Plane Strain Elastostatics: Mixed Finite Element Computer Implementation:

We can implement a mixed displacement-pressure finite element approximation of the plane strain elastostatics problem the same way we can implement a primal displacement-only finite element approximation - by forming and assembling element and boundary element matrices and vectors. In the mixed finite element context, we simply need to build element gradient matrices and mass matrices in addition to element stiffness matrices and element and boundary element force vectors.

To begin, suppose we have Lagrange finite element bases $\{N_A\}_{A\in\eta}$ and $\{\tilde{N}_{\tilde{A}}\}_{\tilde{A}\in\tilde{\eta}}$ for displacement and pressure approximations, respectively, associated with displacement nodes $\{\vec{x}_A\}_{A\in\eta}$ and pressure nodes $\{\tilde{\vec{x}}_{\tilde{A}}\}_{\tilde{A}\in\tilde{\eta}}$. Then the finite element displacement and pressure solutions take the form:

$$\vec{u}^h(\vec{x}) = \sum'_{B\in\eta-\eta_{D_1}} u_1^h(\vec{x}_B)\,\vec{e}_1 N_B(\vec{x}) + \sum'_{B\in\eta-\eta_{D_2}} u_2^h(\vec{x}_B)\,\vec{e}_2 N_B(\vec{x}) + \vec{g}^h$$

Unknown DOF

Unknown DOF

$$\vec{g}^h(\vec{x}) = \sum'_{B\in\eta_{D_1}} g_1(\vec{x}_B)\,\vec{e}_1 N_B(\vec{x}) + \sum'_{B\in\eta_{D_2}} g_2(\vec{x}_B)\,\vec{e}_2 N_B(\vec{x})$$

$$p^h(\vec{x}) = \sum'_{\tilde{B}\in\tilde{\eta}} p^h(\tilde{\vec{x}}_{\tilde{B}})\,\tilde{N}_{\tilde{B}}(\vec{x})$$

Unknown DOF

where the unknown degrees of freedom may be attained by solving the linear system:

$$\begin{bmatrix} \underline{\underline{\bar{K}}} & \underline{\underline{G}} \\ -\underline{\underline{G}}^T & \underline{\underline{M}} \end{bmatrix} \begin{bmatrix} \underline{d} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{\bar{F}} \\ \underline{H} \end{bmatrix}$$

where:

$$\bar{K}_{PQ} = \bar{b}(N_B \vec{e}_j, N_A \vec{e}_i) \qquad \text{where } P = ID(A,i),\ Q = ID(B,j)$$

$$G_{P\hat{B}} = -\int_{\Omega} \hat{N}_{\hat{B}} \vec{\nabla} \cdot (N_A \vec{e}_i)\, d\Omega \qquad \text{where } P = ID(A,i)$$

$$M_{\hat{A}\hat{B}} = \int_{\Omega} 1/\lambda\, \hat{N}_{\hat{B}} \hat{N}_{\hat{A}}\, d\Omega$$

$$\bar{F}_P = \ell(N_A \vec{e}_i) - \bar{b}(\vec{g}^h, N_A \vec{e}_i) \qquad \text{where } P = ID(A,i)$$

$$H_{\hat{A}} = -\int_{\Omega} \hat{N}_{\hat{A}} \vec{\nabla} \cdot \vec{g}^h\, d\Omega$$

$$d_Q = u_j^h(\vec{x}_B)$$

$$p_{\hat{B}} = p^h(\hat{x}_{\hat{B}}) \qquad \text{where } Q = ID(B,j)$$

We can express the integrals above as sums of integrals over elements and boundary edges:

$$\bar{K}_{PQ} = \sum_{\Omega^e \in \mathcal{M}^h}' \int_{\Omega^e} \underset{=}{\varepsilon}(N_A \vec{e}_i)^T \bar{\underset{=}{D}} \, \underset{=}{\varepsilon}(N_B \vec{e}_j) \, d\Omega$$

$$G_{P\breve{B}} = \sum_{\Omega^e \in \mathcal{M}^h}' - \int_{\Omega^e} \breve{N}_{\breve{B}} \, \vec{\nabla} \cdot (N_A \vec{e}_i) \, d\Omega$$

