1D Model Problem : Strong and Weak Forms:

To begin our study of finite element methods, we begin by considering a 1D model problem. This will allow us to introduce fundamental concepts without being bogged down by the complexity associated with real-life engineering problems. Our launching point will be the following working definition provided in the previous lecture:

Finite Element Method: A numerical method for approximating the solution of partial differential equations involving:

1. A weak form of the problem at hand, and
2. Approximation of the weak solution using finite element functions.

There are three key words in the above definition: weak form, weak solution, and finite element functions. We will explain what is meant by weak form and weak solution in today's lecture, while finite element functions will be the focus of a future lecture.

To begin, we consider the following 1D Dirichlet boundary value problem:
Find $u : [0,L] \rightarrow \mathbb{R}$ such that:

$$(S) \begin{cases} \left(-\kappa u_{,x}\right)_{,x} = f & \forall x \in (0,L) \\ u(0) = g_0, \quad u(L) = g_L \end{cases}$$

Above, u is the unknown state variable, K is a material modulus, f is a source, g_0 and g_L represent given Dirichlet boundary condition data at points 0 and L respectively, and $\cdot_{,x}$ denotes differentiation with respect to x (e.g., $u_{,x} = \frac{du}{dx}$). A large number of physical problems can be cast into the form given by Problem (S) as depicted by the below table:

Physical Problem	Conservation Principle	State Variable	Material Modulus	Source
Heat Conduction in a Rod	Conservation of Energy	Temperature	Thermal Conductivity	Heat Sources
Deformation of an Elastic Bar	Equilibrium of Forces	Displacement	Young's Modulus	Body Forces
Fluid Flow	Conservation of Momentum	Velocity	Viscosity	Body Forces
Electrostatics	Conservation of Electric Flux	Electric Potential	Dialectric Permittivity	Charge
Flow Through Porous Media	Conservation of Mass	Hydraulic Head	Permeability	Fluid Source

Problem (S) is known as a conservation law because, if we integrate from 0 to L, we find:

$$-Ku_{,x}\Big|_0^L = \int_0^L (-Ku_{,x})_{,x}\, dx = \int_0^L f\, dx$$

Thus, if $f = 0$, then:

$$\sigma(L) = \sigma(0)$$

where $\sigma = -\kappa u_{,x}$ is the flux. That is, the flux is conserved across the domain. For the above tabulated problems:

Physical Problem	Flux	Constitutive Equation $\sigma = -\kappa u_{,x}$
Heat Conduction in a Rod	Heat Flux	Fourier's Law
Deformation of an Elastic Bar	(Negative) Stress	Hooke's Law
Fluid Flow	(Negative) Shear Stress	Stokes' Law
Electrostatics	Electric Flux	Coulomb's Law
Flow Through Porous Media	Flow Rate	Darcy's Law

Problem (S) is often referred to as the strong form of the 1D Dirichlet boundary value problem. This is due to the fact that if a solution to Problem (S) exists, it satisfies the differential equation:

$$(-\kappa u_{,x})_{,x} = f$$

in a pointwise manner or strong sense. Consequently, a solution to Problem (S) is referred to as a strong solution. A strong solution not only exists but is also unique and depends continuously on the problem data κ, f, g_0, and g_L if:

(i) κ is positive (i.e., $\kappa > 0$),

(ii) κ is continuous and continuously differentiable (i.e., $\kappa \in C^1([0,L])$), and

(iii) f is continuous (i.e., $f \in C^0([0,L])$).

A strong solution, if it exists, is necessarily continuous, continuously differentiable, and continuously second differentiable (i.e., $u \in C^2([0,L])$). Note that if there exists a unique strong solution that depends continuously on the problem data, the strong form is said to be well-posed.

Many numerical methods are based on strong forms. For instance, finite difference methods, point collocation methods, and least squares methods are all based on strong forms. However, strong forms are not always well-posed. For instance, the strong form given by Problem (S) is not well-posed if either:

1. κ is not continuously differentiable, or
2. f is not continuous.

