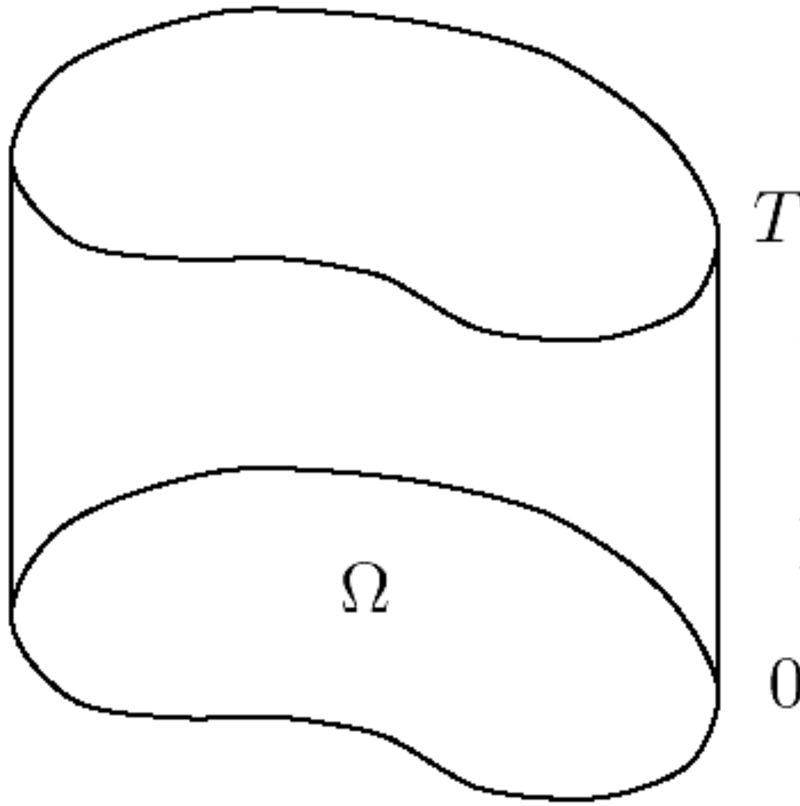
Incompressible Navier-Stokes Equations



$$\mathsf{Q} = \Omega \times (0, T)$$

$$\mathsf{P} = \Gamma \times (0, T)$$

$$\Gamma = \partial\Omega$$

Incompressible Navier-Stokes Equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \left(\boldsymbol{u} \otimes \boldsymbol{u} \right) + \nabla p = \nabla \cdot \left(2\nu \nabla^{s} \boldsymbol{u} \right) + \boldsymbol{f} \quad \text{in Q}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in Q}$$

$$\boldsymbol{u} = \mathbf{g} \qquad \text{on } \mathrm{P}_{in}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on } \mathrm{P}_{wall}$$

$$\left(2\nu \nabla^{s} \boldsymbol{u} \right) \boldsymbol{n} - p\boldsymbol{n} = \mathbf{0} \qquad \text{on } \mathrm{P}_{out}$$

$$\boldsymbol{u}(t = 0^{+}) = \boldsymbol{u}(t = 0^{-}) \quad \text{on } \Omega$$

Variational Semi-Discrete Formulation

Find $\boldsymbol{U} = \{\boldsymbol{u}, p\} \in \boldsymbol{S}$ such that $\forall \boldsymbol{W} = \{\boldsymbol{w}, q\} \in \boldsymbol{V}$

$$B(\boldsymbol{W};\boldsymbol{U}) = B_1(\boldsymbol{W},\boldsymbol{U}) + B_2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{U}) = (\boldsymbol{W},\boldsymbol{F})$$

where

$$B_1(\boldsymbol{W},\boldsymbol{U}) = \left(\boldsymbol{w}, \frac{\partial \boldsymbol{u}}{\partial t}\right)_\Omega + \left(\nabla^s \boldsymbol{w}, 2\nu\, \nabla^s \boldsymbol{u}\right)_\Omega$$
$$+ \left(q, \nabla \cdot \boldsymbol{u}\right)_\Omega - \left(\nabla \cdot \boldsymbol{w}, p\right)_\Omega$$

$$B_2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{U}) = -\left(\nabla \boldsymbol{w}, \boldsymbol{u} \otimes \boldsymbol{u}\right)_\Omega + \left(\left(\boldsymbol{u} \cdot \boldsymbol{n}\right)_+, \boldsymbol{u} \cdot \boldsymbol{w}\right)_{\Gamma_{out}}$$

$$(\boldsymbol{W},\boldsymbol{F}) = (\boldsymbol{w},\boldsymbol{f})_\Omega$$

Variational Multiscale Formulation

Split

$$\boldsymbol{V} = \bar{\boldsymbol{V}} \oplus \boldsymbol{V}'$$

Finite dimensional subspace

Standard FE, spectral or NURBS space, etc.

$\boldsymbol{V} \setminus \bar{\boldsymbol{V}}$

Fine-scales

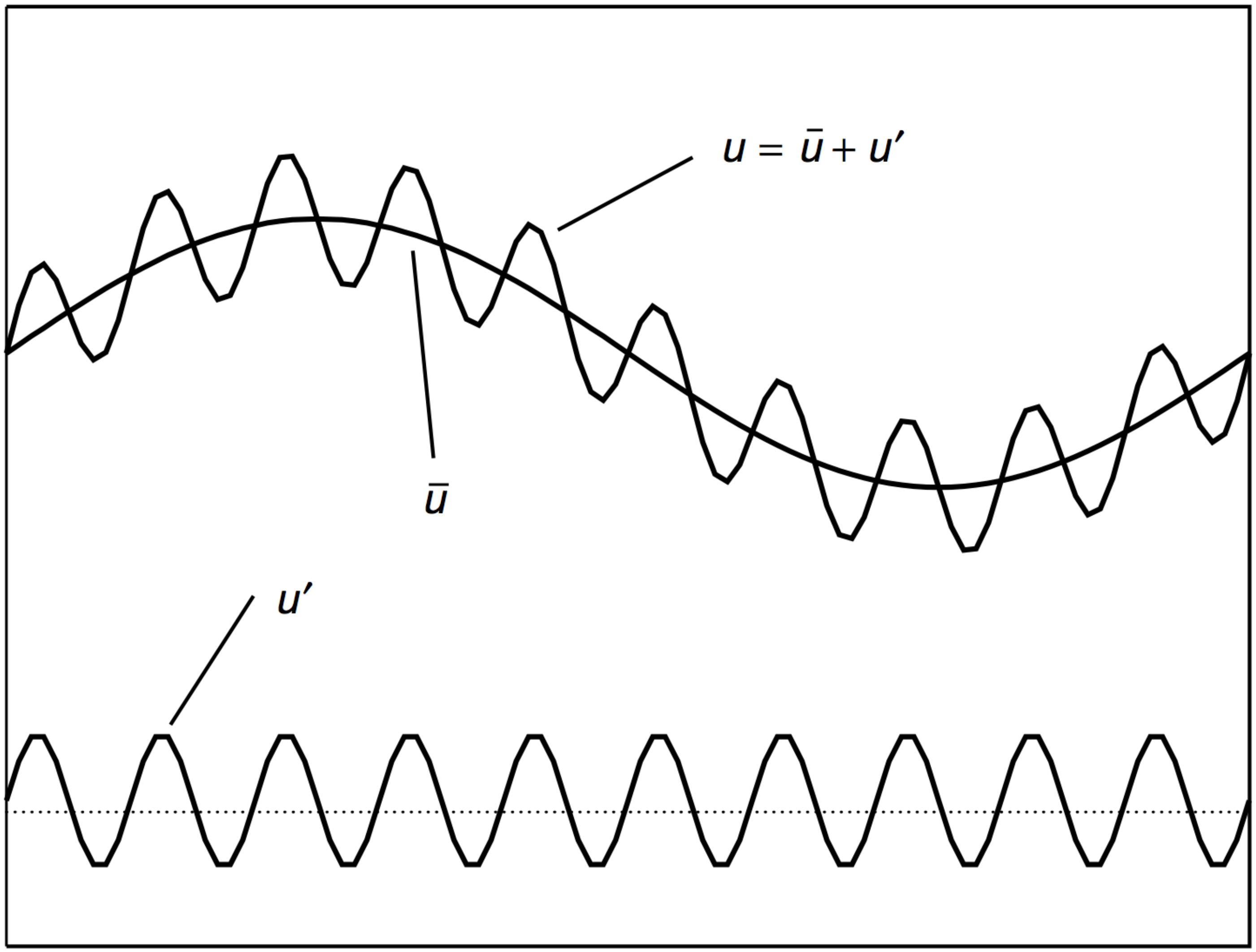
Original problem becomes:

