Boundary Value Problems: The Heat Equation

As an example of solving a PDE posed over the domain defined by a NURBS geometry, let us consider the steady heat equation. In strong form, the problem is:

$$
(S) \left\{ \begin{array}{lrcll}
\text{Find } u: \overline{\Omega} \to \mathbb{R} \text{ s.t.} & & & & \\
& -\vec{\nabla} \cdot (\kappa \vec{\nabla} u) & = & f & \text{in } \Omega \\
& u & = & g & \text{on } \Gamma_D \\
& \kappa\, \partial u / \partial n & = & h & \text{on } \Gamma_N \\
& \kappa\, \partial u / \partial n & = & -\beta (u - u_R) & \text{on } \Gamma_R
\end{array} \right.
$$

Above, we have:

u = The temperature field $u: \overline{\Omega} \to \mathbb{R}$

κ = The thermal conductivity $\kappa: \Omega \to \mathbb{R}^+$ $(\kappa > 0)$

f = The heating $f: \Omega \to \mathbb{R}$

g = The fixed temperature field $g: \Gamma_D \to \mathbb{R}$

h = The fixed heat flux $h: \Gamma_N \to \mathbb{R}$ (flux <u>into</u> Ω)

u_R = The temperature of the surrounding convective medium $u_R: \Gamma_R \to \mathbb{R}$

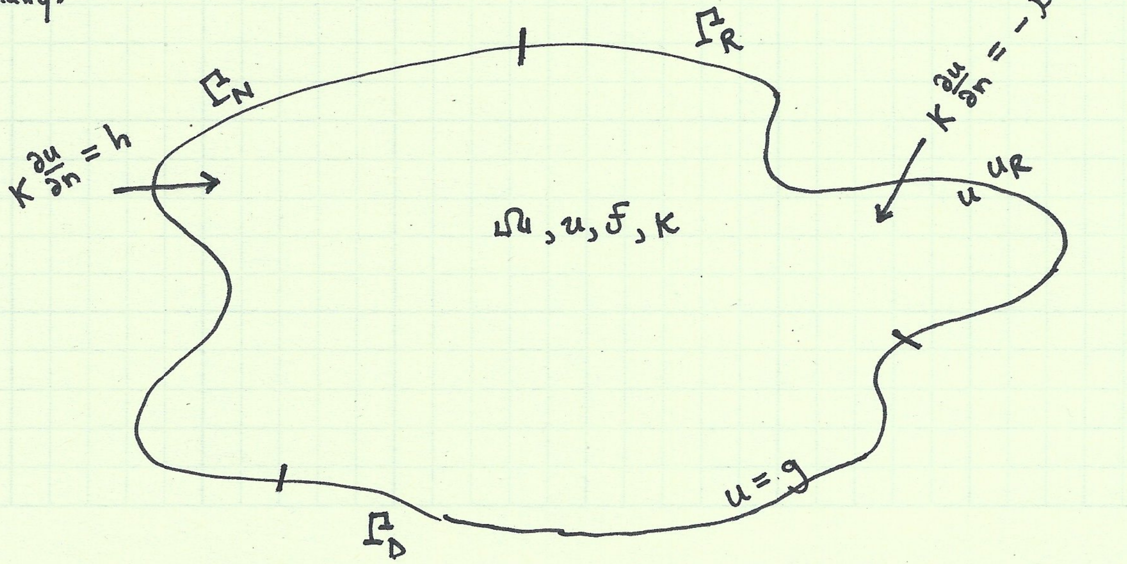
β = The convective heat transfer coefficient $\beta: \Gamma_R \to \mathbb{R}^+$ $(\beta > 0)$

Γ_D = The Dirichlet boundary

Γ_N = The Neumann boundary

Γ_R = The Robin boundary

We require that $\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R} = \Gamma \equiv \partial\Omega$ and $\Gamma_D \cup \Gamma_N \cup \Gamma_R = \emptyset$.
Visually:



Note that if $u_R < u$, then $K\,\partial u/\partial n < 0$ which indicates that there is a heat flux out of the Robin boundary. Hence, the convective medium may be used to cool the material in the domain Ω. For example, the domain Ω may represent a hot solid and the convective medium may be a cool liquid. Cooling of supercomputers occurs in such a fashion.