$$M_{\breve{A}\breve{B}} = \sum_{\Omega^e \in \mathcal{M}^h} \int_{\Omega^e} 1/\lambda \, \breve{N}_{\breve{A}} \breve{N}_{\breve{B}} \, d\Omega$$

$$F_P = \sum_{\Omega^e \in \mathcal{M}^h} \int_{\Omega^e} N_A f_i \, d\Omega - \sum_{\Omega^e \in \mathcal{M}^h} \int_{\Omega^e} \underset{=}{\varepsilon}(N_A \vec{e}_i)^T \bar{\underset{=}{D}} \, \underset{=}{\varepsilon}(\vec{g}^h) \, d\Omega$$

$$+ \sum_{\Gamma^e \in \mathcal{E}^h_{N_i}} \int_{\Gamma^e} N_A h_i \, d\Gamma$$

$$H_{\breve{A}} = - \sum_{\Omega^e \in \mathcal{M}^h} \int_{\breve{\Omega}} \breve{N}_{\breve{A}} \, \vec{\nabla} \cdot \vec{g}^h \, d\Omega$$

Suppose we locally renumber the n_{en} non-zero displacement basis functions over each element and boundary edge using IEN and IENB arrays just as we did for primal displacement-only finite element approximations of the plane strain elastostatics problem, and suppose we also locally renumber the $\breve{n}_{en}$ non-zero pressure basis functions over each element using an IENP array:

$$\breve{A} = \text{IENP}(\breve{a}, e)$$

Global Basis Function Number

Local Basis Function Number

Element Number

Then we can write:

$$\bar{K}_{PQ} = \sum_{e=1}^{nel} \sum_{i=1}^{2} \sum_{a=1}^{n_{en}} \sum_{j=1}^{2} \sum_{b=1}^{n_{en}} \bar{k}^{e}_{\underset{\downarrow\,2(a-1)+i}{p}\,q \,\to\, 2(b-1)+j}$$

$$P = ID(IEN(a,e),i) \qquad Q = ID(IEN(b,e),j)$$

$$G_{P\breve{B}} = \sum_{e=1}^{nel} \sum_{i=1}^{2} \sum_{a=1}^{n_{en}} \sum_{\breve{b}=1}^{\breve{n}_{en}} g^{e}_{\underset{\downarrow\,2(a-1)+i}{p}\,\breve{b}}$$

$$P = ID(IEN(a,e),i) \qquad \breve{B} = IENP(\breve{b},e)$$

$$M_{\breve{A}\breve{B}} = \sum_{e=1}^{nel} \sum_{\breve{a}=1}^{n_{en}} \sum_{\breve{b}=1}^{\breve{n}_{en}} m^{e}_{\breve{a}\breve{b}}$$

$$\breve{A} = IENP(\breve{a},e) \qquad \breve{B} = IENP(\breve{b},e)$$

$$\bar{F}_P = \sum_{e=1}^{nel} \sum_{i=1}^{2} \sum_{a=1}^{n_{en}} \left(f^{e}_{\underset{\downarrow\,2(a-1)+i}{p}} - \sum_{j=1}^{2} \sum_{b=1}^{n_{en}} \bar{k}^{e}_{\underset{\downarrow\,2(a-1)+i}{p}\,\underset{\downarrow\,2(b-1)+j}{q}}\, g_j(\vec{x}_{\underset{\downarrow\,IEN(b,e)}{B}}) \right)$$

$$P = ID(IEN(a,e),i) \qquad O = ID(IEN(b,e),j)$$

$$+ \sum_{e=1}^{nelb} \sum_{i=1}^{2} \sum_{a=1}^{nenb} f^{e,b}_{\underset{\downarrow\,2(a-1)+i}{p}}$$

$$P = ID(IENB(a,e),i)$$

$$H_{\breve{A}} = \sum_{e=1}^{nel} \sum_{\breve{a}=1}^{n_{en}} \sum_{j=1}^{2} \sum_{b=1}^{n_{en}} g^{e}_{\underset{\downarrow\,2(b-1)+j}{q}\,\breve{a}}\; g_j(\vec{x}_{\underset{\downarrow\,IEN(b,e)}{B}})$$

