Incompressible Fluid Flow: The Generalized-α Method and Implementation:

As was the case for hyperelasticity, we can express the Galerkin semi-discrete formulation for the incompressible Navier-Stokes equations in a compact form using the parlance of matrix algebra. Let $\underline{V}$, $\dot{\underline{V}}$, and $\underline{P}$ denote the vectors of control variables for velocity, acceleration, and pressure:

$$\left.\begin{array}{lll} \underline{V} = [v_p] & v_p = (\vec{d}_i)_A & p = ID(A,i) \\ \dot{\underline{V}} = [\dot{v}_p] & \dot{v}_p = ((\vec{d}_i)_A)_{,t} & p = ID(A,i) \end{array}\right\} \begin{array}{l} A = 1,\dots,d, \\ i = 1,\dots,n \end{array}$$

$$\underline{P} = [p_i] \qquad p_i = d_i \qquad i = 1,\dots,n$$

where $\vec{u}^h = \sum_i N_i \vec{d}_i$ and $p^h = \sum_i N_i d_i$. Moreover, let us define the residual vectors $\underline{R}^m$ & $\underline{R}^c$ such that:

Momentum: $\underline{R}^m = [R^m_p] \qquad R^m_p = B_{MS}(\{\hat{e}_A N_i, 0\}, \underline{U}^h) - L_{MS}(\{\hat{e}_A N_i, 0\})$

where, $p = ID(A,i)$

Continuity: $\underline{R}^c = [R^c_i] \qquad R^c_i = B_{MS}(\{\vec{0}, N_i\}, \underline{U}^h) - L_{MS}(\{\vec{0}, N_i\})$

Then (MS) takes the following form:

$$\boxed{(DAE) \left\{ \begin{array}{l} \text{Find } \underline{V}(t), \dot{\underline{V}}(t), \text{ and } \underline{P}(t) \text{ s.t.} \\ \qquad \underline{R}^m(\dot{\underline{V}}(t), \underline{V}(t), \underline{P}(t)) = \underline{0} \\ \qquad \underline{R}^c(\dot{\underline{V}}(t), \underline{V}(t), \underline{P}(t)) = \underline{0} \\ \qquad \underline{V}(0) = \underline{V}_0 \end{array} \right.}$$

(DAE) consists of a set of differential algebraic equations as the time derivative of the pressure field does not appear. This is because the continuity equation is not an evolution equation in the context of incompressible flow.

To solve the system given by (DAE), we employ the generalized-α method. The full statement of the method is as follows:

$$\begin{array}{l} \text{Given } \dot{\underline{V}}_n \text{ and } \underline{V}_n, \text{ find } \dot{\underline{V}}_{n+1}, \underline{V}_{n+1}, \text{ and } \underline{P}_{n+1} \text{ s.t.:} \\ \left.\begin{array}{l} \underline{R}^m(\dot{\underline{V}}_{n+\alpha_m}, \underline{V}_{n+\alpha_f}, \underline{P}_{n+1}) = \underline{0} \\ \underline{R}^c(\dot{\underline{V}}_{n+\alpha_m}, \underline{V}_{n+\alpha_f}, \underline{P}_{n+1}) = \underline{0} \end{array}\right\} \text{Equations of Motion} \\ \left.\begin{array}{l} \dot{\underline{V}}_{n+\alpha_m} = \dot{\underline{V}}_n + \alpha_m(\dot{\underline{V}}_{n+1} - \dot{\underline{V}}_n) \\ \underline{V}_{n+\alpha_f} = \underline{V}_n + \alpha_f(\underline{V}_{n+1} - \underline{V}_n) \end{array}\right\} \text{Intermediate Solution} \end{array}$$

$$\boxed{\underline{V}_{n+1} = \underline{V}_n + \Delta t \left((1-\gamma)\, \dot{\underline{V}}_n + \gamma\, \dot{\underline{V}}_{n+1} \right)} \Big\} \text{ Newmark}$$

Second-order accuracy is attained if:

$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m$$

while unconditional stability requires:

$$\alpha_m \geq \alpha_f \geq \frac{1}{2}$$

Moreover, as the Navier-Stokes equations are first-order in time, an optimal combination of high-frequency and low-frequency dissipation occurs when:

$$\alpha_m = \frac{1}{2} \left(\frac{3 - \rho_\infty}{1 + \rho_\infty} \right), \quad \alpha_f = \frac{1}{1+\rho_\infty}$$

where $\rho_\infty \in [0, 1]$ is the spectral radius of the amplification matrix at infinite time step.

