Fractals Avoiding Fractal Sets

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Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Suppose you have to determine the stability of an operator on function spaces. Thinking geometrically, one can study how the operator acts on indicator functions. The existence of interesting characteristics in sets often gives insight into how the operator behaves on the corresponding indicator function. Proving the operator's boundedness then reduces to determining how large sets can be possessing certain characteristics.

In this paper, we describe methods to find large sets avoiding fine-scale patterns. Important examples include affine configurations, such as sets not containing the vertices of equilateral triangles, sets not containing three term arithmetic progressions, sets not generating particular families of angles, and sets not containing points in a common hyperplane. For these examples, the Lebesgue density theorem implies any set of positive measure contains these patterns. Thus we quantify the size of sets by their Hausdorff dimension.

There are two approaches to the pattern avoidance problem. We get upper bounds by proving sets with large Hausdorff dimension contains patterns. Constructing sets avoiding patterns with large Hausdorff dimension give lower bounds. In this paper, we focus on the *construction problem* for lower bounding pattern avoidance problems.

There are already generic pattern avoidance methods in the literature. We compare our method to them in detail in section 6. But these rely on the non-singularity of the patterns, lying on a smooth function. The novel feature of our method is we can avoid points with patterns lying on an *arbitrary* fractal set, and the Hausdorff dimension of our constructions is still comparable to previous methods.

The *key idea* of our method is the introduction of a new geometric framework for pattern avoidance problems, described in section 1. A simple combinatorial argument, described in section 2, exploited repeatedly in section 3 via a queueing process leads directly to a pattern avoiding set. We believe this new geometric framework should help find further methods in the field. We show this by proving another pattern avoidance result in section 5, assuming extra geometric information on the patterns.

1 A Fractal Avoidance Framework

The common framework used to think about pattern avoidance problems is to specify the pattern as the zero set of some function.

- A set $X \subset \mathbf{R}^d$ contains the vertices of no equilateral triangles if and only if for any three distinct $x, y, z \in X$,
$$f(x, y, z) = d(x, y) + d(y, z) - 2d(x, z) \neq 0$$
- A set $X \subset \mathbf{R}$ contains no three term arithmetic progressions if and only if for any distinct $x, y, z \in X$,
$$f(z, y, z) = (x - y) - (x - z) - 2(x - z) \neq 0$$
- A set $X \subset \mathbf{R}^d$ does not contain a family of angles $\{\alpha_i\}$ if and only if for any distinct $x, y, z \in X$, and i,
$$f(x, y, z) = \frac{(x - z) \cdot (y - z)}{|x - z||y - z|} \neq \cos(\alpha_i)$$
- A set $X \subset \mathbf{R}^d$ does not contain d points in a hyperplane if and only if for any distinct $x_1, \dots, x_d \in X$,
$$f(x_1, \dots, x_d) = \det(x_1, \dots, x_d) \neq 0$$

This leads to a common generic formulation of pattern avoidance problems:

The Configuration Avoidance Problem: Given a function $f : (\mathbf{R}^d)^n \to \mathbf{R}$ as input, find $X \subset \mathbf{R}^d$ with high Hausdorff dimension such that for any *distinct* $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$.

This is the viewpoint of [3] and [5], who give results assuming various regularity conditions on the function f. It is the viewpoint of this paper that the function f contains extraneous information which is not really useful to the problem. The only important information we need to extract from the function f is the geometric structure of it's zero set. If we denote the zero set of f by Z, the problem becomes equivalent to a more flexible framework:

The Fractal Avoidance Problem: Given $Z \subset (\mathbf{R}^d)^n$, find a set $X \subset \mathbf{R}^d$ such that $X^d \cap Z \subset \Delta$, where $\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for some } i \text{ and } j\}$, with as high a Hausdorff dimension as possible.

A natural goal is to solve the generic fractal avoidance problem with minimal assumptions. Thus Z can take the form of an arbitrary fractal, and the only assumptions we place on Z are it's fractal dimension.

