Linear Elasticity: The Strong and Weak Forms

We now consider the application of NURBS-based isogeometric analysis to the problem of linear elasticity. This is a very classical subject with a rich history, and it is one that is particularly well suited to examination by isogeometric analysis - not only for its geometrical accuracy but also for the high quality of the stress fields resulting from the use of C^1-continuous bases. We restrict ourselves at present to the case of linear elastostatics and equilibrium solutions. We will also restrict ourselves to the two-dimensional setting. Such a setting will allow us to address materials subject to either plane strain or plane stress.

In what follows, indices A, B, C, and D may take on values $1, \ldots, d$ where $d = 2$ in the two-dimensional setting. Unlike the problem of heat conduction, in linear elasticity, the solution field will be vector-valued, with u_A referring to the A^{th} component of $\vec{u}$. Moreover, differentiation will be denoted by a comma, e.g.,

$$u_{A,B} = u_{A,x_B} = \frac{\partial u_A}{\partial x_B}$$

We will employ a summation convention applying to A, B, C, D, in which repeated indices imply summation, e.g.,

$$u_{A,BB} = u_{A,11} + u_{A,22}$$

In the case of a general nonsymmetric tensor $\underline{\underline{A}} = [A_{AB}]$, we use parantheses around the indices to denote its symmetric part and square brackets around the indices to denote its skew-symmetric part. Then $A_{AB} = A_{(AB)} + A_{[AB]}$ and:

$$A_{(AB)} = A_{(BA)} \equiv \frac{1}{2}\left(A_{AB} + A_{BA}\right)$$

$$A_{[AB]} = -A_{[BA]} \equiv \frac{1}{2}\left(A_{AB} - A_{BA}\right)$$

Note that if $\underline{\underline{A}}$ is a nonsymmetric tensor and $\underline{\underline{B}}$ is a symmetric tensor, then:

$$A_{ij} B_{ij} = A_{(ij)} B_{ij}$$

$$A_{[ij]} B_{ij} = 0$$

We will use the above properties repeatedly to reduce redundant computations in what follows.

Let σ_{AB} denote (Cartesian) components of the (Cauchy) stress tensor, u_A denote components of the displacement vector, and f_A denote components of the prescribed body force per unit volume. The infinitesimal strain tensor, $\underline{\underline{\varepsilon}} = [\varepsilon_{AB}]$, is defined as:

$$\varepsilon_{AB} = u_{(A,B)} \equiv \frac{1}{2}\left(u_{A,B} + u_{B,A}\right)$$

The constitutive law relating this strain tensor to the aforementioned stress tensor is the generalized Hooke's law, given by:

$$\sigma_{AB} = c_{ABCD}\, \varepsilon_{CD}$$

where the c_{ABCD}'s are the elastic coefficients, which are given functions of $\vec{x}$. If the c_{ABCD}'s are constant throughout, the body is said to be homogeneous. The elastic coefficients are assumed to satisfy:

Symmetry:

$$c_{ABCD} = c_{CDAB} \quad \text{(major symmetry)}$$

$$\left.\begin{aligned} c_{ABCD} &= c_{BACD} \\ c_{ABCD} &= c_{ABDC} \end{aligned}\right\} \quad \text{(minor symmetry)}$$

Positive-Definiteness:

$$c_{ABCD}\,\psi_{AB}\,\psi_{CD} \geq 0$$

$$c_{ABCD}\,\psi_{AB}\,\psi_{CD} = 0 \quad \text{if and only if} \quad \psi_{AB} = 0$$

for all symmetric tensor $\underline{\underline{\psi}}$

The above properties ensure that $\sigma_{AB} = \sigma_{BA}$. Moreover, in the Galerkin setting, they ensure that the system stiffness matrix $\underline{\underline{K}}$ is symmetric and positive-definite, provided, of course, that appropriate displacement boundary conditions are applied.

We can now formally state the strong form of the boundary value problem:

$$(S)\left\{\begin{array}{llll} \text{Find } \vec{u} : \overline{\Omega} \to \mathbb{R}^d \text{ s.t.} & & & \\ \sigma_{AB,B} + f_A = 0 & \text{in } \Omega & A = 1,\dots,d \\ u_A = g_A & \text{on } \Gamma_{D_A} & A = 1,\dots,d \\ \sigma_{AB}\, n_B = h_A & \text{on } \Gamma_{N_A} & A = 1,\dots,d \end{array}\right.$$

Above, we have:

g_A = The prescribed boundary displacement $g_A : \Gamma_{D_A} \to \mathbb{R}$ for $A = 1,\dots,d$

h_A = The prescribed boundary traction $h_A : \Gamma_{N_A} \to \mathbb{R}$ for $A = 1,\dots,d$