For steady state computations, it is advantageous to introduce maximal damping in all modes. Consequently, for such computations, the backward Euler method is preferred. This corresponds to $\gamma = \alpha_m = \alpha_f = 1$.

To solve the generalized-α equations, we utilize a two-phase predictor-multicorrector implementation of Newton's method. We employ the constant velocity/pressure predictor in the following presentation:

Predictor Phase: Set:

$$\underline{V}^0_{n+1} = \underline{V}_n$$

$$\underline{P}^0_{n+1} = \underline{P}_n$$

$$\dot{\underline{V}}^0_{n+1} = \frac{(\gamma - 1)}{\gamma} \dot{\underline{V}}_n$$

Multicorrector Phase: Repeat the following steps for $i = 0, 1, 2, \ldots, i_{max}$ or until convergence is achieved.

1. Evaluate iterates at the intermediate time levels as:

$$\dot{\underline{V}}^i_{n+\alpha_m} = \dot{\underline{V}}_n + \alpha_m \left(\dot{\underline{V}}^i_{n+1} - \dot{\underline{V}}_n \right)$$

$$\underline{V}^i_{n+\alpha_f} = \underline{V}_n + \alpha_f \left(\underline{V}^i_{n+1} - \underline{V}_n \right)$$

2. Use the intermediate solutions to assemble the residuals of the continuity and momentum equations and corresponding tangent matrices in the linear system:

$$\left[\begin{array}{c:c} \overbrace{\dfrac{\partial \underline{R}^{m,i}}{\partial \underline{\dot{V}}_{n+1}}}^{\underline{\underline{K}}^i} & \overbrace{\dfrac{\partial \underline{R}^{m,i}}{\partial \underline{P}_{n+1}}}^{\underline{\underline{G}}^i} \\ \hdashline \underbrace{\dfrac{\partial \underline{R}^{c,i}}{\partial \underline{\dot{V}}_{n+1}}}_{\underline{\underline{D}}^i} & \underbrace{\dfrac{\partial \underline{R}^{c,i}}{\partial \underline{P}_{n+1}}}_{\underline{\underline{L}}^i} \end{array}\right] \left[\begin{array}{c} \Delta \underline{\dot{V}} \\ \hdashline \Delta \underline{P} \end{array}\right] = - \left[\begin{array}{c} \underline{R}^{m,i} \\ \hdashline \underline{R}^{c,i} \end{array}\right]$$

where:

$$\underline{R}^{m,i} = \underline{R}^{m}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

$$\underline{R}^{c,i} = \underline{R}^{c}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

$$\underline{\underline{K}}^{i} = \frac{\partial \underline{R}^{m}}{\partial \underline{\dot{V}}_{n+1}}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

$$= \alpha_m \frac{\partial \underline{R}^{m}}{\partial \underline{\dot{V}}_{n+\alpha m}} + \alpha_f \gamma \Delta t \frac{\partial \underline{R}^{m}}{\partial \underline{V}_{n+\alpha f}}$$

$$\underline{\underline{G}}^{i} = \frac{\partial \underline{R}^{m}}{\partial \underline{P}_{n+1}}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

$$\underline{\underline{D}}^{i} = \frac{\partial \underline{R}^{c}}{\partial \underline{\dot{V}}_{n+1}}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

$$= \alpha_m \frac{\partial \underline{R}^{c}}{\partial \underline{\dot{V}}_{n+\alpha m}} + \alpha_f \gamma \Delta t \frac{\partial \underline{R}^{c}}{\partial \underline{V}_{n+\alpha f}}$$

$$\underline{\underline{L}}^{i} = \frac{\partial \underline{R}^{c}}{\partial \underline{P}_{n+1}}\left(\underline{\dot{V}}^{i}_{n+\alpha m}, \underline{V}^{i}_{n+\alpha f}, \underline{P}^{i}_{n+1}\right)$$

