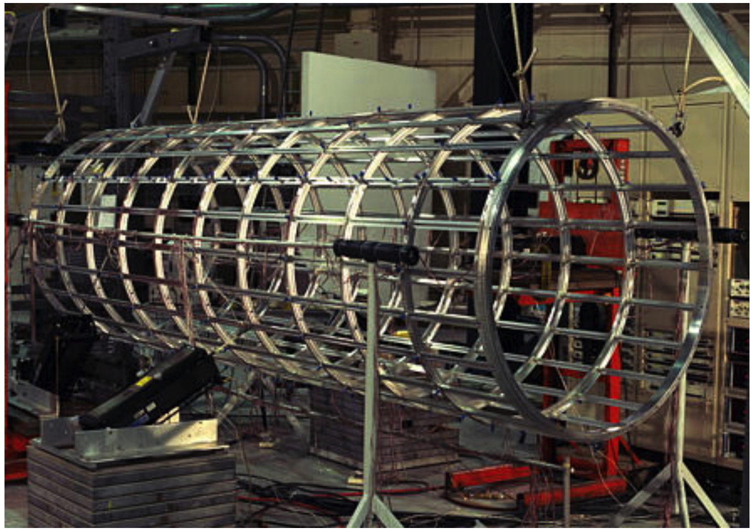
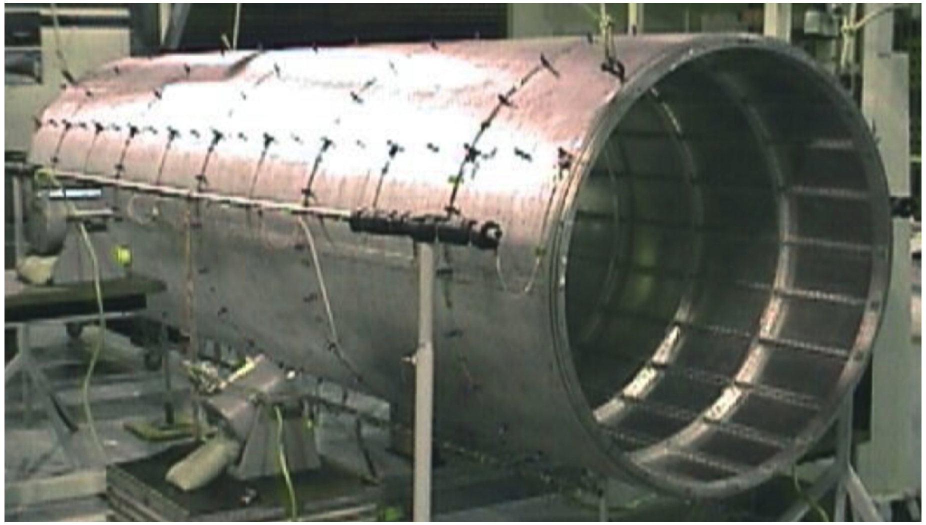
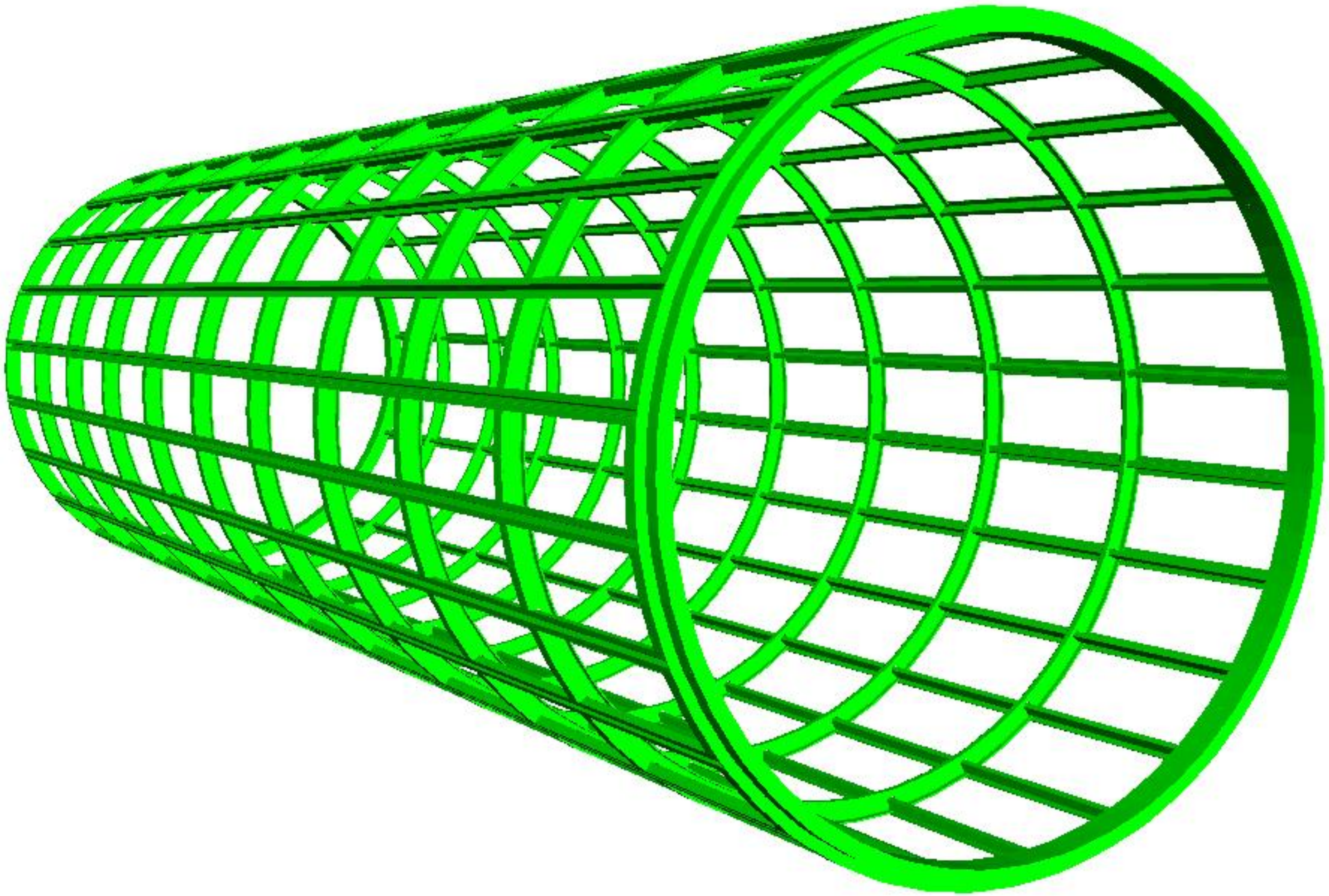
# Structural Vibration Analysis

