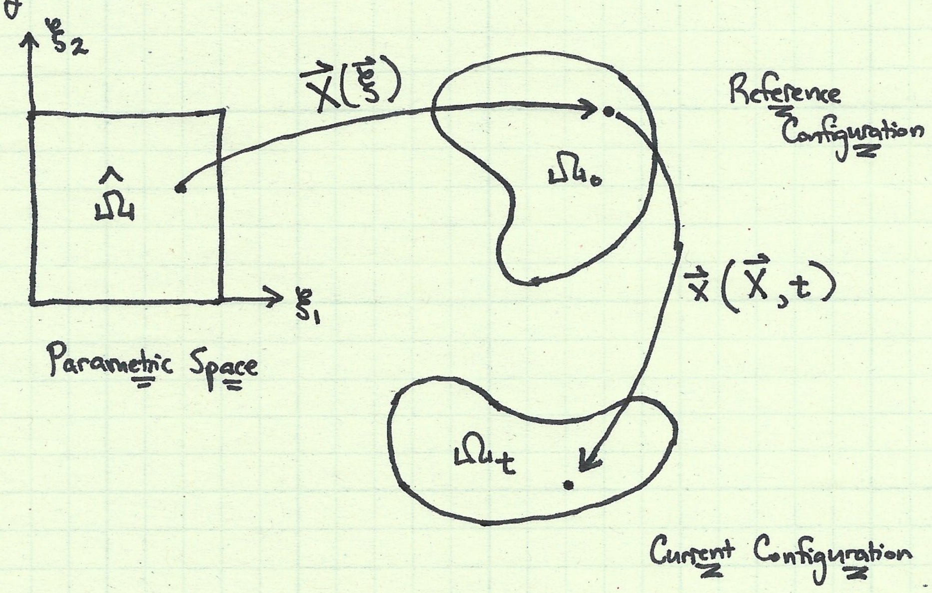
From Linear to Nonlinear: Dynamics of Hyperelastic Bodies:

We are now ready to jump from the class of linear problems to the class of nonlinear problems. As a particular test case, we discuss the application of isogeometric analysis to the study of the dynamics of a hyperelastic body.

In what follows, Ω_0 will denote the material configuration (i.e., the undeformed shape). We assume that Ω_0 is constructed via a NURBS mapping $\vec{X}: \hat{\Omega} \rightarrow \Omega_0$ of the form:

$$\vec{X}(\vec{\xi}) = \sum_{i=1}^{n} \underbrace{\hat{N}_i(\vec{\xi})}_{\text{NURBS Basis Functions}} \underbrace{\vec{P}_i}_{\text{Control Points}}$$

We use the notation Ω_t to denote the current configuration at time $t \in (0,T)$, and we define $\vec{x}(t): \Omega_0 \rightarrow \Omega_t$ to be the position vector which, given a reference position $\vec{X}$ at time $t=0$, returns the position of a particle initially located at $\vec{X}$ at time t. Visually:



Given the current and reference positions $\vec{x}(\vec{X},t)$ and $\vec{X}$, we define the displacement vector $\vec{u}(\vec{X},t)$ as the mapping $\vec{u}(t): \Omega_0 \rightarrow \mathbb{R}^d$ satisfying:

$$\vec{u}(\vec{X},t) = \underbrace{\vec{x}(\vec{X},t)}_{\text{Current position}} - \underbrace{\vec{X}}_{\text{Reference Position}}$$

The velocity and acceleration vectors are then defined as:

$$\vec{v}(\vec{X},t) = \vec{x}_{,t}(\vec{X},t) = \vec{u}_{,t}(\vec{X},t)$$

$$\vec{a}(\vec{X},t) = \vec{x}_{,tt}(\vec{X},t) = \vec{u}_{,tt}(\vec{X},t)$$

Finally, we define the deformation gradient $\vec{\vec{F}}(t) : \Omega_0 \rightarrow \mathbb{R}^{d \times d}$ as:

$$\vec{\vec{F}}(\vec{X},t) = \frac{\partial \vec{x}}{\partial \vec{X}}(\vec{X},t) = \underbrace{\vec{\vec{I}}}_{\text{Identity Tensor}} + \frac{\partial \vec{u}}{\partial \vec{X}}(\vec{X},t)$$

and denote its determinant as $J = \det(\vec{\vec{F}})$. Note that $J(0) = 1$, if $\vec{u}(0) = \vec{0}$.

