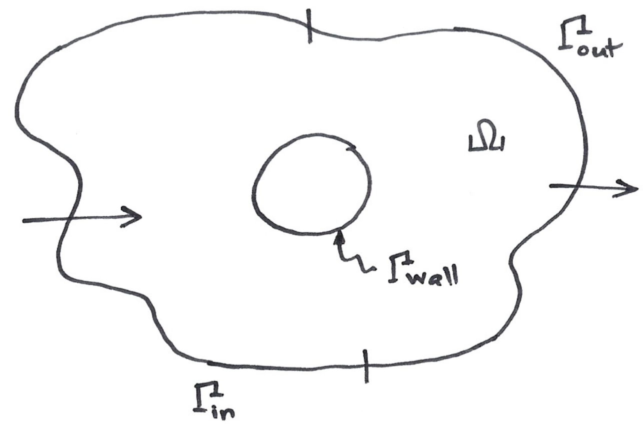
Incompressible Fluid Flow:

Weak Formulation:

$$(W) \begin{cases} \text{Find } \underline{U}(t) = \{\vec{u}(t), p(t)\} \in \mathcal{S}_t \text{ s.t.} \\ \quad B(\underline{W}; \underline{U}(t)) = L(\underline{W}) \quad \forall\, \underline{W} \in \mathcal{V} \;\&\; t \in (0,T) \\ \quad \vec{u}(0) = \vec{u}_0 \end{cases}$$



where:

$\mathcal{S}_t :=$ Set of Trial Solutions Satisfying inflow & wall BCs

$\mathcal{V} :=$ Space of Test Functions satisfying homogeneous inflow & wall BCs

$$B(\underline{W}; \underline{U}) := B_1(\underline{W}, \underline{U}) + B_2(\underline{W}, \underline{U}, \underline{U})$$

$$L(\underline{W}) := \int_\Omega \vec{w} \cdot \vec{f}\, d\Omega$$ Body Force

Unsteady Acceleration / Viscous Diffusion / Pressure Force / Divergence Constraint

$$B_1(\underline{W}, \underline{U}) := \int_\Omega \vec{w} \cdot \vec{u}_{,t}\, d\Omega + \int_\Omega 2\nu\, (\vec{\nabla}^s \vec{w}) : (\vec{\nabla}^s \vec{u})\, d\Omega - \int_\Omega (\vec{\nabla} \cdot \vec{w})\, p\, d\Omega + \int_\Omega q\, (\vec{\nabla} \cdot \vec{u})\, d\Omega$$

Bilinear Form

$$B_2(\underline{W}, \underline{U}, \underline{U}) := -\int_\Omega (\vec{\nabla} \vec{w}) : (\vec{u} \otimes \vec{u})\, d\Omega + \int_{\Gamma_{out}} \vec{w} \cdot \left(\{\vec{u} \cdot \vec{n}\}_+ \vec{u} \right) d\Gamma$$

Trilinear Form

Convective / Advective Acceleration

"Outflow Momentum"

where: $\underline{U} = \{\vec{u}, p\}$

$\underline{W} = \{\vec{w}, q\}$

(Stabilized) VMS Formulation:

Add later

$$
(MS)\quad \begin{cases} \text{Find } \underline{U}^h(t) = \{\vec{u}^h(t), p^h(t)\} \in \mathcal{S}_t^h \text{ s.t.} \\ B_{MS}(\underline{W}^h; \underline{U}^h(t)) = L_{MS}(\underline{W}^h) \quad \forall\, \underline{W}^h \in \mathcal{V}^h \;\&\; t \in (0,T) \\ \vec{u}^h(0) = \vec{u}_0^h \end{cases}
$$

