Topology

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September 21, 2016

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Chapter 1

There and Back Again

Metaphor is at the heart of modern mathematics. Given a complex abstract problem, it is difficult to see beyong the logic. Via abstraction, we obtain analogies laden with intuition. Why try and see an infinite dimensional vector space, when we can see a three dimensional cartesian space which allows us to 'see the problem'. In the context of number theory, linear algebra, and algebraic geometry, abstract algebra is the analogy constructor. In analysis, topology is king, generalizing that intuitive notion of space which encloses calculus, differential geometry, functional, real, and complex analysis under a single umbrella.

Initially, the idea of limits, continuity, and neighbourhood arose in geometry and in the physical sciences, where it connects to the idea of experimental approximation. Since most of the measurements ins ciences are obtained numerically, it is not a surprise that these topological facts mostly remained in the realm of real and complex sciences. However, in the beginning of the 20th century, a general description of these ideas was found from a diverse range of fields.

Example. *It is well understood what we mean when we say a sequence of numbers a_i converges to a point a. However, it becomes less clear what it means for a sequence of real valued functions f_i to converge to a function f. One definition is pointwise convergence: f_i converges to f if $f_i(x)$ converges to $f(x)$ for each x. But then the equality*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x) = \int_{-\infty}^{\infty} \lim_{i\to\infty} f_n(x)$$

need not hold, and does not even make sense in all cases. We could instead use uniform convergence: f_i converges to f if, for any ε, there is m such that for any $n > m$, and for any x, $|f_n(x) - f(x)| < \varepsilon$. If the f_i are integrable, and vanish outside a bounded region, then the equality above holds. Nonetheless, uniform convergence is much more restrictive than pointwise convergence, and neither is universally used over the other. It turns out that changing the definition of sequential convergence for functions corresponds to changing the topological structure of the space of all real-valued functions. Topology gives us the framework to naturally describe these kinds of structures on many classes of mathematical objects.

Example. *Consider the space of all loops in the plane. It is natural to identify a loop with one of its parameterizations λ. We can then use the above definitions of convergence for functions to decide when loops converge to one another, and we ask what conditions on the convergence properties of λ_n allow us to conclude*

$$\lim_{n \to \infty} \int_{\lambda_n} f(z)\, dz = \int_{\lambda} f(z)\, dz$$

a relationship with useful applications in complex analysis

Topology does not just have applications to the interchange of limit processes, but seems to be integral to any part of mathematics which involves 'the infinite'.

Example. *In recreational mathematics, one deals with 'rubber sheet' geometry problems. Given two shapes, if they were made of rubber would it be possible to stretch one shape into the other shape, without tearing or making holes in the objects. A rigorous mathematical definition of 'stretching' most naturally emerges from topology by the definition of a homeomorphism.*

Example. *The central theorem of Galois field theory is that the subgroups of automorphisms of a finite Galois field extension F/E are in a natural bijection with subextensions $F/K/E$ of fields. However, this correspondence does not seem to hold if we allow F/E to be infinite – there tends to be more subgroups then field extensions. Topology provides a 'finiteness' condition, known as compactness, which allows us to specify which groups correspond to subextensions, generalizing the fundamental theorem of Galois theory.*

Seeing as how general topology arose from a diverse range of fields, it is unsurprising to see that there are many definitions which specify the object studied in the field of topology. The following definition is due to the German mathematician Georg Cantor. Though it is not the only way to define what we mean by a topology, it is by far the most common.

1.1 The Topological Space, and other delights

Just as how numbers and rotations are the primary examples which abstract algebra generalizes, the primary example of a topological space, from which all the terminology is obtained, is the Euclidean plane. I'm sure that, at one point in your childhood, you owned a colouring book. In this tome of black and white, you delighted in colouring between in the lines which define a figure. Shapes emphasized by a black silouhette are easily distinguished and filled by a nice blue, green, or pink fluorescent marker. Mathematically, we distinguish between shapes which contain their silouhette and shapes that do not. Those shapes that do contain their 'boundary', as it is formally called, are called closed. The shapes that contain none of their boundary are open. Consider the interior of a circle (not including an outline). At any point on this shape, there is an infinitude of points from your pen tip to the edge of the circle – you'll never finish precisely coloring the shape in, if you don't include the boundary; you'll either overshoot by a tiny bit if you try and get close to the boundary, or undershoot. Sets which do not contain their boundary are known as open. The foundations of topology are obtained by abstracting the qualities that these 'edgeless' shapes possess.

A **topological space** is a set X, together with a **topological structure** Ω, a designated collection of subsets, known as *open sets*, satisfying

1. Both $\varnothing$ and X are open.

2. If $\{A_i\}_{i\in I}$ is an arbitrary family of open sets, then their union $\bigcup_{i\in I} A_i$ is also open.

3. If $\{A_i\}_{i=1}^n$ is a finite collection of open sets, then their intersection $\bigcap_{i=1}^n A_i$ is also open (it is sufficient to show that $A \cap B$ is open if A and B are open).

