Plane Strain Elastostatics: Mixed Strong and Weak Forms:

As noted in the last lectures, the finite element approximations presented so far for the plane strain elastostatics problem suffer from a critical flaw - they exhibit volumetric locking as the Poisson ratio ν approaches 0.5. Volumetric locking is due to the incompressibility (constant volume) constraint being applied to the displacement field as $\nu \to 0.5$, so to alleviate volumetric locking, we must somehow weaken the incompressibility constraint. One approach to weakening the incompressibility constraint is the use of a mixed finite element approximation. In a mixed finite element approximation, one enforces constitutive equations (such as the relationship between stress and strain in plane strain elastostatics) in a weak sense and discretizes variables defined by these constitutive equations (such as stress). In a primal finite element approximation, one instead enforces constitutive equations in a strong sense, so the variables defined by these constitutive equations do not need to be discretized. The finite element approximations presented so far for the plane strain elastostatics problem are examples of primal finite element approximations. As the only unknowns appearing in these approximations are displacement unknowns, we also refer to these as displacement-only finite element approximations. In the next few lectures, we will discuss mixed finite element approximations in which a constitutive equation defining pressure in terms of the divergence of displacement is weakly enforced and pressure is discretized by finite element functions. We refer to such mixed finite element approximations as mixed displacement-pressure finite element approximations. There also exist mixed displacement-stress, displacement-strain, and displacement-stress-strain finite element approximations for the plane strain elastostatics problem. These alternative mixed finite element approximations weakly enforce constitutive equations between stress and strain and/or strain and displacement.

In order to construct a mixed displacement-pressure finite element approximation for the plane strain elastostatics problem, we first need to define suitable mixed displacement-pressure strong and weak forms. To begin, recall that:

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + f_2 = 0$$

for the plane strain elastostatics problem. Then, since:

$$\sigma_{11} = 2\mu u_{1,1} + \lambda(u_{1,1} + u_{2,2})$$
$$\sigma_{22} = 2\mu u_{2,2} + \lambda(u_{1,1} + u_{2,2})$$
$$\sigma_{12} = \sigma_{21} = \mu(u_{1,2} + u_{2,1})$$

we have:

$$-(2\mu u_{1,1})_{,1} - (\mu(u_{1,2} + u_{2,1}))_{,2} - (\lambda(u_{1,1} + u_{2,2}))_{,1} = f_1$$

$$-(\mu(u_{1,2} + u_{2,1}))_{,1} - (2\mu u_{2,2})_{,2} - (\lambda(u_{1,1} + u_{2,2}))_{,2} = f_2$$

Now define the pressure field $p : \Omega \to \mathbb{R}$ via the constitutive equation:

$$\frac{p}{\lambda} + u_{1,1} + u_{2,2} = 0$$

Then:

$$p = -\lambda\,(u_{1,1} + u_{2,2})$$

So:

$$-(2\mu\, u_{1,1})_{,1} - (\mu(u_{1,2} + u_{2,1}))_{,2} + p_{,1} = f_1$$

$$-(\mu(u_{1,2} + u_{2,1}))_{,1} - (2\mu\, u_{2,2})_{,2} + p_{,2} = f_2$$

Note further that:

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 = 2\mu\,(u_{1,1}) n_1 + \lambda\,(u_{1,1} + u_{2,2}) n_1 + \mu\,(u_{1,2} + u_{2,1}) n_2$$

$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 = \mu\,(u_{1,2} + u_{2,1}) n_1 + 2\mu\,(u_{2,2}) n_2 + \lambda\,(u_{1,1} + u_{2,2}) n_2$$

So:

$$t_1 = 2\mu\, u_{1,1} n_1 + \mu\,(u_{1,2} + u_{2,1}) n_2 - p n_1$$

$$t_2 = \mu(u_{1,2} + u_{2,1})n_1 + 2\mu u_{2,2} n_2 - p n_2$$

Collecting the above gives rise to the following mixed displacement-pressure strong form of the plane elastostatics problem:

$$
(S_{mixed}) \begin{cases}
\text{Find } u_1 : \overline{\Omega} \to \mathbb{R},\ u_2 : \overline{\Omega} \to \mathbb{R}, \text{ and } p : \Omega \to \mathbb{R} \text{ such that:} \\
-(2\mu u_{1,1})_{,1} - (\mu(u_{1,2}+u_{2,1}))_{,2} + p_{,1} = f_1 & \text{in } \Omega \\
-(\mu(u_{1,2}+u_{2,1}))_{,1} - (2\mu u_{2,2})_{,2} + p_{,2} = f_2 & \text{in } \Omega \\
p/\lambda + u_{1,1} + u_{2,2} = 0 & \text{in } \Omega \\
u_1 = g_1 & \text{on } \Gamma_{D_1} \\
u_2 = g_2 & \text{on } \Gamma_{D_2} \\
2\mu u_{1,1} n_1 + \mu(u_{1,2}+u_{2,1})n_2 - p n_1 = h_1 & \text{on } \Gamma_{N_1} \\
\mu(u_{1,2}+u_{2,1})n_1 + 2\mu u_{2,2} n_2 - p n_2 = h_2 & \text{on } \Gamma_{N_2}
\end{cases}
$$

Note in the limit as $\lambda \to \infty$ the above strong form becomes:

$$
(\mathring{S}_{mixed}) \begin{cases}
\text{Find } u_1 : \overline{\Omega} \to \mathbb{R},\ u_2 : \overline{\Omega} \to \mathbb{R}, \text{ and } p : \Omega \to \mathbb{R} \text{ such that:} \\
-(2\mu u_{1,1})_{,1} - (\mu(u_{1,2}+u_{2,1}))_{,2} + p_{,1} = f_1 & \text{in } \Omega \\
-(\mu(u_{1,2}+u_{2,1}))_{,1} - (2\mu u_{2,2})_{,2} + p_{,2} = f_2 & \text{in } \Omega \\
u_{1,1} + u_{2,2} = 0 & \text{in } \Omega \\
u_1 = g_1 & \text{on } \Gamma_{D_1} \\
u_2 = g_2 & \text{on } \Gamma_{D_2}
\end{cases}
$$

$$\begin{cases} 2\mu u_{1,1} n_1 + \mu (u_{1,2} + u_{2,1}) n_2 - p n_1 = h_1 & \text{on } \Gamma_{N_1} \\ \mu (u_{1,2} + u_{2,1}) n_1 + 2\mu u_{2,2} n_2 - p n_2 = h_2 & \text{on } \Gamma_{N_2} \end{cases}$$

Thus, the mixed displacement-pressure strong form is well-defined in the limit as $\lambda \rightarrow \infty$. The primal displacement-only strong form, on the other hand, is not. Further note the mixed displacement-stress strong form inherits the constraint:

$$\vec{\nabla} \cdot \vec{u} = u_{1,1} + u_{2,2} = 0$$

in the limit as $\lambda \rightarrow \infty$. The pressure is then the Lagrange multiplier associated with this constraint. This will be discussed in more detail later.

To construct a mixed displacement-pressure weak form, we again turn to the use of weighted residuals. We now have three residuals, two associated with our vector-valued conservation law:

$$R_{u_1}((u_1, u_2, p)) := -(2\mu u_{1,1})_{,1} - (\mu (u_{1,2} + u_{2,1}))_{,2} + p_{,1} - f_1$$

$$R_{u_2}((u_1, u_2, p)) := -(\mu (u_{1,2} + u_{2,1}))_{,1} - (2\mu u_{2,2})_{,2} + p_{,2} - f_2$$

and one associated with our constitutive equation:

$$R_p((u_1, u_2, p)) := p/\lambda + u_{1,1} + u_{2,2}$$

We multiply each by a weighting function and integrate over the domain to arrive at three weighted residuals:

$$\int_\Omega R_{u_1}((u_1, u_2, p))\, w_1 \, d\Omega = 0$$

$$\int_\Omega R_{u_2}((u_1, u_2, p))\, w_2 \, d\Omega = 0$$

$$\int_\Omega R_p((u_1, u_2, p))\, q \, d\Omega = 0$$

or equivalently:

$$\int_\Omega \left(-(2\mu\, u_{1,1})_{,1} - (\mu(u_{1,2} + u_{2,1}))_{,2} + p_{,1}\right) w_1 \, d\Omega = \int_\Omega f_1 w_1 \, d\Omega$$

