Isogeometric Methods: Homework #4

Problem 1:

Write a MATLAB function which solves the two-dimensional linear elasticity problem for a single-patch geometry. The body is assumed to be homogeneous and isotropic. Your function should take the form:

```
function [d] = Linear_Elasticity(p_1,p_2,n_1,n_2,Xi_1,Xi_2,P,w,n_q,problem)
```

where `d` is an array storing the control variables for the discrete displacement field, `p_1` and `p_2` are the polynomial degrees in directions ξ_1 and ξ_2 respectively, `n_1` and `n_2` are the number of basis functions in directions ξ_1 and ξ_2, `Xi_1` and `Xi_2` are the univariate knot vectors in directions ξ_1 and ξ_2, `P` is an array storing the control points for the NURBS surface, `w` is an array storing the weights for the NURBS surface, `n_q` is the number of quadrature points in each direction, and `problem` is an integer corresponding to a problem specification. The Young's modulus E, Poisson ratio ν, prescribed body force $\mathbf{f}$, prescribed boundary displacements g_1 and g_2, prescribed boundary tractions h_1 and h_2, Dirichlet boundaries Γ_{D_1} and Γ_{D_2}, and Neumann boundaries Γ_{N_1} and Γ_{N_2} should all be determined by the integer `problem`. Moreover, the integer `problem` should determine whether the body is subject to plane strain or plane stress. For this homework, you should have three possible values for the integer `problem`, namely:

$$\texttt{problem} = \begin{cases} 1 & \text{for Problem 3} \\ 2 & \text{for Problem 4, Part 1} \\ 3 & \text{for Problem 4, Part 2} \end{cases}$$

Problem 2:

Write a MATLAB function which plots displacement or stress field contour lines over the deformed physical geometry. Your function should allow for the displacement field to be amplified for ease of visualization. That is, your function should plot displacement or stress field contour lines over the geometry defined by:

$$\mathbf{x}_{amp}(\boldsymbol{\xi}) = \sum_{i=1}^{n} \left(\mathbf{P}_i + a\mathbf{d}_i\right) N_i(\boldsymbol{\xi})$$

where a is a user-defined amplification factor. Your function should take the form:

```
function Plot_Elasticity(p_1,p_2,n_1,n_2,Xi_1,Xi_2,P,w,d,E,nu,state,field,amp)
```

where `p_1` and `p_2` are the polynomial degrees in directions ξ_1 and ξ_2 respectively, `n_1` and `n_2` are the number of basis functions in directions ξ_1 and ξ_2, `Xi_1` and `Xi_2` are the univariate knot vectors in directions ξ_1 and ξ_2, `P` is an array storing the control points for the NURBS surface, `w` is an array storing the weights for the NURBS surface, `d` is an array storing the control variables for the discrete displacement field, `E` is the Young's modulus, `nu` is the Poisson ratio, `state` is an integer determining whether the body is subject to plane strain or plane stress (i.e., `state` $= 1$ if the body is in plane strain and `state` $= 2$ if the body is in plane stress), `field` is an integer determining the displacement or stress field to be visualized, and `amp` is the displacement amplification factor

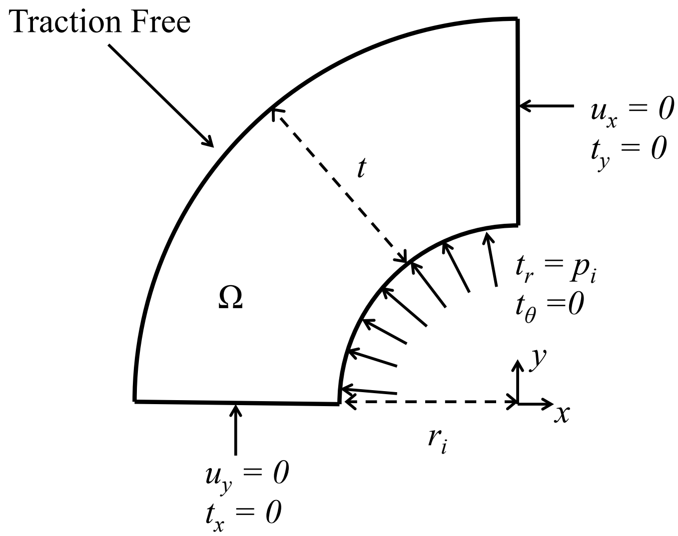

Figure 1: Setup for Problem 3.

a. For this homework, you should have ten possible values for the integer `field`, namely:

$$
\texttt{field} = \begin{cases}
1 & \text{for plotting the } x\text{-displacement field} \\
2 & \text{for plotting the } y\text{-displacement field} \\
3 & \text{for plotting the magnitude of the displacement field} \\
4 & \text{for plotting the } \sigma_{xx} \text{ stress field} \\
5 & \text{for plotting the } \sigma_{xy} \text{ stress field} \\
6 & \text{for plotting the } \sigma_{yy} \text{ stress field} \\
7 & \text{for plotting the } \sigma_{zz} \text{ stress field} \\
8 & \text{for plotting the maximum in-plane principal stress field} \\
9 & \text{for plotting the minimum in-plane principal stress field} \\
10 & \text{for plotting the von Mises stress field}
\end{cases}
$$