You should learn the definition of open-ness by rote. It is the most crucial idea in topology; everything else we shall deal with is defined in terms of open sets and the qualities they possess amongst themselves. Right now, it is an abstract axiomatic set of principles, but as we study topology, an intuitive spatial relationship will emerge.

Together with the definition comes an onslaught of terminology: to learn a field, you must first speak its language. All terminology will be expanded fully later on. This is just to provide an easy reference.

- Elements of a topological space are called **points**.
- A subset is **closed** if its complement is open.
- A **neighbourhood** of a point $x \in X$ is a set enveloping an open set containing x.[1]
- A **limit point** of a subset U of X is a point p such that every open set containing p contains points in $U - \{p\}$. The set of limit points of a subset U is denoted U', and is called the derived set of U.
- If we have two topological structures Δ and Ω, then we say Δ is **finer** than Ω, or Ω is coarser, if $\Omega \subset \Delta$. Spaces are comparable in the same way that rocks, pebbles, and grains of sand are.
- A function $f : X \to Y$ between two topological spaces is **continuous** if $f^{-1}(U)$ is open for every open set U in Y. A homeomorphism is a continuous bijective function whose inverse is continuous.

Topology also defines additional **separation axioms** which ensure your spaces are more reasonable. If your space does not have at least the very low level properties, then be wary for non-intuitive spatial structure.

- A topological space is **Kolmogorov**, or **T0**, if, for any points $x \neq y$, either x has a neighbourhood not containing y, or y has a neighbourhood not containing x. Almost every Topological space 'occuring in real life' has this property, and if it doesn't there is normally a trick, described later, which allows us to remove this property.

[1] Some mathematicians enforce a neighbourhood itself to be open. Normally, this does not cause issues, but when reading other works of topology, make sure you remember which definition the author is using.

- A topological space is **Frechet**, or **T1** if, for any points $x \neq y$, we may find a neighbourhood of x not containing y. Equivalently, a **Frechet** space is one in which points are closed.

- A topological space is **Hausdorff**, or **T2** if, for any points $x \neq y$, we may find disjoint open sets U and V such that $x \in U$, $y \in V$. This is the property which represents 'unique convergence' of sequences, which we shall discuss later. Spaces which are Hausdorff are the 'normal' spaces in topology. Some topologists consider the Hausdorff property in the definition of a topological space, to avoid pathological examples.

- A topological space is **Regular Hausdorff**, or **T3**, if it is Hausdorff, and when C is a closed set not containing a point x, we may find disjoint open sets U and V, such that $C \subset U$, and $x \in V$.

- A topological space is **Normal Hausdorff**, or **T4**, if it is Hausdorff and if two disjoint closed sets C and D can be spearated by disjoint open neighbourhoods U and V, such that $C \subset U$, $D \subset V$. Normal spaces are useful because they are useful for constructing families of continuous functions on spaces – if C and D are disjoint and closed in a normal space X, there exists a continuous function $f : X \to \mathbf{R}$ such that $f(c) = 0$ for $c \in C$, $f(d) = 1$ for $d \in D$.

We defined openness as above because these properties hold on the Euclidean plane (that is, for the 'coloring book topology'). We define a shape to be open if we can never colour up to an edge. Precisely, A shape (a subset of the plane) is open if we may draw a circle around every point in the shape, and all points in the interior of the circle are contained in the shape itself. Intuition should tell you why the properties of open sets hold for this space – we shall define this precisely later. In this topology, the finite assumption of open intersections is crucial. If we take the intersection of an infinite number of open sets, we may no longer have an open set – the boundary may have been stretched too thin. Consider the intersection of the 'infinite venn diagram below'.

INFINITE CIRCLES INTERSECTING ONLY AT A POINT.

Let us define the Euclidean topology precisely.

Example. *The Euclidean topology on* $\mathbf{R}^2$ *is defined such that a set* A *is open if it contains open circles around every point. Precisely, if* $x \in A$*, there is a* $\varepsilon > 0$ *such that the circle*

$$B_\varepsilon(x) = \{y \in \mathbf{R}^2 : d(x,y) < \varepsilon\}$$

is a subset of A*, where* d *is the function measuring the distance between two points. If we swap* 2 *with* n*, we obtain the Euclidean topology in any dimension. These are the standard topologies whenever the real line, plane, and higher dimensional analogues are mentioned.*

Given any set X there are two spatial extremities we can use to form a topology. We either choose the minimal number of open sets, or the maximal number. The 'discrete topology' on X lets the topological structure Ω be equal to the power set $\mathcal{P}(X)$. In this case, every subset of the plane is open, and the axioms of trivially satisfied. The 'lumpy topology' on X has minimum structure, with topological structure $\Omega' = \{\emptyset, X\}$. The fact that these are topologies is verified by trivial set theory. The discrete topology is named based on how closeness factors into topological spaces. 'limit points', as we have defined them above, indicate points which reach infinitely close to a given set. This is as close to distance as we can get in topology – we cannot tell if two points are miles or metres away, but only if they are infinitisimally close. The lumpy topology is named because topologically, every point is a limit point of every set (except in the case where the set is empty).

