Rational B-splines

While B-spline curves, surfaces, and solids comprise a wide set of geometrical objects, many geometries of engineering interest cannot be exactly represented as a B-spline. However, all conic sections (hyperbolas, parabolas, and ellipses) may be represented using a rational quadratic curve, and all quadric surfaces (ellipsoids, spheroids, paraboloids, hyperboloids, cones, and cylinders) also admit a rational quadratic parametrization. It is precisely for this reason why Non-Uniform Rational B-Splines, or NURBS for short, were invented.

NURBS basis functions are readily built from standard B-splines. Namely, given a set of d_s-dimensional B-spline basis functions and a set of weights:

$$w_i \in \mathbb{R}^+ \qquad \text{for } i \in I$$

we define a corresponding set of d_s-dimensional NURBS basis functions as:

$$R_{i,p}(\vec{\xi}) = \frac{w_i N_{i,p}(\vec{\xi})}{w(\vec{\xi})} \qquad \text{for } i \in I$$

where w is the so-called weighting function:

$$w(\vec{\xi}) = \sum_{i \in I} \overset{i^{th}\text{ weights}}{w_i} N_{i,p}(\vec{\xi})$$

As B-spline basis functions comprise a partition of unity, NURBS basis functions simplify to standard B-spline basis functions when all the NURBS weights are equal to one.

Given a 1-D, 2-D, ~~and~~ or 3-D basis, we can then easily define a NURBS curve, surface, or solid as:

$$\vec{C}(\xi) = \sum_{i \in I} \vec{P}_i R_{i,p}(\xi) \qquad d_s = 1$$

$$\vec{S}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i R_{i,p}(\vec{\xi}) \qquad d_s = 2$$

$$\vec{V}(\vec{\xi}) = \sum_{i \in I} \vec{P}_i R_{i,p}(\vec{\xi}) \qquad d_s = 3$$

where $\{\vec{P}_i\}_{i \in I}$ is a corresponding set of control points $\vec{P}_i \in \mathbb{R}^d$.

To understand the influence of the NURBS weights, it helps to consider a concrete example. In what follows, we consider a one-dimensional NURBS basis corresponding to:

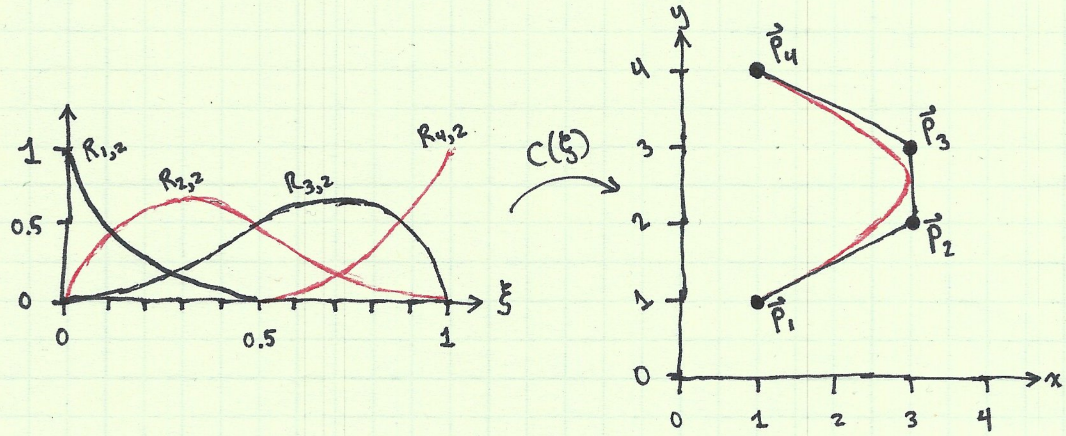
$$p = 2$$
$$n = 4$$
$$\vec{\Xi} = \{0, 0, 0, 0.5, 1, 1, 1\}$$
$$d = 2$$

