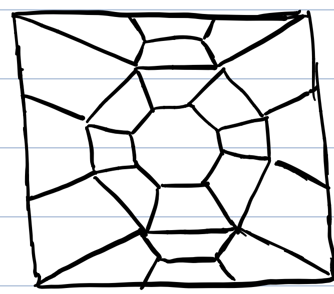
2D Heat Conduction: Quadrilateral Finite Element Approximations:

Just like one can build a triangular finite element approximation over a triangular finite element mesh, one can also build a quadrilateral finite element approximation over a quadrilateral finite element mesh. A quadrilateral mesh is simply a two-dimensional mesh of quadrilateral elements:



Each quadrilateral element has four edges and four vertices, and we say a quadrilateral mesh is conforming if no element vertices lie on another element's edge (i.e., if there are no hanging vertices). We use the same notation for quadrilateral meshes as we did with triangular meshes. For instance, we use $\mathcal{M}^h := \{\Omega^e\}_{e=1}^{n_{el}}$ to denote the mesh of elements. Additionally, a quadrilateral mesh is suitable for finite element analysis provided it satisfies the same requirements listed earlier for a triangular mesh. Defining a space of finite element functions over a quadrilateral finite element mesh, however, is not as straightforward as defining as it is over a triangular finite element mesh. For instance, the space:

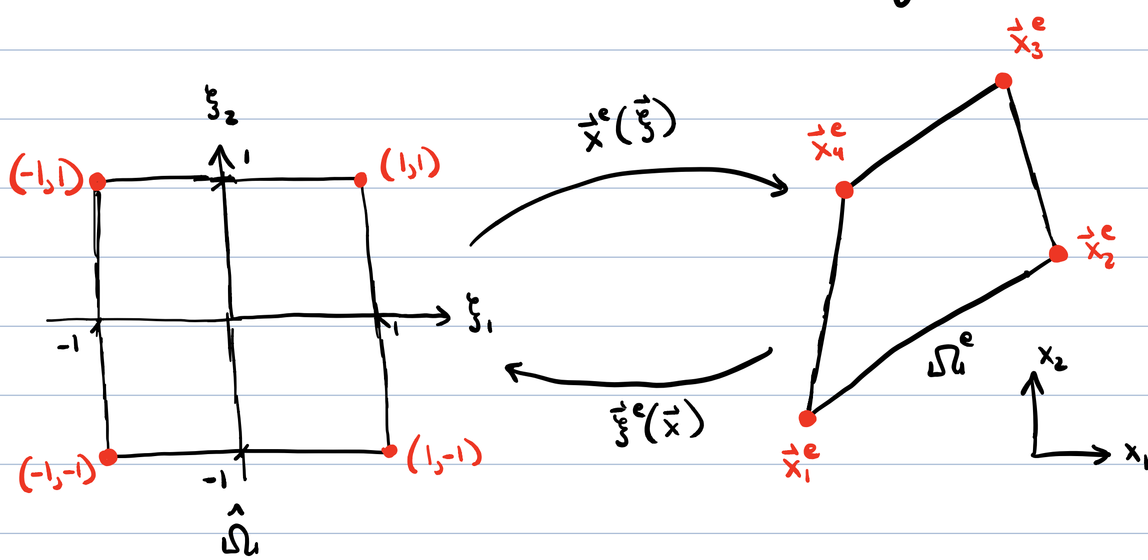
$$\mathcal{P}^k_{cont}(\mathcal{M}^h) := \left\{ v^h \in C^0(\overline{\Omega^h}) : v^h|_{\Omega^e} \in \mathcal{P}^k(\Omega^e) \text{ for } e = 1, \ldots, n_{el} \right\}$$

is generally only composed of global polynomials of degree k unless the mesh has special construction. We describe here two alternate spaces: (i) the space of tensor-product finite element functions of degree k and (ii) the space of serendipity finite element functions of degree k.

To proceed, for any element Ω^e, let $\vec{x}^e : \hat{\Omega} \to \Omega^e$ be defined by:

$$\vec{x}^e(\vec{\xi}) = \vec{x}_1^e \left(\frac{1-\xi_1}{2}\right)\left(\frac{1-\xi_2}{2}\right) + \vec{x}_2^e \left(\frac{1+\xi_1}{2}\right)\left(\frac{1-\xi_2}{2}\right) + \vec{x}_3^e \left(\frac{1+\xi_1}{2}\right)\left(\frac{1+\xi_2}{2}\right) + \vec{x}_4^e \left(\frac{1-\xi_1}{2}\right)\left(\frac{1+\xi_2}{2}\right)$$

where $\hat{\Omega} := (-1,1)^2$ is the parent or reference element and $\{\vec{x}_a^e\}_{a=1}^4$ are the four vertices of Ω^e in counter-clockwise order. Visually:



We denote the inverse of $\vec{x}^e$ as $\vec{\xi}^e$ as displayed above. If element Ω^e is a parallelogram, then $\vec{x}^e$ is an affine map. Otherwise, it is a general bilinear map.

With the above concepts in hand, the space of tensor-product finite element functions of degree k is:

$$\mathcal{Q}^k_{\text{cont}}(\mathcal{M}^h) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{Q}^k(\hat{\Omega}) \text{ for } e = 1, \dots, n_{el} \right\}$$

where:

$$\mathcal{Q}^k(\hat{\Omega}) := \text{span} \left\{ \xi^a \xi^b \right\}_{a,b=0}^{k}$$

is the space of tensor-product polynomials of degree k over the parent element, and the space of serendipity finite element functions of degree k is:

$$\mathcal{S}^k_{\text{cont}}(\mathcal{M}^h) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{S}^k(\hat{\Omega}) \text{ for } e = 1, \dots, n_{el} \right\}$$

where:

$$\mathcal{S}^k(\hat{\Omega}) := \mathcal{P}^k(\hat{\Omega}) + \text{span} \left\{ \xi_1 \xi_2^k, \xi_1^k \xi_2 \right\}$$

is the space of serendipity polynomials of degree k over the parent element.
Note that:

$$\mathcal{P}^k(\hat{\Omega}) \subset \mathcal{S}^k(\hat{\Omega}) \subset \mathcal{Q}^k(\hat{\Omega})$$

So:

$$\mathcal{S}^k_{\text{cont}}(\mathcal{M}^h) \subset \mathcal{Q}^k_{\text{cont}}(\mathcal{M}^h)$$

When $k=1$, the two spaces are equal:

$$\mathcal{S}^1_{\text{cont}}(\mathcal{M}^h) = \mathcal{Q}^1_{\text{cont}}(\mathcal{M}^h)$$

It can also be shown that:

$$\mathcal{P}^k(\Omega^h) \subset \mathcal{Q}^k_{cont}(\mathcal{M}^h)$$

The same is not necessarily true, however, of $\mathcal{S}^k_{cont}(\mathcal{M}^h)$. That is, the space of tensor-product finite element functions of degree k contains all global polynomials of degree k, while the space of serendipity finite element functions, in general, does not. Nonetheless, finite element methods based on both tensor-product and serendipity finite element functions exhibit a convergence rate of k in the H^1-norm when the exact solution is smooth, up to geometry approximation.

