Isogeometric Methods: Homework #3

Problem 1:

Write a MATLAB function which solves the two-dimensional heat conduction problem for a single-patch geometry. Your function should take the form:

```
function [d] = Heat_Conduction(p_1,p_2,n_1,n_2,Xi_1,Xi_2,P,w,n_q,problem)
```

where `d` is an array storing the control variables for the discrete temperature field, `p_1` and `p_2` are the polynomial degrees in directions ξ_1 and ξ_2 respectively, `n_1` and `n_2` are the number of basis functions in directions ξ_1 and ξ_2, `Xi_1` and `Xi_2` are the univariate knot vectors in directions ξ_1 and ξ_2, `P` is an array storing the control points for the NURBS surface, `w` is an array storing the weights for the NURBS surface, `n_q` is the number of quadrature points in each direction, and `problem` is an integer corresponding to a problem specification. The thermal conductivity κ, heating f, fixed temperature field g, fixed heat flux h, convective medium temperature u_R, convective heat transfer coefficient β, Dirichlet boundary Γ_D, Neumann boundary Γ_N, and Robin boundary Γ_R should all determined by the integer `problem`. For this homework, you should have four possible values for the integer `problem`, namely:

$$\texttt{integer} = \begin{cases} 1 & \text{for Problem 3} \\ 2 & \text{for Problem 4, Part 1} \\ 3 & \text{for Problem 4, Part 2} \\ 4 & \text{for Problem 4, Part 3} \end{cases}$$

Problem 2:

Write a MATLAB function which plots the discrete temperature field over the physical geometry. Your function should take the form:

```
function Plot_Temperature(p_1,p_2,n_1,n_2,Xi_1,Xi_2,P,w,d)
```

where `p_1` and `p_2` are the polynomial degrees in directions ξ_1 and ξ_2 respectively, `n_1` and `n_2` are the number of basis functions in directions ξ_1 and ξ_2, `Xi_1` and `Xi_2` are the univariate knot vectors in directions ξ_1 and ξ_2, `P` is an array storing the control points for the NURBS surface, `w` is an array storing the weights for the NURBS surface, and `d` is an array storing the control variables for the discrete temperature field.

Problem 3:

In this problem, you will solve the problem of heat conduction through the thickness of a copper pipe with no internal heating. Assuming that the temperature field is isotropic in the z-direction, the problem definition is as depicted in Figure 1. Symmetries have been exploited to reduce the problem size by a factor of four. For the problem in consideration, the physical and geometric

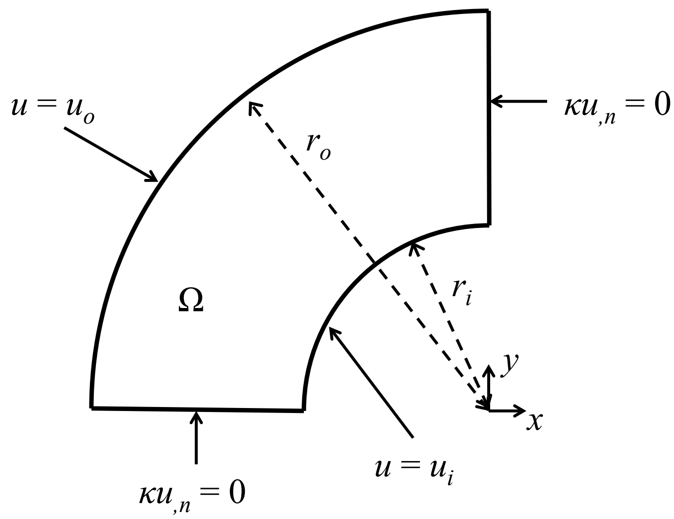

Figure 1: Setup for Problem 3.

parameters are as follows:

$$\begin{aligned}
\kappa &= 385 \ \frac{\mathrm{W}}{\mathrm{m}\cdot\mathrm{K}} \\
u_o &= 70 \ ^\circ\mathrm{C} \\
u_i &= 200 \ ^\circ\mathrm{C} \\
r_o &= 50 \ \mathrm{mm} \\
r_i &= 40 \ \mathrm{mm}
\end{aligned}$$

This problem has an exact axisymmetric analytical solution, which may be expressed in polar coordinates as follows:

$$u(r) := \frac{u_o \ln(r/r_i) - u_i \ln(r/r_o)}{\ln(r_o/r_i)}$$

Solve the above heat conduction problem using the MATLAB function `Heat_Conduction` you wrote for Problem 1. Plot the resulting temperature field over the physical geometry using the MATLAB function `Plot_Temperature` you wrote for Problem 2. Confirm that the exact solution is obtained under the limit of mesh refinement (i.e., uniform knot insertion) for both polynomial degrees $p = 2$ and $p = 3$. To do so, it is sufficient to show that the L^2-norm of the error:

$$\|u - u^h\|_{L^2(\Omega)} = \left(\int_\Omega \Big(u(r) - u^h(r) \Big)^2 dr \right)^{1/2}$$

goes to zero in the limit of mesh refinement. What is the rate of convergence in terms of the number of degrees of freedom?

