Large Salem Sets Avoiding Configurations

Jacob Denson

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1 Introduction

Geometric measure theory explores the relationship between the geometry of subsets of Euclidean spaces, and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a particular subset of Euclidean space. In this paper, we focus on the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a pattern, we construct a family of compact sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel subset E of $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$ is it's *Fourier dimension*; given a finite Borel measure μ, it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$ is the supremum of all $s \in [0, d]$ such that $\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty$. The Fourier dimension of a Borel set E is then the supremum of $\dim_{\mathbf{F}}(\mu)$, where μ ranges over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most pattern avoiding sets constructed in the literature are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem. We prove two results, one avoiding 'rough' patterns, in the sense of [1], and the second avoiding patterns formed by a family of smooth equations. Our methods are also generic; we consider a metric space $\mathcal{X}_\beta$ which essentially consists of all subsets of $\mathbf{T}^d$ with Fourier dimension β, and show that for any appropriate choice of β, the family of all Salem sets in this metric space which avoid a particular pattern are the complement of a set of first category in $\mathcal{X}_\beta$.

Theorem 1.1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $E \subset \mathbf{E}$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n - 1/2}, d\right)$$

such that for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$. Moreover, we can define a metric structure on the space $\mathcal{X}_\beta$ of all pairs (E, μ), where μ is supported on E and $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, such that the set of all (E, μ) such that E is Salem and avoids the pattern generated by Z is the complement of a set of first category in $\mathcal{X}_\beta$.

Remark 1.2. *Theorem 1.1 is an attempt to strengthen the main result of [1] to construct a set with a Fourier dimension bound rather than a Hausdorff dimension bound, albeit under a weaker dimension bound than that obtained in [1]. Unlike in [1], the case of Theorem 1.1 when $0 \leqslant \alpha < d/2$ is still interesting, since it is non-obvious that the trivial construction $[0,1]^d - \pi(Z)$ is necessarily a Salem set, where $\pi(x_1, \ldots, x_n) = x_1$ is projection onto the first coordinate. That this set is not necessarily Salem is hinted at in Example 8 of [3], where it is shown that there exists a compact set $E \subset [0,1]$ such that $\dim_{\mathbf{M}}(E) < 1$ and $\dim_{\mathbf{F}}([0,1] - E) < 1$. Setting $Z = E \times \{0\} \cup \{0\} \times E$ shows that neither subtracting projections onto the first nor the second coordinate gives a Salem set.*

Under the stronger assumption that Z is a countable union of hypersurfaces satisfying a certain geometric condition (that these hypersurfaces are transverse to any axis-oriented hyperplane), we are able to recover the dimension bound originally proved in the Hausdorff dimension setting in [4].

Theorem 1.3. *Suppose $Z \subset \mathbf{E}^n$ is a countable union of sets of the form*

$$\{(y, x) \in U \times V : y = f(x_1, \ldots, x_{n-1})\}$$

where U is an open set of $\mathbf{E}$, V is an open set of $\mathbf{E}^{n-1}$, and $f : V \to U$ is a smooth map such that for each $k \in \{1, \ldots, n-1\}$, the matrix

$$D_{x_k} f = \left(\frac{\partial f_i}{\partial x_k^j} \right)$$

is invertible at any collection of distinct values $x_1, \ldots, x_{n-1}$. Then there exists a compact Salem set $E \subset \mathbf{E}$ with dimension

$$\beta = \frac{d}{n-1}$$

such that for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$. Again, if $\mathbf{E} = \mathbf{T}^d$, then the family of Salem sets E avoiding Z in $\mathcal{X}_\beta$ is the complement of a set of first category.

Remark 1.4. *If $n = 2$, the avoidance problem for a continuous function $f : V \to U$ is essentially trivial. If there exists $x \in \mathbf{T}^d$ such that $f(x) \neq x$, there there exists an open set U around x such that $U \cap f(U) = \varnothing$. Then U has full Fourier dimension, and avoids solutions to the equation $y = f(x)$. On the other hand, if $f(x) = x$ for all x, then there are no distinct x and y in $[0,1]$ such that $y = f(x)$, and so the problem is also trivial. On the other hand, it is less trivial that a generic set avoids this pattern, which Theorem 1.3 proves.*

Because we are working with *compact* sets avoiding patterns, working in the domain $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d$. But working in $\mathbf{T}^d$ has several technical advantages over $\mathbf{R}^d$, which is why in this paper we have chosen to work with the pattern avoidance pattern in this setting. Given a Borel measure μ on $\mathbf{R}^d$, we define the Fourier dimension $\dim_{\mathbf{F}}(\mu)$ of μ to be the supremum of all $s \in [0, d]$ such that $\sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x)\, d\mu^*(x) = \int_{\mathbf{R}^d} f(x)\, d\mu(x), \tag{1.1}$$

then $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. A proof is given in Lemma 39 of [2]. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$ (this can be done, for instance, by equating Frostman measure conditions). Using these results, one can reduce the study of patterns on $\mathbf{R}^{dn}$ to patterns on $\mathbf{T}^{dn}$, which we can identify with $[0,1]^{dn}$, and thus obtain analogous results to Theorems 1.1 and 1.3 in the domain $\mathbf{R}^d$ instead of $\mathbf{T}^d$.

It is expected that Theorem 1.3 is tight for general patterns Z. If E is Salem and has dimension $d/(n-1)$, then $f(E^n)$ is a subset of $\mathbf{T}^{d(n-1)}$ with nonempty interior, because

$$\begin{aligned}\int e^{-2\pi i \xi \cdot y} df_*(\mu^{\otimes})(y) &= \int e^{-2\pi i \xi \cdot f(x)} d\mu(x_1) \dots d\mu(x_n)\\ &= \lim_{k \to \infty} \int e^{-2\pi i \xi \cdot f(x)} \phi_k^{\otimes}(x) dx\\ &= \lim_{k \to \infty} \int e^{-2\pi i \xi \cdot (x_1 + \dots + x_n)} \det(D_{x_1} f) \phi_k(g(z, x_2, \dots, x_n)) \phi_k^{\otimes}(x)\, dx\end{aligned}$$

where $f(g(z, x_2, \dots, x_n), x_2, \dots, x_n) = z$. On the other hand, for patterns with richer structure this result is certainly non-optimal. For instance, in BLAH a Salem set in $\mathbf{R}$ of dimension one is constructed avoiding solutions to the equation $x_3 = 2x_2 - x_1$; our techniques only guarantee the existence of a Salem set of dimension 1/2.

2 Examples

3 ADJAWIOJ

3.1 Isosceles Triangles

$f(t_1, t_2, t_3) = |\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2$

$$\gamma(t_1) = (t_1, f(t_1))$$

$$|t_1 - t_2|^2 + |f(t_1) - f(t_2)|^2 = |t_2 - t_3|^2 + |f(t_2) - f(t_3)|^2$$

3.2 Sets Avoiding Arithmetic Equations

In [7], it is shown that for each $n > 0$, there exists a set $E \subset \mathbf{T}$ with Fourier dimension $1/(n-1)$ such that for any $x_1, \dots, x_n \in E$ and any integers $m_1, \dots, m_n \in \mathbf{Z}$, not all zero, $m_1 x_1 + \dots + m_n x_n \neq 0$. We note that the set

$$Z_n = \{(x_1, \dots, x_n) \in \mathbf{T}^{dn} : m_1 x_1 + \dots + m_n x_n = 0 \text{ for some nonzero } m \in \mathbf{Z}^n\} \tag{3.1}$$

is a countable union of hyperplanes in $\mathbf{T}^{dn}$. Our methods directly apply to the set

$$Z_m' = \left\{ (x_1, \dots, x_n) \in \mathbf{T}^{dn} : \begin{array}{c}\text{there are integers } m_1, \dots, m_n \in \mathbf{Z}, \text{ all of which}\\ \text{nonzero, such that } m_1 x_1 + \dots + m_n x_n = 0.\end{array} \right\}.$$

For each set of nonzero $m_1, \dots, m_n$, the condition $m_1 x_1 + \dots + m_n x_n = 0$ holds if and only if $x_n = (-m_1/m_n) x_1 + \dots + (-m_{n-1}/m_n) x_{n-1}$, so that Z_m' is a countable union of sets formed by equations to which Theorem 1.3 applies. In particular, the proof of this theorem will show that if $\beta_m = d/(m-1)$, then for any $\beta \leqslant \beta_m$, a generic element of $\mathcal{X}_\beta$ avoids Z'_m. In particular, taking finite intersections, a generic element of $\mathcal{X}_{\beta_n}$ avoids the pattern $Z_n = \bigcup_{m=2}^n Z'_m$, so we recover Korner's result. Our proof also gives a d dimensional generalization, which cannot be addressed by Korner's techniques. The arguments in this paper are heavily inspired by the techniques of [7], but augmented with novel applications of probabilistic concentration inequalities, which enables us to push the results of [7] to a much more general family of patterns. In particular, our arguments show that the results of that paper do not depend on the rich arithmetic structure of the pattern Z in (3.1), but rather on the dimension of the hypersurfaces which define the pattern Z.

