Boundary Value Problems: Putting It All Together

We can now put everything together to initialize, construct, solve, and analyze our system.

Step 1: Initialize the Problem

Set $p_1, p_2, n_1, n_2, \Xi_1, \Xi_2, \{\vec{P}_i\}_{i=1}^{n}, \{w_i\}_{i=1}^{n}$ ← NURBS Data

Call Extract And Localize $(n_1, n_2, p_1, p_2, \Xi_1, \Xi_2, \{\vec{P}_i\}_{i=1}^{n}, \{w_i\}_{i=1}^{n})$ to obtain n_{el}, $\{\underline{\underline{C}}^e\}_{e=1}^{n_{el}}$, IEN, $\{\{\vec{P}_a^{b,e}\}\}$, $\{\{w_a^{b,e}\}\}$

Set $n_q, \{\tilde{\xi}_q\}_{q=1}^{n_q}, \{w_q\}_{q=1}^{n_q}$ ← Quadrature Data

Construct BC, Neumann, and Robin arrays
Construct $\underline{g}$ (vector w/ Dirichlet data)
} ← BC Data

Step 2: Construct the Matrix System

Set $\underline{\underline{K}} = \underline{\underline{0}}$ and $\underline{F} = \underline{0}$

for $e = 1, \dots, n_{el}$

  Call Element Formation $(e, n_q, \{\tilde{\xi}_q\}_{q=1}^{n_q}, \underline{\underline{C}}^e, \{\vec{P}_a^{b,e}\}_{a=1}^{n_{loc}}, \{w_a^{b,e}\}_{a=1}^{n_{loc}}, \{w_a^e\}_{a=1}^{n_{loc}}, \{\tilde{w}_q\}_{q=1}^{n_q}$, Neumann, Robin$)$ to obtain $\underline{\underline{k}}^e$ and $\underline{f}^e$

  Call Element Assembly $(e, \underline{\underline{k}}^e, \underline{f}^e$, IEN, BC, $\underline{g}, \underline{\underline{K}}, \underline{F})$ to obtain updated $\underline{\underline{K}}$ and $\underline{F}$

endloop

(*)

Step 2.1: Account for BC's

for $i = 1, \dots, n$
  if $BC(i) = 1$
    Set $K_{ii} = 1$
    Set $F_i = g_i$
  endif
endloop

Step 3: Solve the Matrix System

Solve $\underline{d} = \underline{\underline{K}}^{-1}\underline{F}$

Step 4: Analyze Results

The functions Extract And Localize, Element Formation, and Element Assembly are described in detail in the proceeding pages.

Extract And Localize:

Input: $n_1, n_2, p_1, p_2, \vec{\Xi}_1, \vec{\Xi}_2, \{\vec{P}_i\}_{i=1}^{n}, \{w_i\}_{i=1}^{n}$

Output: $n_{el}, \{\underline{\underline{C}}^e\}_{e=1}^{n_{el}}, IEN, \left\{\{\vec{P}_a^{b,e}\}_{a=1}^{n_{loc}}\right\}_{e=1}^{n_{el}}, \left\{\{w_a^{b,e}\}_{a=1}^{n_{loc}}\right\}_{e=1}^{n_{el}}$

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function  Extract And Localize (n_1, n_2, p_1, p_2, Ξ_1, Ξ_2, {P_i}_{i=1}^n, {w_i}_{i=1}^n)
begin

     Call Extraction 2D (n_1, n_2, p_1, p_2, Ξ_1, Ξ_2, ~~{P_i}_{i=1}^n~~, ~~{w_i}_{i=1}^n~~) to obtain
          n_el and {C^e}_{e=1}^{n_el}

     Call IEN 2D (n_1, n_2, p_1, p_2, Ξ_1, Ξ_2) to obtain IEN

     Call Extract Geometry (d', p_1, p_2, n_el, {C^e}, IEN, {P_i}, {w_i}) to obtain:
                            ↖2
               {{P_a^{b,e}}_{a=1}^{n_loc}}_{e=1}^{n_el}  and  {{w_a^{b,e}}_{a=1}^{n_loc}}_{e=1}^{n_el}
                                                                  ↳ From
                                                                    Homework 2
     for e=1,...,n_el
          for a=1,...,n_loc
               Set i = IEN(a,e)
               Set P_a^e = P_i
               Set w_a^e = w_i
          endloop
     endloop

     return n_el, {C^e}_{e=1}^{n_el}, IEN, {{P_a^{b,e}}_{a=1}^{n_loc}}_{e=1}^{n_el}, {{w_a^{b,e}}_{a=1}^{n_loc}}_{e=1}^{n_el}

end
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Element Formation:

Input: $e, n_q, \{\tilde{\xi}_q\}_{q=1}^{n_q}, \underline{\underline{C}}^e, \{\vec{P}_a^{b,e}\}_{a=1}^{n_{loc}}, \{w_a^{b,e}\}_{a=1}^{n_{loc}},$
$\{w_a^e\}_{a=1}^{n_{loc}}, \{\tilde{w}_q\}_{q=1}^{n_q}$, Neumann, Robin

