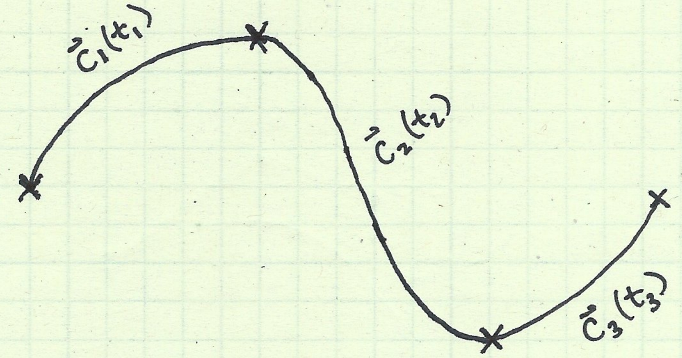
# Composite Bézier Curves

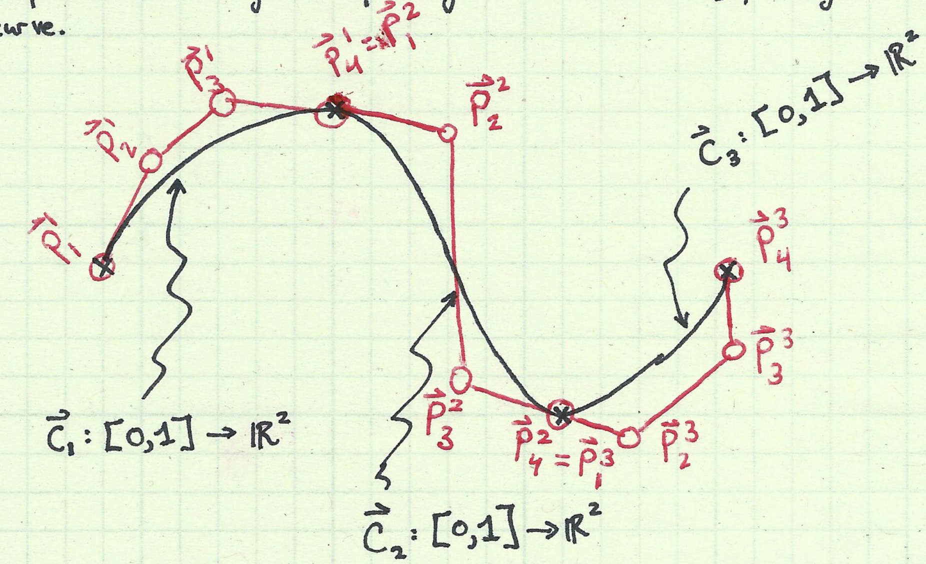
While Bézier curves have distinct advantages over polynomial interpolation with regards to shape design, they harbor several disadvantages as well:

- A high polynomial degree is needed to satisfy a large number of constraints.
- A high polynomial degree is needed to accurately fit some complex shapes.
- Each control point affects the Bézier curve in a global manner.
- Bézier curves cannot exhibit kinks or discontinuities.

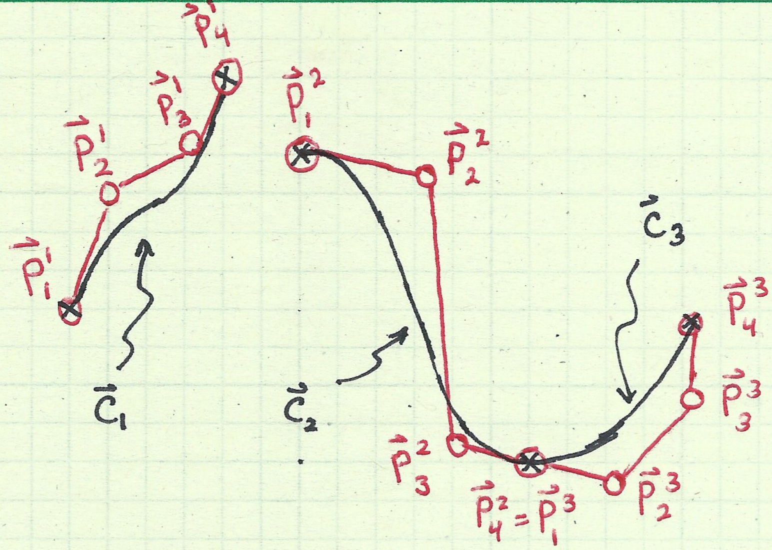
One obvious solution to overcoming the above shortcomings is to use curves which are piecewise polynomial. For example, consider the following curve consisting of three cubic segments:



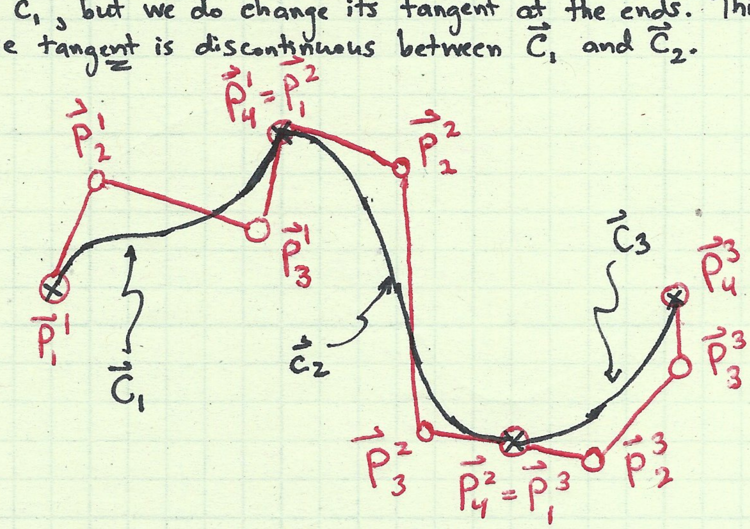

We can represent each segment separately as a Bézier curve, yielding a composite Bézier curve.




Now, suppose we move control point $\vec{P}_4^1$. Then, curve $\vec{C}_1$ changes while $\vec{C}_2$ and $\vec{C}_3$ remain the same, resulting in a discontinuous curve.




Alternatively, if we instead move control point $\vec{P}_3^1$, we do not change the endpoints of curve $\vec{C}_1$, but we do change its tangent at the ends. This results in a composite curve whose tangent is discontinuous between $\vec{C}_1$ and $\vec{C}_2$.




So, we see that we do not maintain positional and tangential continuity of the composite curve when manipulating the control points. In order to do so, we need to apply the following constraints:

Positional Continuity Between $\vec{C}_1$ and $\vec{C}_2$:

$\vec{P}_4^1$ must be equal to $\vec{P}_1^2$.

Tangential Continuity Between $\vec{C}_1$ and $\vec{C}_2$:

$\vec{P}_3^1$, $\vec{P}_4^1$, $\vec{P}_1^2$, and $\vec{P}_2^2$ must be colinear.

Positional Continuity Between $\vec{C}_2$ and $\vec{C}_3$:

$\vec{P}_4^2$ must be equal to $\vec{P}_1^3$.

Tangential Continuity Between $\vec{C}_2$ and $\vec{C}_3$:

$\vec{P}_3^2$, $\vec{P}_4^2$, $\vec{P}_1^3$, and $\vec{P}_2^3$ must be colinear.

The above constraints are a direct consequence of the endpoint interpolation and tangency properties of Bézier curves.

Designers usually desire to preserve positional and tangential continuity, also known as $G^0$ and $G^1$ continuity, in shape design of curves and surfaces. In addition, they also seek to preserve continuity of curvature, or $G^2$ continuity. This is more difficult to achieve than $G^0$ and $G^1$ continuity for Bézier curves, and much more difficult to achieve in the setting of Bézier surfaces.

To summarize, there are two primary shortcomings of composite Bézier curves:

1) It is difficult to enforce various levels of continuity between segments.
2) Redundant data must be stored. For a three segment $G^1$ Bézier curve, only 8 degrees-of-freedom exist but there are 12 control points.

To circumvent the above issues, B-splines were invented. They are the cornerstone of modern computer aided geometric design and the basic building blocks of isogeometric methods.