4 Notation

- Given a metric space X, $x \in X$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. We let δ_x denote the Dirac delta measure at x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let
$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$
denote the *ε-thickening* of the set E.

- A subset of a metric space X is of *first category*, or *meager* in X if it is the countable union of closed sets with empty interior, and is *comeager* if it is the complement of such a set. We say a property holds *quasi-always*, or a property is *generic* in X if the set of points in X satisfying that property is comeager. The Baire-category theorem shows that any comeager set in a complete metric space is dense.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let
$$|x| = \min\{|x+n| : n \in \mathbf{Z}\},$$
and for $x \in \mathbf{T}^d$, we let
$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$
The canonical metric on $\mathbf{T}^d$ is then given by $d(x,y) = |x-y|$, for $x, y \in \mathbf{T}^d$. For an axis-oriented cube Q in $\mathbf{T}^d$, we let $2Q$ be the cube in $\mathbf{T}^d$ with the same centre and twice the sidelength.

- Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0,d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as
$$H^\alpha_\delta(E) = \inf\left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } 0 < \varepsilon_i \leqslant \delta \text{ for all } i \geqslant 1 \right\}.$$
The α dimensional Hausdorff measure of E is equal to
$$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0, d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0, d]$ such that $H^s(E) = 0$. Frostman's lemma says that if we define the Hausdorff dimension $\dim_{\mathbf{H}}(\mu)$ of a finite Borel measure μ as the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ \mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{E}, \varepsilon > 0 \right\} < \infty, \tag{4.1}$$

then $\dim_{\mathbf{H}}(E)$ is the supremum of $\dim_{\mathbf{H}}(\mu)$, over all Borel probability measures μ supported on E. This is analogous to the definition of the Fourier dimension of a set E given in the introduction.

For a measurable set $E \subset \mathbf{E}$, we let $|E|$ denote it's Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{r \to 0} d - \log_r |E_r|.$$

Thus $\underline{\dim}_{\mathbf{M}}(E)$ is the largest number such that for $\alpha < \underline{\dim}_{\mathbf{M}}(E)$, there exists a decreasing sequence $\{r_i\}$ with $\lim_{i \to \infty} r_i = 0$ and $|E_{r_i}| \leqslant r_i^{d-\alpha}$ for each i.

- In this paper we will need to employ probabilistic concentration bounds several times. In particular, we use *McDiarmid's inequality*, trivially modified from the standard theorem to work with complex-valued functions. Let $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. Let $\{X_1, \dots, X_N\}$ be an independant family of random variables, and consider a function $f : \mathbf{E}^N \to \mathbf{C}$. Suppose that for each $i \in \{1, \dots, N\}$, there exists a constant $A_i > 0$ such that for any $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N \in \mathbf{E}$, and for each $x_i, x_i' \in \mathbf{E}$,

$$|f(x_1, \dots, x_i, \dots, x_N) - f(x_1, \dots, x_i', \dots, x_N)| \leqslant A_i.$$

Then McDiarmid's inequality guarantees that for all $t \geqslant 0$,

$$\mathbf{P}\left(|f(X_1, \dots, X_N) - \mathbf{E}(f(X_1, \dots, X_N))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-2t^2}{A_1^2 + \dots + A_N^2} \right).$$

The complex-valued extension we have just stated is proved easily from the real-valued case by taking a union bound to the inequality for the real and imaginary values of f. Proofs of McDiarmid's inequality are given in many probability texts, for instance, in Theorem 3.11 of [6].

A special case of McDiarmid's inequality is *Hoeffding's Inequality*. The version of Hoeffding's inequality we use states that if $\{X_1, \dots, X_N\}$ is a family of independant random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

$$\mathbf{P}\left(|X_1 + \dots + X_N - \mathbf{E}(X_1 + \dots + X_N)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-t^2/2}{A_1^2 + \dots + A_N^2} \right).$$

- Throughout this paper, we will need to consider a standard mollifier. So we fix a smooth, non-negative function $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$ and

$$\int_{\mathbf{T}^d} \phi(x)\, dx = 1.$$

For each $r \in (0,1)$, we can then define $\phi_r \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_r(x) = \begin{cases} r^{-d}\phi(x/r) & : |x| < r, \\ 0 & : \text{otherwise.} \end{cases}$$

The following standard properties hold:

(1) For each $r \in (0,1)$, ϕ_r is a non-negative smooth function with

$$\int_{\mathbf{T}^d} \phi_r(x)\, dx = 1, \tag{4.2}$$

and $\phi_r(x) = 0$ for $|x| \geqslant r$.

(2) For any $r \in (0,1)$,

$$\|\widehat{\phi}_r\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1. \tag{4.3}$$

(3) For each $\xi \in \mathbf{Z}^d$,

$$\lim_{r \to 0} \widehat{\phi}_r(\xi) = 1. \tag{4.4}$$

(4) For each $T > 0$, and for all $r > 0$ and any non-zero $\xi \in \mathbf{Z}^d$,

$$|\widehat{\phi}_r(\xi)| \lesssim_T r^{-T}|\xi|^{-T}. \tag{4.5}$$

5 Overview

6 A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task and establish a set of tools for obtaining convergence in this metric space. In later sections we will fix a specific choice of β to avoid a particular pattern. But in this section we let β be an arbitrary element of $(0,d]$. Our approach in this section is heavily influenced by [7]. However, we employ a Frechét space construction instead of the Banach space used in [7], which enables us to use softer estimates in our arguments:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

$$d_{\mathbf{H}}(E,F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

then $(\mathcal{E}, d_{\mathbf{H}})$ forms a complete metric space.

- We let $M(\beta/2)$ consist of the class of all finite Borel measures μ on $\mathbf{T}^d$ such that for each $\varepsilon \in (0, \beta/2]$, the quantity

$$\|\mu\|_{M(\beta/2-\varepsilon)} = \sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon}$$

is finite. Then $\|\cdot\|_{M(\beta/2-\varepsilon)}$ is a seminorm on $M(\beta/2)$, and the collection of all such seminorms for $\varepsilon \in (0, \beta/2]$ gives $M(\beta/2)$ the structure of a Frechét space. Under this topology, a sequence of probability measures $\{\mu_k\}$ converges to a probability measure μ in $M(\beta/2)$ if and only if for any $\varepsilon > 0$, $\lim_{k\to\infty} \|\mu_k - \mu\|_{M(\beta/2-\varepsilon)} = 0$.

We now let $\mathcal{X}_\beta$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(\beta/2)$, where μ is a probability measure such that $\text{supp}(\mu) \subset E$. Then $\mathcal{X}_\beta$ is a closed subset of $\mathcal{E} \times M(\beta/2)$ under the product metric, and thus a complete metrizable space. We remark that for any $\varepsilon > 0$ and $(E, \mu) \in \mathcal{X}_\beta$,

$$\lim_{|\xi|\to\infty} |\xi|^{\beta/2-\varepsilon} |\widehat{\mu}(\xi)| = 0, \tag{6.1}$$

which follows because $\|\mu\|_{M(\beta/2-\varepsilon/2)}$ is finite. Thus $\dim_{\mathbf{F}}(\mu) \geqslant \beta$ for each $(E, \mu) \in \mathcal{X}_\beta$.

Combined with out quantitative approach to the problem, the next lemma enables us to work with smooth measures, without loss of generality, for the remainder of the paper.

Lemma 6.1. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Proof. Consider $(E, \mu) \in \mathcal{X}_\beta$. For each $r \in (0, 1)$, consider the convolved measure $\mu_r = \mu * \phi_r$. Then $\mu_r \in C^\infty(\mathbf{T}^d)$. We claim that $\lim_{r\to 0}(E_r, \mu_r) = (E, \mu)$. Since $\text{supp}(\mu_r) \subset E_r$, we find that $d_{\mathbf{H}}(E, E_r) \leqslant r$, and so $\lim_{r\to 0} E_r = E$. Now fix $\varepsilon_1 \in (0, \beta/2]$ and $\varepsilon > 0$. For each $\xi \in \mathbf{Z}^d$, $|\widehat{\mu_r}(\xi)| = |\widehat{\phi_r}(\xi)||\widehat{\mu}(\xi)|$, so

$$|\xi|^{\beta/2-\varepsilon_1} |\mu_r(\xi) - \mu(\xi)| = |\xi|^{\beta/2-\varepsilon_1} |\widehat{\phi_r}(\xi) - 1||\widehat{\mu}(\xi)|. \tag{6.2}$$

We control (6.2) over large frequencies using the term $|\widehat{\mu}(\xi)|$, and by small frequencies using the term $|\widehat{\phi_r}(\xi) - 1|$. Since $(E, \mu) \in \mathcal{X}_\beta$, we can apply (6.1) to find $R > 0$ such that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1} |\widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{6.3}$$