Problem 4:

In this problem, you will solve the problem of heat conduction through a wafer. The wafer is being heated due to the transfer of heat from an internal microchip and cooled due to convective heat transfer from the wafer to a surrounding convective medium. Assuming that the temperature field is isotropic in the z-direction, the problem definition is as depicted in Figure 2. Symmetries have been exploited to reduce the problem size by a factor of four. For the problem in consideration,

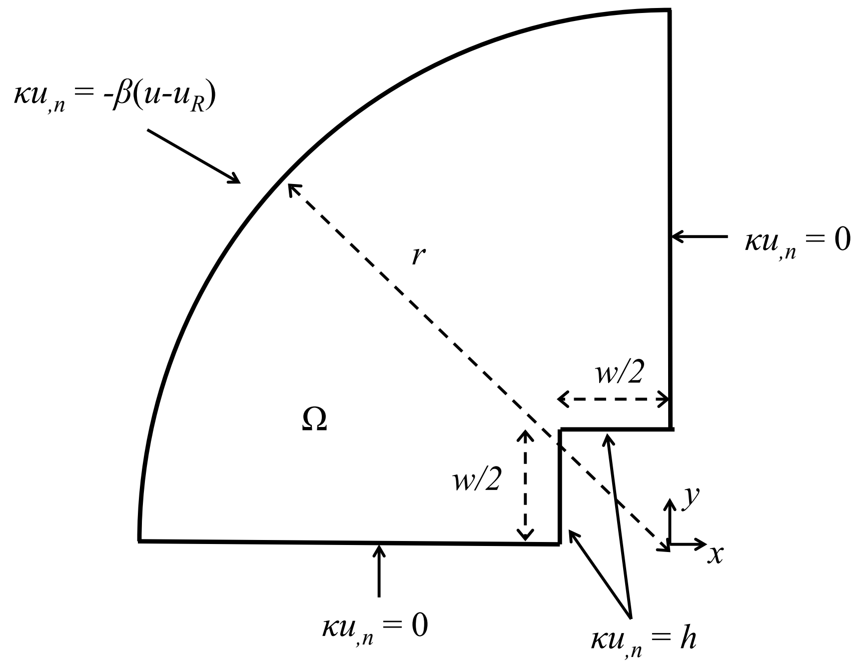

Figure 2: Setup for Problem 4.

the geometric parameters are as follows:

$$r = 12.5 \text{ mm}$$
$$w = 5 \text{ mm}$$

the temperature of the surrounding convective medium is:

$$u_R = 30\ ^\circ\text{C}$$

and the applied heat flux from the microchip is:

$$h = 50{,}000\ \frac{\text{W}}{\text{m}^2}$$

Part 1: Assume that the wafer is made of silicon and the surrounding convective medium is forced air. Then, the thermal conductivity is:

$$\kappa = 157\ \frac{\text{W}}{\text{m} \cdot \text{K}}$$

and the convective heat transfer coefficient is:

$$\beta = 250\ \frac{\text{W}}{\text{m}^2 \cdot \text{K}}$$

Solve the resulting heat conduction problem using the MATLAB function `Heat_Conduction` you wrote for Problem 1. Plot the resulting temperature field over the physical geometry using the MATLAB function `Plot_Temperature` you wrote for Problem 2, and determine the average temperature of the wafer. Use mesh refinement to ensure that your results are accurate. For example, you may ensure that your computed value for the average temperature of the wafer is within one tenth of a degree or less of the actual average temperature. Explain your methodology for determining your results are sufficiently accurate.

Part 2: Now assume that the surrounding convective medium is forced air but the wafer is made of silicon dioxide. Then, the thermal conductivity is:

$$\kappa = 1.4 \, \frac{\mathrm{W}}{\mathrm{m} \cdot \mathrm{K}}$$

and the convective heat transfer coefficient is:

$$\beta = 250 \, \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}}$$

Solve the resulting heat conduction problem using the MATLAB function `Heat_Conduction` you wrote for Problem 1. Plot the resulting temperature field over the physical geometry using the MATLAB function `Plot_Temperature` you wrote for Problem 2, and determine the average temperature of the wafer. Compare your results with the results of Part 1.

Part 3: Now assume that the wafer is made of silicon but the surrounding convective medium is still air. Then, the thermal conductivity is:

$$\kappa = 157 \, \frac{\mathrm{W}}{\mathrm{m} \cdot \mathrm{K}}$$

and the convective heat transfer coefficient is:

$$\beta = 10 \, \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{K}}$$

Solve the resulting heat conduction problem using the MATLAB function `Heat_Conduction` you wrote for Problem 1. Plot the resulting temperature field over the physical geometry using the MATLAB function `Plot_Temperature` you wrote for Problem 2, and determine the average temperature of the wafer. Compare your results with the results of Part 1 and Part 2.

Bonus: Let us again assume that the wafer is made of silicon and the surrounding convective medium is forced air. However, let us now suppose that we are able to choose the temperature of the convective medium. Determine the *maximum* temperature u_R that ensures that the temperature *does not exceed* 70 °C *anywhere* within the wafer.