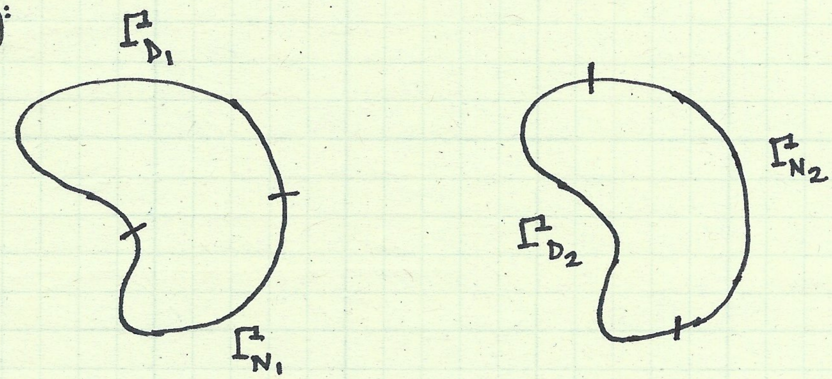
Γ_{D_A} = The Dirichlet boundary for $A = 1,\dots,d$

Γ_{N_A} = The Neumann boundary for $A = 1,\dots,d$

As in the setting of heat conduction, we require that $\overline{\Gamma_{D_A} \cup \Gamma_{N_A}} \equiv \Gamma \equiv \partial\Omega$ and that $\Gamma_{D_A} \cap \Gamma_{N_A} = \emptyset$ for $A = 1,\dots,d$. Note that this yields d decompositions of the boundary:

$$\Gamma = \overline{\Gamma_{D_A} \cup \Gamma_{N_A}} \qquad A = 1,\dots,d$$

Visually:



(S) is sometimes referred to as the mixed boundary-value problem of linear elastostatics. Under appropriate hypotheses on the data, (S) possesses a unique solution.

In practice, it is important to deal with somewhat more complicated boundary-condition specifications involving normal and tangential displacement and traction specifications. For the sake of simplicity of exposition, we ignore such specifications in what follows.

As in the setting of heat conduction, we also develop a corresponding weak formulation for the problem of linear elasticity. For us to proceed, we need a set of trial solutions and a space of weighting functions for each direction $A = 1, \ldots, d$. These are defined as follows:

$$\mathcal{S}_A := \left\{ u_A : \overline{\Omega} \rightarrow \mathbb{R} : u_A \in H^1(\Omega) \text{ and } u_A \big|_{\Gamma_{D_A}} = g_A \right\}$$

$$\mathcal{V}_A := \left\{ w_A : \overline{\Omega} \rightarrow \mathbb{R} : w_A \in H^1(\Omega) \text{ and } w_A \big|_{\Gamma_{D_A}} = 0 \right\}$$

Proceeding as in the setting of heat conduction, we then multiply (S) by a weighting function and integrate by parts to obtain a variational form of the problem. The resulting weak form is as follows:

$$(W) \begin{cases} \text{Find } u_A \in \mathcal{S}_A, \ A = 1, \ldots, d, \text{ s.t.} \\[1em] \displaystyle\int_{\Omega} w_{(A,B)} \, \sigma_{AB} \, d\Omega = \int_{\Omega} w_A f_A \, d\Omega + \sum_{A=1}^{d} \left(\int_{\Gamma_{N_A}} w_A h_A \, d\Gamma \right) \\[1em] \hfill \forall w_A \in \mathcal{V}_A, \ A = 1, \ldots, d \end{cases}$$

As in the setting of heat conduction, we can rewrite (W) in a more concise form. Define:

$$\mathcal{S} := \left\{ \vec{u} : u_A \in \mathcal{S}_A \right\}$$

$$\mathcal{V} := \left\{ \vec{w} : w_A \in \mathcal{V}_A \right\}$$

and:

$$a(\vec{w},\vec{u}) := \int_{\Omega} w_{(A,B)}\, c_{ABCD}\, u_{(C,D)}\, d\Omega$$

$$L(\vec{w}) := \int_{\Omega} w_A f_A \, d\Omega + \sum_{A=1}^{d} \left(\int_{\Gamma_{h_A}} w_A h_A \, d\Gamma \right)$$

Then (W) may be stated as follows:

$$(W) \begin{cases} \text{Find } \vec{u} \in \mathcal{S} \text{ s.t.} \\ \qquad a(\vec{w},\vec{u}) = L(\vec{w}) \qquad \forall \vec{w} \in \mathcal{V} \end{cases}$$

In the solid mechanics literature, (W) is sometimes referred to as the principle of virtual work, or principle of virtual displacements, w_A being the virtual displacements. This is because $L(\vec{w})$ represents the work done by the body force $\vec{f}$ times the virtual displacement $\vec{w}$, plus boundary traction contributions, and $a(\vec{w},\vec{u})$ represents the stress work. Consequently, (W) is a statement of static equilibrium in which the virtual work contributions are in balance.