Geometry

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Part I

Euclid

I'm writing these notes so that I can understand Euclidean geometry better. We'll build up the axioms from the ground up, so I can understand Euclid's work from the ground up. Thus these notes probably won't be useful for someone trying to understand Euclid themselves, because it's just my ramblings about the subject.

Chapter 1

Book I

Basic Euclidean geometry consists of three objects: Points, Lines (both finite lines with endpoints, an infinite lines with no extremities), and Circles (defined by a point and a radius). Classically, these objects were seen as distinct, but with the power of set theory, it is easier to model lines and circles as sets of points. This has the advantage of making things notationally simple. These is no real logical difference between switching to this notation – any theorem provable in one system is provable in the other. However, we'll avoid from using set theory too much, to avoid making the exposition too austere.

Euclid was the first to pioneer the axiomatic method in mathematics. However, the philosophy behind his proofs was different to ours. At the end of the day, his arguments attack a particular model of the planar geometry find in our world, and he proves things like a physicist, adopted some methods of proof not explicitly stated in his assumptions. This causes problems for us when we try and look at his proofs from a modern day perspective. We will eventually look at other logical systems for geometry, but for now a naive approach will be most useful.

Most of Euclid's proofs concern constructions of certain figures in the planes. Rather than a proof of existence, Euclid literally builds these figures from the ground up. In the early parts of the text these figures will all be defined by a simple curve consisting of straight lines, so that we may describe such a figure by the sequence of points which define the figure. If $X_1, \dots, X_n$ are points, then $X_1 \dots X_n$ will denote the figure obtained by drawing the line $X_1 X_2$, then $X_2 X_3$, and so on, finishing off by drawing $X_n X_1$. Two such figures will be considered equal if we may obtain the points of one from the points of the other by performing a cycle permutation of the points. For instance, a **triangle** is just a sequence of distinct points ABC, and $ABC = BCA = CAB$, and we can abuse the notation, denoting a line between two points A and B as $AB = BA$. The question of whether this is a unique description of such a line is settled by the first axiom of geometry.

Axiom 1. *There is a unique straight line between any pair of points, having those points as endpoints.*

Euclid does not assume that the straight line which exists between the points is unique, but later he uses the fact that a finite line is defined by its endpoints, so we can only assume that he really wants this fact to hold. In order to discuss the lengths of lines, we shall be required to discuss circles at points, and so we introduce the second axiom.

Axiom 2. *A circle may be described with any centre and radius.*

A circle is *defined* by its centre and radius, so the circle which exists by this axiom is unique. Note that circles with a different radii and the same centre may still be equal. Indeed, this happens exactly when the two radii have the same length, a concept we will very shortly discuss.

Euclid defines an **equilateral triangle** as 'a triangle whose three sides are equal', which he really means as saying the *magnitude*, or length, of the sides are equal. In Euclid's synthetic geometry, there do not exist real numbers to assign length to, and as is well known most Greek's did not even believe in irrational numbers. But we shall find that we can get away with much of the theory of magnitude without ever mentioning the concept of a number, which gives a certain sense of satisfaction.

Right now, we only need equality in the length of lines, and we shall discuss a very aggreeable manner in checking equality. If we have two lines AB and AC with a common point, we can check if they have equal length by checking if the circles constructed with centre A and radii AB and AC are equal. This gives us an equivalence relation on the set of lines extending out from A. We shall require that this equivalence relation describes exactly the set of circles with centre A, so that a point C lies on the circle with centre A and radius AB if and only if the length of AC is equal to the length of AB.

Axiom 3. *If C lies on the circle with radius A and radii AB, then AC has the same length as AB.*

In order to generalize equality of length of arbitrary lines, we just make the relation transitive. The relation is already reflexive and symmetric, so this generates an equivalence relation on the set of all lines in the plane. Thus we see that the only basic way to check if two lines AB and CD are equal is to form a sequence of lines beginning at B, and ending at C, which are all equal to one another as lines extending from the same basepoint.