Unfortunately, both conditions above are quite common in engineering practice. For instance, if we are interested in heat conduction in a rod and the rod is made of two different materials, then the thermal conductivity is not only not continuously differentiable but it is also not continuous. Likewise, if we are interested in the deformation of an elastic bar and the bar is subject to point loads, then the body force is not only not continuous but it is also not a function at all - instead, it is a so-called generalized function (in particular, it is a linear combination of Dirac delta functions).

Fortunately, we can often weaken the strong form to recover well-posedness when the problem data is not sufficiently smooth. Consider, for instance, the situation when:

$$K = \begin{cases} K^+ & \text{if } x \geq x^* \\ K^- & \text{if } x < x^* \end{cases}$$

and f is continuous. Such a situation arises, for instance, when considering heat conduction of a rod of two different materials. Let $\varepsilon > 0$. If we integrate our differential equation of interest from $x^* - \varepsilon$ to $x^* + \varepsilon$ and use the fundamental theorem of calculus, we find:

$$\left(-K u_{,x}\right)\Big|_{x^*-\varepsilon}^{x^*+\varepsilon} = \int_{x^*-\varepsilon}^{x^*+\varepsilon} \left(-K u_{,x}\right)_{,x} dx = \int_{x^*-\varepsilon}^{x^*+\varepsilon} f\, dx$$

If we let $\varepsilon \to 0$, we obtain:

$$(-\kappa u_{,x})((x^*)^+) = (-\kappa u_{,x})((x^*)^-)$$

where:

$$(-\kappa u_{,x})((x^*)^+) = \lim_{\varepsilon \to 0^+} (-\kappa u_{,x})(x^* + \varepsilon)$$

$$(-\kappa u_{,x})((x^*)^-) = \lim_{\varepsilon \to 0^+} (-\kappa u_{,x})(x^* - \varepsilon)$$

The above equation states that heat flux is conserved across x^*. However, as κ is discontinuous across x^*, so must be $u_{,x}$. In particular, the above equation dictates that:

$$\lim_{\varepsilon \to 0^+} u_{,x}(x+\varepsilon) = \frac{\kappa^-}{\kappa^+} \lim_{\varepsilon \to 0^+} u_{,x}(x-\varepsilon)$$

If we also enforce continuity of temperature across x^*, we arrive at the following weakened strong form: Find $u : [0,L] \to \mathbb{R}$ such that:

$$(S_{\text{weakened}}) \quad \begin{cases} (-\kappa u_{,x})_{,x} = f & \forall x \in (0,L) \setminus \{x^*\} \\ (-\kappa u_{,x})((x^*)^+) = (-\kappa u_{,x})((x^*)^-) \\ u((x^*)^+) = u((x^*)^-) \\ u(0) = g_0 \\ u(L) = g_L \end{cases}$$

Unfortunately, the more complex κ and f become, the more complex the weakened strong form becomes.

An alternative approach to arrive at a weakened form of the problem at hand is to consider so-called weighted residuals. A weighted residual is simply the integral of the residual:

$$R(u) = (-\kappa u_{,x})_{,x} - f$$

multiplied by a weighting or test function w over the entire domain:

$$\int_0^L R(u)\, w \, dx$$

Since $R(u) = 0$ pointwise for a strong solution, it holds that:

$$\int_0^L R(u)\, w \, dx = 0$$

for all possible weighting functions for such a solution. When a strong solution does not exist, we can use weighted residuals to arrive at a weakened problem statement. In fact, we actually used weighted residuals to arrive at the weakened strong form ($S_{weakened}$) above. In particular, we used the weighting functions:

$$w_\varepsilon = \begin{cases} 1 & \text{if } x^* - \varepsilon < x < x^* + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

