2D Heat Conduction: Strong and Weak Forms:

During the first several weeks of the semester, we focused on finite element analysis of a 1D model problem so we could key in one fundamental concepts such as strong forms, weak forms, finite element approximations, and element formation and assembly without worrying about additional complications introduced by real-world science and engineering applications. Today, we turn to one of those real-world applications: steady two-dimensional heat conduction.

To begin, let $\Omega \subset \mathbb{R}^2$ be a domain of interest, an open subset of $\mathbb{R}^2$. The temperature field $T : \overline{\Omega} \to \mathbb{R}$ satisfies the partial differential equation:

$$\vec{\nabla} \cdot \vec{q} = f$$

where $\vec{q}$ is the heat flux and $f : \Omega \to \mathbb{R}$ is the internal heating. Above, $\vec{\nabla} \cdot \vec{q}$ denotes the divergence of $\vec{q}$, defined as:

$$\vec{\nabla} \cdot \vec{q} = \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2}$$

where q_1 is the component of $\vec{q}$ in the x_1-direction and q_2 is the component of $\vec{q}$ in the x_2-direction. We will use comma notation to denote differentiation again (e.g., $u_{,i} = u_{,x_i} = \partial u / \partial x_i$), so we can write the above as:

$$\vec{\nabla} \cdot \vec{q} = q_{1,1} + q_{2,2}$$

We assume the heat flux is defined by Fourier's law of heat conduction:

$$\vec{q} = -K \vec{\nabla} T$$

where $K : \Omega \to \mathbb{R}^+$ is the thermal conductivity, which is allowed to vary throughout the domain of interest. Above, $\vec{\nabla} T$ denotes the gradient of temperature, defined as:

$$\vec{\nabla} T = \begin{bmatrix} T_{,1} \\ T_{,2} \end{bmatrix}$$

Thus, we have:

$$q_1 = -K\, T_{,1} \qquad \text{and} \qquad q_2 = -K\, T_{,2}$$

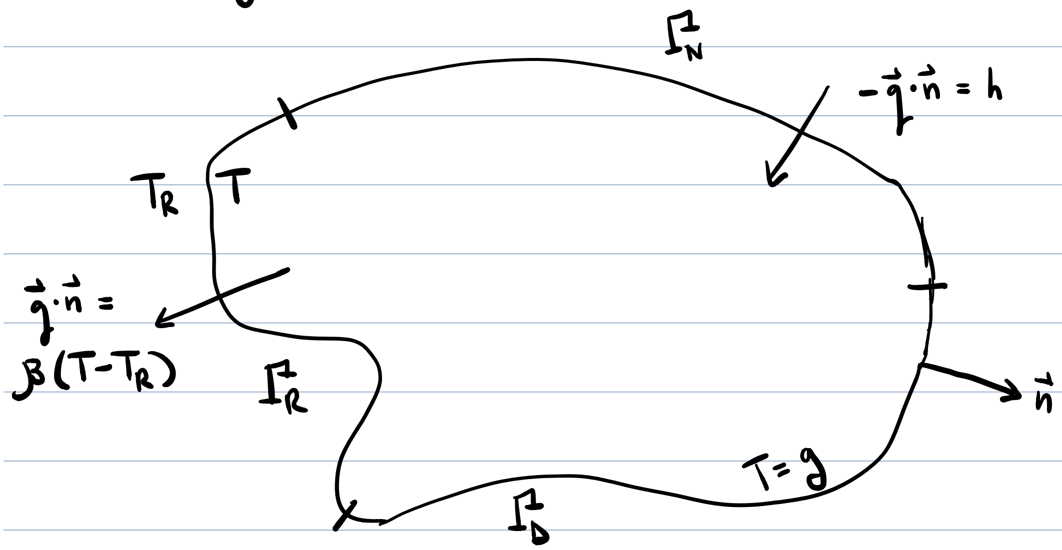
Fourier's law of heat conduction essentially states that heat goes from regions of high temperature to regions of low temperature. The temperature field is also subject to boundary conditions along the boundary of the domain of interest. We split the boundary, denoted Γ, into a Dirichlet boundary Γ_D, a Neumann boundary Γ_N, and a Robin boundary Γ_N. These boundaries must be non-overlapping:

$$\Gamma_D \cap \Gamma_N = \emptyset, \quad \Gamma_N \cap \Gamma_R = \emptyset, \quad \Gamma_R \cap \Gamma_D = \emptyset$$

and they must combine to form the full boundary:

$$\Gamma = \overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R}$$

Visually:



Note that while the domain of interest is drawn as a single body with no holes above, generally it may be comprised of multiple components, some or all with one or more holes. Moreover, while the Dirichlet, Neumann, and Robin boundaries are drawn as connected curves above, generally they consist of several disconnected sub-boundaries.

