Fractals Avoiding Fractal Sets

Jacob Denson Malabika Pramanik Josh Zahl

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Abstract

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Consider a geometric point configuration, such as three points forming an isoceles triangle in the plane, or four points lying in a two dimensional plane in $\mathbf{R}^3$. A natural problem is to determine the minimum size a collection of points must be before we can guarantee a subselection lies in a specified configuration. More precisely, this paper gives techniques to constructs large sets avoiding configurations, thus providing lower bounds for a general class of configuration avoidance problems. We call the construction of such sets a *configuration avoidance problem*.

It is often the case that we can find arbitrarily large finite sets not containing these configurations. On the other hand, every set of positive measure contains the configuration. This is the case for the examples we described, and many other examples, especially when the configurations are affine invariant. Thus we must use finer analytical measures of size for infinite sets, and the standard in this field is given by the Hausdorff dimension. This paper gives techniques to construct sets with large Hausdorff dimension avoiding a very general class of configurations. As a continuous analogue to Ramsey theory, what makes these problems interesting is seeing how the continuous setting differs from its discrete counterpart.

There are already general pattern avoidance methods in the literature. We compare our method to them in section 6. But these rely on the non-singular nature of the configurations. The novel feature of our method is we can avoid configurations which have an *arbitrary* fractal quality to them. Meanwhile, the Hausdorff dimension obtained is comparable to that from the more restricted techniques.

A key idea to our method is the introduction of a novel geometric framework for pattern avoidance problems, described in section 1. We believe this new framework should help develop further methods in the field, some of which we are currently developing for publication in a later paper. The essence of our approach relies on a simple combinatorial argument, described in section 3, which can be applied once we have discretized the problem.

1 A Fractal Avoidance Framework

One way to think about generic pattern avoidance methods is to specify the pattern as the zero set of a function. For example,

- A set $X \subset \mathbf{R}^d$ contains the vertices of no isosceles triangles if and only if for any three distinct $x, y, z \in X$,

  $$f(x, y, z) = d(x, y) - d(y, z) \neq 0$$

  If $f(x, y, z) = 0$, then the line segments xy and zy form the legs of the isoceles triangle.

- A set $X \subset \mathbf{R}^d$ does not contain $d + 1$ points in a lower dimensional hyperplane if and only if for any distinct $x_0, \dots, x_d \in X$,

  $$f(x_0, \dots, x_d) = \det(x_1 - x_0, \dots, x_d - x_0) \neq 0$$

  If $\det(x_1 - x_0, \dots, x_d - x_0) = 0$, the vectors $x_1 - x_0, \dots, x_d - x_0$ are not linearly independant, and thus span a plane of dimension smaller than d.

This method of description is summarized by a general framework.

The Configuration Avoidance Problem: Given a function $f : (\mathbf{R}^d)^n \to \mathbf{R}$, find $X \subset \mathbf{R}^d$ such that for any *distinct* $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$, with as high a Hausdorff dimension as possible.

The functional framework is common in the literature. For instance, it is the viewpoint behind the methods of [2] and [3], who give results assuming various regularity conditions on the function f. Here, we take the perspective that f gives extraneous information irrelevant to the problem at hand. It's real importance is suggesting geometric properties of the zero set Z. Once Z is taken as the primary object to study, the configuration avoidance problem becomes equivalent to another framework. It is the viewpoint of this paper that this framework is more flexible to work with, and thinking in terms of this framework leads to new general avoidance methods.

The Fractal Avoidance Problem: Given $Z \subset (\mathbf{R}^d)^n$, find $X \subset \mathbf{R}^d$ such that if $x_1, \dots, x_n \in X$ are *distinct*, $(x_1, \dots, x_n) \notin Z$, with as high a Hausdorff dimension as possible.

As is suggested by the fact that only the Cartesian product of distinct values of $x_1, \dots, x_n \in X$ are required to avoid Z, the diagonal set Δ of points $(x_1, \dots, x_n) \in (\mathbf{R}^d)^n$ such that $x_i = x_j$ for some $i \neq j$ plays a crucial role in the problem. If a point in Δ has a neighbourhood not intersecting Z, then by placing X^n in this neighbourhood we find a full dimensional solution to the fractal avoidance problem. So in order for the problem to be nontrivial, Z must be dense in a neighbourhood of Δ. The structure of Z around Δ therefore plays a major part in finding solutions to the fractal avoidance problem.