Neglect: Influence of fine-scale unsteady acceleration & viscous diffusion

where:

$$
\mathcal{S}_t^h := \left\{ \underline{U}^h(t) = \{\vec{u}^h(t), p^h(t)\} \in \mathcal{S}_t : \vec{u}^h = \sum_{i=1}^n N_i(\vec{x})\,\vec{d}_i(t),\; p^h = \sum_{i=1}^n N_i(\vec{x})\,d_i(t) \right\}
$$

$$
\mathcal{V}^h := \left\{ \underline{W}^h = \{\vec{w}^h, q^h\} \in \mathcal{V} : \vec{w}^h = \sum_{i=1}^n N_i(\vec{x})\,\vec{c}_i,\; q^h = \sum_{i=1}^n N_i(\vec{x})\,c_i \right\}
$$

$$
\begin{aligned}
B_{MS}(\underline{W}^h; \underline{U}^h) := B(\underline{W}^h; \underline{U}^h) &- \int_{\Omega'} \left((\vec{u}^h \cdot \vec{\nabla})\vec{w}^h + \vec{\nabla} q^h \right) \cdot \vec{u}'\, d\Omega' \\
&- \int_{\Omega'} (\vec{\nabla} \cdot \vec{w}^h)\, p'\, d\Omega' \\
&- \int_{\Omega'} (\vec{\nabla} \vec{w}^h) : (\vec{u}' \otimes \vec{u}^h)\, d\Omega' \\
&- \int_{\Omega'} (\vec{\nabla} \vec{w}^h) : (\vec{u}' \otimes \vec{u}')\, d\Omega'
\end{aligned}
$$

SUPG/PSPG Stabilization

Integrated over Elements

grad-div Stabilization

Non-classical Stabilization

EXPLAIN LATER

where: $\underline{U}^h = \{\vec{u}^h, p^h\}$

$\underline{W}^h = \{\vec{w}^h, q^h\}$

$$
L_{MS}(\underline{W}^h) := L(\underline{W}^h)
$$

Model for Fine-Scales:

Inspired by Residual-Based Stabilization!

$$\vec{u}' \approx -\tau_m \, r_m(\vec{u}^h, p^h)$$

$$p' \approx -\tau_c \, r_c(\vec{u}^h)$$

Pointwise Residuals!

Stabilization Parameters

Residuals:

Momentum: $$r_m(\vec{u}^h, p^h) := \vec{u}^h_{,t} + (\vec{u}^h \cdot \vec{\nabla})\vec{u}^h - \vec{\nabla}\cdot(2\nu\vec{\nabla}^s\vec{u}^h) + \vec{\nabla}p^h - \vec{f}$$

Continuity: $$r_c(\vec{u}^h) := \vec{\nabla}\cdot\vec{u}^h$$

Stabilization Parameters:

$$\tau_m := \left(\left(\frac{2}{\Delta t}\right)^2 + \vec{u}^h \cdot \underline{\underline{G}}\,\vec{u}^h + C_I \nu^2 \, \underline{\underline{G}} : \underline{\underline{G}} \right)^{-1/2}$$

Unsteady   Advection   Diffusion

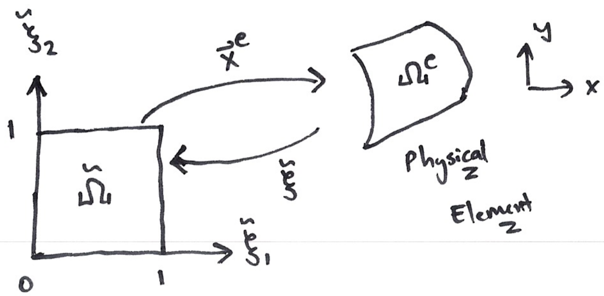
C_I: Inverse Constant $p = 2$: $\boxed{C_I = 36}$

$$\tau_c := \left(\underline{g} \cdot \tau_m \underline{g} \right)^{-1}$$

Metrics:

$$\underline{\underline{G}} = 4 \left(\frac{\partial \vec{\xi}}{\partial \vec{x}^e}\right)^T \left(\frac{\partial \vec{\xi}}{\partial \vec{x}^e}\right)$$

$$\underline{g} = [g_A] \quad w/ \quad g_A = 2 \underbrace{\sum_{B=1}^{d} \left(\frac{\partial \vec{\xi}}{\partial \vec{x}^e}\right)_{BA}}_{\text{Column Sum}}$$



