Proofs in Three Bits of Less

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What Are Interactive Proofs?

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- More versatile than classical proofs – we can ask questions like

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  to recover classical proofs.
- We can measure a theorem's difficulty by how many questions it takes for us to become convinced.

Watch Out For Tricks!

The verifier *cannot* trust the prover. The could either be making mistakes, or maliciously trying to trick you into thinking the proof is correct! The question

"Is your proof correct?"

Is useless. Questions must be checkable!
(Origins in Cryptography)

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- **Perfect Soundness**: If a statement is false, *no* series of answers to the questions I give will convince me the statement is true. You can't trick me!

The Catch!

Proofs of very short statements can be arbitrary long. Very inefficient. Consider the theorem

> " This Theorem Cannot Be Proved In Less Than a Googleplex Symbols "

A 53 character theorem with a HUGE proof.

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- **Imperfect Completeness**: If a statement is true, there exists a series of answers to the questions which convince you the truth with probability $\geq 1/2$.
- **Imperfect Soundness**: If a statement is false, *every* series of answers to the questions I give will fail to convince me the statement is true with probability $\geq 1 - \varepsilon$. It's unlikely you'll trick me!

The PCP Theorem

Theorem (1998, Arora, Lund, Motwani, Sudan, Szegedy)

There exists a universal constant K such that, for any "feasibly checkable" mathematical problem, we can check the problem with perfect soundness and imperfect completeness in K questions.

Theorem (1997, Håstad)

If you only want imperfect completeness, you can set $K = 3$!

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- Volunteers known as 'miners' verify correctness in huge ledgers in exchange for small amounts of currency.
- The PCP theorem has been proposed as a way to make fraud detection in Bitcoin 'scalable' – in exchange for accidently identifying true claims as false every so often, we only need a constant number of queries to become convinced of a transaction.

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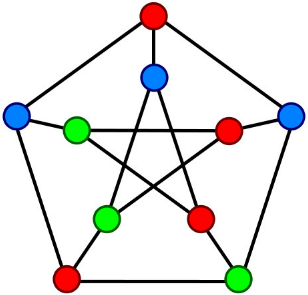
- $O(|V|)$ questions to determine if a graph has a 3-coloring. The PCP theorem says we can encode a coloring of a graph in such a way that, if the graph fails to be colorable, it will be displayed globally in the encoding – we can detect it with $O(1)$ samples.



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- We can't determine the factorization, just that the factorization exists.
- Fourier Analysis is useful for globalizing local properties – L^∞ norm of f is local, but can be controlled by L^1 norm of $\hat{f}$, which is a global property.

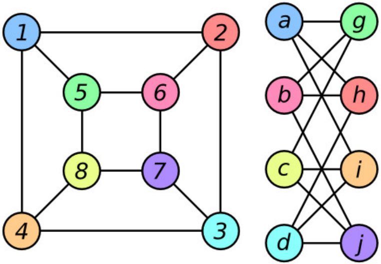
Sushi Break + An Analogy

https://dynermite.wordpress.com/2011/05/04/adorable/

- Any questions so far?

First Example: PCP Checker For Graph Nonisomorphism

- Two graphs are isomorphic if they can be obtained from one another by relabeling vertices. We take two graphs G_0 and G_1 on n vertices, and ask if they are *not* isomorphic to each other.



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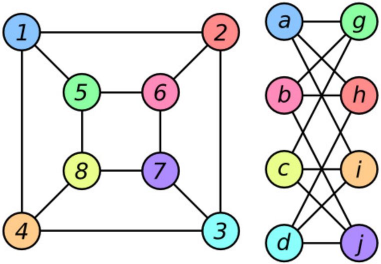
- A proof that two graphs are isomorphic is short – just give an isomorphism. A proof that two graphs are not isomorphic is *very difficult*: check by cases. We give a 1 query proof.



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The Checker

- Ask questions of the following variety, for some graph H

  " If H is isomorphic to G_0, output 0. If H is isomorphic to G_2, output 0. Otherwise, output an arbitrary result. "

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- To obtain a question at random, pick an index $i \in \{0, 1\}$ and a permutation $\pi \in S_n$ uniformly at random. Let $H = \pi(G_i)$. We are convinced of the proof if answer to our question corresponds to the correct answer $H \cong G_i$.

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- If G_0 *is* isomorphic to G_1, then H is independent of i, so $\mathbf{P}(i = 0|H) = \mathbf{P}(i = 1|H) = 1/2$, and so regardless of the answer the prover gives, they have a 50% chance of getting the answer wrong.