We are free to pick any subset of X we want. Because Hausdorff dimension is a local property of the set, we can always assume X lies in a bounded region of space. Once this region is fixed, only a bounded portion of Z could ever intersect X^d, allowing us to use compactness arguments. This is necessary to our method, and because of this, we work with a local version of the fractal avoidance framework, to which the general fractal avoidance problem can always be reduced by fixing a region, and changing coordinates.

The Local Fractal Avoidance Problem: Given $Z \subset ([0,1]^d)^n$, find $X \subset [0,1]^d$ such that if $x_1, \dots, x_n \in X$ are *distinct*, $(x_1, \dots, x_n) \notin Z$.

Since we are the first to introduce the fractal avoidance problem, a natural goal is to solve the general problem with minimal assumptions on Z. Here the only restriction we place on Z is it's fractal dimension.

Theorem 1. *If Z is the countable union of sets with lower Minkowski dimension upper bounded by α, and $\alpha \geqslant d$, then there is X solving the local fractal avoidance problem for Z with*

$$\dim_{\mathbf{H}}(X) = \frac{dn - \alpha}{n-1} = \frac{\operatorname{codim}_{\mathbf{H}}(Z)}{n-1}$$

See section 2 for a precise definition of lower Minkowski dimension we use.

Remark. *If Z is the union of sets with dimension $\alpha < d$, the set obtained from $[0,1]^d$ by removing the projections of Z onto each coordinate has full Hausdorff dimension and trivially solves the local fractal avoidance problem. Thus we need not consider these parameters in our theorem.*

The aim of the theorem is to explore a general method for solving configuration avoidance problems. For particular configurations, it is highly likely that additional geometric features can be exploited to obtain avoiding sets with a much larger Hausdorff dimension. But our goal is not to find tight results for particular configurations. Instead, we want to show how much a single geometric feature of the configuration, in this case a bound on the dimension of the set Z, can be exploited. Other general results focus on other geometric features of the problem. For instance, the method of [3] explores what happens when Z is a low dimensional algebraic variety. As more and more methods are revealed which exploit various features of the problem, we will be able to obtain tight results for particular configurations with ease, by synthesizing various approaches which exploit the most important geometric features of the configuration.

2 Notation

Here we provide a concise summary of all the non-standard notation and terminology we use to describe our configuration avoidance method.

- For a length L, we let $\mathcal{B}(L, d)$ denote the partition of $\mathbf{R}^d$ into the family of all half open cubes with corners on the lattice $(\mathbf{Z}/L)^d$, i.e.

$$\mathbf{B}(L, d) = \{[a_1, b_1) \times \cdots \times [a_d, b_d) : a_i, b_i \in \mathbf{Z}/L\}$$

  If the dimension d is clear, or it's emphasis is unnecessary, we abbreviate $\mathcal{B}(L, d)$ as $\mathcal{B}(L)$. By a $\mathcal{B}(L)$ *cube*, we mean an element of $\mathcal{B}(L)$, and by a $\mathcal{B}(L)$ *set*, we mean a union of $\mathcal{B}(L)$ cubes.

- If E is a point set, then $\mathcal{B}(E, L)$ is the family of all $\mathcal{B}(L)$ cubes intersecting E, i.e. $\mathcal{B}(E, L) = \{I \in \mathcal{B}(L) : I \cap E \neq \varnothing\}$.

- The *lower Minkowski dimension* of a compact set E is defined as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{L \to 0} \frac{|\mathcal{B}(E, L)|}{\log(1/L)}$$

  Thus there is $L_k \downarrow 0$ with $|\mathcal{B}(E, L_k)|$ only slightly larger than $(1/L_k)^{\underline{\dim}_{\mathbf{M}}(E)}$.

- Adopting the terminology of [5], we say a collection of sets $U_1, U_2, \dots$ is a *strong cover* of some set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k.

- Given a cube $I \in \mathcal{B}(L, dn)$, we can consider it's d dimensional sides $I_k \in \mathcal{B}(L, d)$, which decompose I as a Cartesian product $I_1 \times \cdots \times I_n$. We say the cube is *non diagonal* if the intervals $I_1, \dots, I_n$ are distinct from one another.