$$w_1 = w_2 = w_4 = 1 \qquad w_3 \text{ varies}$$

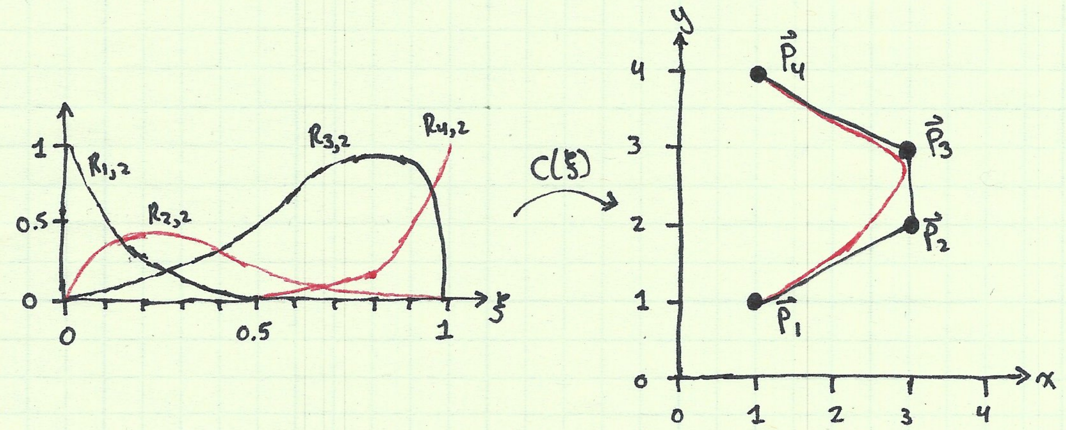
We also define a quadratic NURBS curve from the basis using control points:

$$\vec{P}_1 = (1,1)$$
$$\vec{P}_2 = (3,2)$$
$$\vec{P}_3 = (3,3)$$
$$\vec{P}_4 = (1,4)$$

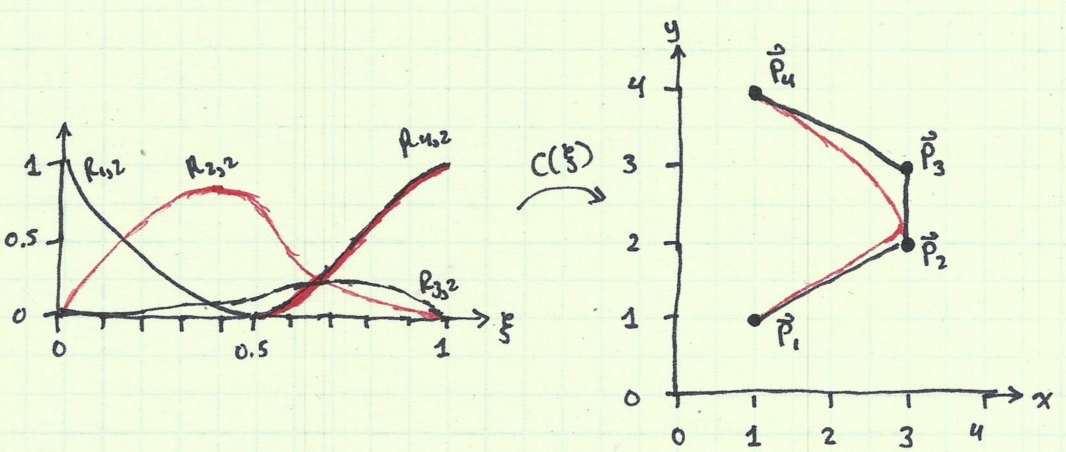
For $w_3 = 1, 5,$ and $1/5$, we have included plots of the resulting NURBS basis functions and curves below:



$w_3 = 1$



$w_3 = 5$

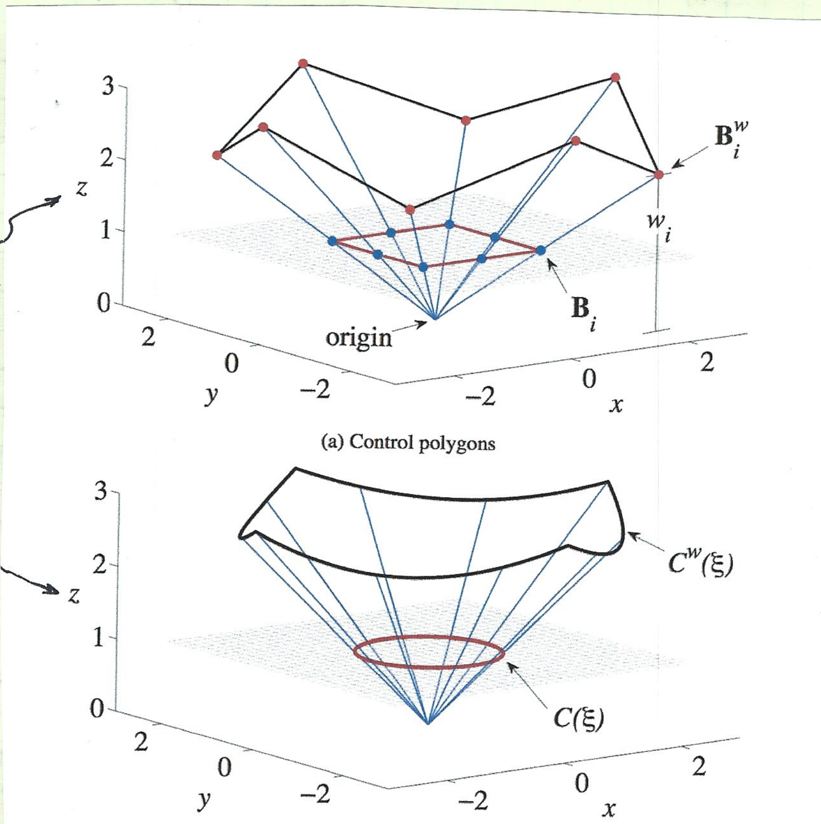


$w_3 = 1/5$

Note that increasing and decreasing w_3 increases and decreases the value of $R_{3,2}$, respectively. Moreover, increasing the value of w_3 pulls the curve toward $\vec{P}_3$, and when $w_3 \to \infty$, the curve will pass through $\vec{P}_3$. On the other hand, decreasing the value of w_3 pushes the curve away from $\vec{P}_3$, and when $w_3 = 0$, control point $\vec{P}_3$ has no impact on the curve.

To better understand the geometric structure of a NURBS curve, surface, or solid and the influence of the NURBS weights, it helps to view a NURBS entity in $\mathbb{R}^d$ as the projective transformation of a standard B-spline entity in $\mathbb{R}^{d+1}$. In what follows, we reference the below figure:

Axis associated with the weights



(a) Control polygons

(b) Curves

The above figure illustrates how a circle in $\mathbb{R}^2$ may be constructed by the projective transformation of a piecewise quadratic B-spline in $\mathbb{R}^3$. The transformation is applied by projecting every point in the curve onto the $z=1$ plane by a ray through the origin. We obtain the control points for the NURBS curve by performing exactly the same projective transformation to the control points of the B-spline curve. In this context, the B-spline, $\vec{C}^w(\xi)$, is called the projective curve with its associated projective control points, while the terms curve and control points are reserved for the NURBS objects $\vec{C}(\xi)$ and $\vec{P}_i$, respectively.

With a given B-spline curve $\vec{C}^w(\xi)$ and its associated projective control points $\vec{P}_i^w$, the control points and weights for a NURBS curve are obtained as follows:

$$(\vec{P}_i)_j = (\vec{P}_i^w)_j / w_i, \quad j = 1, \ldots, d$$

$$w_i = (\vec{P}_i^w)_{d+1}$$

and the NURBS curve is defined as:

$$(\vec{C}(\xi))_j = (\vec{C}^w(\xi))_j / w(\xi), \quad j = 1, \ldots, d$$

$$w(\xi) = (\vec{C}^w(\xi))_{d+1}$$

As suggested previously, we see that projective transformation of a piecewise polynomial curve does indeed result in a piecewise rational curve.

From the standpoint of projective geometry, we easily see that if the weight of a single control point is dragged toward infinity, then the NURBS curve is pulled toward that control point. Similarly, if the weight is dragged toward zero, then the NURBS curve is no longer influenced by that control point. However, if one were to multiply all of the projective control points by a constant (the simplest affine transformation of the projective curve), the resulting NURBS curve would be unchanged. This is because each point of the projective curve would move along its ray through the origin, but not onto a different ray. Consequently, while the projective transformation of a projective curve results in a unique NURBS curve, this transformation is not an injection.

It should be mentioned the previous discussion also applies to the construction of NURBS surfaces and volumes, but it is much more difficult to visualize projective transformation in such a setting.