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- If original graph has m vertices, can be encoded in $O(m)$ bits.
- HUGE increase in size of encoding to obtain globality. The PCP theorem guarantees that we only increase our encoding by $O(n)$. The PCP theorem doesn't apply to the graph nonisomorphism problem because it isn't feasibly checkable – we only know it to be checkable in exponential time, by cases.

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- Perfect completeness: If f is linear, we are always convinced.
- Local soundness: If f convinces us it is linear with probability $1 - \varepsilon$, there is a linear function g such that $d(f, g) \leq \varepsilon$, where

$$d(f, g) = \frac{\{x \in \mathbf{F}_2^n : f(x) \neq g(x)\}}{2^n}$$

is the **Hamming distance** between two functions.

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- Take $Y \in \mathbf{F}_2^n$ uniformly at random. Then $f(x+Y) - f(Y) = g(x)$ with probability $\geq 1 - 2\varepsilon$.
- Even though linearity can fail locally, we can think of f as an encoding of g if it is close enough, so essentially, we can view all functions which are close to linear functions – we've moved from local error to global error.

The Big PCP Test: Quadratic Equations

- Consider the problem of determining whether a series of quadratic equations over $\mathbf{F}_2$ is solvable, e.g.

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- Since this problem is **NP Hard**, all feasibly checkable problems are reducible to this problem, so a PCP theorem for this problem gives a PCP checker for all problems. However, we won't only use a linear number of questions.

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- These values will enable us to check quadratic equations in very few queries.

The Test

- We start by picking subsets S and T, S', T', uniformly at random, and ask if

$$\sum_{i \in S} x_i + \sum_{i \in T} x_i = \sum_{i \in S \Delta T} x_i$$

$$\sum_{(i,j) \in S'} x_i x_j + \sum_{(i,j) \in T'} x_i x_j = \sum_{(i,j) \in S' \Delta J'} x_i x_j$$

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- Note this is the BLR test if the maps $S \mapsto \sum x_i$ and $S' \mapsto \sum x_i x_j$ are *linear*, where we view $2^{[n]}$ and $2^{[n] \times [n]}$ as vector spaces over $\mathbf{F}_2$ with $S + T := S \Delta T$.

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- If a particular answer passes with high probability, then it is close to an *actual* linear operator on $2^{[n]}$ and $2^{[n] \times [n]}$, and all such linear operators are given by some x. Local correction means we can calculate values of a real x.

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- If the quadratic equations are

$$\sum_{(i,j)\in T_k} x_i x_j = b_k$$

for some $k \in \{1, \ldots, m\}$, then pick a random subset $S = \{i_1, \ldots, i_l\} \subset [m]$, set $T = T_{i_1} \Delta \ldots \Delta T_{i_l}$ and test if

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- If a single equation fails to be satisfied, then the linear function on left doesn't equal the linear function on right, and therefore disagrees on half the inputs. Fails half the time.

Why Do We Care? Complexity Theory

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- It is an open problem whether we can factor an integer fast – i.e. is integer factorization in **P**. However, if you give me a set of factors, I can determine if they are the factors of a particular integer just by multiplying the integers together – simple!
- The current state of the art factorization algorithm runs in time proportional to

$$O\left(\exp\sqrt[3]{\frac{64}{9}n(\log n)^2}\right)$$

  where n is the bit number to represent the number we factor.

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Theorem (Canada, Cook, 1971; USSR, Levin, 1969)

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- The first **NP** hard problem discovered was 3-SAT. Given a logical formula, e.g.

$$(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_1 \vee x_7) \wedge (x_1 \vee x_2 \vee x_4)$$

a conjunction of three variable disjunctions, can we give variables truth values to make the statement true?

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- A random assignment satisfies $7/8$ of the clauses, and we can derandomize to get $7/8$ approximation algorithm.

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- Each possible random query in PCP theorem can be viewed as a clause of a 3SAT problem, that separates incorrect solutions ($\leq 1/2$ clauses satisfiable), from correct clauses ($1 - \varepsilon$ of clauses solvable).

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- Each possible random query in PCP theorem can be viewed as a clause of a 3SAT problem, that separates incorrect solutions ($\leq 1/2$ clauses satisfiable), from correct clauses ($1-\varepsilon$ of clauses solvable).
- Håstad used this to show that for any $\varepsilon > 0$, if there is a $7/8+\varepsilon$ approximation algorithm for $3SAT$, then $\mathbf{P} = \mathbf{NP}$. Thus the 7/8 approximation given before is best possible.

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- References:

  - Arora, Barak; Computational Complexity.
  - O' Donell; Analysis of Boolean Functions.
  - Bellare, Coppersmith, Håstad, Kiwi, Sudan; Linearity Testing in Characteristic Two.