

# Fractals avoiding Fractal Sets

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- ▶ What is the largest Hausdorff dimension of a subset of  $\mathbf{R}^d$  such that the distances between any two points is irrational?
- ▶ Given any equation  $f$ , what is the largest Hausdorff dimension of  $X \subset \mathbf{R}^d$  such that for any distinct  $x_1, \dots, x_n \in X$ ,  $f(x_1, \dots, x_n) \neq 0$ .

## Two Observations

- Problems can be summarized as finding  $X$  such that  $X^n$  avoids a given set  $Z$ , except for 'repeated coordinate points'. Let

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- If  $Z = f^{-1}(0)$ ,  $X$  avoids zeroes of  $f$  for distinct values if and only if  $X^n \cap Z \subset \Delta$ .



# The Generic Problem

- **Fractal Avoidance Problem:** Given  $Z \subset \mathbf{R}^{nd}$ , find  $X \subset \mathbf{R}^d$  with large Hausdorff dimension such that  $X^n \cap Z \subset \Delta$ .

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- ▶ Mathé (2012): If  $Z \subset \mathbf{R}^{nd}$  is an algebraic hypersurface specified by a degree  $r$  polynomial in  $nd$  variables with rational coefficients, then we can find  $X$  solving the fractal avoidance problem for  $Z$  with dimension  $d/r$ . This is independent of  $n$ .

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- ▶ Pramanik and Fraser (2016): If  $Z$  is a smooth hypersurface of dimension  $nd - d$ , we can find  $X$  with dimension  $d/(n - 1)$ .

Increasing the Difficulty...

*What if the Patterns are  
Fractally Specified...*

# Main Result

## Theorem

*If  $Z$  is the countable union of sets with lower Minkowski dimension bounded by  $\alpha \geq d$ , we can find  $X$  with  $X^n \cap Z \subset \Delta$  and*

$$\dim_{\mathbf{H}}(X) = \frac{nd - \alpha}{n - 1} = \frac{\text{codim}(Z)}{n - 1}$$

# Low Rank Avoidance

## Theorem

*If we have countably many sets  $Z_i \subset \mathbf{R}^{n_i d}$  with linear transformations  $T_i : \mathbf{R}^{n_i d} \rightarrow \mathbf{R}^{k_i d}$  with rational coordinates such that  $T_i(Z_i)$  has lower Minkowski dimension  $\beta_i$ , we can find  $X$  with  $X^{n_i} \cap Z_i \subset \Delta$  for each  $i$  and*

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- The hypothesis says  $Z$  is coverable efficiently by lower dimensional thickened hyperplanes. Result should also extend when each  $Z_i$  is efficiently covered by thickened pencils of low degree algebraic surfaces, i.e.  $f(Z)$  has low dimension where  $f$  is a polynomial map.

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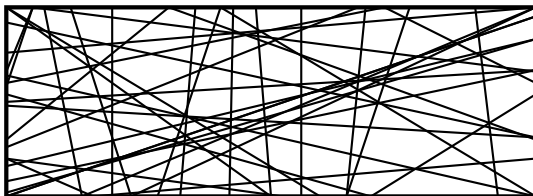
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- ▶ Want to push the  $2k - 1$  to  $k - 1$ , at least for  $k \geq 2$ . Know this is true for certain families of examples.

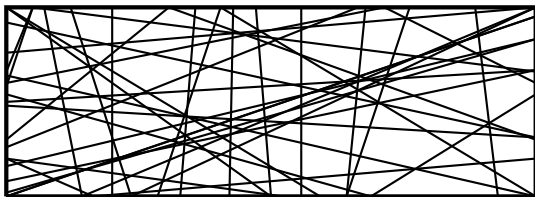


# Advantages of Our Method



- ▶ More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin'  $Z$ , we get stable effects on the Hausdorff dimension of  $X$ .

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- ▶ More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin'  $Z$ , we get stable effects on the Hausdorff dimension of  $X$ .
- ▶ Uncountable unions of regular sets are allowed!

# Applications

- ▶ Given a subset  $Y$  of  $\mathbf{R}^d$  which is the countable union of sets with Minkowski dimension  $\alpha$ , we can find a  $\mathbf{Q}$  vector subspace  $X$  of  $\mathbf{R}^d$  with Hausdorff dimension  $d - \alpha$  disjoint from  $Y$ .

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- ▶ We can find a full dimensional subset of  $\mathbf{R}^d$  avoiding the zero sets of all polynomials with rational coefficients of the form  $f(y \cdot x)$  with  $x \in X^n, y \in \mathbf{Q}^n$ . No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.

Two key ideas:

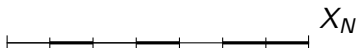
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- ▶ Discretization of Scales.
- ▶ Random Dissection.