To arrive at a general weakened form of the problem at hand, we need to define an entire space of weighting or test functions $\mathcal{V}$

and a corresponding set of trial solutions $\mathcal{S}$ so the corresponding weakened form of the problem is well-posed: Find $u \in \mathcal{S}$ such that:

$$\int_0^L R(u)\, w\, dx = 0$$

for all $w \in \mathcal{V}$. In particular, we need to be sure the integrals:

$$\int_0^L (-K u_{,x})_{,x}\, w\, dx = \int_0^L (-K_{,x} u_{,x} - K u_{,xx})\, w\, dx$$

and:

$$\int_0^L f w\, dx$$

are well-defined for all possible $u \in \mathcal{S}$ and $w \in \mathcal{V}$. To ensure this, we can turn to the so-called Cauchy-Schwarz inequality. This inequality states that if two functions f_1, f_2 are square-integrable over $(0, L)$:

$$\int_0^L (f_1)^2 dx < \infty$$

$$\int_0^L (f_2)^2 dx < \infty$$

then their product is also integrable and satisfies:

$$\left| \int_0^L f_1 f_2\, dx \right| < \left(\int_0^L (f_1)^2 dx \right)^{1/2} \left(\int_0^L (f_2)^2 dx \right)^{1/2}$$

It follows from the Cauchy-Schwarz inequality that if:

(i) $\max_{0 \leq x \leq L} |K| < \infty$ and $\max_{0 \leq x \leq L} |K_{,x}| < \infty$,

(ii) f is square integrable,

(iii) w is square integrable, and

(iv) $u_{,x}$ and $u_{,xx}$ are square integrable

then the integrals:

$$\int_0^L \left(-K u_{,x}\right)_{,x} w \, dx \qquad \text{and} \qquad \int_0^L f w \, dx$$

are well-defined. This inspires the following selections for $\mathcal{V}$ and $\mathcal{S}$:

$$\mathcal{V} := L^2((0,L))$$

$$\mathcal{S} := \left\{ v \in H^2((0,L)) : v(0) = g_0 \text{ and } v(L) = g_L \right\}$$

where:

$$L^2((0,L)) := \left\{ v : (0,L) \to \mathbb{R} : \int_0^L v^2 \, dx < \infty \right\}$$

is the space of square-integrable functions and:

$$H^2((0,L)) := \left\{ v \in L^2((0,L)) : v_{,x}, v_{,xx} \in L^2((0,L)) \right\}$$

is the space of square-integrable functions with square-integrable first and second derivatives. Note that the boundary conditions $u(0) = g_0$ and $u(L) = g_L$ are embedded into the definition of $\mathcal{S}$. We thus say the boundary conditions are strongly enforced. With the above selections, we arrive at the following weakened form of the problem at hand:

$$(W_{strong}) \begin{cases} \text{Find } u \in \mathcal{S} \text{ such that:} \\ \displaystyle\int_0^L (-K u_{,x})_{,x} \, w \, dx = \int_0^L f w \, dx \\ \text{for all } w \in \mathcal{V}. \end{cases}$$

We refer to the above as a weak form of the problem at hand. For reasons to be made more clear later, we in particular refer to it as the strong weak form. Problem (W_{strong}) is referred to as a weak form for three particular reasons:

Reason 1: Solutions to Problem (S) satisfy $R(u) = 0$ in a pointwise manner while solutions to Problem (W_{strong}) satisfy $R(u) = 0$ in a weakened sense.

Reason 2: Solutions to Problem (W_{strong}) are generally less smooth than solutions to Problem (S).

Reason 3: Problem (W_{strong}) is well-posed under weakened conditions as compared with Problem (S). In particular, Problem (W_{strong}) is well-posed if:

(i) $K > 0$,

(ii) $\max_{0 \le x \le L} |K| < \infty$ and $\max_{0 \le x \le L} |K_{,x}| < \infty$, and

(iii) $f \in L^2((0,L))$.