Example. *Let* X *be the ray* $[0,\infty)$*, and let* Ω *consist of* $\emptyset$*,* X*, and all rays* (a,∞) *with* $a \geqslant 0$*. Then* Ω *is a topological structure, which can only distinguish a point from another point if it lies further down the plane (but not the converge).*

Example. *Let* X *be a plane. Let* Σ *consist of* $\emptyset$*,* X*, and all open disks with center at the origin. This is a topological structure, which only measures convergence to the origin, but cannot distinguish any other point.*

The most important example of a topological space is $\mathbf{R}$, the space of real numbers. Many other examples will stem from this space. The topological structure consists of the empty set, and unions of intervals (a,b), where $a < b$ can be infinite. We have $\mathbf{R} = (-\infty,\infty)$. Its a bit fiddly, but you should be able to show that the intersection of two intervals is the union of intervals, and thus that the intersection property holds in general. This topology is the standard topology on $\mathbf{R}$.

Example. *Show that every open set in* $\mathbf{R}$ *can be broken into the disjoint union of open intervals. Bonus points: Show that this union is countable (every interval contains a rational number)!*

Closed sets of $\mathbf{R}$ are not so simple to classify. Later on, we will see the Cantor set, a very strange closed set of $\mathbf{R}$ defying expectations of what a closed set really looks like.

Example. *Let X be a set, and let Ω consist of all subsets of X whose complement is finite, and $\emptyset$. Show that Ω is a topological structure, called the cofinite topology. It is natural to study this topology in the study of algebraic topology, which studies zero sets to algebraic curves. If we define a set to be closed if it the zero set of a one-dimensional polynomial, then the resulting topology is exactly the cofinite topology.*

1.2 Closed topological sets

A set in a topological space X is closed if it is the complement of an open set. A closed interval $[a,b]$ is closed in $\mathbf{R}$, since $[a,b]^c = (-\infty,a) \cup (b,\infty)$. To specify the axioms of a topology, we could have taken as a primitive notion closedness instead of openness. Since

$$\Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c$$

$$\Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c)$$

we could have taken a topology as a collection of closed sets such that arbitrary intersections and finite unions of closed sets are closed.

Example. *Prove that $[0,1)$ is not closed nor open in* $\mathbf{R}$, *yet it is the countable union of closed sets and the intersection of open ones. The former type is known as an F_σ sets (σ stands for sum), and the latter G_δ (δ for durchschnitt, the german word for intersection).*

Example. *Prove that $\{0\} \cup \{\frac{1}{n} : n \in \mathbf{Z}\}$ is closed in* $\mathbf{R}$.

Closed sets are seen as 'containing the points' infinitely close to each other. Here we establish what exactly this means. Recall that x is a limit point of a set A if every open set containing x contains some point in A other than x. Denote the set of limit points by C'.

Example. *C is a closed set if and only if the set C' is a subset of C.*

Proof. Let C be a closed set. Then C^c is open, so if x is not in C, x is not a limit point of C. We have shown what we needed to show. Now suppose C' is contained in C, where C is an arbitrary set. If x is not an element of C, then it follows that there is an open set U_x containing x not containing any point in C. The union

$$\bigcup_{x \notin C} U_x$$

is an open set containing no elements of C, and any point not in C. Thus the union is just C^c, and we have shown C^c is open, hence C is closed. $\square$

1.2.1 An excursion – Cantor's set

This excursion shows us that weird closed sets exist even in the most simple topologies – here the real numbers $\mathbf{R}$.

Example. *Consider the Cantor set K defined below:*

$$K = \left\{ x \in \mathbf{R} : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ with } a_k \in \{0, 2\} \right\}$$

Then K can be inductively defined as $\bigcap_{i=1}^{\infty} K_i$, where $K_0 = [0, 1]$, and

$$K_{i+1} = \frac{K_i}{3} \cup \left(\frac{2}{3} + \frac{K_i}{3} \right)$$

Then K is closed, because $C/3$ and $\frac{2}{3} + C/3$ are closed when C is closed.

Proof. Let us show first that $K \subset \bigcap_{i=1}^{\infty} K_i$. Here is an arbitrary decimal expansion in the Cantor set

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where each $a_k = 0$ or 2. Then $x \in K_0 = [0, 1]$, since

$$0 = \sum_{k=1}^{\infty} \frac{0}{3^k} \leqslant \sum_{k=1}^{\infty} \frac{a_K}{3^k} \leqslant \sum_{k=1}^{\infty} \frac{2}{3^k} = 2(\frac{1}{1 - 1/3} - 1) = 1$$

For an induction, assume $K \subset K_n$. Suppose $a_1 = 0$. Then

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} \cdots = \frac{1}{3}(a_2 + \frac{a_2}{3} + \dots) = y/3$$

where $y \in K$, so $y \in K_n$ by hypothesis. Then $x \in K_n/3 \subset K_{n+1}$. If $a_1 = 2$,

$$x = 2/3 + \frac{1}{3}(\frac{a_2}{3} + \frac{a_3}{3^2} + \dots) = 2/3 + y/3$$

where $y \in K$. Therefore $y \in K_n$, and $x \in 2/3 + K_n/3 \subset K_{n+1}$. This exactly proves that $K \subset K_{n+1}$. By induction, we have shown $K \subset \bigcap_{i=1}^{\infty} K_i$.