Over the Dirichlet boundary, the temperature is set to a particular applied temperature $g : \Gamma_D \to \mathbb{R}$:

$$T = g$$

Over the Neumann boundary, the heat flux entering the domain is set to a particular applied heat flux $h : \Gamma_N \to \mathbb{R}$:

$$-\vec{q}\cdot\vec{n} = h$$

Above, $\vec{n}$ is the unit outward facing normal along the boundary of the domain. Over the Robin boundary, the heat flux leaving the domain is governed by Newton's law of cooling:

$$\vec{q} \cdot \vec{n} = \beta (T - T_R)$$

where $\beta : \Gamma_R \to \mathbb{R}^+$ is the convective heat transfer coefficient along the Robin boundary and $T_R : \Gamma_R \to \mathbb{R}$ is the temperature of the surrounding convective medium. If $T > T_R$, then $\vec{q} \cdot \vec{n} > 0$ along the Robin boundary, resulting in cooling of the body. Hence, the convective medium may be used to cool the material of the domain. For instance, the domain may represent a hot body and the convective medium may be a moving cool liquid. Cooling of supercomputers occurs in such a fashion.

With the above in hand, the strong form of our problem is as follows:

$$(S) \quad \left\{ \begin{array}{ll} \text{Find } T : \bar{\Omega} \to \mathbb{R} \text{ such that:} & \\ \vec{\nabla} \cdot q = f & \text{in } \Omega \\ T = g & \text{on } \Gamma_D \\ -\vec{q} \cdot \vec{n} = h & \text{on } \Gamma_N \\ \vec{q} \cdot \vec{n} = \beta (T - T_R) & \text{on } \Gamma_R \end{array} \right.$$

Before proceeding, it should be noted that the Dirichlet, Neumann, and Robin boundaries are named as such as Dirichlet, Neumann, and Robin boundary conditions are applied over these respective boundaries. Dirichlet boundary conditions involve function values, Neumann boundary conditions involves derivatives, and Robin boundary conditions involve both.

Now, to approximate the solution of the steady two-dimensional heat conduction problem, we need a suitable weak form. To arrive at such a weak form, we first multiply the residual:

$$R(T) = \vec{\nabla} \cdot \vec{q} - f$$

by a suitably smooth weighting function $w : \Omega \to \mathbb{R}$ and integrate over the domain to obtain:

$$\int_{\Omega} \vec{\nabla} \cdot \vec{q} \, w \, d\Omega = \int_{\Omega} f w \, d\Omega$$

Inspired by the standard weak form for the previously studied 1D model problem, we integrate the left term above by parts, resulting in:

$$-\int_{\Omega} \vec{q} \cdot \vec{\nabla} w \, d\Omega + \int_{\Gamma} \vec{q} \cdot \vec{n} \, w \, d\Gamma = \int_{\Omega} f w \, d\Omega$$

To proceed, we break the boundary integral above into a sum of boundary integrals over the Dirichlet, Neumann, and Robin boundaries:

$$\int_{\Gamma} \vec{q}\cdot\vec{n}\, w \, d\Gamma = \int_{\Gamma_D} \vec{q}\cdot\vec{n}\, w \, d\Gamma + \int_{\Gamma_N} \vec{q}\cdot\vec{n}\, w \, d\Gamma + \int_{\Gamma_R} \vec{q}\cdot\vec{n}\, w \, d\Gamma$$

Over the Neumann boundary, we can invoke the Neumann boundary condition to write:

$$\int_{\Gamma_N} \vec{q}\cdot\vec{n}\, w \, d\Gamma = -\int_{\Gamma_N} h\, w \, d\Gamma$$

Over the Robin boundary, we can invoke the Robin boundary condition to write:

$$\int_{\Gamma_R} \vec{q}\cdot\vec{n}\, w \, d\Gamma = \int_{\Gamma_R} \beta\,(T - T_R)\, w \, d\Gamma$$

Inspired by the standard weak form for the previously studied 1D model problem, we set $w|_{\Gamma_D} = 0$, yielding:

$$\int_{\Gamma_D} \vec{q}\cdot\vec{n}\, w \, d\Gamma = 0$$

Finally, noting that $\vec{q} = -K \vec{\nabla} T$, we have:

$$-\int_{\Omega} \vec{q} \cdot \vec{\nabla} w \, d\Omega = \int_{\Omega} \kappa \vec{\nabla} T \cdot \vec{\nabla} w \, d\Omega$$

Collecting the above identities, we find that:

$$\int_{\Omega} \kappa \vec{\nabla} T \cdot \vec{\nabla} w \, d\Omega + \int_{\Gamma_R} \beta T w \, d\Gamma = \int_{\Omega} f w \, d\Omega + \int_{\Gamma_N} h w \, d\Gamma + \int_{\Gamma_R} \beta T_R w \, d\Gamma$$

