Plane Strain Elastostatics: Strong and Weak Forms:

We now proceed forward to our next application: plane strain elastostatics. While steady two-dimensional heat conduction is governed by a scalar-valued partial differential equation, plane strain elastostatics is instead governed by a vector-valued partial differential equation. This introduces additional complications which must be taken into account in a finite element analysis. In particular, finite element analysis of plane strain elastostatics yields vector-valued finite element solutions.

To begin, recall that a three-dimensional elastic body in static equilibrium satisfies:

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 = 0$$
$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 = 0$$

where the σ_{ij}'s are the components of the (Cauchy) stress tensor and the f_i's are the components of the body force (per unit volume) vector. When the body is composed of isotropic material and subject to small deformation, the stress components are equal to:

$$\sigma_{11} = 2\mu\,\varepsilon_{11} + \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$\sigma_{22} = 2\mu\,\varepsilon_{22} + \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$\sigma_{33} = 2\mu\,\varepsilon_{33} + \lambda\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$
$$\sigma_{12} = \sigma_{21} = 2\mu\,\varepsilon_{12} = 2\mu\,\varepsilon_{21}$$
$$\sigma_{13} = \sigma_{31} = 2\mu\,\varepsilon_{13} = 2\mu\,\varepsilon_{31}$$

$$\sigma_{23} = \sigma_{32} = 2\mu\,\varepsilon_{23} = 2\mu\,\varepsilon_{32}$$

where λ and μ are the Lamé parameters and the ε_{ij}'s are the components of the (infinitesimal) strain tensor. The strain components are equal to:

$$\varepsilon_{11} = u_{1,1}$$

$$\varepsilon_{22} = u_{2,2}$$

$$\varepsilon_{33} = u_{3,3}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right)$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(u_{1,3} + u_{3,1}\right)$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(u_{2,3} + u_{3,2}\right)$$

where the u_i's are the components of the displacement vector. The Lamé parameters are typically computed using the Young's modulus E and Poisson ratio ν:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

A body is said to be in plane strain when all of the strains associated with a particular direction, referred to as the out-of-plane direction, are zero. Taking the x_3-direction to be the out-of-plane direction, plane strain occurs when:

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

Then, the stress components are equal to:

$$\sigma_{11} = 2\mu\, \varepsilon_{11} + \lambda\,(\varepsilon_{11} + \varepsilon_{22})$$

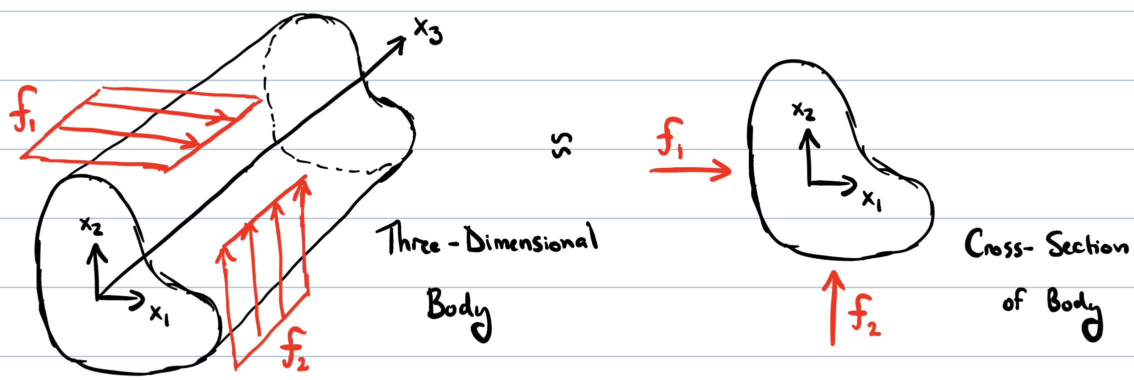
$$\sigma_{22} = 2\mu\, \varepsilon_{22} + \lambda\,(\varepsilon_{11} + \varepsilon_{22})$$

$$\sigma_{12} = 2\mu\, \varepsilon_{12}$$

$$\sigma_{33} = \lambda\,(\varepsilon_{11} + \varepsilon_{22})$$

$$\sigma_{13} = \sigma_{31} = 0$$

Plane strain occurs when one dimension of a body, that associated with the out-of-plane direction, is much larger than the dimensions in the other directions. If the in-plane cross-section of the body is uniform, the loading on the body is purely in the in-plane direction and does not vary in the out-of-plane direction, and motion in the out-of-plane direction is constrained, then plane strain results. Plane strain enables one to perform a two-dimensional analysis along a body's cross-section rather than a full three-dimensional analysis across the body:



In particular, the in-plane displacement components u_1 and u_2 are only functions of the in-plane directions x_1 and x_2 for a body in plane strain, and these components may be found by solving the in-plane equilibrium equations:

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + f_2 = 0$$

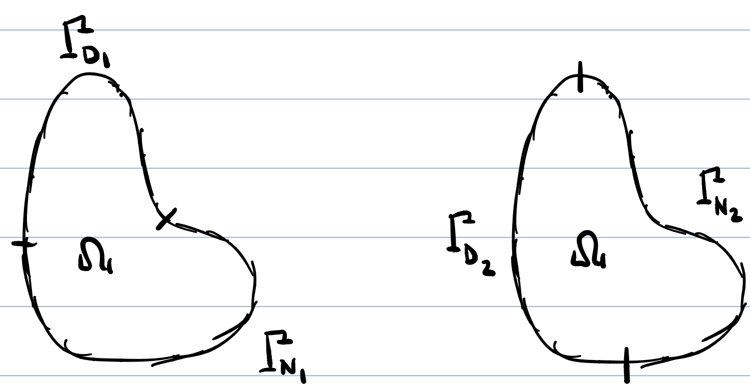
together with appropriate boundary conditions. Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a body in plane-strain. Then, at every point along the boundary of Ω, denoted Γ, two boundary conditions may be applied, one associated with each in-plane direction. Thus, we split the boundary twice, once into a Dirichlet boundary Γ_{D_1} and Neumann boundary Γ_{N_1} such that:

$$\Gamma_{D_1} \cap \Gamma_{N_1} = \phi \qquad \overline{\Gamma_{D_1} \cup \Gamma_{N_1}} = \Gamma$$

and once into a Dirichlet boundary Γ_{D_2} and Neumann boundary Γ_{N_2} such that:

$$\Gamma_{D_2} \cap \Gamma_{N_2} = \phi \qquad \overline{\Gamma_{D_2} \cup \Gamma_{N_2}} = \Gamma$$

Visually:



Along Γ_{D_1}, the in-plane displacement component u_1 is set to a particular prescribed displacement $g_1 : \Gamma_{D_1} \to \mathbb{R}$:

$$u_1 = g_1$$

while along Γ_{D_2}, the in-plane displacement component u_2 is set to a particular prescribed displacement $g_2 : \Gamma_{D_2} \to \mathbb{R}$:

$$u_2 = g_2$$

Along Γ_{N_1} and Γ_{N_2}, components of the traction vector $\vec{t} = \vec{\vec{\sigma}} \cdot \vec{n}$ are instead specified. Along Γ_{N_1}, the in-plane traction component $t_1 = \sigma_{11} n_1 + \sigma_{12} n_2$ is set to a particular prescribed traction $h_1 : \Gamma_{N_1} \to \mathbb{R}$:

$$t_1 = h_1$$

while along Γ_{N_2}, the in-plane traction component $t_2 = \sigma_{21} n_1 + \sigma_{22} n_2$ is set to a particular prescribed traction $h_2 : \Gamma_{N_2} \to \mathbb{R}$:

$$t_2 = h_2$$

Collecting the above, we have arrived at the following strong form for the plane strain elastostatics problem:

$$(S) \left\{ \begin{array}{l} \text{Find } u_1 : \bar{\Omega} \to \mathbb{R} \text{ and } u_2 : \bar{\Omega} \to \mathbb{R} \text{ such that:} \\ \qquad \sigma_{11,1} + \sigma_{12,2} + f_1 = 0 \qquad \text{in } \Omega \\ \qquad \sigma_{21,1} + \sigma_{22,2} + f_2 = 0 \qquad \text{in } \Omega \\ \qquad u_1 = g_1 \qquad \text{on } \Gamma_{D_1} \\ \qquad u_2 = g_2 \qquad \text{on } \Gamma_{D_2} \end{array} \right.$$

$$
\begin{array}{lr}
t_1 = h_1 & \text{on } \Gamma_{N_1} \\
t_2 = h_2 & \text{on } \Gamma_{N_2}
\end{array}
$$

The plane strain assumption is employed in the analysis of dams, tunnels, and other geotechnical works, so the above strong form can be utilized to analyze these structures.