3 Avoidance at Discrete Scales

Because the only feature we know of Z is it's dimension, we are essentially forced to solve the problem by looking at a discretized version of the problem. We replace Z by a union of cubes W with very fine sidelength S. Then our goal is to take a set E, which is a union of cubes with coarse sidelength L cubes, and carve out a union of sidelength S cubes F such that F^n is disjoint from W, except along a discretization of the diagonal.

In order to ensure the set X we construct has the right Hausdorff dimension after applying the discretization argument at many scales, it is integral that in the discrete setting the mass of F is spread uniformly over E. We can achieve this by trying to include a equal portion of mass in each sidelength R subcube of E, for some intermediary scale $L > R > S$. The next lemma shows this is almost possible, and that the most effective way to do this is by a random inclusion process.

Lemma 1. *Fix three dyadic lengths $L > R > S$. Let E be a $\mathcal{B}(L)$ set in $\mathbf{R}^d$, and W a $\mathcal{B}(S)$ set in $\mathbf{R}^{nd}$. Then there exists a $\mathcal{B}(S)$ set $F \subset E$, such that for any distinct $I_1, \dots, I_n \in \mathcal{B}(F, S)$, $I_1 \times \dots \times I_n \notin \mathcal{B}(W, S)$. Furthermore, for all but at most $|\mathcal{B}(W,S)|(S/R)^{dn}$ of the cubes $I \in \mathcal{B}(E,R)$, $|\mathcal{B}(E,S) \cap \mathcal{B}(I,S)| = 1$.*

Proof. Form a random $\mathcal{B}(S)$ set $U \subset E$ by selecting, from each $\mathcal{B}(R)$ subcube of E, a single $\mathcal{B}(S)$ subcube and adding it to U. Thus the probability that any element in $\mathcal{B}(E,S)$ is added to U is $(S/R)^d$. Since any two $\mathcal{B}(S)$ subcubes of U lie in distinct elements of $\mathcal{B}(R)$, the only chance that a $\mathcal{B}(S)$ subcube I of W is a subset of U^n is if $I_1, \dots, I_n$ all lie in separate cubes in $\mathcal{B}(R)$. Then they each have an independant chance of being added to U, and so

$$\mathbf{P}(I \subset U^n) = \mathbf{P}(I_1 \subset U) \dots \mathbf{P}(I_n \subset U) = (S/R)^{dn}$$

If M denotes the number of $\mathcal{B}(S)$ subcubes I of W contained in U^n, then

$$\mathbf{E}(M) = \sum_{I \in \mathcal{B}(W,S)} \mathbf{P}(I \subset U^n) = |\mathcal{B}(W,S)|(S/R)^{dn}$$

If, for each $\mathcal{B}(S)$ subcube I of W contained in U^n, we remove I_1 from U, we obtain a set F with $I_1 \times \dots \times I_n$ disjoint from W for any distinct $I_1, \dots, I_n \in \mathcal{B}(F,S)$. The set F contains a cube from all but M sidelength R cubes. In particular, we can select some nonrandom choice of U such that $M \leqslant |\mathcal{B}(W,S)|(S/R)^{dn}$, which gives the required set F originally desired. ☐

Remark. *This discrete lemma is the core of our avoidance technique. The remaining argument is fairly modular, and can be applied with any other discrete avoidance technique to yield a solution to the fractal avoidance problem. In particular, if the geometry of Z yields properties about it's discretization strong enough to discard fewer intervals in the argument above, one can apply the remaining parts of our paper to yield a set X with a larger Hausdorff dimension.*

Because the choice of F is uniform over E, in our construction we can allow the gap between L and R to be arbitrarily large. However, the gap between R and S can only be 'polynomially large', i.e. we can only have $R = S^\lambda$ for some fixed $\lambda \in (0,1)$. The size of λ is directly related to the Hausdorff dimension of the set X we construct (the larger the better!). If the set we are trying to avoid has fractal dimension α, we will be able to obtain a bound $|\mathcal{B}(Z,S)| \leqslant S^{-\gamma}$ for some γ converging to α in the limit. In the next corollary, we calculate precisely how large we can let λ be given this bound on the number of cubes we have to avoid, so that we include a $\mathcal{B}(S)$ cube in F for more than half of the $\mathcal{B}(R)$ cubes.

