Useful Formulas for the Analysis of Algorithms

This appendix contains a list of useful formulas and rules that are helpful in the mathematical analysis of algorithms. More advanced material can be found in [Gra94], [Gre07], [Pur04], and [Sed96].

Properties of Logarithms

All logarithm bases are assumed to be greater than 1 in the formulas below; $\lg x$ denotes the logarithm base 2, $\ln x$ denotes the logarithm base $e = 2.71828\ldots$; x, y are arbitrary positive numbers.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^y = y \log_a x$
4. $\log_a xy = \log_a x + \log_a y$
5. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
6. $a^{\log_b x} = x^{\log_b a}$
7. $\log_a x = \dfrac{\log_b x}{\log_b a} = \log_a b \log_b x$

Combinatorics

1. Number of permutations of an n-element set: $P(n) = n!$
2. Number of k-combinations of an n-element set: $C(n, k) = \dfrac{n!}{k!(n-k)!}$
3. Number of subsets of an n-element set: 2^n

Important Summation Formulas

1. $\displaystyle\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$ (l, u are integer limits, $l \le u$); $\displaystyle\sum_{i=1}^{n} 1 = n$

2. $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$

3. $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$

4. $\displaystyle\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1}n^{k+1}$

5. $\displaystyle\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1}-1}{a-1}$ ($a \ne 1$); $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

6. $\displaystyle\sum_{i=1}^{n} i 2^i = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n2^n = (n-1)2^{n+1} + 2$

7. $\displaystyle\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$, where $\gamma \approx 0.5772\ldots$ (Euler's constant)

8. $\displaystyle\sum_{i=1}^{n} \lg i \approx n \lg n$

Sum Manipulation Rules

1. $\displaystyle\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$

2. $\displaystyle\sum_{i=l}^{u} (a_i \pm b_i) = \sum_{i=l}^{u} a_i \pm \sum_{i=l}^{u} b_i$

3. $\displaystyle\sum_{i=l}^{u} a_i = \sum_{i=l}^{m} a_i + \sum_{i=m+1}^{u} a_i$, where $l \le m < u$

4. $\displaystyle\sum_{i=l}^{u} (a_i - a_{i-1}) = a_u - a_{l-1}$