$$\breve{A} = IENP(\breve{a},e) \qquad O = ID(IEN(b,e),j)$$

Where:

$$\bar{k}^{e}_{pq} := \int_{\Omega^e} \varepsilon(N^e_a\, \vec{e}_i)^T\, \bar{\underline{\underline{D}}}\, \varepsilon(N^e_b\, \vec{e}_j)\, d\Omega \qquad \begin{aligned} p &= 2(a-1)+i \\ q &= 2(b-1)+j \end{aligned}$$

$$g^e_{p\hat{b}} := -\int_{\Omega^e} \hat{N}^e_{\hat{b}}\ \vec{\nabla}\cdot\left(N^e_a \vec{e}_i\right) d\Omega \qquad p = 2(a-1)+i$$

$$m^e_{\hat{a}\hat{b}} := \int_{\Omega^e} 1/\lambda\ \hat{N}^e_{\hat{a}}\ \hat{N}^e_{\hat{b}}\ d\Omega$$

$$f^e_p := \int_{\Omega^e} N^e_a\ f_i\ d\Omega \qquad p = 2(a-1)+i$$

$$f^{e,b}_p := \int_{\Gamma^e \cap \Gamma^h_{N_i}} N^{e,b}_a\ h_i\ d\Omega \qquad p = 2(a-1)+i$$

are element and boundary element matrix and vector entries. We can approximate these entries using our usual two tricks:

Trick #1: We pull integrals back to a parent element or boundary element.

Trick #2: We use quadrature to approximate parent element integrals.

This yields the approximations:

$$\bar{k}^e_{pq} \approx \sum_{l=1}^{n_q} \left(\left(\underline{\varepsilon}\left(N^e_a \vec{e}_i\right)\right)\left(\vec{x}^e(\tilde{\xi}_l)\right)\right)^T \left(\bar{\underline{\underline{D}}}\left(\vec{x}^e(\tilde{\xi}_l)\right)\right)\left(\left(\underline{\varepsilon}\left(N^e_b \vec{e}_j\right)\right)\left(\vec{x}^e(\tilde{\xi}_l)\right)\right) j^e(\tilde{\xi}_l)\ \hat{w}_l$$

for $p = 2(a-1)+i$ and $q = 2(b-1)+j$

$$g^e_{p\hat{b}} \approx -\sum_{l=1}^{n_q} \hat{N}^e_{\hat{b}}\left(\vec{x}^e(\tilde{\xi}_l)\right)\left(\vec{\nabla}\cdot\left(N^e_a \vec{e}_i\right)\right)\left(\vec{x}^e(\tilde{\xi}_l)\right) j^e(\tilde{\xi}_l)\ \hat{w}_l$$

for $p = 2(a-1)+i$

$$m^e_{\tilde{a}\tilde{b}} \approx \sum_{l=1}^{n_q} 1/\lambda(\vec{x}^e(\hat{\xi}_l))\, \hat{N}^e_{\tilde{a}}(\vec{x}^e(\hat{\xi}_l))\, \hat{N}^e_{\tilde{b}}(\vec{x}^e(\hat{\xi}_l))\, j^e(\hat{\xi}_l)\, \hat{w}_l$$

$$f^e_p \approx \sum_{l=1}^{n_q} N^e_a(\vec{x}^e(\hat{\xi}_l))\, f_i(\vec{x}^e(\hat{\xi}_l))\, j^e(\hat{\xi}_l)\, \hat{w}_l \quad \text{for } p = 2(a-1)+i$$

$$f^{e,b}_p \approx \begin{cases} \sum_{l=1}^{n_{qb}} N^{e,b}_a(\vec{x}^{e,b}(\hat{\xi}^b_l))\, h_i(\vec{x}^{e,b}(\hat{\xi}^b_l))\, j^e_{\Gamma}(\hat{\xi}^b_l)\, \hat{w}^b_l & \text{if } \Gamma^e \in \mathcal{E}^h_{N_i} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } p = 2(a-1)+i$$

Provided the local basis functions can be described as push forwards of shape functions defined over the parent element, we can express local basis function values and derivatives in terms of shape function values and derivatives and the entries of the Jacobian matrix.