Boundary Value Problems: Galerkin's Method

Galerkin's method consists of constructing finite-dimensional approximations of $\mathcal{S}$ and $\mathcal{V}$, denoted as $\mathcal{S}^h$ and $\mathcal{V}^h$ respectively. Strictly speaking, these will be subsets:

$$\mathcal{S}^h \subset \mathcal{S}$$

$$\mathcal{V}^h \subset \mathcal{V}$$

We will further characterize $\mathcal{S}^h$ by requiring that if we have a given function $g^h \in \mathcal{S}^h$, such that $g^h|_{\Gamma_D^1} = g$, then for every $u^h \in \mathcal{S}^h$, there exists a unique $v^h \in \mathcal{V}^h$ such that:

$$u^h = v^h + g^h$$

Then, Galerkin's method is stated as follows:

$$(G)\begin{cases} \text{Find } u^h = v^h + g^h, \text{ where } v^h \in \mathcal{V}^h, \text{ such that:} \\ \quad a(w^h, u^h) = L(w^h) \qquad \forall w^h \in \mathcal{V}^h \end{cases}$$

We can rewrite the variational form of Galerkin's method as:

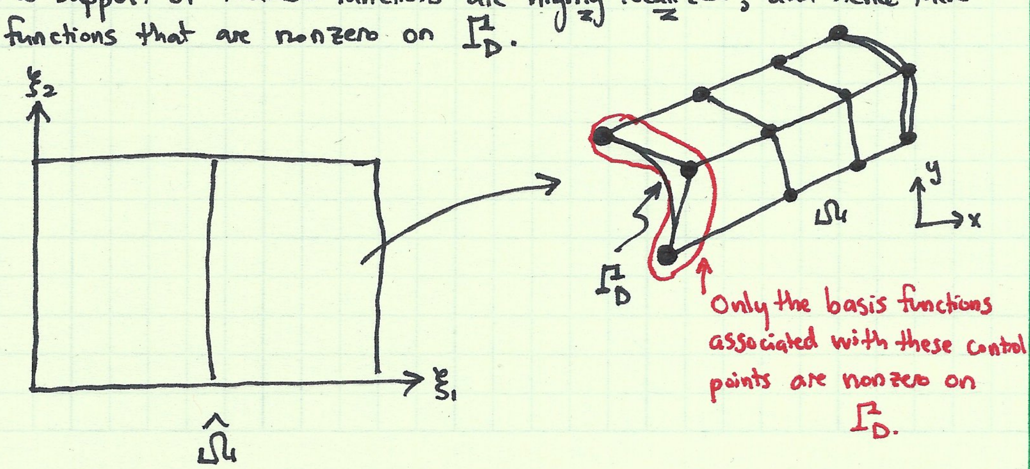
$$\underbrace{a(w^h, v^h)}_{\text{Unknown Information}} = \underbrace{L(w^h) - a(w^h, g^h)}_{\text{Known Information}}$$

In isogeometric analysis, we identify $\mathcal{V}^h$ and $\mathcal{S}^h$ as NURBS spaces:

$$\mathcal{V}^h := \left\{ w^h \in \mathcal{V} : \ w^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, c_i \right\}$$

$$\mathcal{S}^h := \left\{ u^h \in \mathcal{S} : \ u^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, d_i \right\}$$

Recall that the support of NURBS functions are highly localized, and hence there are very few functions that are nonzero on Γ_D^1.



Without loss of generality, let us suppose that there exists an integer $n_{eq} < n$ such that:

$$N_i \big|_{\Gamma_D} = 0 \qquad \forall\, i = 1, \ldots, n_{eq}$$

Then:

$$\mathcal{V}^h := \Big\{ w^h : w^h = \sum_{i=1}^{n_{eq}} N_i(\vec{x})\, c_i \Big\}$$

Completely free

Moreover, we can choose g^h such that $g_1 = \ldots = g_{n_{eq}} = 0$:

$$g^h(\vec{x}) = \sum_{i=n_{eq}+1}^{n} N_i(\vec{x})\, g_i$$

Then, we have that:

$$u^h(\vec{x}) = \sum_{i=1}^{n} N_i(\vec{x})\, d_i = \sum_{i=1}^{n_{eq}} N_i(\vec{x}) \underbrace{d_i}_{\text{Unknown}} + \sum_{i=n_{eq}+1}^{n} N_i(\vec{x}) \underbrace{g_i}_{\text{Known}} = \overset{\in \mathcal{V}^h}{v^h} + g^h$$

If we plug our expressions for u^h and w^h into the Galerkin formulation, we obtain:

$$a\Big(\sum_{i=1}^{n_{eq}} N_i c_i, \sum_{j=1}^{n_{eq}} N_j d_j \Big) = L\Big(\sum_{i=1}^{n_{eq}} N_i c_i \Big) - a\Big(\sum_{i=1}^{n_{eq}} N_i c_i, g^h \Big)$$

Exploiting linearity, we find:

$$\sum_{i=1}^{n_{eq}} \underbrace{c_i}_{\text{Free}} \Big(\sum_{j=1}^{n_{eq}} a(N_i, N_j)\, d_j - L(N_i) + a(N_i, g^h) \Big) = 0$$

As c_i is arbitrary for $i = 1, \ldots, n_{eq}$, we have:

$$\sum_{j=1}^{n_{eq}} a(N_i, N_j)\, d_j = L(N_i) - a(N_i, g^h) \quad \text{for } i = 1, \ldots, n_{eq}$$

Now, suppose we define:

$$K_{ij} = a(N_i, N_j)$$
$$F_i = L(N_i) - a(N_i, g^h)$$

and:

$$\underline{\underline{K}} = [K_{ij}]$$

$$\underline{F} = [F_i]$$

$$\underline{d} = [d_i]$$

Then we have the matrix problem:

$$\boxed{\underline{\underline{K}}\,\underline{d} = \underline{F}}$$

for the unknown control variables $\underline{d} = [d_i]$. Due to the finite element method's historical origins in finite element analysis of structures, we call:

$$\underline{\underline{K}} = \text{stiffness matrix}$$

$$\underline{d} = \text{displacement vector}$$

$$\underline{F} = \text{force vector}$$

We solve for the control variables as:

$$\underline{d} = \underline{\underline{K}}^{-1}\underline{F}$$

and then we finally obtain our desired discrete approximation of the temperature field as:

$$u^h(\vec{x}) = \sum_{i=1}^{n_{eq}} N_i(\vec{x})\underbrace{d_i}_{\text{Solved for!}} + \sum_{i=n_{eq}+1}^{n} N_i(\vec{x})\underbrace{g_i}_{\text{Dirichlet data}}$$