There is not one finite element method. Rather, there exist entire classes of finite element methods. While finite element method eludes precise definition, we proceed with the following working definitions:

Finite Element Analysis: Analysis of an engineering problem using a **finite element method**.

Finite Element Method: A numerical method for approximating the solution of PDEs involving:

1. A **weak form** of the problem at hand, and
2. Approximation of the **weak solution** using **finite element functions**.

Finite Element Function: A piecewise smooth function with respect to a **mesh**.

Mesh: A decomposition of a domain into **finite elements**.

Finite Element: Either:

1. A basic geometric object such as a **triangle** or **quadrilateral**, or
2. A basic geometric object and **local finite element basis functions** and **DOFs**.

Finite Element Approximation Space: A **vector space** of finite element functions.

Finite Element Basis Functions: A basis spanning a finite element approximation space.

Finite Element DOFs: Coefficients of a finite element basis expansion of a finite element function.

The above definitions are not meant be precise or even universally accepted. In fact, many matrix methods introduced in the 1940s and 1950s, including the famous direct stiffness method, are often said to be finite element methods even though they do not involve the numerical solution of PDEs. This has to do with the fact that concepts such as element formation and assembly, integral components of the direct stiffness method, are commonly thought to be defining characteristics of a finite element method, especially in the engineering community. However, most modern state-of-the-art finite element methods fall into the class of methods encompassed by the above definitions.

A large variety of finite element methods have been proposed over the years, including:

- Conforming and Non-Conforming Finite Element Methods
- Mixed and Hybrid Finite Element Methods
- Stabilized and Multiscale Finite Elements Methods
- Generalized, Meshless, and Immersed Finite Element Methods
- Discontinuous and Continuous-Discontinuous Finite Element Methods
- Isogeometric Finite Element Methods

That being said, our discussion will center largely on the most basic finite element methods, those arising from an application of the Bubnov-Galerkin method.

Finite element analysis has origins in both the engineering and mathematics communities. The origins of finite element analysis in the engineering community lie with the contributions of Hrennikoff, McHenry, Kron, and Levy. In 1941, Hrennikoff solved plane elasticity problems by breaking up the domain into little finite pieces whose stiffness was approximated using bars, beams, and springs. A similar "lattice analogy" was developed by McHenry in 1943. At roughly the same time, Kron developed a "method of tearing" for the generation of global systems from large numbers of individual components, and he applied this method to a number of systems of engineering interest in the 1940s and 1950s. Finally, Levy developed the famous "direct stiffness method" for aircraft structures in 1953. In this method, a complex structural assembly is broken into simple structural members, the stiffness of these structural members is characterized in terms of internal and external forces, and member stiffness relations are then assembled into a system stiffness relation. This relation is finally solved for the system response. However, the direct stiffness method is not a numerical method for solving PDEs. Rather, it is a method for automating the analysis of structural assemblies. Nonetheless, thousands of engineering analyses employing the direct stiffness method and similar matrix methods are still conducted on a daily basis, and formation and assembly of element systems into a global system is a key component of modern finite element codes.

The origins of finite element analysis in the mathematics community are frequently traced to a paper of Courant in 1943 in which piecewise linear approximations on triangular meshes are employed to solve a Dirichlet BVP. However, the approximation of variational problems on a mesh of triangles goes back much further. For instance, in 1851, Schellbach proposed a finite-element-like solution to Plateau's problem of determining the surface of minimum area enclosed by a given curve. Even further back in 1696, Leibnitz, the father of modern calculus, employed a piecewise linear approximation of Brachistochrone curves, curves of fastest descent between two points. Courant's work also relies on earlier work by Rayleigh, Ritz, and Galerkin on the development of approximation methods of variational minimization problems.

In the early 1950s, Argyris began to put all of the ideas put forward by the engineering and mathematics communities together into what some call a primitive finite element method. In particular, Argyris combined Kron's "method of tearing" with variational methods of approximation, a fundamental step in the development of modern finite element methods. In 1956, Turner, Clough, Martin, and Topp employed both a local approximation of the PDEs of linear elasticity and an element-by-element assembly process in what is likely the first modern finite element method. Not too much later, in 1960, Clough dubbed this method as a "finite element method", the first time this term was introduced in the literature[1].