# Discretization of Scales



- Construct  $X$  as a Cantor set by limits of interval sets  $X_N$ .

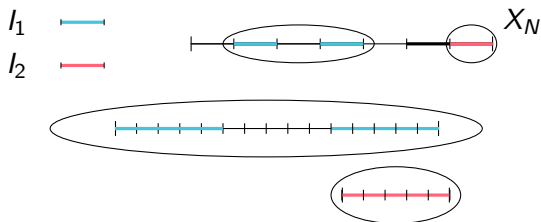


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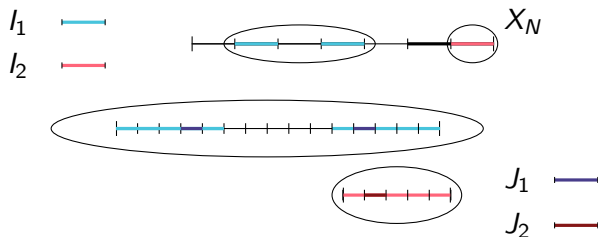
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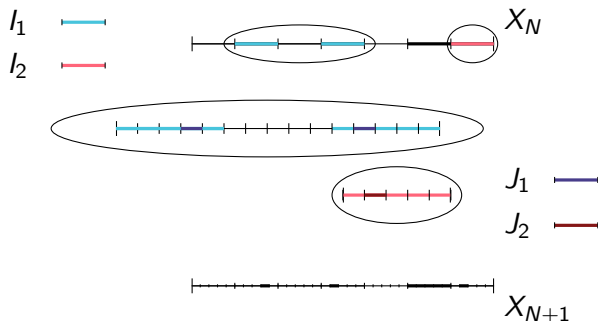
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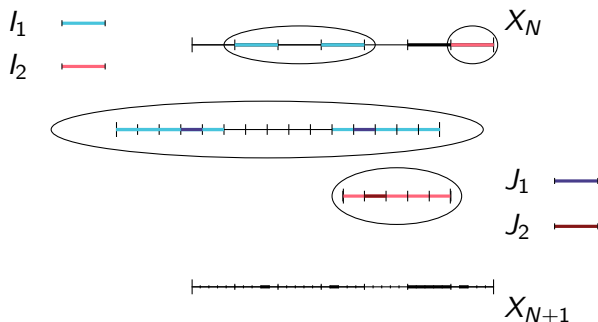
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- Construct  $X$  as a Cantor set by limits of interval sets  $X_N$ .
- **Discrete Problem:** Given disjoint unions of length  $L$  intervals  $I_1, \dots, I_n$ , find  $J_i$  for each  $i$  containing a part of each interval in  $I_i$  such that  $J_1 \times \dots \times J_n$  is disjoint from  $Z$ .

# Queueing

- ▶ Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers  $I_1, \dots, I_n$  of the set  $X$  gives a fractal avoiding set  $X$ .

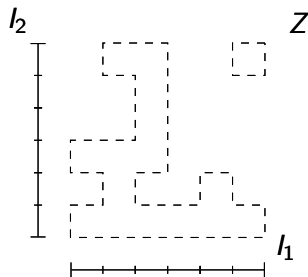
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- ▶ Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers  $I_1, \dots, I_n$  of the set  $X$  gives a fractal avoiding set  $X$ .
- ▶ **Reason:** If we consider distinct  $x_1, \dots, x_n \in X$ , there are intervals  $I_1, \dots, I_n$  with  $x_1 \in I_1, \dots, x_n \in I_n$  considered at some scale. Then  $x_1 \in J_1, \dots, x_n \in J_n$ , and so  $(x_1, \dots, x_n) \in J_1 \times \dots \times J_n$  cannot be contained in  $Z$ .



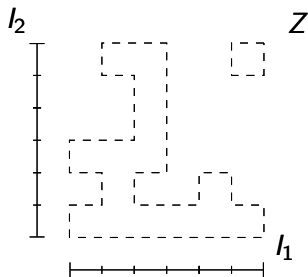


# Exploiting Randomness



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- ▶ How do we prove the discrete scale argument?
- ▶ Aside from  $Z$ 's dimension, we have little structural knowledge.
- ▶ Random choices of the  $J_k$  avoid  $Z$  effectively.
- ▶ We essentially obtain for all but a fraction  $o(1)$  of the length  $L$  intervals in  $I_k$ ,  $J_k$  contains a length  $L^\beta$  section of each, where  $\beta = d(n-1)/(nd-\alpha)$ . This ratio gives the Hausdorff dimension bound  $(nd-\alpha)/(n-1)$  for  $X$ .

So What's Next?

