Multi-Dimensional B-splines

Multi-dimensional B-splines are simply built through tensor products of one-dimensional B-splines. To build a d_s-dimensional B-spline, we need:

$$p_j = \text{polynomial degree in direction } j$$

$$n_j = \text{no. of basis functions in direction } j \qquad \underline{\text{where}}:\ j = 1, \ldots, d_s$$

$$\Xi_j = \{ \xi_1^{(j)}, \xi_2^{(j)}, \ldots, \xi_{n_j+p_j+1}^{(j)} \} = \text{knot vector in direction } j$$

Note that a subscript j or superscript (j) is used to denote direction j. From the above, we construct a set of one-dimensional B-spline basis functions for each $j = 1, \ldots, d_s$:

$$\{ N_{i_j, p_j}^{(j)} \}_{i_j=1}^{n_j} \qquad \text{for } j = 1, \ldots, d_s$$

Let us define the multi-indices:

$$i = (i_1, i_2, \ldots, i_{d_s}) \qquad \text{where } i_j = 1, \ldots, n_j \text{ w/ } j = 1, \ldots, d_s$$

$$p = (p_1, p_2, \ldots, p_{d_s})$$

and let us define I to be the set of all permissible multi-indices:

$$I = \{ i = (i_1, i_2, \ldots, i_{d_s}) : i_j = 1, \ldots, n_j \text{ for } j = 1, \ldots, d_s \}$$

Then, for each multi-index $i \in I$, we define a multi-dimensional B-spline basis function as:

$$N_{i,p}(\vec{\xi}) = \prod_{j=1}^{d_s} N_{i_j, p_j}^{(j)}(\xi_j)$$

$$= N_{i_1, p_1}^{(1)}(\xi_1) \cdots N_{i_{d_s}, p_{d_s}}^{(d_s)}(\xi_{d_s})$$

where $\vec{\xi} \in \mathbb{R}^{d_s}$ is the parametric coordinate $(\xi_1, \xi_2, \ldots, \xi_{d_s})$. We immediately see that two- and three-dimensional B-spline basis functions are defined as follows:

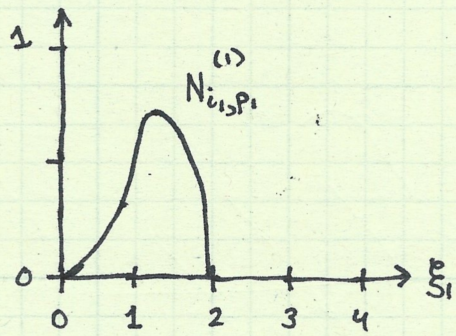
<u>2-D Basis Functions</u>: $N_{i,p}(\vec{\xi}) = N_{i_1,p_1}^{(1)}(\xi_1)\, N_{i_2,p_2}^{(2)}(\xi_2)$

<u>3-D Basis Functions</u>: $N_{i,p}(\vec{\xi}) = N_{i_1,p_1}^{(1)}(\xi_1)\, N_{i_2,p_2}^{(2)}(\xi_2)\, N_{i_3,p_3}^{(3)}(\xi_3)$

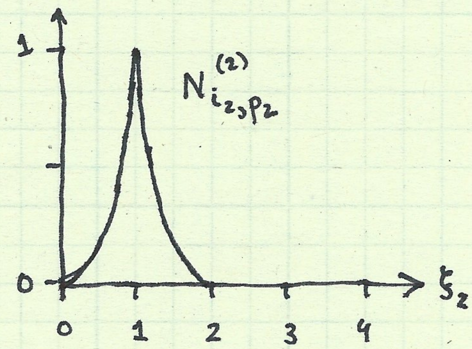
In order to better understand the definition and structure of multi-dimensional B-splines, it helps to consider a two-dimensional example. Let:

$p_1 = 2$	$p_2 = 2$
$n_1 = 7$	$n_2 = 5$
$\Xi_1 = \{0, 0, 0, 1, 2, 2, 3, 4, 4, 4\}$	$\Xi_2 = \{0, 0, 0, 1, 1, 2, 3, 4\}$