It can be readily be shown that:

$$\dim\left(\mathcal{Q}^k(\hat{\Omega})\right) := (k+1)^2$$

$$\dim\left(\mathcal{S}^k(\hat{\Omega})\right) := \begin{cases} 4 & \text{for } k=1 \\ \dfrac{(k+2)(k+1)}{2} + 2 & \text{for } k \geq 2 \end{cases}$$

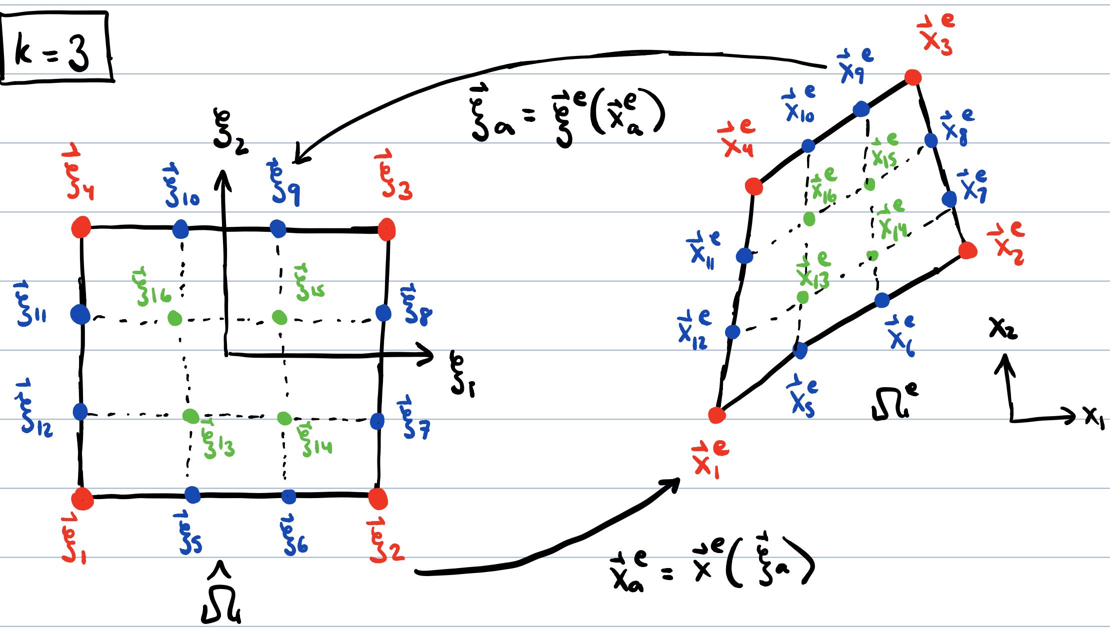
and using a similar technique as employed before for triangular finite element approximation spaces, it can be shown that:

$$\dim\left(\mathcal{Q}^k_{cont}(\mathcal{M}^h)\right) := n_{el} * (k-1)^2 + n_{edge} * (k-1) + n_{vertex}$$

$$\dim\left(\mathcal{S}^k_{cont}(\mathcal{M}^h)\right) := \begin{cases} n_{vertex} & \text{for } k=1 \\ n_{el} * \dfrac{(k-2)(k-3)}{2} + n_{edge} * (k-1) + n_{vertex} & \text{for } k \geq 2 \end{cases}$$

These dimension counts then guide the construction of suitable basis functions over the quadrilatral mesh. As we have done before, we first choose degrees of freedom and then infer corresponding basis functions. For $Q^k_{cont}(\mathcal{M}^h)$, we place one node at every vertex, $(k-1)$ nodes along every edge, and $(k-1)^2$ nodes within every element such that the element pull backs of the nodes to the parent element are equi-spaced:

$\boxed{k=3}$



Note that above, we have locally renumbered the nodes attached to the element according to the following rules:

Rule 1: The first four nodes, $\{\vec{x}^e_a\}_{a=1}^4$, are the vertex nodes, ordered in counter-clockwise fashion.

Rule 2: The next $3*(k-1)$ nodes, $\{\vec{x}^e_a\}_{a=5}^{4k}$, are the edge nodes, also ordered in counter-clockwise fashion. The first of these nodes, $\vec{x}_5^e$, is taken to be the first to the right of $\vec{x}_1^e$.

<u>Rule 3</u>: The remaining nodes, $\{\vec{x}_a^e\}_{a=4k+1}^{n_{en}}$, are the element nodes. These nodes are organized into concentric quads. The nodes in the outmost quad are numbered first, using Rules 1 and 2 unless there is only one node, followed by the next outermost quad and so on. The first node in every concentric quad is taken to be that closest to $\vec{x}_1^e$.

The parent element nodes $\{\vec{\xi}_a\}_{a=1}^{n_{en}}$ are numbered in similar fashion. As was done previously, the relationship between local and global node numbers is stored in an <u>element connectivity</u>:

$$A = IEN(a,e)$$

Global Node Number   Local Node Number

The degrees of freedom are then chosen as finite element function values at the nodes, $\{y^h(\vec{x}_A)\}_{A=1}^{n_{nod}}$, and the corresponding basis functions $\{N_A\}_{A=1}^{n_{nod}}$ are defined by the interpolation property:

$$N_A(\vec{x}_B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}$$

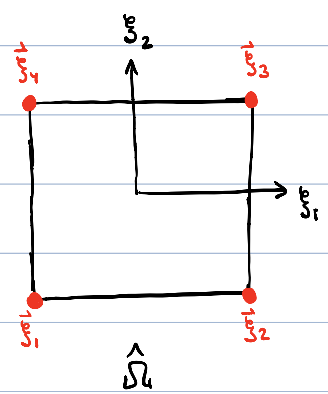
and these basis functions can be described locally in terms of <u>shape functions</u> over the parent element:

$$N_A(\vec{x}) = \begin{cases} \hat{N}_a(\vec{\xi}^{\,e}(\vec{x})) & \text{if there is an } a \text{ such that } A = IEN(a,e) \\ 0 & \text{otherwise} \end{cases}$$

where $\{\hat{N}_a\}_{a=1}^{n_{en}}$ are defined by the interpolation property:

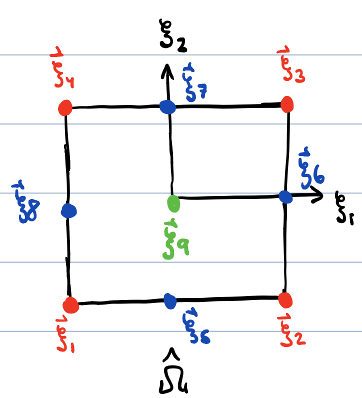
$$\hat{N}_a(\vec{\xi}_b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