Find $\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{U}'$, $\bar{\boldsymbol{U}} \in \bar{\boldsymbol{V}}$, $\boldsymbol{U}' \in \boldsymbol{V}'$, such that

$$(\bar{\boldsymbol{U}}) \qquad B(\bar{\boldsymbol{W}}; \bar{\boldsymbol{U}} + \boldsymbol{U}') = \left(\bar{\boldsymbol{W}}; \boldsymbol{F}\right) \qquad \forall \bar{\boldsymbol{W}} \in \bar{\boldsymbol{V}}$$

$$(\boldsymbol{U}') \qquad B(\boldsymbol{W}'; \bar{\boldsymbol{U}} + \boldsymbol{U}') = \left(\boldsymbol{W}'; \boldsymbol{F}\right) \qquad \forall \boldsymbol{W}' \in \boldsymbol{V}'$$

Variational Multiscale (VMS)



Variational Multiscale Method

Exact Theory

$$(\overline{\boldsymbol{U}}) \qquad B(\overline{\boldsymbol{W}};\overline{\boldsymbol{U}}+\boldsymbol{U}') = \left(\overline{\boldsymbol{W}},\boldsymbol{F}\right)$$

$$(\boldsymbol{U}') \qquad \begin{cases} B(\boldsymbol{W}';\overline{\boldsymbol{U}}+\boldsymbol{U}') = \left(\boldsymbol{W}',\boldsymbol{F}\right) \\ \boxed{\boldsymbol{U}' = \mathrm{F}'(\bar{\boldsymbol{u}},\boldsymbol{R}'(\overline{\boldsymbol{U}}))} \\ \left(\boldsymbol{W}',\boldsymbol{R}'(\overline{\boldsymbol{U}})\right) = B(\boldsymbol{W}';\overline{\boldsymbol{U}}) - \left(\boldsymbol{W}',\boldsymbol{F}\right) \\ \boldsymbol{R}'(\overline{\boldsymbol{U}}) = \text{Residual of coarse scales} \end{cases}$$

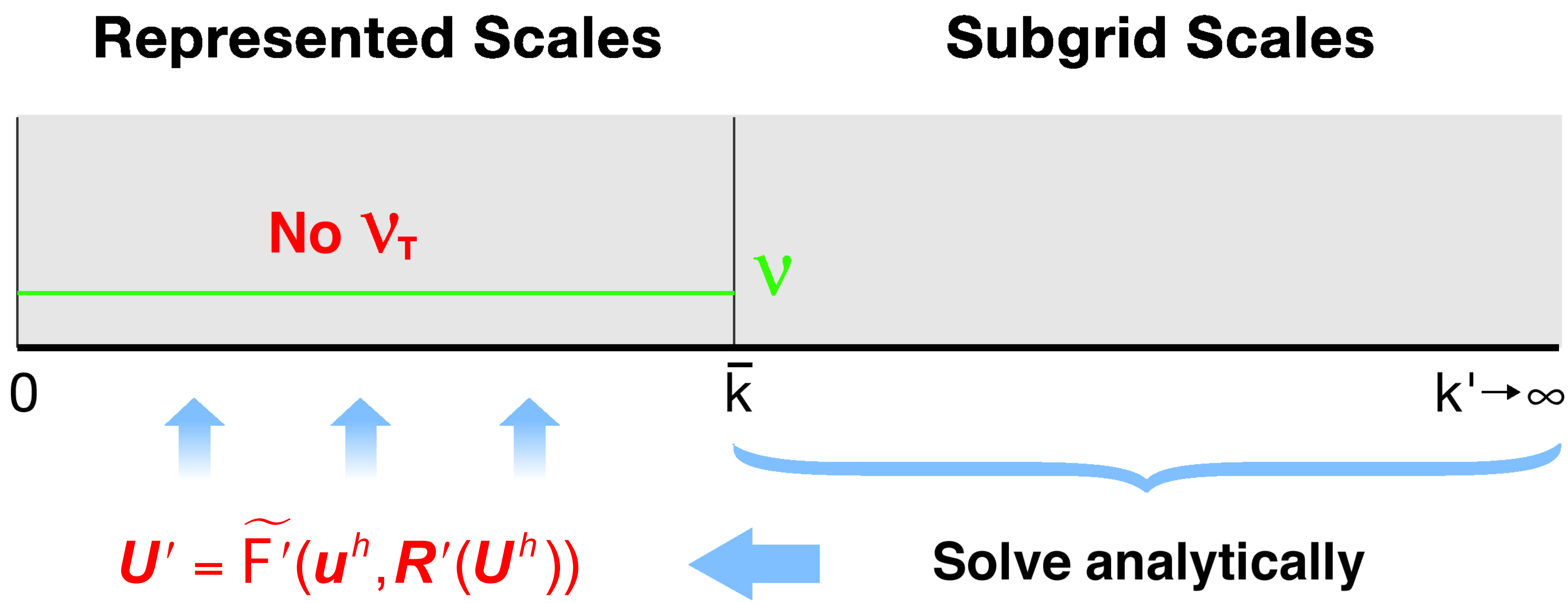
Turbulence Modeling Theory

Set $\overline{\boldsymbol{U}} \equiv \boldsymbol{U}^h$ and $\overline{\boldsymbol{W}} \equiv \boldsymbol{W}^h$ in $(\overline{\boldsymbol{U}})$,

$$(\boldsymbol{U}^h) \qquad B(\boldsymbol{W}^h,\boldsymbol{U}^h + \widetilde{\mathrm{F}'}(\boldsymbol{u}^h,\boldsymbol{R}'(\boldsymbol{U}^h))) = \left(\boldsymbol{W}^h,\boldsymbol{F}\right)$$

where $\widetilde{\mathrm{F}'} \approx \mathrm{F}'$ is the only approximation.

