Fractals avoiding Fractal Sets

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- How large can a subset of $\mathbf{R}^d$ be such that the angles formed from any three points in X are irrational?

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  - The distances between two pairs x, y and z, w are non distinct when $f(x, y; z, w) = d(x, y) - d(z, w) = 0$.
  - The angles formed by three points are irrational when $f(x, y, z) = (x - z)(y - z)/|x - z||y - z| \notin \cos(\mathbf{Q})$.

General Results

- **Configuration Avoidance Problem**: Given $f : [0,1]^{nd} \to \mathbf{R}$, find $X \subset \mathbf{R}^d$ with high Hausdorff dimension such that for any distinct $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$.

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- Mathé (2017) proved that if f is a polynomial of degree r, we can find X with dimension d/r.
- Pramanik and Fraser (2018) proved that if f is smooth and nonsingular, we can find X with dimension $d/(n-1)$.

Increasing the Difficulty

What if the zero sets of the function are also fractals...

Main Result

Theorem

If the zero set of f is α dimensional, we can find X with

$$\dim_{\mathbf{H}}(X) = \frac{nd - \alpha}{n-1} = \frac{codim_{\mathbf{H}}(f^{-1}(0))}{n-1}$$

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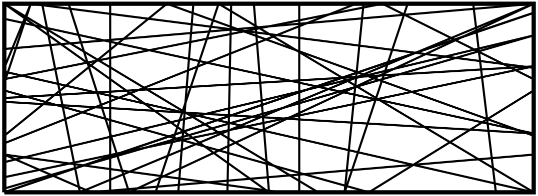
- Extension of Pramanik and Fraser's result (2018), who proved the case where f is smooth and nonsingular, so $f^{-1}(0)$ is a smooth surface of dimension $nd - d$.
- Shows smoothness is only needed to get a bound on the Hausdorff dimension. Not needed anywhere else.

Applications



- More robust result of Pramanik and Fraser, showing that we can 'thicken' or 'thin' the zero set of our function with stable effects on the Hausdorff dimension of X.

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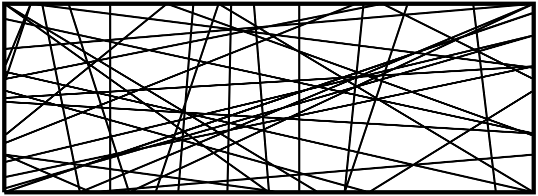

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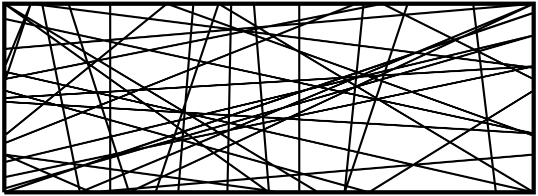

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- Uncountable unions of regular sets are allowed!
- General results in this direction assume f is regular. We don’t assume anything about f.

Applications

- Given a continuous $\gamma : [0,1] \to \mathbf{R}^d$, finding $X \subset [0,1]$ such that $\gamma(x), \gamma(y), \gamma(z)$ do not form an 'arithmetic progression' on the curve, in the sense that

  $$(\gamma(x) - \gamma(y)) - (\gamma(y) - \gamma(z)) = 0$$

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  Pramanik and Fraser can avoid smooth curves. We can now avoid any curve given a Hausdorff dimension bound. Potentially even a Hilbert curve?
- Given a subset Y of $\mathbf{R}^d$ with dimension α, we can find a subset X of dimension $d - \alpha$ such that $X + X$ and $X - X$ are disjoint from Y. I hope to extend this shortly with the additional criterion that the set X we construct is a rational vector space, so for instance $X + X + X$ is disjoint from Y. Then $X \oplus Y$ is a full dimensional subspace of $\mathbf{R}$.

The Method

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- Random Disection.

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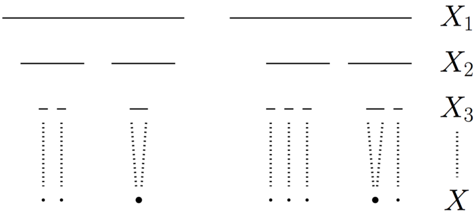
- We can view this as a 'fine scale property', since it involves the infinite digit expansion of a number.
- Scale discretization changes fine-scale configurations into an infinite sequence of discrete configurations which are easier to achieve.
- **Discrete Problem**: Given X_n, find X_{n+1} such that no element of X_{n+1} has 1 as the n'th digit in it's expansion.

