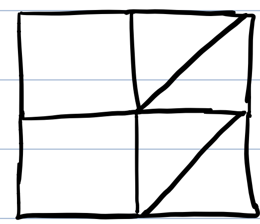
2D Heat Conduction: Mixed Element, Variable Degree, and Non-conforming Meshes:

It is quite common to employ mixed meshes of triangular and quadrilateral elements in practice:



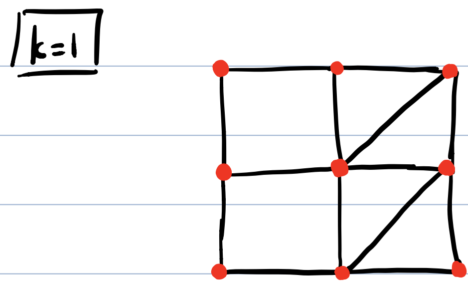
For instance, it is common to utilize quadrilateral elements in the boundary layer and triangular elements in the core region in a fluid flow simulation. One can easily construct finite element approximation spaces over these mixed meshes by combining concepts introduced over the last few lectures. A finite element approximation space of degree k over a mixed element mesh $\mathcal{M}^h := \{\Omega^e\}_{e=1}^{n_{el}}$ takes the form:

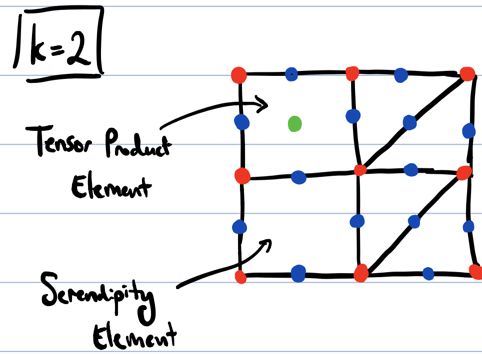
$$\mathcal{M}^k_{cont}(\mathcal{M}^h) := \left\{ v^h \in C^0(\bar{\Omega}^h) : \begin{array}{l} v^h \circ \vec{x}^e \in \mathcal{P}^k(\hat{\Omega}_{tri}) \text{ for triangular elements} \\ v^h \circ \vec{x}^e \in \mathcal{Q}^k(\hat{\Omega}_{quad}) \text{ for tensor product elements} \\ v^h \circ \vec{x}^e \in \mathcal{S}^k(\hat{\Omega}_{quad}) \text{ for serendipity elements} \end{array} \right\}$$

It is easily shown that:

$$\dim\left(\mathcal{M}^k_{cont}(\mathcal{M}^h)\right) = \frac{(k-1)(k-2)}{2} * \left(\begin{array}{c}\text{Number of} \\ \text{Triangular Elements}\end{array}\right) + (k-1)^2 * \left(\begin{array}{c}\text{Number of Tensor} \\ \text{Product Elements}\end{array}\right)$$
$$+ \frac{(k-2)(k-3)}{2} * \left(\begin{array}{c}\text{Number of} \\ \text{Serendipity Elements}\end{array}\right) + (k-1) * n_{edge} + n_{vertex}$$

So nodal degrees of freedom can be attached to vertices, elements, and edges exactly as done previously for pure triangular and quadrilateral meshes:

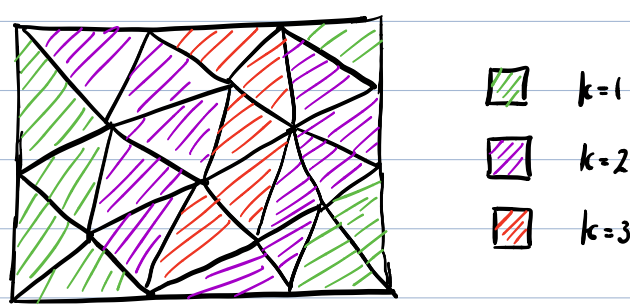




and corresponding Lagrange basis functions can be expressed in terms of triangular, tensor product, and serendipity shape functions over triangular, tensor product, and serendipity elements using element maps and an element connectivity:

$$N_A(\vec{x}) = \begin{cases} \hat{N}_a(\vec{\xi}^e(\vec{x})) & \text{if there is an } a \text{ such that } A = IEN(a,e) \\ 0 & \text{otherwise} \end{cases}$$