This is due to the Babuška-Lax-Milgram theorem.

We will refer to a solution u to Problem (W strong) as a weak solution and in particular a strong weak solution.

While Problem (W_{strong}) is well-posed for more general K and f than Problem (S), it is still not well-posed for all K and f of engineering interest. For instance, it is not well-posed for discontinuous K, as seen heat conduction of a rod of two different materials. Thus, we desire an even further weakened problem statement.

To arrive at a weaker form of the problem at hand, note that for a strong solution u and a sufficiently smooth weighting function w:

$$
\begin{aligned}
0 &= \int_0^L R(u)\, w\, dx \\
&= \int_0^L (-K u_{,x})_{,x}\, w\, dx - \int_0^L f w\, dx \\
&= \int_0^L K u_{,x} w_{,x}\, dx - (K u_{,x} w)\Big|_0^L \\
&\quad - \int_0^L f w\, dx
\end{aligned}
$$

The above result follows by application of integration by parts to the integral:

$$\int_0^L (-K u_{,x})_{,x} \, w \, dx$$

If $w(0) = w(L) = 0$, it follows then that:

$$\int_0^L K u_{,x} w_{,x} \, dx = \int_0^L f w \, dx$$

We can use the above equality, which may be viewed as an alternative representation of the weighted residual equation:

$$\int_0^L R(u) \, w \, dx = 0$$

to arrive at an alternate weak form. In particular, note the integral

$$\int_0^L K u_{,x} w_{,x} \, dx$$

is well-defined if:

(i) $\max_{0 \leq x \leq L} |K| < \infty$,

(ii) $u_{,x}$ is square-integrable, and

(iii) $w_{,x}$ is square-integrable.

This inspires the following selections for $\mathcal{V}$ and $\mathcal{S}$:

$$\mathcal{V} := \left\{ v \in H^1((0,L)) : v(0) = v(L) = 0 \right\}$$

$$\mathcal{S} := \left\{ v \in H^1((0,L)) : v(0) = g_0, \, v(L) = g_L \right\}$$

where:

$$H^1((0,L)) := \{ v \in L^2((0,L)) : v_{,x} \in L^2((0,L)) \}$$

is the space of square-integrable functions with square-integrable derivative. Note that all functions $w \in \mathcal{V}$ satisfy $w(0) = w(L) = 0$ by construction. This is to ensure the integration by parts formula:

$$\int_0^L K u_{,x} \, w_{,x} \, dx = \int_0^L (-K u_{,x})_{,x} \, w \, dx$$

holds for strong solutions u and weighting functions $w \in \mathcal{V}$. Note also again that boundary conditions are strongly enforced in the definition of $\mathcal{S}$. With the above selections, we arrive at the following further weakened form of the problem at hand:

$$(W_{\text{standard}}) \begin{cases} \text{Find } u \in \mathcal{S} \text{ such that:} \\ \displaystyle\int_0^L K u_{,x} \, w_{,x} \, dx = \int_0^L f w \, dx \\ \text{for all } w \in \mathcal{V}. \end{cases}$$

For reasons also to be made clear later, we refer to the above as the standard weak form or simply the weak form of the problem at hand, and we refer to solutions to the above as standard weak solutions or simply weak solutions.

Standard weak solutions satisfy $R(u) = 0$ in an even weaker sense than strong weak solutions. In particular, standard weak solutions satisfy

$$\int_0^L R(u)\, w \, dx = 0$$

for all $w \in \mathcal{V}$ in an integrated-by-parts or distributional sense. Standard weak solutions are also generally less smooth than strong weak solutions. Finally, Problem (W standard) is well-posed under weaker conditions than Problem (W strong). By the Babuška-Lax-Milgram theorem, Problem (W standard) is well-posed if:

(i) $K > 0$,

(ii) $\max_{0 \le x \le L} |K| < \infty$, and

(iii) $f \in L^2((0,L))$.

