

Structural Vibrations and Wave Propagation

We now turn our attention to the study of dynamic systems. To begin, we discuss the problems of structural vibrations and wave propagation in the context of linear elastodynamics.

The strong form of the linear elastodynamics initial-boundary value problem is as follows:

$$(S) \left\{ \begin{array}{ll} \text{Find } \vec{u}: \bar{\Omega}_t \times [0, T] \rightarrow \mathbb{R}^d \text{ s.t.} & \\ \rho u_{A,tt} = \sigma_{AB,B} + f_A & \text{in } \bar{\Omega}_t \times (0, T) \quad A=1, \dots, d \\ u_A = g_A & \text{on } \Gamma_{D_A}^1 \times (0, T) \quad A=1, \dots, d \\ \sigma_{AB} n_B = h_A & \text{on } \Gamma_{N_A}^1 \times (0, T) \quad A=1, \dots, d \\ \vec{u}(\vec{x}, 0) = \vec{u}_0(\vec{x}) & \text{in } \bar{\Omega}_t \\ \vec{u}_{,t}(\vec{x}, 0) = \vec{v}_0(\vec{x}) & \text{in } \bar{\Omega}_t \end{array} \right.$$

Above, we have:

u_A = The A^{th} component of the displacement field \vec{u}

σ_{AB} = The components of the stress tensor

f_A = The A^{th} component of the prescribed body force per unit volume

ρ = The density of the body $\rho: \bar{\Omega}_t \rightarrow \mathbb{R}^+$

g_A = The prescribed boundary displacement $g_A: \Gamma_{D_A}^1 \rightarrow \mathbb{R}$

h_A = The prescribed boundary traction $h_A: \Gamma_{N_A}^1 \rightarrow \mathbb{R}$

\vec{u}_0 = The prescribed initial displacement $\vec{u}_0: \bar{\Omega}_t \rightarrow \mathbb{R}^d$

\vec{v}_0 = The prescribed initial velocity $\vec{v}_0: \bar{\Omega}_t \rightarrow \mathbb{R}^d$

We now construct a weak form for (S). For each time $t \in (0, T)$, we define the set of trial solutions as:

$$\mathcal{S}_t := \left\{ \vec{u}(\cdot, t) \in (H^1(\bar{\Omega}_t))^d : u_A(\vec{x}, t) = g_A(\vec{x}, t) \text{ for } \vec{x} \in \Gamma_{D_A}^1 \right\}$$

Moreover, as in the setting of linear elasticity, we define:

$$a(\vec{w}, \vec{u}) := \int_{\bar{\Omega}_t} w_{(A,B)} c_{ABCD} u_{(C,D)} \, d\Omega_t$$

$$L(\vec{w}) := \int_{\Omega} w_A f_A d\Omega + \sum_{A=1}^d \left(\int_{I_{NA}} w_A h_A dI^2 \right)$$

where any dependence on time is implicitly understood. Finally, for two vector-valued functions \vec{w} and \vec{u} we define:

$$(\vec{w}, \vec{u}) := \int_{\Omega} w_A u_A d\Omega$$

With the above notation established, the weak form for (S) is as follows:

$$\left\{ \begin{array}{l} \text{Find } \vec{u}(t) \in \mathcal{D}_t \text{ s.t.} \\ (\vec{w}) \quad \begin{aligned} (\vec{w}, p \vec{u}_{tt}) + a(\vec{w}, \vec{u}) &= L(\vec{w}) \quad \forall \vec{w} \in \mathcal{V} \\ (\vec{w}, p \vec{u}(0)) &= (\vec{w}, p \vec{u}_0) \quad \forall \vec{w} \in \mathcal{V} \\ (\vec{w}, p \vec{u}_{st}(0)) &= (\vec{w}, p \vec{v}_0) \quad \forall \vec{w} \in \mathcal{V} \end{aligned} \end{array} \right.$$

Above, as in the setting of linear elasticity, we have: $\xrightarrow{\text{No time-dependence!}}$

$$\mathcal{V} := \left\{ \vec{v} \in (H^1(\Omega))^d : v_A(\vec{x}) = 0 \text{ for } \vec{x} \in I_{DA} \right\}$$

The problems of structural vibrations and wave propagation concern the propagation of perturbations. More specifically, these problems address the following questions:

Structural Vibrations: How do perturbations in the initial conditions propagate in time?

Wave Propagation: How do perturbations in the boundary conditions and body force propagate in space?