We use the above integral identity to construct a suitable weak form of the problem at hand. As previously discussed in this class, the key step in this construction is selecting an appropriate set of trial solutions $\mathcal{S}$ and an appropriate space of test functions $\mathcal{V}$ such that all of the above integrals are well-defined. Since both function values and derivatives appear in the above integrals, we require that both $\mathcal{S}$ and $\mathcal{V}$ be subsets of:

$$H^1(\Omega) := \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \right\}$$

where:

$$L^2(\Omega) := \left\{ v : \Omega \rightarrow \mathbb{R} : \int_{\Omega} v^2 \, d\Omega < \infty \right\}$$

We further enforce that trial solutions $v \in \mathcal{S}$ meet the Dirichlet boundary condition $v|_{\Gamma_D} = g$. We do not directly enforce trial solutions to satisfy the Neumann and Robin boundary conditions as the derivatives of functions

$v \in H^1(\Omega)$ are not necessarily well-defined along the boundary of the domain. However, we already employed the Neumann and Robin boundary conditions to arrive at the final integral identity above. Finally, as previously mentioned, we enforce that weighting functions $v \in \mathcal{V}$ satisfy $v|_{\Gamma_D} = 0$. Thus we have that:

$$\mathcal{S} := \left\{ v \in H^1(\Omega) : v|_{\Gamma_D} = g \right\}$$

$$\mathcal{V} := \left\{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \right\}$$

Note that for the space $\mathcal{S}$ to be well-defined, g must be sufficiently smooth. In particular, it must be equal to the trace of a function in $H^1(\Omega)$. A deeper discussion of traces is beyond the scope of this class, but it is sufficient to enforce that g be a continuous and bounded function with bounded first derivatives. For instance, g may be a continuous piecewise polynomial defined over the Dirichlet boundary.

With the above selections for $\mathcal{S}$ and $\mathcal{V}$, each of the integrals appearing the final integral identity derived above are well-defined provided the functions K, f, h, β, and T_R are sufficiently smooth. For instance, one may select $f \in L^2(\Omega)$, $h \in L^2(\Gamma_N)$, $T_R \in L^2(\Gamma_N)$, and K and β bounded from above. Even weaker conditions on K, f, h, β, and T_R may be enforced, but we will not discuss these further here.

With all of the above in hand, the following is a potential weak form for the problem at hand:

$$
(W) \begin{cases} \text{Find } T \in \mathcal{S} \text{ such that:} \\ \qquad b(T,w) = \ell(w) \quad \text{for all } w \in \mathcal{V} \\ \text{where:} \\ \qquad b(T,w) = \int_{\Omega} \kappa \vec{\nabla} T \cdot \vec{\nabla} w \, d\Omega + \int_{\Gamma_R} \beta T w \, d\Gamma \\ \qquad \ell(w) = \int_{\Omega} f w \, d\Omega + \int_{\Gamma_N} h w \, d\Gamma + \int_{\Gamma_R} \beta T_R w \, d\Gamma \end{cases}
$$

Above, $b(\cdot,\cdot)$ is a bilinear form and $\ell(\cdot)$ is a linear form. The bilinear form is symmetric and κ and β are positive, so the above weak form yields a symmetric positive definite stiffness matrix after discretization. We will discuss this in more detail in the next lecture.

The above weak form is well-posed if κ is bounded from below by a positive value, much like the previously considered 1D model problem. In fact, the above weak form is the 2D analogue of the standard weak form for the previously considered 1D model problem. Analogues to the strong and weak weak forms also exist, but they will not be considered here.

Note that in the above weak form, the Dirichlet boundary condition $T|_{\Gamma_D} = g$ is strongly enforced by the choice of the set of trial solutions, while the Neumann and Robin boundary conditions are weakly enforced through the weak form itself. We thus refer to the Dirichlet boundary condition as an essential boundary condition since it must be applied to all trial solutions and the Neumann and Robin boundary conditions as natural boundary conditions since they are naturally applied through the weak form.

As a final comment, since the bilinear form appearing in the above weak form is symmetric, the weak form coincides with the variational form associated with a particular minimization problem. In particular, the solution $u \in \mathcal{S}$ of Problem (W) is also a solution of:

(M) Find:

$$T = \underset{v \in \mathcal{S}}{\operatorname{argmin}} \; E(v)$$

where:

$$E(v) = \frac{1}{2}\int_{\Omega} \kappa |\vec{\nabla} v|^2 \, d\Omega + \frac{1}{2}\int_{\Gamma_R} \beta v^2 \, d\Gamma - \int_{\Omega} f v \, d\Omega - \int_{\Gamma_N} h v \, d\Gamma - \int_{\Gamma_R} \beta T_R v \, d\Gamma$$