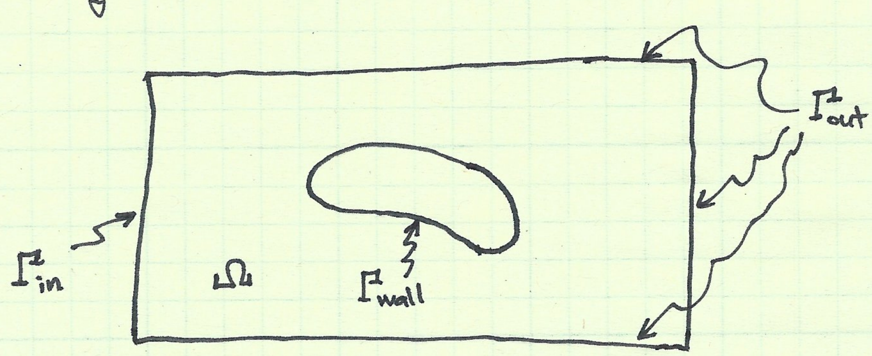
Incompressible Fluid Flow: The Strong and Weak Forms:

We finally turn to the problem of simulating incompressible fluid flow. Many of the biggest challenges in computational mechanics arise when trying to model the behavior of fluids. This is in part due to the wide range of scales present in such problems, and also to the fact that these scales frequently interact with each other in complex ways. Failure to properly represent these interactions can result in inaccurate and/or unstable calculations. The keys to success when performing computational fluids analysis are accuracy and robustness. To ensure both accuracy and robustness, one must utilize both a stable method and a set of approximation functions well-equipped to approximate solutions to the Navier-Stokes equations. NURBS seem to comprise an ideal basis for fluid mechanical applications, while variational multiscale (VMS) analysis yields a set of ideal methodologies. Both of these ingredients will be discussed at length here.

Before proceeding, let us recall the equation set of interest: the incompressible Navier-Stokes equations. In what follows, $\vec{u}$ will denote the velocity vector and p will denote the pressure divided by the fluid density. The domain of interest will be denoted as Ω. We allow Ω to have holes and voids, which will allow us to consider external flows about bluff bodies. We decompose the boundary of Ω into three parts:

$$\overline{\Gamma_{in} \cup \Gamma_{out} \cup \Gamma_{wall}} \equiv \Gamma \equiv \partial\Omega$$

where $\Gamma_{in} \cap \Gamma_{out} = \phi$, $\Gamma_{in} \cap \Gamma_{wall} = \phi$, and $\Gamma_{out} \cap \Gamma_{wall} = \phi$. Γ_{in} will denote the inflow boundary, Γ_{out} will denote the outflow boundary, and Γ_{wall} will denote solid walls. Visually:



We are interested in solving the incompressible flow problem over a time interval $(0,T)$. Thus, we see that $\vec{u} : \overline{\Omega} \times [0,T] \rightarrow \mathbb{R}^d$ and $p : \Omega \times (0,T) \rightarrow \mathbb{R}$ where d is the number of spatial dimensions. As in the setting of linear elasticity, indices A, B, C, and D may take on values $1, \ldots, d$, and we shall use u_A to refer to the A^{th} component of $\vec{u}$. Moreover, differentiation will be denoted by a comma, e.g.,

$$u_{A,B} = u_{A,x_B} = \frac{\partial u_A}{\partial x_B}$$

$$u_{A,t} = \frac{\partial u_A}{\partial t}$$

and we employ summation convention applying to indices A, B, C, and D.

Over the inflow boundary, the flow velocity is specified. We utilize the vector $\vec{g}$ to denote this inflow velocity, with components $g_A \in \mathbb{R}$ for $A = 1, \ldots, d$. Visually:

Inflow BC
$$\begin{cases} \vec{u} = \vec{g} & \text{on } \Gamma_{in} & \underline{\text{Vector}}\ \underline{\text{Form}} \\ (\vec{u})_A = u_A = g_A & \text{on } \Gamma_{in} \quad \text{for } A = 1, \ldots, d & \underline{\text{Indicial}}\ \underline{\text{Form}} \end{cases}$$

Over the outflow boundary, a traction-free boundary condition is typically applied. To state this condition, we first must define the stress tensor σ_{AB}. For a Newtonian fluid, we have:

$$\sigma_{AB} \equiv \rho\,(\underbrace{2\nu u_{(A,B)}}_{\text{Shear Stress}} - \underbrace{p\,\delta_{AB}}_{\text{Normal Stress}})$$

where ρ is the fluid density and ν is the kinematic viscosity. As in the setting of linear elasticity, parantheses around indices of a general tensor indicates its symmetric part. Thus:

$$u_{(A,B)} = u_{(B,A)} = \frac{1}{2}\left(u_{A,B} + u_{B,A}\right)$$

In vector form, we write the stress as:

$$\vec{\vec{\sigma}} \equiv \rho\,(2\nu \vec{\nabla}^s \vec{u} - p\,\vec{\vec{I}})$$

