

Salem Sets Avoiding Rough Configurations

Jacob Denson

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Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension t if it has Hausdorff dimension t , and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty.$$

Our goal in these notes is to obtain, for each set $Z \subset \mathbf{R}^{dn}$ with Minkowski dimension s , a Salem set $X \subset \mathbf{R}^d$ with dimension

$$\frac{nd - s}{n},$$

such that for each set of n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We rely on a random selection approach, like in our paper on rough configurations, to obtain such a result.

1 Concentration Inequalities

Define a convex function $\psi_2 : [0, \infty) \rightarrow [0, \infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \leq 1\}.$$

The family of random variables $\psi_2(L)$ are known as *subgaussian random variables*. Here are some important properties:

- (Gaussian Tails): If $\|X\|_{\psi_2(L)} \leq A$, then for each $t \geq 0$,

$$\mathbf{P}(|X| \geq t) \leq 10 \exp(-t^2/10A^2).$$

- (Bounded Variables are Subgaussian): For any random X ,

$$\|X\|_{\psi_2(L)} \leq 10\|X\|_{L^\infty}.$$

- (Hoeffding's Inequality): If X_1, \dots, X_N are *independent* variables, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leq 10 \left(\|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2 \right)^{1/2}.$$

The Orlicz norm is a convenient notation to summarize calculations involving the principle of concentration of measure. Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leq A$ as vary rarely deviating outside the interval $[-A, A]$.

2 A Family of Cubes

Fix integer-valued sequences $\{K_m : m \geq 1\}$ and $\{M_m : m \geq 1\}$, and then set $N_m = K_m M_m$. We then define two sequences of real numbers $\{l_m : m \geq 0\}$ and $\{r_m : m \geq 0\}$, by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each $m, d \geq 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \dots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \dots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $i \in \Sigma_m^d$, we define a vector $a_i \in (l_m \mathbf{Z})^d$ by setting

$$a_i = i_0 + \sum_{k=1}^m i_{2k-1} \cdot r_k + i_{2k} \cdot l_k$$

Then each string $i \in \Sigma_m^d$ can be identified with the sidelength l_m cube

$$Q_i = \prod_{j=1}^d [a_{ij}, a_{ij} + l_m].$$

centered at a_i . Similarly, for each string $i \in \Pi_m^d$, we define a vector $a \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leq j \leq d$,

$$a_i = i_0 + \left(\sum_{k=1}^{m-1} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k \right) + i_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R_i = \prod_{j=1}^d [a_{ij}, a_{ij} + r_m].$$

We let $\mathcal{Q}_m^d = \{Q_i : i \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R_i : i \in \Pi_m^d\}$. Here are some important properties of this collection of cubes:

- For each m , \mathcal{Q}_m^d and \mathcal{R}_m^d are covers of \mathbf{R}^d .
- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.
- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q . Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R .

We say a set $E \subset \mathbf{R}^d$ is \mathcal{Q}_m discretized if it is a union of cubes in \mathcal{Q}_m^d , and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarly, we say a set $E \subset \mathbf{R}^d$ is \mathcal{R}_m discretized if it is a union of cubes in \mathcal{R}_m^d , and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{i \in \Sigma_m^d : Q_i \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{i \in \Pi_m^d : R_i \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j , and a third index $k \in \Pi_m^d$, such that $R_k \cap Q_i, R_k \cap Q_j \neq \emptyset$.

3 A Family of Mollifiers

We now consider a family of mollifiers, which we will use to smooth out the Fourier transform of the measures we study.

Lemma 1. *There exists a non-negative, C^∞ function ψ supported on $[-1, 1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi(x) dx = 1, \quad (3.1)$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x + n) = 1. \quad (3.2)$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0, 1]$, such that $\alpha(1/2 + x) = \alpha(1/2 - x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \leq \alpha(x) \leq 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|).$$

Symmetry considerations imply that $\int_{\mathbf{R}^d} \alpha(x) + \beta(x) = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x + m) + \beta(x + m) = 1. \quad (3.3)$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

then ψ_0 satisfies the required constraints, at least in the one dimensional case. In general, define $\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d)$. \square

Fix some choice of ψ given by Lemma 1. Since ψ is C^∞ and compactly supported, then for each $t \in [0, \infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\hat{\psi}(\xi)| < \infty. \quad (3.4)$$