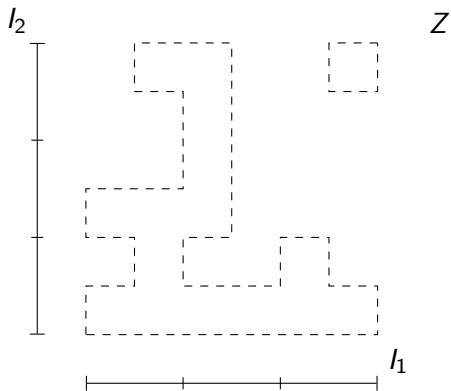
# Extension to Hausdorff Dimension

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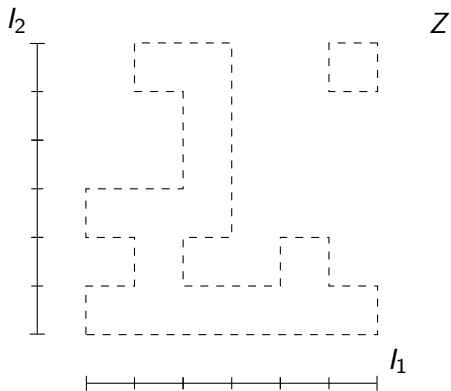
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- ▶ There is no obvious reason why our techniques should fail when  $Z$  has *Hausdorff dimension*  $\alpha$  rather than a Minkowski dimension bound.
- ▶ We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

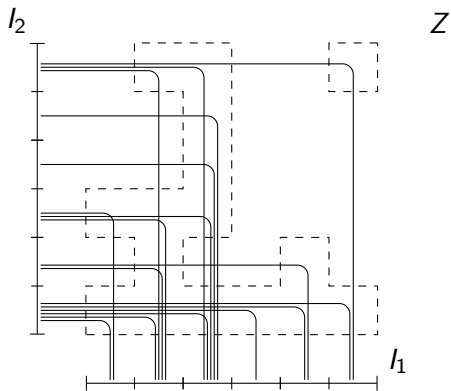
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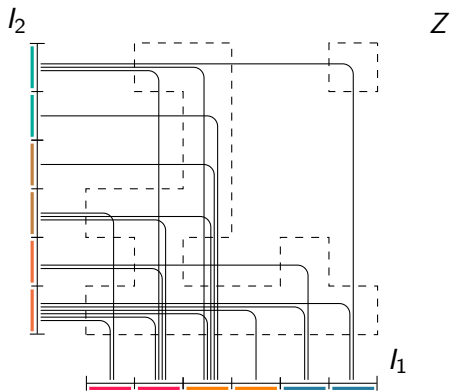


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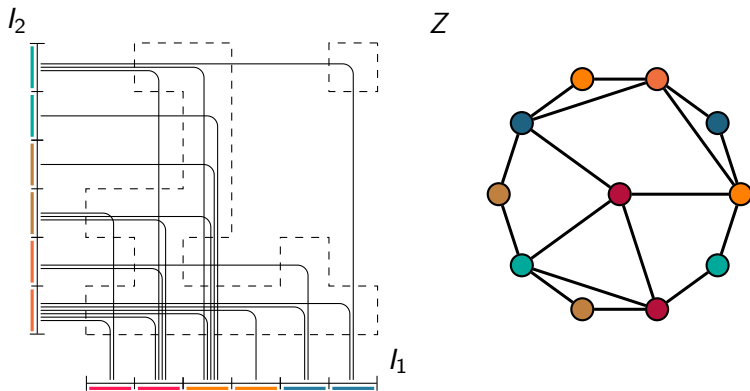




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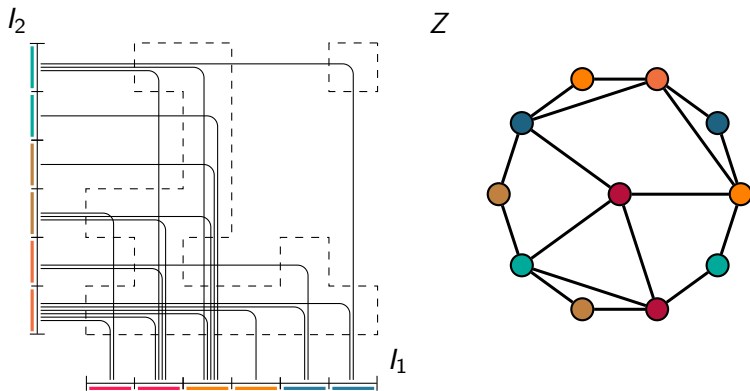


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- ▶ The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- ▶ We are looking to using other methods on hypergraphs to improve the bound when  $Z$  has certain structural properties (like for the low rank result).

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