Plane Strain Elastostatics: Mixed Galerkin Approximations:

In the last lecture, we derived mixed displacement-pressure strong and weak forms for the plane strain elastostatics problem. To arrive at locking-free discretizations, we employ mixed displacement-pressure Galerkin approximations.

To begin, suppose $\mathcal{V}^h$ is a finite-dimensional subspace of $\mathcal{V}$, Q^h is a finite-dimensional subspace of Q, and $\vec{g}^h \in \mathcal{A}$. Then, a mixed displacement-pressure (Bubnov-) Galerkin approximation of the mixed displacement-pressure weak form presented in the last lecture using $\mathcal{V}^h$, Q^h, and $\vec{g}^h$ reads:

(G_{mixed}) Find $\vec{u}^h \in \mathcal{V}^h + \vec{g}^h$ and $p^h \in Q^h$ such that:

$$\bar{b}(\vec{u}^h, \vec{w}^h) - \int_\Omega p^h \vec{\nabla} \cdot \vec{w}^h \, d\Omega + \int_\Omega q^h \vec{\nabla} \cdot \vec{u}^h \, d\Omega + \int_\Omega \frac{1}{\lambda} p^h q^h \, d\Omega = \ell(\vec{w}^h)$$

for all $\vec{w}^h \in \mathcal{V}^h$ and $q^h \in Q^h$ where:

$$\bar{b}(\vec{u}^h, \vec{w}^h) = \int_\Omega \underline{\varepsilon}(\vec{w})^T \, \underline{\underline{D}} \, \underline{\varepsilon}(\vec{u}) \, d\Omega$$

$$\ell(\vec{w}^h) = \int_{\Omega} \vec{w}^h \cdot \vec{f} \, d\Omega + \int_{\Gamma_{N_1}} w_1^h h_1 \, d\Gamma + \int_{\Gamma_{N_2}} w_2^h h_2 \, d\Gamma$$

A mixed displacement-pressure Galerkin approximation of the plane strain elastostatics is necessarily well-posed for finite λ, but it is not guaranteed to be well-posed in the limit as $\lambda \to \infty$. A mixed displacement-pressure Galerkin approximation is well-posed as $\lambda \to \infty$ if and only if the finite-dimensional displacement and pressure test spaces satisfy a particular compatibility condition, the so-called Babuška-Brezzi or inf-sup condition. To motivate this condition, it helps to express a mixed displacement-pressure Galerkin approximation as a matrix system.

Following our earlier notation for primal displacement-only finite element approximations, suppose that $\mathcal{V}^h$ admits a basis:

$$\{ N_A \vec{e}_i \}_{A \in \eta - \eta_{D_i}, \, i=1,2}$$

and suppose that $\mathrm{ID}(A,i)$ maps each $A \in \eta - \eta_{D_i}$ for $i=1,2$ to a unique equation number between 1 and n_{eq} where:

$$n_{eq} := \dim(\mathcal{V}^h)$$

Then each $\vec{w}^h \in \mathcal{V}^h$ admits the form:

$$\vec{w}^h(\vec{x}) = \sum_{i=1}^{2} \sum_{A \in \eta - \eta_{D_i}} (\vec{w}^h_A)_i \, \vec{e}_i \, N_A(\vec{x})$$

and the Galerkin displacement solution $\vec{u}^h \in \mathcal{V}^h + \vec{g}^h$ to Problem $(\mathcal{G}_{mixed})$ admits the form:

Displacement Degrees of Freedom

$$\vec{u}^h(\vec{x}) = \sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} (\vec{u}^h_B)_j \, \vec{e}_j \, N_B(\vec{x}) + \vec{g}^h(\vec{x})$$

