Algebraic Topology

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Chapter 1

Homotopy

To verify two topological spaces are homeomorphic, we need only find a single homeomorphism. On the contrary, verifying two topological spaces are not homeomorphic is much more tricky; we need to show that every function from one space to the other is not a homeomorphism. One trick is to find fundamental topological properties which distinguish two topological spaces. Connectedness, Compactness, and Hausdorffiness are all preserved by homeomorphism, so two spaces in which these properties differ cannot be homeomorphic. Algebraic topology consists of deep techniques to distinguish between spaces by finding distinguished properties in the space.

It shall turn out that most interesting spatial invariants are also invariant under a type of topological equivalence more general that homeomorphism. Consider two continuous functions f and g between topological spaces X and Y[1]. Though f might not be equal to g, they may be in some sense topologically equal – we may be able to deform one to the other in a continuous fashion. Two continuous functions $f, g : X \to Y$ are **homotopic**, written $f \simeq g$, if there exists a continuous map $H : [0,1] \times X \to Y$, such that for all $x \in X$, $H(0,x) = f(x)$, and $H(1,x) = g(x)$. We say that H is a homotopy. H is a homotopy **relative to** $A \subset X$ if $H(t,a) = a$ for all $a \in A$. If H exists, $f \simeq_A g$. Homotopy forms an equivalence relation on continuous maps, which we use to form the homotopy category **Toph**, which is obtained as a quotient category of **Top**.

[1] From now on, we shall assume all functions continuous.

Example. *Any two $\mathbf{R}^n$-valued functions f and g defined on the same domain are homotopic. The map*

$$H(t,x) = tf(x) + (1-t)g(x)$$

is a homotopy between f and g. Thus the spaces $\mathbf{R}^n$ are the terminal objects in the homotopy category.

Two spaces are 'homotopy isomorphic' if they are isomorphic in **Toph** – that is, X is homotopic to Y, written $X \simeq_h Y$ if there is a map $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f$ and $f \circ g$ are homotopic to the identities on X and Y. g is known as the **homotopy inverse** of f. Since each $\mathbf{R}^n$ is a terminal object, they are all homotopy equivalent.

Example. *The n-sphere S^n is homotopy equivalent to $\mathbf{R}^{n+1} - \{0\}$. The embedding $i : S^n \to \mathbf{R}^{n+1} - \{0\}$ has a homotopy inverse $f : \mathbf{R}^{n+1} - \{0\} \to S^n$, defined by*

$$f(v) = v/\|v\|$$

One just deforms the norm to show that $f \circ i$ is homotopic to id_{S^n}, and expands the norm to show $i \circ f$ is homotopic to $id_{\mathbf{R}^n}$.

A retraction is a map $r : X \to X$ for which $r^2 = r$. If $Y = \mathrm{im}(X)$, and r is a homotopy isomorphism, then we say X **deformation retracts** to Y. The deformation retraction is **strong** if $X \simeq_Y Y$. We showed that $\mathbf{R}^{n+1} - \{0\}$ strong deformation retracts to S^n. A space is **contractible** or **null-homotopic** if it is homotopic to a point.

Example. *A subset X of a vector space is* **star-shaped** *if there is $x \in X$ such that if $y \in X$, the line segment between x and y is contained in X. Then X is (strongly) contractible to $\{x\}$, by the map*

$$H(t,y) = ty + (1-t)x$$

which shows every star shaped space is final.

Example. *Consider a connected graph Γ. Then Γ is a* **tree** *if there are no cycles: paths $(v_0, \dots, v_n)$ of distinct edges such that v_n connects to v_0. In a tree, fix a vertex v. For any vertex w, there is a unique path $(w, k_1, \dots, k_n, v)$ of distinct vertices connecting w and v. Take m to be the longest such length of a path. We may identify the path from w to v with the interval $[0, n+2]$, by taking* *a trajectory $c_w : [0, n+2] \to \Gamma$ travelling at uniform velocity from w to v, passing over each vertex at each integer mark. Extend c_w to $[0, \infty]$ by defining $c_w(n+2+t) = v$ for $t > 0$. Define $H_w : [0, m] \times (w, k_1, \dots k_n, v) \to \Gamma$ by*

$$H_w(t, c_w(u)) = c_w(t+u)$$

If w and u are vertices, then H_w agress with H_u on the intersection of their domain, so we may put all the maps together to obtain a homotopy between the identity and a point on Γ, showing Γ is contractible.