Variational Multiscale Method



Residual-Driven Turbulence Modeling

Fine-scale approximation:

$$\boldsymbol{u}' = -\boldsymbol{\tau}_M \boldsymbol{r}_M \quad \text{and} \quad p' = -\tau_C \, r_C$$

$$\boldsymbol{\tau}_M = \left(\left(\frac{2}{\Delta t} \right)^2 + \boldsymbol{u}^h \cdot \boldsymbol{G} \boldsymbol{u}^h + C_I \nu^2 \ \mathrm{tr}\left(\boldsymbol{G} \cdot \boldsymbol{G} \right) \right)^{-1/2}$$

$$\boldsymbol{\tau}_C = \left(\boldsymbol{g} \cdot \boldsymbol{\tau}_M \boldsymbol{g} \right)^{-1}$$

where

$$\boldsymbol{G} = \frac{\partial \xi}{\partial \boldsymbol{x}}^T \frac{\partial \xi}{\partial \boldsymbol{x}} \quad \text{and} \quad g_i = \sum_{j=1}^{3} G_{ij}$$

Discrete "Conservative" Form

$$
\begin{aligned}
&\left(\boldsymbol{w}^h, \frac{\partial \boldsymbol{u}^h}{\partial t}\right)_\Omega \\
&\quad + \left(\nabla^s \boldsymbol{w}^h, 2\nu\, \nabla^s \boldsymbol{u}^h\right)_\Omega + \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{w}^h, p^h\right)_\Omega \\
&\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}^h\right)_\Omega + \left(\left(\boldsymbol{u}^h \cdot \boldsymbol{n}\right)_+, \boldsymbol{u}^h \cdot \boldsymbol{w}^h\right)_{\Gamma_{out}} \\
&\quad - \left(\boldsymbol{w}^h, \boldsymbol{f}\right)_\Omega
\end{aligned}
\Bigg\} \text{ Galerkin terms}
$$

$$
\begin{aligned}
&- \left(\nabla \cdot \left(2\nu\, \nabla^s \boldsymbol{w}^h\right) + \nabla q^h, \boldsymbol{u}'\right)_{\tilde{\Omega}} - \left(\nabla \cdot \boldsymbol{w}^h, p'\right)_{\tilde{\Omega}} \\
&- \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}' \otimes \boldsymbol{u}^h + \boldsymbol{u}^h \otimes \boldsymbol{u}' + \boldsymbol{u}' \otimes \boldsymbol{u}'\right)_{\tilde{\Omega}}
\end{aligned}
\Bigg\} \text{ Multiscale Terms}
$$

Fine-scale approximation: $\boldsymbol{u}' = -\boldsymbol{\tau}_M\, \boldsymbol{r}_M$ and $p' = -\tau_C\, r_C$

Comparison with Classical Stabilized Methods

Coarse-scale equation

$$0 = B(\boldsymbol{W}^h, \boldsymbol{U}^h) - \left(\boldsymbol{W}^h, \boldsymbol{F}\right) \qquad \leftarrow \text{Galerkin terms}$$

$$+ B_1(\boldsymbol{W}^h, \boldsymbol{U}') + B_2(\boldsymbol{W}^h, \boldsymbol{U}', \boldsymbol{U}^h) \qquad \leftarrow \text{Classical stabilization}$$

$$+ B_2(\boldsymbol{W}^h, \boldsymbol{U}^h, \boldsymbol{U}') + B_2(\boldsymbol{W}^h, \boldsymbol{U}', \boldsymbol{U}') \qquad \leftarrow \text{Non-classical stabilization}$$

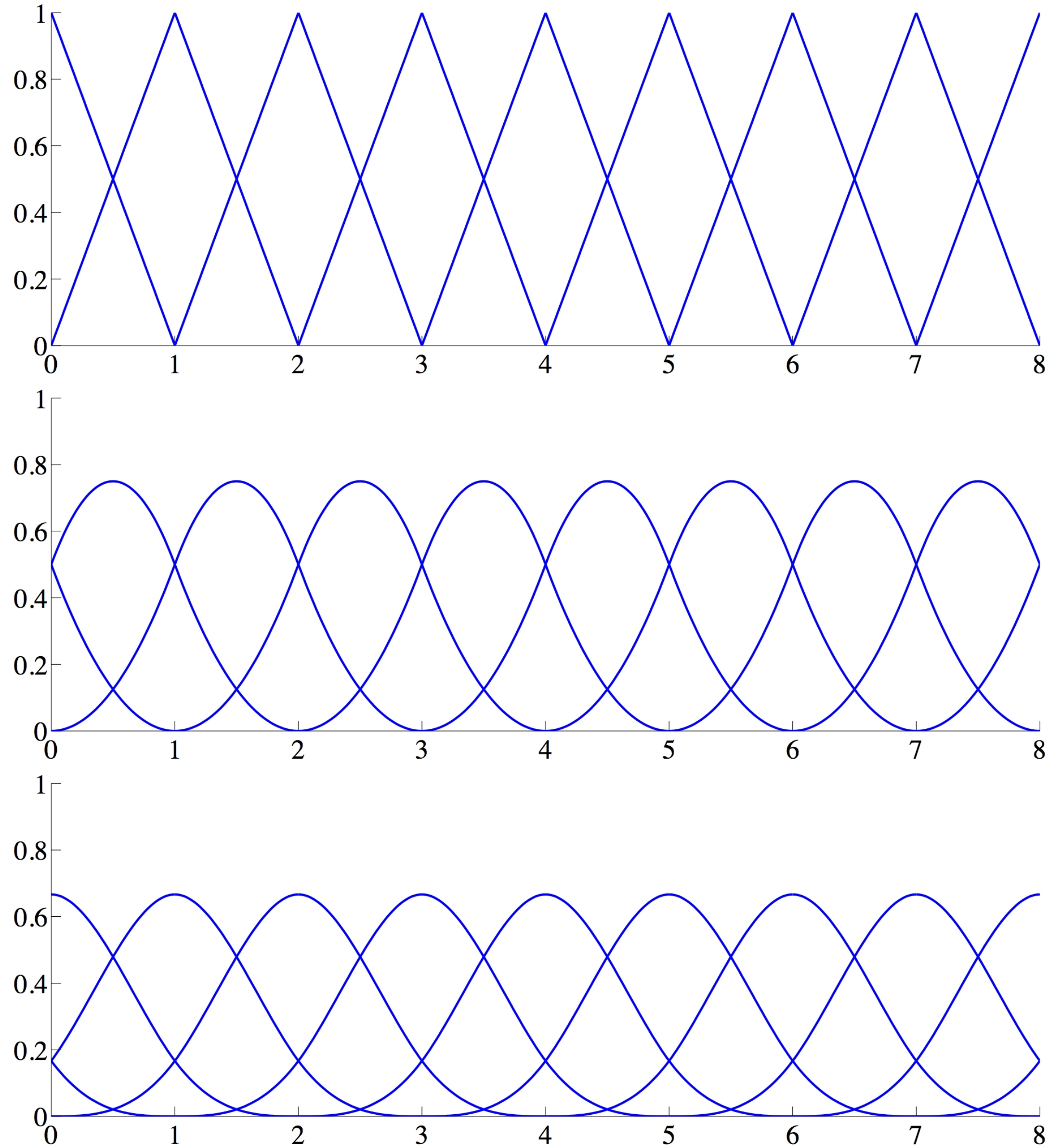
Remarks

- These terms omitted in classical stabilized methods
- Extension of the ideas of SUPG, GLS and MS
- ***No eddy viscosity***