Parent Element

Special Case: $\boxed{\Omega^e \equiv \text{cube}}$

$$\tau_m := \left(\left(\frac{2}{\Delta t}\right)^2 + \left(\frac{2|\vec{u}^h|}{h}\right)^2 + C_I \left(\frac{4\nu}{h^2}\right)^2 \right)^{-1/2}$$

$$\tau_c := \frac{h^2}{\tau_m}$$

Unsteady Dom.: $\tau_m = \Delta t / 2$

Conv. Dom.: $\tau_m = h / 2|\vec{u}^h|$

Diff. Dom.: $\tau_m = \frac{h^2}{4\nu} C_I^{-1/2}$

$\Omega^e \equiv$ cube:

$$\frac{\partial \vec{\xi}}{\partial \vec{x}^e} = h^{-1} \underline{\underline{I}}$$

VMS Formulation as a System of Differential Algebraic Equations (DAEs):

Let $\underline{V}$, $\dot{\underline{V}}$, and $\underline{P}$ denote the vectors of control variable DOF of velocity, acceleration, and pressure:

$$\underline{V} = [v_P] \qquad v_P = (\vec{d}_i)_A \qquad P = ID(A,i)$$

$$\dot{\underline{V}} = [\dot{v}_P] \qquad \dot{v}_P = ((\vec{d}_i)_A)_{,t} \qquad P = ID(A,i)$$

$$\underline{P} = [p_i] \qquad p_i = d_i$$

Moreover, define the residual vectors $\underline{R}^m$ and $\underline{R}^c$ s.t.

Momentum:
$$\underline{R}^m = [R^m_P] \qquad R^m_P = B_{MS}(\{\hat{e}_A N_i, 0\}, \underline{u}^h) - L_{MS}(\{\hat{e}_A N_i, 0\})$$

Continuity:
$$\underline{R}^c = [R^c_i] \qquad R^c_i = B_{MS}(\{\vec{0}, N_i\}, \underline{u}^h) - L_{MS}(\{\vec{0}, N_i\})$$

Then (MS) takes the form:

$$(\text{DAE}) \left\{ \begin{array}{l} \text{Find } \underline{V}(t), \dot{\underline{V}}(t), \text{ and } \underline{P}(t) \text{ s.t.} \\ \qquad \underline{R}^m(\dot{\underline{V}}(t), \underline{V}(t), \underline{P}(t)) = \underline{0} \\ \qquad \underline{R}^c(\dot{\underline{V}}(t), \underline{V}(t), \underline{P}(t)) = \underline{0} \\ \qquad \underline{V}(0) = \underline{V}_0 \end{array} \right.$$

Algebraic System for Pressure

Differential System for Velocity

Generalized - α Method for (DAE):

not needed!

Given $\dot{\underline{V}}_n$, $\underline{V}_n$, (and $\underline{P}_n$) find $\dot{\underline{V}}_{n+1}$, $\underline{V}_{n+1}$, and $\underline{P}_{n+1}$ s.t.:

$$\underline{R}^m\left(\dot{\underline{V}}_{n+\alpha_m}, \underline{V}_{n+\alpha_f}, \underline{P}_{n+1}\right) = \underline{0}$$

$$\underline{R}^c\left(\dot{\underline{V}}_{n+\alpha_m}, \underline{V}_{n+\alpha_f}, \underline{P}_{n+1}\right) = \underline{0}$$

Equations of Motion

$$\underline{V}_{n+1} = \underline{V}_n + \Delta t\left((1-\gamma)\dot{\underline{V}}_n + \gamma\dot{\underline{V}}_{n+1}\right)$$