# NASA Aluminum Testbed Cylinder (ATC)

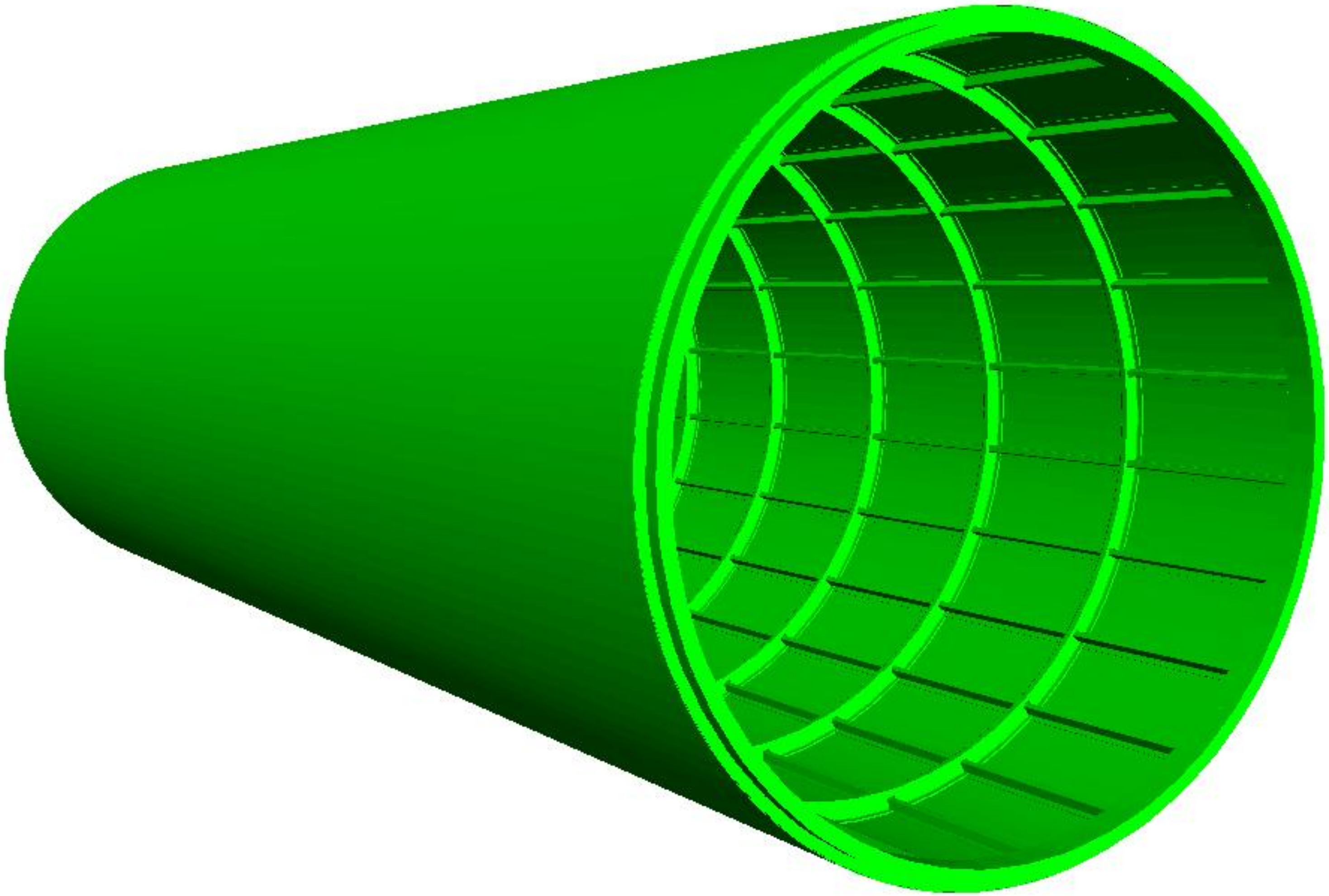




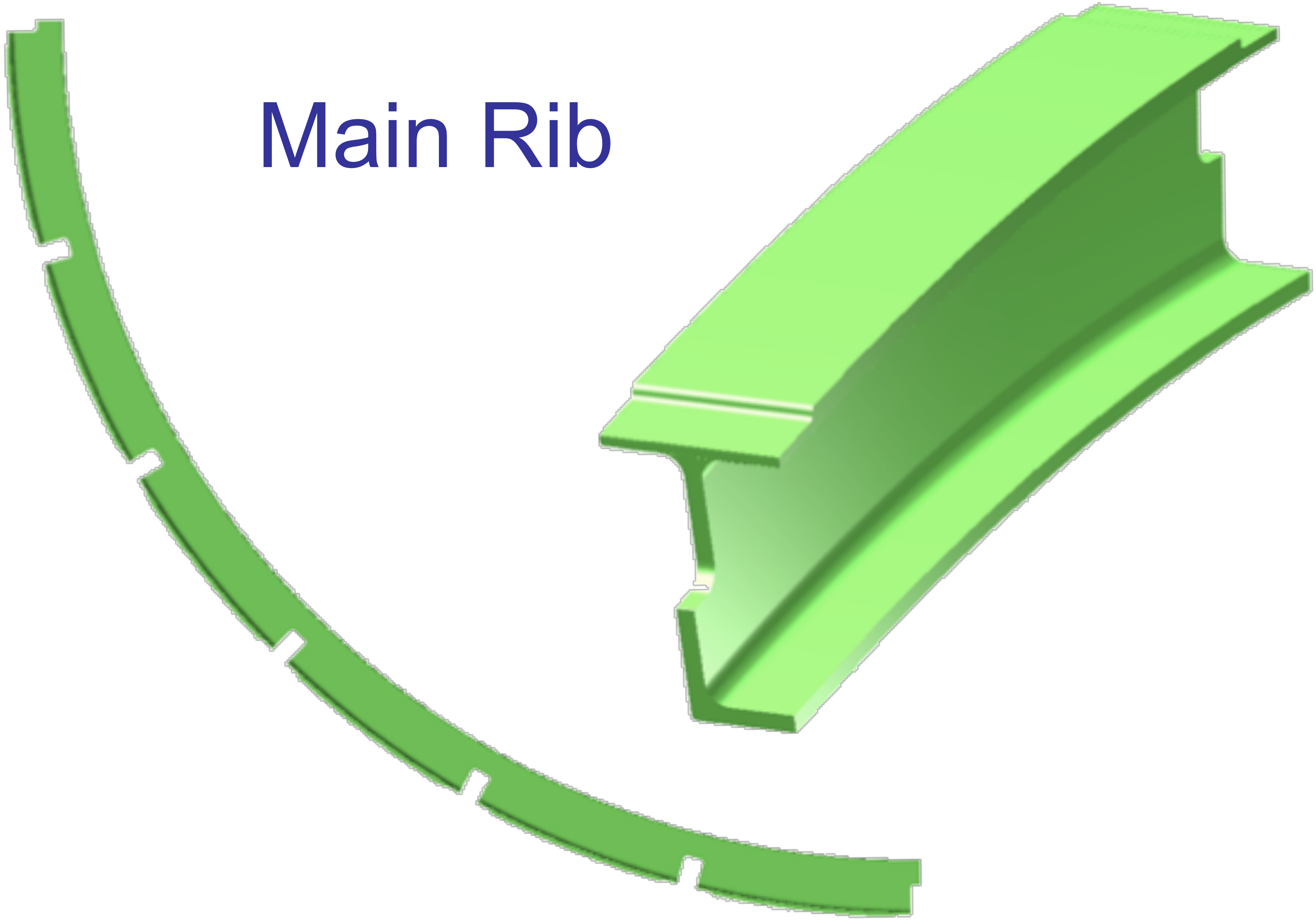
# NASA ATC Frame

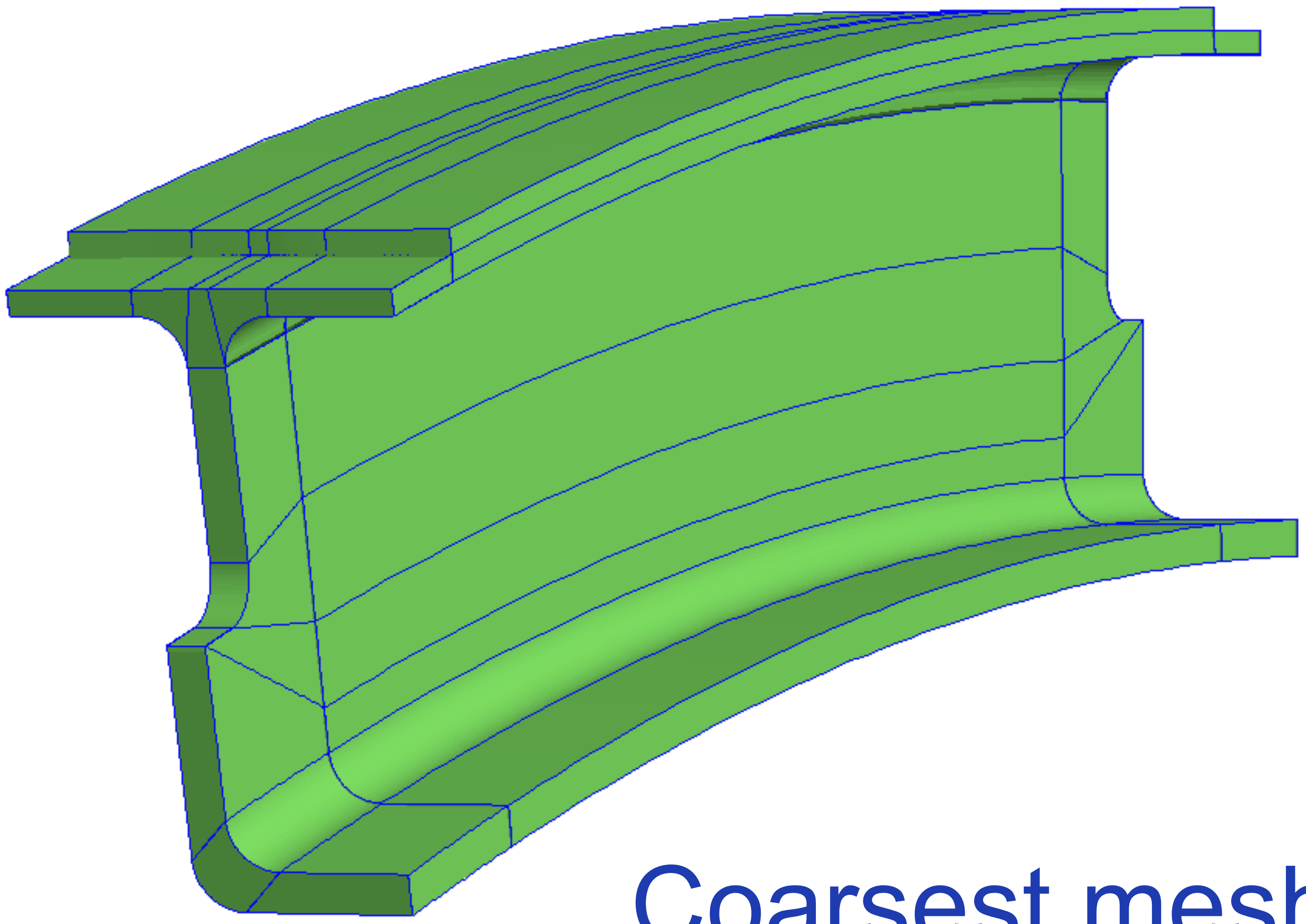


# NASA ATC Frame and Skin

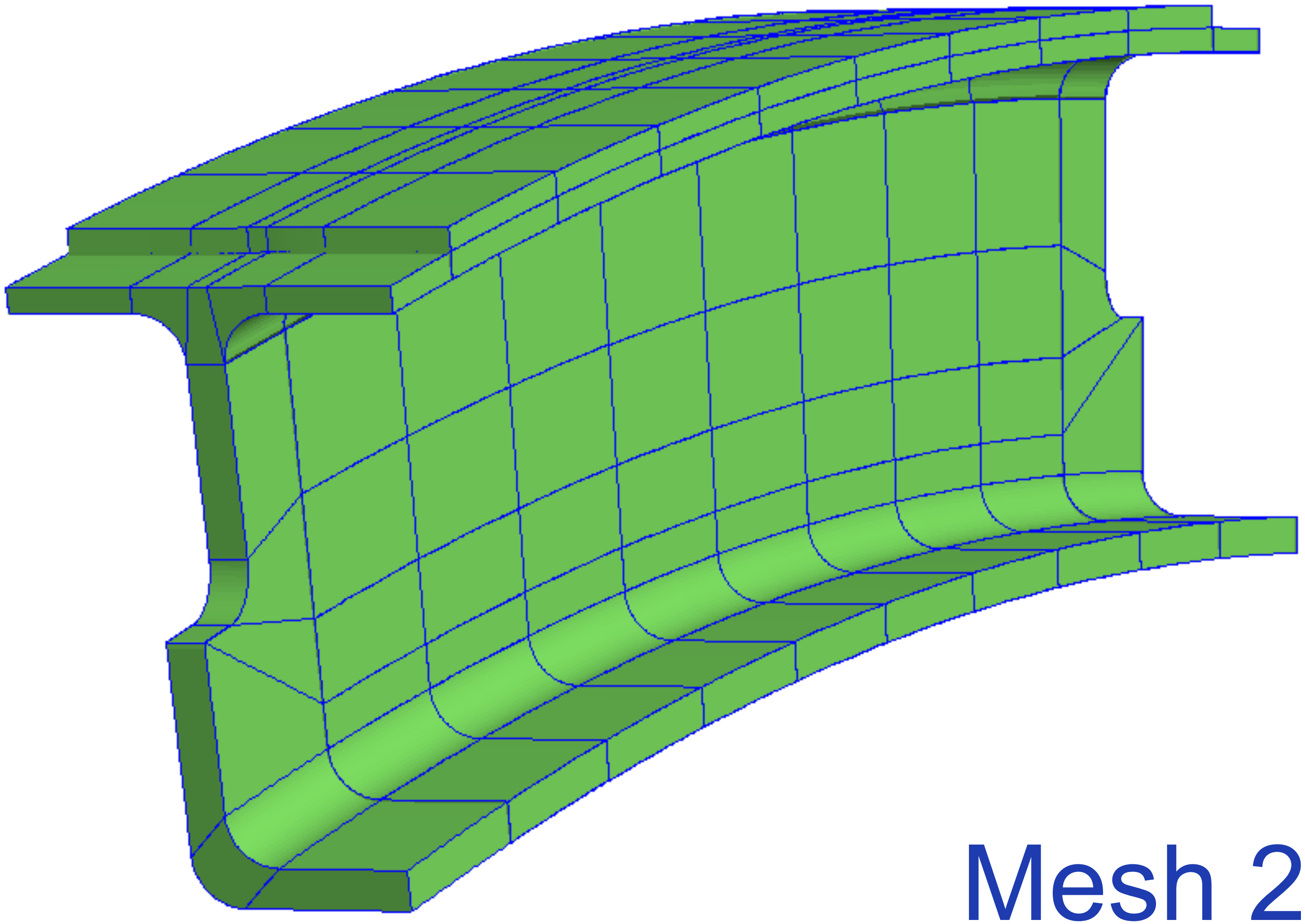


# Main Rib

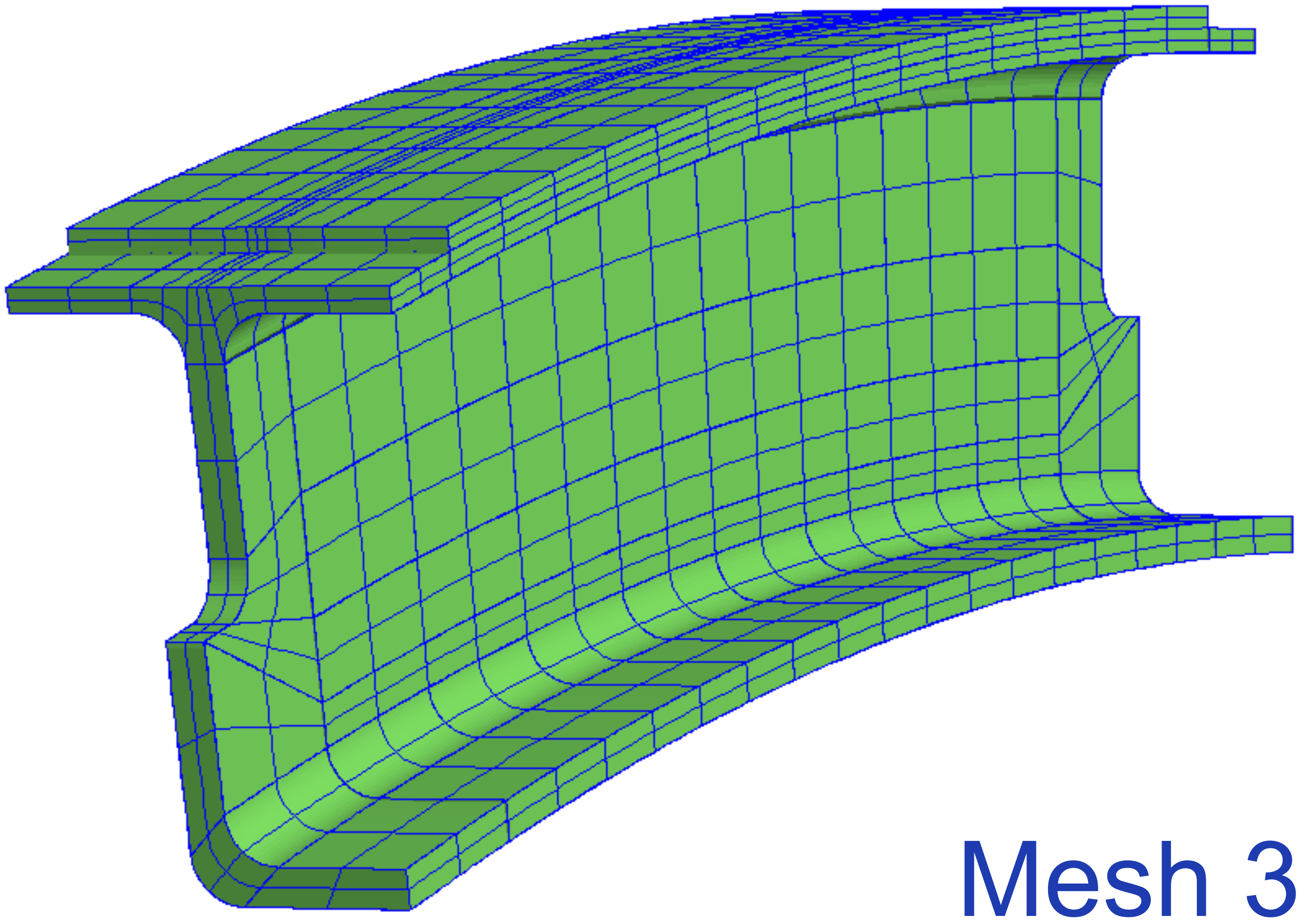




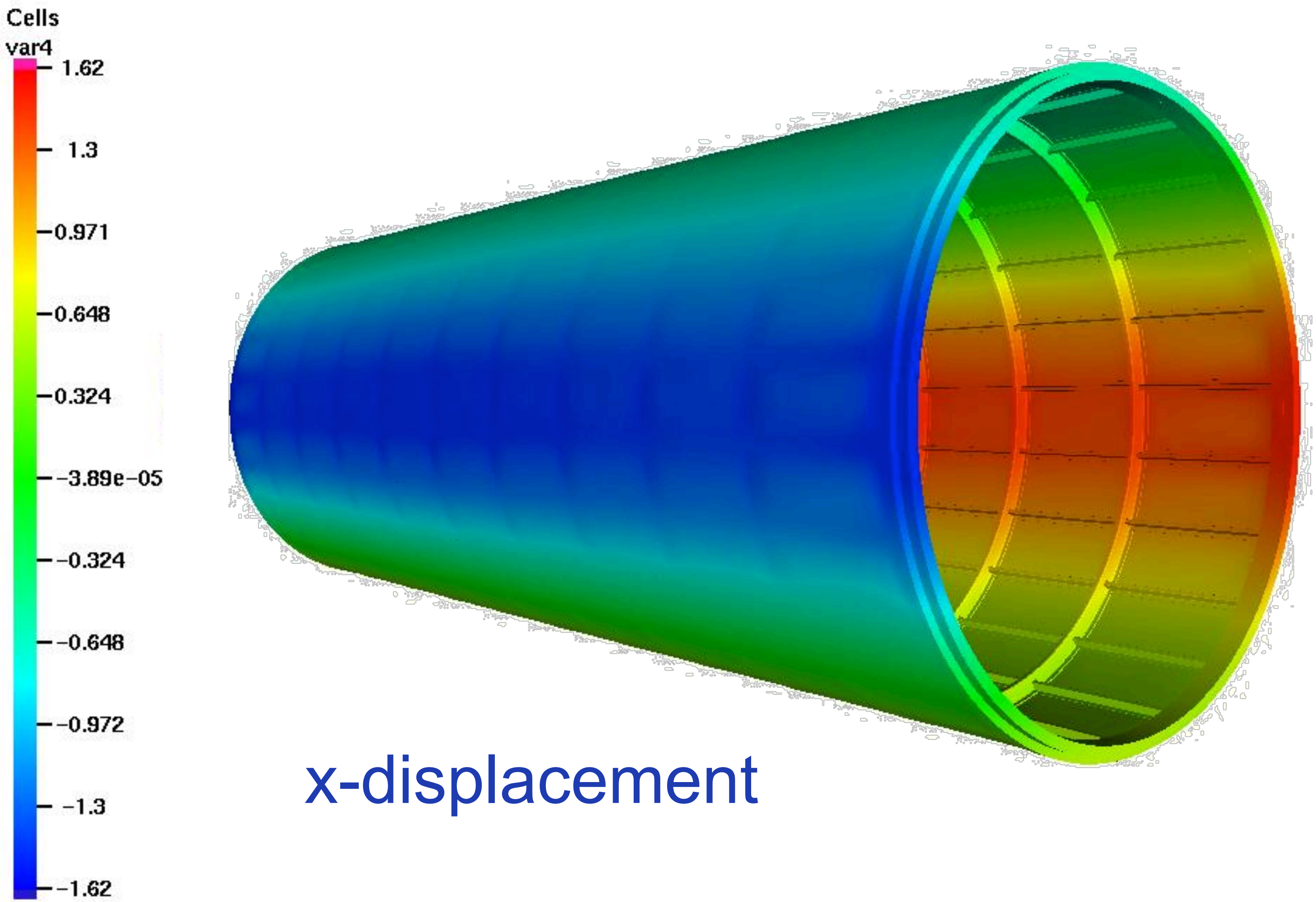
Coarsest mesh
15° segment of main rib



Mesh 2

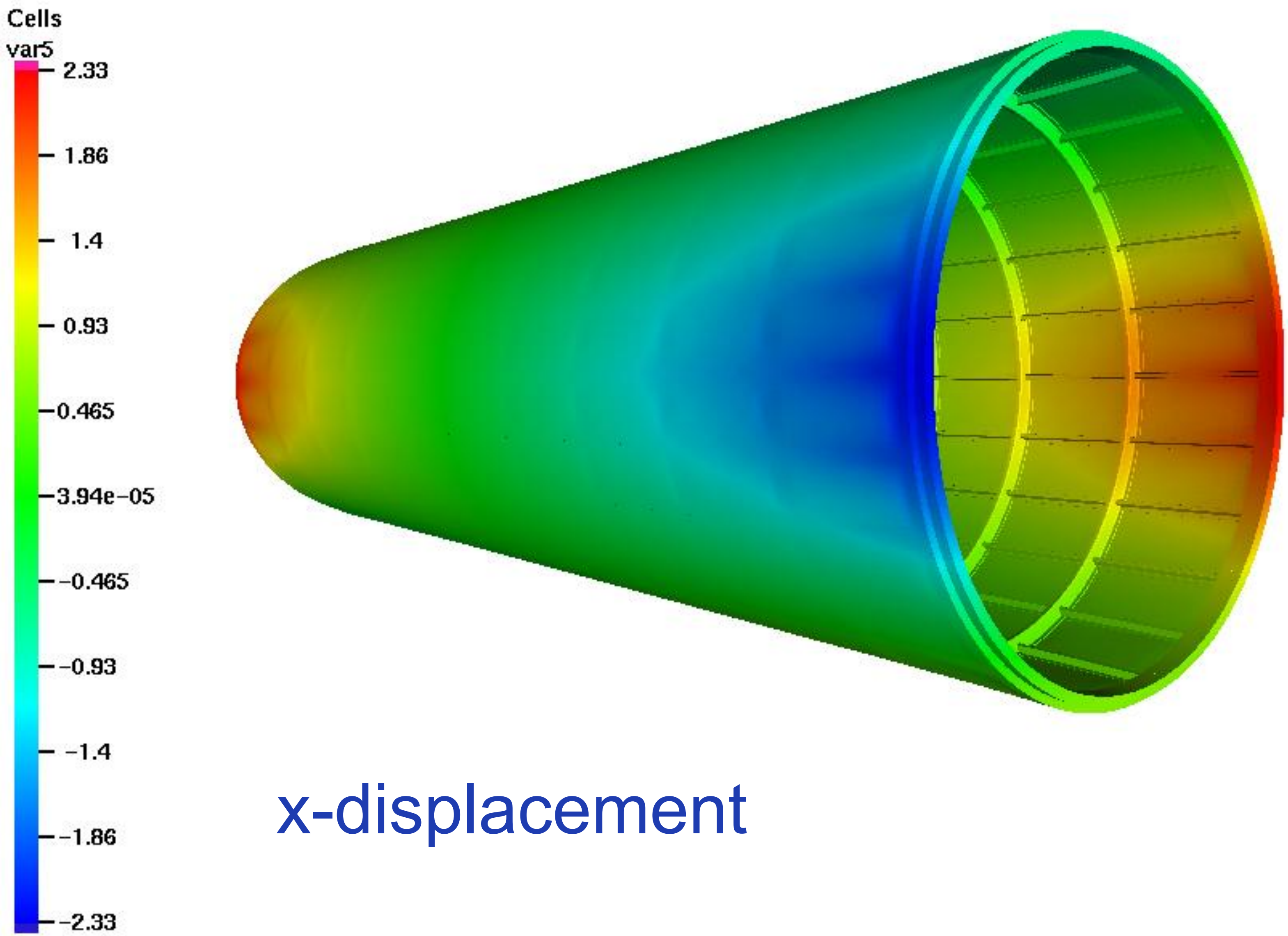

Mesh 3

# First Rayleigh Mode



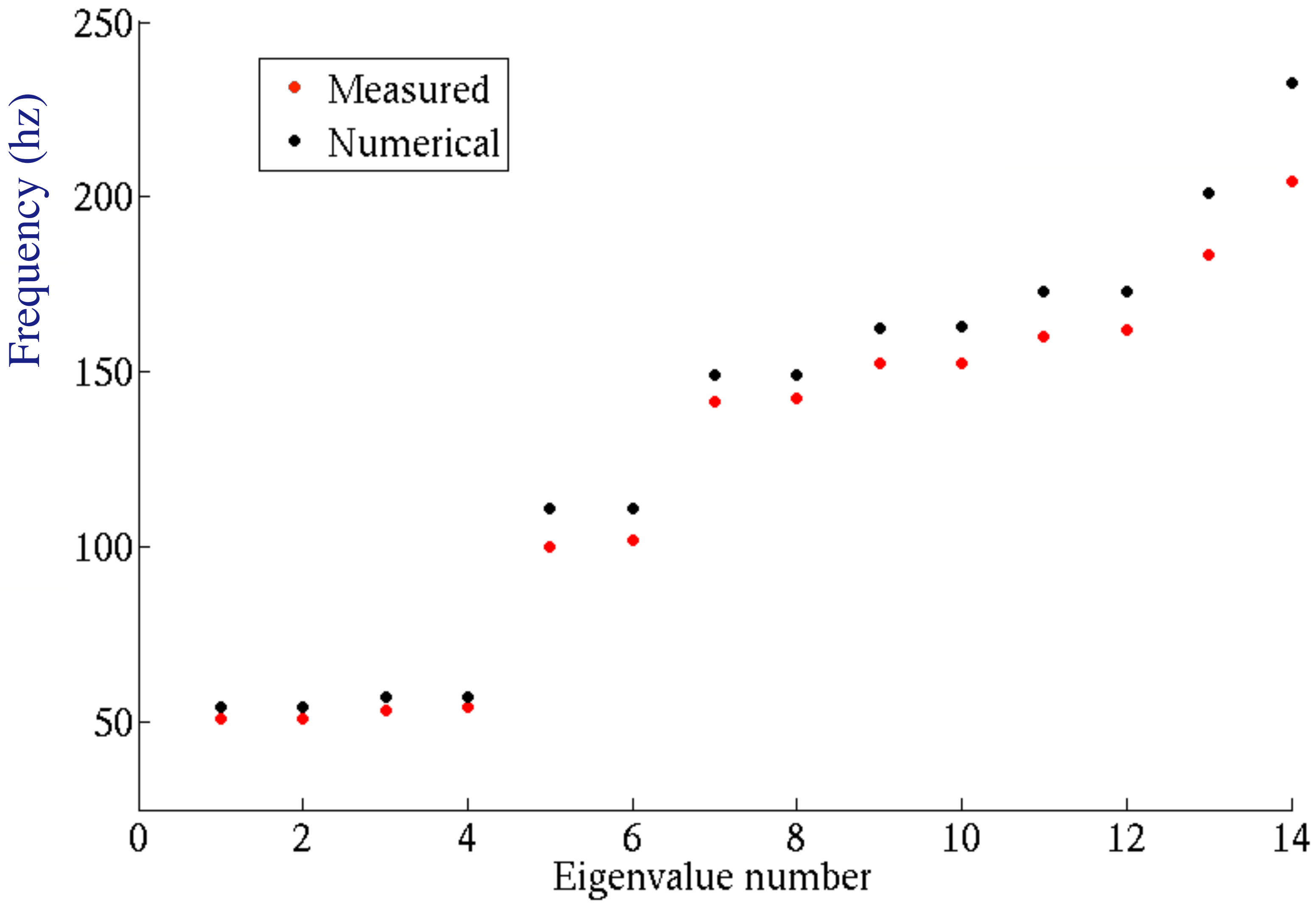

x-displacement

# First Love Mode




x-displacement

# ATC Frame and Skin



# Vibration of a Finite Elastic Rod with Fixed Ends

Eigenvalue problem:

Find all $\{u, \lambda\}$, $u : [0,1] \rightarrow \mathrm{R}$, $\lambda = \omega^2 \in \mathrm{R}^+$, s.t.

$$\boxed{u_{,xx} + \lambda u = 0, \quad u(0) = u(1) = 0}$$

Natural frequencies:

$$\omega_i = (\lambda_i)^{1/2} = i\pi, \quad i = 1, 2, 3, \ldots, \infty$$

Frequency errors:

$$\omega_i^h / \omega_i$$

# Galerkin Method for the Eigenvalue Problem

B. Galerkin

Variational form:

$$V = \left\{ w : \; w \in H^1(0,1), \; w(0) = w(1) = 0 \right\}$$

Find all $\{u, \lambda\}$, $u \in V$, $\lambda = \omega^2 \in \mathrm{R}^+$, s.t.

$$\boxed{B(w,u) = \lambda (w,u) \quad \forall w \in V}$$

where

$$B(w,u) = \int_0^1 \frac{dw}{dx} \; \frac{du}{dx} \; dx$$

$$(w,u) = \int_0^1 w \; u \; dx$$

Galerkin method:

$$V^h = \left\{ w^h : \; w^h = \sum_{i=1}^{n} N_i c_i, \; N_i \in V, \; c_i \in \mathrm{R} \right\} \subset V$$

