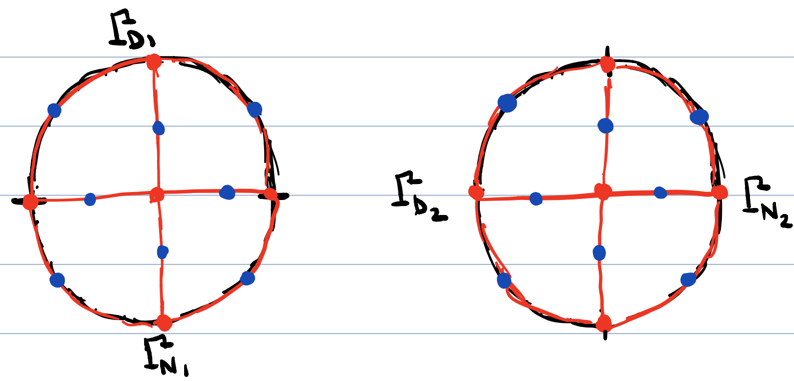
Criterion 3: All nodes within or on the ends of a boundary edge lie either in $\bar{\Gamma}_{D_2}$ or $\bar{\Gamma}_{N_2}$.

Visually:

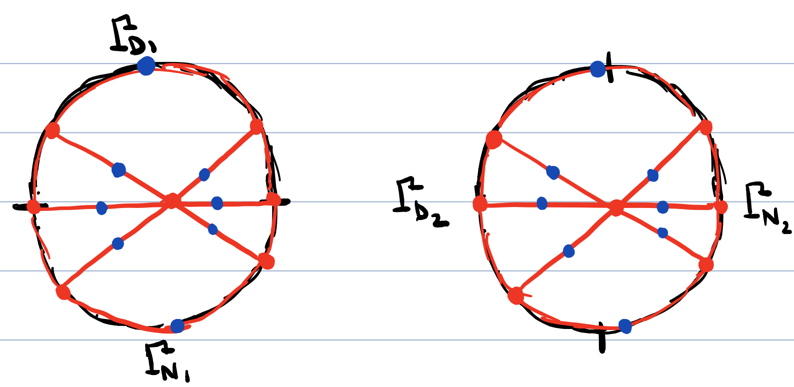
$\boxed{k = 2}$



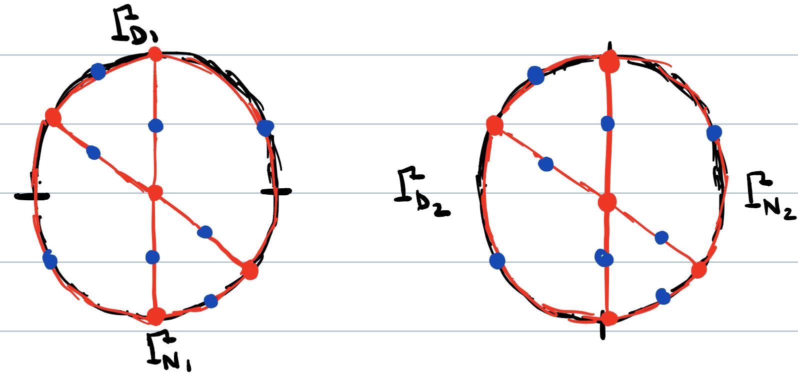
Domain of Interest



Mesh Satisfying All Three Criteria



Mesh Satisfying Criteria 1 & 2 But Not 3



Mesh Satisfying Criteria 1 & 3 But Not 2

If the above hold, then we can define the following Dirichlet and Neumann boundary meshes:

$$\mathcal{E}^h_{D_1} := \left\{ \Gamma^e \in \mathcal{E}^h_{\Gamma} : \vec{x}_A \in \overline{\Gamma}_{D_1} \text{ for all } \vec{x}_A \in \overline{\Gamma^e} \right\}$$

$$\mathcal{E}^h_{N_1} := \left\{ \Gamma^e \in \mathcal{E}^h_{\Gamma} : \vec{x}_A \in \overline{\Gamma}_{N_1} \text{ for all } \vec{x}_A \in \overline{\Gamma^e} \right\}$$

$$\mathcal{E}^h_{D_2} := \left\{ \Gamma^e \in \mathcal{E}^h_{\Gamma} : \vec{x}_A \in \overline{\Gamma}_{D_2} \text{ for all } \vec{x}_A \in \overline{\Gamma^e} \right\}$$