Combining (6.2), (6.3), and (4.3), for $|\xi| \geqslant R$ we find that

$$|\xi|^{\beta/2-\varepsilon_1} |\mu_r(\xi) - \mu(\xi)| \leqslant 2\varepsilon. \tag{6.4}$$

On the other hand, (4.4) shows that there exists $r_0 > 0$ such that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon} |\widehat{\phi_r}(\xi) - 1| \leqslant \varepsilon. \tag{6.5}$$

The (L^1, L^∞) bound for the Fourier transform implies that $|\widehat{\mu}(\xi)| \leqslant \mu(\mathbf{T}^d) = 1$, which combined with (6.5) gives that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1} |\mu_r(\xi) - \mu(\xi)| \leqslant \varepsilon. \tag{6.6}$$

Putting together (6.4) and (6.6), shows that for $r \leqslant r_0$, $\|\mu_r - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant 2\varepsilon$. Since ε and ε_1 were arbitrary, $\lim_{r\to 0} \mu_r = \mu$, completing the proof. $\square$

Remark 6.2. *Let*

$$\tilde{\mathcal{X}}_\beta = \{(E, \mu) \in \mathcal{X}_\beta : \mathit{supp}(\mu) = E\}.$$

Suppose $(E_0, \mu_0) \in \tilde{\mathcal{X}}_\beta$. Then, in the proof above, one may let E_r be equal to $supp(\mu_r)$, since it follows from this that $d_{\mathbf{H}}(E_0, E_r) \leqslant r$. This means that the set of pairs $(E, \mu) \in \tilde{\mathcal{X}}_\beta$ with $\mu \in C^\infty(\mathbf{T}^d)$ are dense in $\tilde{\mathcal{X}}_\beta$.

The reason we must work with $\mathcal{X}_\beta$ rather than the smaller space $\tilde{\mathcal{X}}_\beta \subset \mathcal{X}_\beta$ is that $\tilde{\mathcal{X}}_\beta$ is not a closed subset of $\mathcal{X}_\beta$, and so is not a complete metric space. However, as a consolation, quasi-all elements of $\mathcal{X}_\beta$ belong to $\tilde{\mathcal{X}}_\beta$, so that one can think of $\mathcal{X}_\beta$ and $\tilde{\mathcal{X}}_\beta$ as being equal 'generically'.

Lemma 6.3. *For quasi-all $(E,\mu) \in \mathcal{X}_\beta$, supp$(\mu) = E$.*

Proof. For each closed cube $Q \subset \mathbf{T}^d$, let

$$A(Q) = \{(E,\mu) \in \mathbf{T}^d : (E \cap Q) = \varnothing \text{ or } \mu(Q) > 0\}.$$

Then $A(Q)$ is an open set. If $\{Q_k\}$ is a sequence enumerating all cubes with rational corners in $\mathbf{T}^d$, then

$$\bigcap_{k=1}^\infty A(Q_k) = \{(E,\mu) \in \mathcal{X}_\beta : \operatorname{supp}(\mu) = E\}.$$

Thus it suffices to show that $A(Q)$ is dense in $\mathcal{X}_\beta$ for each closed cube Q. Thus we fix $(E_0,\mu_0) \in \mathcal{X}_\beta - A(Q)$, $\varepsilon_1 \in (0,\beta/2]$, and $\varepsilon > 0$, and try and find $(E,\mu) \in A(Q)$ with $d_{\mathbf{H}}(E,E_0) \leqslant \varepsilon$ and $\|\mu_0 - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Applying Lemma 6.1, we may assume without loss of generality that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Because $(E_0,\mu_0) \in \mathcal{X}_\beta - A(Q)$, we know $E \cap Q \neq \varnothing$ and $\mu(Q) = 0$. Find a smooth probability measure ν supported on $E_\varepsilon \cap Q$ and, for $t \in (0,1)$, define $\mu_t = (1-t)\mu_0 + t\nu$. Then $\operatorname{supp}(\mu_t) \subset E_\varepsilon$, so if we let $E = \operatorname{supp}(\nu) \cup \operatorname{supp}(\mu)$, then $d_{\mathbf{H}}(E,E_0) \leqslant \varepsilon$. Clearly $(E,\mu_t) \in A(Q)$ for $t > 0$. And

$$\|\mu_t - \mu_0\|_{M(\beta/2-\varepsilon)} \leqslant t\left(\|\mu_0\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)}\right),$$

so if we choose $t \leqslant \varepsilon(\|\mu\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)})^{-1}$ we find $\|\mu_t - \mu\|_{M(\beta/2-\varepsilon)} \leqslant \varepsilon$. Since ε was arbitrary, we conclude $A(Q)$ is dense in $\mathcal{X}_\beta$. □

Combining Lemma 6.3 with Remark 6.2 gives the following simple corollary.

Corollary 6.4. *The family of (E,μ) with supp$(\mu) = E$ and $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Our main way of constructing approximations to $(E_0,\mu_0) \in \mathcal{X}_\beta$ is to multiply μ_0 by a smooth function $f \in C^\infty(\mathbf{T}^d)$. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which contribute to the formation of a pattern we are trying to avoid. As long as μ_0 is appropriately smooth, and the Fourier transform of f decays appropriately quickly, the next lemma shows that $f\mu_0 \approx \mu_0$.

Lemma 6.5. *Consider a finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function $f \in C^\infty(\mathbf{T}^d)$. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(\beta/2)} \lesssim_d \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}.$$

Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu_0}$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\xi|^{\beta/2}|\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| = |\xi|^{\beta/2}\left|\sum_{\eta \neq \xi} \widehat{f}(\xi - \eta)\widehat{\mu_0}(\eta)\right|. \tag{6.7}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi-\eta| \leqslant 2|\xi|$, so

$$|\xi|^{\beta/2}|\widehat{f}(\xi-\eta)| \leqslant \|f\|_{M(\beta/2)}|\xi|^{\beta/2}|\xi-\eta|^{-\beta} \leqslant 2^{\beta/2}\|f\|_{M(\beta/2)} \lesssim_d \|f\|_{M(\beta/2)}. \tag{6.8}$$

Thus the bound (6.8) implies

$$\begin{aligned} |\xi|^{\beta/2}\left|\sum_{0\leqslant|\eta|\leqslant|\xi|/2}\widehat{f}(\xi-\eta)\widehat{\mu_0}(\eta)\right| &\lesssim_{\mu_0,d} \|\mu_0\|_{M(d+1)}\|f\|_{M(\beta/2)}\left(1+\sum_{0<|\eta|\leqslant|\xi|/2}\frac{1}{|\eta|^{d+1}}\right) \\ &\lesssim_d \|\mu_0\|_{M(d+1)}\|f\|_{M(\beta/2)} \leqslant \|\mu_0\|_{M(3d/2)}\|f\|_{M(\beta/2)}. \end{aligned} \tag{6.9}$$

On the other hand, for all $\eta \neq \xi$,

$$|\widehat{f}(\xi-\eta)| \leqslant \|f\|_{M(\beta/2)}|\xi-\eta|^{-\beta} \leqslant \|f\|_{M(\beta/2)}. \tag{6.10}$$

Thus we calculate that

$$\begin{aligned} |\xi|^{\beta/2}\left|\sum_{\substack{|\eta|>|\xi|/2\\ \eta\neq\xi}}\widehat{f}(\xi-\eta)\widehat{\mu_0}(\eta)\right| &\lesssim_{d,\mu_0} \|\mu_0\|_{M(3d/2)}\|f\|_{M(\beta/2)}\cdot|\xi|^{\beta/2}\sum_{|\eta|>|\xi|/2}\frac{1}{|\eta|^{3d/2}} \\ &\lesssim_d \|\mu_0\|_{M(\beta/2)}\|f\|_{M(\beta/2)}. \end{aligned} \tag{6.11}$$

Combining (6.7), (6.9) and (6.11) completes the proof. □

Remark 6.6. *In particular, we note that this lemma implies that* $1-\mu(\mathbf{T}^d) \lesssim_{d,\mu_0} \|f\|_{M(0)}$.

The bound in Lemma 6.5, if $\|f\|_{M(\beta/2)}$ is taken appropriately small, also implies that the Hausdorff distance between the supports of μ and μ_0 is small.