Our objective is to find an approximate solution $u^h \approx u$. To do so, we rely on Galerkin's method. Thus, we need a weak, or variational, counterpart to the strong form of the heat equation. To do so, we need to characterize two classes of functions. The first is to be composed of candidate, or trial, solutions. From the onset, these functions will be required to satisfy the Dirichlet boundary condition of (S). Indeed, we define the space of trial solutions as follows:
collection

$$\mathcal{S} := \left\{ u : \bar{\Omega} \to \mathbb{R} : u \in H^1(\Omega) \text{ and } u|_{\Gamma_D} = g \right\}$$

Above, $H^1(\Omega)$ is the Sobolev space:

$$H^1(\Omega) := \left\{ u : \bar{\Omega} \to \mathbb{R} : u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega) \right\}$$

where:

$$L^2(\Omega) := \left\{ u \;\middle|\; \int_\Omega u^2 \, d\Omega < +\infty \right\}$$

Thus, we see that trial solutions are those functions who are square-integrable with square-integrable derivatives and who also satisfy the Dirichlet boundary condition of (S). We expect the temperature field u to be a member of $\mathcal{S}$. Our regularity requirements on $\mathcal{S}$ will be better understood momentarily.

The second collection of functions we need are the weighting functions. The collection of weighting functions is very similar to $\mathcal{S}$, except that we have the homogeneous counterpart of the Dirichlet boundary condition:

$$\mathcal{V} := \left\{ w : \bar{\Omega} \to \mathbb{R} : w \in H^1(\Omega) \text{ and } w|_{\Gamma_D} = 0 \right\}$$

The reason for this homogeneous Dirichlet boundary condition will also be made clear momentarily.

We now take our strong form and multiply through by an arbitrary weighting function $w \in \mathcal{V}$. Integrating over Ω, we obtain:

$$\int_\Omega \left(-w \vec{\nabla} \cdot (K \vec{\nabla} u) \right) d\Omega = \int_\Omega w f \, d\Omega$$

We now integrate the term on the left by parts, resulting in:

$$\int_{\Omega} \left(-w\,\vec{\nabla}\cdot\left(K\vec{\nabla}u\right)\right) d\Omega = \int_{\Omega} K\,\vec{\nabla}w\cdot\vec{\nabla}u\, d\Omega - \int_{\Gamma} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma$$

We now write:

$$\int_{\Gamma} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma = \int_{\Gamma_D} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma + \int_{\Gamma_N} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma + \int_{\Gamma_R} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma$$

As $w|_{\Gamma_D} = 0$, we have:

$$\int_{\Gamma_D} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma = 0$$

As $K\frac{\partial u}{\partial n}\big|_{\Gamma_N} = h$, we have:

$$\int_{\Gamma_N} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma = \int_{\Gamma_N} w\,h\, d\Gamma$$

As $K\frac{\partial u}{\partial n}\big|_{\Gamma_R} = -\beta(u-u_R)$, we have:

$$\int_{\Gamma_R} w\,K\,\frac{\partial u}{\partial n}\, d\Gamma = -\int_{\Gamma_R} \beta\, w\,(u-u_R)\, d\Gamma$$

Thus:

$$\int_{\Omega} K\,\vec{\nabla}w\cdot\vec{\nabla}u\, d\Omega + \int_{\Gamma_R} \beta\, w\, u\, d\Gamma =$$

Heat Transfer Across Γ_R due to Convection

$$\int_{\Omega} w\,f\, d\Omega + \int_{\Gamma_N} w\,h\, d\Gamma + \int_{\Gamma_R} \beta\, w\, u_R\, d\Gamma$$

Internal Heating in Ω

Heat Transfer Across Γ_N

This gives the weak form of the heat equation problem:

$$(W) \begin{cases} \text{Find } u \in \mathcal{S} \text{ s.t.} \\ \quad a(w,u) = L(w) \quad \forall w \in \mathcal{V} \\ \text{where:} \\ \quad a(w,u) := \int_{\Omega} \kappa \vec{\nabla} w \cdot \vec{\nabla} u \, d\Omega + \int_{\Gamma_R} \beta w u \, d\Gamma \\ \quad L(w) := \int_{\Omega} w f \, d\Omega + \int_{\Gamma_N} w h \, d\Gamma + \int_{\Gamma_R} \beta w u_R \, d\Gamma \end{cases}$$

Above, $a(\cdot,\cdot)$ is a bilinear form and $L(\cdot)$ is a linear form. Moreover, $a(\cdot,\cdot)$ is symmetric and positive definite, that is:

$$a(w,u) = a(u,w)$$

$$a(w,w) > 0 \quad \forall w \in \mathcal{V} \text{ s.t. } w \neq 0$$

These two properties will yield a well-conditioned matrix system in the context of finite element analysis.

Moreover, note that the Dirichlet boundary condition of (S) is enforced strongly while the Neumann and Robin boundary conditions of (S) are enforced weakly. That is, the Dirichlet condition is enforced on $\mathcal{S}$, the trial solutions, while the Neumann and Robin conditions are enforced through the variational formulation.

A solution to (W) is denoted as a weak solution, and it is easily shown that (W) admits a unique weak solution. We also now see why $H^1(\Omega)$ was the appropriate space with which to work. Despite the fact that (S) required u to have well-defined second derivatives, (W) only requires that first derivatives are square integrable.