Now we rescale the mollifier. For each $m > 0$, we let

$$\psi_m(x) = l_m^{-d} \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (3.1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \quad (3.5)$$

Equation (3.2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \quad (3.6)$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi}_m(\xi) = \widehat{\psi}(l_m \xi), \quad (3.7)$$

In particular, (3.7) implies that for each $t \geq 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}_m(\xi)| |\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)| |\xi|^t. \quad (3.8)$$

Thus, uniformly in m , $\widehat{\psi}_m$ decays sharply outside of the box $[-l_m^{-1}, l_m^{-1}]^d$, a manifestation of the Heisenberg uncertainty principle.

4 Discrete Lemma

We now consider a discrete form of the Fourier bound argument, which we can apply iteratively to obtain a Salem set avoiding configurations.

Lemma 2. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn - s)/2)$. Let $T \subset [0, 1]^d$ be a non-empty, \mathcal{Q}_m discretized set, and let μ_T be a smooth probability measure compactly supported on T . Let $B \subset \mathbf{R}^{dn}$ be a non-empty, \mathcal{Q}_{m+1} discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}. \quad (4.1)$$

Then there exists a large constant $C(\mu_T, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geq C(\mu_T, n, d, s, \varepsilon), \quad (4.2)$$

and

$$M_{m+1}^{\frac{s}{dn-s} + c\varepsilon} \leq K_{m+1} \leq 2M_{m+1}^{\frac{s}{dn-s} + c\varepsilon}, \quad (4.3)$$

where

$$c = \frac{6dn}{(dn - s)^2},$$

then there exists a \mathcal{Q}_{m+1} discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) If $L = \sup_{m \in \mathbf{Z}^d} |m|^{\frac{dn-s}{2n}-a\varepsilon} |\widehat{\mu}_T(m)|$, then

$$\begin{aligned} & \sup_{m \in \mathbf{Z}^d} |m|^{a\varepsilon - \frac{dn-s}{2n}} |\widehat{\mu}_S(m)| \\ & \leq (1 + 1/2^m) (\max(1, L) + 10^d M_{m+1}^{-\varepsilon}), \end{aligned}$$

where

$$a = \frac{3d + 2dn - 2s}{dn}.$$

Remark 3. To make the statement of (2) more clean, we have hidden the explicit constant $C(\mu_T, n, d, s, \varepsilon)$. But this constant can be made explicit by determining which value $C(\mu_T, n, d, s, \varepsilon)$ must be chosen to ensure that (4.2) implies (4.5), (4.10), (4.16), (4.25), (4.26), (4.27), (4.36), and (4.37) are all satisfied.

Proof of Lemma 2. First, we describe the construction of the set S , and the measure μ_S . For each $i \in \Pi_{m+1}^d$, let j_i be a random integer vector chosen from $[K_{m+1}]^d$, such that the family $\{j_i : i \in \Pi_{m+1}^d\}$ is independent. Then it is certainly true for any $j \in [K_{m+1}]^d$ that

$$\mathbf{P}(j_i = j) = K_{m+1}^{-d}. \quad (4.4)$$

Define a measure ν_S such that, for each $x \in \mathbf{R}^d$,

$$d\nu_S(x) = r_{m+1}^d \sum_{i \in \Pi_{m+1}^d} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

We then normalize, defining

$$\mu_S = \frac{\nu_S}{\nu_S(\mathbf{R}^d)}.$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then S is \mathcal{Q}_{m+1} discretized, μ_S is supported on S , and $S \subset T$. Our goal is to show that, with non-zero probability, some choice of $\{j_i\}$ yields a set S satisfying Properties (A) and (B) of the Lemma.

In our calculations, it will help us to decompose the measure ν_S into components roughly supported on sidelength r_{m+1}^d cubes. For each $i \in \Pi_{m+1}(T)$, define a measure ν_i such that for each $x \in \mathbf{R}^d$,

$$d\nu_i(x) = r_{m+1}^d \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

Then $\nu_S = \sum_{i \in \Pi_{m+1}^d(T)} \nu_i$. We shall split the proof of Properties (A) and (B) of Lemma 2 into several, more manageable lemmas.