Lemma 6.7. *Fix a probability measure* $\mu_0 \in C^\infty(\mathbf{T}^d)$*. For any* $\varepsilon > 0$*, there exists* $\delta > 0$ *depending on* μ_0*,* ε*, and* d*, such that if* $\mu \in C^\infty(\mathbf{T}^d)$*,* $supp(\mu) \subset supp(\mu_0)$*, and* $\|\mu_0-\mu\|_{M(\beta/2)} \leqslant \delta$*, then* $d_{\mathbf{H}}(supp(\mu), supp(\mu_0)) \leqslant \varepsilon$*.*

Proof. Consider any cover of $\mathrm{supp}(\mu_0)$ by a family of radius $\varepsilon/3$ balls $\{B_1,\dots,B_N\}$, and for each $i \in \{1,\dots,N\}$, consider a smooth function $f_i \in C_c^\infty(B_i)$ such that there is $s > 0$ with

$$\int f_i(x)d\mu_0(x) \geqslant s. \tag{6.12}$$

for each $i \in \{1,\dots,N\}$. Fix $A > 0$ with

$$\sum_{\xi\neq 0}|\widehat{f}_i(\xi)| \leqslant A \tag{6.13}$$

for all $i \in \{1,\dots,N\}$ as well. Set $\delta = s/2A$. If $\|\mu_0-\mu\|_{M(\beta/2)} \leqslant \delta$, we apply Plancherel's inequality together with (6.12) and (6.13) to conclude that

$$\begin{aligned} \left|\int f_i(x)d\mu(x)\,dx - \int f_i(x)d\mu_0(x)\right| &= \left|\sum_{\xi\in\mathbf{Z}^d}\widehat{f}_i(\xi)\left(\widehat{\mu}(\xi)-\widehat{\mu_0}(\xi)\right)\right| \\ &\leqslant A\|\mu_0-\mu\|_{M(\beta/2)} \leqslant s/2. \end{aligned} \tag{6.14}$$

Thus we conclude from (6.12) and (6.14) that

$$\int f_i(x)d\mu(x)\ dx \geqslant \int f_i(x)d\mu_0(x) - s/2 \geqslant s/2 > 0. \tag{6.15}$$

Since equation (6.15) holds for each $i \in \{1,\dots,N\}$, the support of μ intersects every ball in $\{B_1,\dots,B_N\}$. Combined with the assumption that $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu_0)$, this implies that $d_{\mathbf{H}}(\mu_0,\mu) \leqslant \varepsilon$. □

To obtain a smooth function f to which we can apply Lemmas 6.5 and 6.7, we take a measure η, which is a linear combination of Dirac deltas, and set $f = \eta * \phi_r$. To obtain the appropriate control on $\widehat{f}$, it suffices to have a decay bound for $\widehat{\eta}$ for $|\xi| \leqslant 1/r$, and a weaker bound for $|\xi|$ slightly bigger than $1/r$, which is required before we can take complete advantage of the Fourier decay of the mollifier ϕ_r for $|\xi| \geqslant 1/r$.

Lemma 6.8. *Fix $C > 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, with $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ depending on all these quantities, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad \text{for } 0 < |\xi| \leqslant (1/r) \tag{6.16}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2} \log(1/r)^{1/2} \qquad \text{for } (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{6.17}$$

if we define $f(x) = (\eta * \phi_r)(x)$, then $\|f\|_{M(\beta/2-\varepsilon_2)} \leqslant 2\varepsilon$.

Proof. For each $\xi \in \mathbf{Z}^d$,

$$\widehat{f}(\xi) = \widehat{\eta}(\xi)\widehat{\phi_r}(\xi). \tag{6.18}$$

For $|\xi| \leqslant 1/r$ we combine (6.16), (6.18) and (4.3) to conclude that

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{6.19}$$

If $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$, (4.5) implies $|\widehat{\phi_r}(\xi)| \lesssim_\beta r^{-\beta/2}|\xi|^{-\beta/2}$, which together with (6.17), (6.18), and (4.3), show that for $r \leqslant r_1$,

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_\beta \left(Cr^{\beta/2}\log(1/r)^{1/2}\right)\left(r^{-\beta/2}|\xi|^{-\beta/2}\right) \\ &\leqslant C\log(1/r)^{1/2} \cdot |\xi|^{-\beta/2} \\ &\leqslant Cr^{\varepsilon_2}\log(1/r)^{1/2} \cdot |\xi|^{\varepsilon_2-\beta/2}. \end{aligned} \tag{6.20}$$

Since $Cr^{\varepsilon_2}\log(1/r)^{1/2} \to 0$ as $r \to 0$, so we conclude from (6.20) that there exists $r_1 > 0$ such that for $r \leqslant r_1$ and $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{6.21}$$

If $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$, we apply (4.5) for $T \geqslant \beta/2$ to conclude

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_T r^{-T}|\xi|^{-T} \\ &= r^{-T}|\xi|^{\beta/2-T} \cdot |\xi|^{-\beta/2} \\ &\leqslant r^{-T}(1/r)^{(\beta/2-T)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2} \\ &= r^{\varepsilon_1 T-(\beta/2)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2}. \end{aligned} \tag{6.22}$$

If we choose $T > (\beta/2)(1 + 1/\varepsilon_1)$, then as $r \to 0$, $r^{\varepsilon_1 T - (\beta/2)(1+\varepsilon_1)} \to 0$. Thus we conclude from (6.22) that there exists $r_2 > 0$ satisfying such that for $0 < r \leqslant r_2$ and $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{-\beta/2} \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{6.23}$$

All that remains is to combine (6.19), (6.21), and (6.23), defining $r_0 = \min(r_1, r_2)$. □

Corollary 6.9. *Fix $C > 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ and $\delta > 0$ depending on these quantities, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \delta \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad \text{for } 0 < |\xi| \leqslant (1/r) \tag{6.24}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2} \log(1/r)^{1/2} \qquad \text{for } (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{6.25}$$

if we define $f(x) = (\eta * \phi_r)(x)$, and a probability measure

$$\mu = \frac{f\mu_0}{(f\mu_0)(\mathbf{T}^d)}$$

then $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon$.

Proof. Let

$$\delta = \min\left(\frac{\varepsilon}{2}, \frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right).$$

Thus if we apply Lemmas 6.5 and 6.8, we conclude there exists r_0 such that for $r \leqslant r_0$,

$$\|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon/2, \tag{6.26}$$

and

$$\|f\mu_0 - \mu_0\|_{M(0)} \leqslant \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right). \tag{6.27}$$

Equation (6.27) implies that

$$1 - \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right) \leqslant (f\mu_0)(\mathbf{T}^d) \leqslant 1. \tag{6.28}$$

But now (6.26) and (6.28) show that

$$\begin{aligned}
\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} &\leqslant \|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon)} + \|\mu - f\mu_0\|_{M(\beta/2-\varepsilon)} \\
&\leqslant (\varepsilon/2) + \left(1 - \frac{1}{(f\mu_0)(\mathbf{T}^d)}\right) \|\mu_0\|_{M(\beta/2-\varepsilon)} \\
&\leqslant (\varepsilon/2) + (\varepsilon/2) \leqslant \varepsilon.
\end{aligned}$$

□

A useful technique to find functions with small Fourier coefficients is to apply a random construction. This is because heuristically, a generic function has small Fourier coefficients.

Lemma 6.10. *Fix a positive integer K. Let $X_1, \ldots, X_K$ be independant random variables on $\mathbf{T}^d$, such that for each nonzero $\xi \in \mathbf{Z}^d$,*

$$\sum_{k=1}^{K} \mathbf{E}\left(e^{2\pi i \xi \cdot X_k}\right) = 0. \tag{6.29}$$

Set

$$\eta(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{X_k}(x)$$

and

$$B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| \leqslant K^{100/\beta}\}.$$

Then there exists a constant C depending on β and d, such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Remark 6.11. *In particular,* (6.29) *holds if the $\{X_i\}$ are uniformly distributed on $\mathbf{T}^d$.*

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$, consider the random variable

$$Y(\xi, k) = K^{-1} e^{2\pi i (\xi \cdot X_k)}. \tag{6.30}$$

Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} Y(\xi, k) = \widehat{\eta}(\xi). \tag{6.31}$$

We also note that for each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$,

$$|Y(\xi, k)| = K^{-1}. \tag{6.32}$$

Moreover,

$$\sum_{k=1}^{K} \mathbf{E}(Y(\xi, k)) = 0. \tag{6.33}$$

Since the family of random variables $\{Y(\xi, k)\}$ is independent for a fixed ξ, we can apply Hoeffding's inequality together with (6.31) and (6.32) to conclude that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\eta}(\xi)| \geqslant t\right) \leqslant 2e^{-Kt^2/2}. \tag{6.34}$$

A union bound obtained by applying (6.34) over all $|\xi| \leqslant K^{100/\beta}$ shows that there exists a constant $C \geqslant 1$ depending on d and β such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant t\right) \leqslant \exp\left(C\log(K) - \frac{5Kt^2}{C}\right). \tag{6.35}$$

But then, setting $t = CK^{-1/2}\log(K)^{1/2}$ in (6.35) completes the proof. $\square$

Let us now consider the consequences of the square-root cancellation bound in Lemma 6.10. Given η as in that lemma, consider the smooth function $f = \eta * \phi_r$. The support of f consists of K radius r balls. Provided $K \approx r^{-\beta}$, the support therefore behaves like an r-thickening of a set with Minkowski dimension β. If

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2}\log(K)^{1/2}, \tag{6.36}$$

as is guaranteed with high probability in Lemma 6.10, we actually find that the support of f also behaves like an r-thickening of a set with *Fourier dimension* β as well, i.e. the hypothesis of Lemma 6.8 apply to f. This is one reason why constructions involving some kind of square-root cancellation are often a viable tactic to construct Salem sets, with random examples being an important example of such a construction.

