

Fractals Avoiding Fractal Sets

Jacob Denson

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Our proof depends very little on the Euclidean structure of the plane, and as such, we rephrase the construction as a combinatorial problem on graphs. Recall that an n uniform hypergraph is a collection of *vertices* and *hyperedges*, where a hyperedge is a set of n distinct vertices. We say such a graph is *hy-partite* if we can partition the vertex set into n sets V_1, \dots, V_n , such that each edge in the graph contains exactly one vertex from each set. An *independant set* is a subset of vertices containing no complete set of vertices in any edge. A *coloring* is a partition of the vertex set into finitely many classes, and we call each such class a *color*. The next lemma is a variant of Turán's theorem on the construction of independant sets, generalized for hypergraphs. For technical reasons, we need an extra restriction on the vertex set so it is 'uniformly' chosen over the graph, which is the reason for the coloring.

Lemma 1. *Let G be an n uniform graph together with a coloring partitioning each vertex set into size A independant sets. Then we can find an independant set U containing all but $(n/A^n)|E|$ of the colors.*

Proof. Let U be a vertex set chosen by randomly selecting a vertex of each color. Then each vertex in the graph is selected with probability $1/A$. For each edge $e = (v_1, \dots, v_n)$ in the graph, we know that the vertices v_i all have different colors, so they are added to U independantly, and so the chance that this edge connects vertices in the set we have constructed is

$$\begin{aligned} \mathbf{P}(v_1 \in U, \dots, v_n \in U) \\ = \mathbf{P}(v_1 \in U) \dots \mathbf{P}(v_n \in U) = 1/A^n \end{aligned}$$

Thus if we let E' denote the set of all edges $e = (u_1, \dots, u_n)$ with $u_1, \dots, u_n \in U$, then

$$\mathbf{E}|E'| = \sum_{e \in E} \mathbf{P}(e \in E') = \sum_{e \in E} 1/A^n = \frac{|E|}{A^n}$$

In particular, we may choose a *particular*, nonrandom choice U for which $|E'| \leq |E|/A^n$. If we form a vertex set $W \subset U$ by removing all vertices $u \in U$ which are a vertex in some edge in E' , then W is an independant set containing all but $n|E'| \leq (n/A^n)|E|$ colors. \square

Corollary. *If $|V| \gtrsim N^a$, $|E| \lesssim N^b$, and $A \gtrsim N^c$, where $b < a + c(n-1)$, then as $N \rightarrow \infty$ we can find an independant set containing all but a fraction $o(1)$ of all colors.*

Proof. A simple calculation on the quantities of the previous lemma yields

$$\begin{aligned} \frac{\#(\text{colors removed})}{\#(\text{total colors})} &= \frac{(n/A^n)|E|}{|V|/A} \\ &= \frac{n|E|}{|V|A^{n-1}} \lesssim \frac{N^b}{N^{a+c(n-1)}} \end{aligned}$$

This is $o(1)$ if $b < a + c(n-1)$. \square

We now apply these constructions on graphs to a problem which is quite clearly related to the fractal avoidance problem, and will form our key building block to constructing solutions to the problem. Given a fixed integer N , we subdivide \mathbf{R}^d into a grid of side-length $1/N$ cubes, the collection of such cubes we will denote by \mathcal{I} .

Theorem 1. *Suppose that $\dim_{\mathbf{M}}(Y) < \alpha$. If $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint collections of length $1/N$ cubes in \mathcal{I} , with $|\mathcal{I}_i| \gtrsim N^d$ for each i , then provided $\beta < (n-1)/(n-\alpha)$, we can find a collection of length $1/N^\beta$ cubes $\mathcal{J}_1, \dots, \mathcal{J}_n$ with $\mathcal{J}_1 \times \dots \times \mathcal{J}_n$ disjoint from Y and as $N \rightarrow \infty$, each \mathcal{J}_i contains a cube in all but $o(1)$ of the cubes in \mathcal{I}_i .*

Proof. For sufficiently large N , if we partition \mathbf{R}^{nd} into a grid of length $1/N^\beta$ cubes, and if \mathcal{K} is the collection of all these cubes intersecting Y , then $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Similarly, we partition each \mathcal{I}_i into a grid of length $1/N^\beta$ cubes \mathcal{I}'_i , using these intervals as vertices in a hypartite graph G with a hyperedge between $I_1 \in \mathcal{I}'_1, \dots, I_n \in \mathcal{I}'_n$ if $I_1 \times \dots \times I_n \in \mathcal{K}$. We say two cubes in G are the same color if they are contained in a common cube in \mathcal{I}_i . Since each side-length $1/N$ cube contains $N^{\beta d}/N^d = N^{(\beta-1)d}$ side-length $1/N^\beta$ cubes, each color in G contains $N^{(\beta-1)d}$ vertices. Each vertex set V_i in G contains $|\mathcal{I}'_i| = N^{\beta-1}|\mathcal{I}_i| \gtrsim N^{\beta+(d-1)}$ vertices. Finally the number of edges is bounded above by $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Thus in the

parameters of the previous corollary, $a = \beta + (d - 1)$, $\alpha\beta < \beta + (d - 1) + (\beta - 1)d(n - 1)$, which can be rearranged to give the inequality in the hypothesis. \square
 $b = \alpha\beta$, and $c = d(\beta - 1)$, and in particular, we can find an independent set $\mathcal{J}_1 \subset \mathcal{I}_1, \dots, \mathcal{J}_n \subset \mathcal{I}_n$ containing all but $o(1)$ of the colors provided that s

References

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