Find all $\{u^h, \lambda^h\}$, $u^h \in V^h$, $\lambda^h = (\omega^h)^2 \in \mathrm{R}^+$, s.t.

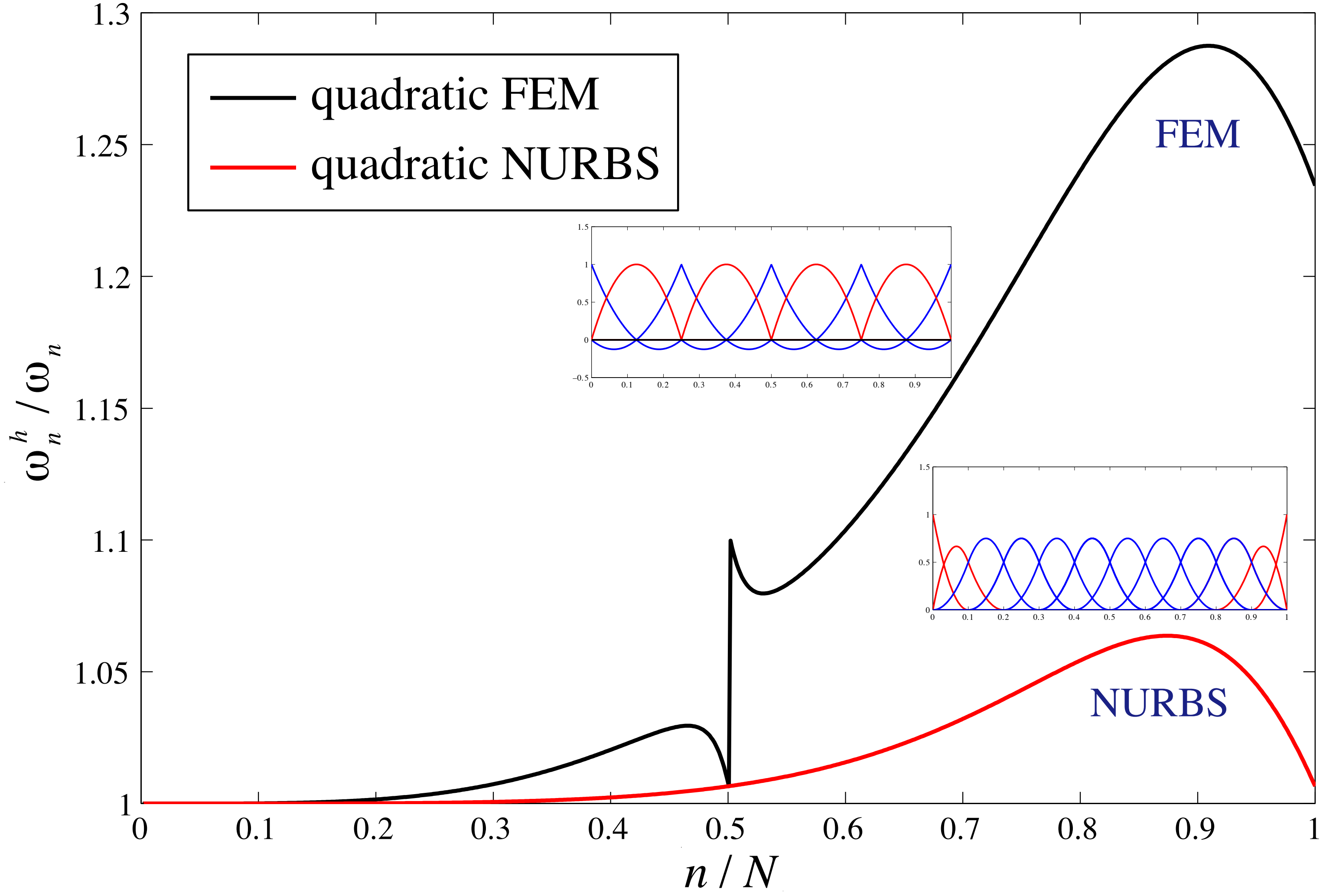
$$\boxed{B(w^h, u^h) = \lambda^h (w^h, u^h) \quad \forall w^h \in V^h}$$

$$\Leftrightarrow \; \mathbf{K}\mathbf{u}_i = \lambda_i^h \; \mathbf{M}\mathbf{u}_i, \; i = 1,2,\ldots,n$$

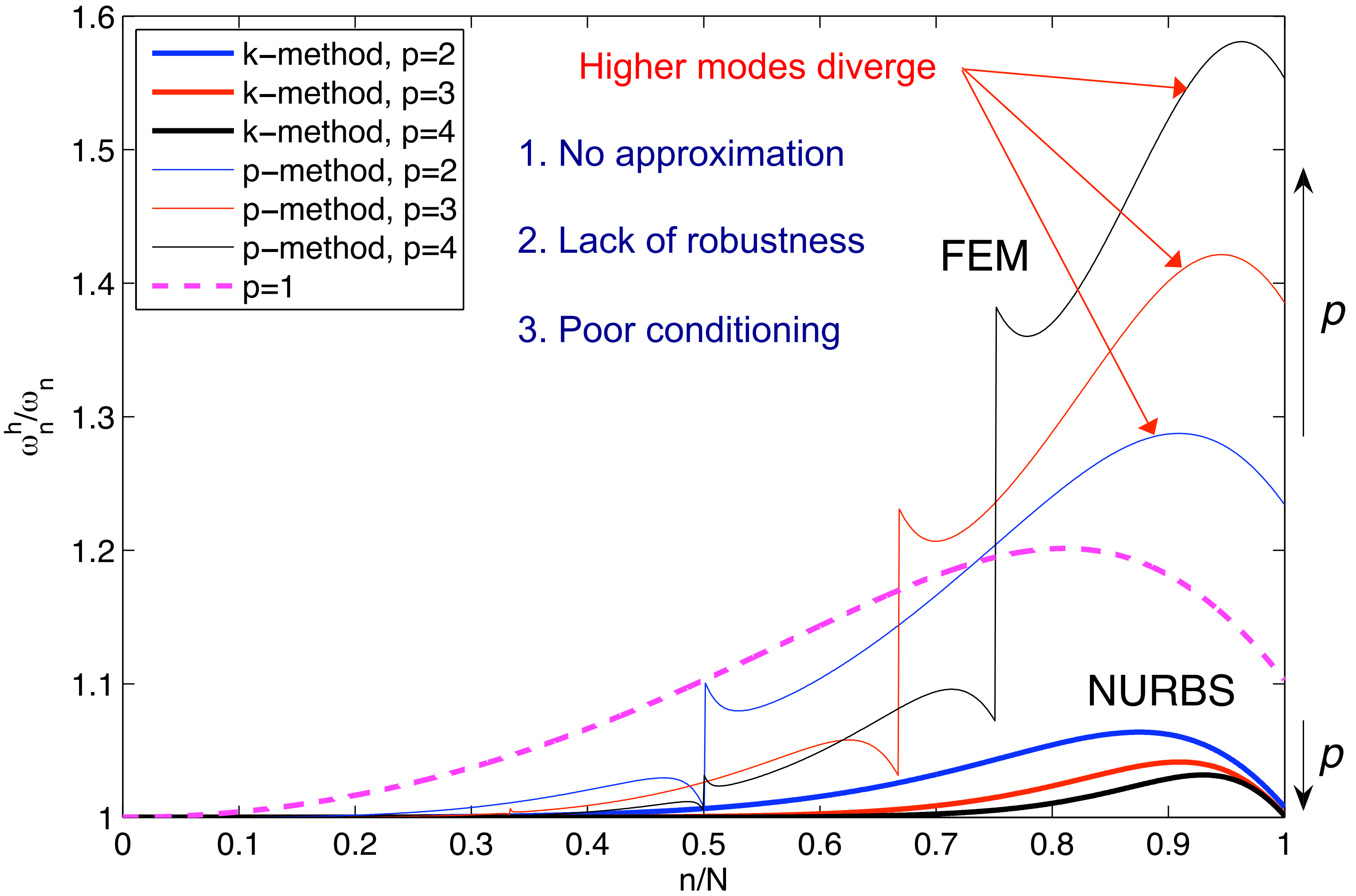
where $\mathbf{K} = \left[ K_{ij} \right]$, $\mathbf{M} = \left[ M_{ij} \right]$

with $K_{ij} = B(N_i, N_j)$, $M_{ij} = (N_i, N_j)$

# Comparison of Quadratic $C^0$ FEM and $C^1$ NURBS Frequency Errors



# Comparison of $C^0$ FEM and $C^{p-1}$ NURBS Frequency Errors



# Pythagorean Eigenvalue Error Theorem

$$B(w,u_l)=\lambda_l(w,u_l),$$
$$0<\lambda_1\le\lambda_2\le\lambda_3\le\ldots,$$
$$(u_l,u_m)=\delta_{lm},$$
$$B(u_l,u_m)=\lambda_l\delta_{lm},$$
$$\|u_l\|=1,\quad \|u_l\|_E^2=\lambda_l$$

$$B(w^h,u_l^h)=\lambda_l^h(w^h,u_l^h),$$
$$0<\lambda_1^h\le\lambda_2^h\le\lambda_3^h\le\ldots,$$
$$(u_l^h,u_m^h)=\delta_{lm},$$
$$B(w^h,u_l^h)=\lambda_l^h\delta_{lm},$$
$$\|u_l^h\|=1,\quad \|u_l^h\|_E^2=\lambda_l^h$$

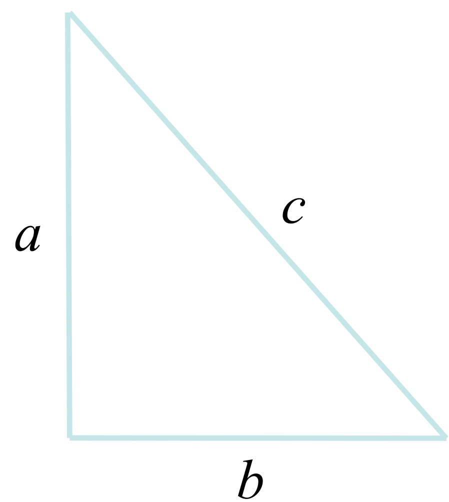
$$a^2+b^2=c^2$$

$$a=\frac{\|u_l^h-u_l\|}{\|u_l\|},\quad b=\left(\frac{\lambda_l^h-\lambda_l}{\lambda_l}\right)^{1/2},\quad c=\frac{\|u_l^h-u_l\|_E}{\|u_l\|_E}$$

$$\boxed{\frac{\|u_l^h-u_l\|^2}{\|u_l\|^2}+\left(\frac{\lambda_l^h-\lambda_l}{\lambda_l}\right)=\frac{\|u_l^h-u_l\|_E^2}{\|u_l\|_E^2}}$$

