NURBS as a Basis for Analysis

Up to this point, we have focused on the use of NURBS for geometric design. We now bring NURBS into the setting of finite element analysis. As a basis for analysis, NURBS generalize and improve upon the traditional piecewise polynomial basis functions, providing unprecedented accuracy and robustness across a wide array of applications. The power of this combination of geometric and analytic capabilities is at the very heart of isogeometric analysis.

The root idea behind the use of NURBS in analysis is the isoparametric concept:

Isoparametric Concept: Use the same basis for the geometry and the analysis.

In classical finite element analysis, the geometric basis is inferred from the finite element basis. That is, the basis chosen to represent the unknown solution field is then used to approximate known geometry. Isogeometric analysis turns this idea on its head and selects a basis capable of exactly representing the known geometry and uses it as a basis for the fields we wish to approximate. In a sense, we are reversing the isoparametric arrow:

Classical FEA:	Geometry	$\Longleftarrow$	Fields
		imposed on	
Isogeometric Analysis:	Geometry	$\Longrightarrow$	Fields

So, how do we define functions using the geometric basis? Suppose that $\{\hat{N}_i(\vec{\xi})\}_{i=1}^n$ constitute a set of NURBS basis functions with associated control points $\{\vec{P}_i\}_{i=1}^n$. We restrict ourselves to the two-dimensional setting, so $\vec{P}_i = \{x_i, y_i\}^T$. The geometry, $\vec{x} : \hat{\Omega} \rightarrow \Omega$, is defined as:

$$\vec{x}(\vec{\xi}) = \sum_{i=1}^{n} \underbrace{\hat{N}_i(\vec{\xi})}_{\text{NURBS Basis Functions}} \underbrace{\vec{P}_i}_{\text{Control Points}}$$

In analogous fashion, we may define other functions over the parametric domain. For example, we may define $\hat{u}^h : \hat{\Omega} \rightarrow \mathbb{R}$ by:

$$\hat{u}^h(\vec{\xi}) = \sum_{i=1}^{n} \underbrace{\hat{N}_i(\vec{\xi})}_{\text{NURBS Basis Functions}} \underbrace{d_i}_{\text{Control Variables}}$$

The superscript "h" denotes we are working with a discrete quantity. Note the coefficients $\{d_i\}_{i=1}^n$ are analogous to control points and hence we call them control variables.

We define the function in the physical space by considering a composition with the inverse of the geometrical mapping. Namely, we define $u^h : \Omega \rightarrow \mathbb{R}$ by:

$$u^h = \hat{u}^h \circ \vec{x}^{-1}$$

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or, more precisely:

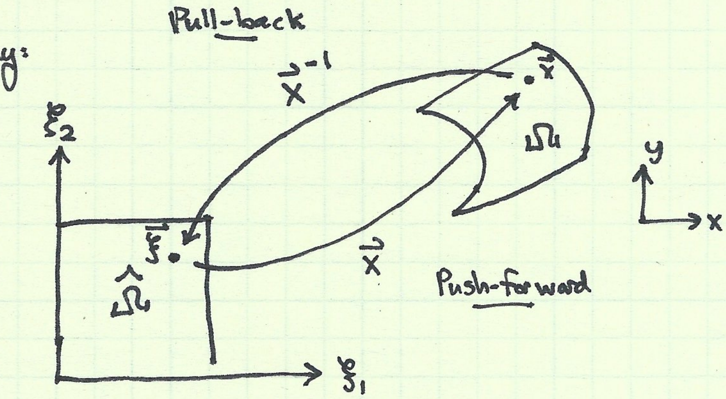
$$u^h(\vec{x}) = \hat{u}^h(\vec{x}^{-1}(\vec{x}))$$

Note that we have defined the function $u^h : \Omega \rightarrow \mathbb{R}$ by pulling back to the parametric domain. Alternatively, we may define $\hat{u}^h : \hat{\Omega} \rightarrow \mathbb{R}$ by pushing forward to the physical domain. Mathematically:

$$u^h(\vec{x}) = \hat{u}^h(\vec{x}^{-1}(\vec{x})) \qquad \text{Pull-back}$$

$$\hat{u}^h(\vec{\xi}) = u^h(\vec{x}(\vec{\xi})) \qquad \text{Push-forward}$$

Visually:



We can also define basis functions in physical space by pulling back to parametric space:

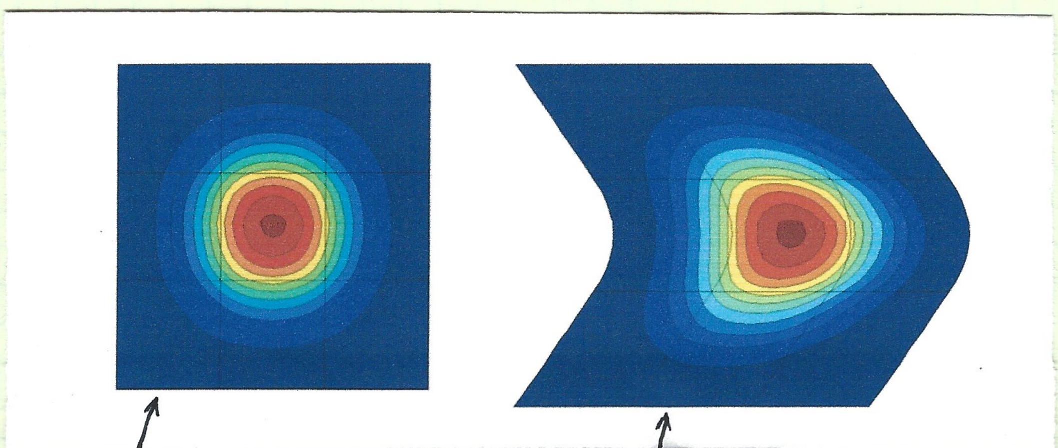
$$N_i = \hat{N}_i \circ \vec{x}^{-1} \qquad i = 1, \ldots, n$$

Then we have:

$$u^h(\vec{x}) = \sum_{i=1}^{n} \hat{N}_i(\vec{x}^{-1}(\vec{x}))\, d_i$$

$$= \sum_{i=1}^{n} N_i(\vec{x})\, \underline{\underline{d_i}}$$

Below, we have visualized a bivariate B-spline and its push-forward for the setting when $p_1 = p_2 = 2$ and $\Xi_1 = \Xi_2 = \{0, 0, 0, 1, 2, 3, 3, 3\}$.



Basis Function $\hat{N}_i(\vec{\xi})$ Basis Function $N_i(\vec{x})$

The basis functions $\{N_i(\vec{x})\}_{i=1}^{n}$ exhibit many of the same properties as the basis functions $\{\hat{N}_i(\vec{\xi})\}_{i=1}^{n}$, including positivity, partition of unity, and enhanced continuity. Moreover, the basis in physical space may be refined via knot insertion or degree elevation. This will be critical in finite element analysis to obtain a desired level of resolution. Recall that refinement leaves the geometry and its parametrization unchanged. Thus, refinement may proceed as needed for analysis without regard for the geometry, which is exact from the coarsest mesh onward.

We finish our discussion here by discussing the approximation power of the NURBS basis. Much like classical finite element analysis, we may derive approximation estimates for NURBS. Define:

$$\mathcal{S}^h := \text{span}\{N_i\}_{i=1}^{n}$$

Then we have:

$$\boxed{\inf_{u^h \in \mathcal{S}^h} \| u - u^h \|_{L^2(\Omega)} \leq C h^s \| u \|_{H^s(\Omega)} \quad \forall u \in H^r(\Omega)}$$

where:

C = constant independent of h, the mesh size

h = mesh size, nominally defined as $\max_e h^e$ where $h^e = \text{diam}(\Omega^e)$ is the mesh size of a given element

$s = \min\{r, p+1\}$ where $p = \min\{p_1, p_2\}$

The above estimates demonstrate that we expect optimal convergence as we refine our mesh via repeated applications of knot insertion. In this manner, we can nail the solution to a problem of interest. While the above estimate is for the L^2-norm, we have analogous estimates for other norms as well.

Remark: The L^2-norm is defined as:

$$\| u \|_{L^2(\Omega)} := \left(\int_{\Omega} u^2 \, d\Omega \right)^{1/2}$$

The Sobolev space $H^s(\Omega)$ is defined as:

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \frac{\partial^{i_1+i_2} u}{\partial x^{i_1} \partial y^{i_2}} \in L^2(\Omega) \text{ for } i_1 + i_2 \leq s \right\}$$

and its norm is defined as:

$$\| u \|_{H^s(\Omega)} := \left(\int_{\Omega} \left(\sum_{\substack{i_1, i_2 \\ i_1 + i_2 \leq s}} \left(\frac{\partial^{i_1+i_2} u}{\partial x^{i_1} \partial y^{i_2}} \right)^2 \right) d\Omega \right)^{1/2}$$