Now we must show, conversely, that $\bigcap K_i \subset K$. Let $x \in \bigcap K_i$. We will build a correct sequence of digits that converge to x. Let $x_0 = 0$, and

$$x_n = \max \left\{ a_k \in \{0,2\} : \sum_{k=0}^{n-1} x_k/3^k + a_k/3^n \leqslant x \right\}$$

Our first claim:

$$\sum_{k=0}^{\infty} x_k/3^k = x$$

We shall prove by induction that if $x \in K_n$, then $x - \sum_{k=1}^{\infty} x_k/3^k < 1/3^n$, If x is in all n, then $x - \sum_{k=1}^{\infty} x_k/3^k$ must be equal to zero, since it is smaller than $1/3^n$ for all n, and greater than or equal to zero. This is trivially true for K_0, since any sum is between 0 and 1, and x is also between 0 and 1. Assume this is true for all $y \in K_n$. Consider $x \in K_{n+1}$. Assume that there is $y = 3x$ in K_n. Then $x_1 = 0$ (since otherwise $x > 1/3$, and $y > 1$), and if we consider the expansion of y, we will see that $y_n = x_{n+1}$. This follows because

$$\sum_{k=1}^{n-1} y_k/3^{k+1} + a/3^n = \sum_{k=1}^{n} x_k/3^k + a/3^{n+1} \leqslant x$$

holds if and only if the inequality

$$3\left(\sum_{k=1}^{n-1} x_k/3^k + a_k/3^n\right) = \sum_{k=1}^{n-1} y_k/3^k + a_k/3^n \leqslant 3x = y$$

holds, assuming for an induction that $y_k = x_{k+1}$ for $k \leqslant n-1$. If

$$y - \sum_{k=1}^{\infty} y_k/3^k < 1/3^n$$

then $y/3 - \sum_{k=1}^{\infty} y_k/3^{k+1} = x - \sum_{k=1}^{n} x_k/3^k < 1/3^{n+1}$. If $x = 2/3 + y$, where $y \in K_n$, the same technique establishes the inequality. Our claim is thus proved. $\square$

A subset Y of a topological space X is dense if $\overline{Y} = X$. This means that any element of X can be approximated by an element in Y. The cardinality of a dense subset naturally corresponds to the cardinality of a basis for the space, because we can obtain a dense subset by picking an element of every open set.

1.3 The Basis of a space

Normally, a topology is not given via specifying every single open set in the topology. Since open sets are constructed from other open sets, we may specify some archetypal sets, and provided these sets are sufficient to describe a topology, define a topology in terms of them.

Let X be a set, and $\{\Omega_i\}_{i \in I}$ a family of topological structures on X. Consider the intersection of all structures,

$$\Delta = \bigcap_{i \in I} \Omega_i$$

Surely $\varnothing$ and X itself are elements of Δ, since they are an element of each Ω_i. If $\{A_i\}_{i \in I} \in \Omega_i$ for each i, then surely $\bigcup_{i \in I} A_i \in \Omega_i$, so that if $\{A_i\}_{i \in I} \in \Delta$, we also have $\bigcup_{i \in I} A_i \in \Delta$. Similarily, if $\{A_i\}_{i \in I} \in \Delta$, then $\bigcap_{i=1}^{n} A_i \in \Delta$, so that Δ is a topological space. This fact will allow us to generate structures from generating sets, as is done in many areas of mathematics.

If X is a set, and D is a family of subsets, then we may consider

$$\Delta = \{\Omega \in \mathcal{P}(X) : D \subset \Omega \text{ and D is a topological space}\}$$

Taking $\Phi = \bigcap \Delta$, we obtain the coarsest topological space containing D, called the topology generated by D. If Ω is a topological space containing D, then $\Phi \subset \Delta$ also, by the construction above.

When D has certain nice properties, the topological space generated by D is much simpler to work with:

Definition. Let D be a subset of a topological space X. Suppose that

1. $\varnothing, X \in D$.
2. If A and B are in D, and $A \cap B$ is non-empty, containing a point x, then there is a set C in D containing x with $C \subset A, B$.

In this case, the topological space Φ generated by D has a following property. A set U is open in Φ if and only if it is the union of sets in D. We call D a **basis** for the topology Φ if it satisfies properties (1) and (2), and a **subbasis** if it is just a generating set.

Exercise 1.1. *Can two distinct topological structures have the same base? That is, does the base of a topology uniquely define a topology.*

Proof. No. Let X be a set, and Δ, Ω two topological structures with the same base. Then Δ must be finer than Ω, since Δ is the minimal structure containing the base. Similarily, Ω must be finer than Δ, so we conclude the two structures are equal. □

Exercise 1.2. *Prove that there is no minimal topological basis for* $\mathbf{R}$.