Now suppose that $\mathcal{Q}^h$ admits a basis:

$$\{\hat{N}^h_{\hat{A}}\}_{\hat{A} \in \hat{\eta}}$$

where:

$$\hat{\eta} := \{1, \ldots, \hat{n}_{eq}\}$$

$$\hat{n}_{eq} := \dim(\mathcal{Q}^h)$$

Then each $q^h \in \mathcal{Q}^h$ admits the form:

$$q^h(\vec{x}) = \sum_{\hat{A} \in \hat{\eta}} q^h_{\hat{A}} \, \hat{N}^h_{\hat{A}}(\vec{x})$$

and the Galerkin pressure solution $p^h \in \mathcal{Q}^h$ to Problem $(\mathcal{G}_{mixed})$

admits the form:

Pressure Degrees of Freedom

$$p^h(\vec{x}) = \sum_{\tilde{B} \in \tilde{\eta}} p^h_{\tilde{B}} \tilde{N}_{\tilde{B}}(\vec{x})$$

Plugging in the above expressions for $\vec{w}^h$, $\vec{u}^h$, q^h, and p^h into Problem (G_{mixed}) and exploiting bilinearity yields:

$$\sum_{i=1}^{2} \sum_{A \in \eta - \eta_{D_i}} (\vec{w}^h_A)_i \left(\begin{array}{l} \displaystyle\sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} \bar{b}(N_B \vec{e}_j, N_A \vec{e}_i)(\vec{u}^h_B)_j \\ \displaystyle - \sum_{\tilde{B} \in \tilde{\eta}} \left(\int_\Omega \tilde{N}_{\tilde{B}} \vec{\nabla} \cdot (N_A \vec{e}_i) \, d\Omega \right) p^h_{\tilde{B}} \\ - \ell(N_A \vec{e}_i) + \bar{b}(\vec{g}^h, N_A \vec{e}_i) \end{array} \right)$$

$$+ \sum_{\tilde{A} \in \tilde{\eta}} q^h_{\tilde{A}} \left(\begin{array}{l} \displaystyle\sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} \left(\int_\Omega \tilde{N}_{\tilde{A}} \vec{\nabla} \cdot (N_B \vec{e}_j) \, d\Omega \right) (\vec{u}^h_B)_j \\ \displaystyle + \sum_{\tilde{B} \in \tilde{\eta}} \left(\int_\Omega \tfrac{1}{\lambda} \tilde{N}_{\tilde{B}} \tilde{N}_{\tilde{A}} \, d\Omega \right) p^h_{\tilde{B}} \\ \displaystyle + \int_\Omega \tilde{N}_{\tilde{A}} \vec{\nabla} \cdot \vec{g}^h \, d\Omega \end{array} \right) = 0$$

Since the $(\vec{w}_A^h)_i$'s and $q_{\tilde{A}}^h$'s above are arbitrary, we have:

$$\sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} \bar{b}(N_B \vec{e}_j, N_A \vec{e}_i)(\vec{u}_B^h)_j - \sum_{\tilde{B} \in \tilde{\eta}} \left(\int_\Omega \tilde{N}_{\tilde{B}} \vec{\nabla} \cdot (N_A \vec{e}_i) \, d\Omega \right) p_{\tilde{B}}^h$$

$$= \ell(N_A \vec{e}_i) - \bar{b}(\vec{g}^h, N_A \vec{e}_i) \quad \text{for } A \in \eta - \eta_{D_i}, \; i = 1, 2$$

and:

$$\sum_{j=1}^{2} \sum_{B \in \eta - \eta_{D_j}} \left(\int_\Omega \tilde{N}_{\tilde{A}} \vec{\nabla} \cdot (N_B \vec{e}_j) \, d\Omega \right) (\vec{u}_B^h)_j + \sum_{\tilde{B} \in \tilde{\eta}} \left(\int_\Omega 1/\lambda \, \tilde{N}_{\tilde{B}} \tilde{N}_{\tilde{A}} \, d\Omega \right) p_{\tilde{B}}^h$$

$$= - \int_\Omega \tilde{N}_{\tilde{A}} \vec{\nabla} \cdot \vec{g}^h \, d\Omega \quad \text{for } \tilde{A} \in \tilde{\eta}$$