To construct a weak form, we again employ weighted residuals. However, in the context of plane strain elastostatics, we have two residuals instead of one:

$$R_1((u_1,u_2)) = -(\sigma_{11,1} + \sigma_{12,2} + f_1)$$

$$R_2((u_1,u_2)) = -(\sigma_{21,1} + \sigma_{22,2} + f_2)$$

We multiply each by a weighting function and integrate over the domain to arrive at two weighted residuals:

$$\int_\Omega R_1((u_1,u_2))\, w_1 \, d\Omega = 0$$

$$\int_\Omega R_1((u_1,u_2))\, w_1 \, d\Omega = 0$$

or equivalently:

$$-\int_\Omega (\sigma_{11,1} + \sigma_{12,2})\, w_1 \, d\Omega = \int_\Omega f_1 \, w_1 \, d\Omega$$

$$-\int_{\Omega} \left(\sigma_{21,1} + \sigma_{22,2}\right) w_2 \, d\Omega = \int_{\Omega} f_2 w_2 \, d\Omega$$

We integrate the left hand sides of both expressions above by parts, resulting in:

$$\int_{\Omega} \left(\sigma_{11} w_{1,1} + \sigma_{12} w_{1,2}\right) d\Omega - \int_{\Gamma} \left(\overbrace{\sigma_{11} n_1 + \sigma_{12} n_2}^{t_1}\right) w_1 \, d\Gamma = \int_{\Omega} f_1 w_1 \, d\Omega$$

$$\int_{\Omega} \left(\sigma_{21} w_{2,1} + \sigma_{22} w_{2,2}\right) d\Omega - \int_{\Gamma} \left(\overbrace{\sigma_{21} n_1 + \sigma_{22} n_2}^{t_2}\right) w_2 \, d\Gamma = \int_{\Omega} f_2 w_2 \, d\Omega$$

We write:

$$\int_{\Gamma} t_1 w_1 = \int_{\Gamma_{D_1}} t_1 w_1 \, d\Gamma + \int_{\Gamma_{N_1}} \overbrace{t_1}^{h_1} w_1 \, d\Gamma$$

$$\int_{\Gamma} t_2 w_2 = \int_{\Gamma_{D_2}} t_2 w_2 \, d\Gamma + \int_{\Gamma_{N_2}} \overbrace{t_2}^{h_2} w_2 \, d\Gamma$$

If we enforce $w_1\big|_{\Gamma_{D_1}} = 0$ and $w_2\big|_{\Gamma_{D_2}} = 0$, we then have by summation:

$$\int_{\Omega} \left(\sigma_{11} w_{1,1} + \sigma_{12} w_{1,2} + \sigma_{21} w_{2,1} + \sigma_{22} w_{2,2}\right) d\Omega = \int_{\Omega} \left(f_1 w_1 + f_2 w_2\right) d\Omega + \int_{\Gamma_{N_1}} h_1 w_1 \, d\Gamma + \int_{\Gamma_{N_2}} h_2 w_2 \, d\Gamma$$

As:

$$\sigma_{11} = 2\mu\,\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22}) = 2\mu\, u_{1,1} + \lambda(u_{1,1} + u_{2,2})$$
$$\sigma_{22} = 2\mu\,\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{22}) = 2\mu\, u_{2,2} + \lambda(u_{1,1} + u_{2,2})$$
$$\sigma_{12} = \sigma_{21} = 2\mu\,\varepsilon_{12} = 2\mu\,\varepsilon_{21} = \mu(u_{1,2} + u_{2,1})$$

we equivalently have:

$$\int_{\Omega} \Big(2\mu\,(u_{1,1}\,w_{1,1} + u_{2,2}\,w_{2,2}) + \mu\,(u_{1,2} + u_{2,1})(w_{1,2} + w_{2,1}) + \lambda\,(u_{1,1} + u_{2,2})(w_{1,1} + w_{2,2})\Big)\, d\Omega$$

$$= \int_{\Omega} (f_1 w_1 + f_2 w_2)\, d\Omega + \int_{\Gamma^{2}_{N_1}} h_1 w_1\, d\Gamma^{2} + \int_{\Gamma^{2}_{N_2}} h_2 w_2\, d\Gamma^{2}$$