Proof. Let D be a basis for $\mathbf{R}$, and let $U \in D$ be a set in D not equal to $\varnothing$ or X, which is therefore open in $\mathbf{R}$ and contains an interval (a,b). Consider $D - U$. The first property of a basis is satisfied, and if x is contained both in two subsets A and B in D, and if it were the case that $x \in U \subset A \cap B$ (without loss of generality, $x \in (a,b)$), then since $(a+\varepsilon, b-\varepsilon)$ is the union of sets in D, there must be some $U' \in D - U$ containing x, since $(a+\varepsilon, b-\varepsilon) \subsetneq U$. Thus $D - U$ is a basis, and generates a topological space. Obviously, D is finer than $D - U$. Let $U = \bigcup (a_i, b_i)$. There exists δ such that, for each $0 < \varepsilon < \delta$, $(a_i + \varepsilon, b_i - \varepsilon)$ is open in $\mathbf{R}$ and is not equal to U, so $(a_i + \varepsilon, b_i - \varepsilon) = \bigcup U_i$, for some $U_i \in D - U$. But then

$$U = \bigcup (a_i, b_i) = \bigcup_{i \in I} \bigcup_{\delta > \varepsilon > 0} \bigcup_{j \in J_i} U_{i,\varepsilon,j}$$

And U is still open in the topology generated by $D - U$. □

A basis for a discrete space X is just X itself, since any open set in a discrete space is the union of some points in X. A basis for $\mathbf{R}$ is the set of all intervals (a, b).

Exercise 1.3. *Show that two bases D and D' generate the same topological structure if every element of D' is the union of D, and vice versa.*

1.4 Topology and Convergence

Analysis takes topology and uses it to study limit operations. For instance, when we initially began studying the topology of $\mathbf{R}$, we defined convergence (with ε's and δ's), and then proceeded to define open and closed sets as a corollary. Most of the applications of topology to analysis deal with convergence, so it is natural to wonder whether it is possible to define all topologies by the convergent sequences that result.

Definition. Let Ω and Ψ be two topologies on a single set X. We will say Ω and Ψ are **sequentially equivalent** when the sequences that converge in (X, Ω) are exactly the same as those in (X, Ψ), and to exactly the same points.

Does it follow that, when two topologies Ω and Ψ are sequentially equivalent, $\Omega = \Psi$? The next example shows this is, unfortunately, not the case.

Example. *Let X consist of all countable ordinals, together with the first uncountable ordinal, denoted ω_1. Let Ω be the order topology on X, and let Ψ be the topology generated by $\Omega \cup \{\{\omega_1\}\}$. Surely $\Psi \neq \Omega$, since $(\omega, \omega_1] \neq \{\omega_1\}$ for any choice of ω, yet Ω is sequentially equivalent to Ψ. Take any sequence of ordinals $\{x_i\}_{i \in \mathbf{N}}$ in X. Since the relative topologies on $[0, \omega_1)$ generated by Ω and Ψ are the same, if $x < \omega_1$, then $x_i \to x$ in (X, Ω) if and only if the sequence converges to x in (X, Ψ). Suppose that $x_i \to \omega_1$ in (X, Ψ). This means precisely that there is some $m \in \mathbf{N}$ such that, for $n > m$, $x_n = \omega_1$. Suppose that this is not true of some sequence $\{x_i\}_{i \in \mathbf{N}}$, that is, $x_k \neq \omega_1$ for arbitrarily large integers k. Then we may select some subsequence $\{y_i\}_{i \in \mathbf{N}}$, such that $y_k \neq \omega_1$ for any k.*

But then, as the countable union of countable ordinals, $\bigcup_{i\in\mathbf{N}} y_i$ is countable, so $\omega_1 > \bigcup_{i\in\mathbf{N}} y_i \geqslant y_k$, for any k, and thus $y_i \not\to \omega_1$ in Ω, hence $x_i \not\to \omega_1$.

Topologists were exiled from sequential paradise as soon as arbitrary unions of open sets were allowed to be open, since this allowed uncountability to enter into topology. It is still possible to find salvation, nonetheless. One solution is to remain in a class of topologies which are determined by their convergent sequences.

Definition. Let (X,Ω) be an arbitrary topological space. We say $U \subset X$ is **sequentially open** if every sequence which converges to $x \in U$ is eventually in U. We say $C \subset X$ is **sequentially closed** if it contains all sequential limits. A **sequential space** is a space (X,Ω) where sequentially open sets are open, or equivalently, if all sequentially closed sets are closed.