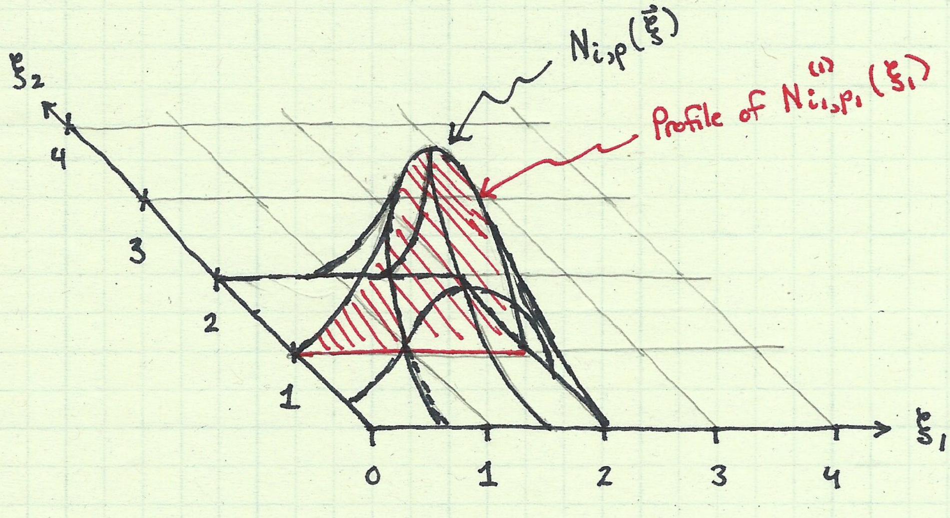
Choosing $i_1 = 3$, we see that $N^{(1)}_{i_1,p_1}(\xi_1)$ looks like:



Choosing $i_2 = 3$, we see that $N^{(2)}_{i_2,p_2}(\xi_2)$ looks like:



Thus, defining $i = (i_1, i_2) = (3,3)$ and $p = (p_1, p_2)$, we see that two-dimensional B-spline basis function $N_{i,p}(\vec{\xi})$ is formed from a tensor product of the one-dimensional B-spline basis functions: N



Make the following observations:

- For each slice $\xi_2 = c$, the basis function $N_{i,p}(\vec{\xi}) = N_{i,p}(\xi_1, c)$ looks like a scaled version of basis function $N^{(1)}_{i_1,p_1}(\xi_1)$.
- For each slice $\xi_1 = c$, the basis function $N_{i,p}(\vec{\xi}) = N_{i,p}(c, \xi_2)$ looks like a scaled version of basis function $N^{(2)}_{i_2,p_2}(\xi_2)$.

- The basis function $N_{i,p}(\vec{\xi})$ is pointwise nonnegative.
- The basis function $N_{i,p}(\vec{\xi})$ has support only over the subdomain $[0,2]^2$.

Indeed, multi-dimensional B-spline basis functions inherit most of the properties of one-dimensional B-splines, including:

- Positivity: $N_{i,p}(\vec{\xi}) \geq 0 \quad$ for $i \in I$
- Local Support: $N_{i,p}(\vec{\xi})$ is nonzero only over the subdomain $\bigotimes_{j=1}^{d_s} [\xi^{(j)}_{i_j}, \xi^{(j)}_{i_j+p_j+1})$.
- At a knot of multiplicity k in direction ℓ, basis function profiles along direction ℓ are $C^{p_\ell - k}$-continuous.

  * To be more precise, suppose that ξ represents a knot of multiplicity k in direction ℓ. Then the one-dimensional function:

  $$f(\xi_\ell) = N_{i,p}(\xi_1, \ldots, \xi_\ell, \ldots, \xi_{d_s}),$$

  wherein ξ_j is fixed for $j \neq \ell$, is $C^{p_\ell - k}$-continuous at $\xi_\ell = \xi$.

- Partition of Unity:

$$\sum_{i \in I} N_{i,p}(\vec{\xi}) = \sum_{i \in I} \prod_{j=1}^{d_s} N^{(j)}_{i_j, d_j}(\xi_j)$$

$$= \prod_{j=1}^{d_s} \sum_{i_j=1}^{n_j} N^{(j)}_{i_j, d_j}(\xi_j)$$

$$= 1 \qquad \text{for } \xi \in \bigotimes_{j=1}^{d_s} [\xi^{(j)}_1, \xi^{(j)}_{n_j+p_j+1})$$

Above, $\otimes$ indicates the tensor product operator as applied to domains. For example, $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}^2$.