Corollary. *Consider the scenario from the last lemma. Suppose R is the closest dyadic number to S^λ, $|\mathcal{B}(Z,S)| \leqslant S^{-\gamma}$ for $\gamma \geqslant d$, and $|E| \leqslant 1/2$. If*

$$0 < \lambda \leqslant \frac{dn - \gamma}{d(n-1)} - O\left(A \log_S |E|\right)$$

then E contains a $\mathcal{B}(S)$ cube from all but a fraction $1/2^A$ of the cubes in $\mathcal{B}(E,R)$.

Proof. The inequality for λ implies

$$dn - \gamma - \lambda d(n-1) \geqslant O\left(\log_S |E|\right)$$

Since R is within a factor of two from S^λ, we compute

$$\begin{aligned}
&\frac{|\{I \in \mathcal{B}(E,R) : \mathcal{B}(I,S) \cap \mathcal{B}(F,S) = \varnothing\}|}{|\mathcal{B}(E,R)|} \\
&\quad \leqslant \frac{|\mathcal{B}(Z,S)|(S/R)^{dn}}{|E|R^{-d}} \leqslant |E|^{-1} S^{dn-\gamma} R^{-d(n-1)} \\
&\quad \leqslant |E|^{-1} S^{dn-\gamma} (S/2)^{-\lambda d(n-1)} \leqslant 2^{\lambda d(n-1)} |E|^{-1} S^{O(A \log_S |E|)} \\
&\quad = 2^{\lambda d} |E|^{O(1)-1} \leqslant 2^{\lambda d + 1 - O(A)} \leqslant 1/2^A
\end{aligned}$$

The last inequality was obtained by picking the $O(A) \geqslant 2dA$. □

4 Fractal Discretization

Now we apply the discrete technique we just described to obtain an actual fractal avoidance set. We consider a decreasing sequence of dyadic scales L_k. The fact that Z is the countable union of sets with Minkowski dimension α implies that we can find an efficient *strong cover* of Z by cubes restricted to lie at the dyadic scales L_k.

Lemma 2. *Let Z be a countable union of sets, each with lower Minkowski dimension bounded by α, and consider any positive sequence ε_k converging to zero. Then there is a decreasing sequence of lengths $L_1, L_2, \dots$, and $\mathcal{B}(L_k)$ sets Z_k such that Z is strongly covered by the sets Z_k and $|\mathcal{B}(Z_k, L_k)| \leqslant 1/L_k^{\alpha+\varepsilon_k}$.*

Proof. Let Z be the union of sets Y_i with $\underline{\dim}_{\mathbf{M}}(Y_i) \leqslant \alpha$ for each k. Consider any sequence $m_1, m_2, \dots$ of integers which repeats each integer infinitely often. If, at the kth step of the argument, we find a $\mathcal{B}(L_k)$ set Z_k covering Y_{m_k}, then we will have found a strong cover for Z. Doing this is quite easy. Since $\underline{\dim}_{\mathbf{M}}(Y_{m_k}) \leqslant \alpha$, there are arbitrarily small lengths L such that $|\mathcal{B}(Y_{m_k}, L)| \leqslant 1/L^{\alpha+\varepsilon_k}$. In particular, we may fix such a length L smaller than the choices of $L_1, \dots, L_{k-1}$. This length will then be our point L_k, and the union of the cubes in $\mathcal{B}(Y_{m_k}, L)$ will form Z_k. □

Remark. *In the proof, we are free to make L_k arbitrarily small in relation to the previous parameters $L_1, \dots, L_{k-1}$ we have chosen. For instance, later on when calculating the Hausdorff dimension, we will assume that $L_{k+1} \leqslant L_k^{k^2}$, and the argument above can be easily modified to incorporate this inequality.*

We can now construct X by avoiding the various discretizations of Z at each scale. The aim is to find a nested family of discretized sets $X_0 \supset X_1 \supset X_2 \supset \dots$ with $X = \lim X_k$. One condition that guarantees that X solves the fractal avoidance problem is that X_k^n is disjoint from *non diagonal* cubes in Z_k.