One may visualize homotopy for lower dimensional spaces in the following way. Let $f : X \to Y$ be a continuous map. Define the **mapping cylinder**

$$Z = (X \times [0,1]) \coprod Y/\sim$$

where $(x, 1) \sim f(x)$. If f is a homotopy equivalence, then Z retracts to both $\pi(X \times \{0\}) = \tilde{X}$ and $\pi(Y) = \tilde{Y}$. Thus two spaces are homotopic if and only if they are both deformation retracts of a bigger space.

The fact that homotopy is an equivalence relation will allow us to distill functions between spaces to their fundamental properties. We need to specialize our definition for it to be more of more use to us.

Definition. Two paths in X are path homotopic if they have the same start and end point, and are homotopic to each other.

Let f and g be two paths in X, where the end point of f is the start point of g. Then we may compose the two paths to form a new path $f * g$, defined by

$$(f * g)(x) = f(2x) : x \in [0, 1/2] g(2x-1) : x \in [1/2, 1]$$

By the pasting lemma, this function is a path which connects the start point of f to the end point of g. Unfortunately, concatenation is not associative, we do not have that $f*(g*h) = (f*g)*h$. These paths are homotopic to each other, however, and moving to path homotopy classes makes the definition much simpler.

Theorem 1.1. *Let f be path homotopic to f', and g path homotopic to g'. Then $f * g$ is homotopic to $f' * g'$.*

Proof. Let F be the path homotopy from f to f', and G the path homotopy from g to g'. Define a homotopy H between $f * g$ and $f' * g'$ by

$$H(\cdot, y) = F(\cdot, y) * G(\cdot, y)$$

More specifically

$$H(x,y) = \begin{cases} F(2x, y) & \text{if } x \in [0, 1/2] \\ G(2x-1, y) & \text{if } x \in [1/2, 1] \end{cases}$$

The pasting lemma guarentees this function is a new homotopy. $\square$

We now consider homotopy classes of paths, so when we talk about a path f, we are really talking about all paths homotopic to f.

Theorem 1.2. $[f] * ([g] * [h]) = ([f] * [g]) * [h]$.

Chapter 2

Appendix: CW complexes

We will not be able to approach algebraic topology on all the topological spaces, so we restrict ourselves to nice spaces. Manifolds are nice, but we can get away with a more general construction. These are the CW-complexes, described in detail in the appendix.

Start with a discrete set X_0. Then, inductively construct $X_{n+1} = X_n \cup_{f_n} Y_n$, where Y_n is the disjoint union of some number of unit disks $\mathbf{D}^n$ in $\mathbf{R}^n$, and $f_n : \partial Y_n \to X_{n-1}$ is a continuous attaching map. Then we have canonical maps $f_{ij} : X_i \to X_j$, for $i \leqslant j$. The direct limit

$$\varinjlim X_i = \left(\coprod_{i=1}^{\infty} X_i \right) /\sim$$

where $x \sim f_{ij}(x)$ for each $x \in X_i$, is known as a **CW Complex**. If the Y_i is empty for $i > n$, then $X_n \cong \varinjlim X_i$. If Y_n is the largest non-empty set, then we call the CW complex n-dimensional. More generally, we shall call anything homeomorphic to a CW complex a CW complex.