Collecting in-plane displacements, forces, and weighting functions into vectors:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and defining:

Strain Vector $$\underset{\sim}{\varepsilon}(\vec{u}) = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{bmatrix}$$

Virtual Strain Vector $$\underset{\sim}{\varepsilon}(\vec{w}) = \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{bmatrix}$$

and:

$$\underline{\underline{D}} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

We can write:

$$\int_{\Omega} \underline{\varepsilon}(\vec{w})^T \underline{\underline{D}}\, \underline{\varepsilon}(\vec{u})\, d\Omega = \int_{\Omega} \vec{w} \cdot \vec{f}\, d\Omega + \int_{\Gamma_{N_1}} w_1 h_1\, d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2\, d\Gamma$$

We use the above integral identity to build a weak form. We choose:

$$\mathcal{V} := \left\{ \vec{v} \in \left(H^1(\Omega)\right)^2 : \; v_1\big|_{\Gamma_{D_1}} = 0 \text{ and } v_2\big|_{\Gamma_{D_2}} = 0 \right\}$$

as the space of (vector-valued) weighting functions and:

$$\mathcal{S} := \left\{ \vec{v} \in \left(H^1(\Omega)\right)^2 : \; v_1\big|_{\Gamma_{D_1}} = g_1 \text{ and } v_2\big|_{\Gamma_{D_2}} = g_2 \right\}$$

as the set of (vector-valued) trial solutions so that Dirichlet boundary conditions as strongly enforced. With these choices, we arrive at the following weak form for the plane strain elastostatics problem:

Find $\vec{u} \in \mathcal{S}$ such that:

$$b(\vec{u}, \vec{w}) = \ell(\vec{w}) \qquad \text{for all } \vec{w} \in \mathcal{V}$$

where:

$$(W)\left\{\begin{aligned} b(\vec{u},\vec{w}) &= \int_{\Omega} \underline{\varepsilon}(\vec{w})^T \underline{\underline{D}}\, \underline{\varepsilon}(\vec{u})\, d\Omega \\ \ell(\vec{w}) &= \int_{\Omega} \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Gamma_{N_1}} w_1 h_1\, d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2\, d\Gamma \end{aligned}\right.$$

Above, $b(\cdot,\cdot)$ is a bilinear form and $\ell(\cdot)$ is a linear form. The bilinear form is symmetric and the Lamé parameters are positive for most materials, so the above weak form gives rise to a symmetric positive definite stiffness matrix after discretization just like for steady two-dimensional heat conduction. The weak form is also well-posed provided the Lamé parameters are bounded above and below by positive numbers, and in fact, it is well-posed for Poisson ratios ν between -1 and 0.5 provided the Young's modulus E is bounded above and below by positive numbers. Finally, note that Dirichlet boundary conditions are enforced strongly and Neumann boundary conditions are enforced weakly in the above weak form, just as was the case for our weak form for steady two-dimensional heat conduction.

The above weak form coincides with the variational form of the following minimization problem:

$$(M)\left\{\begin{aligned} &\text{Find:} \\ &\qquad \vec{u} = \underset{\vec{v}\in\mathcal{S}}{\operatorname{argmin}}\; E(\vec{v}) \end{aligned}\right.$$

where:

$$E(\vec{v}) = \frac{1}{2} b(\vec{v},\vec{v}) - \ell(\vec{v})$$

The above is commonly referred to as the principle of total minimum potential energy. It states that the displacement of a plane strain elastic body in static equilibrium is that which minimizes the sum of internal potential energy:

$$\frac{1}{2} b(\vec{u},\vec{u})$$

and external potential energy:

$$-\ell(\vec{u})$$

The variational form is alternatively referred to as the principle of virtual work. It states that the displacement of a plane strain elastic body in static equilibrium is that for which the virtual work due to internal stresses:

$$b(\vec{u},\vec{w})$$

and the virtual work due to external forces:

$$\ell(\vec{w})$$

are in balance for any virtual displacement $\vec{w}$ of the body from static equilibrium. Thus, $\mathcal{S}$ is often called the set of admissible displacements and $\mathcal{V}$ is often called the space of virtual displacements.