Theorem 1.1. *Any finite line lies on an equilateral triangle.*

Proof. To prove the existence of an equilateral triangle at a line AB, Euclid constructs the circle with radius AB and centre A, and the circle with centre B and radius AB, and considers their point of intersection C. Since C lies on the first circle, AB has the same length as AC, and since C lies on the second circle, CB has the same length as AB. But then the lines AB, BC, and CA describe an equilateral triangle, and so ABC is the triangle required. □

There is only one problem remaining in this proof. There is nothing saying that the two circles given will have a common point of intersection. We could describe an axiom which supplies us with such a point, but this axiom would probably be more general than the theorem itself. Indeed, the existence of a point on the intersection of two circles with the same radius but different centres is equivalent to the theorem we set out to prove. Thus we shall have to settle on the fact that theorem one must be treated as an assumption from our current viewpoint.

Theorem 1.2. *Given a point A and line BC, to construct a line extending from A with the same length as BC.*

Proof. Construct an equilateral triangle ABD on the line AB. Then construct the circle with centre B and radius BC. Find an intersection point E on the circle which either lies on the line BD, or extends the line, and then construct the circle with centre B and radius BE. Extend the line DA from the extremity A to an intersection point F on the circle. We claim AF has the same length as BC. Indeed, the length of DF is the sum of the length of DA and AF, and the length of DE is the sum of DB and BE. Since the length of DF is equal to DE, since they both lie on the same circle extending from D, and the length of DA is equal to the length of DB, we may subtract to conclude that the length of AF is the same as the length of BE. But BE has the same length of BC, which is all that is required to show AF has the same length as BC. □

Chapter 2

Analytic Geometry

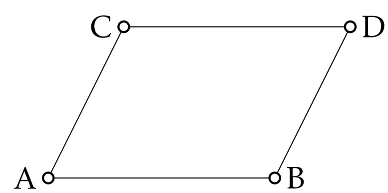
It is sometimes interesting to weaken the axioms of geometry to see that a more general class of models results. In this chapter, we study geometries satisfying three axioms

- Through any two distinct points there is a unique line between them. Given two points X and Y, we shall let XY denote the unique line through X and Y.
- Given a line and some point, there is a unique line through the point parallel to the line (A line is parallel to another line if they don't intersect, or if they are equal to one another).
- There exist three non colinear points in the geometry.

Euclidean geometry is an extension of this axiom set, so any model of Euclidean geometry is automatically a model of this weaker axiom set, and we call the study of models of this axiom set **affine geometry**.

Example. *Let K be a field, and consider the geometry whose points are elements of K^2, and whose lines are the affine span $v + Kw = \{v + xw : x \in K\}$, for a nonzero $w \in K^2$. It is easy to see that every line through a vector v can be written as $v + Kw$, and for two lines $v + Kw_0$ and $v + Kw_1$, either the two lines intersect only at v, or the two lines are equal and w_0 is a scalar multiple of w_1. This tells us that lines intersecting at two or more common points are equal, and therefore there is a unique line $v + K(w - v)$ between any two points v and w. A line $v_0 + Kw_0$ is parallel to a line $v_1 + Kw_1$ if and only if w_0 is a scalar multiple of w_1, and for any point v_1 not on a line $v_0 + Kw$, the line $v_1 + Kw$ is* *a line containing v_1 and parallel to $v_0 + Kw$, and this is the unique such line. Finally, the points $(0,1)$, $(1,0)$, and $(0,0)$ are non colinear, so K^2 is a model of affine geometry.*

Our main result will be that these geometries describe all the possible models of affine geometry. That is, every model of affine geometry can be associated with a copy of K^2, where lines in one space map to lines in the other. This shows that over the class of affine geometries, 'Cartesian analytic geometry' suffices to verify any result. To begin with this construct, consider any affine geometry. We will give a field structure to any line OI in the geometry, with which we may identify the geometry as the vector space $(OI)^2$. We shall define an algebraic structure in which O is the additive identity, and I is the multiplicative identity.