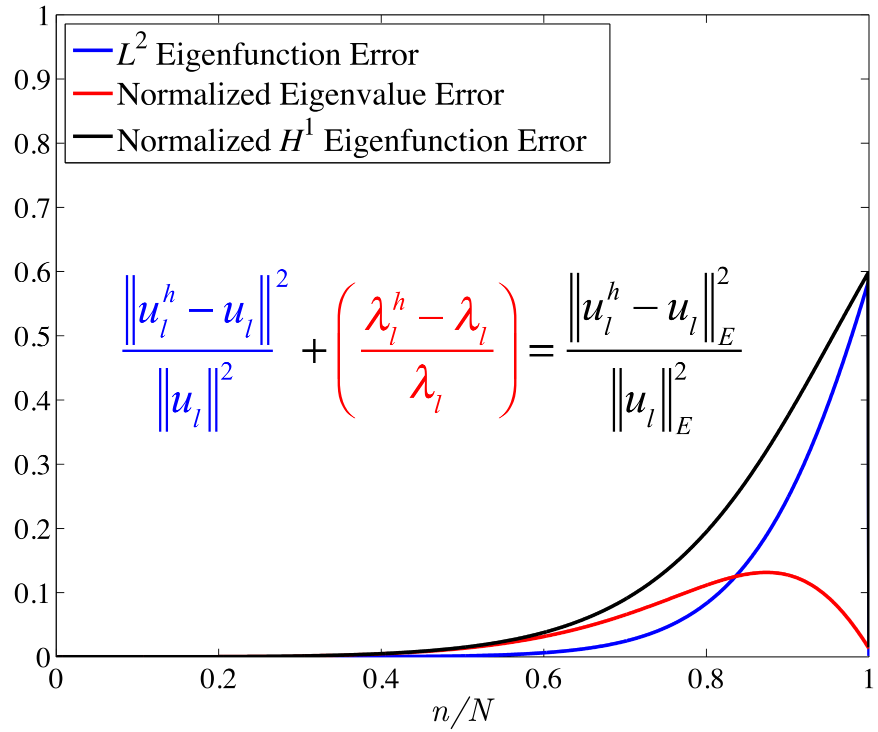
$$\frac{\|u_l^h-u_l\|^2}{\|u_l\|^2}\le\frac{\|u_l^h-u_l\|_E^2}{\|u_l\|_E^2}$$

$$\left(\frac{\lambda_l^h-\lambda_l}{\lambda_l}\right)\le\frac{\|u_l^h-u_l\|_E^2}{\|u_l\|_E^2}$$



# Eigenvalue and Eigenfunction Errors

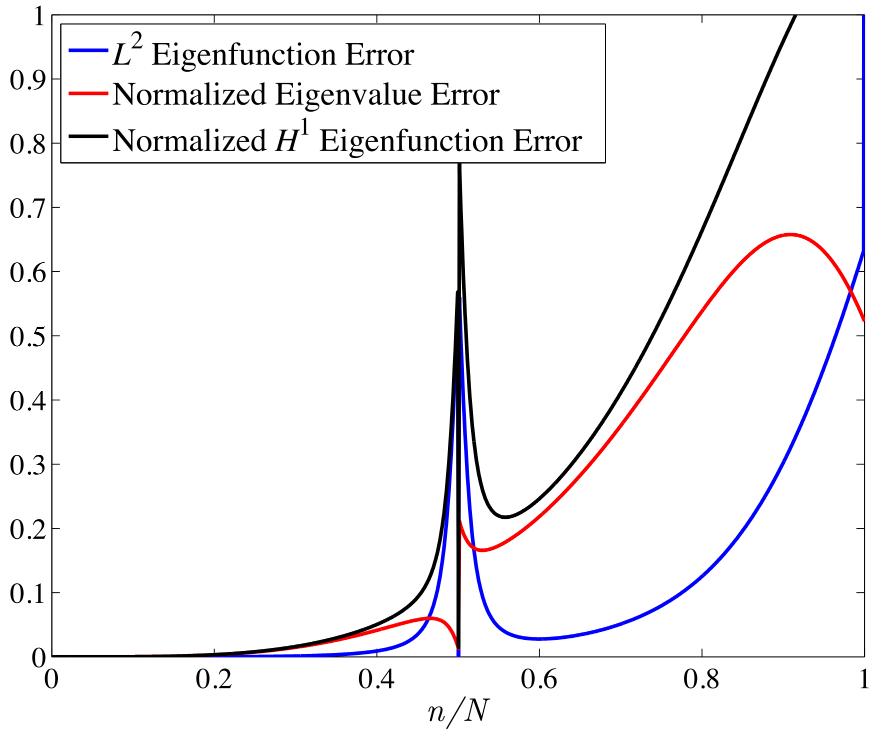
## $C^1$-continuous functions ($p$=2)



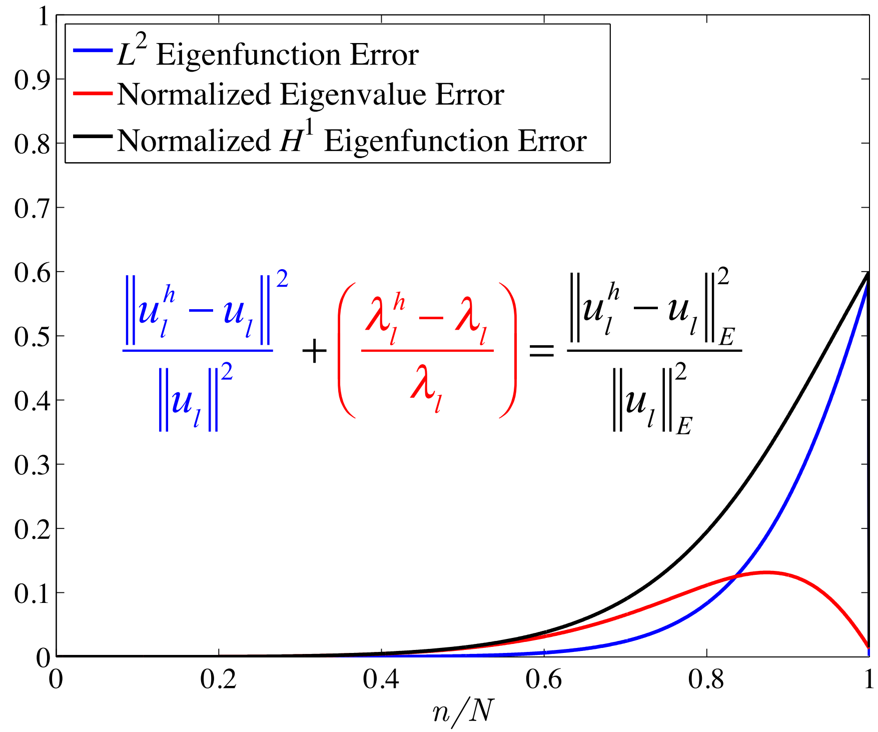

Hughes, Evans & Reali, 2014

# Eigenvalue and Eigenfunction Errors

## $C^0$-continuous functions ($p$=2)




## $C^1$-continuous functions ($p$=2)




Hugnes, Evans & Reali, 2014

# Eigenvalue and Eigenfunction Errors

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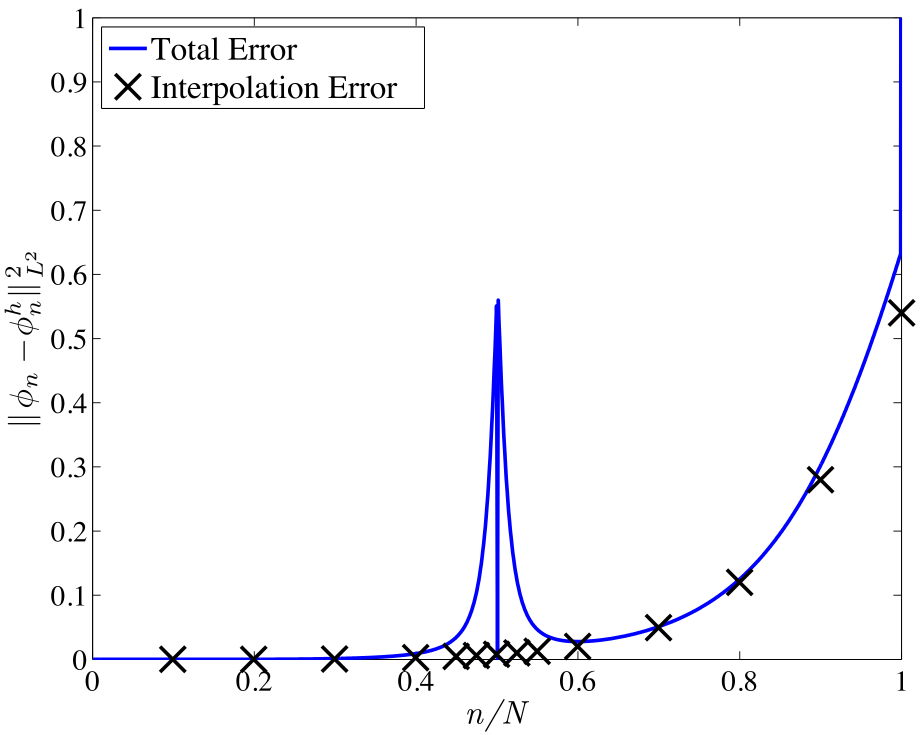
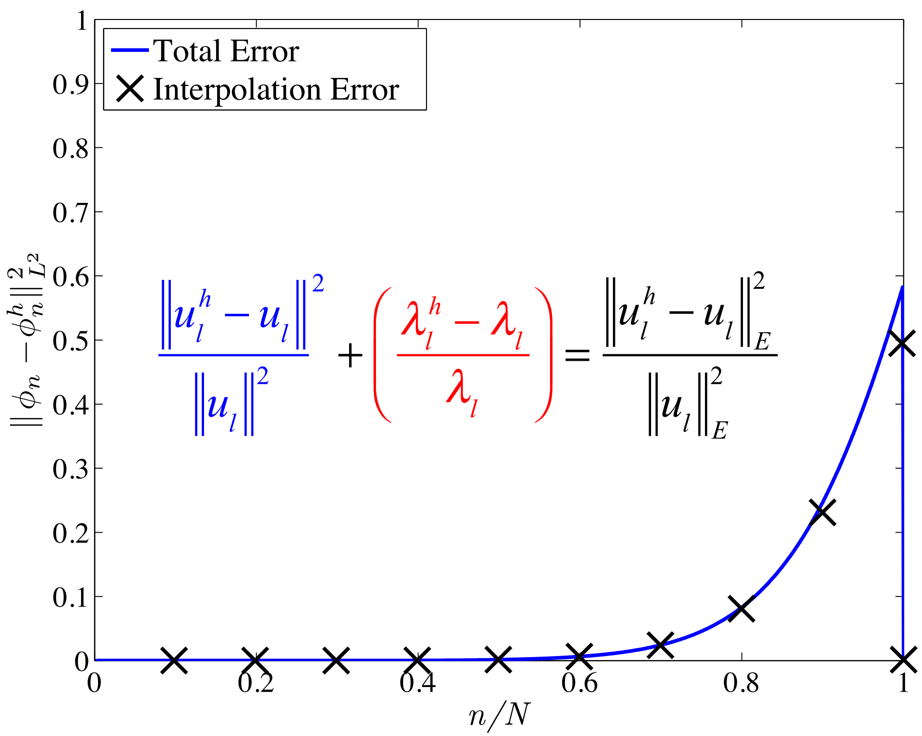