It will be easier to express the equations of hyperelasticity using vector notation as it will allow us to differentiate between material and spatial derivatives. With this in mind, the strong form of the initial-boundary value problem governing the dynamics of a hyperelastic body is:

$$(S)\left\{\begin{array}{lll} \text{Find } \vec{u}(t) : \overline{\Omega}_0 \rightarrow \mathbb{R}^d \text{ for all } t \in [0,T] \text{ s.t.} & & \\ \rho_0 \vec{u}_{,tt} = \vec{\nabla}_{\vec{X}} \cdot (\vec{\vec{P}}) + \vec{f}_0 & & \text{in } \Omega_0 \times (0,T) \\ u_A = g_A & & \text{on } \Gamma_{0,D_A} \times (0,T) \\ & & A = 1,\ldots,d \\ P_{AB} N_B = h_A & & \text{on } \Gamma_{0,N_A} \times (0,T) \\ & & A = 1,\ldots,d \\ \vec{u}(\vec{X},0) = \vec{0} & & \text{in } \Omega_0 \\ \vec{u}_{,t}(\vec{X},0) = \vec{v}_0(\vec{X}) & & \text{in } \Omega_0 \end{array}\right.$$

where:

ρ_0 = The density of the body in the material configuration $\rho_0 : \Omega_0 \rightarrow \mathbb{R}^+$

$\vec{\vec{P}}$ = The first Piola-Kirchhoff stress tensor

$\vec{f}_0$ = The prescribed body force per unit volume in the material configuration

g_A = The prescribed boundary displacement in the material configuration

h_A = The prescribed boundary traction in the material configuration

$\vec{v}_0$ = The prescribed initial velocity

The first Piola-Kirchhoff stress $\vec{\vec{P}}$ is related to the true Cauchy stress $\vec{\vec{\sigma}}$ in the current configuration via the Piola transform:

$$\vec{\vec{P}}(\vec{X},t) = J(\vec{X},t)\, \vec{\vec{\sigma}}(\vec{x}(\vec{X},t),t)\, \vec{\vec{F}}^{-T}(\vec{X},t)$$

The boundary traction in the material configuration is related to the true traction in the current configuration in an analogous fashion. It should be noted that $\vec{N}$ denotes the normal in the material configuration while $\vec{n}$ denotes the normal in the current configuration. We have:

$$\vec{n} = \frac{J \vec{\vec{F}}^{-T} \vec{N}}{\| J \vec{\vec{F}}^{-T} \vec{N} \|} \quad \Rightarrow \quad \vec{\vec{\sigma}} \vec{n} = \frac{\vec{\vec{P}} \vec{N}}{\| J \vec{\vec{F}}^{-T} \vec{N} \|}$$

Thus, it holds that:

$$\int_{\Gamma_0^2} \vec{\vec{P}} \vec{N} \, d\Gamma_0^2 = \int_{\Gamma_t^2} \vec{\vec{\sigma}} \vec{n} \, d\Gamma_t^2$$

The first Piola-Kirchoff stress tensor is generally a nonlinear function of the displacement vector. Typically, we model the second Piola-Kirchhoff stress tensor:

$$\vec{\vec{S}} = \vec{\vec{F}}^{-1} \vec{\vec{P}} = J \vec{\vec{F}}^{-1} \vec{\vec{\sigma}} \vec{\vec{F}}^{-T}$$

as it is a symmetric tensor analogous to the Cauchy stress tensor. Common models include the Fung, Mooney-Rivlin, Ogden, Saint Venant-Kirchoff, and Neo-Hookean models. Hyperelastic bodies are ones in which the stress-strain relationship derives from a strain energy density function. For example, neo-Hookean models express $\vec{\vec{S}}$ as the gradient of an elastic potential:

$$\vec{\vec{S}} = 2 \frac{\partial \Psi}{\partial \vec{\vec{C}}}$$

where $\vec{\vec{C}} = \vec{\vec{F}}^T \vec{\vec{F}}$ is the Cauchy-Green deformation tensor. While the proceeding discussion is general in nature, we will employ the above model in discussing implementational details.