Along a domain boundary, the traction is defined as:

Denotes A^{th} component of outward normal $\vec{n}$.

$$t_A = \sigma_{AB} n_B = \rho\,(2\nu u_{(A,B)} n_B - p n_A) \qquad \underline{\text{Indicial}}\ \underline{\text{Form}}$$
$$\vec{t} = \vec{\vec{\sigma}}\vec{n} = \rho\,(2\nu (\vec{\nabla}^s \vec{u})\vec{n} - p\vec{n}) \qquad \underline{\text{Vector}}\ \underline{\text{Form}}$$

Thus, a traction-free boundary condition corresponds to:

$$t_A = 0 \quad \text{for } A = 1, \ldots, d \qquad \underline{\text{Indicial}}\ \underline{\text{Form}}$$
$$\vec{t} = \vec{0} \qquad \underline{\text{Vector}}\ \underline{\text{Form}}$$

Typically, a traction-free condition at the outflow is stable. However, this is not the case for reentrant flow. To accomodate this case, we modify the traction to account for the reentrant momentum. Namely, we set:

$$t_A = \rho\left(\{\vec{u}\cdot\vec{n}\}_-\right) u_A$$

where:

$$\{\vec{u}\cdot\vec{n}\}_- = \begin{cases} \vec{u}\cdot\vec{n} = u_A n_A & \text{if } \vec{u}\cdot\vec{n} < 0 \\ 0 & \text{otherwise} \end{cases}$$

With the above in mind, we may state the outflow boundary condition as:

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Outflow BC
$$\begin{cases} 2\nu(\vec{\nabla}^s\vec{u})\vec{n} - p\vec{n} = (\{\vec{u}\cdot\vec{n}\}_-)\vec{u} & \text{on } \Gamma_{out} & \underline{\text{Vector}}\ \underline{\text{Form}} \\ 2\nu\, u_{(A,B)} n_B - p n_A = (\{\vec{u}\cdot\vec{n}\}_-) u_A & \text{on } \Gamma_{out} \text{ for } A=1,\ldots,d & \underline{\text{Indicial}}\ \underline{\text{Form}} \end{cases}$$

Finally, over the solid walls, the no-penetration and no-slip conditions are applied. Visually:

Wall BC
$$\begin{cases} \vec{u} = \vec{0} & \text{on } \Gamma_{wall} & \underline{\text{Vector}}\ \underline{\text{Form}} \\ u_A = 0 & \text{on } \Gamma_{wall} \text{ for } A=1,\ldots,d & \underline{\text{Indicial}}\ \underline{\text{Form}} \end{cases}$$

To close our system, we need conservation statements for mass and momentum over the domain. For a general, compressible flow, these take the form:

Conservation of Momentum
$$\begin{cases} \frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla}\cdot(\rho\vec{u}\otimes\vec{u}) = \vec{\nabla}\cdot\vec{\vec{\sigma}} + \rho\vec{f} & \text{in } \Omega & \underline{\text{Vector}}\ \underline{\text{Form}} \\ \frac{\partial}{\partial t}(\rho u_A) + \frac{\partial}{\partial x_B}(\rho u_A u_B) = \frac{\partial}{\partial x_B}(\sigma_{AB}) + \rho f_A & \text{in } \Omega \text{ for } A=1,\ldots,d & \underline{\text{Indicial}}\ \underline{\text{Form}} \end{cases}$$

Conservation of Mass
$$\begin{cases} \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 & \text{in } \Omega & \underline{\text{Vector}}\ \underline{\text{Form}} \\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_B}(\rho u_B) = 0 & \text{in } \Omega & \underline{\text{Indicial}}\ \underline{\text{Form}} \end{cases}$$

where $\vec{f} : \Omega \times (0,T) \to \mathbb{R}^d$ is the body force per unit mass. For an incompressible flow, the above conservation laws simplify considerably. In vector form, we have:

Conservation Laws for Incompressible Flow
$$\begin{cases} \frac{\partial\vec{u}}{\partial t} + \vec{\nabla}\cdot(\vec{u}\otimes\vec{u}) = \vec{\nabla}\cdot(2\nu\vec{\nabla}^s\vec{u}) - \vec{\nabla}p + \vec{f} & \text{in } \Omega \\ \vec{\nabla}\cdot\vec{u} = 0 & \text{in } \Omega \end{cases}$$

Collecting the above results, we obtain the following strong form for the incompressible flow problem:

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$$\begin{cases} \text{Find } \vec{u}(t) : \bar{\Omega} \to \mathbb{R}^d \text{ for all } t \in [0,T] \text{ and } p(t) : \Omega \to \mathbb{R} \text{ for all } t \in (0,T) \text{ s.t.} \\ \vec{u}_{,t} + \vec{\nabla}\cdot(\vec{u}\otimes\vec{u}) = \vec{\nabla}\cdot(2\nu\vec{\nabla}^s\vec{u}) - \vec{\nabla}p + \vec{f} \quad \text{in } \Omega\times(0,T) \\ \vec{\nabla}\cdot\vec{u} = 0 \quad \text{in } \Omega\times(0,T) \\ \vec{u} = \vec{g} \quad \text{on } \Gamma_{in}\times(0,T) \\ 2\nu(\vec{\nabla}^s\vec{u})\vec{n} - p\vec{n} = (\{\vec{u}\cdot\vec{n}\}_-)\vec{u} \quad \text{on } \Gamma_{out}\times(0,T) \\ \vec{u} = \vec{0} \quad \text{on } \Gamma_{wall}\times(0,T) \end{cases}$$

$$\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x}) \quad \text{in } \Omega$$

where above $\vec{u}_0$ is the initial velocity field.

The weak form of the incompressible flow problem is constructed in the usual fashion: multiply the strong form by weighting functions, integrate over the domain, integrate by parts, and enforce outflow boundary conditions weakly and inflow and wall boundary conditions strongly. There aresome subtle differences of note, however. For instance, we have separate weighting functions for conservation of momentum and mass. Thus, we highlight all the steps in what follows.

To begin, we first need to introduce a set of trial solutions and a space of weighting functions. For the velocity field, we define:

$$\mathcal{S}_{u_t} := \left\{ \vec{u}(t) : \bar{\Omega} \to \mathbb{R}^d : \vec{u}(t) \in (H^1(\Omega))^d, \ \vec{u}|_{\Gamma_{in}} = \vec{g}, \ \vec{u}|_{\Gamma_{wall}} = \vec{0} \right\}$$

$$\mathcal{V}_w := \left\{ \vec{w} \in (H^1(\Omega))^d : \vec{w}|_{\Gamma_{in}} = \vec{0}, \ \vec{w}|_{\Gamma_{wall}} = \vec{0} \right\}$$

Note that inflow and wall boundary conditions are strongly enforced. For the pressure field,

$$\mathcal{S}_{p_t} := \left\{ p(t) : \Omega \to \mathbb{R} : p(t) \in L^2(\Omega) \right\}$$

$$\mathcal{V}_q := \left\{ q \in L^2(\Omega) \right\}$$

With the above definitions established, we define the composite set of trial solutions as:

$$\mathcal{S}_t := \mathcal{S}_{u_t} \times \mathcal{S}_{p_t}$$

and the composite space of weighting functions as:

$$\mathcal{V} := \mathcal{V}_w \times \mathcal{V}_q$$

We may then write our composite solution as $\underline{U} = \{\vec{u}, p\} \in \mathcal{S}_t$ for each time $t \in (0,T)$ and our composite weighting function as $\underline{W} = \{\vec{w}, q\} \in \mathcal{V}$.

To construct a weak form, we multiply the conservation of momentum equation by $\vec{w}$ and the conservation of mass equation by q and integrate over the domain:

$$\int_\Omega \vec{w} \cdot \Big\{ \underbrace{\vec{u}_{,t}}_{\text{unsteady acceleration}} + \underbrace{\vec{\nabla} \cdot (\vec{u} \otimes \vec{u})}_{\text{advective acceleration}} - \underbrace{\vec{\nabla} \cdot (2\nu \vec{\nabla}^s \vec{u})}_{\text{viscous diffusion}} + \underbrace{\vec{\nabla} p}_{\text{pressure force}} - \underbrace{\vec{f}}_{\text{body force}} \Big\} \, d\Omega + \int_\Omega q \underbrace{\vec{\nabla} \cdot \vec{u}}_{\text{incompressibility}} \, d\Omega = 0$$