Lemma 1.1. *Every first-countable space is sequential.*

Proof. Let (X,Ω) be a first countable space. Suppose C is a sequentially closed set. Fix some limit point x of C. Let $\{V_i\}_{i\in\mathbf{N}}$ be a countable neighbourhood base of x. Let $W_1 = V_1$, and $W_{n+1} = W_n \cap V_{n+1}$. Then $\{W_i\}_{i\in\mathbf{N}}$ is a countable neighbourhood base such that $i < j$ implies $W_i \supset W_j$. Since x is a limit point of C, $W_i \cap C$ is non-empty, for any i. We may therefore define a choice function $s : \mathbf{N} \to C$ such that $s_i \in W_i \cap C$ for any integer i. By construction, $s_i \to x$, so $x \in C$. It follows that C is closed. $\square$

Corollary 1.2. *Every metric space is sequential.*

Exercise 1.4. *Let X be an arbitrary set, and (X,Ω), (X,Ψ) two sequential spaces. Ω and Ψ are sequentially equivalent if and only if $\Omega = \Psi$.*

1.5 Nets

In functional analysis, topologies are most naturally formed by describing the convergent sequences in the topology. Unfortunately, the cardinality of some spaces can be too big, so that sequences, being countable, are not fine enough to get into the 'cracks' of the space. The trick to fixing this is to allow 'uncountable sequences', which are large enough to completely describe the topology of a space.

Recall that a **directed set** is a preordering $(A, \leq)$ such that, for any two $x, y \in A$, there exists $z \in A$ such that $x, y \leq z$. A **net** or **Moore-Smith sequence** is a function from a directed set to an arbitrary set. We shall often denote the value of a net $\mathfrak{a} : A \to Y$ at $\alpha \in A$ by $\mathfrak{a}_\alpha$, to mimic sequential notation. Nets behave like sequences, except the directed set can have an arbitrarily large cardinality, allowing them to describe spaces which are not topologically 'countable'.

Example. $\mathbf{N}$ *is a directed set with the standard ordering. A sequence is just a net whose domain is* $\mathbf{N}$.

Example. $\mathbf{R}$ *is a directed set. A net defined on* $\mathbf{R}$ *is very strange – there are (uncountably) infinitely many points before and after any point in the domain of the net.*

Example. $K = \{0, 1, 2\}$ *is a directed set. A net defined on K is very strange – there are only three elements in these 'sequences'!*

Topological spaces are normally defined using open sets. The main connection between open sets and convergent nets occurs in the following construction of a net.

Example. *Let x be a point in a topological space, and let $\mathcal{U}$ be the collection of all neighbourhoods of x. Define $U \leq V$ if $V \subset U$. Then $\mathcal{U}$ forms a directed set, since if U and V are arbitrary neighbourhoods of x, $U, V \leq U \cap V$.*

Let X be a topological space. A net $\mathfrak{a} : A \to X$ is **eventually** in a set $Y \subset X$ if there is $\alpha \in A$ such that, for $\beta \geq \alpha$, $\mathfrak{a}_\alpha \in Y$. A net is **frequently** in a set Y if for any α, there is $\beta > \alpha$ for which $\mathfrak{a}_\beta \in Y$. $\mathfrak{a}$ **converges** to a point $a \in X$ if it is eventually in every neighbourhood of a. We need only check this for every element of a basis, or even just for elements of a subbasis.

Theorem 1.3 (Monotone Convergence Theorem)**.** *If $\mathfrak{a} : A \to \mathbf{R}$ is non-decreasing, then $\mathfrak{a} \to \sup\{\mathfrak{p}_\alpha : \alpha \in A\}$, and if $\mathfrak{a}$ is non-increasing, then $\mathfrak{p} \to \inf\{\mathfrak{p}_\alpha : \alpha \in A\}$.*

Proof. We shall prove the non-decreasing case. Let $C = \sup\{\mathfrak{a}_\alpha : \alpha \in A\}$. If $C < \infty$, for each $\varepsilon > 0$, pick $\alpha \in A$ with $C - \varepsilon < \mathfrak{a}_\alpha \leqslant C$. If $\beta > \alpha$, then

$$C - \varepsilon < \mathfrak{a}_\alpha < \mathfrak{a}_\beta \leqslant C$$

So $\mathfrak{a}$ is eventually in $(C-\varepsilon, C+\varepsilon)$. If $C=\infty$, one verifies that $\mathfrak{a}$ is eventually in $(N,\infty]$, for any N, so that $\mathfrak{a}\to\infty$. This theorem is easily seen to generalize from nets into arbitrary ordered sets containing suprema and infima. $\square$

Example. *Calculating the Riemann integral can be seen as finding the limit of a certain net. Let f be a real-valued function defined on an interval $[a,b]$. Consider the set of all possible finite partitions of $[a,b]$ - that is, finite increasing sequences $(P_1,\dots,P_n)$ where $P_1=a$ and $P_n=b$. If P and Q are two partitions, we define $P\leq Q$ to mean every point in P is also in Q. Define two nets on this directed set:*

$$\mathbf{L}(P_1,\dots,P_n)=\sum_{i=1}^{n}(P_{i+1}-P_i)\inf\{f(x):x\in[P_i,P_{i+1}]\}$$

$$\mathbf{U}(P_1,\dots,P_n)=\sum_{i=1}^{n}(P_{i+1}-P_i)\sup\{f(x):x\in[P_i,P_{i+1}]\}$$

Both nets are monotone, hence they converge to some extended real value:

$$\mathbf{L}\to\mathbf{L}\int_a^b f\qquad\qquad \mathbf{U}\to\mathbf{U}\int_a^b f$$

We say f is integrable if $\mathbf{L}\int_a^b f=\mathbf{U}\int_a^b f$, and define the shared value to be the integral of f, denoted $\int_a^b f$. Thus we see that properties of Riemann integration are easily obtained from properties of nets.