$$\int_\Omega \left(-(\mu(u_{1,2} + u_{2,1}))_{,1} - (2\mu\, u_{2,2})_{,2} + p_{,2}\right) w_2 \, d\Omega = \int_\Omega f_2 w_2 \, d\Omega$$

$$\int_\Omega \left(P/\lambda + u_{1,1} + u_{2,2}\right) q \, d\Omega = 0$$

Integrating the left hand sides of the first two weighted residuals above yields:

$$\int_\Omega \left(2\mu\, u_{1,1} w_{1,1} + \mu(u_{1,2} + u_{2,1}) w_{1,2} - p\, w_{1,1}\right) d\Omega$$

$$- \int_{\Gamma^2} \left(2\mu\, u_{1,1} n_1 + \mu(u_{1,2} + u_{2,1}) n_2 - p n_1\right) w_1 \, d\Gamma^2$$

$$= \int_\Omega f_1 w_1 \, d\Omega$$

$$\int_{\Omega} \left(\mu(u_{1,2}+u_{2,1})\, w_{2,1} + 2\mu\, u_{2,2}\, w_{2,2} - p\, w_{2,2} \right) d\Omega$$

$$- \int_{\Gamma} \left(\mu(u_{1,2}+u_{2,1})\, n_1 + 2\mu\, u_{2,2}\, n_2 - p n_2 \right) w_2 \, d\Gamma$$

$$= \int_{\Omega} f_2\, w_2 \, d\Omega$$

If we enforce $w_1 \big|_{\Gamma_{D_1}} = 0$ and $w_2 \big|_{\Gamma_{D_2}} = 0$, then:

$$\int_{\Gamma} \left(2\mu\, u_{1,1}\, n_1 + \mu(u_{1,2}+u_{2,1})\, n_2 - p n_1 \right) w_1 \, d\Gamma = \int_{\Gamma_{N_1}} h_1\, w_1 \, d\Gamma$$

$$\int_{\Gamma} \left(\mu(u_{1,2}+u_{2,1})\, n_1 + 2\mu\, u_{2,2}\, n_2 - p n_2 \right) w_2 \, d\Gamma = \int_{\Gamma_{N_2}} h_2\, w_2 \, d\Gamma$$

Then:

$$\int_{\Omega} \left(2\mu\, u_{1,1}\, w_{1,1} + \mu(u_{1,2}+u_{2,1})\, w_{1,2} - p\, w_{1,1} \right) d\Omega$$

$$= \int_{\Omega} f_1\, w_1 \, d\Omega + \int_{\Gamma_{N_1}} h_1\, w_1 \, d\Gamma$$

$$\int_{\Omega} \left(\mu(u_{1,2}+u_{2,1})\, w_{2,1} + 2\mu\, u_{2,2}\, w_{2,2} - p\, w_{2,2} \right) d\Omega$$

$$= \int_{\Omega} f_2\, w_2 \, d\Omega + \int_{\Gamma_{N_2}} h_2\, w_2 \, d\Gamma$$

$$\int_{\Omega} \left(p/\lambda + u_{1,1} + u_{2,2} \right) q \, d\Omega = 0$$

Summing the above yields:

$$\int_{\Omega_h} \Big(2\mu \left(u_{1,1} w_{1,1} + u_{2,2} w_{2,2} \right) + \mu \left(u_{1,2} + u_{2,1} \right)\left(w_{1,2} + w_{2,1} \right) - p \left(w_{1,1} + w_{2,2} \right) + q \left(u_{1,1} + u_{2,2} \right) + \frac{1}{\lambda} p q \Big) \, d\Omega_h$$

$$= \int_{\Omega_h} \left(f_1 w_1 + f_2 w_2 \right) d\Omega + \int_{\Gamma^h_{N_1}} h_1 w_1 \, d\Gamma^h + \int_{\Gamma^h_{N_2}} h_2 w_2 \, d\Gamma^h$$