Discretization of Scales



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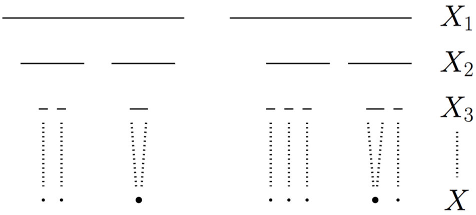

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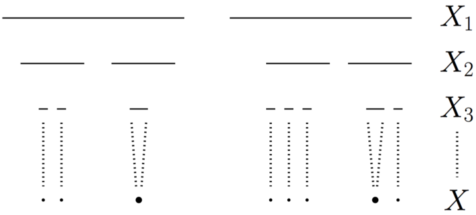

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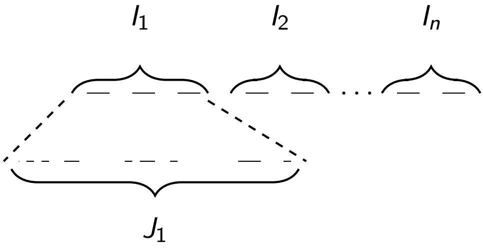
- **Discrete Configuration Problem**: If $f(x_1, \ldots, x_n) = 0$ for $x_1, \ldots, x_n \in X_n$, then some x_i and x_j occur in a common interval.
- If the interval lengths in X_n tend to zero as $n \to \infty$, and $x_1, \ldots x_n \in X$ have $f(x_1, \ldots, x_n) = 0$, then there is i, j such that $|x_i - x_j| = o(1)$, so $x_i = x_j$.

Discrete Scale Problem



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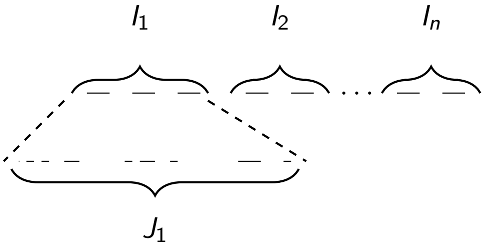

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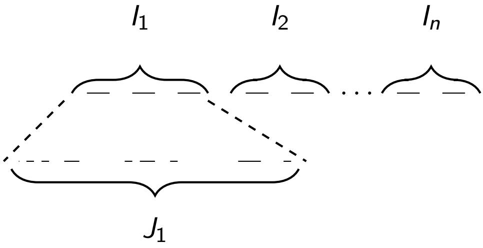

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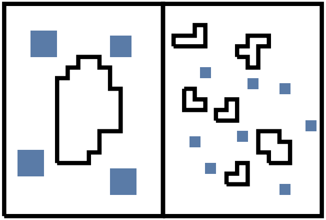
- Using a queueing procedure, if the subsets we choose cover our entire set infinitely often, we still avoid the finte scale configuration. See Pramanik and Fraser (2018) for the same technique applied to their problem.

Exploiting Randomness



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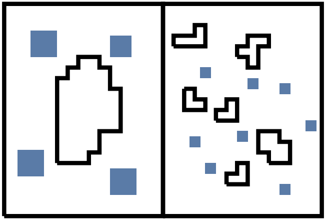

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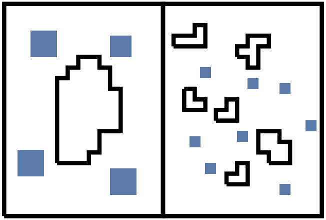

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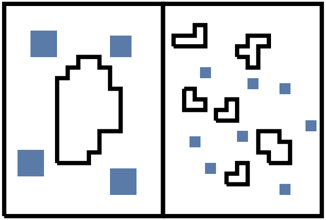

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- Random choices of the J_k avoid $f^{-1}(0)$ effectively.
- We obtain that for all but $o(1)$ of these sections, J_k contains a length $1/N^{\beta}$ section, where $\beta = d(nd - \alpha)/(n-1)$. This ratio gives the Hausdorff dimension bound $(nd - \alpha)/(n-1)$ for X.

Conclusion

So What's Next?

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- Our techniques 'should' work when $f^{-1}(0)$ has *Hausdorff dimension* α.
- We are trying to use hyperdyadic coverings to obtain this.

Analogies with Hypergraphs

- Partition $I_1, \ldots, I_n$ into length $1/N^\beta$ sections. Form a hypergraph whose vertices are the sections, and add a hyperedge between $K_1 \subset I_1, \ldots, K_n \subset I_n$ if $K_1 \times \cdots \times K_n$ intersects $f^{-1}(0)$.

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- Our discrete configuration problem reduces to finding independant sets in hypergraphs.
- The method we use essentially generalizes Turan's theorem to hypergraphs, and is tight?
- We are looking to using other methods on hypergraphs to improve the bound when $f^{-1}(0)$ has certain structural properties.

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