Incompressible Fluid Flow: Variational Multiscale Formulation:

Recall that instabilities can arise due to the failure of a numerical method to represent all of the scales present in an incompressible flow problem. In a finite-dimensional setting, it is simply impossible to capture all of the features of a system, and that which is missing can have a deleterious effect on the ability to accurately model the scales that are otherwise within reach. In order to cope with this dilemma, we invoke the variational multiscale (VMS) method in the current context. In the VMS approach, a model is prescribed for incorporating missing unresolved fine-scale effects into numerical problems governing coarse-scale behavior. This renders an otherwise unstable method stable.

VMS was originally proposed by Hughes in the context of advection-diffusion problems in the two seminal papers:

T.J.R. Hughes, "Multiscale Phenomena: Green's Functions, the Dirichlet-to-Neumann Formulation, Subgrid Scale Models, Bubbles and the Origins of Stabilized Methods," CMAME, 127: 387-401. (1995)

T.J.R. Hughes, G.R. Feijóo, L. Mazzei, J.-B. Quincy, "The Variational Multiscale Method - A Paradigm for Computational Mechanics," CMAME, 166-3-24. (1998)

Since these papers, VMS has remained an active and rich area of research. For a general overview of VMS, refer to the review paper:

T.J.R. Hughes, G. Scovazzi, L.P. Franca, "Multiscale & Stabilized Methods", Encyclopedia of Computational Mechanics.

Here, we will employ the residual-based VMS approach to multiscale modeling. This approach was first proposed for incompressible fluid flow in:

Y. Bazilevs, V.M. Calo, J.A. Cottrell, T.J.R. Hughes, A. Reali, G. Scovazzi, "Variational Multiscale Residual-Based Turbulence Modeling for Large Eddy Simulation of Incompressible Flows," CMAME, 197: 173-201. (2007)

VMS deserves to be the topic of a class onto itself, so we will only briefly mention details of its derivation in what follows. Instead, we will focus on implementation.

The starting point of any VMS formulation is a multiscale direct-sum decomposition of the set of trial solutions and space of test functions into coarse-scale and fine-scale subspaces. To this effect, let us first assume that we can write the solution $\underline{u}$ as $\underline{u}(t) = \underline{v}(t) + \underline{g}(t)$ where $\underline{v}(t) \in \mathcal{V}$ and $\underline{g}(t)|_{\Gamma_{in}} = \{\vec{g}(t), 0\}$. Then we split:

$$\mathcal{V} = \bar{\mathcal{V}} \oplus \mathcal{V}'$$

Infinite-Dimensional Subspace

$\mathcal{V}/\bar{\mathcal{V}}$

Finite-Dimensional Subspace

Finite Element, Spectral, or NURBS Space

We refer to $\bar{\mathcal{V}}$ as the coarse-scale subspace and $\mathcal{V}'$ as the fine-scale subspace. For each $\underline{V} \in \mathcal{V}$, we can write $\underline{V} = \bar{\underline{V}} + \underline{V}'$ where $\bar{\underline{V}} \in \bar{\mathcal{V}}$ and $\underline{V}' \in \mathcal{V}'$. To obtain a unique decomposition, we require the aid of a linear projection operator $\mathcal{P}$ that gives $\bar{\underline{U}} = \mathcal{P}\underline{U}$ and $\underline{U}' = (\mathcal{I} - \mathcal{P})\underline{U}$ for $\underline{U} \in \mathcal{V}$.

We now split our solution into coarse-scale and fine-scale components. Namely, we write:

$$\underline{U} = \bar{\underline{U}} + \underline{U}' \qquad \text{where:} \quad \bar{\underline{U}} = \bar{\underline{V}} + \underline{G} \ \text{ with } \ \bar{\underline{V}} \in \bar{\mathcal{V}}$$

$$\underline{U}' \in \mathcal{V}'$$

Moreover, we split our weighting function into coarse-scale and fine-scale components. Namely, we write:

$$\underline{W} = \bar{\underline{W}} + \underline{W}' \qquad \text{where:} \quad \bar{\underline{W}} \in \bar{\mathcal{V}}$$

$$\underline{W}' \in \mathcal{V}'$$

Then, our weak formulation becomes:

Multiscale Version of (W)

Find $\underline{U}(t) = \bar{\underline{U}}(t) + \underline{U}'(t)$, with $\bar{\underline{U}}(t) = \bar{\underline{V}}(t) + \underline{G}(t)$ and $\bar{\underline{V}}(t) \in \bar{\mathcal{V}}$, $\underline{U}'(t) \in \mathcal{V}'$, such that:

$$\boxed{\text{Coarse-Scale}} \quad B(\bar{\underline{W}}; \bar{\underline{U}}(t) + \underline{U}'(t)) = L(\bar{\underline{W}}) \quad \forall \bar{\underline{W}} \in \bar{\mathcal{V}} \ \& \ t \in (0,T)$$

$$\boxed{\text{Fine-Scale}} \quad B(\underline{W}'; \bar{\underline{U}}(t) + \underline{U}'(t)) = L(\underline{W}') \quad \forall \underline{W}' \in \mathcal{V}' \ \& \ t \in (0,T)$$

$$\vec{u}(0) = \vec{u}_0$$

The cornerstone of the VMS approach is the following idea: Solve the fine-scale problem for $\underline{U}'$ in terms of $\bar{\underline{U}}$ and insert the resulting expression into the coarse-scale problem. If we re-express the fine-scale problem as:

$$\boxed{\text{Fine-Scale}} \quad B^*_{\bar{\underline{U}}}(\underline{W}'; \underline{U}'(t)) = L^*_{\bar{\underline{U}}}(\underline{W}') \qquad \forall \underline{W}' \in \mathcal{V}' \ \& \ t \in (0,T)$$

where:

$$B^*_{\bar{\underline{U}}}(\underline{W}'; \underline{U}'(t)) \equiv B_1(\underline{W}', \underline{U}') + B_2(\underline{W}', \bar{\underline{U}}, \underline{U}') + B_2(\underline{W}', \underline{U}', \bar{\underline{U}}) + B_2(\underline{W}', \underline{U}', \underline{U}')$$

Time-Dependence Implicit

$$L^*_{\bar{\underline{U}}}(\underline{W}') \equiv L(\underline{W}') - B(\underline{W}'; \bar{\underline{U}}) = \int_{\Omega} \underline{W}' \cdot \underline{R}'(\bar{\underline{U}}) \, d\Omega$$

and $\underline{R}'(\bar{\underline{U}})$ is the residual of the coarse-scales, we see we can formally solve for $\underline{U}'$ as:

$$\underline{U}' = \mathcal{F}'(\bar{\underline{U}}, \underline{R}'(\bar{\underline{U}}))$$

Nonlinear Functional

A coarse-scale formulation is then obtained by plugging the above expression back into the coarse-scale problem:

$$\boxed{\text{Coarse-Scale}} \quad B(\underline{\overline{W}}; \underline{\overline{U}}(t) + \underline{\mathcal{F}}'(\underline{\overline{U}}, \underline{R}'(\underline{\overline{U}}))) = L(\underline{\overline{W}})$$

$$\forall \underline{\overline{W}} \in \overline{\mathcal{V}} \;\&\; t \in (0,T)$$

The above formulation is closed in that there are no unspecified parameters. Unfortunately, $\underline{\mathcal{F}}'$ is unknown and must be approximated. We model the fine-scales by a scaling parameter multiplying the residual of the coarse-scales. More specifically, given $\underline{U}^h \approx \underline{\overline{U}}$, we write:

$$\underline{U}' \approx \underset{\approx}{\tau}\, \underline{R}'(\underline{U}^h)$$

where $\underset{\approx}{\tau}$ is a 4×4 matrix in three spatial dimensions for $d = 3$. Here, we have used the superscript h to denote we are working with an approximate numerical solution rather than the exact projected solution $\overline{U} = PU$. Typically, we define $\underset{\approx}{\tau}$ as:

$$\underset{\approx}{\tau} = \text{diag}(\underbrace{\tau_m, \ldots, \tau_m}_{d \text{ times}}, \tau_c)$$