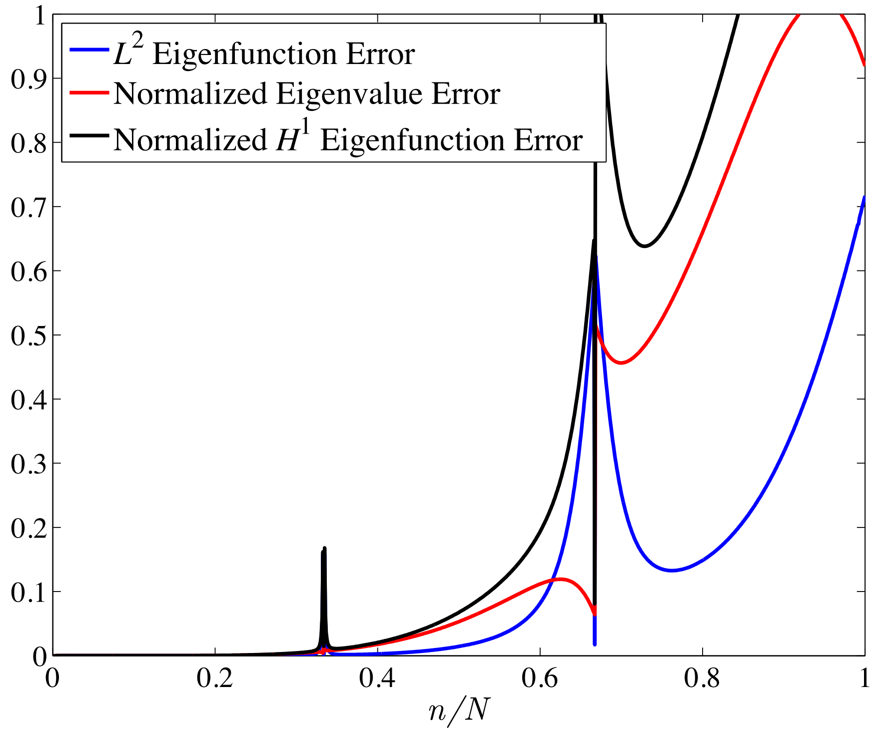

## $C^1$-continuous functions ($p$=2)



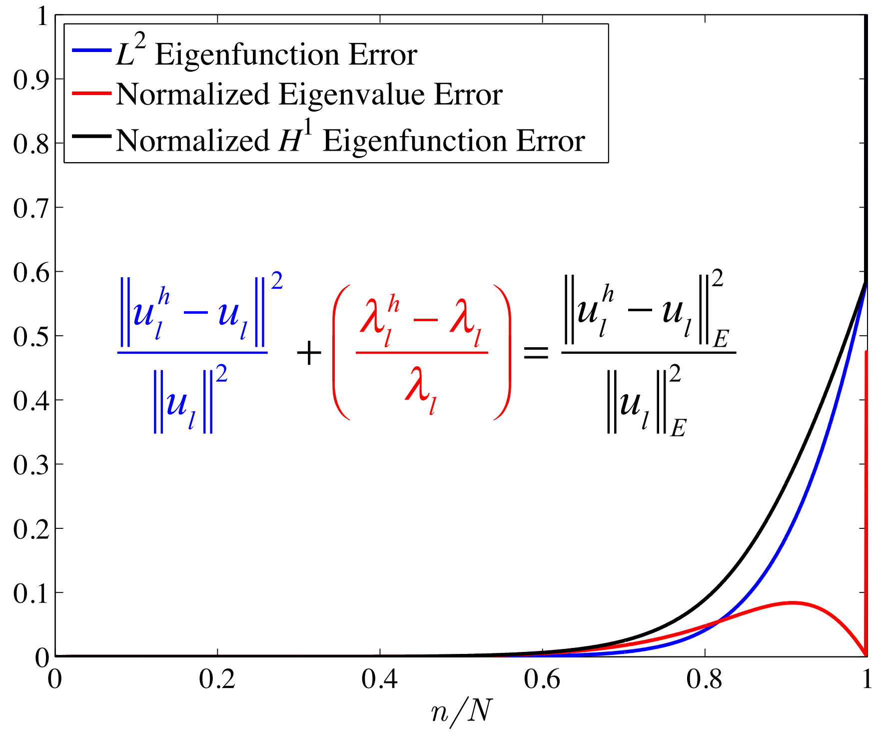

Hughes, Evans & Reali, 2014

# Eigenvalue and Eigenfunction Errors

## $C^0$-continuous functions ($p$=3)



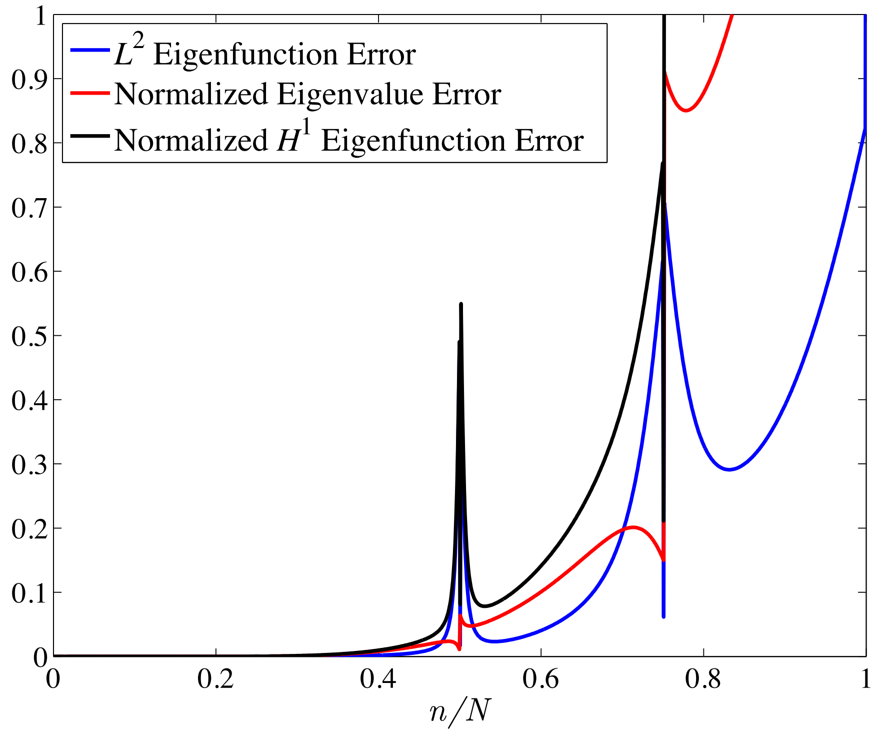

## $C^2$-continuous functions ($p$=3)



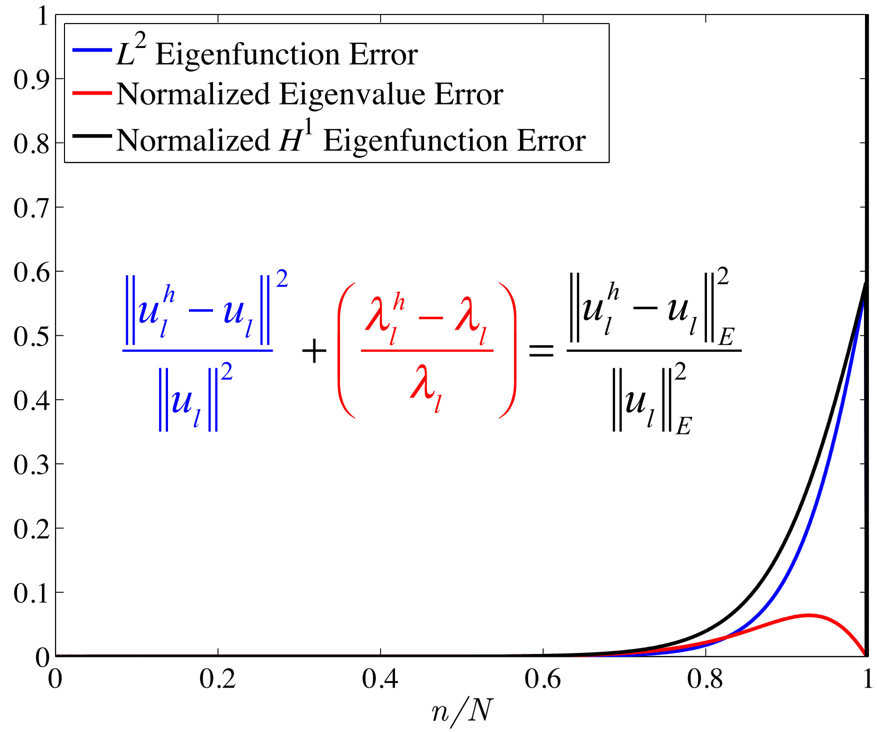

Hughes, Evans & Reali, 2014

# Eigenvalue and Eigenfunction Errors

## $C^0$-continuous functions ($p$=4)

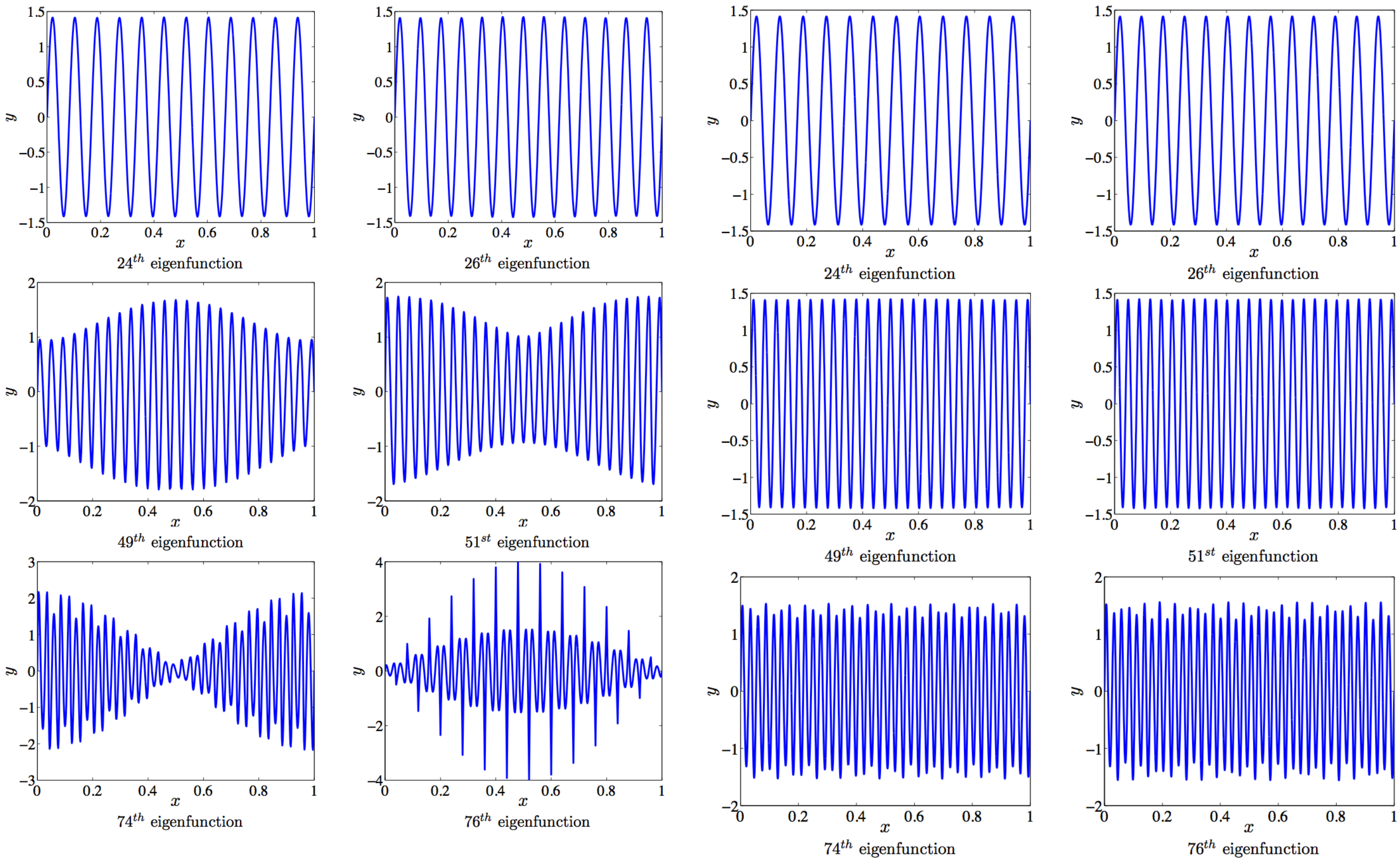



## $C^3$-continuous functions ($p$=4)




Hughes, Evans & Reali, 2014

## $C^0$-continuous functions ($p$=4)    $C^3$-continuous functions ($p$=4)




Hughes, Evans & Reali, 2014

# Wave Propagation Analysis

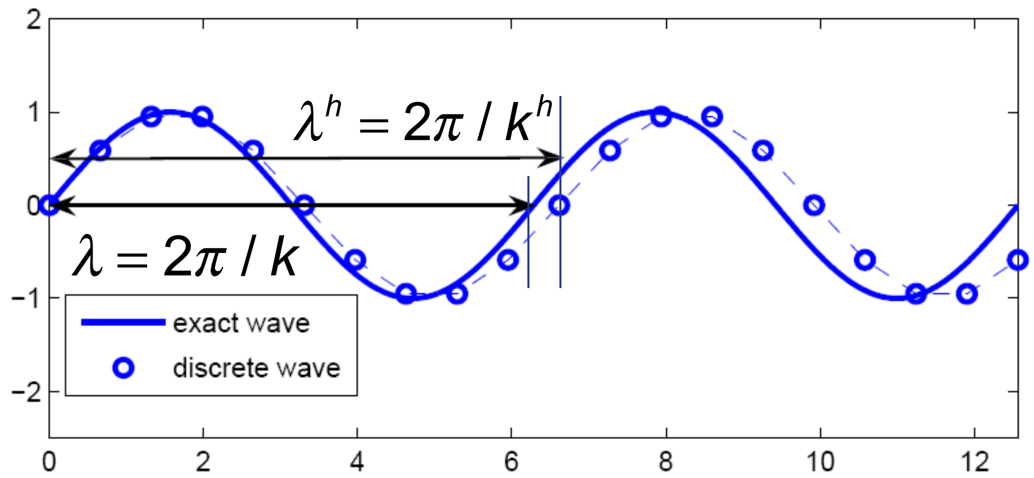
# Wave Propagation in an Infinite Domain

Helmholtz equation:

$$u_{,xx} + k^2 u = 0 \quad \text{for} \quad x \in (-\infty, +\infty)$$

Wave number: $k$

Phase error: $k / k^h$



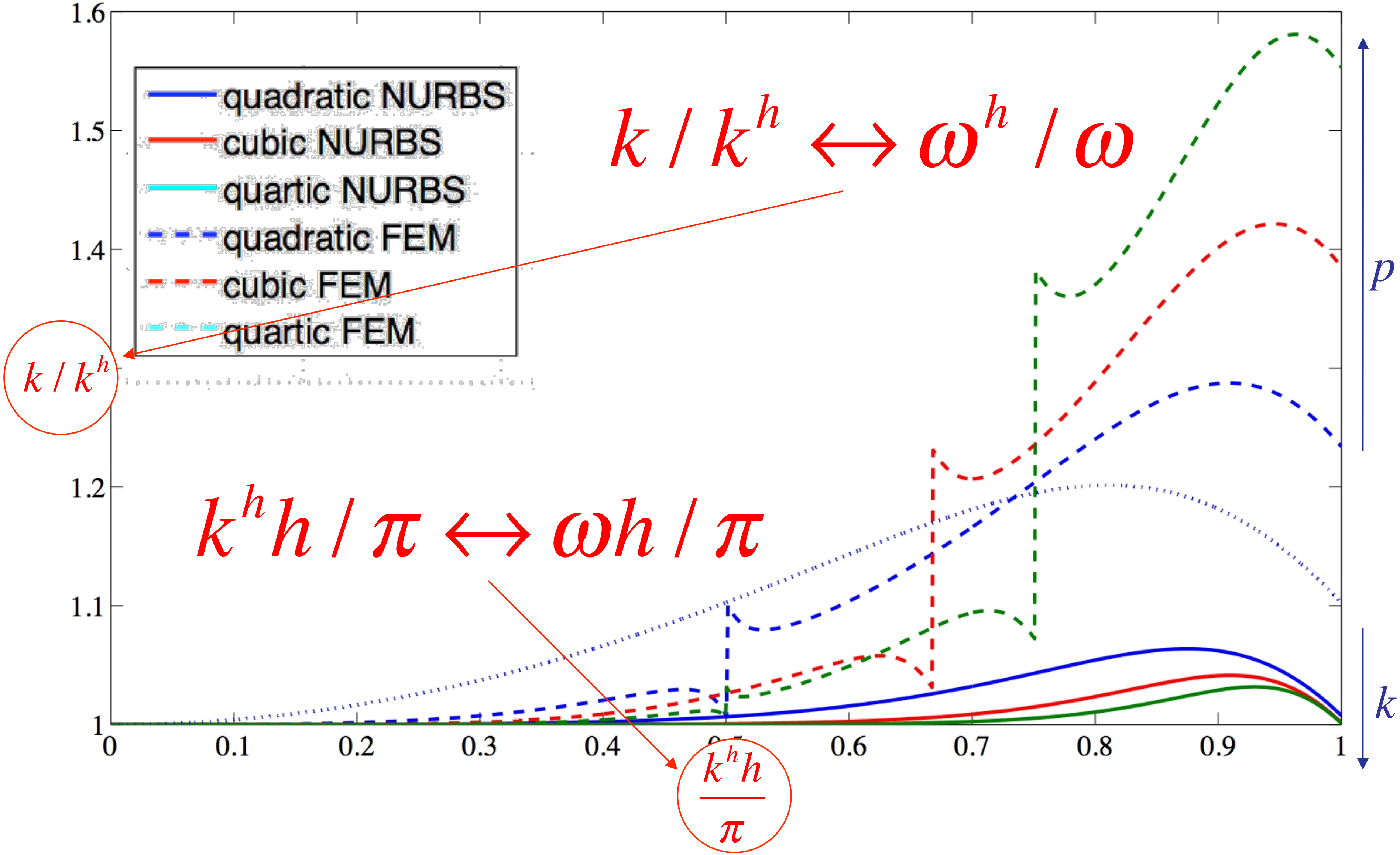
# Duality Principle

- Frequency errors in vibrations and phase errors in wave propagation are related by a change of variables:

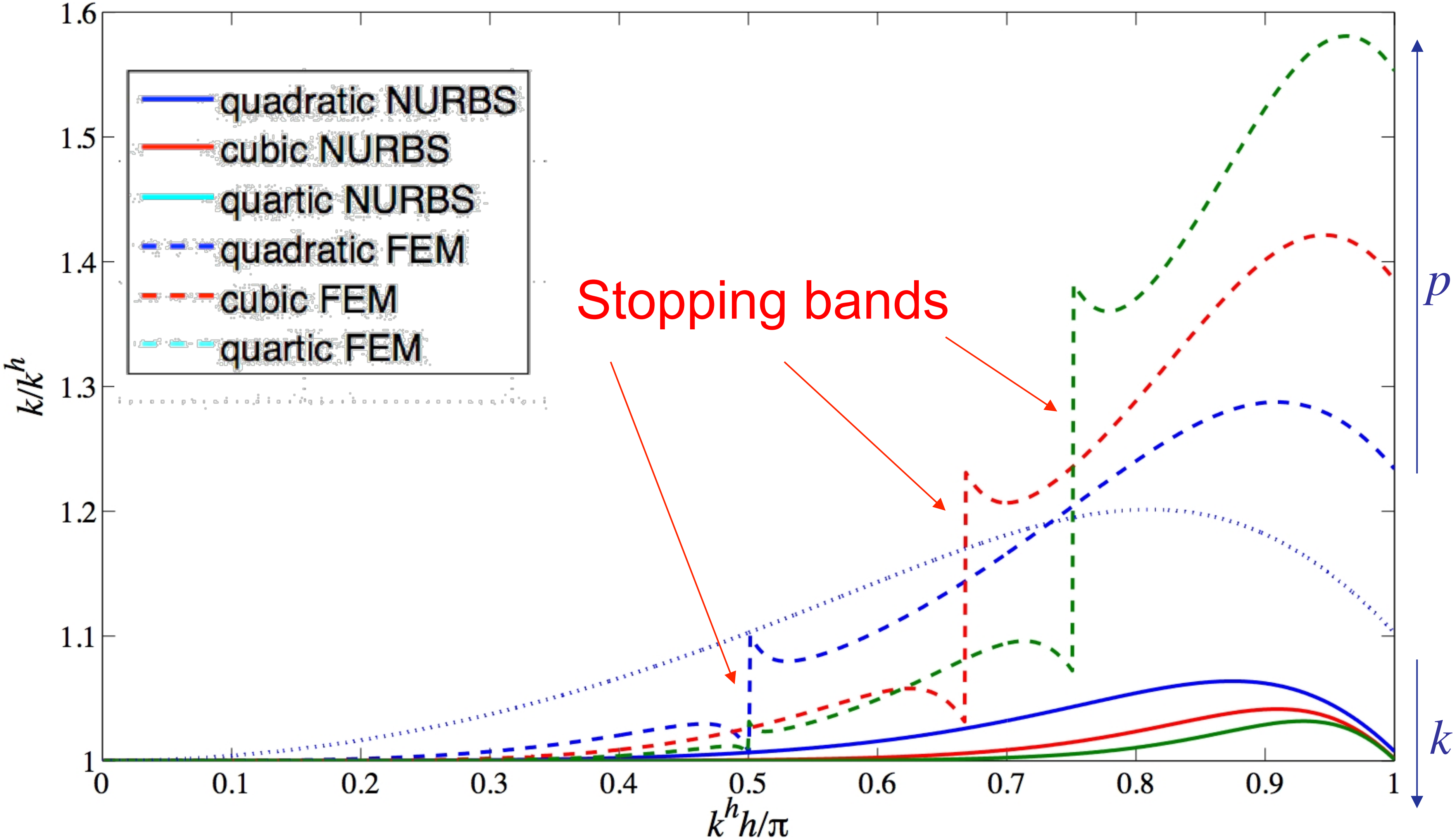
$$k / k^h \leftrightarrow \omega^h / \omega$$

