1D Model Problem: Minimization and Variational Forms:

In the last lecture, we mentioned that the standard weak form is equivalent to the so-called variational form of our 1D model problem. To see this, we first have to discuss the so-called minimization form of our 1D model problem. This minimization form involves so-called functionals, or functions of functions.

To begin, let:

$$\mathcal{S} := \left\{ v \in H^1((0,L)) : v(0) = g_0 \, , v(L) = g_L \right\}$$

be our space of trial solutions, as with the standard weak form. We consider the functional $E : \mathcal{S} \to \mathbb{R}$ defined as:

$$E(v) := \frac{1}{2} \int_0^L \kappa (v_{,x})^2 dx - \int_0^L f v \, dx$$

In the context of structural analysis, E is referred to as the total potential energy, the integral:

$$\frac{1}{2} \int_0^L \kappa (v_{,x})^2 dx$$

is referred to as the elastic strain energy and the integral:

$$- \int_0^L f v \, dx$$

is referred to as the potential energy associated with external loadings. For E to be well-defined, we require $f \in L^2((0,1))$ and $\max_{0 \le x \le L} |K| < \infty$. The minimization form of our 1D model problem is defined as follows:

$$(M) \quad \left\{ \text{Find } u := \underset{v \in \mathcal{S}}{\operatorname{argmin}} \; E(v) \right.$$

It is likely not clear yet that Problem (M) is indeed a form of our 1D model problem, but this will become clear later. If $K > 0$, the functional E is strictly convex, so it has a unique minimizer. To find this minimizer, we use the usual trick of calculus - take a derivative and set it equal to zero! However, there remains a question: How do we take a derivative of E? It's not just a function - it's a function of a function! To define a notion of derivative for functionals, we take inspiration from vector calculus: The directional derivative of a function $f: \mathbb{R}^n \to \mathbb{R}$ in direction $\vec{v} \in \mathbb{R}^n$ is:

$$\vec{\nabla}_{\vec{v}} f(\vec{x}) := \lim_{\varepsilon \to 0} \frac{f(\vec{x} + \varepsilon \vec{v}) - f(\vec{x})}{\varepsilon} = \frac{\partial}{\partial \varepsilon} f(\vec{x} + \varepsilon \vec{v}) \Big|_{\varepsilon = 0}$$

Similarly, we can define the first variation of a functional $\Phi: \mathcal{S} \to \mathbb{R}$ in the direction of a function w as:

$$\delta_w \Phi(v) := \lim_{\varepsilon \to 0} \frac{\Phi(v+\varepsilon w) - \Phi(v)}{\varepsilon} = \frac{\partial}{\partial \varepsilon} \Phi(v+\varepsilon w)\Big|_{\varepsilon=0}$$

Now above, we evaluate Φ at $v + \varepsilon w$, so we require that the sum $v + \varepsilon w \in \mathcal{S}$. This means necessarily that w must be in:

$$\mathcal{V} := \{ v \in H^1((0,L)) : v(0) = v(L) = 0 \}$$

the space of test functions associated with our standard weak form! Note w must satisfy homogeneous Dirichlet boundary conditions so that $(v+\varepsilon w)(a) = g_a$ and $(v+\varepsilon w)(b) = g_b$. Since we perturb v in the direction of w, we refer to w as a variation of v, and we refer to $\mathcal{V}$ as the space of variations in the context of the minimization form. Likewise, $\mathcal{S}$ is referred to as the set of admissible solutions or, in the context of structural analysis, the set of admissible displacements.

With a proper definition of derivative in hand, we see the solution $u \in \mathcal{S}$ of Problem (M) must satisfy:

$$\delta_w E(u) = 0$$

for all $w \in \mathcal{V}$. The first variation satisfies the same properties as a standard derivative operator, including the product rule and the chain rule, so it follows that:

$$\delta_w E(u) = \delta_w \left(\frac{1}{2} \int_0^L \kappa (u_{,x})^2 dx - \int_0^L f u \, dx \right)$$

$$= \frac{1}{2} \int_0^L \kappa \, \delta_w\left((u_{,x})^2\right) dx - \int_0^L f \, \delta_w(u) \, dx$$

$$= \int_0^L \kappa \, u_{,x} \, \delta_w(u_{,x}) \, dx - \int_0^L f \, \delta_w(u) \, dx$$

