Salem Sets Avoiding Rough Configurations

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1 Introduction

Geometric measure theory explores the relationship between the geometry of subsets of Euclidean spaces, and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a particular subset of Euclidean space. In this paper, we focus on the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a 'rough pattern', in the sense of [3], we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel set $E \subset \mathbf{R}^d$ is it's *Fourier dimension*; given a finite Borel measure μ on $\mathbf{R}^d$, it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$, is the supremum of all $s \in [0, d]$ such that

$$\sup\left\{|\widehat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d\right\} < \infty. \tag{1.1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most classical fractal sets are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem.

Theorem 1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset [0,1]^d$ with dimension*

$$\beta = \min\left(\frac{nd-\alpha}{n-1/2}, d\right)$$

such that for any distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

Remark 1. *Theorem 1 is an attempt to strengthen the main result of [3] to give a Fourier dimension bound, albeit under a weaker dimension bound. Unlike in [3], the case of Theorem 1 when $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $[0,\pi]^d - \pi(Z)$ is not necessarily a Salem set, where $\pi(x_1, \dots, x_n) = x_1$ is projection onto the first coordinate. For instance, in Example 8 of [6] it is shown that there exists a compact set $E \subset [0,1]$ such that* $\dim_{\mathbf{M}}(E) < 1$ *and* $\dim_{\mathbf{F}}([0,1] - E) < 1$*. Setting $Z = E \times \{0\} \cup \{0\} \times E$ shows that neither* *subtracting projections onto the first nor the second coordinate gives the required Fourier dimension bounds.*

Because we are working with *compact* sets avoiding patterns, working in the domain $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, and working in this space has several advantages over the Euclidean case. For a finite measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.2}$$

We can then define the Fourier dimension $\dim_{\mathbf{F}}(E)$ of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Since $\mathbf{T}^d$ has a natural metric space structure, we can define the Hausdorff dimension of sets on $\mathbf{T}^d$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1.1) is equivalent to the more discrete condition

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.3}$$

A proof is given in Lemma 39 of [4]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x)\, d\mu^*(x) = \int_{\mathbf{R}^d} f(x)\, d\mu(x), \tag{1.4}$$

then (1.3), together with the Poisson summation formula, implies $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$. Thus Theorem 1 is clearly equivalent to Theorem 2, which is it's periodic variant.

Theorem 2. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset \mathbf{T}^d$ with dimension*

$$\beta = \min\left(\frac{dn - \alpha}{n - 1/2}, d\right)$$

such that for any distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

We construct the set in Theorem 2 by relying on Baire-category arguments. Thus we consider a complete metric space $\mathcal{X}_\beta$, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β, and for distinct $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$, in the sense that the set of pairs (E, μ) which do not satisfy these properties is a set of first category in $\mathcal{X}_\beta$. It thus follows that the consequences of Theorem 2 holds in a 'generic' sense for elements of $\mathcal{X}_\beta$.

Once we have setup the appropriate metric space $\mathcal{X}_\beta$, our approach is quite similar to the construction in [3], relying on a random selection procedure, which is now exploited to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach as used in [3], is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [3]. But the Baire category argument makes our proof much simpler to read, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern.

2 Notation

- Given a metric space Ω (here either $\mathbf{R}^d$ or $\mathbf{T}^d$), $x \in \Omega$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let

  $$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

  denote the *ε-thickening* of the set E. A subset of Ω is of *first category* in Ω if it is the countable union of closed sets with nonempty interior. We say a property holds *quasi-always*, or a property is *generic* in Ω if the set of points in Ω failing to satisfy that property form a set of first category.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let

  $$|x| = \min\{|x+n| : n \in \mathbf{Z}\},$$

  and for $x \in \mathbf{T}^d$, we let

  $$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

  The canonical metric on $\mathbf{T}^d$ is then given by $d(x,y) = |x-y|$, for $x, y \in \mathbf{T}^d$. For a cube Q in $\mathbf{T}^d$, we let $2Q$ be the cube in $\mathbf{T}^d$ with the same centre and twice the sidelength.

- Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0,d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as

  $$H^\alpha_\delta(E) = \inf \left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } \varepsilon_i \leqslant \delta \text{ for all } i \in \mathbf{N} \right\}.$$

  The α dimensional Hausdorff measure of E is equal to

  $$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

  The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0,d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0,d]$ such that $H^s(E) = 0$.

- Suppose $\mathbf{E} = \mathbf{R}^d$ or $\mathbf{E} = \mathbf{T}^d$, and for a measurable set E, we let $|E|$ denote it's Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as

  $$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon|.$$

- In this paper we will need to employ concentration bounds several times. In particular, we use *McDiarmid's inequality*, trivially modified from the standard theorem to work with complex-valued functions. Let $\{X_1, \dots, X_N\}$ be an independant family of random variables, and consider a measurable function $f : \mathbf{R}^N \to \mathbf{C}$. Suppose that for each $i \in \{1, \dots, N\}$, there exists a constant $A_i > 0$ such that for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N \in \mathbf{R}$, and for each $x_i, x_i' \in \mathbf{R}$,

  $$|f(x_1, \dots, x_i, \dots, x_N) - f(x_1, \dots, x_i', \dots, x_N)| \leqslant A_i.$$

  Then McDiarmid's inequality guarantees that for all $t \geqslant 0$,

  $$\mathbf{P}\left(|f(X_1, \dots, X_N) - \mathbf{E}(f(X_1, \dots, X_N))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-2t^2}{A_1^2 + \dots + A_N^2}\right).$$

  The complex-valued extension we have just stated is proved easily from the real-valued case by taking a union bound to the inequality for the real and imaginary values of f.

  A special case of McDiarmid's inequality is *Hoeffding's Inequality*. The version of Hoeffding's inequality we use states that if $\{X_1, \dots, X_N\}$ is an independent family of mean-zero random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

  $$\mathbf{P}\left(|X_1 + \dots + X_N| \geqslant t\right) \leqslant 4 \exp\left(\frac{-t^2}{2(A_1^2 + \dots + A_N^2)}\right).$$

  Proofs of McDiarmid's inequality are given in many probability textbooks, for instance, in Theorem 3.11 of [5].

- Throughout this paper, we will need to consider a standard mollifier. So we fix a smooth, non-negative function $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$ and

  $$\int_{\mathbf{T}^d} \phi(x)\, dx = 1.$$

  For each $r \in (0,1)$, we can then define $\phi_r \in C^\infty(\mathbf{T}^d)$ by writing

  $$\phi_r(x) = \begin{cases} r^{-d}\phi(x/r) & : |x| < r, \\ 0 & : \text{otherwise.} \end{cases}$$

  The following standard properties hold for this choice of mollifier $\{\phi_r\}$:

  (1) For each $r \in (0,1)$, ϕ_r is a smooth probability density, and $\phi_r(x) = 0$ for $|x| \geqslant r$.