Lemma 3. *If for each k, X_k^n avoids non-diagonal cubes in Z_k, then X solves the fractal avoidance problem for Z.*

Proof. Let $z \in Z$ be given with $z_1, \dots, z_n$ are distinct. Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i, j \text{ such that } w_i = w_j\}$$

Then $d(\Delta, z) > 0$. The point z is covered by cubes in infinitely many of collections Z_{k_m}. For suitably large N, the cube I in $\mathcal{B}(L_{k_N})$ containing z is disjoint from Δ. But this means that I is non diagonal, and so $z \notin X_N^d$. In particular, z is not an element of X^n. □

It is now simple to see how we must work at the discrete scales. First, we see $X_0 = [0, 1/2]^d$, so that $|X_0| \leqslant 1/2$. To obtain X_{k+1} from X_k, we apply the discrete argument. We set $E = X_k$ and $W = Z_{k+1}$, with scales $L = L_k$ and $S = L_{k+1}$. We know that we can choose $\gamma = \alpha + \varepsilon_k$, and also pick $R = R_{k+1}$ the closest dyadic number to L_{k+1}^λ, where

$$\lambda = \beta_{k+1} = \frac{dn - \alpha}{d(n-1)} - \frac{\varepsilon_{k+1}}{d(n-1)} - O(k \log_{L_{k+1}} |X_k|)$$

The discrete lemma then constructs a set F with F^n avoiding non diagonal cubes in Z_{k+1}, and containing a $\mathcal{B}(L_{k+1})$ subcube from all but a fraction $1/2^{2k+d+2}$ of the $\mathcal{B}(R_{k+1})$ cubes in I. We set $X_{k+1} = F$. Repeatedly doing this builds an infinite sequence of the X_k. Since X_k^n avoids Z_k, X is a solution to the fractal avoidance problem. It now remains to calculate the Hausdorff dimension of X.

5 Dimension Bounds

We now show that the set X has the expected Hausdorff dimension we need. At the discrete scale L_k, X looks like a $d\beta_k$ dimensional set. If the lengths L_k rapidly converge to zero, then we can ensure $\beta_k \to \beta$, where

$$\beta = \frac{dn - \alpha}{d(n-1)}$$

Then, in the limit X looks $d\beta$ dimensional on the discrete scales, which is the Hausdorff dimension we want. It then suffices to interpolate this result to get a $d\beta$ dimensional behaviour at all intermediary scales. We won't be penalized here by making the gaps between discrete scales too large, because the uniform way that we have selected cubes in consecutive scales implies that between the scales L_k and L_{k+1}^β, X behaves like a full dimensional set. The remainder of this section fills in the details to this argument.

Lemma 4. *$\beta_k \to \beta$.*

Proof. It suffices to show that the error terms in β_k become neglible over time, i.e. we must show

$$\frac{\varepsilon_{k+1}}{d(n-1)} + O(k \log_{L_{k+1}} |X_k|) = o(1)$$

Since $\varepsilon_{k+1} \to 0$, the term corresponding to it converges to zero for free. On the other hand, we need the lengths to tend to zero rapidly to make the other error term decay to zero. Since $L_{k+1} \leqslant L_k^{k^2}$, we find

$$k \log_{L_{k+1}} |X_k| \leqslant \frac{k \log L_k}{\log L_{k+1}} \leqslant \frac{k \log L_k}{k^2 \log L_k} = \frac{1}{k}$$

Thus both error terms tend to zero. □

The most convenient way to look at the dimension of X at various scales is to use Frostman's lemma. To understand the behaviour of X, we construct a non-zero measure μ supported on X such that for all $\varepsilon > 0$, for all lengths L, and for all $I \in \mathcal{B}(L)$, $\mu(I) \leqslant L^{d\beta - \varepsilon}$. We can then understand the behaviour of X at the scale L by looking at μ's behaviour when restricted to cubes at the particular scale, i.e. cubes in $\mathcal{B}(L)$.