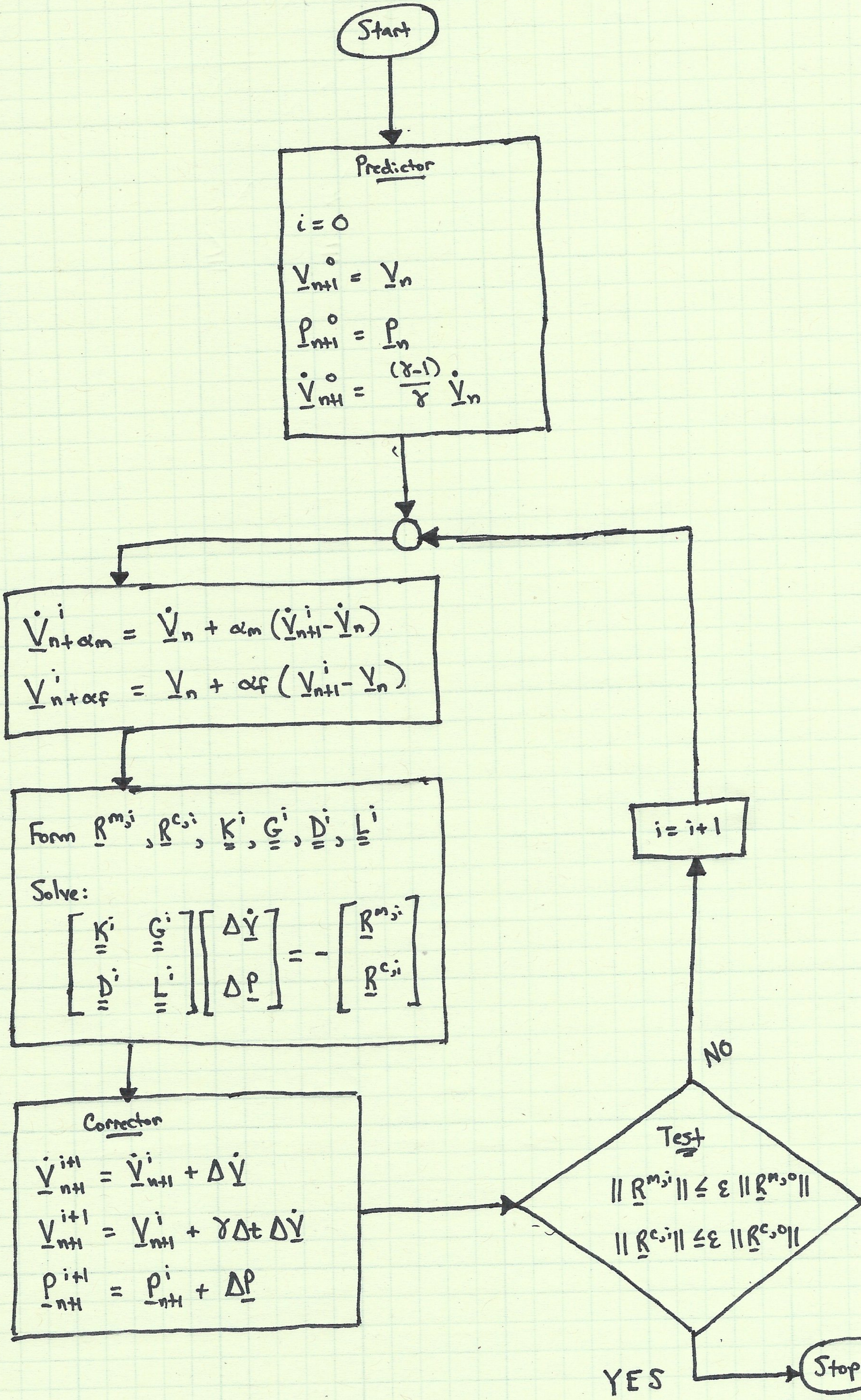
3. Having solved the linear system, update the iterates as:

$$\underline{\dot{V}}^{i+1}_{n+1} = \underline{\dot{V}}^{i}_{n+1} + \Delta \underline{\dot{V}}$$

$$\underline{V}^{i+1}_{n+1} = \underline{V}^{i}_{n+1} + \gamma \Delta t \, \Delta \underline{\dot{V}}$$

$$\underline{P}^{i+1}_{n+1} = \underline{P}^{i}_{n+1} + \Delta \underline{P}$$

A flow chart of the preceding predictor-multicorrector algorithm is given on the following page.