$$k^h h / \pi \leftrightarrow \omega h / \pi$$

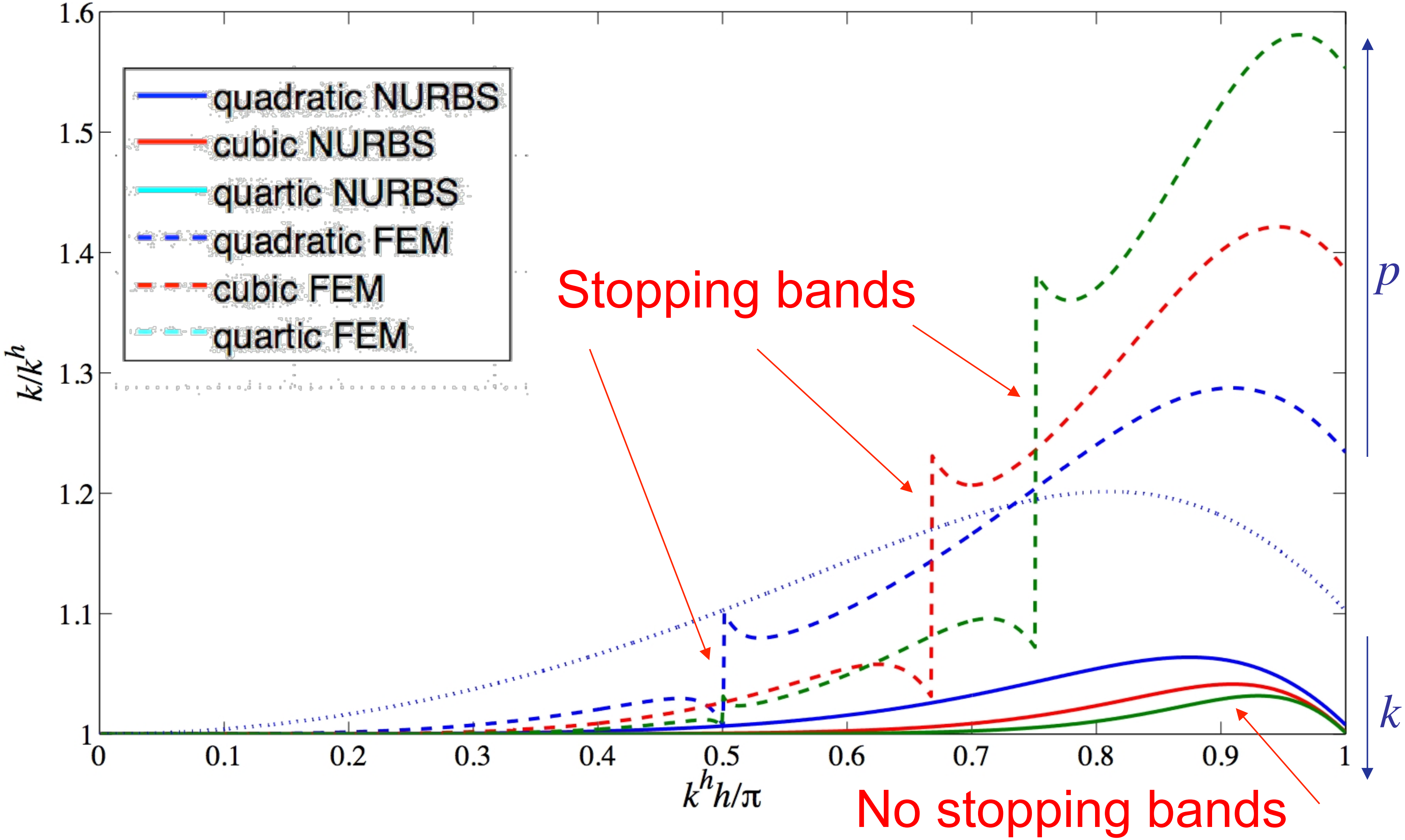
# Duality of Frequency and Phase Errors



# Helmholtz Equation Phase Error



# Helmholtz Equation Phase Error

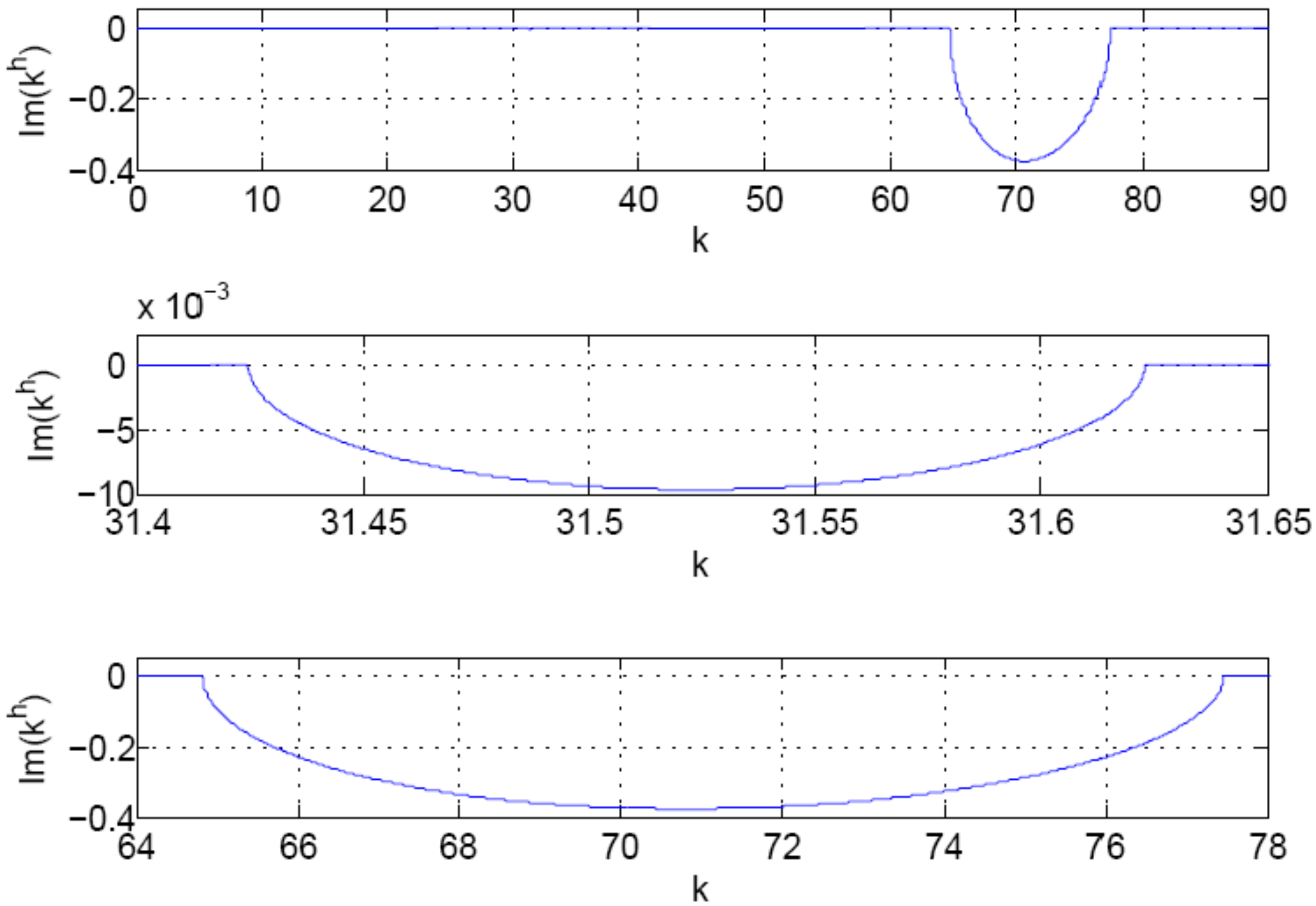


# Wave Propagation (Leszek's problem)

Helmholtz equation in 1D,
Dirichlet boundary conditions

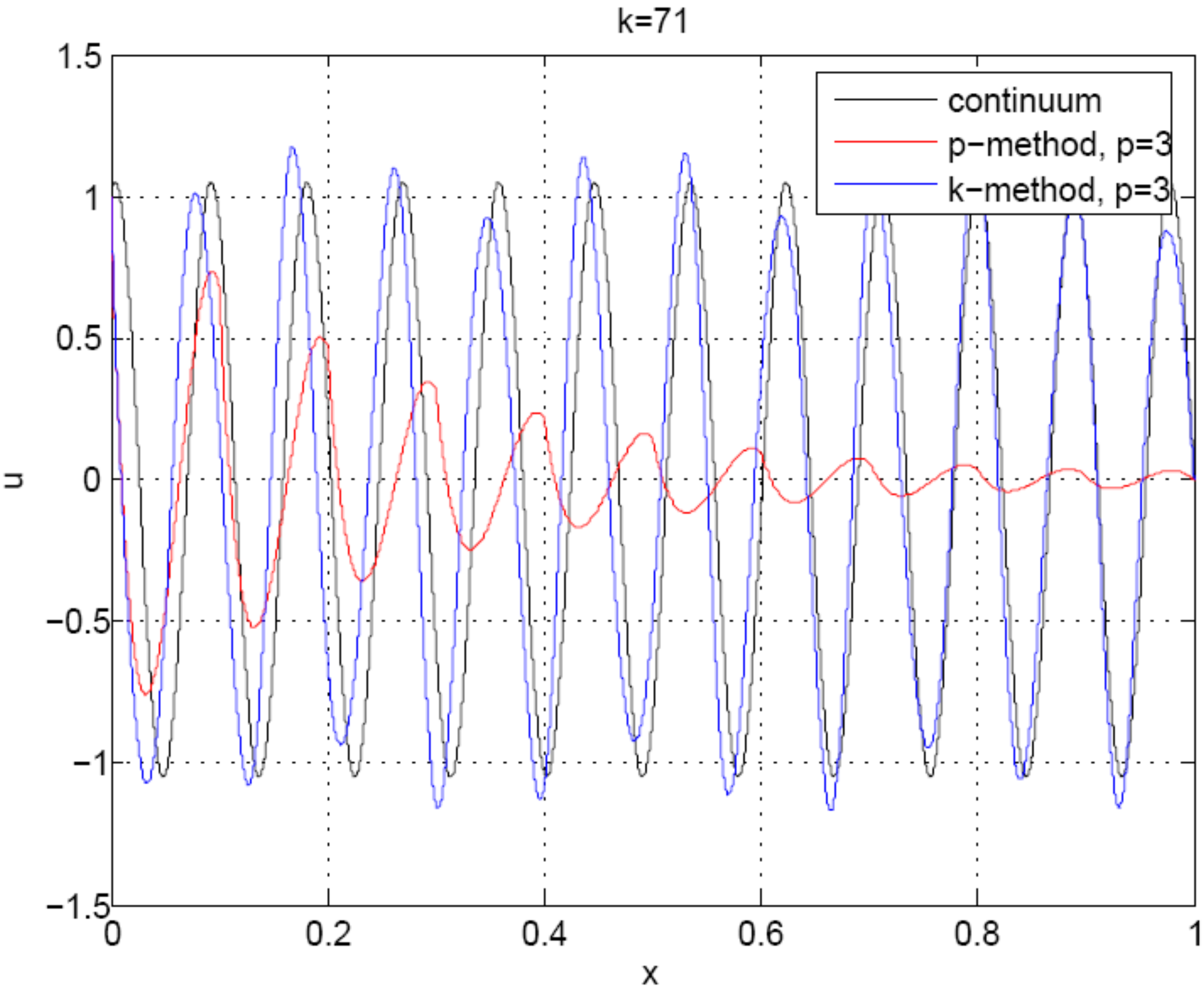
(31 control points for $p = 3$)

# $p$-method stopping bands for $p$ = 3



# $k = 71$; $p = 3$

## (inside the 2nd $p$-method stopping band)



# Wave Propagation in an Infinite Domain

Model Problem:

$$u''(x)+k^2u(x)=0 \text{ on } (0,1)$$
$$u(0)=1$$
$$u'(1)-iku(1)=0$$

Exact Solution:

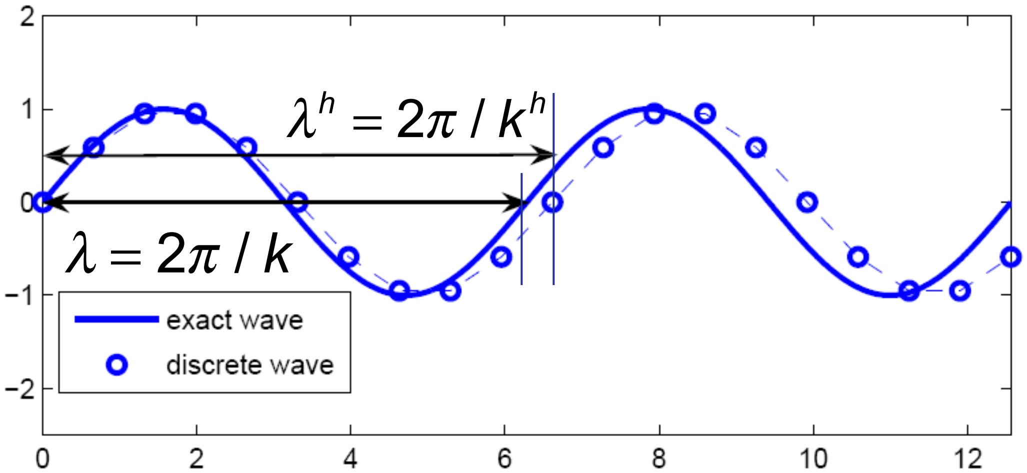
$$u(x)=\exp(ikx)$$

# Dispersion Error and Pollution

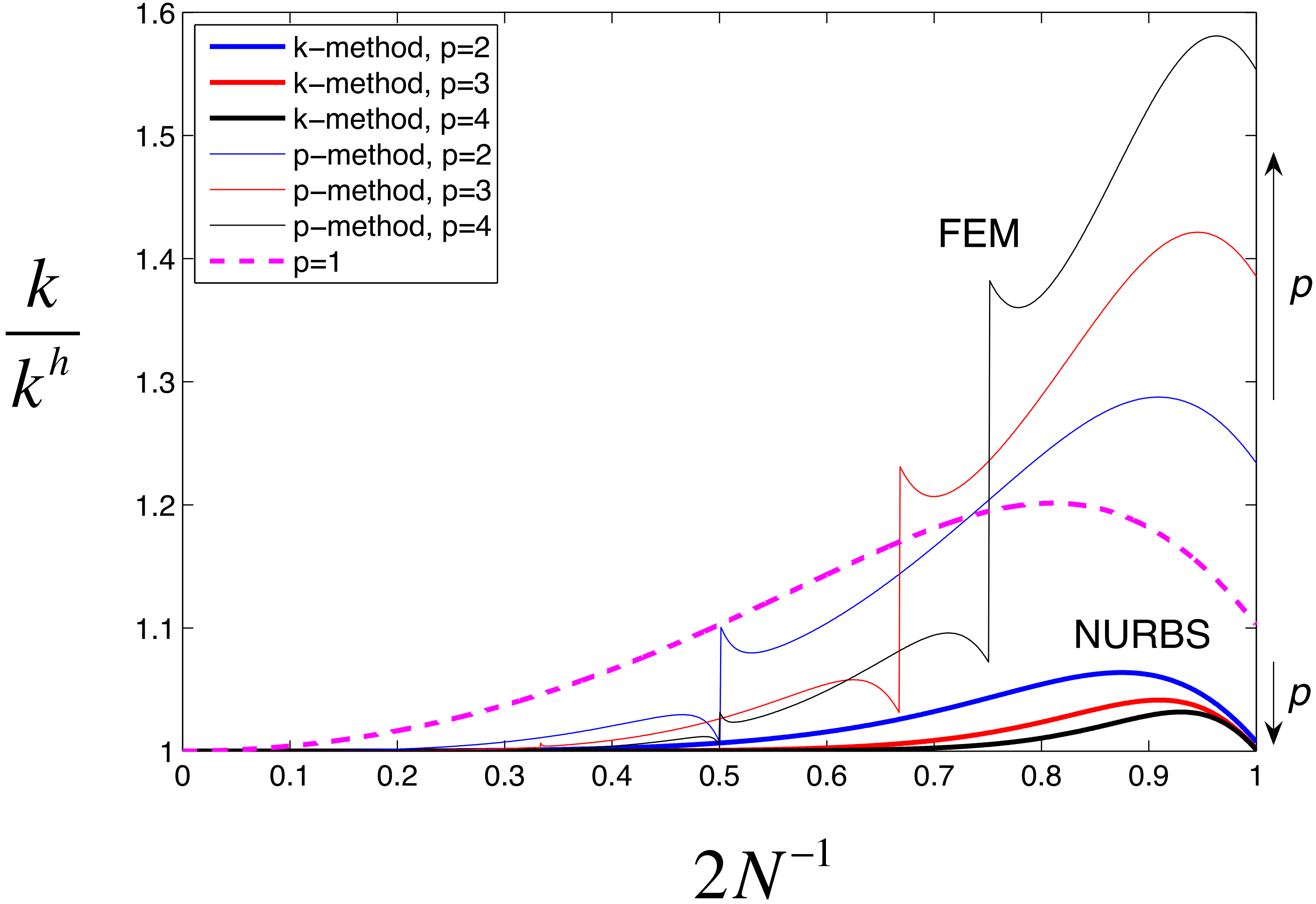
As k grows, dispersion leads to a growth in error:

$$\frac{\lambda^h - \lambda}{\lambda} \sim \left(kh\right)^{p+1} h^{p-1}$$

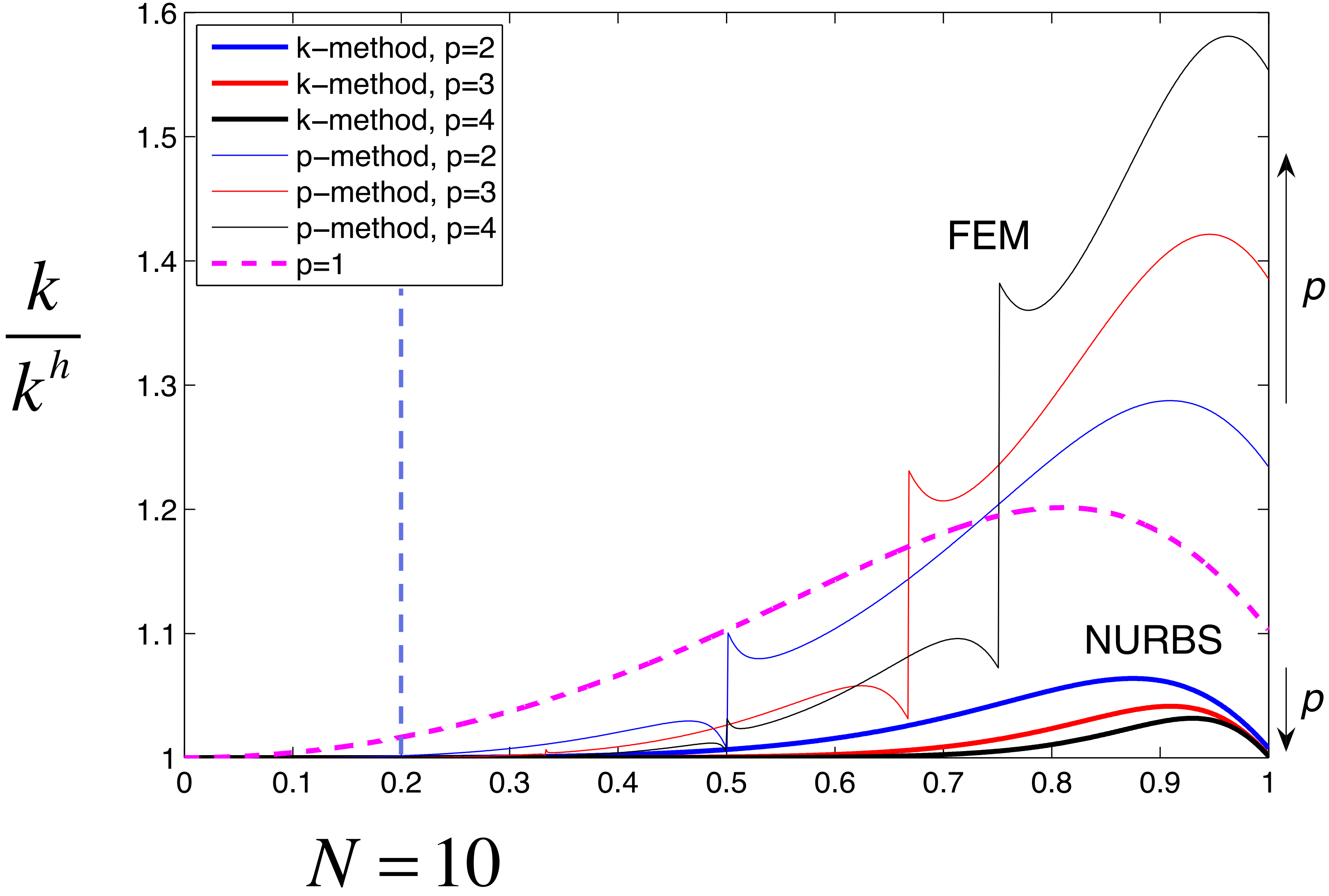
$$\left\| u - u^h \right\| \sim \left(kh\right)^2 h^{p-1} \left(1 + k\right)$$