For reference, recall that the von Mises stress field is equal to:

$$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2\right)\right]}$$

Problem 3:

In this problem, you will solve the problem of linear elastic deformation of a thick walled steel cylindrical pressure vessel subject to internal pressure loading. Assume that the ends of the cylinder are capped, resulting in a plane strain stress state. The resulting problem definition is as depicted in Figure 1. Symmetries have been exploited to reduce the problem size by a factor of four. For the problem in consideration, the physical and geometric parameters are as follows:

$$
\begin{aligned}
E &= 200 \text{ GPa} \\
\nu &= 0.3 \\
p_i &= 30 \text{ MPa} \\
r_i &= 75 \text{ mm} \\
t &= 15 \text{ mm}
\end{aligned}
$$

This problem has an exact axisymmetric analytical solution, which may be expressed in polar coordinates as follows:

$$\begin{aligned}
\sigma_{rr}(r) &:= -p_i \left[\frac{(r_o/r)^2 - 1}{(r_o/r_i)^2 - 1} \right] \\
\sigma_{\theta\theta}(r) &:= p_i \left[\frac{(r_o/r)^2 + 1}{(r_o/r_i)^2 - 1} \right] \\
\sigma_{r\theta}(r) &:= 0 \\
u_r(r) &:= p_i \left[\frac{1+\nu}{E} \right] \left[\frac{r}{(r_o/r_i)^2 - 1} \right] \left[(1-2\nu) + (r_o/r)^2 \right] \\
u_\theta(r) &:= 0
\end{aligned}$$

where $r_o = r_i + t$. Solve the linear elasticity problem detailed above using the MATLAB function `Linear_Elasticity` you wrote for Problem 1. Plot both components of the resulting displacement field and the two in-plane principal stress fields over the deformed physical geometry using the MATLAB function `Plot_Elasticity` you wrote for Problem 2. You may choose any value you desire for the amplification factor `amp`. Confirm that the exact solution is obtained under the limit of mesh refinement (i.e., uniform knot insertion) for both polynomial degrees $p = 2$ and $p = 3$. To do so, it is sufficient to show that the L^2-norm of the error for both components of the displacement field, i.e.,

$$\begin{aligned}
\|u_x - u_x^h\|_{L^2(\Omega)} &= \left(\int_\Omega \left(u_x(x,y) - u_x^h(x,y) \right)^2 dxdy \right)^{1/2} \\
\|u_y - u_y^h\|_{L^2(\Omega)} &= \left(\int_\Omega \left(u_y(x,y) - u_y^h(x,y) \right)^2 dxdy \right)^{1/2}
\end{aligned}$$

goes to zero in the limit of mesh refinement. What is the rate of convergence in terms of the number of degrees of freedom?

Problem 4:

In this problem, you will solve the problem of linear elastic deformation of a high strength concrete arch tunnel subject to a vertical pressure loading along its top surface and a homogeneous vertical body force due to gravity. Assume that the tunnel is much longer than its width or height, resulting in a plane strain stress state. The tunnel opening shape is assumed to be a half circle. The resulting problem definition is as depicted in Figure 2. For the problem in consideration, the physical and geometric parameters are as follows:

$$\begin{aligned}
E &= 30 \text{ GPa} \\
\nu &= 0.2 \\
\rho &= 2400 \text{ kg/m}^3 \\
p &= 300 \text{ kPa} \\
r_t &= 5 \text{ m} \\
l_a &= 3 \text{ m} \\
h_c &= 1.5 \text{ m}
\end{aligned}$$

We will consider two different prescriptions of boundary conditions in this problem, corresponding to Parts 1 and 2 respectively.