Output: $\underline{\underline{k}}^e, \underline{f}^e$

function Element Formation $(e, n_q, \{\tilde{\xi}_q\}_{q=1}^{n_q}, \underline{\underline{C}}^e, \{\vec{P}_a^{b,e}\}_{a=1}^{n_{loc}}, \{w_a^{b,e}\}_{a=1}^{n_{loc}},$
$\{w_a^e\}_{a=1}^{n_{loc}}, \{\tilde{w}_q\}_{q=1}^{n_q}$, Neumann, Robin)

begin

Initialize $\underline{\underline{k}}^e = \underline{\underline{0}}$ and $\underline{f}^e = \underline{0}$

for $q_1 = 1, \ldots, n_q$
  for $q_2 = 1, \ldots, n_q$

Needs additional input

Call Shape Function $(e, \tilde{\xi}_{q_1}, \tilde{\xi}_{q_2})$ to obtain $R_a^e, \frac{\partial}{\partial x} R_a^e, \frac{\partial}{\partial y} R_a^e,$
$\vec{x}$, and $\underline{\underline{J}}$ at $\vec{x}^e(\tilde{\xi}_{q_1}, \tilde{\xi}_{q_2})$

Set $j = \det(\underline{\underline{J}})$, $K_{loc} = K(\vec{x})$, $f_{loc} = f(\vec{x})$

for $a = 1, \ldots, n_{loc}$
  for $b = 1, \ldots, n_{loc}$

Update $k_{ab}^e = k_{ab}^e + K_{loc} * ((\vec{\nabla}_{\vec{x}} R_a^e) \cdot (\vec{\nabla}_{\vec{x}} R_b^e)) * \tilde{w}_{q_1} * \tilde{w}_{q_2} * j$

  endloop

$\boxed{\text{Update } f_a^e = f_a^e + R_a^e * f_{loc} * \tilde{w}_{q_1} * \tilde{w}_{q_2} * j}$

endloop

  endloop
endloop

for side $= 1, \ldots, 4$

if Neumann (side, e) = 1 or Robin (side, e) = 1
  for $q = 1, \ldots, n_q$

Set the following:

$$\tilde{\xi}_1 = \begin{cases} \tilde{\xi}_q & \text{if side} = 1,3 \\ 1 & \text{if side} = 2 \\ 0 & \text{if side} = 4 \end{cases} \qquad \tilde{\xi}_2 = \begin{cases} \tilde{\xi}_q & \text{if side} = 2,4 \\ 1 & \text{if side} = 3 \\ 0 & \text{if side} = 1 \end{cases}$$

TOPS

Set $\hat{\underline{t}} = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if} \quad \text{side} = 1,3 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if} \quad \text{side} = 2,4 \end{cases}$

Call Shape Function$(e, \tilde{\xi}_1, \tilde{\xi}_2)$ to obtain R_a^e, $\frac{\partial}{\partial x} R_a^e$, $\frac{\partial}{\partial y} R_a^e$, $\vec{x}$, and $\underline{\underline{J}}$ at $\vec{x}^e(\tilde{\xi}_1, \tilde{\xi}_2)$

Set $j_\Gamma = |\underline{\underline{J}}\, \hat{\underline{t}}|$

if Neumann (side, e) = 1

Set $h_{loc} = h(\vec{x})$

for $a = 1, \ldots, n_{loc}$

Update $f_a^e = f_a^e + R_a^e * h_{loc} * \hat{w}_q * j_\Gamma$

endloop

elseif Robin (side, e) = 1

Set $\beta_{loc} = \beta(\vec{x})$, $u_{R,loc} = u_R(\vec{x})$

for $a = 1, \ldots, n_{loc}$

for $b = 1, \ldots, n_{loc}$

Update $k_{ab}^e = k_{ab}^e + \beta_{loc} * R_a^e * R_b^e * \hat{w}_q * j_\Gamma$

endloop

Update $f_a^e = f_a^e + \beta_{loc} * R_a^e * u_{R,loc} * \hat{w}_q * j_\Gamma$

endloop

endif

endloop

endif

endloop

return $\underline{\underline{k}}^e$ and $\underline{f}^e$

end

Element Assembly: Input: e, $\underline{\underline{k}}^e$, $\underline{f}^e$, IEN, BC, g, $\underline{\underline{K}}$ & $\underline{F}$ (to be updated)

Output: $\underline{\underline{K}}$ & $\underline{F}$ (updated)

function Element Assembly (e, $\underline{\underline{k}}^e$, $\underline{f}^e$, IEN, BC, g, $\underline{\underline{K}}$, $\underline{F}$)
begin
    for $a = 1, \ldots, n_{loc}$
        [redacted]
        Set $i = \text{IEN}(a,e)$
        if $BC(i) = 0$
            for $b = 1, \ldots, n_{loc}$
                Set $j = \text{IEN}(b,e)$
                if $BC(j) = 0$
                    Update $K_{ij} = K_{ij} + k^e_{ab}$
                else
                    Update $F_i = F_i - k^e_{ab}\, g_j$
                endif
            endloop
            ~~Set~~ Update $F_i = F_i + f^e_a$
        endif
    endfor

    return updated $\underline{\underline{K}}$ and $\underline{F}$
end