For example:



$$\hat{N}_1(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$
$$\hat{N}_2(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$
$$\hat{N}_3(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$
$$\hat{N}_4(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

for $k=1$, and:



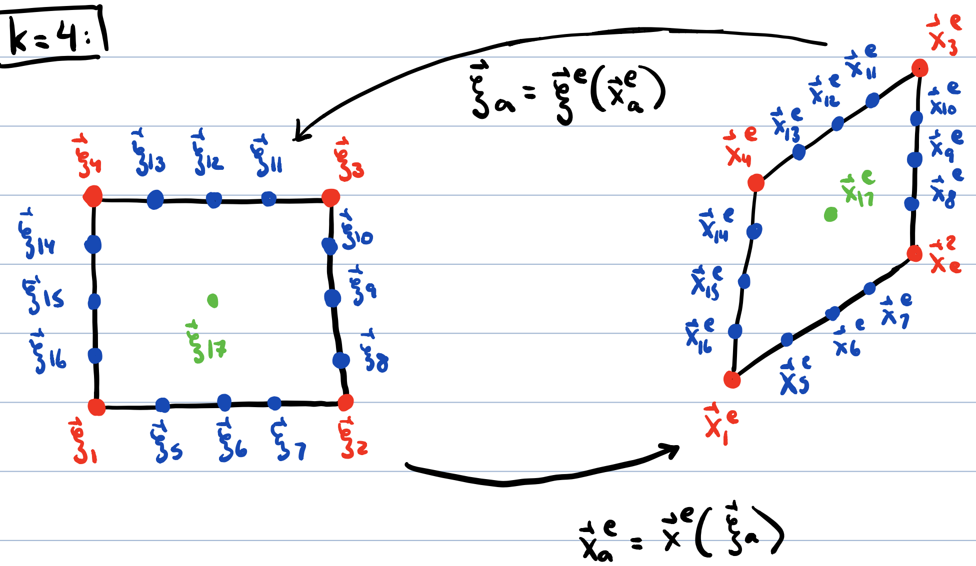
$$\hat{N}_1(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(-\xi_1)(1-\xi_2)(-\xi_2)$$
$$\hat{N}_2(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(\xi_1)(1-\xi_2)(-\xi_2)$$
$$\hat{N}_3(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(\xi_1)(1+\xi_2)(\xi_2)$$
$$\hat{N}_4(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(-\xi_1)(1+\xi_2)(\xi_2)$$

$$\hat{N}_5(\vec{\xi}) = \frac{1}{2}(1+\xi_1)(1-\xi_1)(1-\xi_2)(-\xi_2)$$
$$\hat{N}_6(\vec{\xi}) = \frac{1}{2}(1+\xi_1)(\xi_1)(1+\xi_2)(1-\xi_2)$$
$$\hat{N}_7(\vec{\xi}) = \frac{1}{2}(1+\xi_1)(1-\xi_1)(1+\xi_2)(\xi_2)$$
$$\hat{N}_8(\vec{\xi}) = \frac{1}{2}(1-\xi_1)(-\xi_1)(1+\xi_2)(1-\xi_2)$$
$$\hat{N}_9(\vec{\xi}) = (1+\xi_1)(1-\xi_1)(1+\xi_2)(1-\xi_2)$$

for $k=2$. Plots of these shape functions are included at the end of these notes. Note the basis functions $\{N_A\}_{A=1}^{n_{nod}}$ are locally supported and form a partition of unity, and for $k=1$, they are non-negative. As they are interpolatory, we call them Lagrange basis functions, and we call the degrees of freedom nodal degrees of freedom. Note also that the shape functions are tensor products of univariate shape functions, hence why we refer to the finite element functions as tensor-product finite element functions.

For $\mathcal{S}^k_{cont}(\mathcal{M}^h)$, we instead place one node at every vertex, $(k-1)$ nodes along every edge, and, if $k \geq 2$, $(k-2)(k-3)/2$ nodes within each element. It is common to choose the nodes along edges to be equi-spaced, but there is no particular preferred way to select the locations of nodes within elements. To ensure that basis functions can be written in terms of generic shape functions, however, one should at least ensure the element pull backs of the nodes are always the same locations on the parent element:

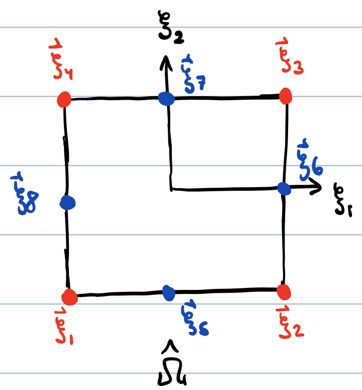
$\boxed{k=4:}$



The Lagrange basis functions $\{N_A\}_{A=1}^{n_{nod}}$ corresponding to nodal degrees of freedom $\{v^h(x_A)\}_{A=1}^{n_{nod}}$ can then be written in terms of generic shape functions $\{\hat{N}_a\}_{a=1}^{n_{en}}$ over the parent element defined by interpolation:

$$\hat{N}_a(\vec{\xi}_b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

For $k=2$:



$$\hat{N}_1(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1-\xi_2)(-\xi_1-\xi_2-1)$$
$$\hat{N}_2(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1-\xi_2)(\xi_1-\xi_2-1)$$
$$\hat{N}_3(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1+\xi_2)(\xi_1+\xi_2-1)$$
$$\hat{N}_4(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1+\xi_2)(-\xi_1+\xi_2-1)$$
$$\hat{N}_5(\vec{\xi}) = \frac{1}{2}(1-\xi_1^2)(1-\xi_2)$$
$$\hat{N}_6(\vec{\xi}) = \frac{1}{2}(1+\xi_1)(1-\xi_2^2)$$
$$\hat{N}_7(\vec{\xi}) = \frac{1}{2}(1-\xi_1^2)(1+\xi_2)$$
$$\hat{N}_8(\vec{\xi}) = \frac{1}{2}(1-\xi_1)(1-\xi_2^2)$$

Plots of these shape functions are given at the end of these notes.

Before proceeding, it is useful to examine an algorithm for determining shape functions that satisfy an interpolation property. Suppose that we have a set of nodes $\{\vec{\xi}_a\}_{a=1}^{n_{en}}$ and we seek shape functions $\{\hat{N}_a\}_{a=1}^{n_{en}}$ comprising a basis for a local approximation space $\mathcal{P}$ that satisfy:

$$\hat{N}_a(\vec{\xi}_b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

If $\mathcal{P}$ admits a basis $\{B_c\}_{c=1}^{n_{en}}$, then we can express each shape function as:

$$\hat{N}_a(\vec{\xi}) = \sum_{c=1}^{n_{en}} \underbrace{C_{ac}}_{\text{Unknown Coefficients}} B_c(\vec{\xi})$$

Then:

$$\sum_{c=1}^{n_{en}} C_{ac} B_c(\vec{\xi}_b) = \hat{N}_a(\vec{\xi}_b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

or:

$$\underline{\underline{C}}\;\underline{\underline{B}} = \underline{\underline{I}}$$

where $B_{cb} = B_c(\vec{\xi}_b)$ and $\underline{\underline{I}}$ is the identity matrix.