Lemma 4. *If*

$$M_{m+1} \geq \left(3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}\right)^2, \quad (4.5)$$

then almost surely, $|\nu_S(\mathbf{R}^d) - 1| \leq M_{m+1}^{-1/2}$.

Proof. If $j_0, j_1 \in [K_{m+1}]^d$, then

$$|a_{ij_0} - a_{ij_1}| = |j_0 - j_1| \cdot l_{m+1} \leq (\sqrt{d} K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1},$$

which, together with (3.5), implies

$$\begin{aligned} & \left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_0}) \mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_1}) \mu_T(x) \right| \\ & \leq r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) |\mu_T(x + a_{ij_0}) - \mu_T(x + a_{ij_1})| \\ & \leq \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1}(x) = \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad (4.6)$$

Thus (4.6) implies that almost surely, for each i ,

$$|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))| \leq \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \quad (4.7)$$

Furthermore, (3.6) implies

$$\begin{aligned}
\sum_{i \in \Pi_{m+1}^d} \mathbf{E}(\nu_i(\mathbf{R}^d)) &= r_{m+1}^d \sum_{(i,j) \in \Sigma_{m+1}^d} \mathbf{P}(j_i = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij}) \mu_T(x) dx \\
&= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left(\sum_{(i,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij}) \right) \mu_T(x) dx \\
&= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} = 1.
\end{aligned} \tag{4.8}$$

For all but at most $3^d \cdot r_{m+1}^{-d}$ indices i , $\nu_i = 0$ almost surely. Thus we can apply the triangle inequality together with (4.7) and (4.8) to conclude that almost surely,

$$\begin{aligned}
|\nu_S(\mathbf{R}^d) - 1| &= \left\| \sum_{i \in \Pi_{m+1}^d} [\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))] \right\|_{L^\infty} \\
&\leq \sum_{i \in \Pi_{m+1}^d} \|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \\
&\leq 3^d \sqrt{d} \cdot r_{m+1}^{-d} r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \\
&= 3^d \sqrt{d} \cdot r_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \\
&= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.
\end{aligned} \tag{4.9}$$

Thus (4.5) and (4.9) imply that almost surely, $|\nu_S(\mathbf{R}^d) - 1| \leq M_{m+1}^{-1/2}$. \square

Lemma 5. *If*

$$M_{m+1} \geq (10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)})^{1/\varepsilon}, \tag{4.10}$$

then

$$\mathbf{P}(S \text{ does not satisfies Property (A)}) \leq 1/10.$$

Proof. For any cube $Q_{ij} \in \Sigma_{m+1}^d$, there are at most 3^d pairs $(i_0, j_0) \in \Sigma_{m+1}^d$ such that $Q_{i_0 j_0} \cap Q_{ij} \neq \emptyset$, and so a union bound together with (4.4) gives

$$\mathbf{P}(Q_{ij} \in \mathcal{Q}_{m+1}(S)) \leq \sum_{Q_{i_0 j_0} \cap Q_{ij} \neq \emptyset} \mathbf{P}(j_{i'} = j') \leq 3^d K_{m+1}^{-d}. \tag{4.11}$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_{i_1 j_1} \times \cdots \times Q_{i_n j_n} \in$

$\mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_{i_k j_k} \in S\}$ are independent from one another, which together with (4.11) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_{i_1 j_1} \in S) \dots \mathbf{P}(Q_{i_n j_n} \in S) \leq 3^{dn} K_{m+1}^{-dn}. \quad (4.12)$$

Taking expectations over all cubes in B , and applying (4.1) and (4.12) gives

$$\begin{aligned} \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leq \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\ &\leq l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\ &= 3^{dn} l_m^{-(s+\varepsilon)} \frac{M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}. \end{aligned} \quad (4.13)$$

Since $\varepsilon \leq (dn - s)/2$, we conclude

$$\begin{aligned} (dn - s - \varepsilon) \left(\frac{s}{dn - s} + c\varepsilon \right) &= s + \varepsilon \left(c(dn - s - \varepsilon) - \frac{s}{dn - s} \right) \\ &\geq s + \varepsilon \left(\frac{c(dn - s)}{2} - \frac{s}{dn - s} \right) \\ &= s + \varepsilon \frac{3dn - s}{dn - s} \geq s + 2\varepsilon. \end{aligned}$$