Forced Isotropic Turbulence

- NURBS (B-splines)
  - Linears (standard hexahedral finite elements), C^0
    - Meshes: 32^3, 64^3, 128^3 , 256^3
  - Quadratics, C^1
    - Meshes: 32^3, 64^3, 128^3
  - Cubics, C^2
    - Meshes: 32^3, 64^3
- Uniform mesh in all three directions
- Statistics:
  - Energy spectra
  - Third-order structure functions $S_3 = \left\langle \left(u(x + r) - u(x) \right)^3 \right\rangle$
  - Forcing of three lowest modes of the velocity field at each instant
- Power input kept constant (P_{input} = 62.8)
- Samples taken ~0.4 T_{eddy} *apart*

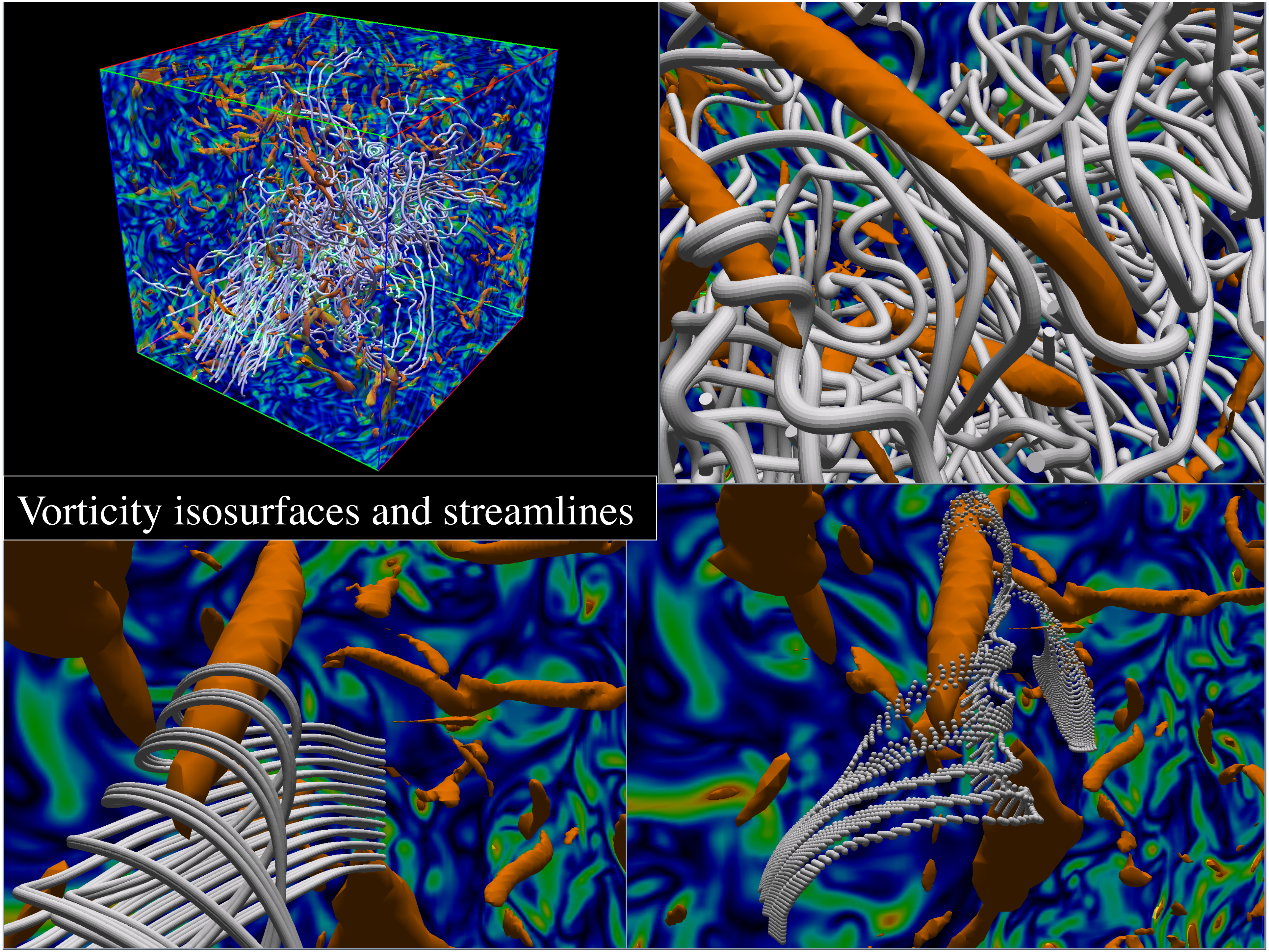
Periodic NURBS (B-spline) Basis Functions