Newmark Formula

$$\dot{\underline{V}}_{n+\alpha_m} = \dot{\underline{V}}_n + \alpha_m\left(\dot{\underline{V}}_{n+1} - \dot{\underline{V}}_n\right)$$

$$\underline{V}_{n+\alpha_f} = \underline{V}_n + \alpha_f\left(\underline{V}_{n+1} - \underline{V}_n\right)$$

Intermediate Solution

Second-Order Accuracy: $\gamma = \frac{1}{2} + \alpha_m - \alpha_f$

Unconditional Stability: $\alpha_m \geq \alpha_f \geq \frac{1}{2}$

Optimal Damping: $\alpha_m = \frac{1}{2}\left(\frac{3-\rho_\infty}{1+\rho_\infty}\right)$, $\alpha_f = \frac{1}{1+\rho_\infty}$

$\rho_\infty \in [0,1]$

Spectral Radius of Amp. Matrix as $\Delta t \to \infty$

For Steady State Computations: Choose: $\gamma = 1$, $\alpha_m = 1$, $\alpha_f = 1$ B.E.

Predictor - Multicorrector Methodology:

Predictor Stage: Set: $\underline{V}^{0}_{n+1} = \underline{V}_n$

$$\underline{\dot{V}}^{0}_{n+1} = \frac{(\gamma - 1)}{\gamma}\,\underline{V}_n$$

$$\underline{P}^{0}_{n+1} = \underline{P}_n$$

$\boxed{\text{Const. Vel. / Press. Predictor}}$

Multicorrector Stage: Repeat the following steps for $i = 0, 1, 2, \ldots, i_{max}$ or until convergence is achieved.

1. Evaluate iterates at the intermediate time levels as:

$$\underline{\dot{V}}^{i}_{n+\alpha_m} = \underline{\dot{V}}_n + \alpha_m\left(\underline{\dot{V}}^{i}_{n+1} - \underline{\dot{V}}_n\right)$$

$$\underline{V}^{i}_{n+\alpha_f} = \underline{V}_n + \alpha_f\left(\underline{V}^{i}_{n+1} - \underline{V}_n\right)$$

$$\underline{P}^{i}_{n+1} \cancel{=} \underline{P}^{i-1}_{n+1}$$

2. Use the intermediate solutions to assemble the residuals of the continuity and momentum equations and corresponding matrices in the linear system:

$$\begin{array}{cc} \underline{\underline{K}}^{i} & \underline{\underline{G}}^{i} \end{array}$$

$$\left[\begin{array}{c|c} \dfrac{\partial \underline{R}^{m,i}}{\partial \underline{\dot{V}}_{n+1}} & \dfrac{\partial \underline{R}^{m,i}}{\partial \underline{P}_{n+1}} \\ \hline \dfrac{\partial \underline{R}^{c,i}}{\partial \underline{\dot{V}}_{n+1}} & \dfrac{\partial \underline{R}^{c,i}}{\partial \underline{P}_{n+1}} \end{array}\right] \left[\begin{array}{c} \Delta \underline{\dot{V}} \\ \hline \Delta \underline{P} \end{array}\right] = - \left[\begin{array}{c} \underline{R}^{m,i} \\ \hline \underline{R}^{c,i} \end{array}\right]$$

$$\begin{array}{cc} \underline{\underline{D}}^{i} & \underline{\underline{L}}^{i} \end{array}$$

where:

$$\underline{R}^{m,i} = \underline{R}^m\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

$$\underline{R}^{c,i} = \underline{R}^c\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

$$\underline{\underline{K}}^i = \frac{\partial \underline{R}^m}{\partial \dot{\underline{V}}_{n+1}}\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

$$\underline{\underline{G}}^i = \frac{\partial \underline{R}^m}{\partial \underline{p}_{n+1}}\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

$$\underline{\underline{D}}^i = \frac{\partial \underline{R}^c}{\partial \dot{\underline{V}}_{n+1}}\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

$$\underline{\underline{L}}^i = \frac{\partial \underline{R}^c}{\partial \underline{p}_{n+1}}\left(\dot{\underline{V}}^i_{n+\alpha_m}, \underline{V}^i_{n+\alpha_f}, \underline{p}^i_{n+1}\right)$$