At this point, we are ready to list the various properties of NURBS basis functions and geometric objects. These are listed below.

Properties of NURBS Basis Functions

- Positivity: $R_{i,p}(\vec{\xi}) \geq 0$ for $i \in I$
- Local Support: $R_{i,p}(\vec{\xi})$ is nonzero only over the subdomain $\bigotimes_{j=1}^{d_s} \left[\xi_{i_j}^{(j)}, \xi_{i_j+p_j+1}^{(j)} \right)$.
- At a knot of multiplicity k in direction l, basis function profiles along direction l are $C^{p_l - k}$-continuous.
- Partition of Unity: $\sum_{i \in I} R_{i,p}(\vec{\xi}) = 1$, for $\vec{\xi} \in \bigotimes_{j=1}^{d_s} \left[\xi_1^{(j)}, \xi_{n_j+p_j+1}^{(j)} \right)$
- If $w_i = c$ for all $i \in I$ where $c \in \mathbb{R}/0$, $R_{i,p} \equiv N_{i,p}$.

Properties of NURBS Curves, Surfaces, and Volumes

- Affine Covariance: To achieve an affine transformation of a NURBS object, simply apply the transformation to the control points, leaving the weights unchanged.
- Strong Convex Hull Property: The NURBS object is contained in the convex hull of its control points. Moreover, if:

$$\xi_j \in \left[\xi_{i_j}^{(j)}, \xi_{i_j+1}^{(j)} \right) \qquad j = 1, \dots, d_s$$

then the geometric object evaluated at $\vec{\xi}$ is contained in the convex hull of control points:

$$\vec{P}_{(i_1 - k_1, \dots, i_{d_s} - k_{d_s})}$$

where $k_j = 0, \dots, p_j$ with $j = 1, \dots, d_s$.

- Projective Invariance: If a projective transformation is applied to a NURBS curve, the result can be constructed from the projective images of its control points and corresponding weights.

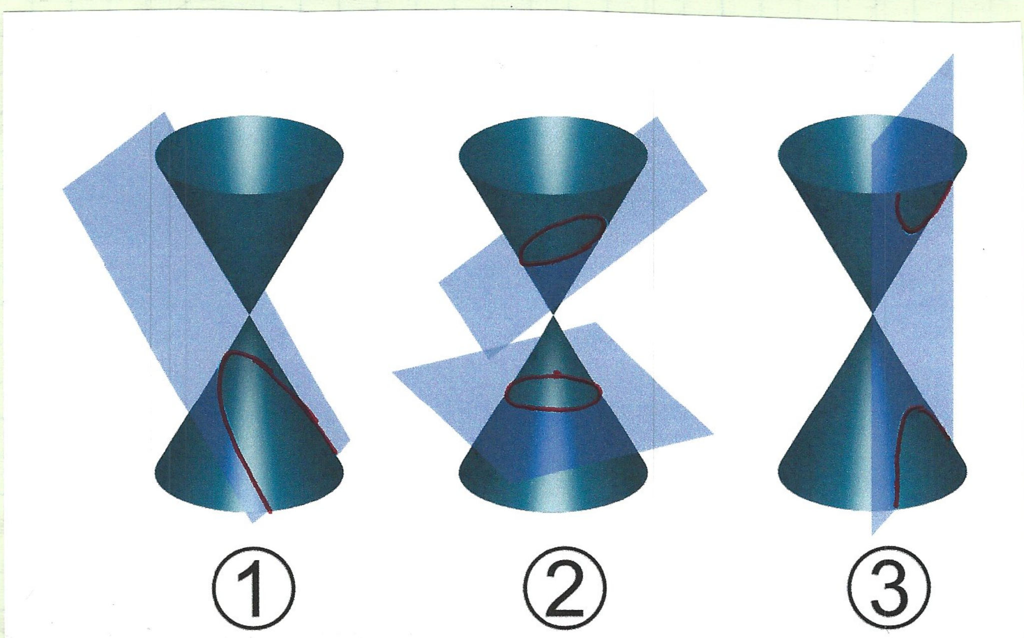
In addition, NURBS curve satisfy the following:

Properties of NURBS Curves

- Variation Diminishing Property.
- If a NURBS curve is built from an open knot vector, it satisfies the endpoint interpolation and tangency properties.