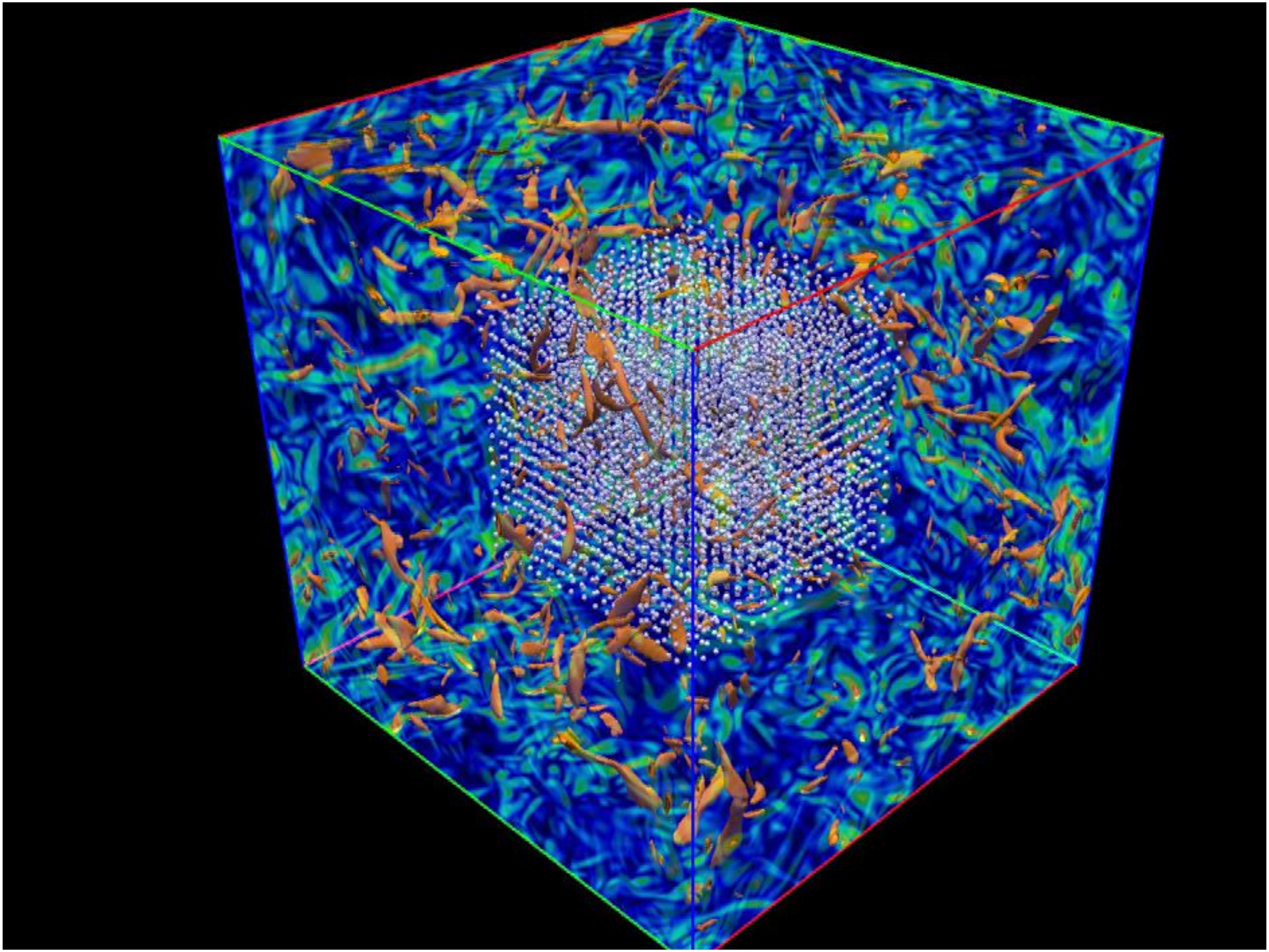


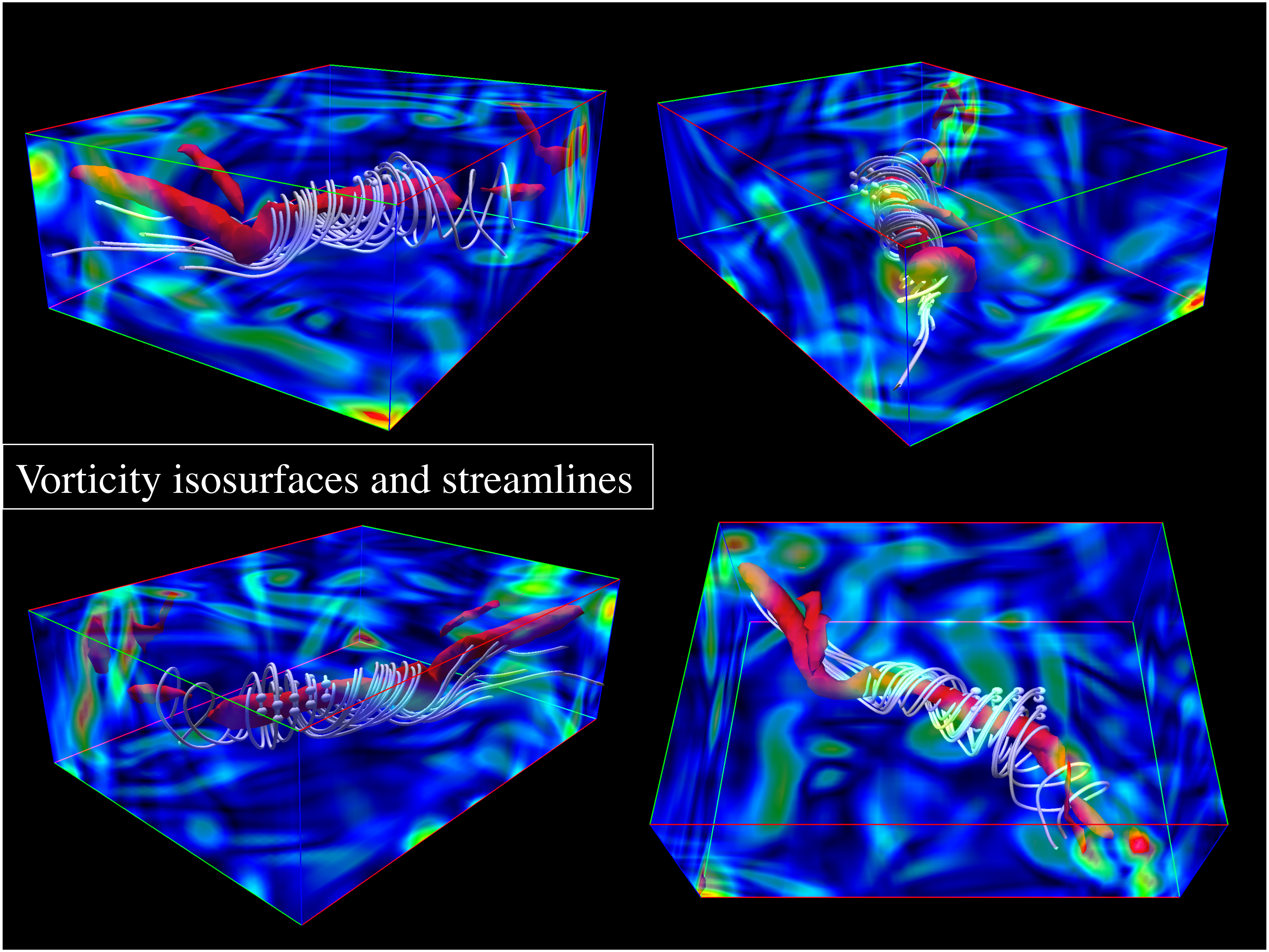
Linears

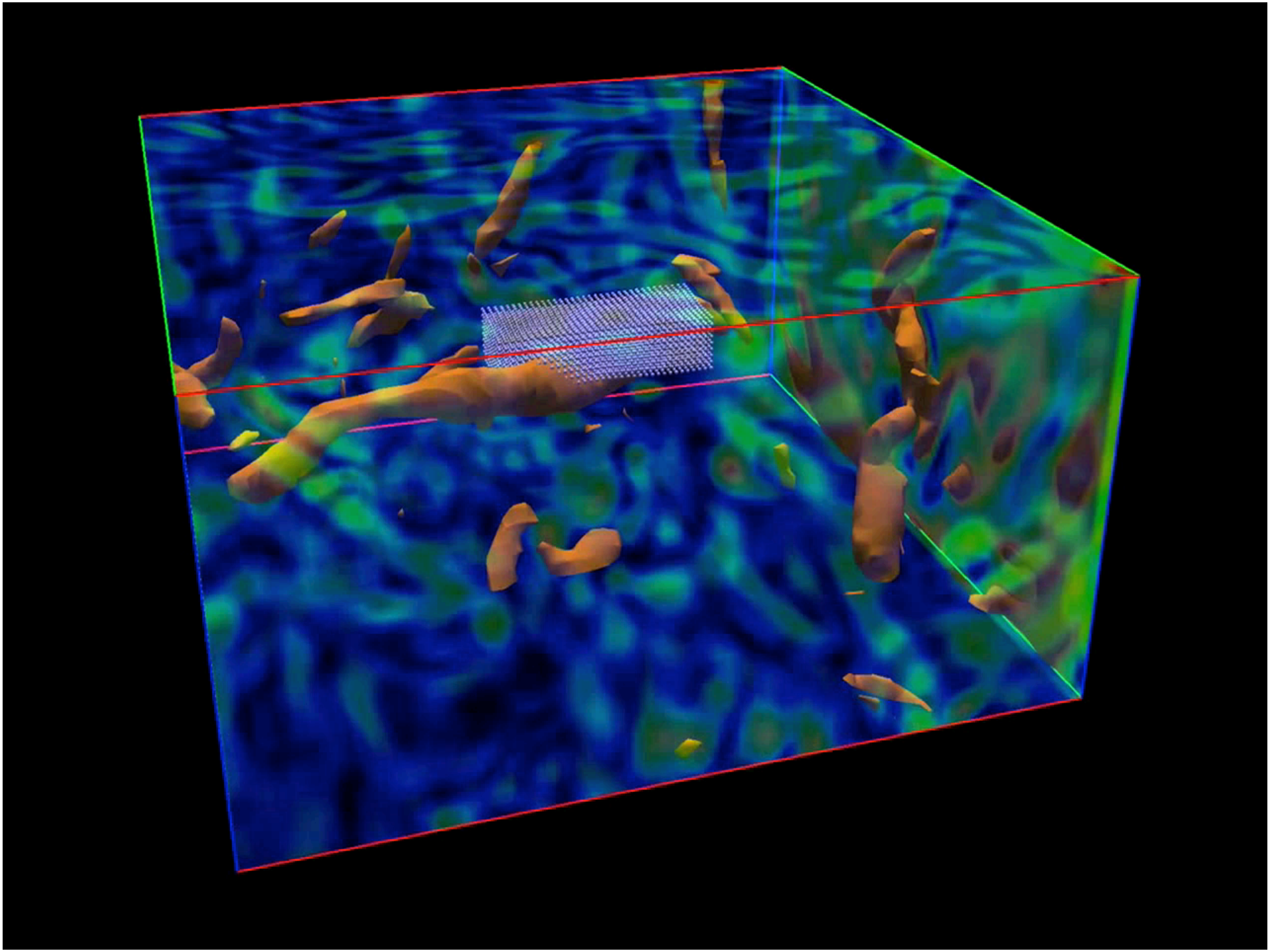
Quadratics