If we then define:

$$\bar{K}_{PQ} = \bar{b}(N_B \vec{e}_j, N_A \vec{e}_i) \quad \text{where } P = ID(A, i), \; Q = ID(B, j)$$

$$G_{P\tilde{B}} = - \int_\Omega \tilde{N}_{\tilde{B}} \vec{\nabla} \cdot (N_A \vec{e}_i) \, d\Omega \quad \text{where } P = ID(A, i)$$

$$M_{\tilde{A}\tilde{B}} = \int_\Omega 1/\lambda \, \tilde{N}_{\tilde{B}} \tilde{N}_{\tilde{A}} \, d\Omega$$

$$\bar{F}_P = \ell(N_A \vec{e}_i) - \bar{b}(\vec{g}^h, N_A \vec{e}_i) \quad \text{where } P = ID(A, i)$$

$$H_A^h = -\int_{\Omega_h} \hat{N}_A^h \, \vec{\nabla} \cdot \vec{g}^h \, d\Omega_h$$

$$d_Q = (\vec{u}_B^h)_j \qquad \text{where } Q = \mathrm{ID}(B, j)$$

$$p_{\hat{B}}^h = p_{\hat{B}}^h$$

then we see the displacement and pressure degrees of freedom may be attained via the solution of the following matrix problem:

$$(L_{mixed}) \begin{cases} \text{Find } \underline{d} \in \mathbb{R}^{n_{eq}} \text{ and } \underline{p} \in \mathbb{R}^{\hat{n}_{eq}} \text{ such that:} \\ \begin{bmatrix} \bar{\underline{\underline{K}}} & \underline{\underline{G}} \\ -\underline{\underline{G}}^T & \underline{\underline{M}} \end{bmatrix} \begin{bmatrix} \underline{d} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \bar{\underline{F}} \\ \underline{H} \end{bmatrix} \end{cases}$$

We refer to $\bar{\underline{\underline{K}}}$ as the stiffness matrix (though note it is different than the stiffness matrix $\underline{\underline{K}}$ appearing before), $\underline{\underline{G}}$ as the gradient matrix, $\underline{\underline{M}}$ as the mass matrix (not to be confused with the mass matrix appearing in the setting of elastodynamics), $\bar{\underline{F}}$ as the force vector (though note it is different than the force vector $\underline{F}$ appearing before), $\underline{H}$ as the source vector, $\underline{d}$ as the displacement vector (this is the same displacement vector as before), and $\underline{p}$ as the pressure vector. The matrix $\underline{\underline{G}}$, as suggested by the naming above, is a discrete gradient operator while the matrix $-\underline{\underline{G}}^T$ is a discrete divergence operator. The discrete gradient and

divergence operators are related via a minus transpose due to the application of integration by parts in defining the discrete gradient operator.

In the above matrix system, $\bar{\underline{\underline{K}}}$ and $\underline{\underline{M}}$ are both symmetric and $\bar{\underline{\underline{K}}}$ is positive-definite. However, the large matrix:

$$\begin{bmatrix} \bar{\underline{\underline{K}}} & \underline{\underline{G}} \\ -\underline{\underline{G}}^T & \underline{\underline{M}} \end{bmatrix}$$

is not symmetric. When λ is finite, $\underline{\underline{M}}$ is positive-definite and hence invertible. Noting that:

$$\bar{\underline{\underline{K}}}\,\underline{d} + \underline{\underline{G}}\,\underline{p} = \bar{\underline{F}}$$
$$-\underline{\underline{G}}^T\underline{d} + \underline{\underline{M}}\,\underline{p} = \underline{H}$$

We see we can solve the second equation above for $\underline{p}$:

$$\underline{p} = \underline{\underline{M}}^{-1}\left(\underline{H} + \underline{\underline{G}}^T\underline{d}\right)$$

and insert this new expression for $\underline{p}$ into our first equation, attaining:

$$\underbrace{\left(\bar{\underline{\underline{K}}} + \underline{\underline{G}}\,\underline{\underline{M}}^{-1}\underline{\underline{G}}^T\right)}_{\underline{\underline{K}}}\underline{d} = \underbrace{\left(\bar{\underline{F}} - \underline{\underline{G}}\,\underline{\underline{M}}^{-1}\underline{H}\right)}_{\underline{F}}$$