Use repeated applications of chain rule to obtain!

$$\underline{\underline{K}}^* = \alpha_m \frac{\partial \underline{R}^m}{\partial \dot{\underline{V}}_{n+\alpha_m}} + \alpha_f \gamma \Delta t \frac{\partial \underline{R}^m}{\partial \underline{V}_{n+\alpha_f}}$$

$$\underline{\underline{G}}^* = \frac{\partial \underline{R}^m}{\partial \underline{p}_{n+1}}$$

$$\underline{\underline{D}} = \alpha_m \frac{\partial \underline{R}^c}{\partial \dot{\underline{V}}_{n+\alpha_m}} + \alpha_f \gamma \Delta t \frac{\partial \underline{R}^c}{\partial \underline{V}_{n+\alpha_f}}$$

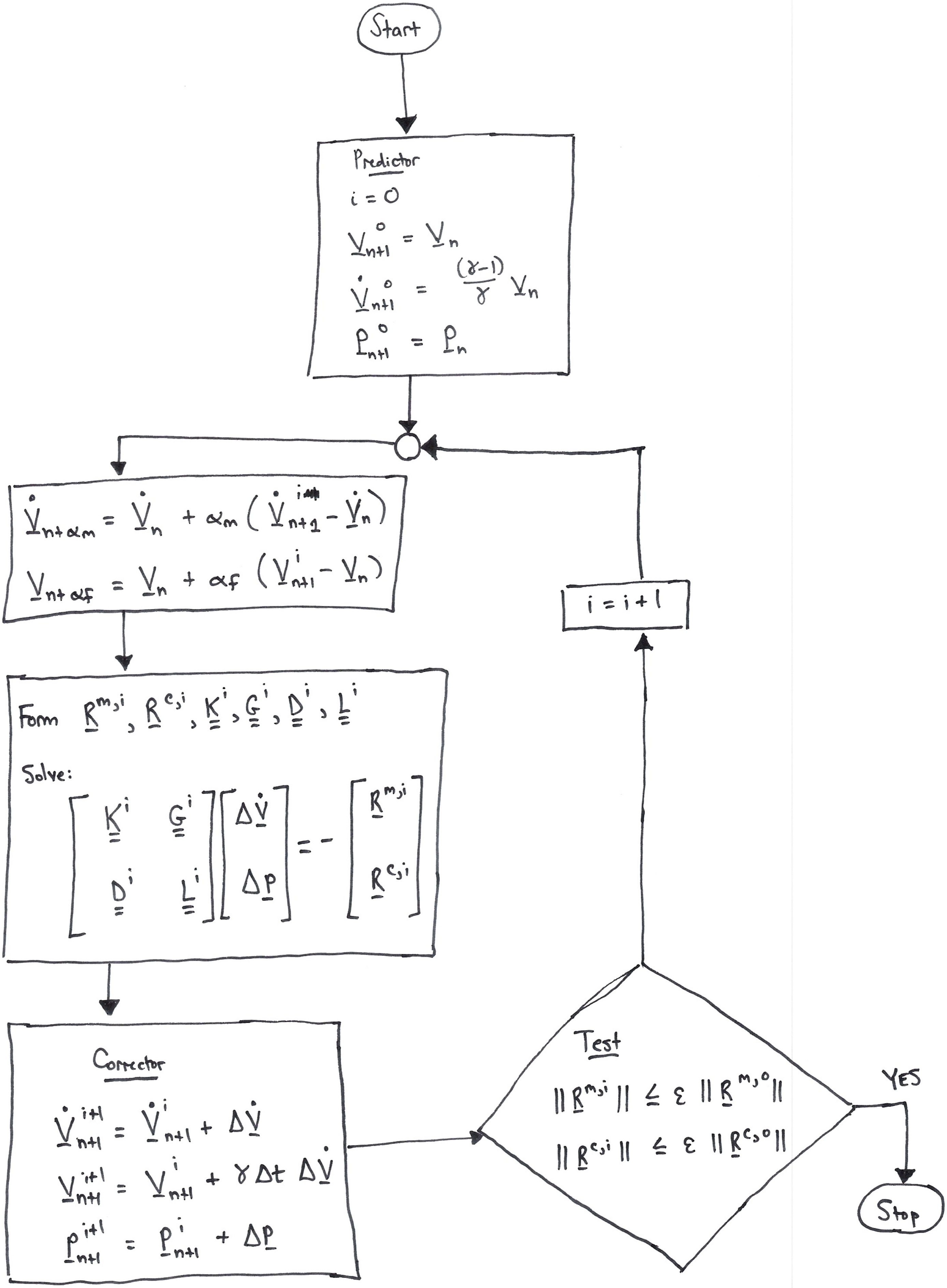
$$\underline{\underline{L}} = \frac{\partial \underline{R}^c}{\partial \underline{p}_{n+1}}$$