Theorem 1. *If $Z \subset (\mathbf{R}^d)^n$ be a set with Minkowski dimension α, then there exists a set X solving the fractal avoidance problem for Z with*

$$\dim_{\mathbf{H}}(X) = \frac{dn - \alpha}{n-1} = \frac{\operatorname{codim}_{\mathbf{H}}(Z)}{n-1}$$

A second goal is to show an example where assuming extra geometric conditions on Z imply a better result, showing this framework extends to further methods in pattern avoidance. We consider the condition where Z has *low rank*, in a certain sense.

Theorem 2. *Let $Z \subset (\mathbf{R}^d)^n$ be a set together with a projection π such that $\pi(Z)$ is α dimensional. Then there exists a set X solving the fractal avoidance problem for Z with dimension*

Because of the lack of any *rigid* geometric information about the set Z, such as smoothness, really the only way we can avoid Z is by covering arguments. This can be neatly summarized as a combinatorial argument on graphs, which we detail in the next section.

2 Avoidance at a Single Scale

We now develop a discrete technique used to construct solutions to the fractal avoidance problem. It depends very little on the Euclidean structure of the plane. As such, we rephrase the construction as a combinatorial problem on graphs.

Recalling definitions, we say an *n uniform hypergraph* is a collection of *vertices* and *hyperedges*, where a hyperedge is a set of n distinct vertices. An *independent set* is a subset of vertices containing no complete set of vertices in any hyperedge of the graph. A *coloring* is a partition of the vertex set into finitely many independent sets, each of which we call a *color*. Such a coloring is *K uniform* if each color class has K elements.

The next lemma is a variant of Turán's theorem on independent sets. For technical reasons, we need an extra restriction on the independent set so it is 'uniformly' chosen over the graph. This is the reason for the introduction of colorings.

Lemma 1. *Let G be an n uniform hypergraph with a K uniform coloring. Then there is an independent set W containing elements from all but $|E|/K^n$ colors.*

Proof. Let U be a random vertex set chosen by selecting a vertex of each color uniformly randomly. Every vertex occurs in U with probability $1/K$. For any edge $e = (v_1, \dots, v_n)$, the vertices v_i all have different colors. Thus they occur in U with independent chances, and so

$$\mathbf{P}(v_1 \in U, \dots, v_n \in U) = \mathbf{P}(v_1 \in U) \dots \mathbf{P}(v_n \in U) = 1/K^n$$

If we let E' denote the edges $e = (u_1, \dots, u_n)$ with $u_1, \dots, u_n \in U$, then

$$\mathbf{E}|E'| = \sum_{e \in E} \mathbf{P}(e \in E') = \sum_{e \in E} 1/K^n = \frac{|E|}{K^n}$$

This means we may choose a *particular*, nonrandom U for which $|E'| \leqslant |E|/K^n$. If we form a vertex set $W \subset U$ by removing, for each $e \in E'$, a vertex in U adjacent to the edge, then W is an independent set containing all but $|E'| \leqslant |E|/K^n$ colors. □

Corollary. *If $|V| \gtrsim N^a$, $|E| \lesssim N^b$, and $K \gtrsim N^c$, where $b < a + c(n-1)$, then as $N \to \infty$ we can find an independent set containing all but a fraction $o(1)$ of the colors.*

Proof. A simple calculation on the quantities of the previous lemma yields

$$\frac{\#(\text{colors removed})}{\#(\text{all colors})} = \frac{|E|/K^n}{|V|/K} = \frac{|E|}{|V|K^{n-1}} \lesssim \frac{N^b}{N^{a+c(n-1)}}$$

This is $o(1)$ if $b < a + c(n-1)$. □

We now apply these constructions to a problem clearly related to the fractal avoidance problem. It will form our key method to construct fractal avoidance solutions. Given an integer N, we subdivide $\mathbf{R}^d$ into a lattice of side length $1/N$ cubes with corners on $\mathbf{Z}^d/N$, the collection of such cubes we will denote by $\mathcal{B}(1/N)$. This grid is used to granularize configuration avoidance.