Consequently, we see that NURBS inherit all of the beneficial properties of B-splines and exhibit more flexibility.

At this stage, let us recall that all conic sections may be represented using a rational quadratic curve. How might one show this? To answer this question, let us recall the definition of a conic section. A conic section is simply a curve defined as the intersection of a cone with a plane:

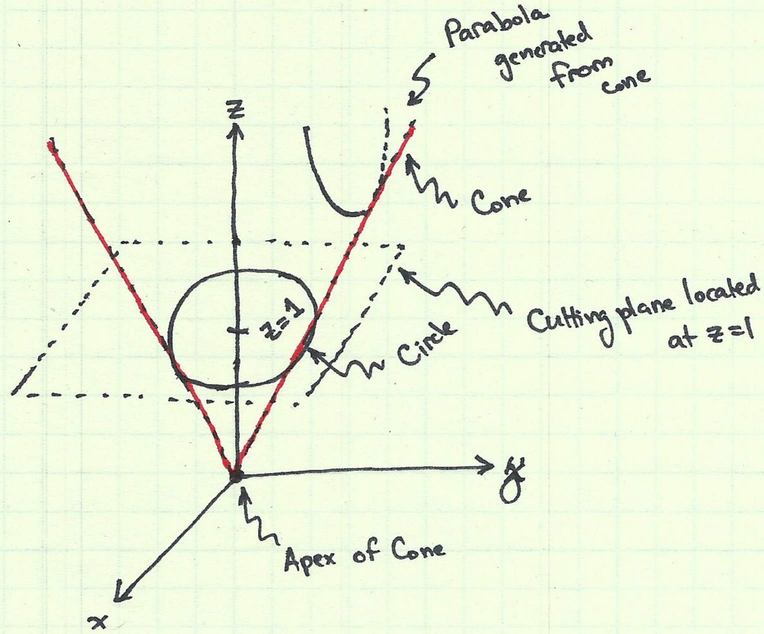


Types of Conic Section: ① Parabola: The conic section generated if the cutting plane is parallel to exactly one generating line of the cone.

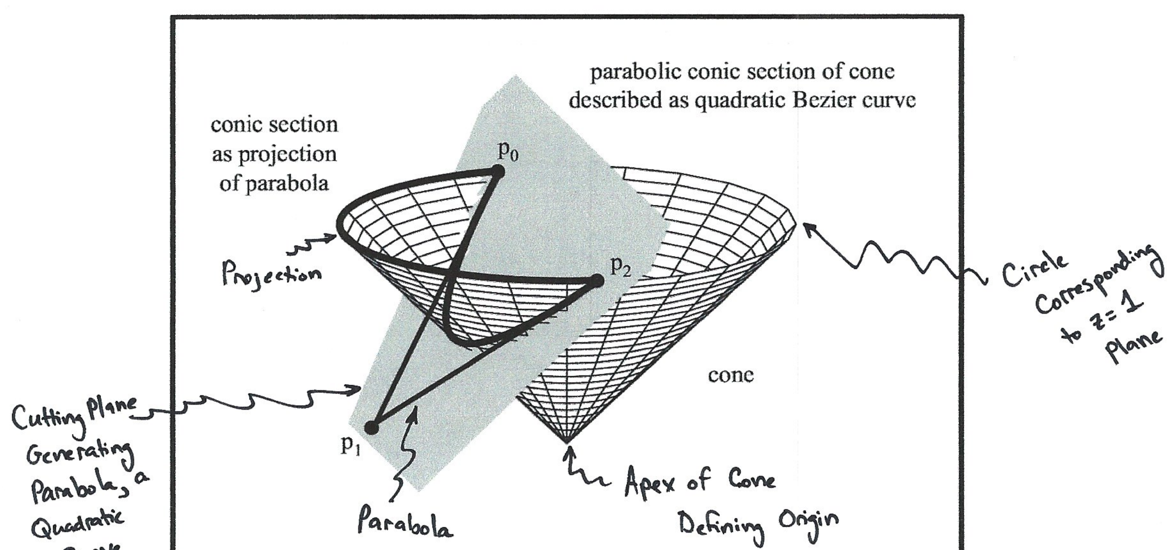
② Ellipse: The conic section generated when the intersection is a closed curve.