A technical result is often useful when calculating limits.

Theorem 1.4 (Iterated Limits)**.** *Let $\mathfrak{b}:D\to X$ be a net converging to b, and for each $\alpha\in D$, let $\mathfrak{a}^\alpha:E_\alpha\to X$ be a net converging to $\mathfrak{b}_\alpha$. Then the net $\mathfrak{a}:D\times(\bigtimes_\alpha E_\alpha)\to X$ defined by*

$$\mathfrak{a}_{(\alpha,v)}=\mathfrak{a}^{\alpha}_{v_\alpha}$$

also converges to b, where $(\alpha,v)\leqslant(\beta,w)$ if $\alpha\leqslant\beta$ and $v_\lambda\leqslant w_\lambda$ for each λ.

Proof. Fix a neighbourhood U of b, and choose β such that for $\alpha>\beta$, $\mathfrak{b}_\alpha\in U$. Then U is a neighbourhood of $\mathfrak{b}_\alpha$, and so there is λ_α such that for $\gamma>\lambda_\alpha$, $\mathfrak{a}^\alpha_\gamma\in U$. If we let $w=\{\lambda_\alpha\}$, then $\mathfrak{a}_{(\beta,w)}\in U$ for $(\beta,w)>(\alpha,v)$. $\square$

The corresponding theorem about sequences is that if $x_i \to x$, and for each i, we have a sequence $y(i)_j$ converging to x_i, then there is an increasing subsequence of the integers k_i for which the sequence $y(i)_{k_i}$ converges to x. It turns out that nets can describe sufficiently all topological properties of space, so that we may define topological spaces in terms of nets.

Lemma 1.5. *Any limit point of a set A is the limit of a net.*

Proof. If a is a limit point, for every neighbourhood U of a, $U \cap A$ is non-empty. Therefore, we may define a choice function $\mathfrak{s}$ on the set of neighbourhoods $\mathcal{U}$ of a, such that $\mathfrak{s}(U) \in U \cap A$, for any neighbourhood U of a. We know $\mathcal{U}$ forms a directed set, so $\mathfrak{s}$ is a net, and the fact that $\mathfrak{s}$ converges to a is almost too obvious: if U is a neighbourhood surrounding a, then for $V \subset U$, $\mathfrak{s}(V) \in U$. $\square$

Corollary 1.6. *$C \subset X$ is closed if and only if nets valued in the set converge only to elements in the set.*

Corollary 1.7. *A set is open if and only if any net converging to a point in the set eventually ends up in the set.*

Suppose (X, Ω) and (X, Γ) are two topological spaces which possess the same convergent nets. Then $\Omega = \Gamma$, since open and closed sets may be identified by convergent nets.

Lemma 1.8. *$f : X \to Y$ is continuous if and only if, when $\mathfrak{a}$ converges to a, $f \circ \mathfrak{a}$ converges to $f(a)$.*

Proof. If f is continuous, and if U is a neighbourhood of $f(a)$, $f^{-1}(U)$ is a neighbourhood of a, so $\mathfrak{a}$ is eventually in $f^{-1}(U)$, and thus $f \circ \mathfrak{a}$ eventually in U. Conversely, let V be an arbitrary neighbourhood of $f(a)$. We must verify that $f^{-1}(V)$ is open. If $\mathfrak{a}$ converges to $v \in f^{-1}(V)$, then $f \circ \mathfrak{a}$ converges to $f(v)$, so $f \circ \mathfrak{a}$ is eventually in V, so $\mathfrak{a}$ is eventually in $f^{-1}(V)$. Thus $f^{-1}(V)$ is open. $\square$

Theorem 1.9. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then a net $\mathfrak{a}$ whose range lies in $\prod X_\alpha$ converges to $\{a_\beta\}$ if and only if $\pi_\alpha \circ \mathfrak{p}$ converges to a_α for each $\alpha \in A$.*

Proof. A subbasis for $\{X_\alpha\}$ consists of the family

$$\{\pi_\alpha^{-1}(U) : \alpha \in A, U \text{ open in } X_\alpha\}$$

If $\mathfrak{a}$ converges to $\{a_\beta\}$, then by continuity, for each α, $\pi_\alpha \circ \mathfrak{a}$ converges to

$$\pi_\alpha(\{a_\beta\}) = a_\alpha$$

Conversely, suppose $\pi_\alpha \circ \mathfrak{a}$ converges to a_α, for each α. Then $\pi_\alpha \circ \mathfrak{a}$ is eventually in each neighbourhood U of a_α, so $\mathfrak{a}$ is eventually in $\pi_\alpha^{-1}(U)$. This implies that $\mathfrak{a}$ is eventually in every element of a subneighbourhood basis of $\{a_\alpha\}$, so $\mathfrak{a}$ converges to $\{a_\alpha\}$. $\square$