We answer both of these questions in succession.

Structural Vibrations

We first address the question of structural vibrations, het:

$$\vec{u}_0(\vec{x}) \leftarrow \vec{u}_0(\vec{x}) + \delta \vec{u}_0(\vec{x})$$

$$\vec{v}_0(\vec{x}) \leftarrow \vec{v}_0(\vec{x}) + \delta \vec{v}_0(\vec{x})$$

$$\vec{u}(\vec{x}, t) \leftarrow \vec{u}(\vec{x}, t) + \delta \vec{u}(\vec{x}, t)$$

It is easily seen that the perturbation $\delta \vec{u}$ satisfies the linear elastodynamics equations with homogeneous Dirichlet & Neumann boundary conditions and zero body force.

Let us suppose with an abuse of notation that:

$$\delta \vec{u}(\vec{x}, t) = \vec{u}(\vec{x}) e^{i\omega t}$$

where:

\vec{u} = The amplitude of the perturbation

ω = The frequency of the perturbation

The amplitude and frequency are unknown and must be solved for. Inserting the above expression into (W), we find we have the following eigenproblem for the amplitude and frequency:

$$(E) \left\{ \begin{array}{l} \text{Find } (\omega, \vec{u}) \in \mathbb{R}^+ \times \mathcal{V} \text{ such that} \\ a(\vec{w}, \vec{u}) - \omega^2 (\vec{w}, \vec{p}\vec{u}) = 0 \quad \forall \vec{w} \in \mathcal{V} \end{array} \right.$$

Problem (E) admits an infinite number of solutions $(\omega_l, \vec{u}_l) \in \mathbb{R}^+ \times \mathcal{V}$, and these may be ordered as to satisfy:

$$0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq \dots \quad \text{Non-Decreasing Eigenvalues}$$

$$(\vec{u}_k, \vec{u}_l) = \delta_{kl} \quad \text{Orthonormal Eigenvectors}$$

If $\omega_l = 0$, then \vec{u}_l corresponds to a rigid body mode (translation or rotation).

The eigenvectors $\{\vec{u}_l\}_{l=1}^{\infty}$ form a basis for $L^2(\Omega)$, and hence we may write:

$$\delta \vec{u}_0 = \sum_{l=1}^{\infty} a_l \vec{u}_l(\vec{x}) \quad \text{where: } a_l = (\delta \vec{u}_0, \vec{u}_l)$$

$$\delta \vec{v}_0 = \sum_{l=1}^{\infty} b_l \vec{u}_l(\vec{x}) \quad \text{where: } b_l = (\delta \vec{v}_0, \vec{u}_l)$$

Then we have the following exact expression for the perturbation $\delta \vec{u}$:

$$\delta \vec{u}(\vec{x}, t) = \sum_{l=1}^{\infty} (a_l \cos(\omega_l t) + \frac{b_l}{\omega_l} \sin(\omega_l t)) \vec{u}_l(\vec{x})$$

From the above, we see that perturbations in the initial conditions propagate exactly as described by the natural frequencies and eigenvectors (or eigenmodes or simply modes) of the body.

Given the importance of the natural frequencies and associated modes, it is desirable to compute them. Galerkin's method is one means by which we may approximate them. The Galerkin approximation to (E) is as follows:

$$(G) \left\{ \begin{array}{l} \text{Find } (\omega^h, \vec{u}^h) \in \mathbb{R}^+ \times \mathcal{Y}^h \text{ such that} \\ a(\vec{w}^h, \vec{u}^h) - (\omega^h)^2 (\vec{w}^h, p\vec{u}^h) = 0 \quad \forall \vec{w}^h \in \mathcal{Y}^h \end{array} \right.$$

The corresponding matrix eigenproblem is: Find the natural frequency $\omega_k^h \in \mathbb{R}^+$ and eigenvector $\underline{\Psi}_k$ s.t.:

$$(\underline{\underline{K}} - (\omega_k^h)^2 \underline{\underline{M}}) \underline{\Psi}_k = \underline{0}$$

where:

$$\text{Stiffness Matrix} \rightarrow \underline{\underline{K}} = [\underline{\underline{K}}_{PQ}] \quad w/ \quad K_{PQ} = a(N_i \hat{e}_A, N_j \hat{e}_B)$$

$$\text{Mass Matrix} \rightarrow \underline{\underline{M}} = [\underline{\underline{M}}_{PQ}] \quad w/ \quad M_{PQ} = \delta_{AB} (N_i, p N_j)$$

$$P = ID(A_{ij})$$

$$Q = ID(B_{ij})$$

The above generalized eigenproblem may be solved using an iterative method such as the Lanczos algorithm.

