Linear Elasticity: Isotropic Bodies, Plane Strain, and Plane Stress

In this class, we will exclusively deal with isotropic bodies. Then we have that:

$$C_{ABCD}(\vec{x}) = \mu(\vec{x}) \left(\delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC} \right) + \lambda(\vec{x})\, \delta_{AB}\delta_{CD}$$

$$A, B, C, D = 1, \ldots, 3$$

where λ and μ are the Lamé parameters. The relationships of λ and μ to E, the Young's modulus, and ν, the Poisson's ratio, are given by:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

Then we see that:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

We may also invert this relationship to find:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

In the setting of plane strain, the strains in the z-direction are considered to be negligible, yielding $\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$. For an isotropic body, this indicates that:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \text{From Above} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 0 \\ 0 \\ 0 \\ \varepsilon_{12} \end{bmatrix}$$

Ignoring the stress components associated with the z-direction, we find:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

Thus, for the setting of plane strain, we have:

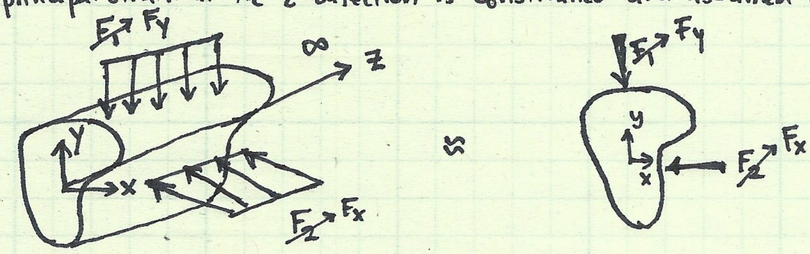
$$\boxed{\vec{\sigma}(\vec{u}) = \vec{\vec{D}}\,\vec{\varepsilon}(\vec{u})}$$

where:

Plane Strain

$$\boxed{\vec{\vec{D}} = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}}$$

Plane strain occurs when the z-direction dimension is very large compared to the others. In this situation, the principal strain in the z-direction is constrained and assumed to be zero.



The plane strain assumption is employed in the analysis of dams, tunnels, and other geotechnical works.

In the setting of plane stress, the stresses in the z-direction are considered to be negligible, yielding $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. For an isotropic body, this implies the compliance relationship:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} & & \\ & \text{From Above} & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ 0 \\ 0 \\ \sigma_{12} \end{bmatrix}$$

Ignoring the strain components associated with the z-direction, we find:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

and the inverse relation is:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & 0 \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & 0 \\ 0 & 0 & E/2(1+\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

Thus, for the setting of plane stress, we have:

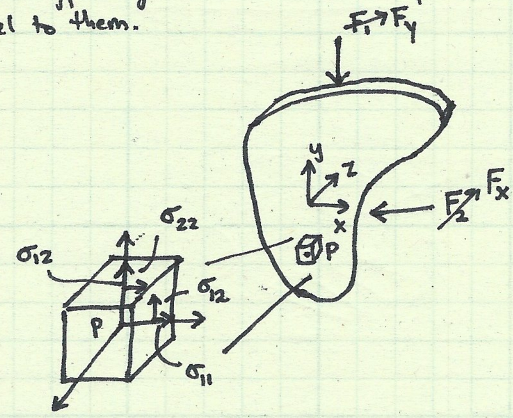
$$\boxed{\vec{\sigma}(\vec{u}) = \vec{\vec{D}}\,\vec{\varepsilon}(\vec{u})}$$

where:

$$\boxed{\vec{\vec{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}$$

Plane Stress

Plane stress typically occurs in thin flat plates that are acted upon only by load forces that are parallel to them.



Plane strain and plane strain are the canonical settings for two-dimensional elasticity, which is also referred to as the plane theory of elasticity. The box below summarizes the findings from above.

Plane Theory of Elasticity

Stress-Strain Relationship: $\vec{\sigma}(\vec{u}) = \vec{\vec{D}}\,\vec{\varepsilon}(\vec{u})$

Stress Vector — Strain Vector

Elastic Coefficients: $D_{IJ} = C_{ABCD}$

Isotropic Body Subject to Plane Strain:

$$\vec{\vec{D}} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Out-of-plane Normal Stress: $\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22})$

Isotropic Body Subject to Plane Stress:

$$\vec{\vec{D}} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & 0 \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & 0 \\ 0 & 0 & E/2(1+\nu) \end{bmatrix}$$

Out-of-plane Normal Strain: $\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$