  (2) For any $r \in (0,1)$,
  $$\|\widehat{\phi_r}\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1. \tag{2.1}$$

  (3) For each $\xi \in \mathbf{Z}^d$,
  $$\lim_{r \to 0} \widehat{\phi_r}(\xi) = 1. \tag{2.2}$$

  (4) For each $T > 0$, and for all $r > 0$ and non-zero $\xi \in \mathbf{Z}^d$,
  $$|\widehat{\phi_r}(\xi)| \lesssim_T r^{-T} |\xi|^{-T}. \tag{2.3}$$

3 A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task. Though in later sections we will specify β as in Theorem 1, in this section we let β be an arbitrary element of $(0, d]$. We proceed as in [2], forming our metric space as a combination of two simpler metric spaces. However, we employ a novel Frechét space construction instead of the Banach space used in [2], which enables us to use softer estimates in our arguments:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

  $$d_H(E,F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

  then $(\mathcal{E}, d_H)$ forms a complete metric space.

- We let $M(\beta/2)$ consist of the class of all finite Borel measures μ on $\mathbf{T}^d$ such that for each $\varepsilon \in (0, \beta/2]$, the quantity

  $$\|\mu\|_{M(\beta/2-\varepsilon)} = \sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon}$$

  is finite. Then $\|\cdot\|_{M(\beta/2-\varepsilon)}$ is a seminorm on $M(\beta/2)$, and the collection of all such seminorms for $\varepsilon \in (0, \beta/2]$ gives $M(\beta/2)$ the structure of a Frechét space. Under this topology, a sequence of probability measures $\{\mu_k\}$ converges to a probability measure μ in $M(\beta/2)$ if and only if for any $\varepsilon > 0$, $\lim_{k\to\infty} \|\mu_k - \mu\|_{M(\beta/2-\varepsilon)} = 0$.

We now let $\mathcal{X}_\beta$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(\beta/2)$, where μ is a probability measure such that $\text{supp}(\mu) \subset E$. Then $\mathcal{X}_\beta$ is a closed subset of $\mathcal{E} \times M(\beta/2)$ under the product metric, and thus a complete metrizable space. We remark that for any $\varepsilon > 0$ and $(E, \mu) \in \mathcal{X}_\beta$,

$$\lim_{|\xi|\to\infty} |\xi|^{\beta/2-\varepsilon} |\widehat{\mu}(\xi)| = 0, \tag{3.1}$$

which follows because $\|\mu\|_{M(\beta/2-\varepsilon/2)}$ is finite. Thus $\dim_{\mathbf{F}}(\mu) \geqslant \beta$ for each $(E, \mu) \in \mathcal{X}_\beta$.

The next lemma allows us to work with smooth measures for the remainder of the paper.

Lemma 1. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Proof. Consider $(E, \mu) \in \mathcal{X}_\beta$. For each $r \in (0,1)$, consider the convolved measure $\mu_r = \mu * \phi_r$. Then $\mu_r \in C^\infty(\mathbf{T}^d)$. If we set $E_r = E \cup \text{supp}(\mu_r)$, then we claim

$$\lim_{r\to 0} (E_r, \mu_r) = (E, \mu). \tag{3.2}$$

Since $\text{supp}(\mu_r) \subset E_r$, we conclude that

$$d_H(E, E_r) \leqslant r. \tag{3.3}$$

Thus $E_r \to E$ as $r \to 0$ under the Hausdorff distance. Now fix $\varepsilon_1 \in (0, \beta/2]$ and $\varepsilon > 0$. For each $\xi \in \mathbf{Z}^d$, $|\widehat{\mu_r}(\xi)| = |\widehat{\phi_r}(\xi)||\widehat{\mu}(\xi)|$, so

$$|\xi|^{\beta/2-\varepsilon_1} |\mu_r(\xi) - \mu(\xi)| = |\xi|^{\beta/2-\varepsilon_1} |\widehat{\phi_r}(\xi) - 1||\widehat{\mu}(\xi)|. \tag{3.4}$$

Since $(E,\mu) \in \mathcal{X}_\beta$, we can apply (3.1) to conclude that there exists R such that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{3.5}$$

Combining (3.4), (3.5), and (2.1), we conclude that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\mu_r(\xi) - \mu(\xi)| \leqslant 2\varepsilon. \tag{3.6}$$

By (2.2), we conclude that there exists $r_0 > 0$ such that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon}|\widehat{\phi}_r(\xi) - 1| \leqslant \varepsilon. \tag{3.7}$$

The (L^1, L^∞) bound for the Fourier transform implies that

$$|\widehat{\mu}(\xi)| \leqslant \mu(\mathbf{T}^d) = 1 \tag{3.8}$$

But from (3.7) and (3.8) applied to (3.4), we find that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\mu_r(\xi) - \mu(\xi)| \leqslant \varepsilon. \tag{3.9}$$

Putting together (3.6) and (3.9), we find that for $r \leqslant r_0$,

$$\|\mu_r - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant 2\varepsilon. \tag{3.10}$$

Since ε and ε_1 were arbitrary, we conclude from (3.10) and (3.3) that $(E_r, \mu_r) \to (E, \mu)$, completing the proof. $\square$

Remark 2. *Let*

$$\tilde{\mathcal{X}}_\beta = \{(E,\mu) \in \mathcal{X}_\beta : supp(\mu) = E\}.$$

Suppose $(E_0, \mu_0) \in \tilde{\mathcal{X}}_\beta$. Then, in the proof above, one may let E_r be equal to $supp(\mu_r)$, since it follows from this that $d_H(E_0, E_r) \leqslant r$. This means that the set of pairs $(E,\mu) \in \tilde{\mathcal{X}}_\beta$ with $\mu \in C^\infty(\mathbf{T}^d)$ are dense in $\tilde{\mathcal{X}}_\beta$.

The reader might be wondering why we don't work with the smaller space $\tilde{\mathcal{X}}_\beta \subset \mathcal{X}_\beta$. The reason is that $\tilde{\mathcal{X}}_\beta$ is not a closed subset of $\mathcal{E} \times M(\beta/2)$, and so is not a complete metrizable space under the product topology. However, as a consolation, quasi-all elements of $\mathcal{X}_\beta$ belong to $\tilde{\mathcal{X}}_\beta$, so that one can think of $\mathcal{X}_\beta$ and $\tilde{\mathcal{X}}_\beta$ as being equal 'generically'.

Lemma 2. *For quasi-all $(E,\mu) \in \mathcal{X}_\beta$, $supp(\mu) = E$.*

Proof. For each closed cube $I \subset \mathbf{T}^d$, let

$$A(I) = \{(E,\mu) \in \mathbf{T}^d : (E \cap I) = \varnothing \text{ or } \mu(I) > 0\}.$$

Then $A(I)$ is an open set. If $\{I_k\}$ is a sequence enumerating all cubes with rational corners in $\mathbf{T}^d$, then

$$\bigcap_{k=1}^{\infty} A(I_k)$$

is the collection of $(E, \mu) \in \mathcal{X}_\beta$ with $\text{supp}(\mu) = E$. Thus it suffices to show that $A(I)$ is dense in $\mathcal{X}_\beta$ for each closed cube I.