As in the infinite dimensional setting, the discrete solutions may be ordered as to satisfy:

$$0 \leq \omega_1^h \leq \omega_2^h \leq \dots \leq \omega_{ndof}^h \quad \text{Non-Decreasing Eigenvalues}$$

$$(\vec{u}_k^h, \vec{u}_l^h) = \delta_{kl} \quad \text{Orthonormal Eigenvectors}$$

and we have that:

$$\omega_i^h \approx \omega_i \quad \text{and} \quad \vec{u}_i^h \approx \vec{u}_i \quad \text{for } i=1, \dots, ndof$$

A natural question to then ask is how well the discrete frequencies and modes. This is the subject matter of several publications, including:

J.A. Cottrell, A. Reali, Y. Bazilevs, and T.J.R. Hughes, "Isogeometric analysis of structural vibrations," CMAME, 196: 2413 - 2430. (2006)

T.J.R. Hughes, A. Reali, and G. Sangalli, "Duality and unified analysis of discrete approximations in structural dynamics and wave propagation: Comparison of p-method finite elements with k-method NURBS," CMAME, 197: 4104 - 4124. (2008)

T.J.R. Hughes, J.A. Evans, and A. Reali, "Finite element and NURBS approximations of eigenvalue, boundary-value, and initial-value problems," CMAME, 272: 290 - 320. (2014)

Some general observations from these works are:

- NURBS eigenvalues and eigenfunctions are accurate across the spectrum.

- Finite element eigenvalues and eigenfunctions exhibit spurious behavior.
- Finite element eigenvalues in the "optical branches" of the spectrum are completely spurious, and these branches diverge with increasing polynomial degree.
- Finite element eigenfunction error spikes between spectrum branches.
- NURBS approximations are in general characterized by a few spurious "outlier modes", but these are finite in number.

The above observations speak to the merit of using isogeometric discretizations in the study of structural vibrations.

Wave Propagation

We now address the question of wave propagation. Let:

$$g_A(\vec{x}, t) \leftarrow g_A(\vec{x}, t) + \delta g_A(\vec{x}) e^{i\omega t} \quad A = 1, \dots, d$$

$$h_A(\vec{x}, t) \leftarrow h_A(\vec{x}, t) + \delta h_A(\vec{x}) e^{i\omega t} \quad A = 1, \dots, d$$

$$f_A(\vec{x}, t) \leftarrow f_A(\vec{x}, t) + \delta f_A(\vec{x}) e^{i\omega t} \quad A = 1, \dots, d$$

$$\vec{u}(\vec{x}, t) \leftarrow \vec{u}(\vec{x}, t) + \delta \vec{u}(\vec{x}) e^{i\omega t}$$

Provided compatible initial conditions are prescribed, it is then easily shown that:

$$\delta \vec{u}(\vec{x}, t) = \vec{u}(\vec{x}) e^{i\omega t}$$

where \vec{u} satisfies:

$$(H) \begin{cases} \text{Find } \vec{u} \in \mathcal{D}_\delta \text{ s.t.} \\ a(\vec{w}, \vec{u}) - \omega^2 (\vec{w}, p\vec{u}) = L_\delta(\vec{w}) \quad \forall \vec{w} \in \mathcal{V} \end{cases}$$

where:

$$\mathcal{D}_\delta := \left\{ \vec{u} \in (H^1(\Omega))^d : u_A(\vec{x}) = \delta g_A(\vec{x}), \vec{x} \in \Gamma_{DA} \right\}$$

$$L_\delta(\vec{w}) := \int_{\Omega} w_A \delta f_A d\Omega_A + \sum_{A=1}^d \left(\int_{\Gamma_N} w_A \delta h_A d\Gamma_A \right)$$

A Galerkin formulation follows much as before, resulting in the following matrix problem:
Find the vector \underline{d} s.t.

$$(\underline{K} - \omega^2 \underline{M}) \underline{d} = \underline{F}_\delta$$

Provided ω is not a natural frequency, the above system admits a unique solution. Note the similarity in form between the above matrix problem and the vibrations matrix problem.

However, the frequency ω is prescribed in the current setting while it is determined in the structural vibrations setting.