To construct a measure μ naturally reflecting the dimension of X, we rely on a variant of the mass distribution principle. This means we take a sequence of measures μ_k, supported on X_k, and then take a weak limit to form a measure μ. We initialize this construction by setting μ_0 to be the uniform measure on $X_0 = [0, 1/2]^d$. We then define μ_{k+1}, supported on X_{k+1}, by modifying the distribution of μ_k. First, we throw away the mass of the $\mathcal{B}(L_k)$ cubes I for which over half of the $\mathcal{B}(I, R_{k+1})$ cubes fail to contain a part of X_{k+1}. For the cubes I for which more than half of the cubes $\mathcal{B}(I, R_{k+1})$ contain a part of X_{k+1}, we distribute the mass of $\mu_k(I)$ uniformly over the subcubes of I in X_{k+1}. This gives a mass function μ_{k+1}. It is easy to see from the cumulative distribution functions of the μ_k that these measures converge to a function μ such that for any $I \in \mathcal{B}(L_k)$, $\mu(I) \leqslant \mu_k(I)$, which is useful for passing from bounds on the discrete measures to bounds on the final measure.

Lemma 5. *If $I \in \mathcal{B}(L_k)$, then*

$$\mu(I) \leqslant \mu_k(I) \leqslant 2^{d+k} \left[\frac{R_k R_{k-1} \dots R_1}{L_{k-1} \dots L_1} \right]^d$$

Proof. Consider $I \in \mathcal{B}(L_{k+1})$, $J \in \mathcal{B}(L_k)$. If $\mu_k(I) > 0$, this means that J contains a $\mathcal{B}(L_k)$ cube in at least half of the $\mathcal{B}(R_N)$ cubes it contains. Thus the mass of J distributes itself evenly over at least $2^{-1}(L_{k-1}/R_k)^d$ cubes, which gives that $\mu_k(I) \leqslant 2(R_k/L_k)^d \mu_{k-1}(J)$. But then expanding this recursive inequality, we obtain exactly the result we need. □

Corollary. *The measure μ is positive.*

Proof. To prove this result, it suffices to show that the total mass of μ_k is bounded below, independantly of k. At each stage k, X_k consists of at most

$$\left[\frac{L_{k-1} \dots L_1}{R_k \dots R_1} \right]^d$$

$\mathcal{B}(L_k)$ cubes. Since only a fraction $1/2^{2k+d+2}$ of the $\mathcal{B}(R_k)$ cubes do not contain an interval in X_{k+1}, it is only for at most a fraction $1/2^{2k+d+1}$ of the $\mathcal{B}(L_k)$ cubes that X_{k+1} fails to contain a $\mathcal{B}(L_{k+1})$ cube from more than half of the $\mathcal{B}(R_{k+1})$ cubes. But this means that we discard a total mass of at most

$$\left(\frac{1}{2^{2k+d+1}}\left[\frac{L_{k-1}\dots L_1}{R_k\dots R_1}\right]^d\right)\left(2^{d+k}\left[\frac{R_k\dots R_1}{L_{k-1}\dots L_1}\right]^d\right) \leqslant 1/2^{k+1}$$

Thus

$$\mu_k(\mathbf{R}^d) \geqslant 1-\sum_{i=0}^{k}\frac{1}{2^{i+1}} \geqslant 1/2$$

This implies $\mu(\mathbf{R}^d) \geqslant 1/2$, and in particular, $\mu \neq 0$. ☐

Ignoring all parameters in the inequality for I which depend on indices $< k$, we 'conclude' that $\mu_k(I) \lesssim R_k^d \lesssim L_k^{\beta_k d}$. The fact that $L_{k+1} \leqslant L_k^{k^2}$ essentially enables us to ignore quantities not depending on previous indices, and obtain a true inequality.

Corollary. *For all $I \in \mathcal{B}(L_k)$, $\mu(I) \leqslant \mu_k(I) \lesssim L_k^{d\beta_k - k^{-1/2}}$.*

Proof. Given ε, we find

$$\begin{aligned}\mu_k(I) &\leqslant 2^{d+k}\left[\frac{R_k\dots R_1}{L_{k-1}\dots L_1}\right]^d \leqslant \left(\frac{2^{2d+k}}{L_{k-1}^{d(1-\beta_{k-1})}\dots L_1^{d(1-\beta_1)}}\right)L_k^{d\beta_k}\\ &\leqslant \left(2^{2d+k}L_k^{\varepsilon}/L_{k-1}^{d(k-1)}\right)L_k^{d\beta_k-\varepsilon} \leqslant \left(2^{2d+k}L_{k-1}^{\varepsilon k^2 - d(k-1)}\right)L_k^{d\beta_k-\varepsilon}\end{aligned}$$

The open bracket term converges to zero so fast that it still tends to zero if ε is not fixed, but is instead equal to $k^{-1/2}$, giving the required inequality. ☐

This is close to the cleanest expression of the $d\beta$ dimensional behaviour at discrete scales. To get a general inequality of this form, we use the fact that our construction distributes uniformly across all intervals.