Consider $(E_0, \mu_0) \in \mathcal{X}_\beta - A(I)$, $\varepsilon_1 \in (0, \beta/2]$, and $\varepsilon > 0$. Our goal is to find $(E, \mu) \in A(I)$ with $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu_0 - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Without loss of generality by Lemma 1 we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$. Because $(E_0, \mu_0) \in \mathcal{X}_\beta - A(I)$, we know $E \cap I \neq \varnothing$ and $\mu(I) = 0$. Find a smooth probability measure ν supported on $E_\varepsilon \cap I$ and, for $t \in (0, 1)$, define $\mu_t = (1-t)\mu_0 + t\nu$. Then $\text{supp}(\mu_t) \subset E_\varepsilon$, so if we let $E = \text{supp}(\nu) \cup \text{supp}(\mu)$, then $d_H(E, E_0) \leqslant \varepsilon$. Clearly $(E, \mu_t) \in A(I)$ for $t > 0$. And

$$\|\mu_t - \mu_0\|_{M(\beta/2-\varepsilon)} \leqslant t\left(\|\mu_0\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)}\right),$$

so if we choose $t \leqslant \varepsilon(\|\mu\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)})^{-1}$ shows $\|\mu_t - \mu\|_{M(\beta/2-\varepsilon)} \leqslant \varepsilon$. Since ε was arbitrary, we conclude $A(I)$ is dense in $\mathcal{X}_\beta$. □

Combining Lemma 2 with Remark 2 gives the following simple corollary.

Corollary 1. *The family of (E, μ) with supp$(\mu) = E$ and $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Our main way of constructing approximations to $(E_0, \mu_0) \in \mathcal{X}_\beta$ is to multiply μ_0 by a smooth function f. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which contribute to the formation of a pattern we are trying to avoid. As long as the Fourier transform of f decays appropriately quickly, we find $f\mu_0 \approx \mu_0$.

Lemma 3. *Consider a smooth finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function $f \in C^\infty(\mathbf{T}^d)$. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(\beta/2)} \lesssim_{d,\mu_0} \|f\|_{M(\beta/2)}.$$

Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu_0}$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\xi|^{\beta/2}|\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| = |\xi|^{\beta/2}\left|\sum_{\eta \neq \xi} \widehat{f}(\xi - \eta)\widehat{\mu_0}(\eta)\right|. \tag{3.11}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi - \eta| \leqslant 2|\xi|$, so

$$|\xi|^{\beta/2}|\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)}|\xi|^{\beta/2}|\xi - \eta|^{-\beta} \leqslant 2^{\beta/2}\|f\|_{M(\beta/2)} \lesssim_d \|f\|_{M(\beta/2)}. \tag{3.12}$$

Since μ_0 is smooth, for any $T \geqslant 0$ and $\xi \in \mathbf{Z}^d$,

$$|\xi|^T|\widehat{\mu_0}(\xi)| \lesssim_{T,\mu_0} 1. \tag{3.13}$$

Thus we can combine the bounds (3.12) and (3.13), with $T = d + 1$, to conclude that

$$|\xi|^{\beta/2}\left|\sum_{0 \leqslant |\eta| \leqslant |\xi|/2} \widehat{f}(\eta)\widehat{\mu_0}(\xi - \eta)\right| \lesssim_{\mu_0,d} \left(1 + \sum_{0 < |\eta| \leqslant |\xi|/2} \frac{1}{|\eta|^{d+1}}\right)\|f\|_{M(\beta)} \lesssim_d \|f\|_{M(\beta)}. \tag{3.14}$$

On the other hand, for all $\eta \neq \xi$,

$$|\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)}|\xi - \eta|^{-\beta} \leqslant \|f\|_{M(\beta/2)}. \tag{3.15}$$

Thus applying (3.13) and (3.15), with $T = 3d/2$, we conclude that

$$|\xi|^{\beta/2} \left| \sum_{\substack{|\eta| > |\xi|/2 \\ \eta \neq \xi}} \widehat{f}(\xi - \eta)\widehat{\mu_0}(\eta) \right| \lesssim_{d,\mu_0} |\xi|^{\beta/2} \sum_{|\eta| > |\xi|/2} \frac{\|f\|_{M(\beta/2)}}{|\eta|^{3d/2}} \lesssim_d \|f\|_{M(\beta/2)}. \tag{3.16}$$

Combining (3.11), (3.14) and (3.16) completes the proof. $\square$

Remark 3. *In particular, we note that this lemma implies that* $\mu(\mathbf{T}^d) \geqslant 1 - O_{d,\mu_0}(\|f\|_{M(0)})$.

The bound in (3), if $\|f\|_{M(\beta/2)}$ is taken appropriately small, also implies that the Hausdorff distance between the supports of μ and μ_0 is not too large.

Lemma 4. *Fix a probability measure* $\mu_0 \in C^\infty(\mathbf{T}^d)$. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *depending on* μ_0 *and* ε, *such that if* $\mu \in C^\infty(\mathbf{T}^d)$ *is any other probability measure and* $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, *then* $d_H(supp(\mu), supp(\mu_0)) \leqslant \varepsilon$.

Proof. Consider any cover of $\text{supp}(\mu_0)$ by a family of radius ε balls $\{B_1, \dots, B_N\}$, and for each $i \in \{1, \dots, N\}$, consider a smooth function $f_i \in C_c^\infty(B_i)$ such that there is $s > 0$ with

$$\int f_i(x) d\mu_0(x) \geqslant s. \tag{3.17}$$

for each $i \in \{1, \dots, N\}$. Fix $A > 0$ with

$$\sum_{\xi \neq 0} |\widehat{f}_i(\xi)| \leqslant A \tag{3.18}$$

for all $i \in \{1, \dots, N\}$ as well. Set $\delta = s/2A$. If $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, we apply Plancherel's inequality together with (3.17) and (3.18) to conclude that

$$\begin{aligned} \left| \int f_i(x) d\mu(x)\, dx - \int f_i(x) d\mu_0(x) \right| &= \left| \sum_{\xi \in \mathbf{Z}^d} \widehat{f}_i(\xi) \left(\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)\right) \right| \\ &\leqslant A\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant s/2. \end{aligned} \tag{3.19}$$

Thus we conclude from (3.17) and (3.19) that

$$\int f_i(x) d\mu(x)\, dx \geqslant \int f_i(x) d\mu_0(x) - s/2 \geqslant s/2 > 0. \tag{3.20}$$

Since equation (3.20) hodls for each $i \in \{1, \dots, N\}$, the support of μ intersects every balls in $\{B_1, \dots, B_N\}$. Combined with the assumption that $\text{supp}(\mu) \subset \text{supp}(\mu_0)$, this implies that $d_H(\mu_0, \mu) \leqslant \varepsilon$. $\square$

Given K points $x_1, \dots, x_K \in \mathbf{T}^d$, consider the smooth function

$$f(x) = \frac{1}{K} \sum_{i=1}^{K} \phi_r(x - x_i).$$

The support of f consists of K radius r balls. Provided $K \approx r^{-\beta}$, the support of f behaves like an r-thickening of a β dimensional set. The next lemma shows that if there is enough square root cancellation in the exponential sum

$$\frac{1}{K}\sum_{i=1}^{K} e^{2\pi i x_i \cdot \xi},$$

then f has the appropriate Fourier decay of a discretized β dimensional set.