Problem (H) admits solutions with very complicated behavior. In general, the solution \vec{u} may be decomposed into a primary, or pressure, wave \vec{u}_p and a secondary, or shear, wave \vec{u}_s . Visually:

$$\vec{u} = \vec{u}_p + \vec{u}_s$$

\downarrow \downarrow
Primary Wave Secondary Wave

The primary wave is irrotational:

$$\vec{\nabla} \times \vec{u}_p = \vec{0}$$

while the secondary wave is isovoluminous:

$$\vec{\nabla} \cdot \vec{u}_s = 0$$

For a homogeneous, isotropic body, the primary and secondary waves satisfy the Helmholtz equation:

$$-\vec{\nabla}^2 \vec{u}_p - k_p^2 \vec{u}_p = \vec{0} \quad k_p = \frac{\omega}{c_p}$$

$$-\vec{\nabla}^2 \vec{u}_s - k_s^2 \vec{u}_s = \vec{0} \quad k_s = \frac{\omega}{c_s}$$

where the primary and secondary wavespeeds are:

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$c_s = \sqrt{\frac{\mu}{\rho}}$$

For simplicity, δF was assumed to be zero above. Far from the source in an infinite medium, the waves are known to take the form:

$$\vec{u}_p = u_p \hat{e}_p e^{-k_p \cdot \vec{x}}$$

$$\vec{u}_s = u_s \hat{e}_s e^{-k_s \cdot \vec{x}}$$

where:

u_p, u_s := Amplitudes of waves

\hat{e}_p, \hat{e}_s := Direction of displacements

\vec{k}_p, \vec{k}_s := Vector wavenumbers w/ magnitude k_p, k_s

The Galerkin approximation to (H) admits a similar solution, albeit with a different set of wavenumbers k_p^h and k_s^h . Hence, a natural question one might ask is: How well do the discrete wavenumbers approximate the exact wavenumbers?

The above question was addressed in the paper:

T. J. R. Hughes, A. Reali, and G. Sangalli, "Duality and unified analysis...", CMAME, 197: 4104 - 4124, (2008).

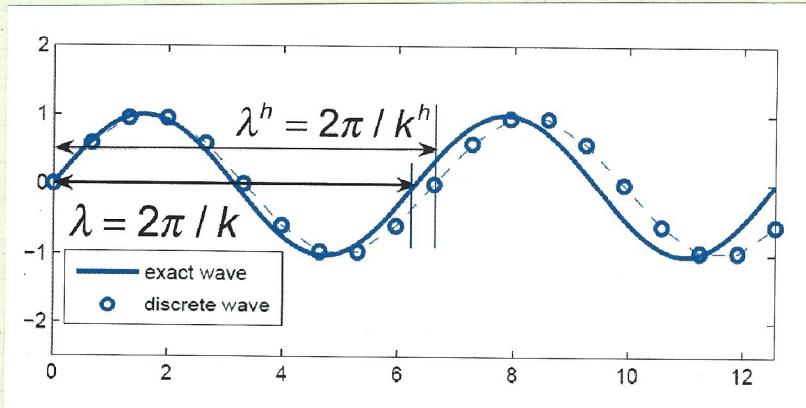
In this paper, the scalar Helmholtz equation:

$$-\Delta u - k^2 u = 0$$

and its discretization:

$$(\underline{K} - k^2 \underline{M}) \underline{d} = \underline{0} \quad \text{w/ } \underline{K} \text{ & } \underline{M} \text{ being 1-D stiffness and mass}$$

were analyzed. Hughes et. al. noticed the above equation was identical to the discrete vibration equation for an elastic rod under the substitution $\omega^h \leftrightarrow k$. Furthermore, if we exchange ω^h and k and ω and k^h , we achieve a duality between spectrum analysis of vibrations and dispersion analysis of wave propagation in the domain where k^h is real. An immediate consequence is that the discrete wavelength, $\lambda^h = 2\pi/k^h$, is necessarily longer than the exact wavelength, $\lambda = 2\pi/k$:



Hughes et. al. also showed that you could plot the ratio k/k^h vs. $2/N$ where N is the number of DOF per wavelength.* Such a plot is provided on the next page. Note that the ratio is much more well-behaved for smooth NURBS than for standard FEM. Moreover, standard FEM admits complex wave-numbers between branches. The non-zero imaginary part of k^h produces an amplitude modulation of the discrete solutions which is an unphysical feature of the numerical solution.

* To be more specific, N is the number of DOF per discrete wavelength. For NURBS:

$$N = \frac{\lambda^h}{h}$$

