Linear Elasticity: Alternative Constructions of the ID Array

Before proceeding, let us return to the topic of the ID array. In particular, we are interested in its construction. Recall that we have:

$$P = ID(A, i)$$

Equation No.

DOF No.

Basis Function No.

Unknown No.

$$Q = ID(B, j)$$

We have two obvious means of constructing the ID array: (i) grouping together unknowns by basis function number, and (ii) grouping together unknowns by DOF number.

Option 1: Grouping Unknowns by Basis Function Number

In this setting, the displacement vector takes the following form:

$$\underline{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{2n-1} \\ d_{2n} \end{bmatrix} = \begin{bmatrix} (\vec{d}_1) \\ (\vec{d}_2) \\ \vdots \\ (\vec{d}_n) \end{bmatrix} \rightarrow \quad ID(B,j) = \overset{(j-1)*d+B}{\cancel{(d-1)*j + B}}$$

$$j = 1, \ldots, n$$
$$B = 1, \ldots, d$$

The corresponding stiffness matrix looks like:

$$\underline{\underline{K}} = \begin{bmatrix} (\underline{\underline{k}})_{11} & (\underline{\underline{k}})_{12} & \cdots & (\underline{\underline{k}})_{1n} \\ (\underline{\underline{k}})_{21} & (\underline{\underline{k}})_{22} & \cdots & (\underline{\underline{k}})_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (\underline{\underline{k}})_{n1} & (\underline{\underline{k}})_{n2} & \cdots & (\underline{\underline{k}})_{nn} \end{bmatrix}$$

where $\underline{\underline{k}}_{ij}$ is a $d \times d$ matrix associated with basis functions i and j. Option 1 leads to a stiffness matrix with enhanced sparsity properties (i.e., a tighter bandwidth) than the stiffness matrix resulting from Option 2.

Option 2: Grouping Unknowns by DOF Number

In this setting, the displacement vector takes the following form:

$$\underline{d} = \begin{bmatrix} d_1 \\ \vdots \\ \underline{d_n} \\ d_{n+1} \\ \vdots \\ d_{2n} \end{bmatrix} = \begin{bmatrix} (\vec{d}_1)_1 \\ \vdots \\ \underline{(\vec{d}_n)_1} \\ (\vec{d}_1)_2 \\ \vdots \\ (\vec{d}_n)_2 \end{bmatrix} = \begin{bmatrix} \underline{d}_1 \\ \hline \underline{d}_2 \end{bmatrix} \rightarrow ID(B,j) = \overset{(B-1)*n+j}{\cancel{(n-1)*B+j}}$$

$j = 1, \ldots, n$
$B = 1, \ldots, d$

The corresponding stiffness matrix looks like:

$$\underline{\underline{K}} = \begin{bmatrix} \underline{\underline{K}}_{11} & \underline{\underline{K}}_{12} \\ \underline{\underline{K}}_{21} & \underline{\underline{K}}_{22} \end{bmatrix}$$

where $\underline{\underline{K}}_{AB}$ is an nxn matrix associated with DOF A and B. The global system looks like:

$$\begin{bmatrix} \underline{\underline{K}}_{11} & \underline{\underline{K}}_{12} \\ \underline{\underline{K}}_{21} & \underline{\underline{K}}_{22} \end{bmatrix} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} = \begin{bmatrix} \underline{F}_1 \\ \underline{F}_2 \end{bmatrix}$$

so we see that Option 2 leads to a physics-based ordering scheme for the matrix system emanating from Galerkin's method. The resulting linear system can be efficiently solved using physics-based linear solvers and preconditioners.