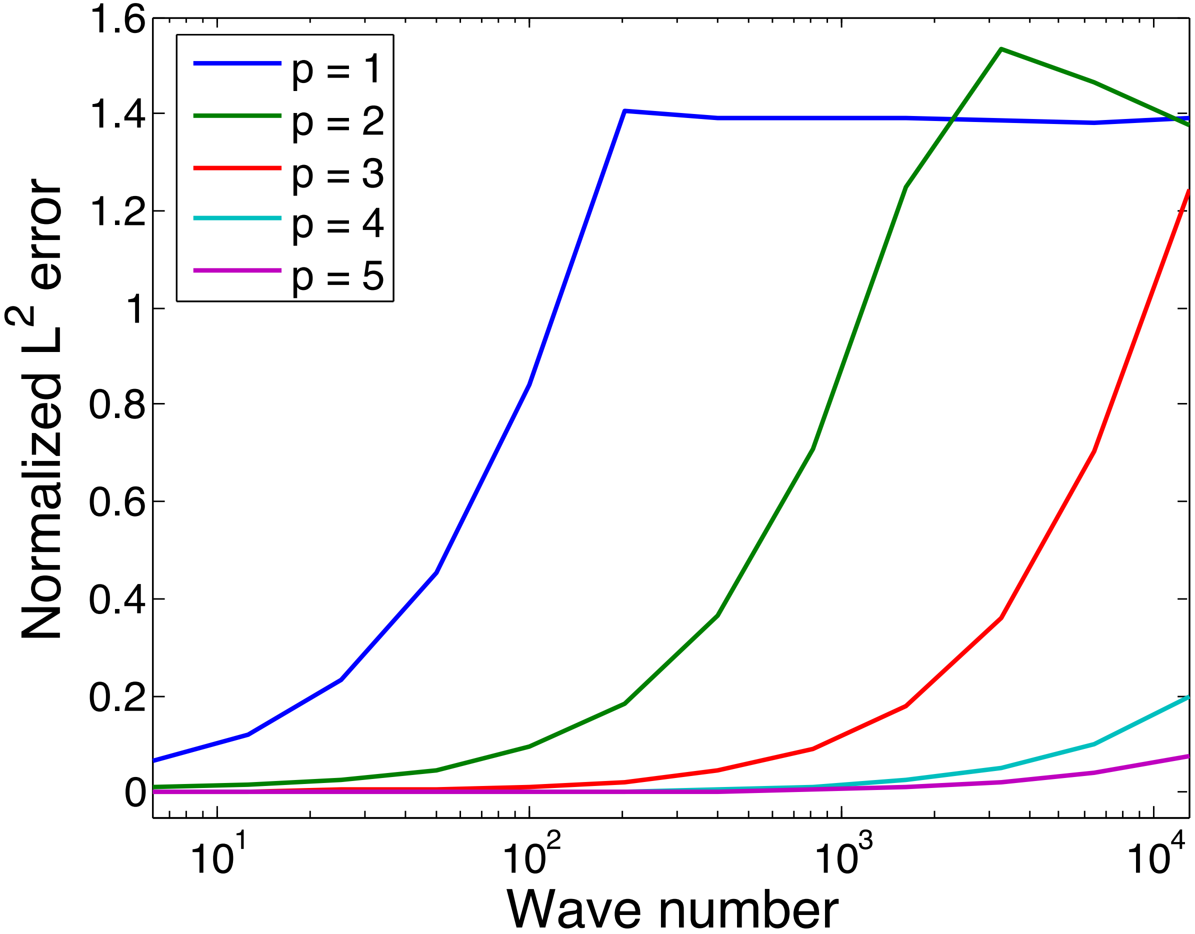
# Dispersion Error and Pollution



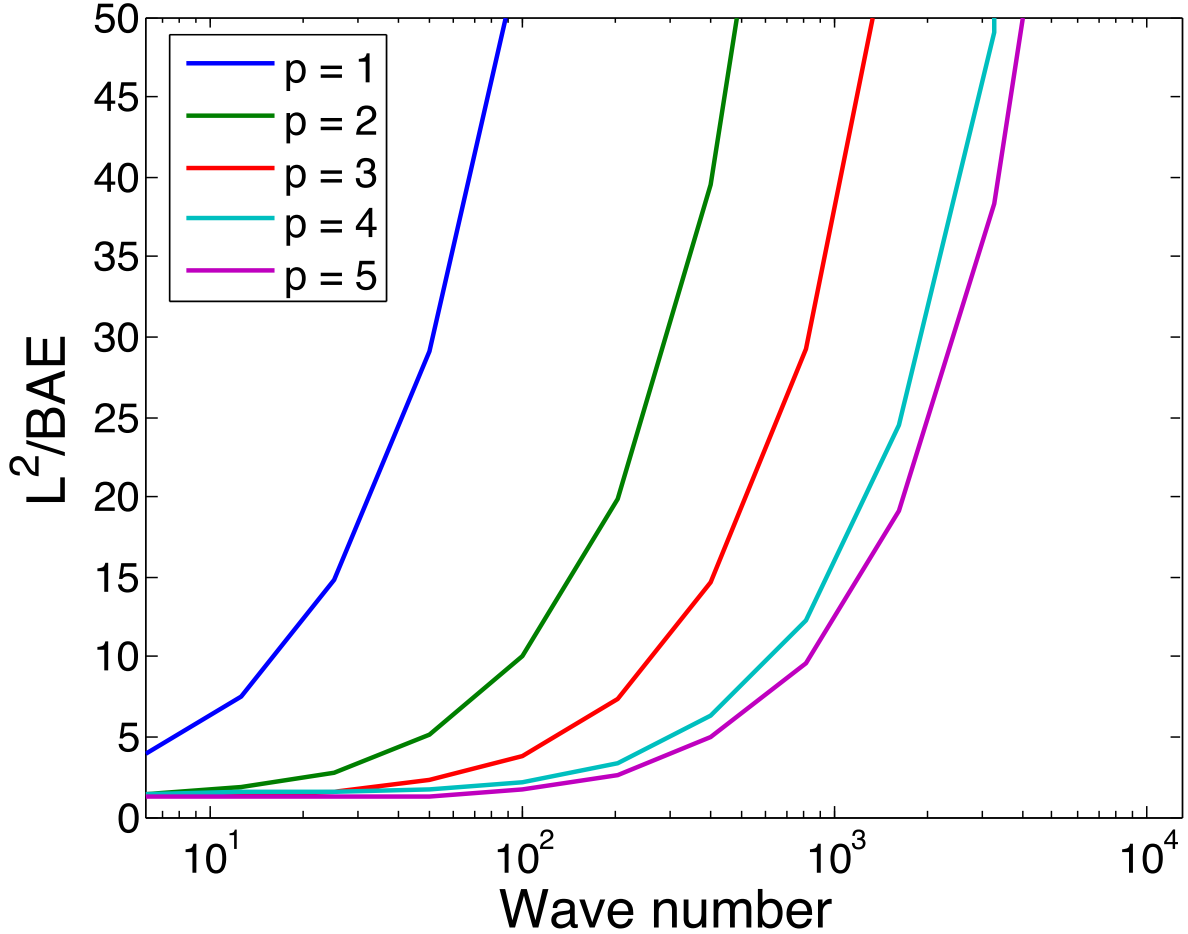
# Dispersion Error and Pollution



# Pollution: FEM

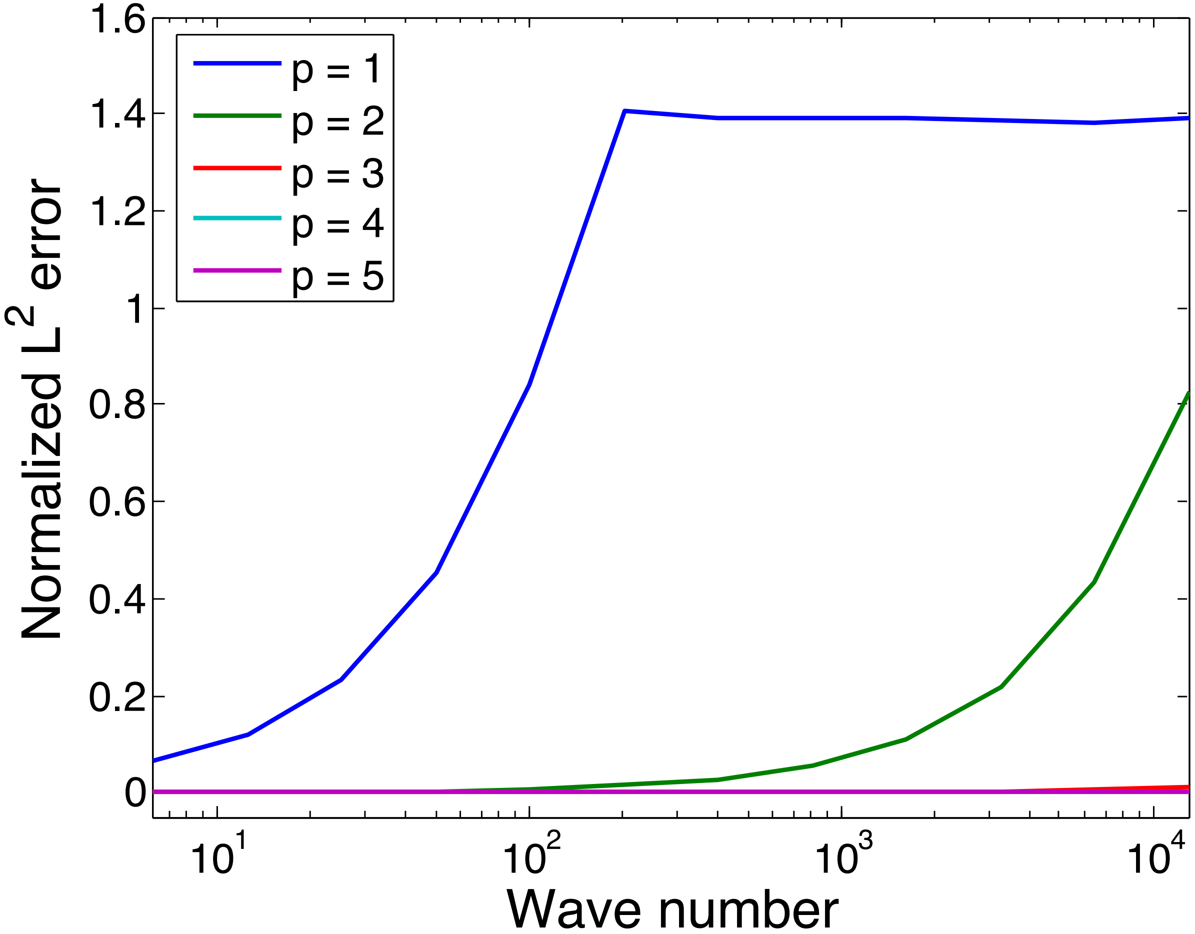


# Pollution: FEM

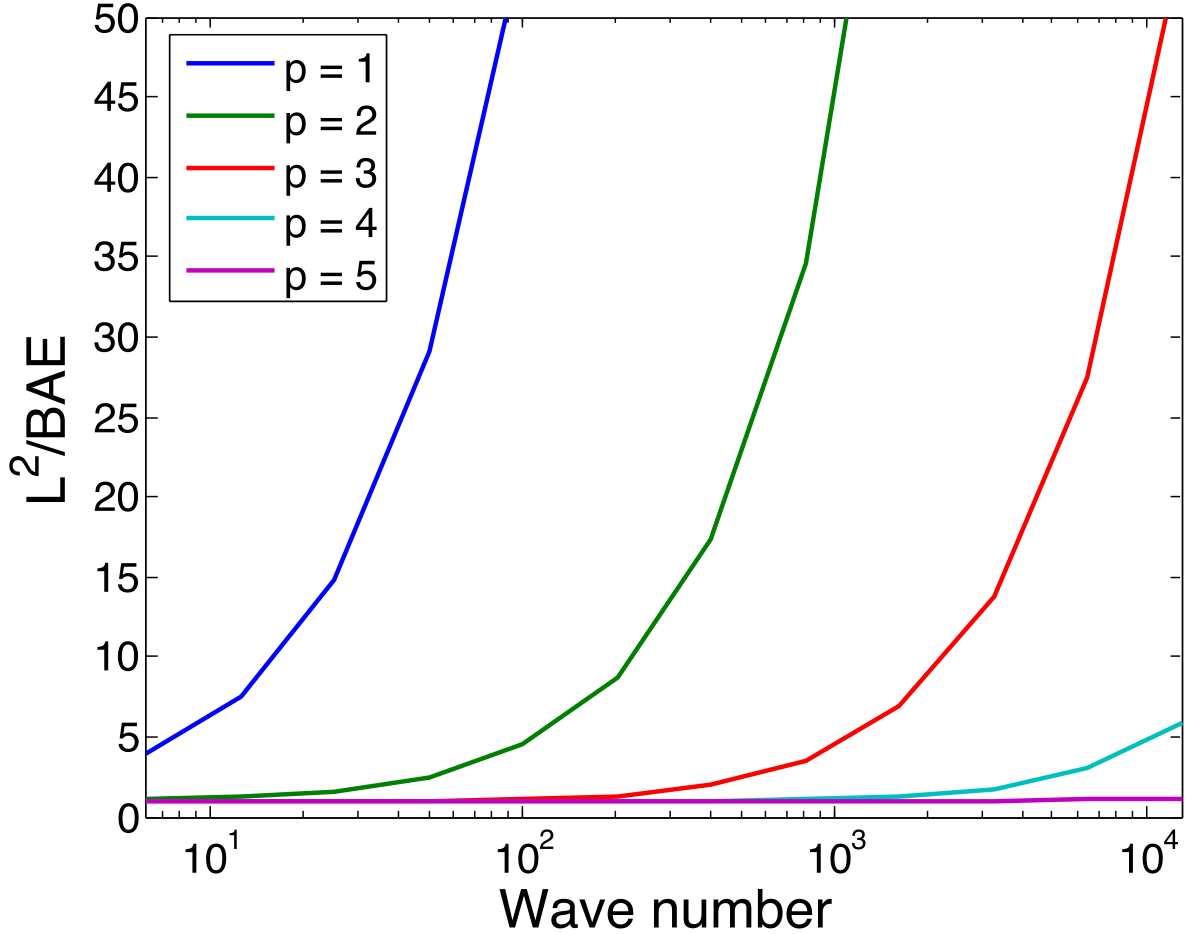



BAE: Best Approximation Error

# Pollution: NURBS

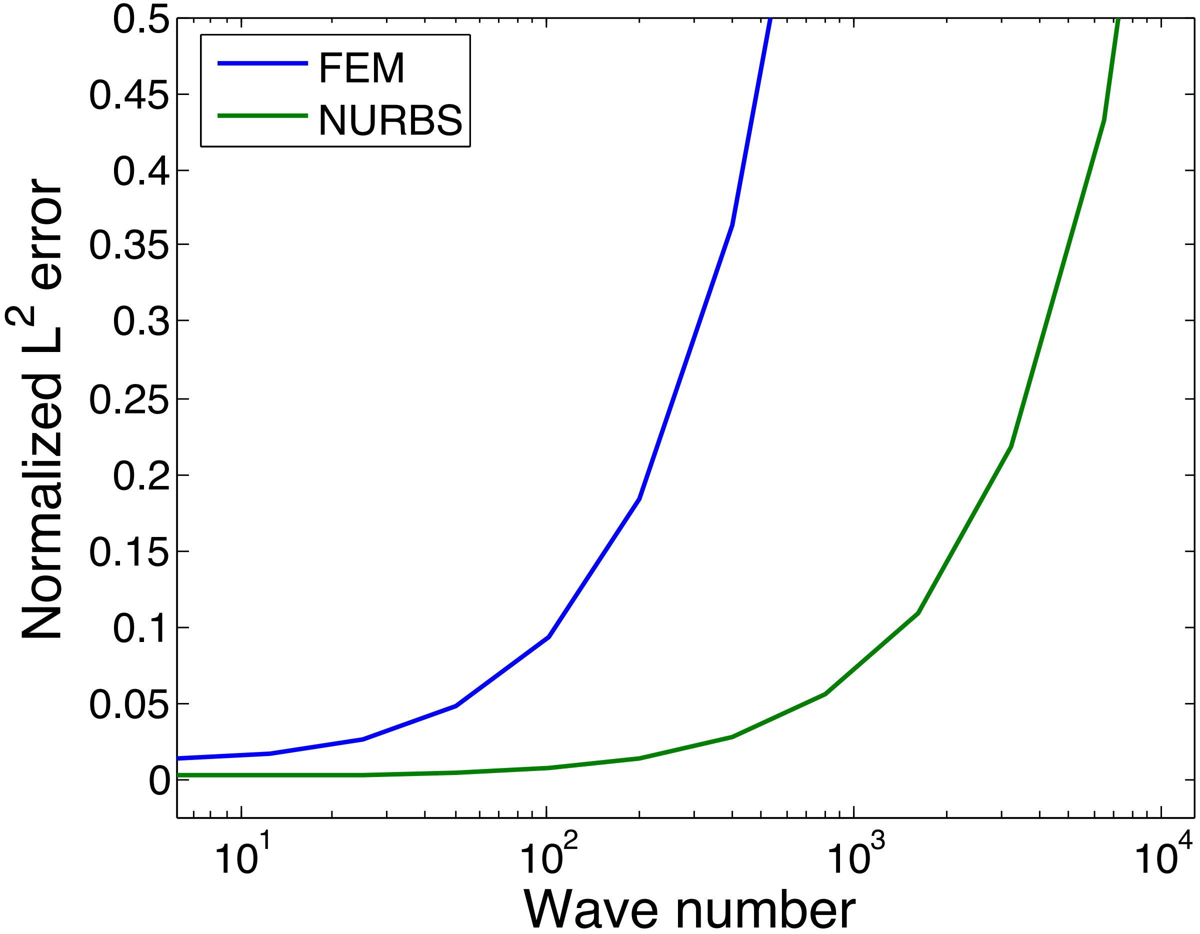


# Pollution: NURBS

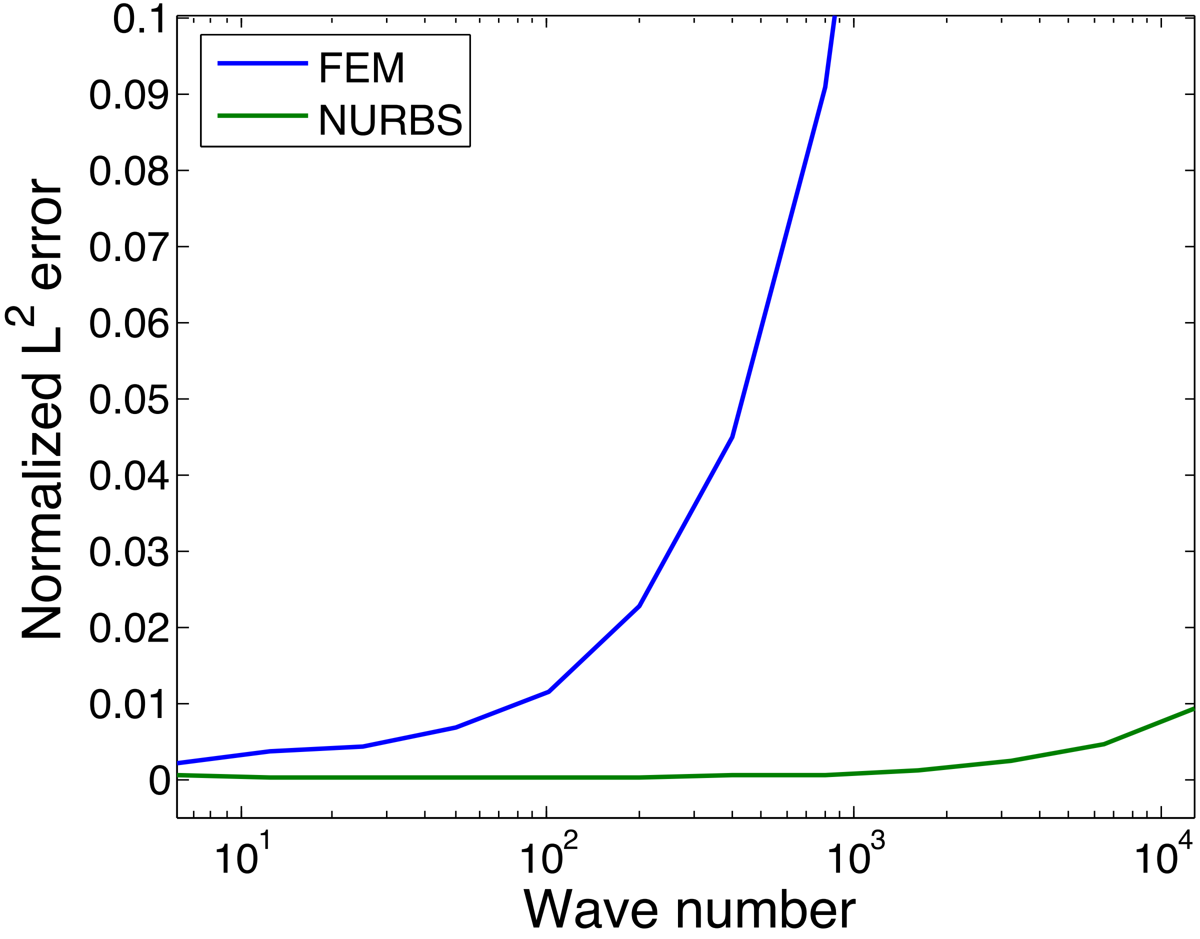



BAE: Best Approximation Error

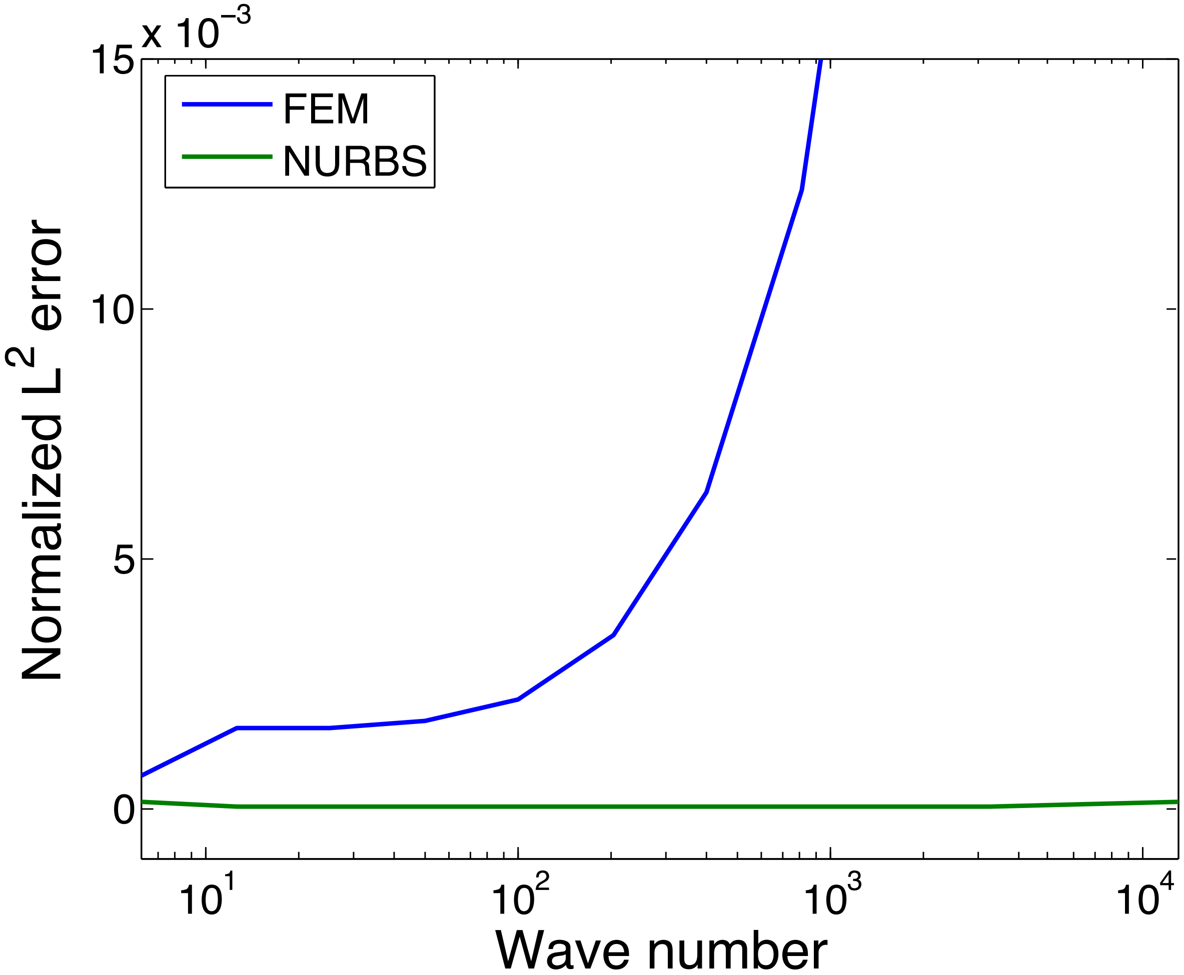
# Pollution: Degree 2 Comparison



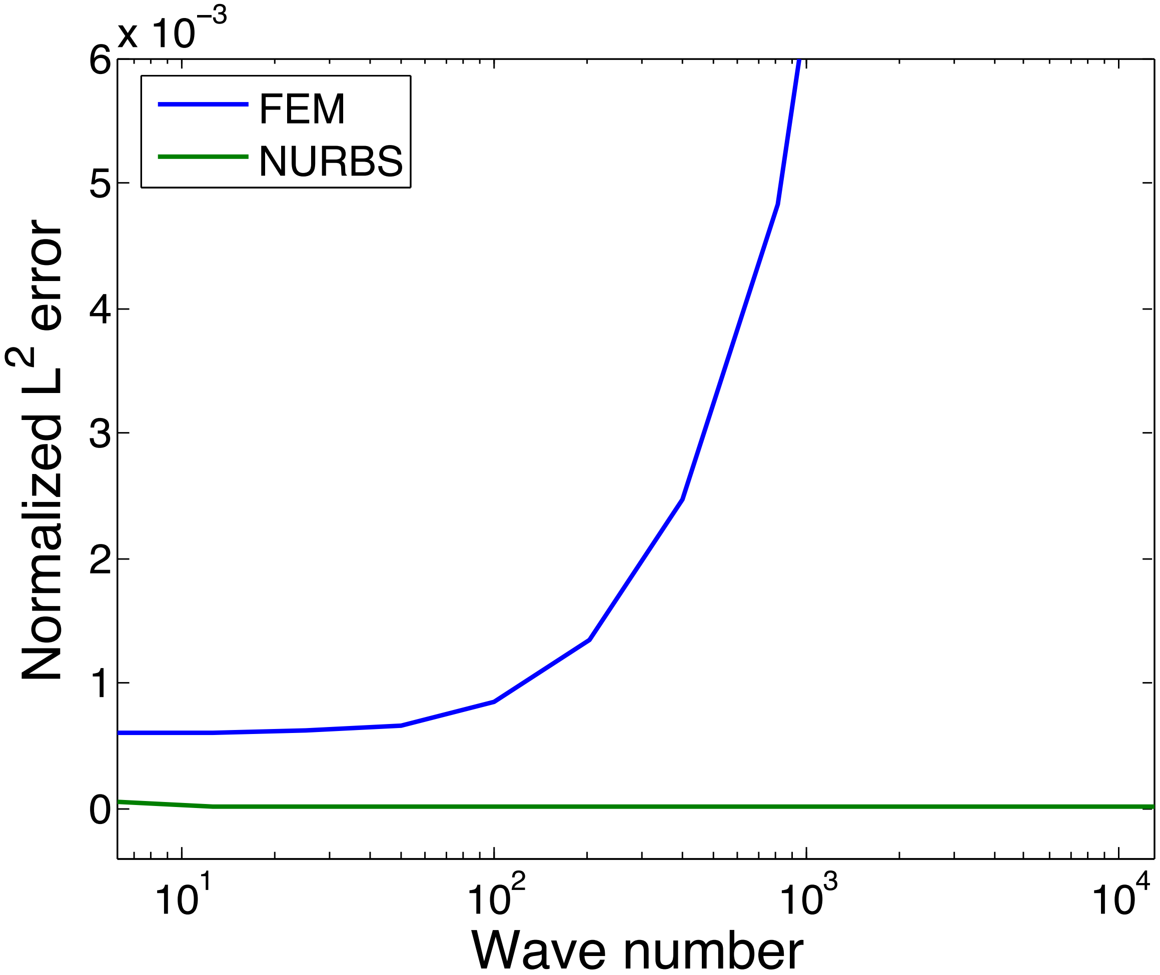
# Pollution: Degree 3 Comparison



# Pollution: Degree 4 Comparison



# Pollution: Degree 5 Comparison



# Tabulated Errors at $k = 4112\pi$

| Method | Normalized L2 Error | L2 Error / BAE |
|---|---|---|
| FEM, p = 1 | 1.39e0 | 90.21 |
| NURBS, p = 1 | 1.39e0 | 90.21 |
| FEM, p = 2 | 1.38e0 | 149.32 |
| NURBS, p = 2 | 8.27e-1 | 524.74 |
| FEM, p = 3 | 1.24e0 | 399.15 |
| NURBS, p = 3 | 9.16e-2 | 54.74 |
| FEM, p = 4 | 1.96e-1 | 195.31 |
| NURBS, p = 4 | 1.05e-4 | 5.81 |
| FEM, p = 5 | 7.67e-2 | 152.57 |
| NURBS, p = 5 | 2.31e-6 | 1.17 |