Applying (4.3) together with this bound, we conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geq M_{m+1}^{(dn-s-\varepsilon)(\frac{s}{dn-s} + c\varepsilon)} \geq M_{m+1}^{s+2\varepsilon}.$$

Combined with (4.10), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leq \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^\varepsilon} \leq 1/10. \quad (4.14)$$

We can then apply Markov's inequality with (4.13) and (4.14) to conclude

$$\begin{aligned} \mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \emptyset) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geq 1) \\ &\leq \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) \\ &\leq 1/10. \end{aligned} \quad \square$$

Lemma 6. Set $D = \{m \in \mathbf{Z}^d : |m| \leq 10l_{m+1}^{-1}\}$. Then if

$$K_{m+1} \leq M_{m+1}^{\frac{2dn}{dn-s}}, \quad (4.15)$$

and

$$M_{m+1} \geq \exp \left(\frac{10^7 (3dn - s)d^2}{dn - s} \right), \quad (4.16)$$

then

$$\mathbf{P} \left(\|\widehat{\nu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \leq 1/10$$

Proof. For each $i \in \Pi_{m+1}^d$, and $m \in \mathbf{Z}$, define $X_{im} = \widehat{\nu}_i(m) - \widehat{\mathbf{E}(\nu_i)}(m)$. Applying (3.2) gives

$$\begin{aligned} \sum_{i \in \Pi_{m+1}^d} \widehat{\mathbf{E}(\nu_i)}(m) &= \sum_{i \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} \psi_{m+1}(x - a_{ij}) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} d\mu_T(x) = \widehat{\mu}_T(m). \end{aligned} \quad (4.17)$$

For each i and m , the standard (L^1, L^∞) bound on the Fourier transform, combined with (4.7), shows

$$\begin{aligned} \|X_{im}\|_{\psi_2(L)} &\leq 10 \|X_{im}\|_{L^\infty} \\ &\leq 10 [\|\nu_i(\mathbf{R}^d)\|_{L^\infty} + \mathbf{E}(\nu_i)(\mathbf{R}^d)] \\ &\leq 10^2 \left(\mathbf{E}(\nu_i)(\mathbf{R}^d) + r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (4.18)$$

For a fixed m , the family of random variables $\{X_{im}\}$ are independent. Furthermore, $\sum X_{im} = \widehat{\nu}_S(m) - \widehat{\mathbf{E}(\nu_S)}(m)$. Equations (3.6) and (4.4) imply that

$$\begin{aligned} \mathbf{E}(\widehat{\nu}_S(m)) &= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} \left(\sum_{(i,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij}) \right) d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \widehat{\mu}_T(m) = \widehat{\mu}_T(m). \end{aligned} \quad (4.19)$$

Hoeffding's inequality, together with (4.18) and (4.19), imply that

$$\begin{aligned} \|\widehat{\nu}(m) - \widehat{\mu}_T(m)\|_{\psi_2(L)} &\leq 10^3 \sqrt{d} \left(\left(\sum \mathbf{E}(\nu_i)(\mathbf{R}^d)^2 \right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (4.20)$$

Equation (3.5) shows

$$\begin{aligned}\mathbf{E}(\nu_i)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int \psi_{m+1}(x - a_{ij}) d\mu_T(x) \\ &\leq r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned}\tag{4.21}$$

Combining (4.20) and (4.21) gives

$$\|\widehat{\nu}(m) - \widehat{\mu}_T(m)\|_{\psi_2(L)} \leq 10^3 \sqrt{d} \left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right] r_{m+1}^{d/2}.\tag{4.22}$$

We can then apply a union bound over $D = \{m \in \mathbf{Z}^d : |m| \leq 10l_{m+1}^{-1}\}$, which has cardinality at most $10^{d+1}l_{m+1}^{-d}$, together with (4.22) to conclude that

$$\begin{aligned}\mathbf{P} \left(\|\widehat{\nu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \\ \leq 10^{d+2} \cdot l_{m+1}^{-d} \exp \left(-\frac{\log(M_{m+1})^2}{10^7 d} \right) \\ = 10^{d+2} l_m^{-d} \exp \left(d \log(M_{m+1} K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d} \right).\end{aligned}\tag{4.23}$$