Lemma 6.12. *Fix $\varepsilon, \varepsilon_1 > 0$ and $C > 0$ with $\varepsilon_1 \leqslant \beta/2$. Then there exists $r_0 > 0$ and $C' > 0$ depending on these quantities such that for $r \leqslant r_0$, if $K \geqslant (1/C)r^{-\beta}$ and η is a Borel probability measure with*

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2}\log(K)^{1/2},$$

then

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon|\xi|^{\varepsilon_1-\beta/2} \qquad \text{for } |\xi| \leqslant (1/r)$$

and

$$|\widehat{\eta}(\xi)| \leqslant C'r^{\beta/2}\log(1/r)^{1/2} \qquad \text{for } (1/r) \leqslant |\xi| \leqslant (1/r)^2.$$

Proof. The function $x \mapsto x^{-1/2}\log(x)^{1/2}$ is decreasing for sufficiently large x. Thus if r_0 is chosen appropriately small, then for $|\xi| \leqslant (1/r)$ we find

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2} \leqslant Cr^{\varepsilon_2}\log((1/C)r^{-\beta})^{1/2}\cdot|\xi|^{\varepsilon_2-\beta/2} \leqslant \varepsilon\cdot|\xi|^{\varepsilon_2-\beta/2}. \tag{6.37}$$

If r_0 is appropriately small, then

$$(1/r)^2 \leqslant C^{2/\beta}K^{2/\beta} \leqslant K^{100/\beta}$$

Thus for $(1/r) \leqslant |\xi| \leqslant (1/r)^2$, $\xi \in B$, and so we find there is $C' > 0$ such that

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2} \leqslant Cr^{\beta/2}\log((1/C)r^{-\beta})^{1/2} \leqslant C'r^{\beta/2}\log(1/r)^{1/2}. \tag{6.38}$$

Together, (6.37) and (6.38) give the conclusions of the Lemma. □

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}_\beta$ to be as low as possible. For each $(E,\mu) \in \mathcal{X}_\beta$, the condition that $\mu \in M(\beta/2)$ implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus it is natural to expect that for quasi-all $(E,\mu) \in M(\beta/2)$, the set E has both Hausdorff dimension and Fourier dimension equal to β.

Lemma 6.13. *For quasi-all $(E,\mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since when $\beta = d$, E is a Salem set for any $(E, \mu) \in \mathcal{X}_\beta$, and thus the result is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, E has Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{X} : H^\alpha_\delta(E) < s\}$. Then $A(\alpha, \delta, s)$ is an open subset of $\mathcal{X}_\beta$, and

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(\beta + 1/n, 1/m, 1/k)$$

is precisely the family of $(E, \mu) \in \mathcal{X}_\beta$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha, \delta, s)$ is dense in $\mathcal{X}_\beta$ for $\alpha \in (\beta, d)$ and $\delta, s > 0$. Fix $(E_0, \mu_0) \in \mathcal{X}_\beta$, $\alpha \in (\beta, d)$, $\delta > 0$, $s > 0$, and $\varepsilon_1 > 0$. We aim to show that for each $\varepsilon > 0$, there exists $(E, \mu) \in A(\alpha, \delta, s)$ such that $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. Without loss of generality, in light of Lemma 6.1, we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a small value r, and then find an integer K such that $r^{-\beta} \leqslant K \leqslant r^{-\beta} + 1$. Lemma 6.10 shows that there exists a constant C depending on β and d, as well as K points $x_1, \dots, x_K \in \mathbf{T}^d$ such that if

$$\eta(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{X_k}(x),$$

then for each $|\xi| \leqslant (1/r)^{1+1/\beta} \leqslant K^{1/\beta+1}$,

$$|\widehat{\eta}(\xi)| \leqslant C K^{-1/2} \log(K)^{1/2}. \tag{6.39}$$

Equation (6.39) shows η satisfies the hypotheses of Lemma 6.12, and the result of Lemma 6.12 can then be fed into Corollary 6.9 to conclude that for any $\delta > 0$, there exists $r_1 > 0$ such that if $r \leqslant r_1$, if

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k=1}^{K} \phi_r(x - x_k) \right) \mu_0(x),$$

and if

$$\mu = \mu_1 / \mu_1(\mathbf{T}^d),$$

then

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{6.40}$$

If δ is chosen appropriately, Lemma 6.7 implies

$$d_{\mathbf{H}}(\operatorname{supp}(\mu), \operatorname{supp}(\mu_0)) \leqslant \varepsilon. \tag{6.41}$$

Note that μ is supported on K balls of radius r. Thus for $r \leqslant \delta$,

$$H^\alpha_\delta(\operatorname{supp}(\mu)) \leqslant K r^\alpha \leqslant (r^{-\beta} + 1) r^\alpha = r^{\alpha - \beta} + r^\alpha. \tag{6.42}$$

Since $\alpha > \beta$, (6.42) implies that there is $r_2 > 0$ depending on α, β, and s such that for $r \leqslant r_2$, $H^\alpha_\delta(\operatorname{supp}(\mu)) \leqslant s$. This means $(\operatorname{supp}(\mu), \mu) \in A(\alpha, \delta, s)$, and since $\varepsilon > 0$ was arbitrary, we see that we have proved what was required. $\square$

This concludes the setup to the proof of Theorems 1.1 and 1.3. All that remains is to show that quasi-all elements of $\mathcal{X}_\beta$ avoid the given set Z; just as with some of the proofs we have given in this section, the advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

7 Random Avoiding Sets for Rough Patterns

We begin by proving Theorem 1.1, which requires simpler calculations than Theorem 1.3. In the last section, our results held for an arbitrary $\beta \in (0, d]$. But in this section, we assume

$$\beta = \min\left(d, \frac{dn - \alpha}{n - 1/2}\right),$$

which will enable us to generically avoid the pattern Z described in Theorem 1.1. The construction here is very similar to the construction in [1], albeit in a Baire category setting and with modified parameters to ensure a Fourier dimension bound rather than just a Hausdorff dimension bound.

Lemma 7.1. *Suppose $\mathbf{E} = \mathbf{R}^d$ or $\mathbf{E} = \mathbf{T}^d$, and let $Z \subset \mathbf{E}^n$ is a countable union of compact sets, each with lower Minkowski dimension at most α. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.*

Proof. The set $Z \subset \mathbf{R}^{dn}$ is the countable union of sets with lower Minkowski dimension at most α. For a closed set $W \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, and $s > 0$, consider the set

$$B(W, s) = \left\{ (E, \mu) \in \mathcal{X}_\beta : \begin{array}{c} \text{for all } x_1, \dots, x_n \in E \text{ such that} \\ |x_i - x_j| \geqslant s \text{ for } i \neq j,\ (x_1, \dots, x_n) \notin W \end{array} \right\}.$$

If $(E_0, \mu_0) \in B(W, s)$, then the compactness of E_0 implies that the set

$$F = \{(x_1, \dots, x_n) \in E_0^n : |x_i - x_j| \geqslant s \text{ for } i \neq j\}$$

is also compact. Since W is also closed, hence compact, there exists $\varepsilon > 0$ such that if $(x_1, \dots, x_n) \in F$, then $d((x_1, \dots, x_n), W) > \varepsilon$. It follows that if $d_{\mathbf{H}}(E_0, E) \leqslant \varepsilon$, then for any measure μ supported on E, $(E, \mu) \in B(W, s)$. Thus $B(W, s)$ is an open subset of $\mathcal{X}_\beta$. If Z is a countable union of closed sets $\{Z_k\}$ with lower Minkowski at most α, then clearly the set

$$\bigcap_{k=1}^\infty \bigcap_{n=1}^\infty B(Z_k, 1/n)$$

consists of the family of sets (E, μ) such that for distinct $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$. Thus it suffices to show that $B(W, s)$ is dense in $\mathcal{X}_\beta$ for any $s > 0$, and any closed set W with lower Minkowski dimension at most α.