We now integrate the advective acceleration, viscous diffusion, and pressure force terms by parts, resulting in:

$$\int_{\Omega} \vec{w}\cdot\vec{u}_{,t}\, d\Omega - \int_{\Omega} (\vec{\nabla}^s\vec{w}) : (\vec{u}\otimes\vec{u})\, d\Omega + \int_{\Gamma_{in}} \vec{w}\cdot\left(\{\vec{u}\cdot\vec{n}\}\,\vec{u}\right) d\Gamma$$

$$+ \int_{\Gamma_{out}} \vec{w}\cdot\left(\{\vec{u}\cdot\vec{n}\}\,\vec{u}\right) d\Gamma + \int_{\Gamma_{wall}} \vec{w}\cdot\left(\{\vec{u}\cdot\vec{n}\}\,\vec{u}\right) d\Gamma$$

$$+ \int_{\Omega} 2\nu(\vec{\nabla}^s\vec{w}) : (\vec{\nabla}^s\vec{u})\, d\Omega - \int_{\Gamma_{in}} \vec{w}\cdot\left(2\nu(\vec{\nabla}^s\vec{u})\vec{n}\right) d\Gamma$$

$$- \int_{\Gamma_{out}} \vec{w}\cdot\left(2\nu(\vec{\nabla}^s\vec{u})\vec{n}\right) d\Gamma - \int_{\Gamma_{wall}} \vec{w}\cdot\left(2\nu(\vec{\nabla}^s\vec{u})\vec{n}\right) d\Gamma$$

$$- \int_{\Omega} \vec{\nabla}\cdot\vec{w}\; p\, d\Omega + \int_{\Gamma_{in}} \vec{w}\cdot(p\vec{n})\, d\Gamma$$

$$+ \int_{\Gamma_{out}} \vec{w}\cdot(p\vec{n})\, d\Gamma + \int_{\Gamma_{wall}} \vec{w}\cdot(p\vec{n})\, d\Gamma$$

$$- \int_{\Omega} \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Omega} q\, \vec{\nabla}\cdot\vec{u} = 0$$

We now enforce boundary conditions. Namely, we set:

$$\vec{w} = \vec{0} \quad \text{on } \Gamma_{in} \;\&\; \Gamma_{wall} \qquad \text{Strong BC}$$

$$2\nu(\vec{\nabla}^s\vec{u})\vec{n} - p\vec{n} = \left(\{\vec{u}\cdot\vec{n}\}_-\right)\vec{u} \quad \text{on } \Gamma_{out} \qquad \text{Weak BC}$$

$$= \begin{cases} \vec{u}\cdot\vec{n} & \text{if } \vec{u}\cdot\vec{n} > 0 \\ 0 & \text{otherwise} \end{cases}$$

This results in:

$$\int_{\Omega} \vec{w}\cdot\vec{u}_{,t}\, d\Omega - \int_{\Omega} (\vec{\nabla}^s\vec{w}) : (\vec{u}\otimes\vec{u})\, d\Omega + \int_{\Gamma_{out}} \vec{w}\cdot\left(\{\vec{u}\cdot\vec{n}\}_+\,\vec{u}\right) d\Gamma$$

$$+ \int_{\Omega} 2\nu(\vec{\nabla}^s\vec{w}) : (\vec{\nabla}^s\vec{u})\, d\Omega - \int_{\Omega} \vec{\nabla}\cdot\vec{w}\; p\, d\Omega - \int_{\Omega} \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Omega} q\,\vec{\nabla}\cdot\vec{u}$$

$$= 0$$

Now define:

$$B(\underline{W};\underline{U}) \equiv B_1(\underline{W},\underline{U}) + B_2(\underline{W},\underline{U},\underline{U})$$

$$L(\underline{W}) \equiv \int_\Omega \vec{w}\cdot\vec{f}\, d\Omega$$

where $\underline{U} = \{\vec{u}, p\}$, $\underline{W} = \{\vec{w}, q\}$, and:

$$B_1(\underline{W},\underline{U}) \equiv \int_\Omega \vec{w}\cdot\vec{u}_{,t}\, d\Omega + \int_\Omega 2\nu\, (\vec{\nabla}^s\vec{w}) : (\vec{\nabla}^s\vec{u})\, d\Omega$$

Bilinear Form

$$- \int_\Omega (\vec{\nabla}\cdot\vec{w})\, p\, d\Omega + \int_\Omega q\, (\vec{\nabla}\cdot\vec{u})\, d\Omega$$

$$B_2(\underline{W},\underline{U},\underline{U}) \equiv - \int_\Omega (\vec{\nabla}^s\vec{w}) : (\vec{u}\otimes\vec{u})\, d\Omega$$

Trilinear Form

$$+ \int_{\Gamma_{out}} \vec{w}\cdot\left(\{\vec{u}\cdot\vec{n}\}_+\, \vec{u}\right) d\Gamma$$

Then the weak form of the incompressible flow problem is as follows:

$$\boxed{(W)\ \begin{cases} \text{Find } \underline{U}(t) = \{\vec{u}(t), p(t)\} \in \mathcal{S}_t \text{ s.t.} \\ \quad B(\underline{W};\underline{U}(t)) = L(\underline{W}) \quad \forall\, \underline{W}\in\mathcal{V} \ \& \ t\in(0,T) \\ \quad \vec{u}(0) = \vec{u}_0 \end{cases}}$$

$\equiv$