Lemma 5. *Fix $C > 0$, $r > 0$ and $\varepsilon_1 > 0$. Consider an integer $K \geqslant (1/C) \cdot r^{-\beta}$, and let $x_1, \dots, x_K \in \mathbf{T}^d$ be such that for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant (1/r)^{1+\varepsilon_1}$,*

$$\left| \frac{1}{K}\sum_{i=1}^{K} e^{2\pi i x_i \cdot \xi} \right| \leqslant C K^{-1/2} \log(K)^{1/2}. \tag{3.21}$$

Then for each $\varepsilon > 0$ and $\varepsilon_2 \in (0, \beta/2)$, there exists $r_0 > 0$ depending only on C, β, ε, ε_1, and ε_2, such that if $r \leqslant r_0$, and we define

$$f(x) = \frac{1}{K}\sum_{i=1}^{K} \phi_r(x - x_i)$$

then $\|f\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon$.

Proof. Set

$$D(x) = \frac{1}{K}\sum_{i=1}^{K} \delta(x - x_i),$$

then (3.21) is equivalent to the property that for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{D}(\xi)| \leqslant C K^{-1/2} \log(K)^{1/2}. \tag{3.22}$$

Noting that $f = D * \phi_r$, we conclude that

$$|\widehat{f}| = |\widehat{D}||\widehat{\phi_r}|. \tag{3.23}$$

If $r_1 = (10C)^{-1/\beta}$ and $r \leqslant r_1$, then the function $x \mapsto x^{-1/2}\log(x)^{1/2}$ is decreasing for $x \geqslant (1/C)r^{-1/\beta}$. Thus for $0 < |\xi| \leqslant 1/r$ and $r \leqslant r_1$, we combine (3.22), (3.23) and (2.1) together with the bound $K \geqslant (1/C)r^{-\beta}$ to conclude that

$$\begin{aligned} |\widehat{f}(\xi)| &\leqslant \left[C K^{-1/2} \log(K)^{1/2} |\xi|^{\beta/2-\varepsilon_2} \right] |\xi|^{\varepsilon_2-\beta/2} \\ &\leqslant \left[C^{1+\beta/2} r^{\beta/2} \log((1/C)r^{-\beta})^{1/2} (1/r)^{\beta/2-\varepsilon_2} \right] |\xi|^{\varepsilon_2-\beta/2} \\ &\leqslant \left[C^{1+\beta/2} r^{\varepsilon_2} \log((1/C)r^{-\beta})^{1/2} \right] |\xi|^{\varepsilon_2-\beta/2}. \end{aligned} \tag{3.24}$$

As $r \to 0$, $C^{1+\beta/2} r^{\varepsilon_2} \log((1/C)r^{-\beta})^{1/2} \to 0$, so we conclude from (3.24) that there exists $r_2 \leqslant r_1$ depending on C, β, ε_2, and ε, such that for $r \leqslant r_2$ and $0 < |\xi| \leqslant (1/r)$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon |\xi|^{\varepsilon_2-\beta/2}. \tag{3.25}$$

If $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$, (2.3) implies $|\widehat{\phi}_r(\xi)| \lesssim_\beta r^{-\beta/2}|\xi|^{-\beta/2}$, which together with (3.22), (3.23), and the bound $K \geqslant (1/C)r^{-\beta}$, show that for $r \leqslant r_1$,

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_\beta \left(CK^{-1/2}\log(K)^{1/2}r^{-\beta/2}|\xi|^{-\varepsilon_2}\right)|\xi|^{\varepsilon_2-\beta/2} \\ &\leqslant \left(C^{1+\beta/2}\log((1/C)r^{-\beta})^{1/2}r^{\varepsilon_2(1+\varepsilon_1)}\right)|\xi|^{\varepsilon_2-\beta/2}. \end{aligned} \tag{3.26}$$

Again, we find that as $r \to 0$, $C^{1+\beta/2}\log((1/C)r^{-\beta})^{1/2}r^{\varepsilon_2} \to 0$, so we conclude from (3.26) that there exists r_3 depending on C, β, ε, ε_1 and ε_2, such that if $r \leqslant r_3$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon|\xi|^{\varepsilon_2-\beta/2}. \tag{3.27}$$

If $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$, we apply (2.3) for $T \geqslant \beta/2$ together with the bound $K \geqslant (1/C)r^{-\beta}$ to conclude

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_T [r^{-T}|\xi|^{\beta/2-T}]|\xi|^{-\beta/2} \\ &\leqslant \left[r^{-T}(1/r)^{(\beta/2-T)(1+\varepsilon_1)}\right]|\xi|^{-\beta/2} \\ &= \left[r^{\varepsilon_1 T-(\beta/2)(1+\varepsilon_1)}\right]|\xi|^{-\beta/2}. \end{aligned} \tag{3.28}$$

If we choose $T > (\beta/2)(1 + 1/\varepsilon_1)$, then as $r \to 0$, $r^{\varepsilon_1 T-(\beta/2)(1+\varepsilon_1)} \to 0$. Thus we conclude from (3.28) that there exists a large integer r_4 depending on C, β, ε, ε_1, and ε_2 such that for $r \leqslant r_4$ and $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon|\xi|^{-\beta/2} \leqslant \varepsilon|\xi|^{\varepsilon_2-\beta/2}. \tag{3.29}$$

All that remains is to combine (3.25), (3.27), and (3.29), defining $r_0 = \min(r_1, r_2, r_3, r_4)$. □

Corollary 2. *Consider a smooth finite measure μ_0 on $\mathbf{T}^d$. Fix $C > 0$, $r > 0$ and $\varepsilon_1 > 0$. Consider an integer $K \geqslant (1/C)r^{-\beta}$, and let $x_1, \dots, x_K \in \mathbf{T}^d$ be such that if*

$$D(x) = \frac{1}{K}\sum_{i=1}^{K}\delta(x - x_i),$$

then for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{D}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{3.30}$$

It is then true that for each $\varepsilon_2 \in (0, \beta/2]$, there exists r_0 depending on C, β, d, β, ε, ε_1, and ε_2, such that if $r \leqslant r_0$, and we define

$$f(x) = \frac{1}{K}\sum_{i=1}^{K}\phi_r(x - x_i),$$

and a smooth probability measure

$$\mu = \frac{f\mu_0}{(f\mu_0)(\mathbf{T}^d)},$$

then $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon$.

