Elastodynamics: Semi-Discrete Methods and Matrix Formulation

Now that we have concluded our discussion on structural vibrations and wave propagation, we turn to the problem of approximating the transient behavior of elastodynamic systems. The literature is replete with different approaches, but we will focus our discussion on semi-discrete methods which separate discretization of space and time.

To begin, let us recall the strong form of the linear elastodynamics problem:

$$
(S)\left\{
\begin{array}{lll}
\text{Find } \vec{u} : \bar{\Omega} \times [0,T] \rightarrow \mathbb{R}^d \text{ s.t.} & & \\
\rho u_{A,tt} = \sigma_{AB,B} + f_A & \text{in } \Omega \times (0,T) & A = 1,\dots,d \\
u_A = g_A & \text{on } \Gamma_{D_A} \times (0,T) & A = 1,\dots,d \\
\sigma_{AB} n_B = h_A & \text{on } \Gamma_{N_A} \times (0,T) & A = 1,\dots,d \\
\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x}) & \text{in } \Omega & \\
\vec{u}_{,t}(\vec{x},0) = \vec{v}_0(\vec{x}) & \text{in } \Omega &
\end{array}
\right.
$$

and the associated weak form:

$$
(W)\left\{
\begin{array}{ll}
\text{Find } \vec{u}(t) \in \mathcal{S}_t \text{ s.t.} & \\
(\vec{w}, \rho \vec{u}_{,tt}) + a(\vec{w},\vec{u}) = L(\vec{w}) & \forall \vec{w} \in \mathcal{V} \\
(\vec{w}, \rho \vec{u}(0)) = (\vec{w}, \rho \vec{u}_0) & \forall \vec{w} \in \mathcal{V} \\
(\vec{w}, \rho \vec{u}_{,t}(0)) = (\vec{w}, \rho \vec{v}_0) & \forall \vec{w} \in \mathcal{V}
\end{array}
\right.
$$

In a semi-discrete method, we discretize space using a Galerkin finite element scheme and formulate the problem as though the time were continuous. In particular, we represent the solution as a linear combination of basis functions that depend only on space and coefficients that depend on time. Visually:

$$
\vec{u}^h(\vec{x},t) = \sum_{i=1}^{n} N_i(\vec{x}) \, \vec{d}_i(t)
$$

Spatially Varying Basis Functions   Temporally Varying Control Variables

Formally, we denote the set of discrete trial solutions at time t as $\mathcal{S}_t^h \subset \mathcal{S}$ and we denote the space of discrete test functions as $\mathcal{V}^h \subset \mathcal{V}$. We further characterize $\mathcal{S}_t^h$ by requiring that if we have a given function $\vec{g}^h(t) \in \mathcal{S}_t^h$, then for every function $\vec{u}^h(t) \in \mathcal{S}_t^h$, there exists a unique $\vec{v}^h(t) \in \mathcal{V}^h$ such that:

$$\vec{u}^h(t) = \vec{v}^h(t) + \vec{g}^h(t)$$

Then our semi-discrete method is as follows:

$$(G)\begin{cases} \text{Find } \vec{u}^h(t) = \vec{v}^h(t) + \vec{g}^h(t), \text{ where } \vec{v}^h(t) \in \mathcal{V}^h, \text{ s.t.} \\ (\vec{w}^h, \rho\vec{u}^h_{,tt}) + a(\vec{w}^h, \vec{u}^h) = L(\vec{w}^h) & \forall \vec{w}^h \in \mathcal{V}^h \\ (\vec{w}^h, \rho\vec{u}^h(0)) = (\vec{w}^h, \rho\vec{u}_0) & \forall \vec{w}^h \in \mathcal{V}^h \\ (\vec{w}^h, \rho\vec{u}^h_{,t}(0)) = (\vec{w}^h, \rho\vec{v}_0) & \forall \vec{w}^h \in \mathcal{V}^h \end{cases}$$

Representing $\vec{v}^h$ as:

$$(\vec{v}^h(\vec{x},t))_A = \sum_{i \in \eta - \eta_{g_A}} N_i(\vec{x}) \, (\vec{d}_i(t))_A \qquad A = 1, \dots, d$$

allows us to apply the usual arguments and arrive at a matrix problem. Let:

$$\underline{\underline{M}} = [M_{PQ}] = \text{mass matrix}$$

$$\underline{\underline{K}} = [K_{PQ}] = \text{stiffness matrix}$$

$$\underline{F}(t) = [F_P(t)] = \text{force vector}$$

$$\underline{d}(t) = [d_Q(t)] = \text{displacement vector}$$

$$\underline{d}_0 = [d_{0Q}] = \text{initial displacement vector}$$

$$\dot{\underline{d}}_0 = [\dot{d}_{0Q}] = \text{initial velocity vector}$$

where, with $P = ID(A,i)$ and $Q = ID(B,j)$, we have defined:

$$M_{PQ} = (N_i \hat{e}_A, \rho N_j \hat{e}_B) = \delta_{AB} \int_\Omega N_i \rho N_j \, d\Omega$$

$$K_{PQ} = a(N_i \hat{e}_A, N_j \hat{e}_B)$$

$$F_P(t) = L(N_i \hat{e}_A) - a(N_i \hat{e}_A, \vec{g}^h(t)) - (N_i \hat{e}_A, \rho \vec{g}^h_{,tt}(t))$$

$$d_Q(t) = (\vec{d}_j(t))_B$$

Let us define the intermediate vectors:

$$\tilde{d}_{0P} = (N_i \hat{e}_A, \rho(\vec{u}_0 - \vec{g}^h))$$

$$\dot{\tilde{d}}_{0P} = (N_i \hat{e}_A, \rho(\vec{v}_0 - \vec{g}^h_{,t}))$$

and denote the Q,P entry of the inverse of the mass matrix, $\underline{\underline{M}}^{-1}$, by M^{-1}_{QP}. Then:

$$d_{0Q} = M^{-1}_{QP} \hat{d}_{0P}$$

$$\dot{d}_{0Q} = M^{-1}_{QP} \hat{\dot{d}}_{0P}$$

and we can rewrite (G) as a matrix problem:

$$\boxed{\begin{aligned} \underline{\underline{M}}\, \ddot{\underline{d}}(t) + \underline{\underline{K}}\, \underline{d}(t) &= \underline{F}(t), \qquad t \in (0,T) \\ \underline{d}(0) &= \underline{d}_0 \\ \dot{\underline{d}}(0) &= \dot{\underline{d}}_0 \end{aligned}}$$

The above is a system of ordinary differential equations for the coefficients $d_P(t)$ of the displacement field component $\vec{v}^h(t)$. We have used dots to denote differentiation in time.