Lemma 2.1. *If AB is a line, then for any $C \notin AB$, there is a unique D such that CD is parallel to AB, and BD is parallel to AC.*

Proof. Since $C \notin AB$, there is a unique line through C parallel to AB, and a unique line through B parallel to AC. Since AC is not parallel to AB, the lines we have formed intersect at a unique point, which is the point D required. □

Corollary 2.2. *In the context of the previous theorem, there is a unique bijection $f : AB \to CD$ such that $Xf(X)$ is parallel to AC.*

Proof. Using the previous theorem, for each X there is a point Y such that CY is parallel to AB, and XY is parallel to AC. This implies that $CY = CD$, so $Y \in CD$ and we can *define* a map $f : AB \to CD$ satisfying the property of the lemma, and we have certainly justified it is unique. Using the symmetry of the problem, there is a unique map $g : CD \to AB$ such that $Xg(X)$ is parallel to AC for each X. Then $(g \circ f)(X) = X$, because $f(X)X$ is parallel to AC, and by symmetry, we also find $(f \circ g)(X) = X$, so f really is a bijection. □

Now given O, fix a line L through O. For each $A, B \in L$, fix $M \notin L$, and let L_M be the line through M parallel to L. Then there is a unique point C such that AC is parallel to OM. Similarily, there is a unique point D on L such that BM is parallel to CD. We define $D = A + B$. We require a certain lemma to verify this is well defined.

Lemma 2.3. *If A_0B_0 is parallel to A_1B_1, B_0C_0 is parallel to B_1C_1, and A_0A_1, B_0B_1, and C_0C_1 are all parallel, then A_0C_0 is parallel to A_1C_1.*

Proof. Suppose that A_0C_0 and A_1C_1 meet at a point X. If $C_0 \neq C_1$, then there is a point $Y \neq X$ such that XY is parallel to C_0C_1 □

To verify this is well define, if $N \notin L$ is any other point, let L_N be the line through N parallel to L. Then there is a unique point C' such that AC' is parallel to ON. Then CC' is parallel to MN

Let $f_0 : L \to L_M$ be the map such that $Xf_0(X)$ is parallel to OM. Similarily, define $f_1 : L_M \to L$ be such that $Xf_1(X)$ is parallel to $Bf_0(X)$, and we define $A + B = f_1(f_0(A))$.

Now consider the line OI we fixed. Then for any $A, B \in OI$,

for a fixed $M \notin OI$ there is a unique point C lying on the line through M parallel to OI such that AC is parallel to OM. Now we can use the lemma above in reverse to find a point $D \in OI$ such that CD is parallel to BM, and we define $A + B = D$.

To verify that addition is well defined, fix some $N \notin OI$. Let L_M be the line through M parallel to OI, and L_N the line through N parallel to OI. Then let $f : OI \to L_M$, $g : OI \to L_N$ be the unique bijections constructed by the corollary above. Then $D = f(A)$

Then $g \circ f^{-1} : L_M \to L_N$ is such that $X(g \circ f^{-1})(X)$ is parallel to MN

To verify that D is uniquely defined, fix some $N \notin OI$. Then there is a unique point C' lying on the line through M parallel to OI such that AC' is parallel to ON. Since the line through M parallel to OI is parallel to the line through N parallel to OI, C' is the unique point on the line through N parallel to OI such CC' is parallel to MN.

First, note that for any point $M \notin OI$, there is a unique line through M parallel to OI. Given $X \in OI$, there is a unique line through X parallel to OM. Since OM is not parallel to OI, there is a unique point N lying on the intersection of these lines, thus MN is parallel to OI, and XN is parallel to OM. For any point on OM, and for a fixed X, we can always perform this construction, yielding a map $f : OM \to XN$, with $f(O) = X$, and

$f(M) = N$, and more generally, $f(Y)$ is the unique point such that $Xf(Y)$ is parallel to OM, and $Yf(Y)$ is parallel to OI. It is injective, because if XZ is parallel to OM, and Y_0Z and Y_1Z are both parallel to OI, then by uniqueness of parallel lines we find $Y_0Z = Y_1Z$, and since distinct lines intersect in a unique position, we must have $Y_0 = Y_1$. The surjectivity follows because we may always swap O and X in the theorem to conclude that for any $Z \in XN$, there is a unique point $Y \in OM$ such that YZ is parallel to OI, and then $f(Y) = Z$.

Now given OI, consider $X, Y \in OI$, and fix $M \notin OI$. Consider the unique line through M parallel to OI, and consider the bijection f above with $f(x)$