The 1960s were the formative years of finite element analysis. In this decade, engineers realized that they could construct finite element methods from variational principles, and variationally based methods dominated the literature for almost a decade. Virtually every method proposed relied on the use of Ritz's method for approximating the solutions of variational minimization problems. As most structural problems of interest, including rods, beams, membranes, plates, and shells, may be described as variational minimization problems, this decade saw a variety of finite element methods emerge for tackling such applications. Unfortunately, some of the methods introduced in the 1960s are lost to the sands of time, as they were presented at conferences, including the famous Dayton conferences on finite elements held at AFRL, rather than in journal publications. Various mixed and hybrid finite element methods and C^1-finite element approximations were presented for the first time at the Dayton finite element conference series.

Many other problems of interest, such as those arising in fluid mechanics, may not be described as variational minimization problems. As such, finite element methods for such problems were not proposed during the formative years of finite element analysis. However, by the late 1960s and early 1970s, there emerged the realization that the finite element method could be applied to weak forms of such problems via application of Bubnov-Galerkin and Petrov-Galerkin methods. In particular, Oden proposed the first finite element methods for the full Navier-Stokes equations in the late 1960s[2]. A number of techniques were also introduced in the late 1960s and early 1970s which led to more accurate, efficient, and robust finite element analysis procedures. For instance, the patch test was introduced as a methodology for verifying finite element codes, the frontal technique was introduced to speed up linear system solution, numerical integration techniques were introduced to streamline construction of element stiffness matrices, and loading vectors, and isoparametric finite elements were introduced to enable high-order finite element analysis on curved geometries. Coincidentally, all of these techniques were introduced by Irons, a heavyweight in the field, and many of these techniques remain a key component of finite element codes. Finally, the practice of finite element analysis grew in popularity with practicing engineers in the late 1960s and early 1970s in large part due to the 1967 text on finite element analysis by Zienkiewicz and Cheung.

A comprehensive mathematical theory of finite elements also began to emerge in the late 1960s and early 1970s. Early papers, including papers by Varga and Birkhoff, de Boor, Schwartz, and Wendroff, focused on proving convergence of Hermite-based Ritz-Galerkin approximations of one-dimensional problems. Mathematical analysis of two- and higher-dimensional finite element methods began in 1968 with a series of papers on the subject, including papers by Johnson and McLay, Zlamal, Ciarlet, and Oliveira. Of particular interest was the development of a finite element interpolation theory, which focuses on the question of how well a given finite element interpolation can

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[1] Our own Carlos Felippa wrote one of the first dissertations on finite element analysis under Clough in 1966.

[2] The only finite element class I personally have taken is "Advanced finite element methods" with Oden.

approximate functions in a given function space. In a seminal paper in 1972, Ciarlet and Raviart established a comprehensive interpolation theory for Lagrange and Hermite finite elements. However, an even more groundbreaking paper also emerged in 1972. In this paper, Babuska and Aziz presented the famous inf-sup condition which is a necessary and sufficient requirement for stability and convergence of a given finite element method. In a parallel paper in 1974, Brezzi introduced an equivalent condition for constrained elliptic problems, and for this reason, the inf-sup condition is commonly referred to as the Babuska-Brezzi condition in the literature today[3]. The inf-sup condition is of critical importance in the design of stable and convergent mixed and hybrid finite element methods with application to nearly incompressible solids, plates and shells, and fluid flow. The first textbook on mathematical properties of finite element methods by Strang and Fix appeared shortly after the introduction of the inf-sup condition in 1973, Oden and Reddy published an introduction to the mathematical theory of finite element methods the next year, and Ciarlet published his famous treatise on finite element methods for elliptic problems just two years later. Thus, it perhaps comes as no surprise that a solid foundation for the mathematical theory for finite element methods for linear problems was established by 1980, and in the 1980s, focus shifted toward establishing a mathematical theory for finite element methods for nonlinear problems and even variational inequalities such as those governing contact.

There was a flurry of activity in the 1970s and 1980s dedicated to the development of robust, accurate, and efficient finite element methodologies for plates, shells, and nearly incompressible materials. Primal displacement-based finite element methods are known to exhibit parasitic shear, membrane, and volumetric locking when applied to the analysis of plates, shells, and nearly incompressible materials. A variety of techniques have been introduced to alleviate or remove locking altogether in these methods. Various methods such as selective reduced integration and the B-bar and F-bar methods, originally introduced in the 1970s and 1980s, are now a key component of structural finite element analysis codes. It turns out that such methods may be attained via suitable hybridization of mixed displacement-pressure, displacement-stress, or displacement-stress-strain methods. I will introduce such mixed methods in the latter part of this course, and they are covered in more detail in ASEN 6367 Advanced Finite Element Methods for Plates, Shells, and Solids.