Thus the unknown coefficients are simply given by:

$$\underline{\underline{C}} = \underline{\underline{B}}^{-1}$$

As an example, suppose we want to find shape functions for $Q^1(\hat{\Omega})$ and:

$$\vec{\xi}_1 = (-1,-1), \quad \vec{\xi}_2 = (1,-1), \quad \vec{\xi}_3 = (1,1), \quad \vec{\xi}_3 = (-1,1)$$

We can choose the power basis:

$$B_1(\vec{\xi}) = 1, \quad B_2(\vec{\xi}) = \xi_1, \quad B_3(\vec{\xi}) = \xi_2, \quad B_4(\vec{\xi}) = \xi_1 \xi_2$$

as a basis for $Q^1(\hat{\Omega})$, yielding:

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Then:

$$\underline{\underline{C}} = \underline{\underline{B}}^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

We then find:

$$\hat{N}_1(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$
$$\hat{N}_2(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$
$$\hat{N}_3(\vec{\xi}) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$
$$\hat{N}_4(\vec{\xi}) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

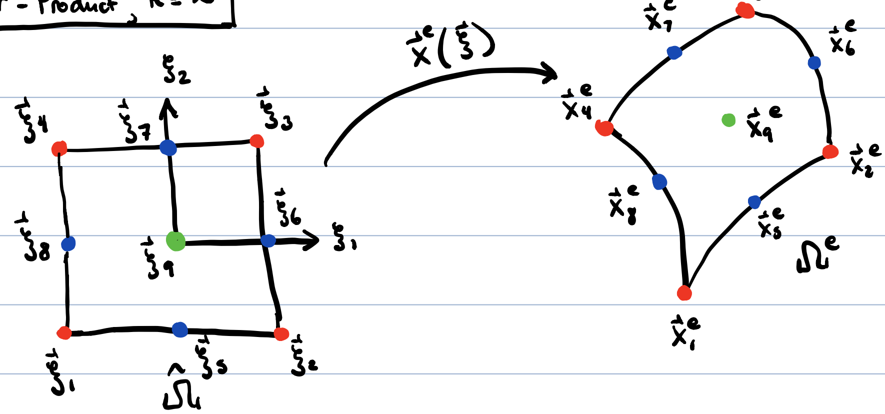
which coincides with what was presented before. It should be noted that $\underline{\underline{B}}$ is an example of (the transpose of) a generalized Vandermonde matrix.

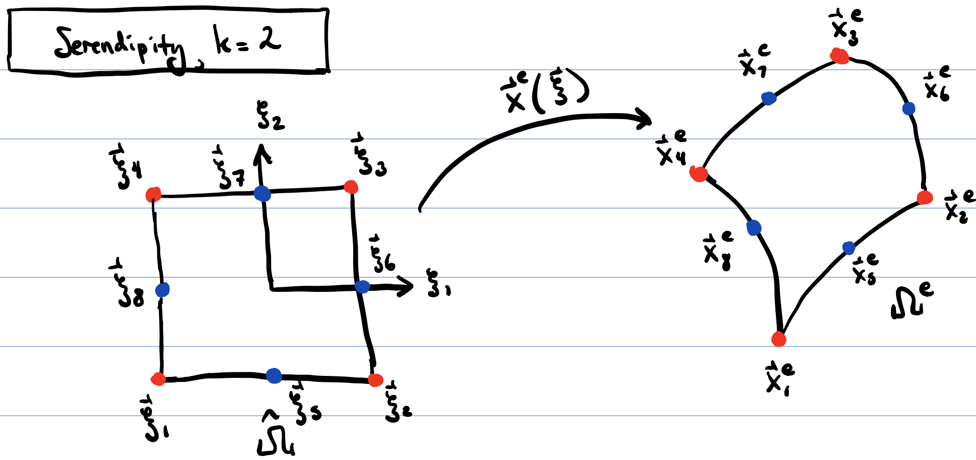
Just as we did in the triangular mesh setting, we can curve a quadrilateral mesh by moving the nodes $\{\vec{x}_A\}_{A=1}^{n_{nod}}$ and replacing the element maps $\vec{x}^e$ with:

$$\vec{x}^e(\vec{\xi}) = \sum_{a=1}^{n_{en}} \vec{x}_a^e \, \hat{N}_a(\vec{\xi})$$

Visually:

$\boxed{\text{Tensor-Product, } k=2}$





We can then define an isoparametric space of tensor-product finite element functions:

$$Q^k_{cont}\left(\mathcal{M}^h; \{x_A\}_{A=1}^{n_{nod}}\right) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in Q^k(\hat{\Omega}) \text{ for } e = 1, \ldots, n_{el} \right\}$$

and an isoparametric space of serendipity finite element functions:

$$\mathcal{S}^k_{cont}\left(\mathcal{M}^h; \{x_A\}_{A=1}^{n_{nod}}\right) := \left\{ v^h \in C^0(\bar{\Omega}^h) : v^h \circ \vec{x}^e \in \mathcal{S}^k(\hat{\Omega}) \text{ for } e = 1, \ldots, n_{el} \right\}$$

Both spaces admit a Lagrange basis $\{N_A\}_{A=1}^{n_{nod}}$ satisfying:

$$N_A(\vec{x}_B) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}$$

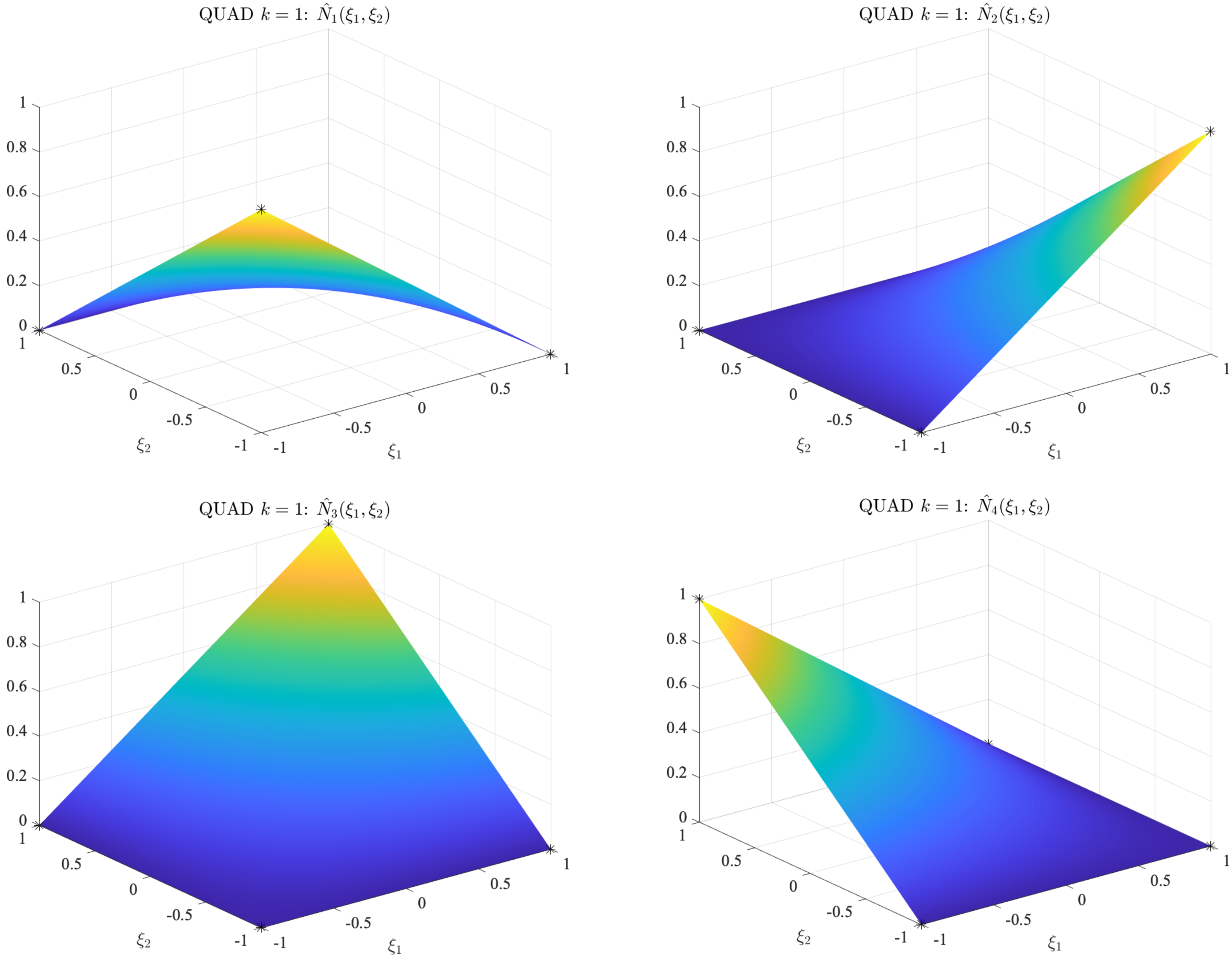
and:

$$N_A(\vec{x}) = \begin{cases} (\hat{N}_a \circ \vec{\xi}^e)(\vec{x}) & \text{if there is an } a \text{ such that } A = IEN(a,e) \\ 0 & \text{otherwise} \end{cases}$$

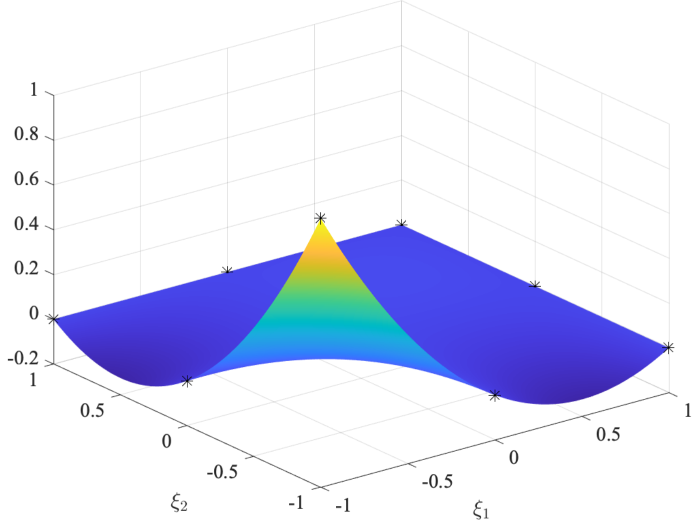
where $\vec{\xi}^e : \Omega^e \to \hat{\Omega}$ is the inverse of $\vec{x}^e : \hat{\Omega} \to \Omega^e$ and nodal degrees of freedom $\{v^h(\vec{x}_A)\}_{A=1}^{n_{nod}}$. The basis functions are locally supported and form a partition of unity, and both isoparametric spaces of tensor-product finite element functions and isoparametric spaces of serendipity finite element functions contain linear polynomials. Finally, both isoparametric isoparametric spaces of tensor-product finite element functions and isoparametric spaces of serendipity finite element functions may be applied to the Galerkin approximation of steady two-dimensional heat conduction, giving rise to finite element solutions of the form:

$$T^h(x) = \sum_{A \in \eta - \eta_D} \underbrace{T^h(x_A)}_{\text{Degree of Freedom}} N_A(\vec{x}) + \underbrace{\sum_{A \in \eta_D} \overbrace{T^h(x_A)}^{g(x_A)} N_A(\vec{x})}_{g^h(\vec{x})}$$

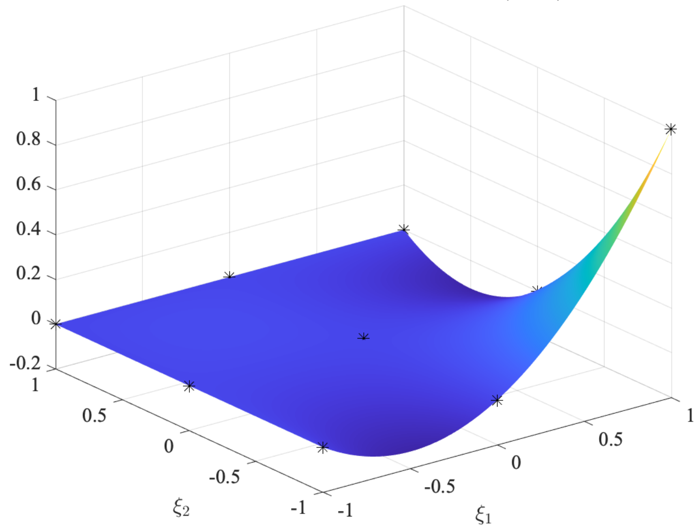
whose degrees of freedom may be attained by solving a linear system of identical form as Problem (L) from the previous lecture.