Since $\bar{\underline{\underline{K}}}$ and $\underline{\underline{M}}$ are symmetric positive-definite, so is $\underline{\underline{K}}$. Thus $\underline{\underline{K}}$ is invertible and $\underline{d}$ is uniquely:

$$\underline{d} = \underline{\underline{K}}^{-1} \underline{F}$$

As $\underline{d}$ is unique, $\underline{p}$ is then uniquely:

$$\underline{p} = \underline{\underline{M}}^{-1} \left(\underline{H} + \underline{\underline{G}}^T \underline{d} \right)$$

When $\lambda \to \infty$, $\underline{\underline{M}} \to \underline{\underline{0}}$, so we cannot solve for $\underline{d}$ and $\underline{p}$ in the above manner. Noting in this case that:

$$\begin{aligned} \bar{\underline{\underline{K}}} \underline{d} + \underline{\underline{G}} \underline{p} &= \bar{\underline{F}} \\ - \underline{\underline{G}}^T \underline{d} &= \underline{H} \end{aligned}$$

We can elect to instead first solve the first equation for $\underline{d}$:

$$\underline{d} = \bar{\underline{\underline{K}}}^{-1} \left(\bar{\underline{F}} - \underline{\underline{G}} \underline{p} \right)$$

and insert this new expression for $\underline{d}$ into the second equation:

$$\underbrace{\left(\underline{\underline{G}}^T \bar{\underline{\underline{K}}}^{-1} \underline{\underline{G}} \right)}_{\underline{\underline{S}}} \underline{p} = \underline{H} + \underline{\underline{G}}^T \bar{\underline{\underline{K}}}^{-1} \bar{\underline{F}}$$

The matrix $\underline{\underline{S}}$ above is the so-called Schur complement of $\bar{\underline{\underline{K}}}$ in the big matrix system:

$$\begin{bmatrix} \bar{\underline{\underline{K}}} & \underline{\underline{G}} \\ -\underline{\underline{G}}^T & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{d} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \bar{\underline{F}} \\ \underline{H} \end{bmatrix}$$

Since the stiffness matrix $\bar{\underline{\underline{K}}}$ is invertible, the Schur complement $\underline{\underline{S}}$ is invertible if and only if the gradient matrix $\underline{\underline{G}}$ is of full column rank, that is, if the columns of $\underline{\underline{G}}$ are linearly independent. Equivalently, $\underline{\underline{S}}$ is invertible if and only if:

$$\underline{\underline{G}}\,\underline{q} \neq \underline{0} \qquad \text{for all} \qquad \underline{0} \neq \underline{q} \in \mathbb{R}^{n_{eq}}$$

To further see this, suppose there is a $\underline{q} \neq \underline{0}$ such that $\underline{\underline{G}}\,\underline{q} = \underline{0}$. Then, if $\underline{p}$ is a solution of:

$$\left(\underline{\underline{G}}^T \bar{\underline{\underline{K}}}^{-1} \underline{\underline{G}}\right) \underline{p} = \underline{H} + \underline{\underline{G}}^T \underline{\underline{K}}^{-1} \bar{\underline{F}}$$

then so is $\underline{p} + \alpha \underline{q}$ for any $\alpha \in \mathbb{R}$. Any $\underline{q} \neq \underline{0}$ such that $\underline{\underline{G}}\,\underline{q} = \underline{0}$ is referred to as a pressure mode, so $\underline{\underline{S}}$ is invertible if there are no pressure modes. Now note that:

$$\underline{\underline{G}}\,\underline{q} \neq \underline{0} \qquad \text{for all} \quad \underline{0} \neq \underline{q} \in \mathbb{R}^{n_{eq}}$$

if and only if for each $0 \neq \underline{q} \in \mathbb{R}^{n_q}$ there exists a $\underline{v} \in \mathbb{R}^{n_q}$ such that:

$$\underline{v}^T \underline{\underline{G}}\, \underline{q} > 0$$

or equivalently for each $0 \neq q^h \in Q^h$ there exists a $\vec{v}^h \in \mathcal{V}^h$ such that:

$$-\int_\Omega q^h \vec{\nabla} \cdot \vec{v}^h \, d\Omega > 0$$

If the above holds, then:

$$\beta^h := \inf_{q^h \in Q^h} \sup_{\vec{v}^h \in \mathcal{V}^h} \frac{-\int_\Omega q^h \vec{\nabla} \cdot \vec{v}^h \, d\Omega}{\|\vec{q}^h\|_{L^2(\Omega)} \|\vec{v}^h\|_{(H^1(\Omega))^2}} > 0$$