Proof. It suffices to use Lemmas 3 and 5 to show that there exists r_0 such that for $r \leqslant r_0$,

$$\|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon/2, \tag{3.31}$$

and

$$\|f\mu_0 - \mu_0\|_{M(0)} \leqslant \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right). \tag{3.32}$$

Equation (3.32) implies that

$$1 - \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right) \leqslant (f\mu_0)(\mathbf{T}^d) \leqslant 1. \tag{3.33}$$

But now (3.31) and (3.33) show that

$$\begin{aligned}\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} &\leqslant \|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon)} + \|\mu - f\mu_0\|_{M(\beta/2-\varepsilon)} \\ &\leqslant (\varepsilon/2) + \left(1 - \frac{1}{(f\mu_0)(\mathbf{T}^d)}\right)\|\mu_0\|_{M(\beta/2-\varepsilon)} \\ &\leqslant (\varepsilon/2) + (\varepsilon/2) \leqslant \varepsilon.\end{aligned}$$

□

A common method to obtain square root cancellation in a sum is to use concentration properties of collections of independant random variables. Since square root cancellation occurs often in random phenomena, it is reasonable to use random constructions to construct Salem sets.

Lemma 6. *Fix a large integer K. Let $X_1, \dots, X_K$ be independant random variables on $\mathbf{T}^d$, such that for each nonzero $\xi \in \mathbf{Z}^d$,*

$$\sum_{k=1}^{K} \mathbf{E}\left(e^{2\pi i \xi \cdot X_k}\right) = 0. \tag{3.34}$$

In particular, (3.34) *holds if the family $\{X_i\}$ are uniformly distributed on $\mathbf{T}^d$. Set*

$$D(x) = \frac{1}{K}\sum_{k=1}^{K} \delta(x - x_k)$$

and

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leqslant K^{1/\beta+1}\}.$$

Then there exists a constant C depending on β and d, such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \dots, K\}$, consider the random variable $Y(\xi, k) = K^{-1}e^{2\pi i(\xi \cdot X_k)}$. Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} Y(\xi, k) = \widehat{D}(\xi). \tag{3.35}$$

We also note that for each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \dots, K\}$,

$$|Y(\xi, k)| = K^{-1}, \tag{3.36}$$

Moreover,

$$\sum_{i=1}^{k} \mathbf{E}(Y(\xi, k)) = 0. \tag{3.37}$$

Since the family of random variables $\{Y(\xi, 1), \dots, Y(\xi, K)\}$ is independent for a fixed non-zero ξ, we can apply Hoeffding's inequality together with (3.35) and (3.36) to conclude that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{D}(\xi)| \geqslant t\right) \leqslant 2e^{-Kt^2/2}. \tag{3.38}$$

A union bound obtained by applying (3.38) over all $|\xi| \leqslant K^{1/\beta+1}$ shows that if

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leqslant K^{1/\beta+1}\},$$

then there exists a constant $C \geqslant 1$ depending on d and β such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \geqslant t\right) \leqslant \exp\left(C \log(K) - \frac{5Kt^2}{C}\right). \tag{3.39}$$

But then, setting $t = CK^{-1/2} \log(K)^{1/2}$ in (3.39) completes the proof. $\square$

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}_\beta$ to be as low as possible. For each $(E, \mu) \in \mathcal{X}_\beta$, the condition that $\mu \in M(\beta/2)$ implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus it is natural to expect that for quasi-all $(E, \mu) \in M(\beta/2)$, the set E has both Hausdorff dimension and Fourier dimension equal to β.

Lemma 7. *For quasi-all $(E, \mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since when $\beta = d$, E is a Salem set for any $E(, \mu) \in \mathcal{X}_\beta$, and thus the result is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, E has Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{X} : H^\alpha_\delta(E) < s\}$. Then $A(\alpha, \delta, s)$ is an open subset of $\mathcal{X}_\beta$, and

$$\bigcap_{n=1}^\infty \bigcap_{m=1}^\infty \bigcap_{k=1}^\infty A(\beta + 1/n, 1/m, 1/k)$$

is precisely the family of $(E, \mu) \in \mathcal{X}_\beta$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha, \delta, s)$ is dense in $\mathcal{X}_\beta$ for $\alpha \in (\beta, d)$ and $\delta, s > 0$. Fix $(E_0, \mu_0) \in \mathcal{X}_\beta$, $\alpha \in (\beta, d)$, $\delta > 0$, $s > 0$, and $\varepsilon_1 > 0$. We aim to show that for each $\varepsilon > 0$, there exists $(E, \mu) \in A(\alpha, \delta, s)$ such that $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Without loss of generality, in light of Lemma 1, we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a small value r, and then find an integer K such that $r^{-\beta} \leqslant K \leqslant r^{-\beta} + 1$. Lemma 6 shows that there exists a constant C depending on β and d, as well as K points $x_1, \dots, x_K \in \mathbf{T}^d$ such that if

$$D(x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - x_k),$$

then for each $|\xi| \leqslant (1/r)^{1+1/\beta} \leqslant K^{1/\beta+1}$,

$$|\widehat{D}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \tag{3.40}$$

Applying Corollary 2 with (3.40), we conclude that there exists r_1 depending on d, β, μ_0, ε, and ε_1 such that if $r \leqslant r_1$, if

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k=1}^{K} \phi_r(x - x_k) \right) \mu_0(x),$$

and if

$$\mu = \mu_1 / \mu_1(\mathbf{T}^d),$$

then

$$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon. \tag{3.41}$$

Note that μ is supported on K balls of radius r. Thus for $r \leqslant \delta$,

$$H^\alpha_\delta(\mathrm{supp}(\mu)) \leqslant Kr^\alpha \leqslant (r^{-\beta} + 1)r^\alpha = r^{\alpha-\beta} + r^\alpha. \tag{3.42}$$

Since $\alpha > \beta$, (3.42) implies that there is r_2 depending on α, β, and s such that for $r \leqslant r_2$,

$$H^\alpha_\delta(\mathrm{supp}(\mu)) \leqslant s. \tag{3.43}$$

Now let

$$E = \mathrm{supp}(\mu) \cup \{y_1, \dots, y_N\},$$

where $\{y_1, \dots, y_N\} \subset E_0$ is a ε-net of E_0. Set $r_0 = \min(r_1, r_2, \delta)$ and suppose $r \leqslant r_0$. Equation (3.43) implies that $H^\alpha_\delta(E) \leqslant s$, so $(E, \mu) \in A(\alpha, \delta, s)$. And since $\mathrm{supp}(\mu) \subset E$,

$$d_H(E, E_0) \leqslant \varepsilon. \tag{3.44}$$

Recalling (3.41), we see that we have proved what was required. □

All that now remains is to show that quasi-all elements of $\mathcal{X}_\beta$ avoid the given set Z; just as with the proof above, the advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

4 Random Avoiding Sets

In the last section, our results held for an arbitrary $\beta \in (0, d]$. But in this section, we assume

$$\beta = \frac{dn - \alpha}{n - 1/2},$$

which will enable us to generically avoid the pattern Z described in Theorem 2. The argument here is very similar to the construction in [3] albeit in the Baire category setting, though we must modify parameters slightly to ensure a Fourier dimension bound rather than a Hausdorff dimension bound.