QUAD $k = 1$: $\hat{N}_1(\xi_1, \xi_2)$
QUAD $k = 1$: $\hat{N}_2(\xi_1, \xi_2)$
QUAD $k = 1$: $\hat{N}_3(\xi_1, \xi_2)$
QUAD $k = 1$: $\hat{N}_4(\xi_1, \xi_2)$
ξ_1
ξ_2

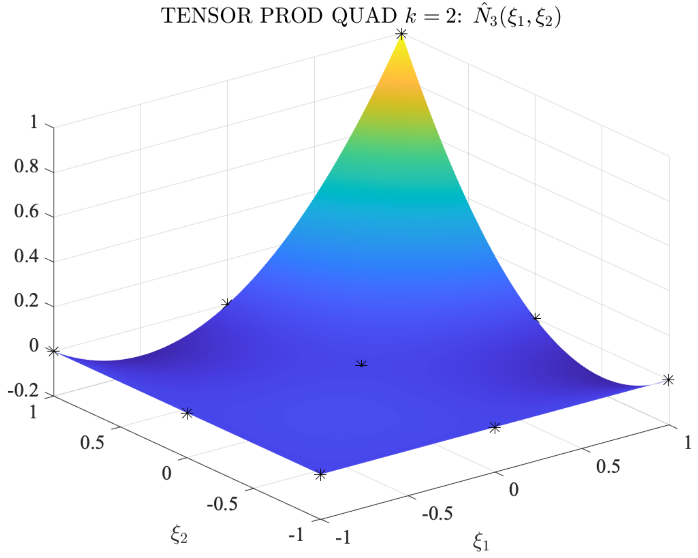
TENSOR PROD QUAD $k = 2$: $\hat{N}_1(\xi_1, \xi_2)$



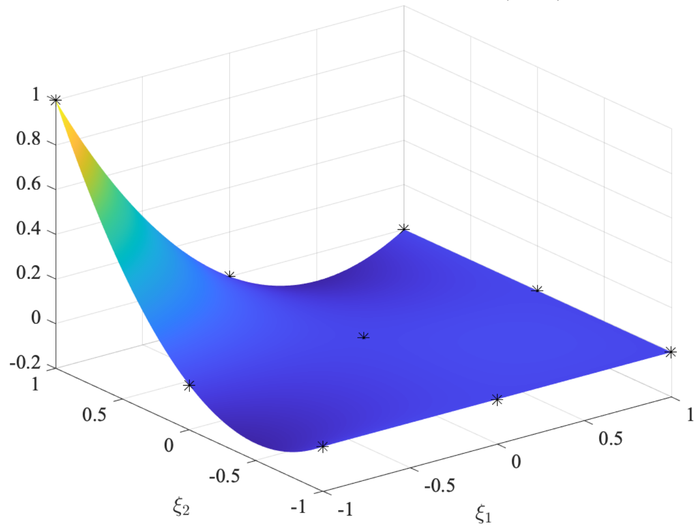
TENSOR PROD QUAD $k = 2$: $\hat{N}_2(\xi_1, \xi_2)$



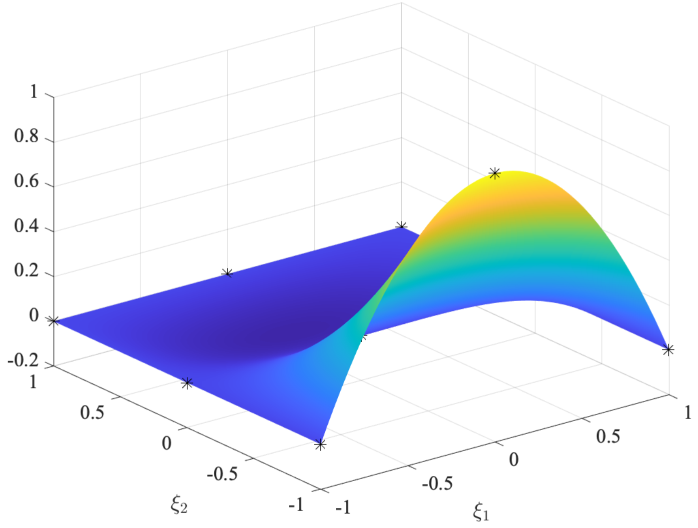
TENSOR PROD QUAD $k = 2$: $\hat{N}_3(\xi_1, \xi_2)$



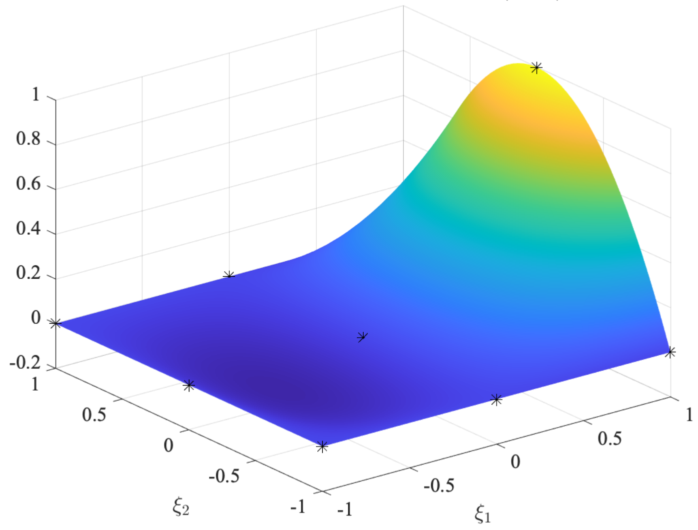
TENSOR PROD QUAD $k = 2$: $\hat{N}_4(\xi_1, \xi_2)$



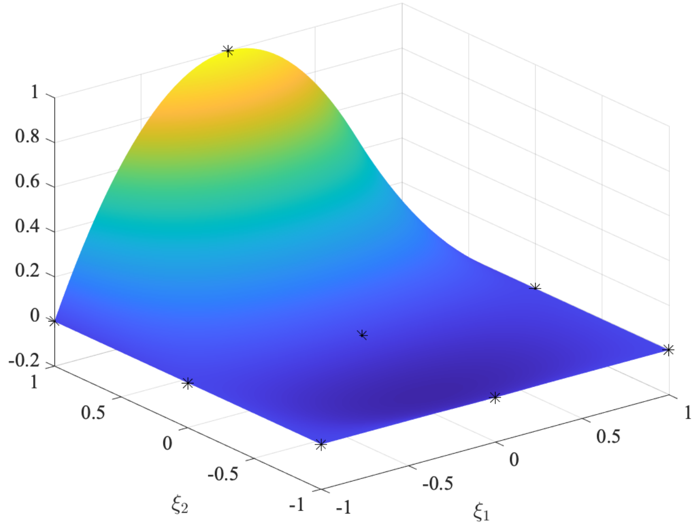
TENSOR PROD QUAD $k = 2$: $\hat{N}_5(\xi_1, \xi_2)$