Cubics

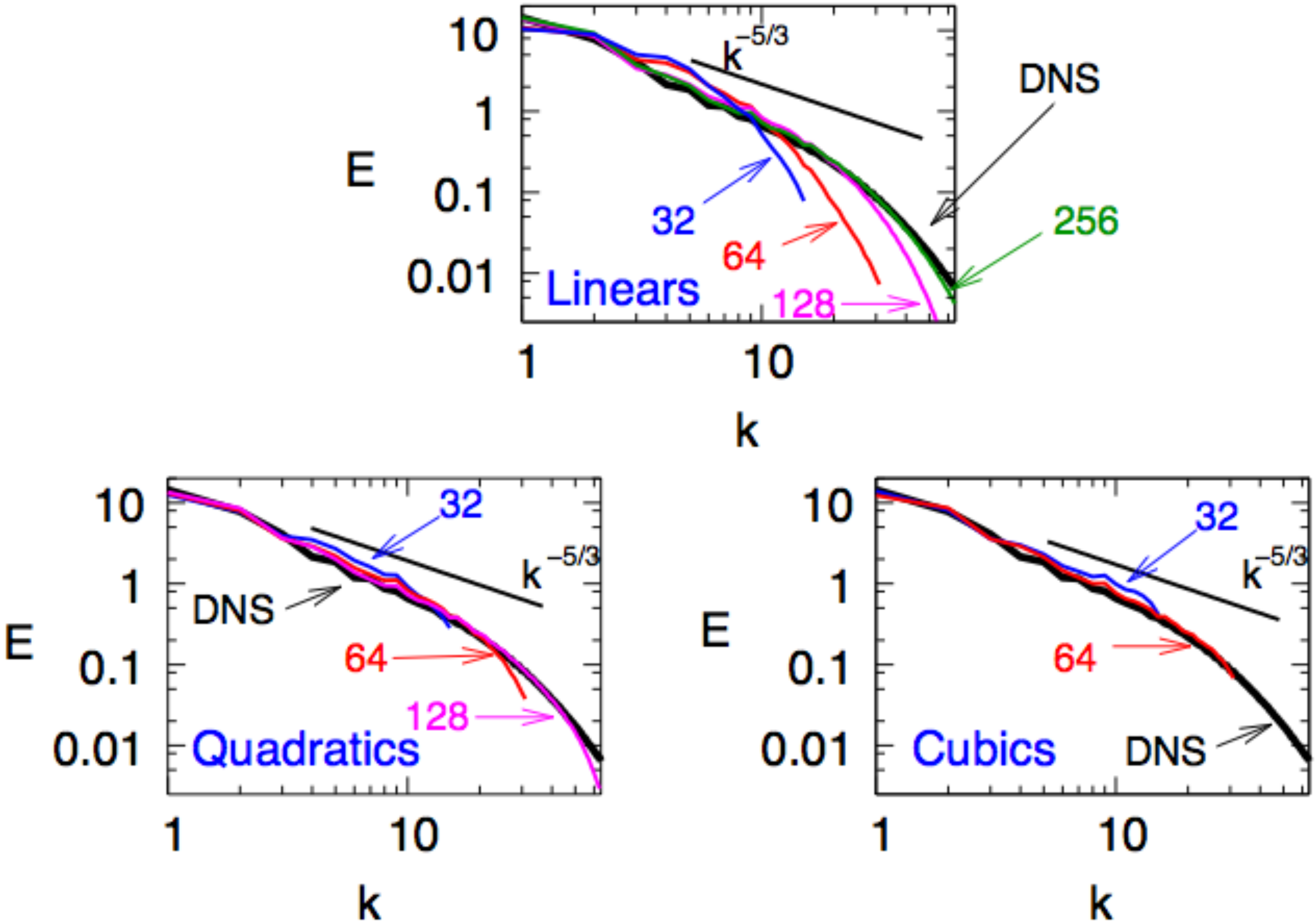

Vorticity isosurfaces and streamlines



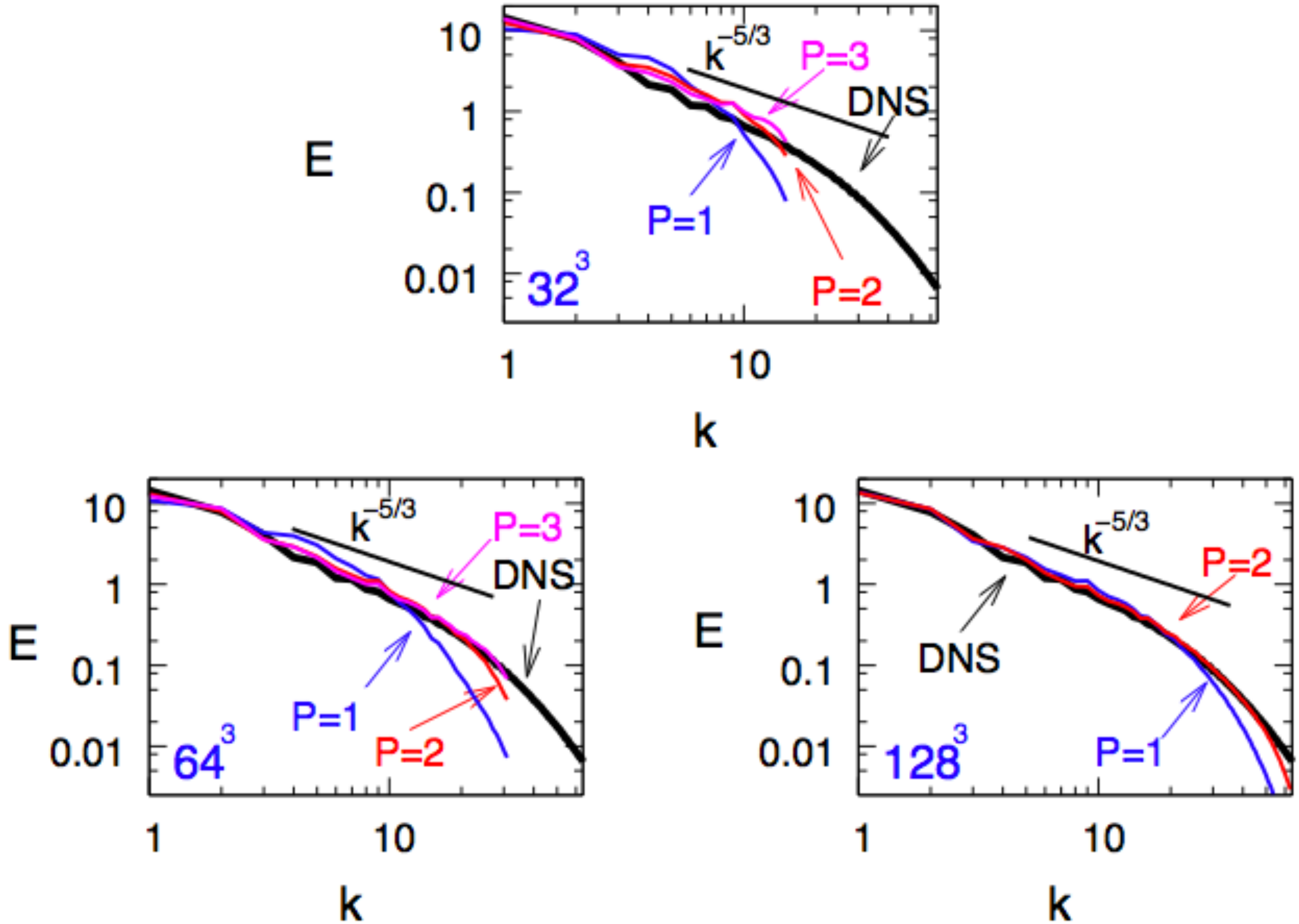