Combined with (4.15) and (4.16), (4.23) implies

$$\mathbf{P} \left(\|\widehat{\nu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \leq 1/10.\tag{4.24}$$

Thus $\widehat{\nu}_S$ and $\widehat{\mu}_T$ are highly likely to differ only by a negligible amount over small frequencies. \square

Lemma 7. *Fix $r > 0$. If*

$$N_{m+1}^d \geq \frac{10 \cdot 2^{d+1+r/2}}{L} A(d+1+r/2),\tag{4.25}$$

$$N_{m+1}^d \geq \frac{10 \cdot 2^{3d}}{(1+r/2)L} A(d+1+r/2),\tag{4.26}$$

and

$$N_{m+1}^d \geq \frac{10 \cdot 2^{3d+r/2+1}}{L} B(d+r/2+1),\tag{4.27}$$

then almost surely, if $|\eta| \geq 10l_{m+1}^{-1}$,

$$|\widehat{\nu}_S(\eta)| \leq \frac{L}{|\eta|^{r/2}}.$$

Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{i \in \Pi_{m+1}^d \\ d(a_i, T) \leq 2r_{m+1}^{-1}}} \delta_{a_{ij_i}}.$$

Then $\nu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\nu}_S = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu}_T$. Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup |\widehat{\mu}_T(\xi)| |\xi|^t < \infty.$$

In light of (3.7), if we define, for each $t > 0$,

$$B(t) = \sup |\widehat{\psi}(\xi)| |\xi|^t,$$

then

$$\sup |\widehat{\psi_{m+1}}(\xi)| |\xi|^t = l_{m+1}^{-t} B(t).$$

The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d , so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leq \alpha(\mathbf{R}^d) \leq 2^d$. Thus

$$|\widehat{\nu}_S(\eta)| \leq 2^d \int |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi. \quad (4.28)$$

If $|\xi| \leq |\eta|/2$, $|\eta - \xi| \geq |\eta|/2$, so for all $t > 0$, and since (3.5) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leq 1$, we find

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \leq \frac{A(t) 2^{t-d}}{|\eta|^{t-d}}. \quad (4.29)$$

Set $t = d + 1 + s/2$. Equation (4.29), together with (4.25), implies

$$\begin{aligned} & \int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \\ & \leq \frac{A(d + 1 + s/2) 2^{1+s/2} |\eta|^{-1}}{|\eta|^{s/2}} \\ & \leq \frac{A(d + 1 + s/2) 2^{1+s/2} l_{m+1}}{|\eta|^{s/2}} \\ & \leq \frac{L}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \end{aligned} \quad (4.30)$$

Conversely, if $|\xi| \geq 2|\eta|$, then $|\eta - \xi| \geq |\xi|/2$, so for each $t > d$,

$$\begin{aligned} \int_{|\xi| \geq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi &\leq \int_{|\xi| \geq 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\ &\leq 2^d \int_{2|\eta|}^{\infty} r^{d-1-t} A(t) 2^t \\ &\leq \frac{4^d A(t)}{t-d} |\eta|^{d-t}. \end{aligned} \quad (4.31)$$

Set $t = d + 1 + s/2$. Equation (4.26), applied to (4.31), allows us to conclude

$$\int_{|\xi| \geq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{L}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \quad (4.32)$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu}_T\|_{L^\infty(\mathbf{R}^d)} \leq 1$ to conclude that

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \quad (4.33)$$

Set $t = d + s/2 + 1$. Then (4.33) and (4.27) imply

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{L}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \quad (4.34)$$

It then suffices to sum up (4.30), (4.32), and (4.34), and apply (4.28). \square

Proof of Lemma 2, Continued. Let us now put all our calculations together. In light of Lemma 5 and Lemma 6, there exists some choice of j_i for each i , and a resultant non-random pair (ν_S, S) such that S satisfies Property (A) of the Lemma, and for any $m \in \mathbf{Z}^d$ with $|m| \leq 10l_{m+1}^{-1}$,

$$|\widehat{\nu}_S(m) - \widehat{\mu}_T(m)| \leq r_{m+1}^{d/2} \log(M_{m+1}). \quad (4.35)$$