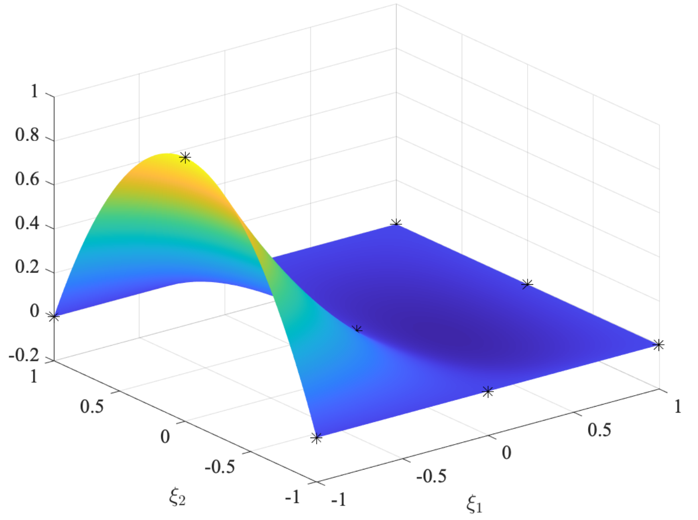
TENSOR PROD QUAD $k = 2$: $\hat{N}_6(\xi_1, \xi_2)$



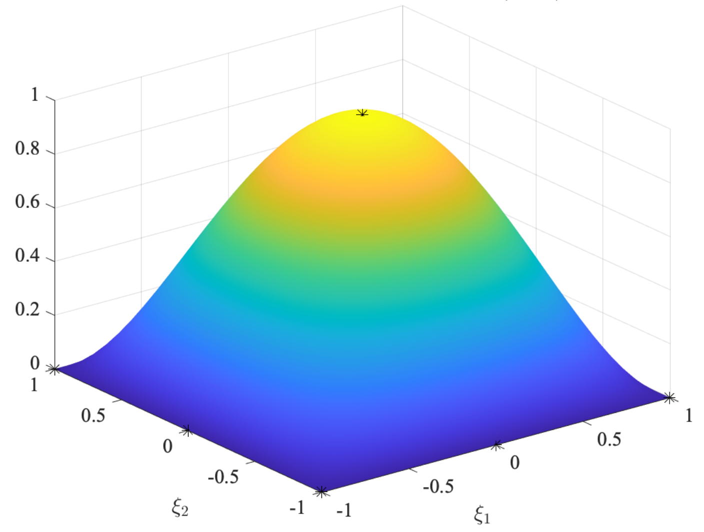
TENSOR PROD QUAD $k = 2$: $\hat{N}_7(\xi_1, \xi_2)$

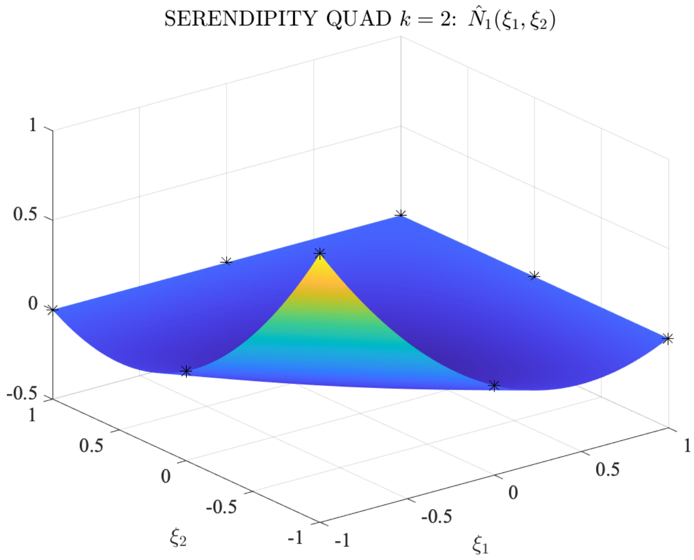


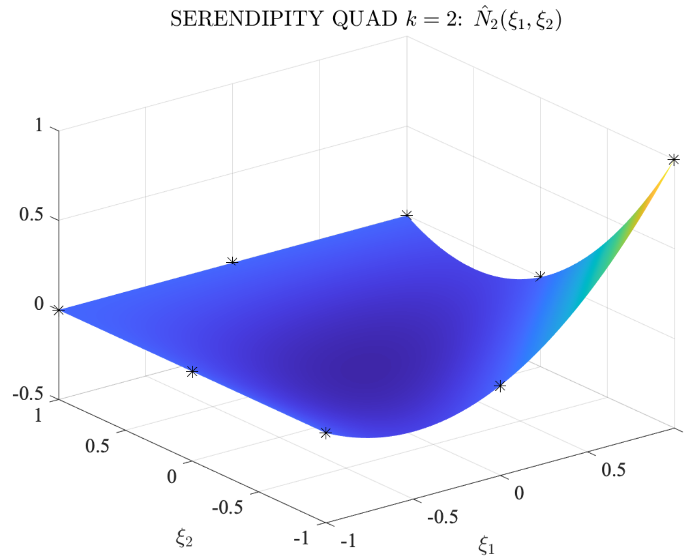
TENSOR PROD QUAD $k = 2$: $\hat{N}_8(\xi_1, \xi_2)$

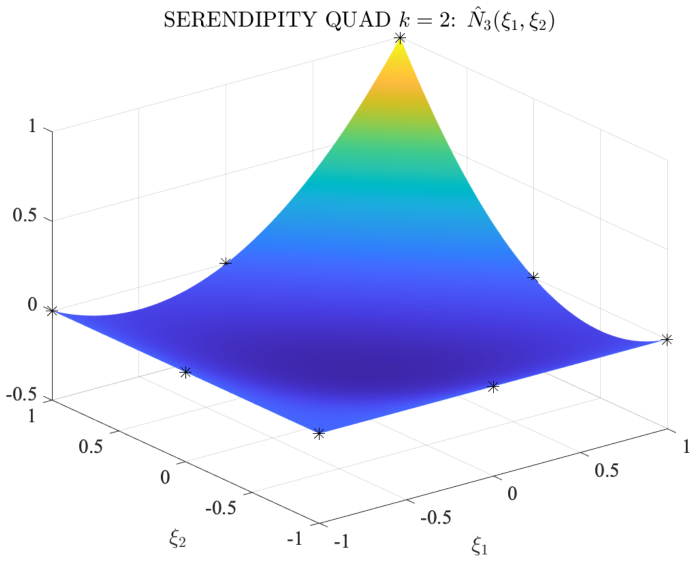


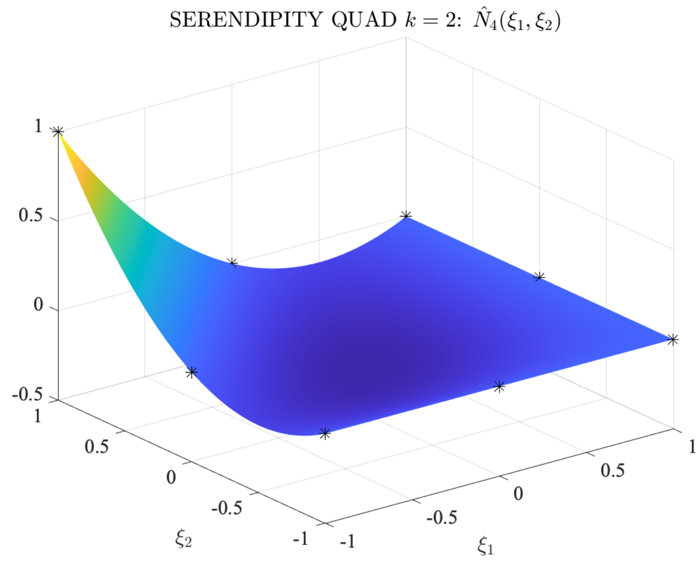
TENSOR PROD QUAD $k = 2$: $\hat{N}_9(\xi_1, \xi_2)$