③ Hyperbola: The conic section generated when the plane intersects both halves of the cone.

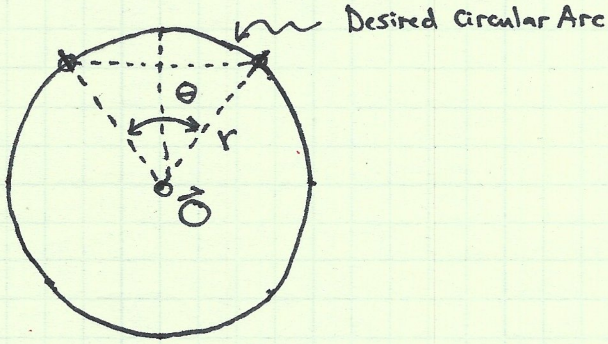
Parabolas may always be represented as a quadratic curve. Hence, to represent an arbitrary conic section, we first associate the apex of the cone with the origin in $\mathbb{R}^3$ and the cutting plane of the desired conic section with $z = 1$. For a circle, we thus have:



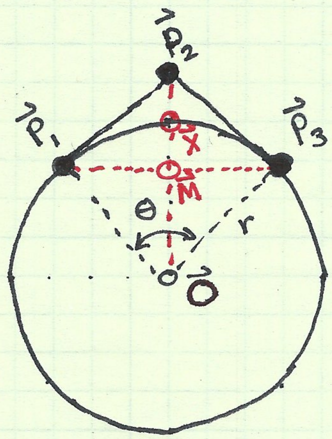
Then, to define a section of the conic section, we utilize a second cutting plane parallel to one of the generating lines of the cone. The intersection of this plane with the cone gives a parabola, which is precisely a quadratic curve in $\mathbb{R}^3$. Finally, to obtain a section of the conic section, we then simply perform a projective transformation of the parabola onto the plane $z=1$, resulting in a rational quadratic curve.



While the above discussion is informative, it is hardly constructive. Thus, let us try to build a NURBS representation of a circular arc. For simplicity, we restrict ourselves to the setting of a quadratic rational Bezier description. Suppose we want to represent the following arc:



Then, roughly speaking, our control point layout should be as follows. By the endpoint and tangency properties of a NURBS curve, $\vec{P}_1$ and $\vec{P}_3$ should be the endpoints of the arc while segments $\vec{P}_1 - \vec{P}_2$ and $\vec{P}_2 - \vec{P}_3$ should be tangent to the arc at $\vec{P}_1$ and $\vec{P}_3$, respectively.



So, the question remains: what should we choose for the weights? We may choose $w_1 = w_3 = 1$, but we must choose w_2 such that $\vec{X}$ lies precisely on the circular arc. A simple calculation yields:

$$\vec{X} = \vec{C}(0.5) = \overbrace{\tfrac{1}{2}(\vec{P}_1 + \vec{P}_3)}^{\vec{M}} + \frac{w}{1+w}\left(\vec{P}_2 - \overbrace{\tfrac{1}{2}(\vec{P}_1 + \vec{P}_3)}^{\vec{M}}\right)$$

$$= \vec{M} + \frac{w}{1+w}\left(\vec{P}_2 - \vec{M}\right)$$

Thus:

Length of segment $\vec{M} - \vec{X}$

$$\frac{|\vec{M}\vec{X}|}{|\vec{M}\vec{P}_2|} = \frac{w_2}{1+w_2}$$

Length of segment $\vec{M} - \vec{P}_2$

However:

$$|\vec{M}\vec{X}| = |\vec{O}\vec{X}| - |\vec{O}\vec{M}| = r - r\cos(\theta/2) = r(1 - \cos(\theta/2))$$

and:

$$|\vec{M}\vec{P}_2| = |\vec{O}\vec{P}_2| - |\vec{O}\vec{M}| = \frac{r}{\cos(\theta/2)} - r\cos(\theta/2) = \frac{r(1-\cos^2(\theta/2))}{\cos(\theta/2)}$$

so:

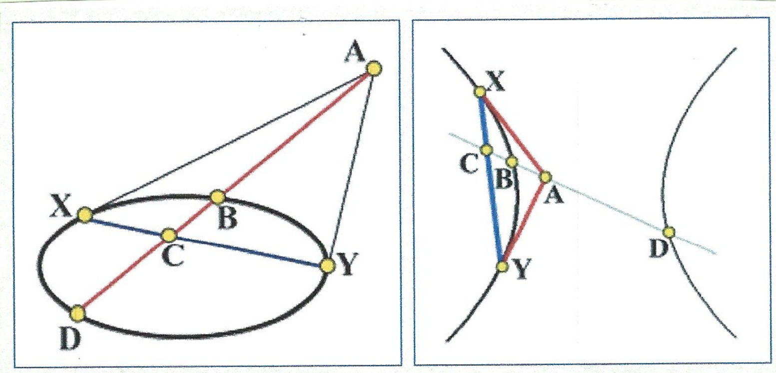
$$\frac{w_2}{1+w_2} = \frac{|\vec{M}\vec{X}|}{|\vec{M}\vec{P}_2|} = \frac{\cos(\theta/2)}{1+\cos(\theta/2)}$$

and hence:

$$w_2 = \cos(\theta/2) \quad \text{to generate a circular arc}$$

Arcs greater than 180° may be constructed from multiple smaller arcs, and the underlying basis must be no more than C^0-continuous where the arcs meet. In this manner, we can construct NURBS parametrizations of a full circle. See Section 4.1 of the IGA text for a more in-depth discussion.

In our preceding example, the weight w_2 provides a natural separation between parabolas, ellipses, and hyperbolas. To see this, consider a point $\vec{A}$ outside of a given conic section. Let us draw two tangents meeting the curve at $\vec{X}$ and $\vec{Y}$ and an arbitrary secant line that meets the chord $\vec{X}\vec{Y}$ at $\vec{C}$ and the conic at $\vec{B}$ and $\vec{D}$ with $\vec{B}$ being inside the triangle $\vec{A}\vec{X}\vec{Y}$ as shown below. If the curve is an ellipse, $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ follow in that order, while if the curve is a hyperbola, the point $\vec{D}$ lies on the other branch of the curve.



(a) Ellipse (b) Hyperbola

A well-known result in projective geometry states that:

$$\frac{|\vec{D}\vec{C}|}{|\vec{C}\vec{B}|} = \frac{|\vec{D}\vec{A}|}{|\vec{A}\vec{B}|}$$

or equivalently:

$$\frac{|\vec{A}\vec{B}|}{|\vec{C}\vec{B}|} = \frac{|\vec{D}\vec{A}|}{|\vec{D}\vec{C}|}$$

Thus:

$$\frac{|\vec{C}\vec{B}|}{|\vec{C}\vec{A}|} = \frac{|\vec{C}\vec{B}|}{|\vec{C}\vec{B}| + |\vec{B}\vec{A}|} = \frac{1}{1 + \frac{|\vec{B}\vec{A}|}{|\vec{C}\vec{B}|}} = \frac{1}{1 + \frac{|\vec{D}\vec{A}|}{|\vec{D}\vec{C}|}}$$

So:

$$\frac{|\vec{C}\vec{B}|}{|\vec{C}\vec{A}|} < \frac{1}{2} \quad \text{for an ellipse as } |\vec{D}\vec{A}| > |\vec{D}\vec{C}|$$

$$\frac{|\vec{C}\vec{B}|}{|\vec{C}\vec{A}|} > \frac{1}{2} \quad \text{for a hyperbola as } |\vec{D}\vec{A}| < |\vec{D}\vec{C}|$$

In our earlier example:

$$\vec{X} = \vec{P}_1$$
$$\vec{Y} = \vec{P}_3$$
$$\vec{A} = \vec{P}_2$$
$$\vec{B} = \vec{X}$$
$$\vec{C} = \vec{M}$$

So:

$$\frac{|\vec{C}\vec{B}|}{|\vec{C}\vec{A}|} = \frac{|\vec{M}\vec{X}|}{|\vec{M}\vec{P}_2|} = \frac{w_2}{1+w_2}$$

Thus:

$$\vec{C}(\xi) = \begin{cases} \text{Parabola} & \text{if } w_2 = 1 \\ \text{Ellipse} & \text{if } w_2 < 1 \\ \text{Hyperbola} & \text{if } w_2 > 1 \end{cases}$$

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