$$\mathcal{E}^h_{N_2} := \left\{ \Gamma^e \in \mathcal{E}^h_{\Gamma} : \vec{x}_A \in \overline{\Gamma}_{N_2} \text{ for all } \vec{x}_A \in \overline{\Gamma^e} \right\}$$

such that:

$$\mathcal{E}^h_{\Gamma} = \mathcal{E}^h_{D_1} \cup \mathcal{E}^h_{N_1} \quad \text{and} \quad \mathcal{E}^h_{D_1} \cap \mathcal{E}^h_{N_1} = \phi$$

and:

$$\mathcal{E}^h_{\Gamma} = \mathcal{E}^h_{D_2} \cup \mathcal{E}^h_{N_2} \quad \text{and} \quad \mathcal{E}^h_{D_2} \cap \mathcal{E}^h_{N_2} = \phi$$

Moreover, we can define the Dirichlet and Neumann finite element boundaries:

$$\Gamma^h_{D_1} := \text{int}\left(\overline{\bigcup_{\Gamma^e \in \mathcal{E}^h_{D_1}} \Gamma^e} \right) \qquad \Gamma^h_{D_2} := \text{int}\left(\overline{\bigcup_{\Gamma^e \in \mathcal{E}^h_{D_2}} \Gamma^e} \right)$$

$$\Gamma^h_{N_1} := \text{int}\left(\overline{\bigcup_{\Gamma^e \in \mathcal{E}^h_{N_1}} \Gamma^e} \right) \qquad \Gamma^h_{N_2} := \text{int}\left(\overline{\bigcup_{\Gamma^e \in \mathcal{E}^h_{N_2}} \Gamma^e} \right)$$

Such that:

$$\Gamma^h = \overline{\Gamma^h_{D_1} \cup \Gamma^h_{N_1}} \quad \text{and} \quad \Gamma^h_{D_1} \cap \Gamma^h_{N_1} = \emptyset$$

and:

$$\Gamma^h = \overline{\Gamma^h_{D_2} \cup \Gamma^h_{N_2}} \quad \text{and} \quad \Gamma^h_{D_2} \cap \Gamma^h_{N_2} = \emptyset$$

Now define:

$$\eta_{D_1} := \left\{ A \in \eta : \vec{x}_A \in \overline{\Gamma^h_{D_1}} \right\}$$

and:

$$\eta_{D_2} := \left\{ A \in \eta : \vec{x}_A \in \overline{\Gamma^h_{D_2}} \right\}$$

where:

$$\eta := \{ 1, 2, \ldots, n_{nod} \}$$

is the set of node numbers. We then select:

$$\mathcal{S}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h; \{ \vec{x}_A \}_{A=1}^{n_{nod}} \right) \right)^2 : v^h_i(\vec{x}_A) = g_i(\vec{x}_A) \text{ for } A \in \eta_{D_i},\ i=1,2 \right\}$$

as the set of finite element trial solutions and:

$$\mathcal{V}^h := \left\{ \vec{v}^h \in \left(\mathcal{P}^k_{cont}\left(\mathcal{M}^h ; \{ \vec{x}_A \}_{A=1}^{n_{nod}} \right) \right)^2 : v_i^h(\vec{x}_A) = 0 \text{ for } A \in \eta_{D,i}, i = 1,2 \right\}$$

as the space of finite element test functions. With these selections, an isoparametric triangular finite element approximation of plane strain elastostatics takes the form:

$$(G) \left\{ \begin{array}{l} \text{Find } \vec{u}^h \in \mathcal{S}^h \text{ such that:} \\ \qquad b^h(\vec{u}^h, \vec{w}^h) = \ell^h(\vec{w}^h) \quad \text{for all } \vec{w}^h \in \mathcal{V}^h \\ \text{where:} \\ \qquad b^h(\vec{u}^h, \vec{w}^h) = \int_{\Omega^h} \underline{\varepsilon}(\vec{w}^h)^T \underline{\underline{D}}\, \underline{\varepsilon}(\vec{u}^h)\, d\Omega \\ \qquad \ell^h(\vec{w}^h) = \int_{\Omega^h} \vec{w}^h \cdot \vec{f}\, d\Omega + \int_{\Gamma^h_{N_1}} w_1^h h_1\, d\Gamma + \int_{\Gamma^h_{N_2}} w_2^h h_2\, d\Gamma \end{array} \right.$$