Now

$$r_{m+1}^{d/2} \log(M_{m+1}) = \left(l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \right) l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}.$$

Equation (4.3) implies

$$\begin{aligned} &l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \\ &= \frac{l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} \log(M_{m+1}) K_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}}{M_{m+1}^{a\varepsilon + \frac{s}{2n}}} \\ &\leq \left[l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} \right] \log(M_{m+1}) M_{m+1}^{\left(\frac{s}{dn-s} + c\varepsilon\right)\left(\frac{dn-s}{2n} - a\varepsilon\right) - a\varepsilon - \frac{s}{2n}}. \end{aligned}$$

Now

$$\begin{aligned}
\left(\frac{s}{dn-s} + c\varepsilon\right) \left(\frac{dn-s}{2n} - a\varepsilon\right) - a\varepsilon &\leq \left[\frac{(dn-s)c}{2n} - \left(\frac{s}{dn-s} + 1\right)a\right] \varepsilon \\
&= \left[\frac{d(3-na)}{(dn-s)}\right] \varepsilon \\
&\leq -2\varepsilon.
\end{aligned}$$

Since $\log(M_{m+1}) \leq (2/\varepsilon)M_{m+1}^{\varepsilon/2}$, if we assume that

$$M_{m+1} \geq \left(l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} (2/\varepsilon)\right)^{2/\varepsilon}, \quad (4.36)$$

then we conclude

$$\begin{aligned}
r_{m+1}^{d/2} \log(M_{m+1}) &\leq \left[l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} \log(M_{m+1}) M_{m+1}^{-\varepsilon}\right] M_{m+1}^{-\varepsilon} l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon} \\
&\leq M_{m+1}^{-\varepsilon} l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}
\end{aligned}$$

Applying Lemma 6, we conclude that if $|m| \leq 10l_{m+1}^{-1}$,

$$\begin{aligned}
|\widehat{\nu}_S(m)| &\leq |\widehat{\nu}_S(m) - \widehat{\mu}_T(m)| + |\widehat{\mu}_T(m)| \\
&\leq r_{m+1}^{d/2} \log(M_{m+1}) + (L) |m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq l_{m+1}^{-\frac{dn-s}{2n} - c\varepsilon + \varepsilon} + (L) |m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq [L + 10^d M_{m+1}^{-\varepsilon}] |m|^{c\varepsilon - \frac{dn-s}{2n}}.
\end{aligned}$$

Lemma 7 with $r = 2c\varepsilon - (dn-s)/n$ implies that for $|m| \geq 10l_{m+1}^{-1}$,

$$|\widehat{\nu}_S(m)| \leq L |m|^{c\varepsilon - \frac{dn-s}{2n}} \leq |m|^{c\varepsilon - \frac{dn-s}{2n}},$$

Thus we conclude that for all $m \in \mathbf{Z}^d$,

$$|\widehat{\nu}_S(m)| \leq (L + 10^d M_{m+1}^{-\varepsilon}) |m|^{c\varepsilon - \frac{dn-s}{2n}}.$$

Applying Lemma 4, as well as a valid assumption that

$$M_{m+1} \geq 4^m, \quad (4.37)$$

we conclude that for all $m \in \mathbf{Z}^d$,

$$\begin{aligned}
|\widehat{\mu}_S(m)| &\leq \frac{L + 10^d M_{m+1}^{-\varepsilon}}{1 - M_{m+1}^{-1/2}} |m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq (1 + M_{m+1}^{-1/2}) [L + 10^d M_{m+1}^{-\varepsilon}] |m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq (1 + 1/2^m) [L + 10^d M_{m+1}^{-\varepsilon}] |m|^{c\varepsilon - \frac{dn-s}{2n}}. \quad \square
\end{aligned}$$

5 Construction of the Salem Set

Let us now construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^{\infty} Z_i$, where Z_i has lower Minkowski dimension at most s for each i . Then choose an infinite sequence $\{i_m : m \geq 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers $\{\varepsilon_m : m \geq 1\}$, with $\varepsilon_m < (dn - s)/2$ for each m .