As we did before in constructing a displacement-only weak form, we define:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\underline{\varepsilon}(\vec{u}) = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{bmatrix} \qquad \underline{\varepsilon}(\vec{w}) = \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{bmatrix}$$

$$\bar{\underline{\underline{D}}} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Then:

$$\int_{\Omega_h} \left(\underline{\varepsilon}(\vec{w})^T \bar{\underline{\underline{D}}} \, \underline{\varepsilon}(\vec{u}) - p \, \vec{\nabla} \cdot \vec{w} + q \, \vec{\nabla} \cdot \vec{u} + \tfrac{1}{\lambda} q p \right) d\Omega_h$$

$$= \int_{\Omega} \vec{w} \cdot \vec{f} \, d\Omega + \int_{\Gamma_{N_1}} w_1 h_1 \, d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2 \, d\Gamma$$

We refer to $\vec{w}$ as the displacement test function and q as the pressure test function in the above integral identity. Now, we use the above integral identity to arrive at a suitable mixed weak form. Like we did before in constructing a displacement-only weak form, we choose:

$$\mathcal{V} := \left\{ \vec{v} \in \left(H^1(\Omega)\right)^2 : \; v_1\big|_{\Gamma_{D_1}} = 0 \;\text{ and }\; v_2\big|_{\Gamma_{D_2}} = 0 \right\}$$

as the space of (vector-valued) displacement weighting functions and:

$$\mathcal{S} := \left\{ \vec{v} \in \left(H^1(\Omega)\right)^2 : \; v_1\big|_{\Gamma_{D_1}} = g_1 \;\text{ and }\; v_2\big|_{\Gamma_{D_2}} = g_2 \right\}$$

as the set of (vector-valued) displacement trial solutions. Since we do not differentiate the pressure solution or pressure test function in the above integral identity, we choose:

$$\mathcal{Q} := L^2(\Omega)$$

as both our space of (scalar-valued) pressure test functions and our set of (scalar-valued) pressure trial solutions. With the above choices, we arrive at the following mixed displacement-pressure weak form for the plane elastostatics problem:

$$
(W_{mixed}) \begin{cases} \text{Find } \vec{u} \in \mathcal{S} \text{ and } p \in Q \text{ such that:} \\[1em] \bar{b}(\vec{u},\vec{w}) - \int_\Omega p \vec{\nabla}\cdot\vec{w}\, d\Omega + \int_\Omega q \vec{\nabla}\cdot\vec{u}\, d\Omega + \int_\Omega \frac{1}{\lambda} pq\, d\Omega \\ \qquad = \ell(\vec{w}) \\[1em] \text{for all } \vec{w} \in \mathcal{V} \text{ and } q \in Q \text{ where:} \\[1em] \bar{b}(\vec{u},\vec{w}) = \int_\Omega \underline{\varepsilon}(\vec{w})^T \underline{\underline{\bar{D}}}\, \underline{\varepsilon}(\vec{u})\, d\Omega \\[1em] \ell(\vec{w}) = \int_\Omega \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Gamma_{N_1}} w_1 h_1\, d\Gamma + \int_{\Gamma_{N_2}} w_2 h_2\, d\Gamma \end{cases}
$$

and in the limit as $\lambda \to \infty$ the above weak form becomes:

$$
(\mathring{W}_{mixed}) \begin{cases} \text{Find } \vec{u} \in \mathcal{S} \text{ and } p \in Q \text{ such that:} \\[1em] \bar{b}(\vec{u},\vec{w}) - \int_\Omega p \vec{\nabla}\cdot\vec{w}\, d\Omega + \int_\Omega q \vec{\nabla}\cdot\vec{u}\, d\Omega = \ell(\vec{w}) \\[1em] \text{for all } \vec{w} \in \mathcal{V} \text{ and } q \in Q. \end{cases}
$$

Since:

$$
\int_\Omega q \vec{\nabla}\cdot\vec{u}\, d\Omega + \int_\Omega \frac{1}{\lambda} pq\, d\Omega = 0
$$

for all $q \in Q$, we see that the weak pressure solution is:

$$p = -\lambda \vec{\nabla} \cdot \vec{u}$$

As:

$$\bar{b}(\vec{u}, \vec{w}) - \int_{\Omega} p \vec{\nabla} \cdot \vec{w} \, d\Omega = \ell(\vec{w})$$

for all $\vec{w} \in \mathcal{V}$, we see the weak displacement solution satisfies:

$$b(\vec{u}, \vec{w}) = \bar{b}(\vec{u}, \vec{w}) + \int_{\Omega} \lambda \vec{\nabla} \cdot \vec{u} \, \vec{\nabla} \cdot \vec{w} \, d\Omega = \ell(\vec{w})$$