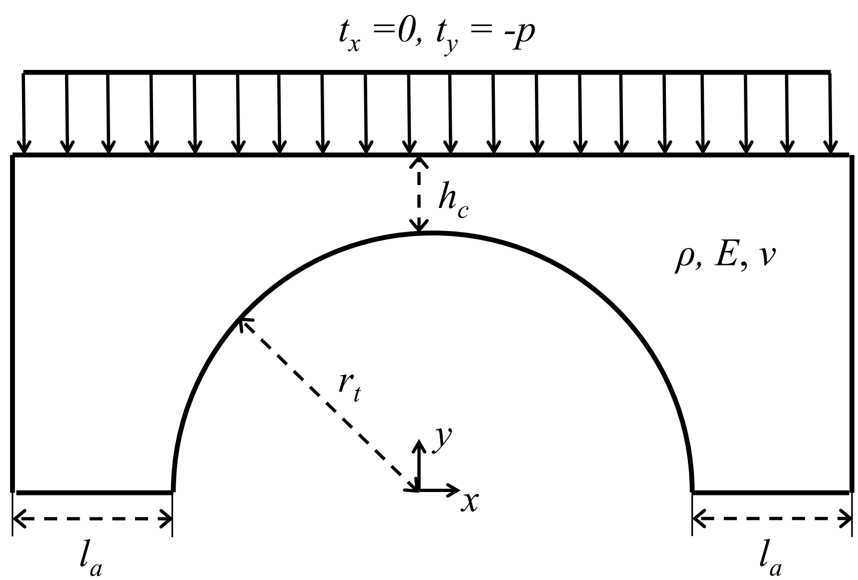

Figure 2: Setup for Problem 4.

Part 1: Assume that the left, right, and bottom surfaces are fixed and the tunnel opening is non-pressurized. Then the resulting boundary condition specification is as depicted in Figure 3. Solve the resulting linear elasticity problem using the MATLAB function `Linear_Elasticity` you wrote for Problem 1. Plot both components of the resulting displacement field and the von Mises stress field over the deformed physical geometry using the MATLAB function `Plot_Elasticity` you wrote for Problem 2. You may choose any value you desire for the amplification factor `amp`. Moreover, determine the *maximum* von Mises stress in the arch. Explain your methodology for determining your results are sufficiently accurate.

Part 2: Now assume that the left, right, and bottom surfaces are on "rollers" and the tunnel opening is non-pressurized. Then the resulting boundary condition specification is as depicted in Figure 4. Solve the resulting linear elasticity problem using the MATLAB function `Linear_Elasticity` you wrote for Problem 1. Plot both components of the resulting displacement field and the von Mises stress field over the deformed physical geometry using the MATLAB function `Plot_Elasticity` you wrote for Problem 2. As in Part 1, determine the maximum von Mises stress in the arch, and compare your results with the results of Part 1.

Bonus: Determine if the geometrical shape of the tunnel opening is optimal in terms of *minimizing* the maximum von Mises stress while still allowing a centerline vertical clearance of 5 meters and a ground-level horizontal clearance of 10 meters. Use the boundary condition specification given in Part 2.

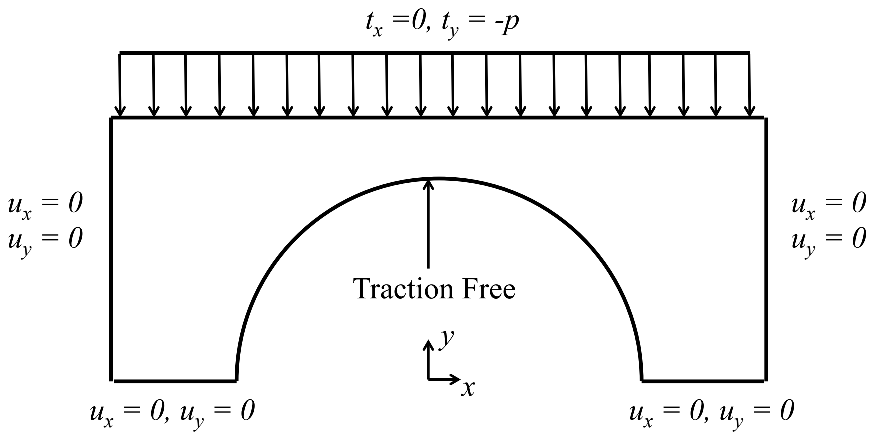

Figure 3: Boundary condition specification for Problem 4 Part 1.

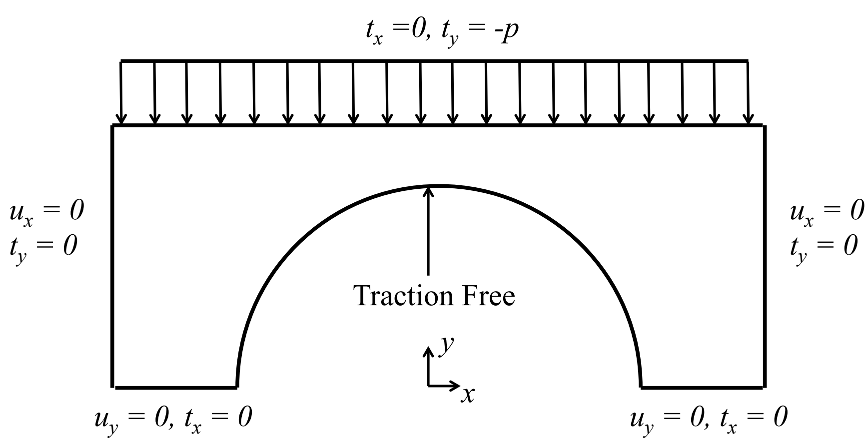

Figure 4: Boundary condition specification for Problem 4 Part 2.