A quick calculation shows $\delta_w(u) = w$, $\delta_w(u_{,x}) = w_{,x}$, and the first variation commutes with differentiation. Thus:

$$\delta_w E(u) = \int_0^L \kappa u_{,x} w_{,x} \, dx - \int_0^L f w \, dx$$

It follows then that the solution $u \in \mathcal{S}$ of Problem (M) is also the solution of:

$$(V) \begin{cases} \text{Find } u \in \mathcal{S} \text{ such that:} \\ \displaystyle\int_0^L \kappa u_{,x} w_{,x} \, dx = \int_0^L f w \, dx \\ \text{for all } w \in \mathcal{V}. \end{cases}$$

Since Problem (V) was obtained via taking a first variation, it is referred to as the variational form of Problem (M). Similarly, the subject of finding the maxima and minima of functionals is referred to as variational calculus.

Note that Problem (V) is identical to the standard weak form from last lecture. Thus, as originally claimed, Problem (M) is indeed an alternate form of the 1D model problem at hand. In structural analysis, it is often more natural to pose problems in terms of the minimization of a total potential energy rather than in terms of a partial differential equation and associated boundary conditions. This is due to the principle of minimum total potential energy. The corresponding variational form is then known as the principle of virtual work. I will discuss these principles further when we discuss elastostatics later in this class.

To continue, I will show two things:

(i) Every solution to Problem (S) is a solution to Problem (V).

(ii) If a solution u to Problem (V) satisfies $u \in C^2([0,L])$, it is a solution to Problem (S).

Showing (i) is easy. Let $u \in C^2([0,L])$ be a solution to Problem (S). Then:

$$\left(-\kappa u_{,x}\right)_{,x} - f = 0 \qquad \forall x \in (0,L)$$

and:

$$\int_0^L \left(\left(-\kappa u_{,x}\right)_{,x} - f\right) w \, dx = 0 \qquad \forall w \in \mathcal{V}$$

By integration by parts, since $w(0) = w(L) = 0$:

$$\int_0^L K u_{,x} w_{,x}\, dx = \int_0^L f w\, dx \qquad \forall w \in \mathcal{V}$$

Showing (ii) is a tad more tricky. Let $u \in \mathcal{S}$ be a solution to Problem (V), and suppose $u \in C^2([0,L])$. Then:

$$\int_0^L K u_{,x} w_{,x}\, dx = \int_0^L f w\, dx \qquad \forall w \in \mathcal{V}$$

By reverse integration by parts, since $w(0) = w(L) = 0$:

$$\int_0^L \left(\left(-K u_{,x} \right)_{,x} - f \right) w\, dx = 0 \qquad \forall w \in \mathcal{V}$$

Note we could only do reverse integration by parts since $u \in C^2([0,L])$.
Now formally choose:

$$w = \phi \left(\left(-K u_{,x} \right)_{,x} - f \right)$$

where $\phi > 0$ and $\phi(0) = \phi(L) = 0$. Then we have:

$$\int_0^L \left(\left(-K u_{,x} \right)_{,x} - f \right)^2 \phi\, dx = 0$$

Since $\phi > 0$, it must follow that $(-K u_{,x})_{,x} = f$ for all points $x \in (0,L)$.

As an aside, note one now can infer why we used "$\mathcal{S}$" to denote the set of trial solutions and "$\mathcal{V}$" to denote the space of test functions. "$\mathcal{S}$" stands for solution, while "$\mathcal{V}$" denotes variation. Note also that while a given problem may have multiple weak forms, if it has a minimization form, it is unique. Likewise, variational forms are also unique.

The minimization and variational forms are powerful tools. However, not every partial differential equation has such forms. For instance, the Navier-Stokes equation governing fluid flow are not associated with a total potential energy minimization principle. In the early days of finite element analysis, finite element methods were constructed based on the minimization and variational forms, and thus it was thought that finite element methods could not be constructed for general partial differential equations not associated with a total potential energy minimization principle. However, as we will discover over the next few lectures, we can build a finite element method based on either weak or variational forms, and fortunately, every partial differential equation has one or more weak forms.