where τ_m and τ_c are carefully selected parameters. This results in the fine-scale approximations:

$$\underline{u}' \approx -\tau_m \underline{r}_m(\overline{u}, \overline{p}) \approx -\tau_m \underline{r}_m(\vec{u}^h, p^h)$$

$$p' \approx -\tau_c r_c(\overline{u}, \overline{p}) \approx -\tau_c r_c(\vec{u}^h, p^h)$$

where $\underline{r}_m$ and r_c are the momentum & continuity residuals:

$$\underline{r}_m(\vec{u}^h, p^h) = \vec{u}^h_{,t} + (\vec{u}^h \cdot \vec{\nabla})\vec{u}^h - \vec{\nabla} \cdot (2\nu \vec{\nabla}^s \vec{u}^h) + \vec{\nabla} p^h - \vec{f}$$

$$r_c(\vec{u}^h, p^h) = \vec{\nabla} \cdot \vec{u}^h$$

It remains to specify our coarse-scale space and the parameters τ_m and τ_c. We define the coarse-scale space using isoparametric NURBS. Namely, we define our coarse-scale set of trial solutions as:

$$\mathcal{S}^h_t \equiv \mathcal{S}^h_{u_t} \times \mathcal{S}^h_{p_t}$$

where:

$$\mathcal{S}^h_{u_t} := \left\{ \vec{u}^h(t) \in \mathcal{S}_{u_t} : \vec{u}^h(\vec{x}, t) = \sum_{i=1}^{n} N_i(\vec{x}) \vec{d}_i(t) \right\}$$

$$\mathcal{S}^h_{p_t} := \left\{ p^h(t) \in \mathcal{S}_{p_t} : p^h(\vec{x}, t) = \sum_{i=1}^{n} N_i(\vec{x}) d_i(t) \right\}$$

and our coarse-scale space of weighting functions as:

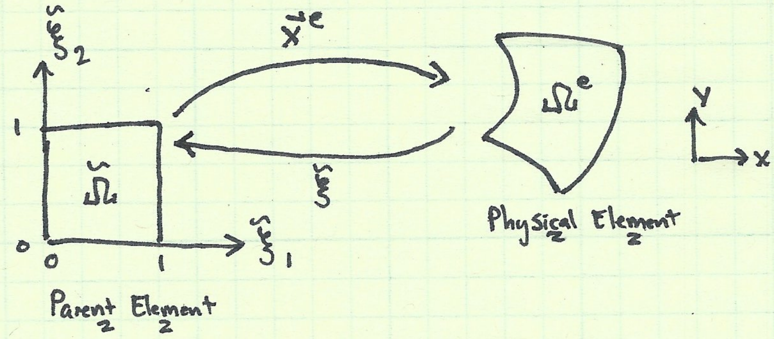
$$\mathcal{V}^h \equiv \mathcal{V}^h_w \times \mathcal{V}^h_q$$

where:

$$\mathcal{V}^h_w := \left\{ \vec{w}^h \in \mathcal{V}_w : \vec{w}^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, \vec{c}_i \right\}$$

$$\mathcal{V}^h_q := \left\{ q^h \in \mathcal{V}_q : q^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, c_i \right\}$$

The parameters τ_m and τ_c are defined using the local geometric map $\vec{x}^e(\tilde{\xi})$ which maps $\tilde{\Omega}$ to Ω^e:



Note that we have assumed that the mapping $\vec{x}^e$ has a well-defined inverse, $\tilde{\xi}$, in the above picture. Let us define the second-rank tensor:

$$\underline{\underline{G}} = 4 \left(\frac{\partial \tilde{\xi}}{\partial \vec{x}^e} \right)^T \left(\frac{\partial \tilde{\xi}}{\partial \vec{x}^e} \right)$$

and vector $\underline{g}$:

$$g = [g_A]$$

$$g_A = 2 \sum_{B=1}^{d} \left(\frac{\partial \tilde{\xi}}{\partial \vec{x}^e} \right)_{BA}$$

As an example of $\partial \tilde{\xi} / \partial \vec{x}^e$, consider the case when the element under consideration is a cube with edge length h. The parent element is scaled such that $\partial \tilde{\xi} / \partial \vec{x}^e = h^{-1} \underline{\underline{I}}$ where $\underline{\underline{I}}$ is the identity matrix.