There was also a flurry of activity in the 1970s and 1980s dedicated to the development of suitable finite element methodologies for transport and fluid flow. The Streamline Upwind Petrov Galerkin (SUPG) method, the most common finite element method for solving transport and fluid flow problems, was first proposed by Brooks and Hughes at a conference in 1978 and later published in Computer Methods in Applied Mechanics and Engineering in 1982. The research group of Hughes also proposed a number of other methodologies for transport and fluid flow problems during the 1980s, including the Galerkin Least Squares (GLS) method, the Pressure Stabilizing Petrov Galerkin (PSPG) method, and grad-div stabilization. These methods are now commonly referred to as stabilized finite element methods as they rely on residual-based stabilization of nominally unstable Bubnov-Galerkin methods. Stabilized finite element methods are a key component of many academic and industrial codes around the world, including here at the University of Colorado Boulder. For instance, Parallel Hierarchical Stabilized Transient Analysis (PHASTA), Ken Jansen's research code, utilizes SUPG/PSPG/grad-div stabilization to tackle the simulation of viscous incompressible fluid flows. I unfortunately will not have time to discuss stabilized finite element methods in this class, but I cover them in detail in ASEN 6519 Stabilized and Multiscale Finite Element Methods. Stabilized finite element methods are also introduced in ASEN 5331 Computational Fluid Dynamics Unstructured Grid.

In the past three decades, a number of exotic finite element methods have been proposed to tackle emerging problems of engineering interest. In the 1990s, a number of discontinuous Galerkin methods were proposed to solve compressible fluid flow problems. Discontinuous Galerkin methods may be thought of as a marriage between finite element methods and finite volume methods. In particular, they inherit the conservation structure of finite volume methods and the high accuracy of finite element methods. Also in the 1990s, a number of so-called meshless finite element methods were introduced. The primary attraction of a meshless finite element method is you do not need to build a mesh to define the underlying finite element basis functions. However, one typically still needs to create a mesh to perform numerical integration of the system tangent matrix and residual vector. Meshless finite element methods may also be applied to challenging problems which are difficult to tackle using classical finite element method, including soil mechanics and crack propagation. In the 2000s, immersed finite element methods grew

[3] Perhaps my biggest research accomplishment is that I have co-authored papers with both Babuska and Brezzi.

rapidly in popularity, and they continue to grow in popularity today. Like meshless finite element methods, one does not need to build a conformal finite element mesh to define the underlying finite element basis functions. Instead, boundary and interface conditions are enforced in a suitably weak manner. There are a large number of different classes of immersed finite element methods, including the finite cell method, CutFEM, and immersogeometric analysis, to name a few. Personally, I have used immersed finite element methods to tackle the simulation of bio-prosthetic heart valves. This problem involves the coupling of three difficult sub-problems: fluid flow through the heart, motion of the valve, and motion of the heart walls. In fact, the fluid flow domain changes topology as the valve opens and closes, rendering classical finite element methods useless when applied to this challenging fluid-thin structure interaction problem. Immersed finite element methods, on the other hand, may be readily applied. Finally, in the 2000s, isogeometric analysis emerged as a popular alternative to classical finite element methods. Isogeometric analysis tightly integrates finite element analysis with computer aided design descriptions of geometry. As such, isogeometric analysis streamlines integrated design-through-analysis engineering tasks such as shape optimization. I cover isogeometric analysis in detail in ASEN 6519 Isogeometric Methods.

As a final note, I should mention there are now a plethora of open- and closed-source finite element codes out in the world today. The first finite element codes emerged in the 1960s and 1970s, and many of these are still widely used today. The first release of the structural finite element analysis code NASTRAN was back in 1968, while the first commercial version of Ansys was released in 1971. LS-DYNA, a structural finite element analysis code popular for automobile crashworthiness studies, was built on the shoulders of DYNA3D, developed by John Hallquist in 1976. Finally, the past two decades has seen the emergence of a number of open-source finite element analysis platforms. The first release of the deal.II finite element library, for instance, was back in 2000, while the first release of FeniCS was back in 2003. Thousands of practitioners have contributed to both these codes since their initial release, and I anticipate thousands more will contribute in the coming years.