Let us begin by fixing a set $W \subset \mathbf{T}^{dn}$ and a pair $(E_0, \mu_0) \in \mathcal{X}_\beta$. We will show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, we can find $(E, \mu) \in B(W, s)$ with $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and

$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. We may assume by Corollary 6.4 that $\operatorname{supp}(\mu_0) = E$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$. Since W has lower Minkowski dimension at most α, we can find arbitrarily small $r \in (0,1)$ such that

$$|W_r| \leqslant r^{dn-\alpha-\varepsilon_1/4}. \tag{7.1}$$

Assume also that r is small enough that we can find an integer $K \geqslant 10$ with

$$r^{-(\beta-\varepsilon_1/2)} \leqslant K \leqslant r^{-(\beta-\varepsilon_1/2)} + 1. \tag{7.2}$$

Let $X_1, \dots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $k_1, \dots, k_n \in \{1, \dots, K\}$, the random vector $X_k = (X_{k_1}, \dots, X_{k_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (7.1) and (7.2) imply that

$$\mathbf{P}(d(X_k, W) \leqslant r) \leqslant |W_r| \leqslant r^{dn-\alpha-\varepsilon_1/4} \lesssim_{d,n,\beta} K^{\frac{-(dn-\alpha-\varepsilon_1/4)}{\beta-\varepsilon_1/2}} \leqslant K^{-(n-1/2)}, \tag{7.3}$$

where we used the calculation

$$\begin{aligned} \frac{dn-\alpha-\varepsilon_1/4}{\beta-\varepsilon_1/2} &= \frac{dn-\alpha}{\beta} + \frac{[(dn-\alpha)/\beta](\varepsilon_1/2) - \varepsilon_1/4}{\beta-\varepsilon_1/2} \\ &\geqslant \frac{dn-\alpha}{\beta} + \frac{(n-1/2)(\varepsilon_1/2) - \varepsilon_1/4}{\beta-\varepsilon_1/2} \\ &\geqslant \frac{dn-\alpha}{\beta} \geqslant n - 1/2. \end{aligned} \tag{7.4}$$

If M_0 denotes the number of indices i such that $d(X_i, W) \leqslant r$, then by linearity of expectation we conclude from (7.3) that there is a constant C depending only on d, n, and β such that

$$\mathbf{E}(M_0) \leqslant (C/10)K^{1/2}. \tag{7.5}$$

Applying Markov's inequality to (7.5), we conclude that

$$\mathbf{P}(M_0 \geqslant CK^{1/2}) \leqslant 1/10. \tag{7.6}$$

Taking a union bound to (7.6) and the results of Lemma 6.10, we conclude that there exists K points $x_1, \dots, x_K \in \mathbf{T}^d$ and a constant C depending only on d, n, and β such that the following two statements hold:

(1) Let S be the set of indices $k_1 \in \{1, \dots, K\}$ with the property that we can find distinct indices $k_2, \dots, k_n \in \{1, \dots, K\}$ such that if $X = (X_{k_1}, \dots, X_{k_n})$, then $d(X, W) \leqslant r$. Then

$$\#(S) \leqslant CK^{1/2}. \tag{7.7}$$

(2) If we define

$$\eta_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k}(x)$$

then for $0 < |\xi| \leqslant K^{100/\beta}$,

$$|\widehat{\eta_0}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \tag{7.8}$$

Thus (7.7) and (7.8) imply that if

$$\eta_1(x) = \sum_{k \notin S} \delta_{x_k}(x),$$

then for each $|\xi| \leqslant K^{1/\beta_0+1}$,

$$|\widehat{\eta_1}(\xi)| \leqslant 2CK^{-1/2}\log(K). \tag{7.9}$$

Since $K \geqslant r^{\varepsilon_1/2-\beta}$, we can apply Lemma 6.12 to η_1, and then apply Corollary 6.9 to the result of Lemma 6.12 to conclude that for any $\delta > 0$, there is $r_0 > 0$ such that if $r \leqslant r_0$ and we define

$$\mu'(x) = \left(\sum_{k \notin S} \phi_{(r/2n^{1/2})}(x - X_k) \right) \mu_0(x),$$

and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, then

$$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{7.10}$$

Choosing δ in accordance with Lemma 6.7, (7.10) also implies that $d_{\mathbf{H}}(\text{supp}(\mu), \text{supp}(\mu_0)) \leqslant \varepsilon$. Since ε and ε_1 are arbitrary, our proof would therefore be complete if we could show that $(\text{supp}(\mu), \mu) \in B(W, s)$.

Consider n points $y_1, \dots, y_n \in \text{supp}(\mu)$, with $|y_i - y_j| \geqslant r$ for any two indices $i \neq j$. We can therefore find distinct indices $k_1, \dots, k_n \in \{1, \dots, K\}$ such that for each $i \in \{1, \dots, n\}$, $|x_{k_i} - y_i| \leqslant (n^{-1/2}/2) \cdot r$, which means if we set $x = (x_{k_1}, \dots, x_{k_n})$ and $y = (y_1, \dots, y_n)$, then

$$|x - y| \leqslant (r/2). \tag{7.11}$$

Since $i_1 \notin S$, $d(x, W) \geqslant r$, which combined with (7.11) implies

$$d(y, W) \geqslant d(x, W) - |x - y| \geqslant r/2. \tag{7.12}$$

Thus in particular we conclude $y \notin W$, which shows $(E, \mu) \in B(W, s)$. $\square$

The Baire category theorem, applied to Lemma 7.1 implies Theorem 1.1, and thus concludes the proof of Theorem 1.1. Before we move onto the next proof, let us discuss where the loss in Theorem 1.1 occurs in our proof, as compared to the Hausdorff dimension bound of [1]. In the proof of Lemma 7.1, in order to obtain the bound (7.9), we were forced to choose the parameter r such that $\#(S) \leqslant K^{1/2}$, so that the trivial bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \leqslant \#(S) \tag{7.13}$$

obtained by the triangle inequality is viable to obtain a Fourier dimension bound. On the other hand, if we were able to justify that with high probability, we could obtain a square root cancellation bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \lesssim \#(S)^{1/2}, \tag{7.14}$$

then we would only need to choose the parameter r such that $\#(S) \lesssim K$, which leads to a set with larger Fourier dimension, matching with the Hausdorff dimension bound obtained in [1]. Under stronger assumptions on the pattern we are trying to avoid (the hypotheses of Theorem 1.3), we are able to obtain some such square root cancellation, though with an additional term that decays fast as $|\xi| \to \infty$, which enables us to obtain the improved dimension bound in the conclusions of that theorem.

8 Concentration Bounds for Smooth Surfaces

In this section we prove Theorem 1.3 using some more robust probability concentration calculations. We set

$$\beta = \frac{d}{n - 3/4}.$$

For this value we will be able to generically avoid the pattern Z described in Theorem 1.3.

Lemma 8.1. *Let $Z \subset \mathbf{T}^{dn}$ satisfy the hypothesis of Theorem 1.3. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, and for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.*

Proof. Fix a set

$$W = \{(x_1, \dots, x_d) \in U \times V : x_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\}$$

as in the statement of Theorem 1.3, where $f \in C^\infty(V)$. For notational convenience, we assume $i = 1$. Given any family of disjoint, closed cubes $R_1, \dots, R_n \subset \mathbf{T}^d$ such that $(R_1 \times \dots \times R_n) \cap W$ is a closed set, we let

$$H(W; R_1, \dots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : (R_1 \times \dots \times R_n) \cap W \cap E^n = \varnothing\}. \tag{8.1}$$

Then $H(W; R_1, \dots, R_n)$ is an open subset of $\mathcal{X}_\beta$. For the purpose of a Baire category argument, our result will follow by showing $H(W; R_1, \dots, R_n)$ is dense in $\mathcal{X}_\beta$ for any family of disjoint cubes $\{R_1, \dots, R_n\}$ such that

- $R_2, \dots, R_n$ all have common sidelength s for some $s > 0$.
- If $Q_1 = 2R_1, \dots, Q_n = 2R_n$, then $Q_1 \subset U$ and $Q_2 \times \dots \times Q_n \subset V$.
- Since f is smooth, we can fix a constant $L \geqslant 0$ such that for any $x_1, x_2 \in Q_2 \times \dots \times Q_n$,

  $$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|.$$

  We require that R_1 has sidelength at most $2^{dn}(L+1)s$ and $f(R_2 \times \dots \times R_n) \subset R_1$.
- $d(R_i, R_j) \geqslant 100 \cdot 2^{dn}(L+1)s$ for each $i, j \in \{1, \dots, n\}$.

As with the proof of Lemma 7.1, we fix $(E_0, \mu_0) \in \mathcal{X}_\beta$ with $\text{supp}(\mu_0) = E_0$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$, and show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, there is $(E, \mu) \in H(W; Q_1, \dots, Q_n)$ with

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon,$$

from which the density property of $H(W; R_1, \dots, R_n)$ follows.

