Elastodynamics: Viscous Damping

Before proceeding on a discussion of time integration, it helps to note where we stand from the standpoint of physics. Through discretizing in space using a Galerkin finite element scheme, we have arrived at the matrix system:

$$\underline{\underline{M}}\,\ddot{\underline{d}} + \underline{\underline{K}}\,\underline{d} = \underline{F}$$

Above, $\underline{\underline{M}}$ is a symmetric, positive-definite matrix accounting for the mass of the system while $\underline{\underline{K}}$ is a symmetric, positive-semidefinite matrix accounting for the stiffness of the system. By analogy to spring systems, we see that our system experiences no damping. This indicates that oscillations due to system disturbances do not dissipate in time. This is rarely realistic. Hence, we often introduce the modified system:

$$\underline{\underline{M}}\,\ddot{\underline{d}} + \underline{\underline{C}}\,\dot{\underline{d}} + \underline{\underline{K}}\,\underline{d} = \underline{F}$$

where $\underline{\underline{C}}$ is the so-called viscous damping matrix. There have been many proposed definitions for $\underline{\underline{C}}$, but the first and perhaps most basic choice is the Rayleigh damping matrix:

$$\underline{\underline{C}} = a\,\underline{\underline{M}} + b\,\underline{\underline{K}}$$

where a and b are parameters. The two constituents of Rayleigh damping are seen to be proportional to mass and stiffness respectively. The parameters a and b may be selected to produce desired damping characteristics. It should be noted that the forcing vector must also be modified in the face of damping. Namely, for Rayleigh damping, we have:

$$\begin{aligned} F_p(t) = L(N_i\hat{e}_A) &- a\,(N_i\hat{e}_A, \vec{g}^h + b\,\vec{g}^h_{,t}) \\ &- (N_i\hat{e}_A, \rho\,(\vec{g}^h_{,tt} + a\,\vec{g}^h_{,t})) \end{aligned}$$

Note that, for Rayleigh damping, we are dealing with the modified equation of motion:

$$\rho\, u_{A,tt} + a\rho\, u_{A,t} = \sigma_{AB,B} + f_A$$

where the generalized Hooke's law is modified to account for the stiffness proportional effect, namely,

$$\sigma_{AB} = c_{ABCD}\left(u_{(C,D)} + b\,u_{(C,D),t}\right)$$

From the above, we see that a and b are viscosity coefficients and our expression for σ_{AB} corresponds to that of a viscoelastic system. That being said, a and b are not usually interpreted as physical parameters but rather chosen to fit simulation results to known data.