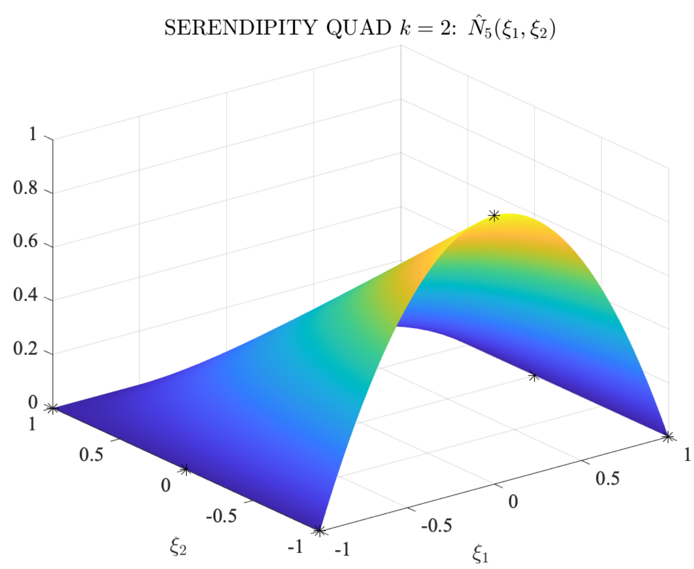


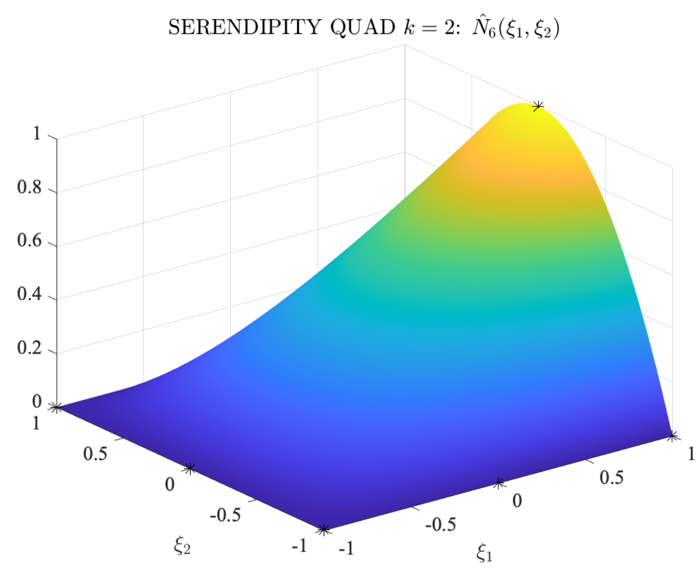

SERENDIPITY QUAD $k = 2$: $\hat{N}_1(\xi_1, \xi_2)$
ξ_1
ξ_2

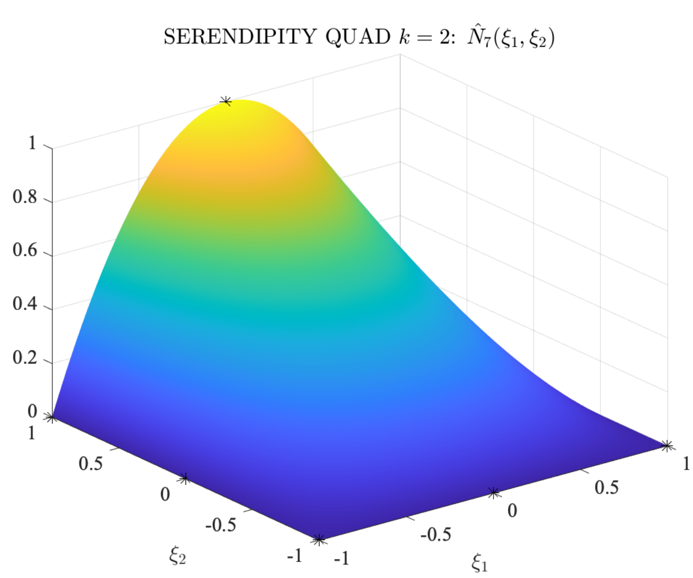

SERENDIPITY QUAD $k = 2$: $\hat{N}_2(\xi_1, \xi_2)$
ξ_1
ξ_2

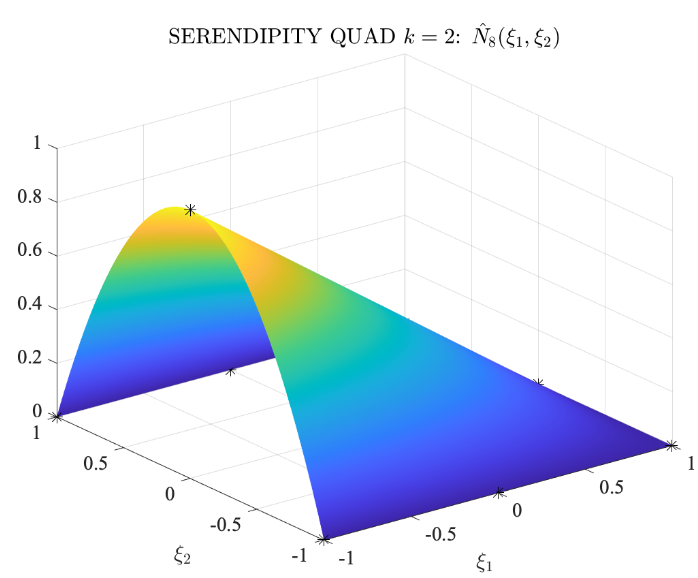

SERENDIPITY QUAD $k = 2$: $\hat{N}_3(\xi_1, \xi_2)$
ξ_1
ξ_2


SERENDIPITY QUAD $k = 2$: $\hat{N}_4(\xi_1, \xi_2)$
ξ_1
ξ_2


SERENDIPITY QUAD $k = 2$: $\hat{N}_5(\xi_1, \xi_2)$
ξ_1
ξ_2


SERENDIPITY QUAD $k = 2$: $\hat{N}_6(\xi_1, \xi_2)$
ξ_1
ξ_2


SERENDIPITY QUAD $k = 2$: $\hat{N}_7(\xi_1, \xi_2)$
ξ_1
ξ_2


SERENDIPITY QUAD $k = 2$: $\hat{N}_8(\xi_1, \xi_2)$
ξ_1
ξ_2