Chapter 1

The Setup

In any physical theory, we must characterize mathematically the *state* of a system (all information describing the situation of a physical system at a particular time), and the *observables*, the functions of a state, which give ways in which the state of a system can be reduced to quantities that can be observed experimentally. For instance, in Hamiltonian classical mechanics, the state of a system is given by a point in a sympletic manifold M, and observables given by functions $f : M \to \mathbf{R}$, which should be continuous if we are to correctly measure these observables up to a small degree of error. The observables are then 'second order' as they are defined in terms of states, but we can also reverse the situation, describing the observables as the C^* algebra $A = C(M)$. The states then become precisely a *positive* linear functional $\phi : A \to \mathbf{R}$ with $\phi(1) = 1$. It is natural in the later quantum mechanics to complexify the C^* algebra A. Then the observables become the *self-adjoint* elements of A, and the states the linear functionals $\phi : A \to \mathbf{C}$ with $\phi(1) = 1$ and with $\phi(X) \geqslant 0$ if $X \geqslant 0$. The Riesz representation theorem allows us to identify an arbitrary positive linear functionals $\phi : A \to \mathbf{R}$ such that $\phi(1) = 1$ with a Borel probability measure μ on M. We then think of an element $X \in A$ as a *random variable* over the probability space (M, μ), because we then have

$$\mathbf{E}_\phi[X] = \int X \, d\mu = \phi(X).$$

Similarily,

$$\sigma_\phi(X)^2 = \mathbf{V}_\phi(X) = \phi(X^2) - \phi(X)^2.$$

The *pure*, deterministic states ϕ can then be identified from general *mixed states* as those states such that $\mathbf{V}_\phi(X) = 0$ for all observables X.

What caused this formulation to fail to explain quantum mechanical phenomena. The most fundamental experimental observation in the theory is the *uncertainty principle*. It is an experimental observation that in any physical system, if $p : A \to \mathbf{C}$ and $q : A \to \mathbf{C}$ are the position and momentum observables, then for any state ϕ,

$$2\sigma_\phi(p)\sigma_\phi(q) \geqslant \hbar,$$

where $\hbar$ is *Planck's constant*. But there are no two observables $\phi : A \to \mathbf{R}$ with this property for all classical states, because $\sigma_\phi(p) = \sigma_\phi(q) = 0$ for any deterministic state. Thus it appears that the only physically possible states ϕ *must be uncertain* in a suitable sense; this is the *uncertainty principle*.

In the standard theory, this is remedied by replacing the observables of a system with elements of an abstract C^* algebra A, and the states with normalized, positive linear functions $\phi : A \to \mathbf{C}$. Each fixed state ϕ then induces an algebra homomorphism Φ from A to the family of random variables in an appropriate probability space, such that $\mathbf{E}(\Phi(X)) = \phi(X)$. Thus one can use the spectral calculus to obtain detailed information about the probability distribution of $\Phi(X)$, since for any continuous $f : \sigma(X) \to \mathbf{C}$, we have $\mathbf{E}(f(X)) = \phi(f(X))$. Note, in particular, that this means that the support of the random variable X is on $\sigma(X)$.

The reason this formulation is useful is that we can theoretically derive the uncertainty principle, provided we are working in a *non-commutative* C^* algebra A. Indeed, if X and Y are any observables with $\phi(X) = \phi(Y) = 0$, we calculate that the matrix

$$M = \begin{pmatrix} \phi(X^2) & (1/2)\phi(i[X,Y]) \\ (1/2)\phi(i[X,Y]) & \phi(Y^2) \end{pmatrix}$$

is positive-semidefinite, since for any $v = (\alpha, \beta)^T \in \mathbf{R}$,

$$v^T M v = \phi(X^2)\alpha^2 + \phi(i[X,Y])\alpha\beta + \phi(Y^2)\beta^2 = \phi((\alpha X - i\beta Y)(\alpha X + i\beta Y)) \geqslant 0.$$

Thus $\det(M) = \phi(X^2)\phi(Y^2) - \phi(i[X,Y])^2/4$ is non-negative, which means that

$$2\sigma_\phi(X)\sigma_\phi(Y) = 2\phi(X^2)^{1/2}\phi(Y^2)^{1/2} \geqslant \phi(i[X,Y]).$$

Thus the uncertainty principle for position and momenta follows immediately if we model these quantities by observables p and q with $[p,q] = -i\hbar$.

Chapter 2

Quantum Information Theory

The simplest unit of information in classical physics is a *bit*, represented by an element of $\{0, 1\}$. We can generalize

and the state of a collection of n bits are represented by an element of $\{0, 1\}^n$. From the quantum perspective, a *quantum bit*, or *qubit*, is represented by an element $\psi = \psi_0 \langle 0| + \psi_1 \langle 1|$ of a two dimension Hermitian product space with orthonormal basis $\{\langle 0|, \langle 1|\}$.

(Gleason)