Linear Elasticity: Voigt Notation

It simplifies our later implementations if we re-write symmetric tensors as vectors, thereby reducing their order. This is often referred to as Voigt notation. For linear elasticity, we write:

$$\vec{\varepsilon}(\vec{u}) := \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{bmatrix} \qquad \text{Strain Vector}$$

$$\vec{\varepsilon}(\vec{w}) := \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,2} \end{bmatrix} \qquad \text{Virtual Strain Vector}$$

$$\vec{\sigma}(\vec{u}) := \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \qquad \text{Stress Vector}$$

We can also rewrite the fourth rank tensor of elastic coefficients as a matrix:

$$\vec{\vec{D}} := \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{symmetric} & & D_{33} \end{bmatrix}$$

where:

$$D_{IJ} = c_{ABCD}$$

where the indices are related by the following table:

I/J	A/C	B/D
1	1	1
2	2	2
3	1	2
3	2	1

Then, we see:

$$\vec{\sigma}(\vec{u}) = \vec{\vec{D}}\,\vec{\varepsilon}(\vec{u}) \qquad \text{Matrix - Vector Product}$$

Moreover:

$$w_{(A,B)}\, c_{ABCD}\, u_{(C,D)} = \left(\vec{\varepsilon}(\vec{w})\right)^T \vec{\vec{D}}\, \vec{\varepsilon}(\vec{u})$$

So we have:

$$\boxed{a(\vec{w},\vec{u}) = \int_{\Omega} \left(\vec{\varepsilon}(\vec{w})\right)^T \vec{\vec{D}}\, \vec{\varepsilon}(\vec{u})\, d\Omega}$$