Fix a family of non-negative bump functions $\psi_0, \psi_1, \dots, \psi_n \in C^\infty(\mathbf{T}^d)$, such that for $i \in \{1, \dots, n\}$, $\psi_i(x) = 1$ for $x \in R_i$, $\psi_i(x) = 0$ for $x \notin Q_i$, and $\psi_0(x) + \cdots + \psi_n(x) = 1$ for $x \in \mathbf{T}^d$. For $i \in \{0, \dots, n\}$, let $A_i = \int \psi_i(x)\, dx$ denote the total mass of ψ_i. Now consider a family of independant random variables $\{X_i(k) : 0 \leqslant i \leqslant n, 1 \leqslant k \leqslant K\}$, where the random variable $X_i(k)$ is continuous with distribution function $A_i^{-1}\psi_i$. Let $r = K^{\varepsilon_1/2 - 1/\beta}$ and then let S be the set of indices $k_1 \in \{1, \dots, K\}$ such that there are indices $k_2, \dots, k_n \in \{1, \dots, K\}$ with the property that

$$|X_1(k_1) - f(X_2(k_2), \dots, X_n(k_n))| \leqslant (L+1)r. \tag{8.2}$$

A simple argument following from (8.2) shows that if $k_1 \notin S$, then for any $k_2, \dots, k_n \in \{1, \dots, K\}$, if $X = (X_1(k_1), \dots, X_n(k_n))$, then $d(X, W) \geqslant r$. Thus if we define

$$\eta = \frac{1}{K}\left(\sum_{i \in \{0,2,\dots,n\}} A_i \sum_{k=1}^{K} \delta_{X_i(k)} + A_1 \sum_{k \notin S} \delta_{X_1(k)} \right),$$

if we define $\mu' = (\eta * \phi_{(r/2n^{1/2})}) \cdot \mu_0$, and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, $E = \operatorname{supp}(\mu)$, then $(E, \mu) \in H(W; R_1, \dots, R_n)$. The remainder of our argument consists of obtaining control on the exponential sum

$$\widehat{\eta}(\xi) = \frac{1}{K} \sum_{i \in \{0,2,\dots,n\}} A_i \sum_{k=1}^{K} e^{-2\pi i \xi \cdot X_i(k)} + \frac{1}{K} A_1 \sum_{k \notin S} e^{-2\pi i \xi \cdot X_1(k)}, \tag{8.3}$$

for nonzero $\xi \in \mathbf{Z}^d$, so that we may apply Corollary 6.9 to bound $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)}$.

To analyze $\widehat{\eta}$, introduce the measures

$$\nu = \frac{1}{K} \sum_{i \in \{0,\dots,n\}} A_i \sum_{k=1}^{K} \delta_{X_i(k)}$$

and

$$\sigma = \nu - \eta = \frac{A_1}{K} \sum_{k \in S} \delta_{X_1(k)}.$$

For non-zero $\xi \in \mathbf{Z}^d$,

$$\mathbf{E}(\widehat{\nu}(\xi)) = \sum_{i=0}^{n} \int \psi_i(x) e^{2\pi i \xi \cdot x}\, dx = \int_{\mathbf{T}^d} e^{2\pi i \xi \cdot x}\, dx = 0. \tag{8.4}$$

Applying Lemma 6.10, we conclude that if $B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| < K^{100/\beta}\}$, then there is $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\nu}\|_{L^\infty(B)} \geqslant CK^{-1/2} \log(K)^{1/2} \right) \leqslant 1/10. \tag{8.5}$$

Analyzing $\widehat{\sigma}(\xi)$ requires a more subtle concentration bound. As a random quantity, $\widehat{\sigma}(\xi)$ is a function of the independant random quantities $\{X_i(k)\}$, and so McDiarmid's inequality presents itself as a useful concentration bound. However, a naive application of McDiarmid's inequality fails, because changing a single random vairable $X_i(k)$ for $i \geqslant 2$ while fixing other random variables can change $\widehat{\sigma}(\xi)$ by as much as $\Omega(1)$, which is far too much to obtain the square root cancellation bounds like we obtained in (8.5). On the other hand, it seems that a single variable $X_i(k)$ only changes $\widehat{\sigma}(\xi)$ by $\Omega(1)$ if the random variables $\{X_1(k)\}$ are configured in a very particular way, which is unlikely to happen. Thus we should expect that adjusting a single random variable $X_i(k)$ does not influence the value of $Y(\xi)$ very much if the quantity $Y(\xi)$ is *averaged* over the possible choices of $\{X_1(k)\}$, and then we can apply McDiarmid's inequality.

Consider the random set Ω of values $x \in Q_1$ such that there are $k_2, \dots, k_n \in \{1, \dots, K\}$ with

$$|x - f(X_2(k_2), \dots, X_n(k_n))| \leqslant r. \tag{8.6}$$

Then

$$\widehat{\sigma}(\xi) = \frac{1}{K} \sum_{k=1}^{K} Z(k). \tag{8.7}$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_1(k)} & : X_1(k) \in \Omega, \\ 0 & : X_1(k) \notin \Omega \end{cases}. \tag{8.8}$$

If Σ is the σ algebra generated by the random variables $\{X_i(k) : i \geqslant 2, k \in \{1, \dots, K\}\}$, then the random variables $\{Z(k)\}$ are *conditionally independant* given Σ. Since we have $|Z(k)| \leqslant 1$ almost surely, Hoeffding's inequality thus implies that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2}\right). \tag{8.9}$$

It is simple to see that

$$\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) = \int_{\Omega} \psi_1(x) e^{2\pi i \xi \cdot x}\, dx. \tag{8.10}$$

Since

$$\Omega = \bigcup \left\{B_r(f(X_2(k_2), \dots, X_n(k_n))) : 1 \leqslant k_2, \dots, k_n \leqslant K\right\}. \tag{8.11}$$

we see that varying each random variable $X_i(k)$ adjusts at most K^{n-2} of the radius r balls forming Ω, and thus varying $X_i(k)$ independantly of the other random variables changes $\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)$ by at most

$$2 \cdot (2r)^d \cdot K^{n-2} = 2^{d+1} \cdot r^d \cdot K^{n-2} \leqslant \frac{2^{d+1}}{K} \leqslant \frac{2^{d+1}}{K}. \tag{8.12}$$

Thus McDiarmid's inequality shows that for any $t \geqslant 0$,

$$\mathbf{P}\left(|\,\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{8.13}$$

Combining (8.9) and (8.13), we conclude that for each $\xi \in \mathbf{Z}^d$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 8 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{8.14}$$

Applying a union bound to (8.14) over all $\xi \in B$ shows that there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10. \tag{8.15}$$

Combining (8.5) and (8.15), we conclude that there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\eta}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/2. \tag{8.16}$$

We have thus reduced the study of $\widehat{\eta}$ to the quantity $\mathbf{E}(\widehat{\sigma}(\xi))$. We break the analysis of $\mathbf{E}(\widehat{\sigma}(\xi))$ into two cases, depending on where $n = 2$. The major difference here is that for $n = 2$ we obtain a full dimensional set, which means some arguments that work for the case $n > 2$ break here. On the other hand, the analysis of patterns when $n = 2$ is more trivial since there is no cartesian product structure here to understand, so the analysis is more simple in this case.

Let us start with the case $n = 2$. Using the fact that the family $\{X_1(k) : 1 \leqslant k \leqslant K\}$ are uniformly distributed, we calculate that

$$\mathbf{E}(\widehat{\sigma}(\xi)) = \mathbf{E}(A_1 \cdot e^{2\pi i \xi \cdot X_1(1)} \cdot \mathbf{I}(1 \in S)) = A_1 \int_{\mathbf{T}^d} \psi_1(x)\, \mathbf{P}\left(1 \in S | X_1(1) = x\right) e^{2\pi i \xi \cdot x}\, dx. \tag{8.17}$$

For each $x \in \mathbf{T}^d$,

$$\mathbf{P}(1 \in S | X_1(1) = x) = 1 - \left(1 - \int_{f^{-1}(B_{(L+1)r}(x))} \psi_2(x)\, dx\right)^K$$

Now let us consider the case $n > 2$. Equation (8.17) continues to hold in this regime, though the analysis of this equation is more complicated. For a set $E \subset \mathbf{T}^{d(n-1)}$, let $A(E)$ denote the event that there exists $k_2, \dots, k_n$ such that $(X_2(k_2), \dots, X_n(k_n)) \in E$. Then

$$\mathbf{P}(1 \in S | X_1(1) = x) = \mathbf{P}(A(f^{-1}(B_{(L+1)r}(x)))).$$

For any cube $Q \in \mathbf{T}^{d(n-1)}$ and any indices $1 \leqslant k_2, \dots, k_n \leqslant K$, set $k = (k_2, \dots, k_n)$ and let $A(Q; k)$ denote the event that $(X_2(k_2), \dots, X_n(k_n)) \in Q$. Then

$$A(Q) = \bigcup_k A(Q; k).$$

For any Q and k,

$$\mathbf{P}(A(Q; k)) = \int_Q \psi_2(x_2) \dots \psi_n(x_n)\, dx_2 \dots dx_n,$$

and so

$$\sum_k \mathbf{P}(A(Q; k)) = K^{n-1} \int_Q \psi_2(x_2) \cdots \psi_n(x_n)\, dx_2 \dots dx_n.$$

An application of inclusion exclusion shows that

$$\left|\mathbf{P}(A(Q)) - K^{n-1} \int_Q \psi_2(x_2) \cdots \psi_n(x_n)\, dx_2 \dots dx_n\right| \leqslant \sum_{k \neq k'} \mathbf{P}(A(Q; k) \cap A(Q; k')).$$