Vorticity isosurfaces and streamlines



Energy Spectra Re_λ=164 (h-refinement)

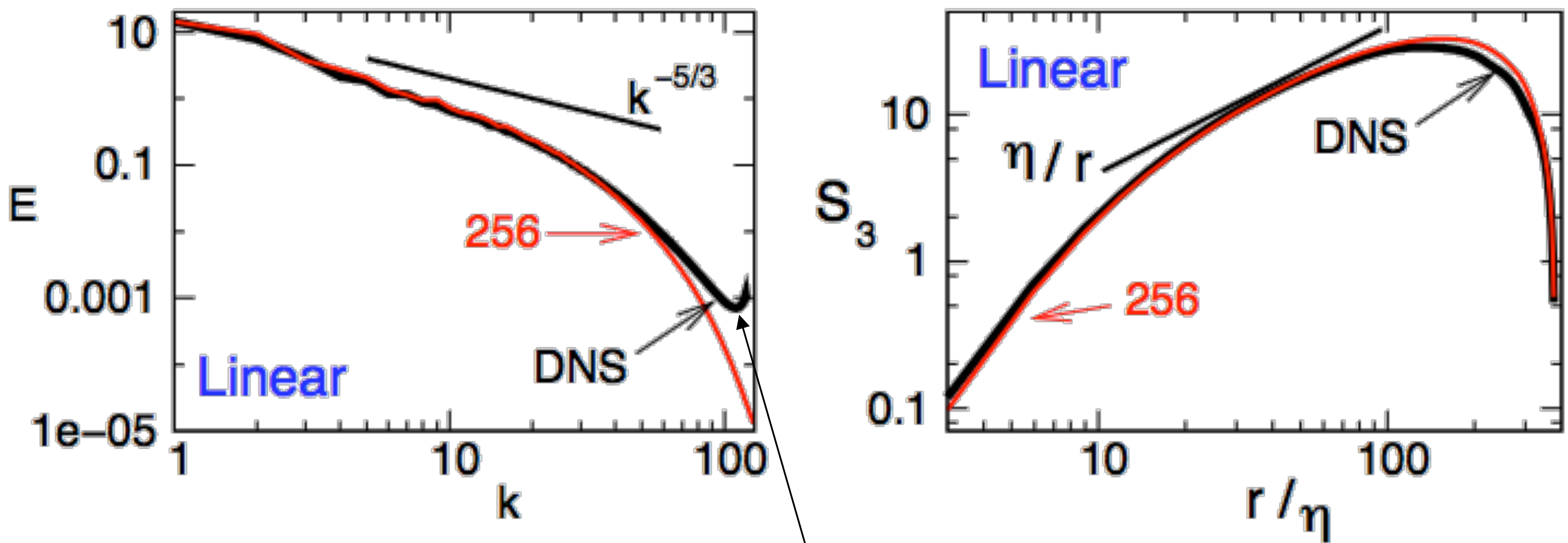


Energy Spectra Re_λ=164 (k-refinement)