Theorem 3. *Suppose $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint collections of cubes in $\mathcal{B}(1/N)$, with $|\mathcal{I}_i| \gtrsim N^d$. If α bounds the lower Minkowski dimension of Y from above, and we have a rational parameter $\beta > d(n-1)/(n-\alpha)$. Then there exists arbitrarily large N such that N^β is an integer, and there exists collections of cubes $\mathcal{J}_1, \dots, \mathcal{J}_n \in \mathcal{B}(1/N^\beta)$ with each cube in $\mathcal{J}_1 \times \dots \times \mathcal{J}_n$ disjoint from Y, and as $N \to \infty$, each $\mathcal{J}_i$ contains cubes in all but a fraction $o(1)$ of cubes in $\mathcal{I}_i$.*

Proof. If $\mathcal{K} \subset \mathcal{B}(1/N^\beta)^n$ is the collection of all cubes in a side length $1/N^\beta$ lattice intersecting Y, then $|\mathcal{K}| \lesssim N^{\alpha\beta}$. We then let $\mathcal{I}_i'$ be all cubes in $\mathcal{B}(1/N^\beta)$ contained in $\mathcal{I}_i$. Considering these cubes as vertices gives us an n uniform hypergraph G with a hyperedge between $I_1 \in \mathcal{I}_1', \dots, I_n \in \mathcal{I}_n'$ if $I_1 \times \dots \times I_n \in \mathcal{K}$. We say two cubes in G are the same color if they are contained in a common cube in $\mathcal{I}_i$.

Using the fact that a side length $1/N$ cube contains $N^{d(\beta-1)}$ side length $1/N^\beta$ cubes, we conclude that G has

$\sum |\mathcal{I}_i| = N^{d(\beta-1)} \sum |\mathcal{I}_i| \gtrsim N^{d\beta}$ vertices. We bound the number of edges in G by $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Finally, the coloring is $N^{d(\beta-1)}$ uniform. Thus in the terminology of the previous corollary, $a = d\beta$, $b = \alpha\beta$, and $c = d(\beta - 1)$, and the inequality in the hypothesis of this theorem is then equivalent to the inequality in the hypothesis of the corollary. Applying the corollary gives the required result. □

The value $d(n-1)/(n-\alpha)$ in the theorem is directly related to the dimension $(n-\alpha)/(n-1)$ we get in our main result. Any improvement on this bound for special cases of the fractal avoidance problem immediately will lead to improvements on the Hausdorff dimension of the set constructed. The fact that our hypergraph result is tight indicates that for the general fractal avoidance problem, our construction gives tight bounds.

3 A Fractal Avoiding Set

We get solutions X to fractal avoidance problems by breaking the problem down into a sequence of discrete configuration problems. The central idea was first used in [3]. We construct X as a limit $\lim X_N$, where X_N is a disjoint union of side length L_N cubes, and subdividing X_N into cubes of side length R_N, then further subdividing these cubes into cubes of smaller side length L_{N+1}, and removing a portion of them, gives X_{N+1}.

At each step N, we consider a disjoint collection of side length R_N cubes $\mathcal{I}_1(N), \dots, \mathcal{I}_n(N) \subset \mathcal{B}(R_N)$, each cube contained in X_N. The main result of the previous section allows us to find a collection of side length $L_{N+1} = R_N^{\beta_N}$ cubes $\mathcal{J}_i(N) \subset \mathcal{I}_i(N)$ with all cubes in $\mathcal{J}_1(N) \times \dots \times \mathcal{J}_n(N)$ disjoint from Y, and where β_N converges to $\beta = d(n-1)/(n-\alpha)$ from above. We then form X_{N+1} from X_N by removing the parts of cubes in $\mathcal{I}_i(N)$ which are not contained in the cubes in $\mathcal{J}_i(N)$. Once we fix parameters and an initial set X_0, we get a sequence $X_0, X_1, \dots$ converging to a set X. We set $X_0 = [0,1]^d$ for simplicity. A simple constraint detailed below is the only requirement to ensure that X is a solution to the fractal avoidance problem.