B-spline surfaces are built through linear combinations of two-dimensional B-spline basis functions. Namely, given a control net $\{\vec{P}_i\}$ of control points:

$$\vec{P}_i \in \mathbb{R}^d \quad \text{for } i \in I, \quad d \geq 2, \quad d_s = 2$$

a tensor product B-spline surface is defined as:

$$\vec{S}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i N_{i,p}(\vec{\xi})$$

or, more explicitly as:

$$\vec{S}(\xi_1, \xi_2) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \vec{P}_{(i_1,i_2)} N^{(1)}_{i_1,p_1}(\xi_1) N^{(2)}_{i_2,p_2}(\xi_2)$$

The above definition makes the tensor-product nature of B-spline surfaces clear. Note that:

$$\vec{S} : \mathbb{R}^{d_s} \to \mathbb{R}^{d}$$

where $d_s = 2$. Many of the properties of a B-spline surface directly emanate from the properties of the underlying B-spline basis. For instance, a B-spline surface satisfies:

- Affine Covariance
- The Strong Convex Hull Property

} Properties of a B-spline Surface

However, there is no multi-dimensional analogue of the variation diminishing property.

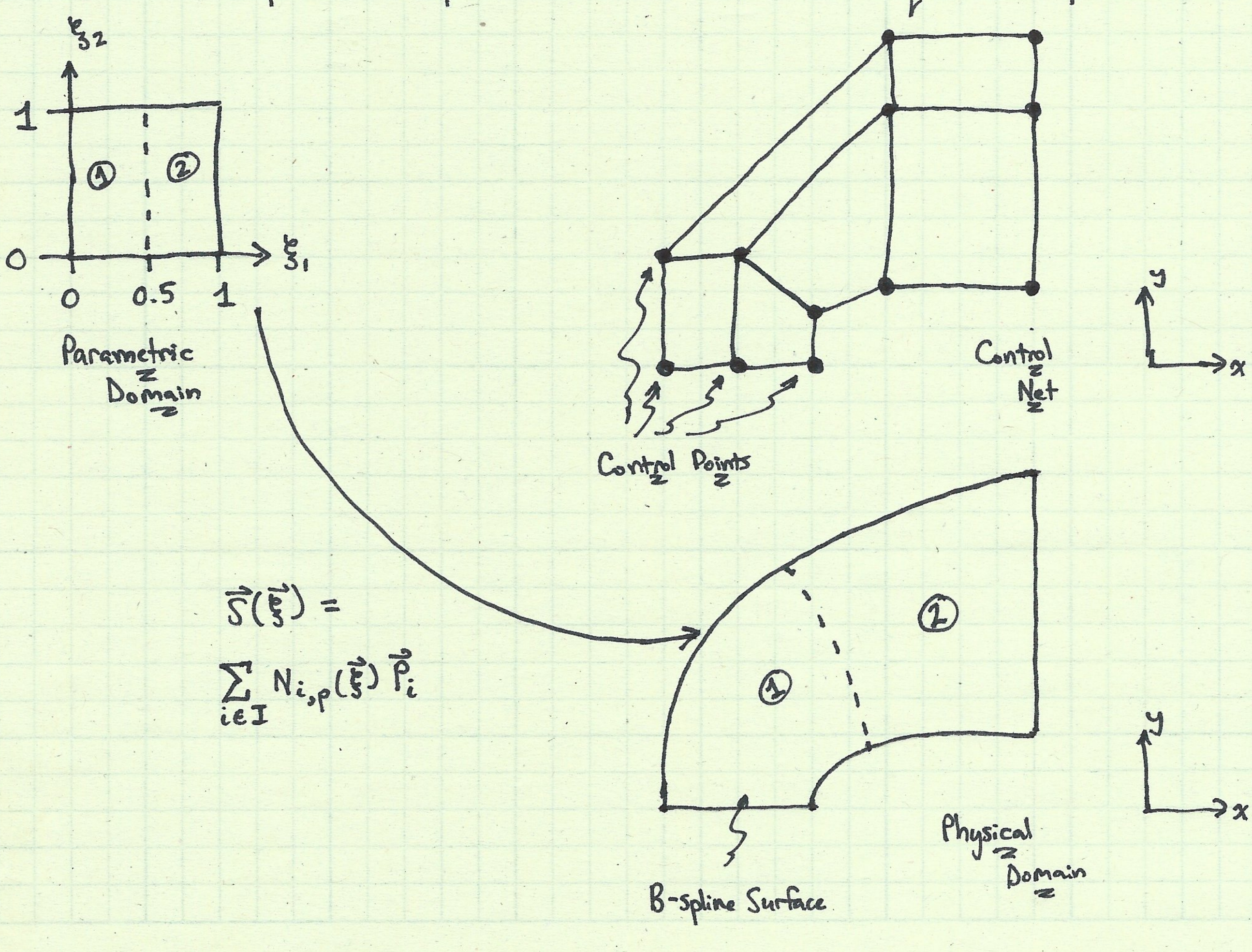
It again helps to consider an example. Let:

$d = 2$ $\qquad \Xi_1 = \{0, 0, 0, 0.5, 1, 1, 1\}$

$p_1 = p_2 = 2$

$n_1 = 4, \; n_2 = 3$ $\qquad \Xi_2 = \{0, 0, 0, 1, 1, 1\}$

Then an example of a B-spline surface associated with these quantities is plotted below:



$$\vec{S}(\vec{\xi}) = \sum_{i \in I} N_{i,p}(\vec{\xi}) \vec{P}_i$$

Note that $\vec{S}$ maps ① in the parametric domain to ① in the physical domain and similarly for ②. In the context of finite element analysis, ① and ② will play the role of elements.

B-Spline solids are built in analogous fashion to B-spline surfaces. Given a control lattice $\{\vec{P}_i\}$ of control points:

$$\vec{P}_i \in \mathbb{R}^d \quad \text{for } i \in I, \quad d \geq 3, \quad d_s = 3$$

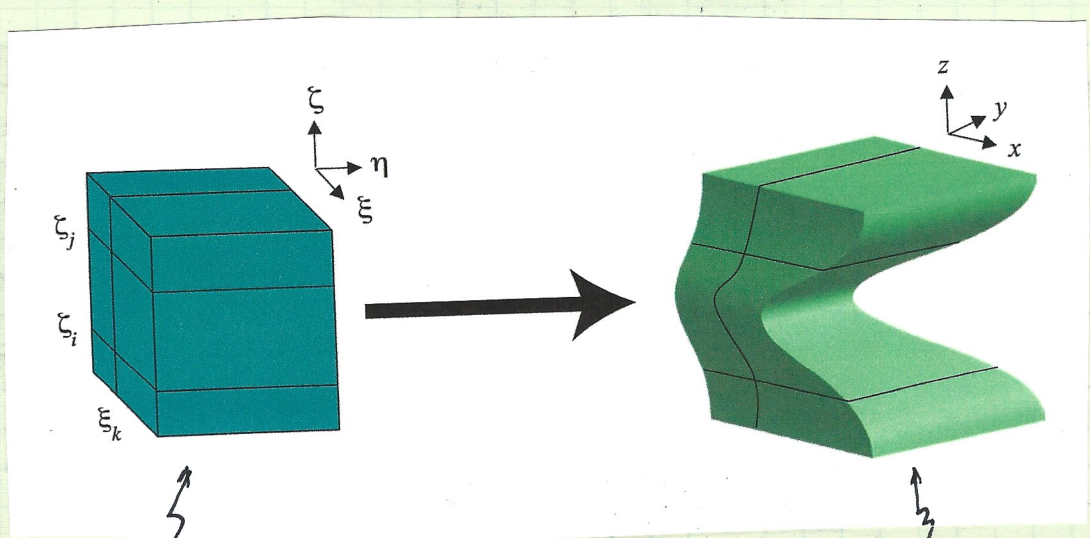
a tensor-product B-spline solid is defined as:

$$\vec{V}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i N_{i,p}(\vec{\xi})$$

or, more explicitly as:

$$\vec{V}(\xi_1, \xi_2, \xi_3) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} \vec{P}_{(i_1,i_2,i_3)} N^{(1)}_{i_1,p_1}(\xi_1) N^{(2)}_{i_2,p_2}(\xi_2) N^{(3)}_{i_3,p_3}(\xi_3)$$

Note that $\vec{V}: \mathbb{R}^{d_s} \to \mathbb{R}^d$ where $d_s = 3$. Like a B-spline surface, the B-spline solid satisfies affine covariance and the convex hull property. A simple example of a B-spline solid is illustrated below:



Parametric Domain

Physical Domain

It should now be clear the advantage of using multi-index notation. Notably, we may represent curves, surfaces, and solids using the same notation:

$$\vec{C}(\xi) = \sum_{i \in I} \vec{P}_i N_{i,p}(\xi), \quad \vec{S}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i N_{i,p}(\vec{\xi}),$$

$$\vec{V}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i N_{i,p}(\vec{\xi})$$

The only difference between these three objects is the specification of $d_s = 1, 2, 3$.