Lemma 8. *Suppose $Z \subset \mathbf{T}^{dn}$ is a countable union of compact sets, each with lower Minkowski dimension at most α. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.*

Proof. The set $Z \subset \mathbf{R}^{dn}$ is the countable union of sets with lower Minkowski dimension at most α. For a closed set $W \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, and $s > 0$, consider the set

$$B(W, s) = \left\{ (E, \mu) \in \mathcal{X}_\beta : \begin{array}{c} \text{for all } x_1, \dots, x_n \in E \text{ such that} \\ |x_i - x_j| \geqslant s \text{ for } i \neq j,\ (x_1, \dots, x_n) \notin W \end{array} \right\}.$$

If $(E_0, \mu_0) \in B(W, s)$, then because E_0 is compact, so too is the set

$$F = \{(x_1, \dots, x_n) \in E_0^n : |x_i - x_j| \geqslant s \text{ for } i \neq j\}$$

Since W is also closed, hence compact, there exists $\varepsilon > 0$ such that if $(x_1, \dots, x_n) \in F$, then $d((x_1, \dots, x_n), W) > \varepsilon$. It follows that if $d_H(E_0, E) \leqslant \varepsilon$, then for any measure μ supported on E, $(E, \mu) \in B(W, s)$. Thus $B(W, s)$ is an open subset of $\mathcal{X}_\beta$. If Z is a countable union of closed sets $\{Z_k\}$ with lower Minkowski at most α, then clearly the set

$$\bigcup_{k=1}^\infty \bigcup_{n=1}^\infty B(Z_k, 1/n)$$

consists of the family of sets (E, μ) such that for distinct $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$. Thus it suffices to show that $B(W, s)$ is dense in $\mathcal{X}_\beta$ for any $s > 0$, and any closed set W with lower Minkowski dimension at most α.

Let us begin by fixing a set $W \subset \mathbf{T}^{dn}$ and a pair $(E_0, \mu_0) \in \mathcal{X}_\beta$. We will show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, we can find $(E, \mu) \in B(W, s)$ with $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. We may assume by Corollary 1 that $\mathrm{supp}(\mu_0) = E$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$. Since W has lower Minkowski dimension at most α, we can find arbitrarily small $r \in (0, 1)$ such that

$$|W_r| \leqslant r^{dn - \alpha - \varepsilon_1/4}. \tag{4.1}$$

Assume also that r is small enough that we can find an integer $K \geqslant 10$ with

$$r^{\varepsilon_1/2 - \beta} \leqslant K \leqslant r^{\varepsilon_1/2 - \beta} + 1. \tag{4.2}$$

Let $X_1, \dots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $k_1, \dots, k_n \in \{1, \dots, K\}$, the random vector $X_k = (X_{k_1}, \dots, X_{k_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (4.1) and (4.2) imply that

$$\begin{aligned}\mathbf{P}(d(X_k, W) \leqslant r) \leqslant |W_r| &\leqslant r^{dn-\alpha-\varepsilon_1/4} \\ &\lesssim_{d,n,\beta} K^{\frac{-(dn-\alpha-\varepsilon_1/4)}{\beta-\varepsilon_1/2}} \\ &= K^{-(n-1/2-\varepsilon_1/4\beta)\left(1+\frac{\varepsilon_1}{2\beta-\varepsilon_1}\right)} \\ &= K^{1/2-n+\varepsilon_1/4\beta-\frac{\varepsilon_1}{2\beta-\varepsilon_1}(n-1/2-6\varepsilon_1/\beta)} \\ &\leqslant K^{1/2-n+\varepsilon_1/4\beta-\frac{\varepsilon_1}{2\beta-\varepsilon_1}} \leqslant K^{1/2-n}.\end{aligned} \tag{4.3}$$

If M_0 denotes the number of indices i such that $d(X_i, W) \leqslant r$, then by linearity of expectation we conclude from (4.3) that there is a constant C depending only on d, n, and β such that

$$\mathbf{E}(M_0) \leqslant (C/10)K^{1/2}. \tag{4.4}$$

Applying Markov's inequality to (4.4), we conclude that

$$\mathbf{P}(M_0 \geqslant CK^{1/2}) \leqslant 1/10. \tag{4.5}$$

Taking a union bound to (4.5) and the result of Lemma 6, we conclude that there exists K points $x_1, \dots, x_K \in \mathbf{T}^d$ and a constant C depending only on d, n, and β such that the following two statements hold:

(1) Let S be the set of indices $k_1 \in \{1, \dots, K\}$ with the property that we can find distinct indices $k_2, \dots, k_n \in \{1, \dots, K\}$ such that if $X = (X_{k_1}, \dots, X_{k_n})$, then $d(X, W) \leqslant r$. Then

$$\#(S) \leqslant CK^{1/2}. \tag{4.6}$$

(2) If we define

$$D_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - x_k)$$

then for $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D_0}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \tag{4.7}$$

Thus (4.6) and (4.7) imply that if

$$D_1(x) = \sum_{k \notin S} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta_0+1}$,

$$|\widehat{D_1}(\xi)| \leqslant 2CK^{-1/2} \log(K). \tag{4.8}$$

Since $K \geqslant r^{\varepsilon_1/2-\beta}$, we can apply Corollary 2 for any $\delta > 0$ in conjunction with (4.8) to find $r_0(q)$ depending only on d, q, n, β, μ_0, ε, δ, and ε_1 such that if $r \leqslant \min(r_0(q), s)$ and we define

$$\mu'(x) = \left(\sum_{k \notin S} \phi_{(1/2n^{1/2})r}(x - X_k) \right) \mu_0(x),$$

and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, then

$$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{4.9}$$

Choosing δ in accordance with Lemma 4, (4.9) also implies that

$$d_H(\mathrm{supp}(\mu), \mathrm{supp}(\mu_0)) \leqslant \varepsilon.$$

Thus all that remains to prove the density statement is to show that if $E = \mathrm{supp}(\mu)$, then $(E, \mu) \in B(W, s)$.