Lemma 2. *Suppose that for any choice of distinct $x_1, \dots, x_n \in X$, there exists N such that $\mathcal{I}_i(N)$ contains a cube containing x_i. Then $X^d \cap Y \subset \Delta$.*

Proof. For then $x_1, \dots, x_n \in X_{N+1}$, so $x_1 \in \mathcal{J}_1(N), \dots, x_n \in \mathcal{J}_n(N)$, so $\mathcal{J}_1(N) \times \dots \times \mathcal{J}_n(N)$ contains a cube containing the tuple $(x_1, \dots, x_n)$, and by assumption this cube is disjoint from Y. Taking contrapositives of this argument shows that if $y \in X^d \cap Y$, then there must be some i and j for which $y_i = y_j$, so $y \in \Delta$. □

We achieve the constraint in the lemma by dynamically choosing parameters subject to a queueing process. The queue will consist of an ordered sequence of tuples $(I_1, \dots, I_n)$, where $I_1, \dots, I_n$ are disjoint cubes. At stage N of the construction, we take off the front tuple $(I_1, \dots, I_n)$ from the queue, and set $\mathcal{I}_i(N)$ to be the set of all cubes in $\mathcal{B}(R_N)$ which are a subset of both I_i and X_N. We then subdivide X_N using these parameters to form the set X_{N+1} as a union of length $L_{N+1} = R_N^\beta$ intervals. After this, for *any* ordered choice of distinct intervals $I_1, \dots, I_n \in \mathcal{B}(L_{N+1})$, with each interval I_i a subset of X_{N+1}, we add the tuple $(I_1, \dots, I_n)$ to the end of the queue.

If $L_N \to 0$, which will of course be the case, then for any distinct choice of $x_1, \dots, x_n \in X$, there exists N and L_N such that $|x_i - x_j| \geqslant 2L_N$ for all $i \neq j$. Thus at stage N of the construction, we push a tuple $(I_1, \dots, I_n)$ to the end of the queue with $x_i \in I_i$, and at a *much much* later stage M of the construction, this tuple pops off the front of the queue, and so $\mathcal{I}_i(M)$ contains a cube containing x_i. Thus we conclude that X is a solution to the fractal avoidance problem.

4 Dimension Bounds

To complete the proof, it suffices to choose the parameters R_N and β_N which lead to the correct Hausdorff dimension bound on X. The actual choice of β_N doesn't matter, only that it is an increasing sequence converging to β in the limit. We also fix a decreasing sequence λ_N such that $\lambda_N \beta_N > d$, used later on in our argument. Since β_N converges to β from above, we can let λ_N tend to $\lambda = (dn - \alpha)/(n-1)$ from below. The fact that the dissection of X_{N+1} for X_N occurs uniformly over the will aid us in annihilating the super-exponentially increasing constants which inherently occur from the exponentially decreasing values of L_N forced upon us.

We rely on the mass distribution principle to construct a probability measure μ supported on X. This enables us to calculate the Hausdorff dimension of X using Frostman's lemma. We begin by putting the uniform probability measure μ_0 on $X_0 = [0,1]^d$. Then, at each stage of the construction, we construct μ_{N+1} from μ_N by taking the mass on a certain side length L_N cube in X_N, and uniformly distributing it's mass over the side length L_{N+1} cubes in $I \cap X_{N+1}$. Using the weak compactness of the unit ball in $L^1(\mathbf{R}^d)^*$, we get a weak limit $\mu = \lim \mu_n$. The fact that X_n contains the support of μ_n implies X supports μ.

It is intuitive that the mass on μ will distribute more thinly at each stage the fatter the cubes we keep. Quantifying this precisely allows us to apply Frostman's lemma. More precisely, we will prove that for each length L interval I, $\mu(I) \lesssim_N L^{\lambda_N}$. Thus Frostman's lemma guarantees that $\dim_{\mathbf{H}}(X) \geqslant \lambda_N$, and taking $\lambda_N \to \lambda$ will complete the proof.

Lemma 3. *For $R_N \gg 0$, if $I \in \mathcal{B}(L_{N+1})$ and $J \in \mathcal{B}(L_N)$,*

$$\mu(I) \leqslant 2(R_N/L_N)^d \mu(J) \quad \mu(I) \leqslant 2^N R_0^{d-\beta_0} \dots R_N^{d-\beta_N} R_N^d$$