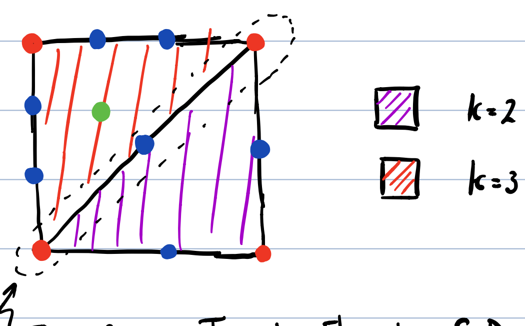
It is also common to use finite element functions of variable polynomial degree over a given mesh:



A finite element approximation space for such a mesh takes the form:

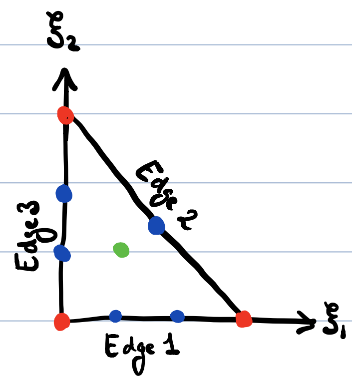
$$\mathcal{M}^{\text{variable}}_{\text{cont}}(\mathcal{M}^h) := \left\{ v^h \in C^0(\bar{\Omega}^h) : \begin{array}{l} v^h \circ \vec{x}^e \in \mathcal{P}^{k_e}(\hat{\Omega}_{tri}) \text{ for triangular elements} \\ v^h \circ \vec{x}^e \in \mathcal{Q}^{k_e}(\hat{\Omega}_{quad}) \text{ for tensor product elements} \\ v^h \circ \vec{x}^e \in \mathcal{S}^{k_e}(\hat{\Omega}_{quad}) \text{ for serendipity elements} \end{array} \right\}$$

where $\{k^e\}_{e=1}^{n_{el}}$ is a set of polynomial degrees for all the elements in the mesh. Unfortunately, it is not as straight-forward to build a basis for a variable degree mesh as it is for a fixed degree mesh. In particular, finite element functions take on the lower of two polynomial degrees along edges between elements of different polynomial degrees k_{lower} and k_{upper}, so only $k_{lower}-1$ nodes may be placed along it:



Edge Between Triangular Elements of Degrees $k=2$ and $k=3$

Consequently, basis functions are not equal to the push forwards of shape functions of degree k_{upper} over the element of degree k_{upper}. To deal with this issue, several different approaches can be taken. First, basis functions can be expressed instead in terms of linear combinations of shape functions of degree k_{upper} over the element of degree k_{upper}. This gives rise to constraint matrices relating local and global degrees of freedom. Second, alternative shape functions can be employed using $k_{lower}-1$ nodes along the shared edge:

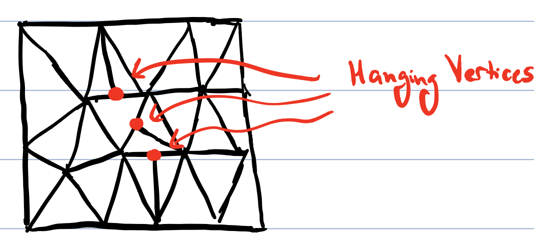


Parent Node Layout for Approximation Space:

$$\mathcal{P} = \left\{ v^h \in \mathcal{P}^3(\hat{\Omega}_h) : v^h\big|_{\text{Edge } 2} \text{ is of degree } 2 \right\}$$

Not too surprisingly, these first two approaches are equivalent. Third, a hierarchical basis may be employed instead of a Lagrange basis. It is much easier to construct variable degree hierarchical basis functions than Lagrange basis functions.

Finally, it is possible to define finite element approximation spaces over non-conforming meshes with hanging vertices:



As in the variable degree setting, basis functions are not equal to the push forwards of canonical shape functions over elements with a hanging vertex. In practice, one instead expresses basis functions as linear combinations of shape functions over such elements using constraint matrices expressing fictitious hanging vertex degrees of freedom in terms of real degrees of freedom.