The weak form of the hyperelastic problem is constructed in the usual fashion: multiply the strong form by a weighting function, integrate over the domain, integrate by parts, and enforce Dirichlet boundary conditions strongly and Neumann boundary conditions weakly. The resulting formulation is as follows:

$$(W) \begin{cases} \text{Find } \vec{u}(t) \in \mathcal{S}_t \text{ s.t.} & \\ \eta(\vec{w}; \vec{u}, \vec{u}_{,tt}) = L(\vec{w}) & \forall \vec{w} \in \mathcal{V} \\ \vec{u}(0) = \vec{0} & \\ (\vec{w}, \rho \vec{u}_{,t}(0))_{\Omega_0} = (\vec{w}, \rho \vec{v}_0)_{\Omega_0} & \forall \vec{w} \in \mathcal{V} \end{cases}$$

where:

$$\mathcal{S}_t := \{ \vec{u}(\cdot, t) \in (H^1(\Omega_0))^d : u_A(\vec{X}, t) = g_A(\vec{X}, t) \text{ for } \vec{X} \in \Gamma_{0,D_A}^2 \}$$

$$\mathcal{V} := \{ \vec{w} \in (H^1(\Omega_0))^d : w_A(\vec{X}) = 0 \text{ for } \vec{X} \in \Gamma_{0,D_A}^2 \}$$

$$\eta(\vec{w}; \vec{u}, \vec{u}_{,tt}) := (\vec{w}, \rho_0 \vec{u}_{,tt})_{\Omega_0} + (\vec{\nabla}_{\vec{X}} \vec{w}, \vec{\vec{P}})_{\Omega_0}$$

$$L(\vec{w}) := (\vec{w}, \vec{f}_0)_{\Omega_0} + \sum_{A=1}^{d} (w_A, h_A)_{\Gamma_{0,N_A}^2}$$

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More explicitly:

$$\eta(\vec{w};\vec{u},\vec{u}_{,tt}) = \int_{\Omega_0} \vec{w}\cdot(\rho_0 \vec{u}_{,tt})\, d\Omega_0 + \int_{\Omega_0} (\vec{\nabla}_{\vec{X}}\vec{w}) : \vec{\vec{P}}\, d\Omega_0 \;{}^{*}$$

$$L(\vec{w}) = \int_{\Omega_0} \vec{w}\cdot\vec{\mathcal{F}}_0 \, d\Omega_0 + \sum_{A=1}^{d} \int_{\Gamma_{0,N_A}} w_A h_A \, d\Gamma_{0,N_A}$$

A Galerkin semi-discrete formulation of (W) is now obvious. Namely, we seek an approximate displacement field of the form:

$$\vec{u}^h(\vec{X},t) = \sum_{i=1}^{n} N_i(\vec{X})\, \vec{d}_i(t)$$

As in the setting of linear elastodynamics, we write $\vec{u}^h(t) \in \mathcal{S}_t$ where $\mathcal{S}_t \subset \mathcal{S}$ is the set of discrete trial solutions and we denote the space of discrete test functions as $\mathcal{V}^h \subset \mathcal{V}$. The resulting semi-discrete method is as follows:

$$(G)\begin{cases} \text{Find } \vec{u}^h(t) \in \mathcal{S}_t^h \text{ s.t.} \\ \eta(\vec{w}^h;\vec{u}^h,\vec{u}^h_{,tt}) = L(\vec{w}^h) \qquad \forall \vec{w}^h \in \mathcal{V}^h \end{cases}$$

$\star$ It should be remarked that here we have defined:

$$(\vec{\nabla}_{\vec{X}}\vec{w}) : \vec{\vec{P}} = \frac{\partial w_A}{\partial X_B} P_{AB}$$