Theorem 2. *If $L \leqslant L_k$ is dyadic and $I \in \mathcal{B}(L)$, then $\mu(I) \lesssim L^{d\beta_k - k^{-1/2}}$.*

Proof. We break our analysis into three cases, depending on the size of L in proportion to L_k and R_k:

- If $R_{k+1} \leqslant L \leqslant L_k$, we can cover I by $(L/R_{k+1})^d$ cubes in $\mathcal{B}(R_{k+1})$. For each of these cubes, because the mass is uniformly distributed over R_{k+1} cubes, we know the mass is bounded by at most $2(R_{k+1}/L_{k+1})^d$ times the mass of a $\mathcal{B}(L_k)$ cube. Thus

$$\begin{aligned}\mu(I) &\lesssim [(L/R_{k+1})^d][2(R_{k+1}/L_k)^d][L_k^{d\beta_k - k^{-1/2}}]\\ &\leqslant 2L^d/L_k^{d+k^{-1/2}-d\beta_k} \leqslant 2L^{d\beta_k - k^{-1/2}}\end{aligned}$$

- If $L_{k+1} \leqslant L \leqslant R_{k+1}$, we can cover I by a single cube in $\mathcal{B}(R_{k+1})$. Each cube in $\mathcal{B}(R_{k+1}, d)$ contains at most one cube in $\mathcal{B}(L_{k+1}, d)$ which is also contained in X_{k+1}, so

$$\mu(I) \lesssim L_{k+1}^{d\beta_{k+1}-(k+1)^{-1/2}} \leqslant L^{d\beta_k - k^{-1/2}}$$

- If $L \leqslant L_{k+1}$, there certainly exists M such that $L_{M+1} \leqslant L \leqslant L_M$, and one of the previous cases yields that

$$\mu(I) \lesssim 2L^{d\beta_M - M^{-1/2}} \leqslant 2L^{d\beta_k - k^{-1/2}}$$

The three bulletpoints address all cases considered in the theorem. □

To use Frostman's lemma, we need the result $\mu(I) \lesssim L^{d\beta_k - k^{-1/2}}$ for an *arbitrary* interval, not just one with $L \leqslant L_k$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. This is because if $L \geqslant L_k$, then $\mu(I)/L^{d\beta_k - k^{-1/2}} \leqslant 1/L_k^{d\beta_k - k^{-1/2}} \lesssim_k 1$, so $\mu(I) \lesssim_k L^{d\beta_k - k^{-1/2}}$ holds automatically for all sufficiently large intervals. Thus we have shown that $\dim_{\mathbf{H}}(X) \geqslant d\beta_k - k^{-1/2}$, and letting $k \to \infty$ gives $\dim_{\mathbf{H}}(X) \geqslant d\beta$. It is also easy to see X has *precisely* this dimension.

Theorem 3. $\dim_{\mathbf{H}}(X) = (dn - \alpha)/(n-1)$.

Proof. X_k is covered by at most

$$\left[\frac{L_{k-1} \dots L_1}{R_k \dots R_1} \right]^d$$

sidelength L_k cubes. It follows that if $\gamma > \beta$, then

$$H_{L_k}^{d\gamma}(X) \leqslant \left[\frac{L_{k-1} \dots L_1}{R_k \dots R_1} L_k^\gamma \right]^d \lesssim \left[\frac{L_{k-1} \dots L_1}{R_{k-1} \dots R_1} L_k^{\gamma - \beta} \right]^d$$

Thus if L_k is suitably small depending on previous constants, which we know to be true from the last corollary, we conclude that as $k \to \infty$, $H^\gamma(X)$ is finite. Since γ was arbitrary, taking it to β allows us to conclude that $\dim_{\mathbf{H}}(X) \leqslant d\beta$. Thus $\dim_{\mathbf{H}}(X) = d\beta$. □

6 Applications

7 Relation to Literature, and Future Work

The technical skeleton of our construction are heavily modelled after [2]. Reading this paper in tandem with ours provides an interesting contrast between the techniques of the function oriented configuration avoidance result, and the fractal avoidance result we use. Because of it's heavy influence on our result, we begin our discussion of the literature with an in depth comparison of our method to theirs.

