Linear Elasticity: Galerkin's Method

To turn the weak statement of the linear elasticity problem into a system of algebraic equations, we again apply Galerkin's method and work in finite-dimensional sets $\mathcal{S}^h \subset \mathcal{S}$ and $\mathcal{V}^h \subset \mathcal{V}$. As before, these sets are defined using the isoparametric NURBS basis, but now with vector-valued control variables. Visually:

$$\mathcal{V}^h := \left\{ \vec{w}^h \in \mathcal{V} : \quad \vec{w}^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, \vec{c}_i \right\}$$

$$\mathcal{S}^h := \left\{ \vec{u}^h \in \mathcal{S} : \quad \vec{u}^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, \vec{d}_i \right\}$$

We further characterize $\mathcal{S}^h$ by requiring that if we have a given function $\vec{g}^h \in \mathcal{S}^h$ such that $(\vec{g}^h)_A|_{\Gamma_{D_A}} = g_A$ for $A = 1, \ldots, d$, then for every $\vec{u}^h \in \mathcal{S}^h$, there exists a unique $\vec{v}^h \in \mathcal{V}^h$ such that:

$$\vec{u}^h = \vec{v}^h + \vec{g}^h$$

Then, Galerkin's method as applied to linear elasticity is stated as follows:

$$(G) \begin{cases} \text{Find } \vec{u}^h = \vec{v}^h + \vec{g}^h, \text{ where } \vec{v}^h \in \mathcal{V}^h, \text{ such that:} \\ a(\vec{w}^h, \vec{u}^h) = L(\vec{w}^h) \qquad \forall \vec{w}^h \in \mathcal{V}^h \end{cases}$$

As in the setting of heat conduction, we can rewrite the variational form of Galerkin's method as:

$$\underbrace{a(\vec{w}^h, \vec{v}^h)}_{\text{Unknown Information}} = \underbrace{L(\vec{w}^h) - a(\vec{w}^h, \vec{g}^h)}_{\text{Known Information}}$$

We can make the above more precise by exploiting the local support of NURBS basis functions. Let us define $\eta = \{1, \ldots, n\}$ to be the set containing the indices of all of the functions in the NURBS basis, and define $\eta_{g_A} \subset \eta$ to be the set containing the indices of all NURBS basis functions that are nonzero on Γ_{D_A} for $A = 1, \ldots, d$:

$$\text{supp}(N_i) \cap \Gamma_{D_A} \neq \emptyset \quad \text{if and only if} \quad i \in \eta_{g_A}$$

Then:

$$\mathcal{V}^h := \left\{ \vec{w}^h : \left(\vec{w}^h(\vec{x})\right)_A = \sum_{i \in \eta - \eta_{g_A}} N_i(\vec{x})\, \underbrace{(\vec{c}_i)_A}_{\text{Completely Free}} , \; A = 1, \ldots, d \right\}$$

Moreover, we can choose $\vec{g}^h(\vec{x})$ such that $(\vec{g}_i)_A = 0$ for $i \in \eta - \eta_{g_A}$:

$$\left(\vec{g}^h(\vec{x})\right)_A = \sum_{i \in \eta_{g_A}} N_i(\vec{x})\, (\vec{g}_i)_A$$

Then we can write the A^{th} component of $\vec{u}^h \in \mathcal{S}^h$ as:

$$\left(\vec{u}^h(\vec{x})\right)_A = \sum_{i \in \eta} N_i(\vec{x})(\vec{d}_i)_A = \underbrace{\overbrace{\sum_{i \in \eta - \eta_{g_A}} N_i(\vec{x})(\vec{d}_i)_A}^{\vec{v}^h}}_{\text{Unknown}} + \underbrace{\overbrace{\sum_{i \in \eta_{g_A}} N_i(\vec{x})(\vec{g}_i)_A}^{\vec{g}^h}}_{\text{Known}}$$

If we plug our expressions for $\vec{u}^h$ and $\vec{w}^h$ into the Galerkin formulation, we obtain:

$$a\left(\sum_{A=1}^{d} \sum_{i \in \eta - \eta_{g_A}} N_i (\vec{c}_i)_A \hat{e}_A, \sum_{B=1}^{d} \sum_{j \in \eta - \eta_{g_B}} N_j (\vec{d}_j)_B \hat{e}_B \right) = L\left(\sum_{A=1}^{d} \sum_{i \in \eta - \eta_{g_A}} N_i (\vec{c}_i)_A \hat{e}_A \right)$$

$$- a\left(\sum_{A=1}^{d} \sum_{i \in \eta - \eta_{g_A}} N_i (\vec{c}_i)_A \hat{e}_A, \vec{g}^h \right)$$

where, in $\mathbb{R}^2$:

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{Unit Vectors}$$

Exploiting linearity, we find:

$$\sum_{A=1}^{d} \sum_{i \in \eta - \eta_{g_A}} (\vec{c}_i)_A \left(\sum_{B=1}^{d} \sum_{j \in \eta - \eta_{g_B}} a(N_i \hat{e}_A, N_j \hat{e}_B)(\vec{d}_j)_B - L(N_i \hat{e}_A) + a(N_i \hat{e}_A, \vec{g}^h) \right)$$

$$= 0$$

As $(\vec{c}_i)_A$ = arbitrary for $i \in \eta - \eta_{g_A}$ and $A = 1, \ldots, d$, we have:

$$\sum_{B=1}^{d} \sum_{j \in \eta - \eta_{g_B}} a(N_i \hat{e}_A, N_j \hat{e}_B)(\vec{d}_j)_B = L(N_i \hat{e}_A) - a(N_i \hat{e}_A, \vec{g}^h)$$

for $i \in \eta - \eta_{g_A}$

$A = 1, \ldots, d$

Note that we have $n_{eq.}$ equations and $n_{eq.}$ unknowns where:

$$n_{eq.} = \sum_{A=1}^{d} |\eta - \eta_{g_A}|$$

Size of the set $\eta - \eta_{g_A}$

Our goal is to obtain a matrix formulation from the above algebraic system. In the setting of heat conduction, this was immediately obvious as we had one equation for each global basis function. Here, we have d equations for each global basis function. Thus, we introduce a new connectivity array, the ID array, which relates a degree-of-freedom number

$A=1,\ldots,d$ and a global function number $i \in \eta - \eta_{g_A}$ and returns equation number P:

$$\underbrace{P}_{\text{Equation Number}} = ID(\underbrace{A}_{\text{DOF Number}},\ \underbrace{i}_{\text{Function Number}})$$

With the ID array defined, we can build the matrix equation:

$$\boxed{\underline{\underline{K}}\,\underline{d} = \underline{F}}$$

where:

$$\underline{\underline{K}} = [K_{PQ}] = \text{stiffness matrix}$$

$$\underline{d} = [d_Q] = \text{displacement vector}$$

$$\underline{F} = [F_P] = \text{force vector}$$

with:

$$P = ID(A,i)$$
$$Q = ID(B,j)$$

such that:

$$K_{PQ} = a(N_i \hat{e}_A, N_j \hat{e}_B)$$

$$F_P = L(N_i \hat{e}_A) - a(N_i \hat{e}_A, \vec{g}^h)$$

and:

$$d_Q = (\vec{d}_j)_B$$