Consider n points $y_1, \dots, y_n \in \mathrm{supp}(\mu)$, with $|y_i - y_j| \geqslant r$ for any two indices $i \neq j$. We can therefore find distinct indices $k_1, \dots, k_n \in \{1, \dots, K\}$ such that for each $i \in \{1, \dots, n\}$,

$$|x_{k_i} - y_i| \leqslant (n^{-1/2}/2) \cdot r. \tag{4.10}$$

If we set $x = (x_{k_1}, \dots, x_{k_n})$ and $y = (y_1, \dots, y_n)$, then (4.10) implies

$$|x - y| \leqslant (r/2). \tag{4.11}$$

Since $i_1 \notin S$, $d(x, W) \geqslant r$, which combined with (4.11) implies

$$d(y, W) \geqslant d(x, W) - |x - y| \geqslant r/2. \tag{4.12}$$

Thus in particular we conclude $y \notin W$, which shows $(E, \mu) \in B(W, s)$. □

Before we move onto the next proof, let us discuss where the loss in Theorem 1 occurs in our proof, as compared to the Hausdorff dimension bound of [3]. In the proof of Lemma 8, in order to obtain the bound (4.8), we were forced to choose the parameter r such that $\#(S) \leqslant K^{1/2}$, so that we can use the trivial bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot x_k)} \right| \leqslant \#(S) \leqslant K^{1/2}. \tag{4.13}$$

On the other hand, if we were able to justify that with high probability, we could obtain a square root cancellation bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot x_k)} \right| \lesssim \#(S)^{1/2}, \tag{4.14}$$

then we would only need to choose the parameter r such that $\#(S) \lesssim K$, which leads to a set with larger Fourier dimension, matching with the relevant Hausdorff dimension bound obtained in [3]. We are able to carry out this square root cancellation calculation completely when the set W we are avoiding has greater regularity, i.e. it is a Lipschitz manifold.

5 The More Difficult Case

Let us now consider the more difficult case. We consider a manifold $W \subset \mathbf{T}^{dn}$ of dimension $dn - d$ with the appropriate regularity condition, and set

$$\beta = \frac{d}{n-1}.$$

Given any family of disjoint, sidelength s cubes $R_1, \dots, R_n \subset \mathbf{T}^d$ such that $d(R_i, R_j) \geqslant 10s$, we let

$$H(W; R_1, \dots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : (R_1 \times \cdots \times R_n) \cap E^n \cap W = \varnothing\}.$$

Then $H(W; R_1, \dots, R_n)$ is an open subset of $\mathcal{X}_\beta$. For the purpose of a Baire category argument, our result will follow by showing $H(W; R_1, \dots, R_n)$ is dense in $\mathcal{X}_\beta$ for each family of cubes $\{R_1, \dots, R_n\}$. Without loss of generality, possibly permuting coordinates if necessary we may assume that $1/s$ is an integer, and that for the family of cubes $\{R_1, \dots, R_n\}$, if $Q_1 = 2R_1, \dots, Q_n = 2R_n$, then there exists a Lipschitz function $f : Q_2 \times \cdots \times Q_n \to \mathbf{R}^d$ such that

$$W \cap (Q_1 \times \cdots \times Q_n) = \{(x_1, \dots, x_d) \in Q_1 \times \cdots \times Q_n : x_1 = f(x_2, \dots, x_d)\}.$$

Fix a constant $L \geqslant 0$ such that for any $x, x' \in Q_2 \times \cdots \times Q_n$,

$$|f(x - x')| \leqslant L|x - x'|. \tag{5.1}$$

As with the previous proof, we fix $(E_0, \mu_0) \in \mathcal{X}_\beta$ with $\operatorname{supp}(\mu_0) = E_0$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$, and show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, there is $(E, \mu) \in H(W; Q_1, \dots, Q_n)$ with $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$.

Consider a family of independant random variables $\{X_i(k) : 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant K\}$, where each $X_i(k)$ is uniformly distributed on Q_i. Set $K_1 = K(s^{-d} - n)$ (an integer since s is a power of two), and consider an additional family of independant random variables $\{X_0(k) : 1 \leqslant k \leqslant K_1\}$, where each $X_0(k)$ is uniformly distributed on $\mathbf{T}^d - Q_1 - \cdots - Q_n$. Let $r = K^{-1/\beta} = 1/K^{\frac{n-1}{d}}$, and then set S be the set of indices $k_1 \in \{1, \dots, K\}$ such that there are indices $k_2, \dots, k_n \in \{1, \dots, K\}$ with the property that

$$|X_1(k_1) - f(X_2(k_2), \dots, X_n(k_n))| \leqslant r/(L+1). \tag{5.2}$$

A simple argument following from (5.2), which we prove in Lemma BLAH, shows that if $k_1 \notin S$, then for any $k_2, \dots, k_n \in \{1, \dots, K\}$, if $X = (X_1(k_1), \dots, X_n(k_n))$, then $d(X, W) \geqslant r$. If we define

$$\mu_k'(x) = \left(\sum_{k=1}^{K_1} \phi_{(1/2n^{1/2})r}(x - X_0(k)) \right)$$

Then for any nonzero $\xi \in \mathbf{Z}^d$,

$$\mathbf{E}\left(\sum_{k=1}^{K_1} e^{2\pi i \xi \cdot X_0(k)} + \sum_{i=1}^{n} \sum_{k=1}^{K} e^{2\pi i \xi \cdot X_i(k)} \right) = 0. \tag{5.3}$$

Let $r = K^{-1/\beta} = 1/K^{\frac{n-1}{d}}$, and then set S be the set of indices $k_1 \in \{1, \dots, K\}$ such that there are indices $k_2, \dots, k_n \in \{1, \dots, K\}$ with the property that

$$|X_1(k_1) - f(X_2(k_2), \dots, X_n(k_n))| \leqslant r.$$

Now for each $\xi \in \mathbf{Z}^d$, let

$$Y_\xi = \sum_{k \in S} e^{2\pi i \xi \cdot X_k}.$$

If we write $\Omega \subset Q_1$ to be the set of values $x \in Q_1$ such that there are $k_2, \dots, k_n \in \{1, \dots, K\}$ such that

$$|x - f(X_2(k_2), \dots, X_n(k_n))| \leqslant r,$$

then

$$Y_\xi = \sum_{k=1}^{K} Z_k$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_1(k)} & : X_1(k) \in \Omega, \\ 0 & : X_1(k) \notin \Omega \end{cases}.$$

If Σ is the σ algebra generated by the random variables $\{X_i(k) : i \geqslant 2, k \in \{1, \dots, K\}\}$, then the random variables $\{Z(k)\}$ are *conditionally independant* (and identically distributed) given Σ. Since we have $|Z(k)| \leqslant 1$ almost surely, Hoeffding's inequality thus implies that for all $t \geqslant 0$,

$$\mathbf{P}\left(|Y_\xi - \mathbf{E}(Y_\xi|\Sigma)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-t^2}{2K}\right).$$