With the metrics $\underline{\underline{G}}$ and g defined, the parameters τ_m and τ_c take the form:

$$\tau_m = \left(\left(\frac{2}{\Delta t} \right)^2 + \vec{u}^h \cdot \underline{\underline{G}}\, \vec{u}^h + C_I \nu^2 \underline{\underline{G}} : \underline{\underline{G}} \right)^{-1/2}$$

$$\tau_c = \left(g \cdot \tau_m\, g \right)^{-1}$$

where Δt is the yet undefined timestep size and C_I is a positive constant, independent of the mesh size, that derives from an element-wise inverse estimate of the form:

$$\int_{\Omega^e} \left|\vec{\nabla}\cdot\left(2\nu\vec{\nabla}^s\vec{w}^h\right)\right|^2 d\Omega^e \le \frac{C_I}{(h^e)^2}\int_{\Omega^e} 2\nu\left(\vec{\nabla}^s\vec{w}^h\right):\left(\vec{\nabla}^s\vec{w}^h\right) d\Omega^e \qquad \forall \vec{w}^h \in \mathcal{V}^h_w$$

where h^e is the local mesh size.

We are now almost ready to state the discrete formulation we will utilize hereafter. However, we make two simplifying assumptions first.

<u>Assumption #1</u>: We assume that $\int_{\Omega} \vec{w}^h\cdot\frac{\partial\vec{u}'}{\partial t}\, d\Omega = 0 \quad \forall \vec{w}^h \in \mathcal{V}^h_w$.

<u>Assumption #2</u>: We assume that $\int_{\Omega} 2\nu\left(\vec{\nabla}^s\vec{w}^h\right):\left(\vec{\nabla}^s\vec{u}'\right) d\Omega = 0 \quad \forall \vec{w}^h \in \mathcal{V}^h_w$.

Now let us define:

SUPG/PSPG Stabilization

$$B_{MS}(\underline{W}^h;\underline{U}^h) := B(\underline{W}^h;\underline{U}^h)$$
$$+ \int_{\Omega}\left(\left(\vec{u}^h\cdot\vec{\nabla}\right)\vec{w}^h + \vec{\nabla}q^h\right)\cdot\left(\tau_m\,\underline{r}_m(\vec{u}^h,p^h)\right) d\Omega$$

grad-div Stabilization

$$+ \int_{\Omega}\left(\vec{\nabla}\cdot\vec{w}^h\right)\left(\tau_c\, r_c(\vec{u}^h,p^h)\right) d\Omega$$
$$+ \int_{\Omega}\left(\vec{\nabla}\vec{w}^h\right):\left(\left(\tau_m\,\underline{r}_m(\vec{u}^h,p^h)\right)\otimes\vec{u}^h\right) d\Omega$$

Non-classical Stabilization

$$- \int_{\Omega}\left(\vec{\nabla}\vec{w}^h\right):\left(\left(\tau_m\,\underline{r}_m(\vec{u}^h,p^h)\right)\otimes\left(\tau_m\,\underline{r}_m(\vec{u}^h,p^h)\right)\right) d\Omega$$

$$L_{MS}(\underline{W}^h) := L(\underline{W}^h)$$

Then our discrete formulation takes the form:

VMS Formulation

$$\boxed{(MS)\begin{cases} \text{Find } \underline{U}^h(t) = \{\vec{u}^h(t), p(t)\} \in \mathcal{S}^h_t \text{ s.t.} \\ B_{MS}(\underline{W}^h;\underline{U}^h) = L_{MS}(\underline{W}^h) \quad \forall\, \underline{W}^h \in \mathcal{V}^h \;\&\; t\in(0,T) \\ \vec{u}^h(0) = \vec{u}^h_0 \end{cases}}$$

where $\vec{u}^h_0$ is a suitably defined initial condition.