We choose our parameters $\{M_m\}$ and $\{K_m\}$ inductively. First, set $X_0 = [0, 1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0 . At the m 'th step of our construction, we have found a set X_{m-1} and a measure μ_{m-1} . We then choose K_m and M_m such that

$$K_m, M_m \geq C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m \geq 2^{m/\varepsilon_m} \quad (5.1)$$

such that

$$M_m^{\frac{s}{dn-s} + c\varepsilon_m} \leq K_m \leq 2M_m^{\frac{s}{dn-s} + c\varepsilon_m},$$

and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in \mathcal{Q}_m , the union of which, we define to be equal to B_m . We can then apply Lemma 2 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a \mathcal{Q}_{m+1} discretized set $S \subset T$, and a measure μ_S supported on S . We define $X_m = S$, and $\mu_m = \mu_S$.

The preceding paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$, and pick an arbitrary measure μ , and some subsequence μ_{i_k} , such that $\mu_{i_k} \rightarrow \mu$ weakly. It then follows from pointwise convergence of the Fourier transform that for each $m \in \mathbf{Z}^d$, and each $\varepsilon > 0$,

$$\sup_{m \in \mathbf{Z}^d} |\widehat{\mu}(m)| |m|^{\frac{dn-s}{2n} - \varepsilon} \leq \limsup_{i \rightarrow \infty} \sup_{m \in \mathbf{Z}^d} |\widehat{\mu}_{i_k}(m)| |m|^{\frac{dn-s}{2n} - \varepsilon}.$$

Fix $\varepsilon > 0$. For each m , define

$$A_{m,\varepsilon} = \sup |\widehat{\mu_M}(m)| |m|^{\frac{dn-s}{2n} - \varepsilon}.$$

Since each measure μ_M is smooth, all these quantities are finite. Since $\varepsilon_m \rightarrow 0$, there is M such that if $m \geq M$, then $a\varepsilon_m \leq \varepsilon$. Property (B) of Lemma (2) thus implies that for each $m \geq M$,

$$A_{m+1,\varepsilon} \leq (1 + 2^{-m})(A_{m,\varepsilon} + 10^d M_{m+1}^{-\varepsilon_{m+1}}).$$

Since $\prod_{m=1}^{\infty} (1 + 2^{-m}) < \infty$, we conclude that

$$\sup_{m \rightarrow \infty} A_{m,\varepsilon} \lesssim 10^d \sum_{m=0}^{\infty} M_{m+1}^{-\varepsilon_{m+1}}.$$

If, we assume (5.1), then the right hand side of the inequality is summable. And so we conclude $\sup_{m \rightarrow \infty} A_{m,\varepsilon} < \infty$ for each $\varepsilon > 0$. Thus X has Fourier dimension $(dn - s)/n$.

It is easy to see from our proof that for each m ,

$$\#(\mathcal{Q}_m(X_m)) = \frac{1}{(M_1 \dots M_m)^d}.$$

Thus

$$\begin{aligned} \log_{1/l_m}(\#(\mathcal{Q}_m(X_m))) &= \frac{d \log(M_1 \dots M_m)}{\log(N_1 \dots N_m)} \\ &\sim \frac{d \log(M_m)}{\log(N_m)} \\ &= \frac{d \log(M_m)}{\log(M_m) + \log(K_m)} \\ &= \frac{d}{1 + \log_{M_m}(K_m)} \\ &\sim \frac{d}{1 + \left(\frac{s}{dn-s} + c\varepsilon\right)} \\ &= \frac{d}{\frac{dn}{dn-s} + c\varepsilon_m}. \end{aligned}$$

Thus

$$\lim_{m \rightarrow \infty} \log_{1/l_m}(\#(\mathcal{Q}_m(X_m))) \leq \frac{dn - s}{n}.$$

In particular, this means X has lower Minkowski dimension at most $(dn - s)/n$, which, together with the Fourier dimension bound, implies that X is Salem.

6 Körner's Work

The last sections gives a first, positive result finding Salem sets avoiding rough configurations. However, given the same assumptions, one can find a

set X with *Hausdorff dimension* $(dn - s)/(n - 1)$ avoiding configurations. Improving the dimension to obtain this result requires a deeper knowledge of the stochastic behaviour of the set $\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n)$, when N_{m+1} is chosen significantly smaller relative to M_{m+1} . In the next section, we provide a summary of an argument due to Körner, which deals with a very similar situation.