Note the finite element solution $\vec{u}^h \in \mathcal{S}^h$ of the above problem admits the form:

$$\vec{u}^h(\vec{x}) = \sum_{B \in \eta} \vec{u}^h(\vec{x}_B) N_B(\vec{x})$$

We can write:

$$\vec{u}^h(\vec{x}) = \sum_{B \in \eta - \eta_{D_1}} u_1^h(\vec{x}_B)\, \vec{e}_1\, N_B(\vec{x}) + \sum_{B \in \eta - \eta_{D_2}} u_2^h(\vec{x}_B)\, \vec{e}_2\, N_B(\vec{x}) + \vec{g}^h$$

Unknown DOF   Unknown DOF

where:

$$\vec{g}^h(\vec{x}) = \sum_{B \in \eta_{D_1}} g_1(\vec{x}_B)\, \vec{e}_1\, N_B(\vec{x}) + \sum_{B \in \eta_{D_2}} g_2(\vec{x}_B)\, \vec{e}_2\, N_B(\vec{x})$$

and:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Unit Vectors

Moreover every finite element weighting function $\vec{w}^h \in \mathcal{V}^h$ takes the form:

$$\vec{w}^h(\vec{x}) = \sum_{A \in \eta - \eta_{D_1}} w_1^h(\vec{x}_A)\, \vec{e}_1\, N_A(\vec{x}) + \sum_{A \in \eta - \eta_{D_2}} w_2^h(\vec{x}_A)\, \vec{e}_2\, N_A(\vec{x})$$

where $\{ w_1^h(\vec{x}_A) \}_{A \in \eta - \eta_{D_1}}$ and $\{ w_2^h(\vec{x}_A) \}_{A \in \eta - \eta_{D_2}}$ are arbitrary. It follows that:

$$\sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} b^h(N_B \vec{e}_j, N_A \vec{e}_i)\, u_j^h(\vec{x}_B) = \ell^h(N_A \vec{e}_i) - b^h(\vec{g}^h, N_A \vec{e}_i)$$

for all $A \in \eta - \eta_{D_i}$ for $i = 1, 2$. Now, since we have an equation for each

$A \in \eta - \eta_{D_i}$ for $i=1,2$, we assign a unique equation number between 1 and n_{eq} to $A \in \eta - \eta_{D_i}$ for $i=1,2$ where $n_{eq} = |\eta - \eta_{D_1}| + |\eta - \eta_{D_2}|$. We likewise assign zero to $A \in \eta_{D_i}$ for $i=1,2$. Just as we did for steady two-dimensional heat conduction, we store these assignments using a destination array:

$$ID(A,i) = \begin{cases} P & \text{if } A \in \eta - \eta_{D_i} \\ 0 & \text{if } A \in \eta_{D_i} \end{cases}$$

Note that for each $P = 1, \ldots, n_{eq}$ there is a unique $i=1,2$ and $A \in \eta - \eta_{D_i}$ such that $ID(A,i) = P$. We can then define $\underline{K}$ to be the $n_{eq} \times n_{eq}$ stiffness matrix with entries:

$$K_{PQ} = b^h(N_B \vec{e}_j, N_A \vec{e}_i) \qquad P = ID(A,i), \; Q = ID(B,j)$$

$\underline{F}$ to be the $n_{eq} \times 1$ force vector with entries:

$$F_P = \ell^h(N_A \vec{e}_i) - b^h(\vec{g}^h, N_A \vec{e}_i) \qquad P = ID(A,i)$$

and $\underline{d}$ to be the $n_{eq} \times 1$ displacement vector with entries:

$$d_Q = u_j^h(\vec{x}_B) \qquad Q = ID(B,j)$$

such that the unknown degrees of freedom may be attained via the solution of the matrix system:

$$(L) \begin{cases} \text{Find } \underline{d} \in \mathbb{R}^{n_{eq}} \text{ such that:} \\ \underline{\underline{K}}\, \underline{d} = \underline{F} \end{cases}$$

To construct the above matrix system, we turn to element formation and assembly just as we did for steady two-dimensional heat conduction. This is the focus of the next lecture. It should be finally noted that while we only constructed an isoparametric triangular finite element approximation of plane strain elastostatics above, isoparametric quadrilateral finite element approximations are constructed in exactly the same way.