In fact, Problem (W standard) is well-posed even for certain generalized functions f. For instance, Problem (W standard) is well-posed when f is a linear combination of Dirac delta functions, and thus it can be used to model deformation of an elastic bar subject to point loads. Generally speaking, Problem (W standard) is well-posed when f is a member of the so-called dual space of $\mathcal{V}$, denoted $\mathcal{V}^*$. Dual spaces are beyond the scope of this class.

Finally, we can attain an even yet weaker form by exploiting the fact that, by two applications of integration by parts, a strong solution u and a sufficiently smooth weighting function w satisfying $w(0) = w(L) = 0$ admit the relationship:

$$
\begin{aligned}
0 &= \int_0^L R(u)\, w \, dx \\
&= \int_0^L (-K u_{,x})_{,x}\, w \, dx - \int_0^L f w \, dx \\
&= \int_0^L K u_{,x} w_{,x} \, dx - \overset{0}{\cancel{(K u_{,x} w)\big|_0^L}} - \int_0^L f w \, dx \\
&= \int_0^L u \, (-K w_{,x})_{,x} \, dx + \underset{K w_{,x}(L) g_L - K w_{,x}(0) g_0}{\cancel{(K w_{,x} u)\big|_0^L}} - \int_0^L f w \, dx
\end{aligned}
$$

Defining:

$$\mathcal{V} := \left\{ v \in H^2((0,L)) : v(0) = v(L) = 0 \right\}$$

$$\mathcal{S} := \left\{ v \in L^2((0,L)) \right\}$$

the above inspires the so-called weak weak form or ultra-weak form of the problem at hand:

$$
(W_{weak}) \begin{cases} \text{Find } u \in \mathcal{S} \text{ such that:} \\[1em] \displaystyle\int_0^L u(-K w_{,x})_{,x} \, dx = \int_0^L f w \, dx - K w_{,x}(L)\, g_L + K w_{,x}(0)\, g_0 \\[1em] \text{for all } w \in \mathcal{V}. \end{cases}
$$

We refer to solutions of the above problem as weak weak solutions or ultra-weak solutions.

Weak weak solutions satisfy $R(u) = 0$ in the weakest sense of all solutions considered here. In fact, weak weak solution do not even satisfy the given boundary conditions in a strong sense. Instead, the boundary conditions are enforced weakly through the weak weak form itself. Weak weak solutions also are generally less smooth than all other solutions considered here. Curiously, however, Problem (W_{weak}) is not well-posed for more general K than Problem $(W_{standard})$. This is because the derivative of K appears in Problem (W_{weak}) while it does not appear in Problem $(W_{standard})$.

Any of the three weak forms appearing here may be used to construct a finite element method for the 1D model problem in consideration. However, it is by far the most common to use the standard weak form. For this reason, we will often simply refer to the standard weak form as "the" weak form for the remainder of this class. In addition to appearing more often in the literature than the strong or weak weak forms, the standard weak form additionally has certain properties that make it attractive from both a theoretical and applied point of view. Chief among these is that the standard weak form is equivalent to the so-called variational form of the problem at hand, a fact that will be discussed in the next lecture.

As a useful reference, a table is given below comparing properties of strong and weak solutions.

Type of Solution	Smoothness of Solution	Boundary Condition Enforcement	Smoothness Required of K	Smoothness Required of F
Strong	C^2	Strong	C^1	C^0
Strong Weak	H^2	Strong	$K, K_{,x}$ Bounded	L^2
Standard Weak	H^1	Strong	K Bounded	$(H^1)^*$
Weak Weak	L^2	Weak	$K, K_{,x}$ Bounded	$(H^2)^*$

As a final parting word, it should be noted that if a strong solution exists, it is necessarily a strong, standard, and weak weak solution, and if a strong, standard, or weak weak solution exists and is also sufficiently smooth, it is also a strong solution. The equivalence between strong and standard weak solutions will be proven in the next lecture.