The above is the famous Babuška-Brezzi condition, also referred to as the inf-sup condition. If the Babuška-Brezzi condition holds, then not only is the Schur complement $\underline{\underline{S}}$ invertible and hence Problem (G mixed) well-posed in the limit as $\lambda \to \infty$, Problem (G mixed) also admits an error estimate of the form:

$$\|\vec{u} - \vec{u}^h\|_{(H^1(\Omega))^2} + \|p - p^h\|_{L^2(\Omega)}$$

$$\leq \left(1 + \frac{\mu_{max}}{\beta^h C_{korn} \mu_{min}}\right)\left(\inf_{\vec{v}^h \in \mathcal{V}^h + g^h} \|\vec{u} - \vec{v}^h\|_{(H^1(\Omega))^2} + \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}\right)$$

in the limit as $\lambda \to \infty$ where:

$$\mu_{max} := \sup_{\vec{x} \in \Omega} \mu(\vec{x})$$
$$\mu_{min} := \inf_{\vec{x} \in \Omega} \mu(\vec{x})$$

Now suppose we have a sequence of mixed displacement Galerkin approximations $\{(\mathcal{V}^h, \mathcal{Q}^h)\}_{h>0}$ such that $\mathcal{V}^h \to \mathcal{V}$ and $\mathcal{Q}^h \to \mathcal{Q}$ as $h \to 0$. Then, if β^h is uniformly bounded from below by a positive number, that is, if there is a constant $\beta > 0$ such that $\beta^h \geq \beta$ for all $h > 0$, it holds that the Galerkin displacement and pressure solutions converge to the weak displacement and pressure solutions. That is, $\vec{u}^h \to \vec{u}$ and $p^h \to p$ as $h \to 0$. We say such a sequence $\{(\mathcal{V}^h, \mathcal{Q}^h)\}_{h>0}$ satisfies a uniform Babuška-Brezzi condition. An identical error estimate to that presented above also holds for finite λ, so sequences $\{(\mathcal{V}^h, \mathcal{Q}^h)\}_{h>0}$ satisfying a uniform Babuška-Brezzi condition are necessarily locking-free. Alternatively, if $\beta^h \to 0$ as $h \to 0$, then convergence is at best attenuated and at worst arrested altogether.

As is likely clear by this point, satisfaction of the Babuška-Brezzi condition is critical to the success of any mixed displacement-pressure Galerkin approximation. Unfortunately, proving the Babuška-Brezzi condition for a given displacement-pressure pair $(\mathcal{V}^h, \mathcal{Q}^h)$ is a mathematically arduous exercise, and proving a uniform Babuška-Brezzi condition for a sequence of displacement-pressure

pairs $\{(\mathcal{V}^h, Q^h)\}_{h>0}$ is an even more difficult exercise. In practice, one often checks if the ratio:

$$r = n_{eq} / \tilde{n}_{eq}$$

is greater than or equal to two for a full sequence of displacement-pressure pairs $\{(\mathcal{V}^h, Q^h)\}_{h>0}$. If $r \geq 2$, then there are at least two displacement degrees of freedom per pressure degree of freedom. While satisfaction of the constraint $r \geq 2$ is not a sufficient nor necessary requirement for satisfaction of the Babuška-Brezzi condition for any given displacement-pressure pair nor does it guarantee a uniform Babuška-Brezzi condition holds for a sequence of displacement-pressure pairs, it is a very easy constraint to check. Note in particular if:

$$r = n_{eq} / \tilde{n}_{eq} < 1$$

then the gradient matrix $\underline{\underline{G}}$ necessarily does not have full column rank as it then has more columns than rows. If:

$$r = n_{eq} / \tilde{n}_{eq} = 1$$

then $\underline{\underline{G}}$ is a square matrix, and if $\underline{\underline{G}}$ has full column rank, then the equation:

$$-\underline{\underline{G}}^T \underline{d} = \underline{H}$$

Fully defines $\underline{d}$. In particular, if $\underline{H} = \underline{0}$, then $\underline{d} = \underline{0}$. Thus, locking can even occur for mixed displacement-pressure Galerkin approximations. In particular, locking necessarily occurs if the number of linearly independent constraints implied by:

$$-\underline{\underline{G}}^T \underline{d} = \underline{H}$$

is equal to the number of displacement unknowns. The number of constraints is precisely:

$$c = \text{row rank of } \underline{\underline{G}}^T = \text{column rank of } \underline{\underline{G}} \leq \hat{n}_{eq},$$

while the number of displacement unknowns is precisely n_{eq}. Thus if locking occurs, it can be alleviated by either (i) decreasing the number of pressure unknowns $\hat{n}_{eq}$ or (ii) increasing the number of displacement unknowns n_{eq}. Both approaches result in weakening the incompressibility constraint. To see this, note that with a mixed displacement-pressure Galerkin approximation, we satisfy the constitutive equation:

$$\frac{1}{\lambda} p + \vec{\nabla} \cdot \vec{u} = 0$$

in a weak sense. In particular, we have:

$$\int_\Omega \tfrac{1}{\lambda} p^h q^h \, d\Omega + \int_\Omega \vec{\nabla} \cdot \vec{u}^h q^h \, d\Omega = 0$$

for all $q^h \in Q^h$. In the limit as $\lambda \to 0$, we have:

$$\int_\Omega \vec{\nabla} \cdot \vec{u}^h q^h \, d\Omega = 0$$

for all $q^h \in Q^h$. If we decrease the size of Q^h, then we satisfy the above equation for less q^h. Thus, we satisfy the constraint $\vec{\nabla} \cdot \vec{u}^h = 0$ in a weaker sense. Alternatively, if we increase the size of $\mathcal{V}^h$, then we satisfy the above equation for just as many q^h, then we introduce additional displacement fields $\vec{v}^h$ such that $\int_\Omega \vec{\nabla} \cdot \vec{v}^h q^h \, d\Omega = 0$ for all $q^h \in Q$ but $\vec{\nabla} \cdot \vec{v}^h \neq 0$. Thus again, we satisfy the constraint $\vec{\nabla} \cdot \vec{u}^h = 0$ in a weaker sense.

It should be noted that when $\Gamma = \Gamma_{D_1} = \Gamma_{D_2}$:

$$\int_\Omega \vec{\nabla} \cdot \vec{v}^h \, d\Omega = \int_\Gamma \vec{v}^h \cdot \vec{n} \, d\Gamma = 0$$

for all $\vec{v}^h \in \vec{\mathcal{V}}^h$. It holds that all $q^h \in \mathbb{R}$ are then pressure modes. However, these pressure modes are physical pressure modes and they do

not pollute the accuracy of a mixed displacement-pressure Galerkin approximation. This is due to the fact that the average of pressure is set by the applied displacement field along the boundary for both the Galerkin pressure solution and the exact pressure solution:

$$\int_{\Omega} \frac{1}{\lambda} p \, d\Omega = \int_{\Omega} \frac{1}{\lambda} p^h \, d\Omega = -\int_{\Gamma^2} \vec{g} \cdot \vec{n} \, d\Gamma^2$$

Non-constant pressure modes, on the other hand, can pollute the accuracy of a mixed displacement-pressure Galerkin approximation. Non-constant pressure modes are called spurious pressure modes as they are non-physical.

Now that we have discussed mixed displacement-pressure Galerkin approximations and their properties at length, we will present mixed displacement-pressure finite element approximations for the plane strain elastostatics problem during the next lecture.