Proof. If I is not a cube in X_{N+1}, then $\mu(I) = \mu_{N+1}(I) = 0$, so the inequality is obviously true. Otherwise, we can find a cube $J \in \mathcal{B}(L_N)$ in $I \cap X_N$. J contains $(L_N/R_N)^d$

side length R_N cubes. Our main discrete result implies that X_{N+1} contains a side length L_{N+1} cube in all but a fraction $o(1)$ of these cubes,. In particular, if we choose R_N sufficiently large, then we know that we keep a side length L_N portion of at least half of these cubes. Thus

$$\begin{aligned}\mu(I) = \mu_{N+1}(I) &\leqslant \frac{\mu_N(J)}{(L_N/R_N)^d/2}\\ &= 2\mu_N(J)(R_N/L_N)^d = 2\mu(J)(R_N/L_N)^d\end{aligned}$$

completing the calculation. Applying this calculation iteratively, we conclude

$$\begin{aligned}\mu(I) &\leqslant 2^N(R_0/L_0)^d(R_1/L_1)^d\dots(R_N/L_N)^d\\ &= 2^N R_0^{d-\beta_0}\dots R_{N-1}^{d-\beta_{N-1}}R_N^d\end{aligned}$$

completing the calculation. □

Corollary. *If $R_N \gg 0$, $\mu(I) \leqslant L_N^{\lambda_N}$ for $I \in \mathcal{B}(L_N)$.*

Proof. We write the inequality in the last problem as

$$\mu(I) \leqslant [2^N R_0^{d-\beta_0}\dots R_{N-1}^{d-\beta_{N-1}}R_N^{d-\lambda_N\beta_N}]L_{N+1}^{\lambda_N}$$

Since $\lambda_N\beta_N > d$, the quantity in the square brackets is $o(1)$ as $R_N \to \infty$. Thus for sufficiently large R_N, we conclude that $\mu(I) \leqslant L_N^{\lambda_N}$. □

This is almost the required inequality, except we have only proven it for intervals at particular scales. To get a general inequality, we use the fact that our construction distributes uniformly across all intervals.

Theorem 4. *If $R_N \gg 0$, then we have $\mu(I) \leqslant 2^{d+1}L^{\lambda_N}$ for all intervals I with side length $L \leqslant L_N$.*

Proof. We break our analysis into three cases, depending on the size of L:

- If $R_N \leqslant L \leqslant L_N$, we can cover I by at most $2^d(L/R_N)^d$ cubes in $\mathcal{B}(R_N)$. For each such cube, we know that the mass on each side length R_N cube is at most $2(R_N/L_N)^d$ times the mass on an element of $\mathcal{B}(L_N)$. Thus

$$\begin{aligned}\mu(I) &\leqslant [2^d(L/R_N)^d][2(R_N/L_N)^d][L_N^{\lambda_N}]\\ &\leqslant \frac{2^{d+1}L^d}{L_N^{d-\lambda_N}} \leqslant 2^{d+1}L^{\lambda_N}\end{aligned}$$

  which gives the required result.

- If $L_{N+1} \leqslant L \leqslant R_N$, we can cover L by at most 2^d cubes in $\mathcal{B}(R_N)$. Each cube in $\mathcal{B}(R_N)$ contains at most one cube in $\mathcal{B}(L_{N+1})$ which is also contained in X_{N+1}, so the bound in the last corollary gives that $\mu(I) \leqslant 2^d L_{N+1}^{\lambda_N} \leqslant 2^d L^{\lambda_N}$.

- If $L \leqslant L_{N+1}$, there certainly exists M such that $L_{M+1} \leqslant L \leqslant L_M$, and one of the previous cases yields that $\mu(I) \leqslant 2^{d+1}L^{\lambda_M} \leqslant 2^{d+1}L^{\lambda_N}$.

This covers all possible situations, completing the proof. □

To use Frostman's lemma, we need the result $\mu(I) \lesssim L^{\lambda_N}$ for an *arbitrary* interval, not just one with $L \leqslant L_N$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. This is because if $L \geqslant L_N$, then $\mu(I)/L^{\lambda_N} \leqslant 1/L_N^{\lambda_N} \lesssim_N 1$, so $\mu(I) \lesssim_N L^{\lambda_N}$ holds automatically for all sufficiently large intervals. Thus all problems with the Hausdorff dimension argument are complete, and we have proven that there is a choice of parameters which constructs a set X with Hausdorff dimension no less than $(nd-\alpha)/(n-1)$ (and by looking at the way we dissect our intervals at each scale, it is easy to see that X has *precisely* this dimension).

5 Low Rank Projection Method

6 Comparison with Other Generic Avoidance Schemes

7 Concluding Remarks

References

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