3. Having solved the linear system, update the iterates as:

$$\dot{\underline{V}}^{i+1}_{n+1} = \dot{\underline{V}}^{i}_{n+1} + \Delta \dot{\underline{V}}$$

$$\underline{V}^{i+1}_{n+1} = \underline{V}^{i}_{n+1} + \gamma \Delta t \, \Delta \dot{\underline{V}}$$

$$\underline{p}^{i+1}_{n+1} = \underline{p}^{i}_{n+1} + \Delta \underline{p}$$


Start
Predictor
$i = 0$
$\underline{V}_{n+1}^{0} = \underline{V}_n$
$\dot{\underline{V}}_{n+1}^{0} = \frac{(\gamma-1)}{\gamma} \underline{V}_n$
$\underline{P}_{n+1}^{0} = \underline{P}_n$
$\dot{\underline{V}}_{n+\alpha_m} = \dot{\underline{V}}_n + \alpha_m \left(\dot{\underline{V}}_{n+1}^{i} - \dot{\underline{V}}_n \right)$
$\underline{V}_{n+\alpha_f} = \underline{V}_n + \alpha_f \left(\underline{V}_{n+1}^{i} - \underline{V}_n \right)$
Form $\underline{R}^{m,i}$, $\underline{R}^{c,i}$, $\underline{\underline{K}}^{i}$, $\underline{\underline{G}}^{i}$, $\underline{\underline{D}}^{i}$, $\underline{\underline{L}}^{i}$
Solve:
$\begin{bmatrix} \underline{\underline{K}}^{i} & \underline{\underline{G}}^{i} \\ \underline{\underline{D}}^{i} & \underline{\underline{L}}^{i} \end{bmatrix} \begin{bmatrix} \Delta \dot{\underline{V}} \\ \Delta \underline{P} \end{bmatrix} = - \begin{bmatrix} \underline{R}^{m,i} \\ \underline{R}^{c,i} \end{bmatrix}$
Corrector
$\dot{\underline{V}}_{n+1}^{i+1} = \dot{\underline{V}}_{n+1}^{i} + \Delta \dot{\underline{V}}$
$\underline{V}_{n+1}^{i+1} = \underline{V}_{n+1}^{i} + \gamma \Delta t \, \Delta \dot{\underline{V}}$
$\underline{P}_{n+1}^{i+1} = \underline{P}_{n+1}^{i} + \Delta \underline{P}$
Test
$\| \underline{R}^{m,i} \| \leq \varepsilon \| \underline{R}^{m,0} \|$
$\| \underline{R}^{c,i} \| \leq \varepsilon \| \underline{R}^{c,0} \|$
YES
Stop
$i = i + 1$

In practice, the consistent tangent matrices are approximated.

