Nodal Domains and Diffusion Processes

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- Georgiev, Mukherjee, *Nodal Geometry, Heat Diffusion, and Brownian Motion*, Anal. PDE. **12** (2017), 133-148.

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- Øksendal, *Stochastic Differential Equations*, Springer, 2003.
- Chung, *Green, Brown, and Probability and Brownian Motion*, World Scientific Publishing Company, 2002.

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Study 'asymptotic geometry' of D_λ as $\lambda \to \infty$.

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Credit: *Yuri Skiba*

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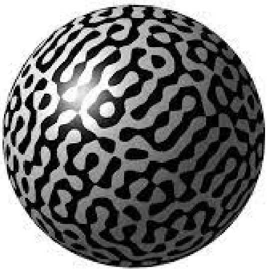

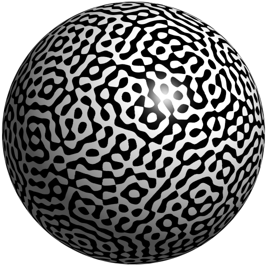

Credit: *Alex Barnett*

Main Result

- **Theorem**: There is $c_M > 0$ such that for any 'good' k-dimensional submanifold Σ of M, then

$$N(\Sigma, c_M/\lambda) = \{x \in M : d(x, \Sigma) < c_M/\lambda\}$$

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- Consider the radius $1/\lambda$ tubular neighborhood

$$T_{1/\lambda}\Sigma = \bigcup\nolimits_{x\in\Sigma}\{v \in (T_x\Sigma)^\perp : |v|_g \leq 1/\lambda\}.$$

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- But no cheating globally!

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  doesn't contain D_λ.
- There is $C_M > 0$ such that $D_\lambda \subset N(Z_\lambda, C_M/\lambda)$, contrasting this result.
- Proof Heuristic: Elliptic methods tend to give $O(1/\lambda)$ localized results. We study stochastic diffusions, which provide cool tools for analyzing eigenfunctions from an elliptic perspective!

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- Related Methods: D_λ satisfies an 'interior cone condition' with angle $O(1/\lambda)$.

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  - As a law predicting future behaviour from present behaviour, i.e. by defining quantities such as

$$\mathbb{E}^x[f(X)] = \mathbb{E}[f(X)|X_0 = x].$$

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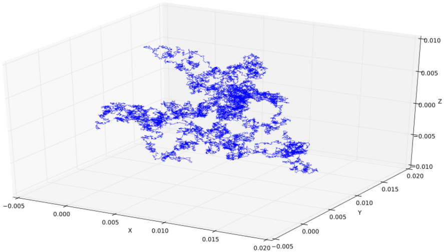
  - For any family of disjoint intervals $I_1, \ldots, I_N \subset [0, \infty)$, with $I_k = [t_k, s_k]$, the random variables $\Delta_{I_k} B$ are independent from one another.



Credit: *Shiyu Ji*

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- For practical purposes, we have

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- Diffuses locally near x faster in directions where $A(x)$ has large eigenvalues.

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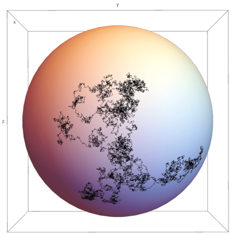
- Can define Itô diffusions on compact Riemannian manifolds M given a section $A : M \to \mathrm{Hom}(TM)$ of positive definite matrices.
- We can define Brownian motion on a Riemannian manifold such that Brownian motion diffuses at a unit speed along geodesics.



Credit: *Ma, Matveev, Pavlyukevich*

Connection to Elliptic Operators

- For any diffusion X, we can associate a semielliptic operator L, the *generator* of X, such that for $f \in C^\infty(M)$,

$$Lf(x) = \partial_t\{\mathbb{E}^x[f(X_t)]\}|_{t=0} = \lim_{t\to 0^+} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}.$$

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- Second order because paths of X are 'half differentiable'.
- For Brownian motion (on $\mathbb{R}^d$ or a manifold M), $L = \Delta/2$.
- 'Morally' apply the Fundamental Theorem of Calculus to get *Dynkin's Formula*

$$\mathbb{E}^x[f(X_T)] = f(x) + \mathbb{E}^x\left[\int_0^T (Lf)(X_s)\, ds\right].$$

Application: Escape Times

- In Dynkin's formula, T can be a 'stopping time', i.e. any $[0, \infty)$ valued function of X which doesn't 'predict the future', i.e. if T stops at a time t, it must only stop because of the properties of X on $[0, T]$, and not behaviour on (T, ∞).

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- If B is Brownian motion on M, escape time will be slower if volume expands (negative curvature) and faster if volume contracts (positive curvature). This is irrelevant for the values $R = O(1/\lambda)$ that we care about.

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  - Can also solve $\partial_t u = Lu$ with $\partial u / \partial \eta = 0$ on ∂D using 'reflection on Brownian motion', but a little more technical with singularities.

The Proof

And now, back to our regularly scheduled programming

- **Theorem**: There is $c_M > 0$ such that for any 'good' k-dimensional submanifold Σ of M, then

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  - $u(x,t) = \mathbb{E}[e_\lambda(B_t)\chi_t]$.

Thanks For Listening!