for all $\vec{w} \in \mathcal{V}$. That is, the weak displacement solution of Problem (W_{mixed}) is also a weak solution of Problem (W). Consequently, the primal displacement-only and mixed displacement-pressure weak forms of the plane strain elastostatics problem yield the same solutions. However, as we will see, properly constructed mixed finite element approximations of the mixed displacement-pressure weak form are free of volumetric locking. One can show well-posedness of Problem (W_{mixed}) for finite λ using the Lax-Milgram theorem. However, one cannot use the Lax-Milgram theorem to show well-posedness of Problem $(\mathring{W}_{mixed})$. Instead, one must use a generalization of the Lax-Milgram theorem, the so-called Babuška-Lax-Milgram theorem. It should further be noted that Problem $(\mathring{W}_{mixed})$ is also the weak form of the problem of Stokes flow or creeping flow. In this context, $\vec{u}$ is the velocity of the fluid and μ is the dynamic viscosity. Stokes flow is fluid flow for which advective inertial forces are dominated by viscous forces.

As previously mentioned, in the limit as $\lambda \to \infty$, pressure becomes the Lagrange multiplier associated with the incompressibility constraint. To see this, recall from the last lecture that the weak displacement solution $\vec{u} \in \mathcal{S}$ of the plane strain elastostatics problem converges to the solution of:

$$
(\mathring{W}) \left\{
\begin{array}{l}
\text{Find } \mathring{u} \in \mathring{\mathcal{S}} \text{ such that:} \\[1em]
\qquad \bar{b}(\mathring{u}, \mathring{w}) = \ell(\mathring{w}) \qquad \text{for all } \mathring{w} \in \mathring{\mathcal{V}} \\[1em]
\text{where:} \\[1em]
\qquad \mathring{\mathcal{S}} := \left\{ \mathring{v} \in \mathcal{S} : \vec{\nabla} \cdot \mathring{v} \equiv 0 \right\} \\[1em]
\qquad \mathring{\mathcal{V}} := \left\{ \mathring{v} \in \mathcal{V} : \vec{\nabla} \cdot \mathring{v} \equiv 0 \right\}
\end{array}
\right.
$$

It is easily shown that the above weak form is the variational form associated with the minimization problem:

$$
(\mathring{M}) \left\{
\begin{array}{l}
\text{Find:} \\[1em]
\qquad \vec{u} = \underset{\vec{v} \in \mathring{\mathcal{S}}}{\operatorname{argmin}} \ \bar{E}(\vec{v}) \\[1em]
\text{where:} \\[1em]
\qquad \bar{E}(\vec{v}) = \frac{1}{2} \bar{b}(\vec{v}, \vec{v}) - \ell(\vec{v})
\end{array}
\right.
$$

By the theory of Lagrange multipliers, the minimizer of $\bar{E}$ in $\mathring{\mathcal{V}}$ may be found from the saddle point of the Lagrangian:

$$\mathcal{L}((\vec{v}, q)) := \bar{E}(\vec{v}) - \int_{\Omega} q \; \vec{\nabla} \cdot \vec{v} \, d\Omega$$

Constraint

Lagrange Multiplier Variable

Namely, if $(\vec{u}, p)$ is the saddle point of the above Lagrangian, then $\vec{u}$ is the solution of Problem $(\mathring{M})$ and p is the Lagrange multiplier associated with the constraint $\vec{\nabla} \cdot \vec{u} = 0$. As it turns out, Problem $(\mathring{W}_{mixed})$ is precisely the variational form attained by taking the first variation of the Lagrangian at the saddle point $(\vec{u}, p)$ in direction $(\vec{w}, -q)$ and setting it to zero. Thus, as previously stated, pressure is the Lagrange multiplier associated with the constraint $\vec{\nabla} \cdot \vec{u} = 0$.

To conclude, it should be noted that pressure as defined here is not equal to the hydrostatic pressure:

$$p_{hydrostatic} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

except in the incompressible limit. However, hydrostatic pressure may be attained from the pressure defined here via:

$$p_{hydrostatic} = p + \frac{2}{3} \mu (u_{1,1} + u_{2,2})$$