Energy Spectra and Third-order Structure Function

Re_λ=164



$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

Energy pile-up for spectral method (cf. Guermond)

Formulation for this Class

$$
\begin{aligned}
&\left(\boldsymbol{w}^h, \frac{\partial \boldsymbol{u}^h}{\partial t}\right)_\Omega \\
&\quad + \left(\nabla^s \boldsymbol{w}^h, 2\nu\, \nabla^s \boldsymbol{u}^h\right)_\Omega + \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{w}^h, p^h\right)_\Omega \\
&\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}^h\right)_\Omega + \left(\left(\boldsymbol{u}^h \cdot \boldsymbol{n}\right)_+, \boldsymbol{u}^h \cdot \boldsymbol{w}^h\right)_{\Gamma_{out}} \\
&\quad - \left(\boldsymbol{w}^h, \boldsymbol{f}\right)_\Omega
\end{aligned}
\Bigg\}\ \text{Galerkin terms}
$$

$$
\begin{aligned}
&- \left(\nabla \cdot \left(2\nu\, \nabla^s \boldsymbol{w}^h\right) + \nabla q^h, \boldsymbol{u}'\right)_{\tilde{\Omega}} - \left(\nabla \cdot \boldsymbol{w}^h, p'\right)_{\tilde{\Omega}} \\
&- \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}' \otimes \boldsymbol{u}^h + \boldsymbol{u}^h \otimes \boldsymbol{u}' + \boldsymbol{u}' \otimes \boldsymbol{u}'\right)_{\tilde{\Omega}}
\end{aligned}
\Bigg\}\ \text{Multiscale Terms}
$$

Fine-scale approximation: $\boldsymbol{u}' = -\boldsymbol{\tau}_M\, \boldsymbol{r}_M$ and $p' = -\tau_C\, r_C$

Formulation for this Class

$$\underbrace{\begin{aligned}&\left(\boldsymbol{w}^h, \frac{\partial \boldsymbol{u}^h}{\partial t}\right)_\Omega \\ &\quad + \left(\nabla^s \boldsymbol{w}^h, 2\nu\, \nabla^s \boldsymbol{u}^h\right)_\Omega + \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{w}^h, p^h\right)_\Omega \\ &\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}^h\right)_\Omega + \left(\left(\boldsymbol{u}^h \cdot \boldsymbol{n}\right)_+, \boldsymbol{u}^h \cdot \boldsymbol{w}^h\right)_{\Gamma_{out}} \\ &\quad - \left(\boldsymbol{w}^h, \boldsymbol{f}\right)_\Omega\end{aligned}}_{\textit{Galerkin terms}}$$

$$\underbrace{\begin{aligned}&- \left(\nabla \cdot \cancel{\left(2\nu\, \nabla^s \boldsymbol{w}^h\right)} + \nabla q^h, \boldsymbol{u}'\right)_{\tilde{\Omega}} - \left(\nabla \cdot \boldsymbol{w}^h, p'\right)_{\tilde{\Omega}} \\ &- \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}' \otimes \boldsymbol{u}^h + \boldsymbol{u}^h \otimes \boldsymbol{u}' + \boldsymbol{u}' \otimes \boldsymbol{u}'\right)_{\tilde{\Omega}}\end{aligned}}_{\textit{Multiscale Terms}}$$

Fine-scale approximation: $\boldsymbol{u}' = -\boldsymbol{\tau}_M\, \boldsymbol{r}_M$ and $p' = -\tau_C\, r_C$

Formulation for this Class

$$\left.\begin{aligned}
&\left(\boldsymbol{w}^h, \frac{\partial \boldsymbol{u}^h}{\partial t}\right)_\Omega \\
&\quad + \left(\nabla^s \boldsymbol{w}^h, 2\nu\, \nabla^s \boldsymbol{u}^h\right)_\Omega + \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_\Omega - \left(\nabla \cdot \boldsymbol{w}^h, p^h\right)_\Omega \\
&\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}^h\right)_\Omega + \left(\left(\boldsymbol{u}^h \cdot \boldsymbol{n}\right)_+, \boldsymbol{u}^h \cdot \boldsymbol{w}^h\right)_{\Gamma_{out}} \\
&\quad - \left(\boldsymbol{w}^h, \boldsymbol{f}\right)_\Omega
\end{aligned}\right\} \text{\textit{Galerkin terms}}$$

$$\left.\begin{aligned}
&\quad - \left(\nabla \cdot \left(\cancel{2\nu\, \nabla^s \boldsymbol{w}^h}\right) + \nabla q^h, \boldsymbol{u}'\right)_{\tilde{\Omega}} - \left(\nabla \cdot \boldsymbol{w}^h, p'\right)_{\tilde{\Omega}} \\
&\quad - \left(\nabla \boldsymbol{w}^h, \cancel{\boldsymbol{u}' \otimes \boldsymbol{u}^h} + \boldsymbol{u}^h \otimes \boldsymbol{u}' + \boldsymbol{u}' \otimes \boldsymbol{u}'\right)_{\tilde{\Omega}}
\end{aligned}\right\} \text{\textit{Multiscale Terms}}$$

Fine-scale approximation: $\boldsymbol{u}' = -\boldsymbol{\tau}_M\, \boldsymbol{r}_M$ and $p' = -\tau_C\, r_C$

Formulation for this Class

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\left.
\begin{aligned}
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\begin{aligned}
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&- \left(\nabla \boldsymbol{w}^h, \cancel{\boldsymbol{u}' \otimes \boldsymbol{u}^h} + \boldsymbol{u}^h \otimes \boldsymbol{u}' + \cancel{\boldsymbol{u}' \otimes \boldsymbol{u}'}\right)_{\tilde{\Omega}}
\end{aligned}
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\begin{aligned}
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&\quad + \left(\nabla^s \boldsymbol{w}^h, 2\nu\, \nabla^s \boldsymbol{u}^h\right)_{\Omega} + \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_{\Omega} - \left(\nabla \cdot \boldsymbol{w}^h, p^h\right)_{\Omega} \\
&\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}^h\right)_{\Omega} + \left(\left(\boldsymbol{u}^h \cdot \boldsymbol{n}\right)_{+}, \boldsymbol{u}^h \cdot \boldsymbol{w}^h\right)_{\Gamma_{out}}
\end{aligned}
\right\} \text{Galerkin terms}
$$

$$
\left.
\begin{aligned}
&\quad - \left(\boldsymbol{w}^h, \boldsymbol{f}\right)_{\Omega} \\
&\quad - \left(\nabla q^h, \boldsymbol{u}'\right)_{\tilde{\Omega}} - \left(\nabla \cdot \boldsymbol{w}^h, p'\right)_{\tilde{\Omega}} \\
&\quad - \left(\nabla \boldsymbol{w}^h, \boldsymbol{u}^h \otimes \boldsymbol{u}'\right)_{\tilde{\Omega}}
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