For each k, k', the quantity $\mathbf{P}(A(Q;k) \cap A(Q;k'))$ depends on the number of indices i such that $k_i = k'_i$. In particular, if $I \subset \{2, \dots, n\}$ is the set of indices where the quantity agrees, then

$$\mathbf{P}(A(Q;k) \cap A(Q;k')) = \left(\prod_{i \in I} \int_{Q_i} \psi_i(x)\, dx \right) \cdot \left(\prod_{i \notin I} \left(\int_{Q_i} \psi_i(x)\, dx \right)^2 \right).$$

In particular, if Q has sidelength l and $\#(I) = m$, then $\mathbf{P}(A(Q;k) \cap A(Q;k')) \lesssim l^{d(2n-m-2)}$. For each m, there are at most K^{2n-m-2} pairs k and k' with $\#(I) = m$. And so provided $l^d \leqslant 1/K$,

$$\sum_{k \neq k'} \mathbf{P}(A(Q;k) \cap A(Q;k')) \lesssim \sum_{m=0}^{n-2} (Kl^d)^{2n-m-2} \lesssim K^n l^{dn}.$$

Thus we conclude that

$$\mathbf{P}(A(Q)) = K^{n-1} \int_Q \psi_2(x_2) \dots \psi_n(x_n) dx_2 \dots dx_n + O(K^n l^{dn}).$$

Now for a particular $x \in \mathbf{T}^d$, let $E = f^{-1}(B_{(L+1)r}(x))$. Since f is a submersion, E is contained in a $O(r)$-thickening of a $d(n-2)$ dimensional surface in $\mathbf{T}^{d(n-1)}$. Applying the Whitney covering lemma, we can find a family of almost disjoint dyadic cubes $\{Q_{ij} : j \geqslant 0\}$ such that

$$E = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{n_i} Q_{ij},$$

where for $i \geqslant 0$, Q_{ij} is a sidelength $r/2^i$ cube, and $n_i \lesssim (r/2^i)^{-d(n-2)}$. It follows that

$$A(E) = \bigcup_{i,j} A(Q).$$

Since $n \geqslant 3$, we calculate that

$$\begin{aligned} \left| \sum_{i,j} \mathbf{P}(A(Q_{ij})) - K^{n-1} \int_E \psi_2(x_2) \dots \psi_n(x_n)\, dx \right| &\lesssim \sum_{i=0}^{\infty} (r/2^i)^{-d(n-2)} \cdot \left(K^n (r/2^i)^{dn} \right) \\ &\lesssim r^{2d} K^n \leqslant K^{-1/2}. \end{aligned}$$

Thus an inclusion exclusion bound implies that

$$\left| \mathbf{P}(A(E)) - K^{n-1} \int_E \psi_2(x_2) \dots \psi_n(x_n)\, dx \right| \lesssim K^{-1/2} + \sum_{(i_1,j_1) \neq (i_2,j_2)} \mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})).$$

The quantity $\mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2}))$ depends on the relation between the various sides of $Q_{i_1 j_1}$ and $Q_{i_2 j_2}$. Without loss of generality, we may assume that $i_1 \geqslant i_2$. If I is the set of indices $2 \leqslant k \leqslant n$ where $Q_{i_1 j_1 k} \subset Q_{i_2 j_2 k}$, and $\#(I) = m$, then

$$\begin{aligned} \mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})) &\lesssim (K(r/2^{i_1})^d)^m \cdot (K(r/2^{i_1})^d \cdot K(r/2^{i_2})^d)^{n-m-1} \\ &= 2^{-d[i_1(n-1) + i_2(n-m-1)]} (Kr^d)^{2n-m-2}. \end{aligned}$$

Since each axis-oriented hyperplane intersects transversally with our hyperplane, if the hyperplane is $d(n-m-1)$ dimensional, then the intersection of the hyperplane with the surface $f^{-1}(x)$ has dimension $d(n-m-2)$, and so for each fixed i_1 and i_2, there are at most

$$O((r/2^{i_1})^{-md}(r/2^{i_2})^{-2d(n-m-2)}) = O(r^{-d(2n-m-4)}2^{-d[mi_1+2(n-m-2)i_2]})$$

pairs $Q_{i_1j_1}$ and $Q_{i_2j_2}$ with $\#(I) = m$. Thus we conclude that

$$\begin{aligned}
&\sum_{(i,j)\neq(i',j')} \mathbf{P}(A(Q_{ij}) \cap A(Q_{i'j'})) \\
&\qquad \lesssim \sum_{m=0}^{n-2} \sum_{i\geqslant i'} 2^{-d[i(n-1)+i'(n-m-1)]}(Kr^d)^{2n-m-2} \cdot r^{-d(2n-m-4)}2^{-d[mi+2(n-m-2)i']} \\
&\qquad \lesssim \sum_{m=0}^{n-2} K^{2n-m-2}r^{2d} \sum_{i\geqslant i'} 2^{-d[i(n+m-1)+i'(3n-3m-5)]} \\
&\qquad \lesssim \sum_{m=0}^{n-2} K^{2n-m-2}r^{2d} \\
&\qquad \lesssim K^n r^{2d} \lesssim K^{-1/2}.
\end{aligned}$$

Thus we conclude that

$$\left|\mathbf{P}(A(E)) - K^{n-1}\int_E \psi_2(x_2)\dots\psi_n(x_n)\; dx_2\dots dx_n\right| \lesssim K^{-1/2},$$

and so as a result,

$$\left|\mathbf{E}(\widehat{\sigma}(\xi)) - K^{n-1}\int_{\mathbf{T}^d}\psi_1(x)\int_{f^{-1}(B_r(x))}\psi_2(x_2)\dots\psi_n(x_n)e^{2\pi i\xi\cdot x_1}\; dx_2\;\dots\; dx_n\; dx_1\right| \lesssim K^{-1/2}.$$

Applying the coarea formula, writing $\psi(x) = \psi_1(x_1)\dots\psi_n(x_n)$, we find

$$\begin{aligned}
\int_{\mathbf{T}^d}\int_{f^{-1}(B_r(x))}\psi(x)e^{2\pi i\xi\cdot x_1}\; dx_2\;\dots\; dx_n\; dx_1 &= \int_{B_r(0)}\int_{\mathbf{T}^d}\int_{f^{-1}(x+v)}\psi(x)e^{2\pi i\xi\cdot x_1}\; dH^{n-2}(x_2,\dots,x_n)\; dx_1\; dv \\
&= \int_{B_r(0)}\int_{\mathbf{T}^{d(n-1)}}\psi(f(x)-v,x)\cdot e^{2\pi i\xi\cdot(f(x)-v)}|Jf(x)|\; dx\; dv \\
&= \int_{B_r(0)}\int_{\mathbf{T}^{d(n-1)}}\tilde{\psi}(x,v)\cdot e^{2\pi i\xi\cdot(f(x)-v)}\; dx\; dv.
\end{aligned}$$

where $\tilde{\psi}(x,v) = \psi(f(x)-v,x)\cdot|Jf(x)|$. Now this is a standard oscillatory integral, and the fact that $Df(x)$ is surjective implies that over all $|v| \leqslant 1$,

$$\left|\int_{\mathbf{T}^{d(n-1)}}\tilde{\psi}(x,v)\cdot e^{2\pi i\xi\cdot(f(x)-v)}\; dx\right| \lesssim_N |\xi|^{-N}.$$

Thus we conclude that

$$\left|\int_{\mathbf{T}^d}\int_{f^{-1}(B_r(x))}\psi(x)e^{2\pi i\xi\cdot x_1}\; dx_2\;\dots\; dx_n\; dx_1\right| \lesssim_N r^d|\xi|^{-N}.$$

In particular, we conclude that

$$|\mathbf{E}(\widehat{\sigma}(\xi))| \lesssim K^{-1/2} + K^{n-1} r^d |\xi|^{-\beta/2} \lesssim K^{-1/2} + K^{-0.25} |\xi|^{-\beta/2}.$$

Thus for any $\delta > 0$, there exists r_1 such that for $r \leqslant r_1$, and any nonzero $\xi \in \mathbf{Z}^d$,

$$|\mathbf{E}(\widehat{\sigma}(\xi))| \leqslant \delta |\xi|^{-\beta/2} \tag{8.18}$$

From (8.16) and (8.18), we conclude there exists a choice of values $\{X_i(k)\}$ such that for any $\xi \in B$,

$$|\widehat{\eta}(\xi)| \leqslant C K^{-1/2} \log(K)^{1/2} + \delta |\xi|^{-\beta/2}. \tag{8.19}$$

Applying Lemma 6.12 and then Corollary 6.9, we conclude that if δ is chosen appropriately, then $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$, which completes the proof. □

9 Salem Sets with non-approximable difference Sets

Analyzing the last proof, you may have noticed that to

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