A common set of approximations is:

$$\underline{\underline{K}} = [K_{PQ}]$$

Unsteady acc. Gal. contrib.

Unsteady acc. SUPG contrib.

$$K_{PQ} = \alpha_m \int_{\Omega} N_i N_j \, d\Omega \, \delta_{AB} + \alpha_m \int_{\Omega} (\vec{u}^h \cdot \vec{\nabla} N_i) \tau_m N_j \, d\Omega \, \delta_{AB}$$

Adv. Gal. Contrib. (Oseen term)

Diff. Gal. contrib. #1

$$+ \alpha_f \gamma \Delta t \int_{\Omega} N_i (\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega \, \delta_{AB} + \alpha_f \gamma \Delta t \int_{\Omega} \nu \vec{\nabla} N_i \cdot \vec{\nabla} N_j \, d\Omega \, \delta_{AB}$$

Diff. Gal. contrib. # 2

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \nu \vec{\nabla} \cdot (N_i \hat{e}_A) \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega$$

Adv. SUPG Contrib.

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \tau_m (\vec{u}^h \cdot \vec{\nabla} N_i)(\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega \, \delta_{AB}$$

Grad-div Stabilization Contrib.

$$+ \alpha_f \gamma \Delta t \int_{\Omega} \tau_c \vec{\nabla} \cdot (N_i \hat{e}_A) \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega$$

$$P = ID(A, i)$$
$$Q = ID(B, j)$$

SUPG Terms:

$$- \int_{\Omega} (\vec{u}^h \cdot \vec{\nabla} \vec{w}^h) \cdot \vec{u}' \, d\Omega$$
$$\Downarrow$$
$$- \int_{\Omega} (\vec{\nabla} \vec{w}^h) : (\vec{u}^h \otimes \vec{u}') \, d\Omega$$

Grad-div Terms:

$$- \int_{\Omega} (\vec{\nabla} \cdot \vec{w}^h) p' \, d\Omega$$

$$\underline{\underline{G}} = [G_{Pj}]$$

Pressure force Gal. Contrib.

Pressure Force SUPG Contrib.

$$G_{Pj} = - \int_{\Omega} \vec{\nabla} \cdot (N_i \hat{e}_A) N_j \, d\Omega + \int_{\Omega} \tau_m (\vec{u}^h \cdot \vec{\nabla} N_i) \hat{e}_A \cdot \vec{\nabla} N_j \, d\Omega$$

$$P = ID(A, i)$$

$$\underline{\underline{D}} = [D_{iQ}]$$

Div. Gal. Contrib.

$$D_{iQ} = \alpha_f \gamma \Delta t \int_{\Omega_h} N_i \ \vec{\nabla} \cdot (N_j \hat{e}_B) \, d\Omega_h$$

Adv. PSPG Contrib.

$$+ \ \alpha_f \gamma \Delta t \int_{\Omega_h} \tau_m \ \vec{\nabla} N_i \cdot \hat{e}_B \ (\vec{u}^h \cdot \vec{\nabla} N_j) \, d\Omega_h$$

Unsteady acc. PSPG Contrib.

$$+ \ \alpha_m \int_{\Omega_h} \tau_m \ \vec{\nabla} N_i \cdot \hat{e}_B N_j \, d\Omega_h$$

$$Q = ID(B, j)$$

PSPG Terms:

$$- \int_{\Omega_h} (\vec{\nabla} q^h) \cdot \vec{u}' \, d\Omega_h$$

$$\underline{\underline{L}} = [L_{ij}]$$

Pressure Force PSPG Contrib.

$$L_{ij} = \int_{\Omega_h} \tau_m \ \vec{\nabla} N_i \cdot \vec{\nabla} N_j \ d\Omega_h$$

Final Remarks:

- ☆ Element-by-Element Assembly of Residual Vectors & Tangent Matrices
- ☆ Formation via Pullback to Parent Element, Beziér extraction, & Gauss Quadrature Approximations of Integrals