It is simple to see that

$$\mathbf{E}(Y_\xi|\Sigma) = \frac{K}{|Q_1|} \int_\Omega e^{2\pi i \xi \cdot x}\, dx = \frac{K}{s^d} \int_\Omega e^{2\pi i \xi \cdot x}\, dx.$$

Since

$$\Omega = \bigcup \left\{B(f(X_2(k_2), \dots, X_n(k_n)), r)\right\}.$$

we see that varying each random variable $X_i(k)$ adjusts at most K^{n-2} of these balls, and thus varying $X_i(k)$ independantly of the other random variables changes $\mathbf{E}(Y_\xi|\Sigma)$ by at most

$$\frac{2^d r^d K^{n-1}}{s} \leqslant \frac{2^d}{s}.$$

Thus McDiarmid's inequality shows that for any $t \geqslant 0$,

$$\mathbf{P}\left(|\,\mathbf{E}(Y_\xi|\Sigma) - \mathbf{E}(Y_\xi)| \geqslant t\right) \leqslant 4 \exp\left(\frac{s}{n2^d} \cdot \frac{-2t^2}{K}\right).$$

Combining BLAH and BLAH, we conclude that

$$\mathbf{P}\left(|Y_\xi - \mathbf{E}(Y_\xi)| \geqslant t\right) \leqslant 8 \exp\left(\min\left(\frac{s}{n2^{d+1}}, 1/8\right) \cdot \frac{-t^2}{K}\right)$$

Now for each $x \in Q_1$, we find that

$$\mathbf{P}(x \in \Omega) = \mathbf{P}(\text{There is } k_2, \dots, k_n \text{ s.t. } |f(X_2(k_2), \dots, X_n(k_n)) - x| \leqslant r).$$

Now

$$\mathbf{E}(Y_\xi) = \mathbf{E}(\mathbf{E}(Y_\xi|\Sigma)) = \frac{K}{s^d} = \frac{K}{s^d} \int_{Q_1} \mathbf{P}(x \in \Omega) e^{2\pi i \xi \cdot x}\, dx.$$

$$u(x) = \frac{|f^{-1}(B(x,r))|}{s^{d(n-1)}},$$

then

$$\mathbf{E}(Y_\xi) = \mathbf{E}(\mathbf{E}(Y_\xi|\Sigma)) = \int_{Q_1} u(x) e^{2\pi i \xi \cdot x}.$$

Using Fubini's theorem, we find that

$$\begin{aligned} \int_{Q_1} u(x)\, dx &= \frac{1}{s^{d(n-1)}} \int_{Q_1} \int_{f^{-1}(B(x,r))} dy\, dx \\ &= \frac{1}{s^{d(n-1)}} \int_{Q_2 \times \dots \times Q_d} \int_{B(f(y),r))} dx\, dy \\ &=\sim r^d \sim K^{1-n} \lesssim K^{-1}. \end{aligned}$$

By permuting coordinate and localizing correctly, we may assume that $W = \{(x_1, \dots, x_n) : x_1 = f(x_2, \dots, x_n)\}$. We have $r \approx K^{-1/\beta}$, and so Ω is a random 'splodge' of K^{n-1} randomly chosen radius $1/K^{(n-1)/d}$ balls, which has total area $O(1)$, so we should expect some overlap, but not too much. It might be useful to assume that $\partial_2 f$ is also bounded from below, since we can probably also treat the case $\partial_2 f$ small separately.

Given that $X_k^i = x_k^i$, the random variable Y_ξ is then the sum of K random variables, which each

Suppose $X_k^i = x_k^i$

If we let Σ be the σ algebra generated , then McDiarmid's inequality tells us that

$$\mathbf{P}\left(|Y_\xi - \mathbf{E}(Y_\xi|\Sigma)| \geqslant t\right)$$

Now if $\#(S) \leqslant K$, then if we individually alter each input, we can adjust Y_ξ by at most $2K$. Thus McDiarmid's inequality gives that

$$\mathbf{P}\left(|Y_\xi - \mathbf{E}(Y_\xi|BLAH)| \geqslant t|BLAH\right) \leqslant 4\exp\left(-\frac{t^2}{2K^{d-1}}\right).$$

Thus Y_ξ deviates from $\mathbf{E}(Y_\xi|BLAH)$ at a rate of $K^{1/2}$ with low probability, and a union bound over all ξ gives a deviation of $K^{1/2}\log(K)^{1/2}$ with low probability. If we can show that $|\mathbf{E}(Y_\xi|BLAH)| \leqslant K^{1/2}$ with high probability, we're be done!.

As a next step, let's calculate $\mathbf{E}(Y_\xi)$. Each $X_1, \dots, X_K$ independantly has a probability of being in S, and the distribution of $e^{2\pi i \xi \cdot X_k}$ given that $k \in S$ has a particular distribution. Thus $\mathbf{E}(Y_\xi|BLAH) = K\,\mathbf{P}(k \in S|BLAH)\,\mathbf{E}(e^{2\pi i \xi \cdot X_k}|BLAH, k \in S)$.

If

$$W_\varepsilon = \{(x,y) \in \mathbf{T}^2 : d(x, K^{-1/2}\,\mathbf{Z}) \leqslant \varepsilon\},$$

then $|W_\varepsilon| \leqslant K^{1/2}\varepsilon$

Let us think about this in a manner discretized as a scale ε. If $\mathbf{P}(k \in S|BLAH) \geqslant K^{-1/2} = \varepsilon^{\beta/2}$, then there are at least $\varepsilon^{\beta/2-1}$ different ε-separated values that X_k can take. Thus $e^{2\pi i X_k}$ can take. Now $e^{2\pi i \xi \cdot X}$ points in a particular direction on $|\xi|\varepsilon^{-1}$

If $k_1 \notin S$, then for any indices $k_2, \dots, k_n \in \{1, \dots, K\}$, if $X = (X^1_{k_1}, \dots, X^n_{k_n})$, then $d(X, W) \geqslant r/(L+1)$. To see this, it certainly follows by definition that

$$|X_1(k_1) - f(X_2(k_2), \dots, X_n(k_n))| \geqslant r.$$

If $Y = (Y_1, \dots, Y_n) \in W \cap (Q_1 \times \cdots \times Q_n)$, then

$$\begin{aligned} r &\leqslant |X_1(k_1) - f(X_2(k_2), \dots, X_n(k_n))| \\ &\leqslant |X_1(k_1) - Y_1| + |Y_1 - f(X_2(k_2), \dots, X_n(k_n)| \\ &= |X_1(k_1) - Y_1| + |f(Y_2, \dots, Y_n) - f(X_2(k_2), \dots, X_n(k_n)| \\ &\leqslant |X_1(k_1) - Y_1| + L|(Y_2, \dots, Y_n) - (X_2(k_2), \dots, X_n(k_n)) \\ &\leqslant (L+1)|X - Y|, \end{aligned}$$

so $|X - Y| \geqslant r/(L+1)$.

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