Start
Predictor
$i = 0$
$\underline{V}^{0}_{n+1} = \underline{V}_n$
$\underline{P}^{0}_{n+1} = \underline{P}_n$
$\dot{\underline{V}}^{0}_{n+1} = \frac{(\gamma-1)}{\gamma} \dot{\underline{V}}_n$
$\dot{\underline{V}}^{i}_{n+\alpha_m} = \dot{\underline{V}}_n + \alpha_m (\dot{\underline{V}}^{i}_{n+1} - \dot{\underline{V}}_n)$
$\underline{V}^{i}_{n+\alpha_f} = \underline{V}_n + \alpha_f (\underline{V}^{i}_{n+1} - \underline{V}_n)$
Form $\underline{R}^{m,i}$, $\underline{R}^{c,i}$, $\underline{\underline{K}}^{i}$, $\underline{\underline{G}}^{i}$, $\underline{\underline{D}}^{i}$, $\underline{\underline{L}}^{i}$
Solve:
$\begin{bmatrix} \underline{\underline{K}}^{i} & \underline{\underline{G}}^{i} \\ \underline{\underline{D}}^{i} & \underline{\underline{L}}^{i} \end{bmatrix} \begin{bmatrix} \Delta \dot{\underline{V}} \\ \Delta \underline{P} \end{bmatrix} = - \begin{bmatrix} \underline{R}^{m,i} \\ \underline{R}^{c,i} \end{bmatrix}$
Corrector
$\dot{\underline{V}}^{i+1}_{n+1} = \dot{\underline{V}}^{i}_{n+1} + \Delta \dot{\underline{V}}$
$\underline{V}^{i+1}_{n+1} = \underline{V}^{i}_{n+1} + \gamma \Delta t \Delta \dot{\underline{V}}$
$\underline{P}^{i+1}_{n+1} = \underline{P}^{i}_{n+1} + \Delta \underline{P}$
Test
$\| \underline{R}^{m,i} \| \leq \varepsilon \| \underline{R}^{m,0} \|$
$\| \underline{R}^{c,i} \| \leq \varepsilon \| \underline{R}^{c,0} \|$
NO
$i = i+1$
YES
Stop

The most difficult part of the above algorithm is construction of the tangent matrices. In practice, the consistent tangent matrices are approximated as follows:

$$\underline{\underline{K}} = [K_{PQ}]$$

Unsteady acc. Gal. term.

Unsteady acc. SUPG term

$$K_{PQ} = \alpha_m \int_{\Omega} N_i N_j \, d\Omega \, \delta_{AB} + \alpha_m \int_{\Omega} (\vec{u}^h \cdot \vec{\nabla} N_i) \tau_m N_j \, d\Omega \, \delta_{AB}$$

Adv. Gal. term. (Oseen)

$$+ \alpha_f \gamma \Delta t \int_{\Omega} N_i (\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega \, \delta_{AB}$$

Diff. Gal. term #1

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \nu \vec{\nabla} N_i \cdot \vec{\nabla} N_j \, d\Omega \, \delta_{AB}$$

Diff. Gal. term #2

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \nu \vec{\nabla} \cdot (N_i \hat{e}_A) \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega$$

Adv. SUPG term

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \tau_m (\vec{u}^h \cdot \vec{\nabla} N_i)(\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega \, \delta_{AB}$$

Grad-div Stab. term

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \tau_c \vec{\nabla} \cdot (N_i \hat{e}_A) \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega$$

$P = ID(A,i)$
$Q = ID(B,j)$

$$\underline{\underline{G}} = [G_{Pj}]$$

Pressure force Gal. term

Pressure Force SUPG term

$$G_{Pj} = -\int_{\Omega} \vec{\nabla} \cdot (N_i \hat{e}_A) N_j \, d\Omega + \int_{\Omega} \tau_m (\vec{u}^h \cdot \vec{\nabla} N_i) \hat{e}_A \cdot \vec{\nabla} N_j \, d\Omega$$

$P = ID(A,i)$

$$\underline{\underline{D}} = [D_{iQ}]$$

Div. Gal. Term

$$D_{iQ} = \alpha_f \gamma \Delta t \int_{\Omega} N_i \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega$$

Adv. PSPG Term

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \tau_M \vec{\nabla} N_i \cdot \hat{e}_B \, (\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega$$

Unsteady Acc. PSPG Term

$$+ \alpha_m \int_{\Omega} \tau_M \vec{\nabla} N_i \cdot \hat{e}_B \, N_j \, d\Omega$$

$$Q = ID(B, j)$$

$$\underline{\underline{L}} = [L_{ij}]$$

Pressure Force PSPG Term

$$L_{ij} = \int_{\Omega} \tau_M \vec{\nabla} N_i \cdot \vec{\nabla} N_j \, d\Omega$$

The above matrices may be constructed via an element-by-element formation and assemby procedure wherein the element residual vectors and tangent matrices are formed using three ingredients: (i) pullback to a master element, (ii) use of Bézier extraction to evaluate basis functions and derivatives, and (iii) Gauss quadrature approximations of integrals.