Example. *A* **graph** *is a 1-dimensional CW complex. It is the simplest example of a CW complex that isn't trivial. Each disk in Y_1 is attached between two points in X_0 to form X_1, and is known as an edge. The ends of an edgemay be attached to the same vertex.*

Example. *The n-sphere S^n is a CW complex. We take $X_0 = \{x_0\}$, and attach $\mathbf{D}^n$ via the trivial map $f : \mathbf{D}^n \to \{x_0\}$. This follows because $S^n \cong \mathbf{D}^n/\partial\mathbf{D}^n$.*

Example. *Real projective space* $\mathbf{RP}^n$ *is a CW complex, which consists of the quotient space of all lines through the origin in Euclidean space.*

$$\mathbf{RP}^n = (\mathbf{R}^{n+1} - \{0\})/(x \sim \lambda x : \lambda \in \mathbf{R} - \{0\}, x \in \mathbf{R}^{n+1} - \{0\})$$

The space may also be described as

$$\mathbf{RP}^n = S^n/(x \sim -x : x \in S^n)$$

First, we notice that we may throw away half the points on the sphere, so that only the top half needs to be considered. Flattening this half sphere, we obtain that

$$\mathbf{RP}^n = \mathbf{D}^n/(x \sim -x : x \in \partial\mathbf{D}^n)$$

But $\partial\mathbf{D}^n \cong S^{n-1}$, *and* $\mathbf{RP}^{n-1}$ *is obtained from* S^{n-1} *by attaching opposite points, so essentially*

$$\mathbf{RP}^n = \mathbf{D}^n \coprod_f \mathbf{RP}^{n-1}$$

where $f : \partial\mathbf{D}^n \to \mathbf{RP}^{n-1}$ *is just the projection map onto the quotient. Since* $\mathbf{RP}^1 \cong S^1$ *is a* 1*-dimensional CW complex, by a recursive construction,* $\mathbf{RP}^n$ *is obtained from an* $n-1$ *dimensional CW complex by attaching a single* n*-dimensional unit disk. It may be of interest to take this to the extreme, and consider*

$$\varinjlim R\mathbf{P}^n = \mathbf{RP}^\infty$$

This CW complex can be seen as the set of lines in $\mathbf{R}^\infty$ *through the origin.*

Example. *One can also consider complex projective space*

$$\mathbf{CP}^n = (\mathbf{C}^{n+1} - \{0\})/(x \sim \lambda x : \lambda \in \mathbf{C} - \{0\}, x \in \mathbf{C}^{n+1} - \{0\})$$

As with real projective space, we can flatten out the quotient to the sphere

$$\mathbf{CP}^n = S^{2n+1}/(x \sim \lambda x : |\lambda| = 1, x \in S^{2n+1})$$

One throws away duplicated points to obtain that the space is really

$$\mathbf{CP}^n = \mathbf{D}^{2n}/(x \sim \lambda x : x \in \partial\mathbf{D}^{2n}, |\lambda| = 1)$$

But, as with the real case, we can write $\mathbf{CP}^n = \mathbf{D}^{2n} \coprod_f \mathbf{CP}^{2n-1}$, *where the map* $f : \partial\mathbf{D}^{2n} \to \mathbf{CP}^{2n-1}$ *is just the projection, since* $\partial\mathbf{D}^{2n} = S^{2n-1}$. *We can then constructively build up a CW complex of* $\mathbf{CP}^1 \cong S^2$ *is a CW complex. It is interesting to note that we construct the CW complex* $\mathbf{CP}^{2n}$ *only using even dimensional disks.*

A **Subcomplex** of a CW complex X is a closed subspace A that is the union of some projections of cells in X. A pair (X, A), where A is a subcomplex of X, is known as a **CW pair**. Particular examples include $(\mathbf{C}P^i, \mathbf{CP}^j)$ and $(\mathbf{RP}^i, \mathbf{RP}^j)$, for $i > j$. S^i is a subcomplex of S^j if we give S^i a different CW structure, since S^{n+1} can be obtained from a CW complex for S^n by attaching two copies of $\mathbf{D}^n$ at the boundary, in the obvious way. We may then consider the sphere

$$S^\infty = \varinjlim S^n$$

and $\mathbf{RP}^\infty$ can be constructed from S^∞ in the obvious way.

The subcategory of CW complexes **CW** in **Top**