dense and sparse modes) to compare the observed running time with the algorithm's theoretical efficiency.

10. *Domino puzzle* A domino is a 2×1 tile that can be oriented either horizontally or vertically. A tiling of a given board composed of 1×1 squares is covering it with dominoes exactly and without overlap. Is it possible to tile with dominoes an 8×8 board without two unit squares at its diagonally opposite corners?

10.4 The Stable Marriage Problem

In this section, we consider an interesting version of bipartite matching called the stable marriage problem. Consider a set $Y = \{m_1, m_2, \ldots, m_n\}$ of n men and a set $X = \{w_1, w_2, \ldots, w_n\}$ of n women. Each man has a preference list ordering the women as potential marriage partners with no ties allowed. Similarly, each woman has a preference list of the men, also with no ties. Examples of these two sets of lists are given in Figures 10.11a and 10.11b. The same information can also be presented by an $n \times n$ ranking matrix (see Figure 10.11c). The rows and columns of the matrix represent the men and women of the two sets, respectively. A cell in row m and column w contains two rankings: the first is the position (ranking) of w in the m's preference list; the second is the position (ranking) of m in the w's preference list. For example, the pair 3, 1 in Jim's row and Ann's column in the matrix in Figure 10.11c indicates that Ann is Jim's third choice while Jim is Ann's first. Which of these two ways to represent such information is better depends on the task at hand. For example, it is easier to specify a match of the sets' elements by using the ranking matrix, whereas the preference lists might be a more efficient data structure for implementing a matching algorithm.

A ***marriage matching*** M is a set of n (m, w) pairs whose members are selected from disjoint n-element sets Y and X in a one-one fashion, i.e., each man m from Y is paired with exactly one woman w from X and vice versa. (If we represent Y and X as vertices of a complete bipartite graph with edges connecting possible marriage partners, then a marriage matching is a perfect matching in such a graph.)

men's preferences

	1st	2nd	3rd
Bob:	Lea	Ann	Sue
Jim:	Lea	Sue	Ann
Tom:	Sue	Lea	Ann

(a)

women's preferences

	1st	2nd	3rd
Ann:	Jim	Tom	Bob
Lea:	Tom	Bob	Jim
Sue:	Jim	Tom	Bob

(b)

ranking matrix

	Ann	Lea	Sue
Bob	$\boxed{2,3}$	1,2	3,3
Jim	3,1	$\boxed{1,3}$	2,1
Tom	3,2	2,1	$\boxed{1,2}$

(c)

FIGURE 10.11 Data for an instance of the stable marriage problem. (a) Men's preference lists; (b) women's preference lists. (c) Ranking matrix (with the boxed cells composing an unstable matching).

A pair (m, w), where $m \in Y$, $w \in X$, is said to be a ***blocking pair*** for a marriage matching M if man m and woman w are not matched in M but they prefer each other to their mates in M. For example, (Bob, Lea) is a blocking pair for the marriage matching $M = \{$(Bob, Ann), (Jim, Lea), (Tom, Sue)$\}$ (Figure 10.11c) because they are not matched in M while Bob prefers Lea to Ann and Lea prefers Bob to Jim. A marriage matching M is called ***stable*** if there is no blocking pair for it; otherwise, M is called ***unstable***. According to this definition, the marriage matching in Figure 10.11c is unstable because Bob and Lea can drop their designated mates to join in a union they both prefer. The ***stable marriage problem*** is to find a stable marriage matching for men's and women's given preferences.

Surprisingly, this problem always has a solution. (Can you find it for the instance in Figure 10.11?) It can be found by the following algorithm.

Stable marriage algorithm

Input: A set of n men and a set of n women along with rankings of the women by each man and rankings of the men by each woman with no ties allowed in the rankings

Output: A stable marriage matching

Step 0 Start with all the men and women being free.

Step 1 While there are free men, arbitrarily select one of them and do the following:

Proposal The selected free man m proposes to w, the next woman on his preference list (who is the highest-ranked woman who has not rejected him before).

Response If w is free, she accepts the proposal to be matched with m. If she is not free, she compares m with her current mate. If she prefers m to him, she accepts m's proposal, making her former mate free; otherwise, she simply rejects m's proposal, leaving m free.

Step 2 Return the set of n matched pairs.

Before we analyze this algorithm, it is useful to trace it on some input. Such an example is presented in Figure 10.12.

Let us discuss properties of the stable marriage algorithm.

THEOREM The stable marriage algorithm terminates after no more than n^2 iterations with a stable marriage output.

PROOF The algorithm starts with n men having the total of n^2 women on their ranking lists. On each iteration, one man makes a proposal to a woman. This reduces the total number of women to whom the men can still propose in the future because no man proposes to the same woman more than once. Hence, the algorithm must stop after no more than n^2 iterations.

		Ann	Lea	Sue	
Free men: Bob, Jim, Tom	Bob	2, 3	$\boxed{1,2}$	3, 3	Bob proposed to Lea
	Jim	3, 1	1, 3	2, 1	Lea accepted
	Tom	3, 2	2, 1	1, 2	

		Ann	Lea	Sue	
Free men: Jim, Tom	Bob	2, 3	$\boxed{1,2}$	3, 3	Jim proposed to Lea
	Jim	3, 1	$\underline{1, 3}$	2, 1	Lea rejected
	Tom	3, 2	2, 1	1, 2	

		Ann	Lea	Sue	
Free men: Jim, Tom	Bob	2, 3	$\boxed{1,2}$	3, 3	Jim proposed to Sue
	Jim	3, 1	1, 3	$\boxed{2,1}$	Sue accepted
	Tom	3, 2	2, 1	1, 2	

		Ann	Lea	Sue	
Free men: Tom	Bob	2, 3	$\boxed{1,2}$	3, 3	Tom proposed to Sue
	Jim	3, 1	1, 3	$\boxed{2,1}$	Sue rejected
	Tom	3, 2	2, 1	$\underline{1, 2}$	

		Ann	Lea	Sue	
Free men: Tom	Bob	2, 3	1, 2	3, 3	Tom proposed to Lea
	Jim	3, 1	1, 3	$\boxed{2,1}$	Lea replaced Bob with Tom
	Tom	3, 2	$\boxed{2,1}$	1, 2	

		Ann	Lea	Sue	
Free men: Bob	Bob	$\boxed{2,3}$	1, 2	3, 3	Bob proposed to Ann
	Jim	3, 1	1, 3	$\boxed{2,1}$	Ann accepted
	Tom	3, 2	$\boxed{2,1}$	1, 2	

FIGURE 10.12 Application of the stable marriage algorithm. An accepted proposal is indicated by a boxed cell; a rejected proposal is shown by an underlined cell.

Let us now prove that the final matching M is a stable marriage matching. Since the algorithm stops after all the n men are one-one matched to the n women, the only thing that needs to be proved is the stability of M. Suppose, on the contrary, that M is unstable. Then there exists a blocking pair of a man m and a woman w who are unmatched in M and such that both m and w prefer each other to the persons they are matched with in M. Since m proposes to every woman on his ranking list in decreasing order of preference and w precedes m's match in M, m must have proposed to w on some iteration. Whether w refused m's proposal or accepted it but replaced him on a subsequent iteration with a higher-ranked match, w's mate in M must be higher on w's preference list than m because the rankings of the men matched to a given woman may only improve on each iteration of the algorithm. This contradicts the assumption that w prefers m to her final match in M. ■

The stable marriage algorithm has a notable shortcoming. It is not "gender neutral." In the form presented above, it favors men's preferences over women's