Our result is a direct generalization of the main result of [2], which says that if $Z \subset (\mathbf{R}^d)^n$ is a smooth surface of dimension $nd-d$, then we can find X with dimension $(n-1)^{-1}$ solving the fractal avoidance problem. Of course, such a Z has Minkowski dimension $nd-d$, and our result achieves the same dimension for X. In response to [2], our result says that the only really necessary feature of a smooth hypersurface to the avoidance problem, aside from other geometric features, is it's dimension. Not only is our result more flexible, enabling the surface Z to have non smooth points, but we can also take advantage of the fact that the surface might have dimension different from $nd-d$. Better yet, we can 'thicken' or 'thin' Z by slightly increasing or decrease the Minkowski dimension, while stably affecting the Hausdorff dimension of the solution X we construct.

The technique leading to this generalization can be compared to a phenomenon that has recently been noticed in the discrete setting, i.e. [4]. There, certain combinatorial problems can be rephrased as abstract problems on hypergraphs, and by doing this one can often generalize the solutions of these problems into analogues on 'sparse versions' of these hypergraphs. One can see our result as a continuous analogue to this phenomenon, where sparsity is represented by the dimension of the set Z we are trying to avoid. One can even view Lemma 1 as a solution to a problem about independant sets in hypergraphs. In particular, we can form a hypergraph by taking the intervals $\mathcal{B}(F,S)$ as vertices, and adding an edge $(I_1,\dots,I_n)$ between n distinct cubes if $I_1 \times \dots \times I_n$ intersects W. Then the union of an independant set of cubes in this graph is precisely a set F with F^n disjoint except on the discretization of the diagonal. And so the goal of Lemma 1 is essentially to find a 'uniformly chosen' independant set in this graph. Thus we even applied the discrete phenomenon at many scales to obtain the continuous version of the phenomenon.

An interesting technique used in [2], and it's predecessor [1], is a Cantor set construction 'with memory'; a queue in their construction algorithm allows storage of particular configurations, to be retrieved and avoided at a much, much later step of the building process. The fact that our result is more general, yet we can discard the queueing method from our proof, is an interesting anomoly. Adding memory to the queueing set is certainly an important trick to remember when thinking of new constructions for fractal avoiding sets. It enables one to restrict the requirements of an analogy to Lemma 1 from carving out an avoiding set F from a single set E, to carving $F_1,\dots,F_n$ out of disjoint sets $E_1,\dots,E_n$, such that $F_1 \times \dots \times F_n$ avoids W. Nonetheless, it makes the construction much more complicated to describe, which makes understanding dimension bounds slightly more complicated, because it's hard to 'grasp' precisely what configuration we are avoiding at each step of the construction. The fact that our algorithm is more general than [2], yet we can discard the queueing method, is an interesting anomoly. We have ideas on how to exploit the fact that we do not use queueing to generalize our theorem to much more wide family of 'dimension α' sets Z, which we plan to publish in a later result.

Aside from [2], another paper that takes the perspective of solving a generic fractal avoidance problem is [3], who finds a solution X to an avoidance problem with Z a degree k hypersurface with Hausdorff dimension d/k. If $k \geqslant n-1$, then our result does better than Mathe's result, so where Mathe's result excels is when Z is a low dimensional hypersurface. Just like how the result of this paper is a sparse analogue of [2], we would like to publish a follow up result giving a sparse analogue to [3]. Just as our result is obtained by assuming Z is covered by a sparse family of cubes, a sprase analogue of [3] would give a result if Z is covered by a sparse family of thickened varieties from a pencil of low degree surfaces. We already have ideas we are refining on how to achieve this.

References

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