Theorem 1.10. *A topological space is Hausdorff if and only if each net converges to no more than one point.*

Proof. If X is Hausdorff, and $x, y \in X$, $x \neq y$, there are disjoint open sets $U, V \in \Omega$ with $x \in U$, $y \in V$. If $\mathfrak{s} : D \to X$ is a net which converges to x, then $\mathfrak{s}$ is eventually in U, so $\mathfrak{s}$ can never be eventually in V, and thus cannot converge to y. Conversely, suppose X is not Hausdorff. Then there are x and y such that every neighbourhood of x is a neighbourhood of y, and vice versa. Let $\mathcal{U}$ be the set of neighbourhoods. If we define an arbitrary choice function $\mathfrak{s} : \mathcal{U} \to X$. Then $\mathfrak{s}$ is eventually in every neighbourhood of x and y, so $\mathfrak{s}$ does not converge to a unique point in X. $\square$

Let $\{x_1, x_2, \dots\}$ be a sequence. A subsequence is $\{x_{i_1}, x_{i_2}, \dots\}$, where $\{i_k\}$ is an increasing sequence of natural numbers. Thus we have $i_k \to \infty$. To work with subnets of nets, we must generalize the way we 'thin' a sequence. This generalization is necessary to find the net-theoretic description of compactness. A subset J of a directed set I is **cofinal** if, for any $\alpha \in I$, there is $\beta \in J$ with $\alpha \leq \beta$. Then J is also a directed set, and if $\mathfrak{s}$ is a net defined on I, then $\mathfrak{s}|_J$ is also a net. If $\mathfrak{s}$ converges to a point s in a topological space X, then it is easy to verify that $\mathfrak{s}|_J$ also converges to s. Cofinality is essential to ensure convergence.

Example. *Consider the directed set* $\mathbf{N} \cup \{\infty\}$, *and define a net*

$$\mathfrak{s}(x) = \begin{cases} x & x \in \mathbf{N} \\ 0 & x = \infty \end{cases}$$

Then $\mathfrak{s}$ *converges to* 0, *yet* $\mathfrak{s}|_{\mathbf{N}}$ *does not converge at all. This results because* $\mathbf{N}$ *is not cofinal in* $\mathbf{N} \cup \{\infty\}$.

In metric spaces, a set is compact if and only if every sequence contains a convergent subsequence. We would like to extend this definition to nets, so that a topological space is compact if and only if every net contains a convergent subnet. To do this, we cannot simply take cofinal restrictions, for there are examples of sequences in topological spaces which contain no convergent subsequence. This problem occurs because the ordering on a cofinal set $I \subset J$ is very restricted. If we desire $\mathfrak{s}|_I$ to converge to x whenever $\mathfrak{s}|_J$ converges to x, we probably want $x \preceq_I y$ to imply $x \preceq_J y$, but the converse is not completely necessary. This motivates the general definition of a subnet. Let $\mathfrak{a} : I \to X$ be a net. a **subnet** is a function $\mathfrak{b} = \mathfrak{a} \circ i$, where $i : J \to I$ is an order preserving map between two directed sets, such that $i(J)$ is cofinal in I.

Example. *We shall define a new ordering on* $\mathbf{N}$. *Define $x \preceq y$ if x and y are both even and $x \leqslant y$, or if x is odd and $y = x+1$. Then the identity preserves the new ordering $\preceq$ in the old ordering* $\mathbf{N}$, *and the range of the injection is cofinal (it is the whole set). If $\{x_i\}$ is a sequence, then, in the new ordering, $\{x_i\}$ is a subnet, which is certainly not a subsequence, for it converges to x if and only if the even values of i converge to x.*

Theorem 1.11. *A topological space is compact if and only if every net in that set has a convergent subnet.*

Proof. Define an accumulation point of a net $\mathfrak{a}$, valued on X, to be the set of all values $x \in X$ such that $\mathfrak{a}$ is frequently in every neighbourhood of x. Suppose that such a sequence $\mathfrak{a}$ has no accumulation points. For each x, we may then pick a neighbourhood U_x which contains no points in some tail of the sequence. Then U_x is a cover, and therefore possesses a finite subcover by the compactness of X, which is impossible because it implies that $\mathfrak{a}_\alpha \notin X$ for some tail $\alpha > \beta$. Thus every net in a compact set contains an accumulation point.

Now we show that for every accumulation point x of a net $\mathfrak{a}$, there is a subnet converging to x. Let J be the set of all pairs (α, U), where α is an index and U is a neighbourhood of x containing x_α. Define $(\alpha, U) \leqslant (\beta, V)$ if $\alpha \leqslant \beta$, $U \leqslant V$. Then the projection $(\alpha, U) \mapsto \alpha$ defines a subnet $\mathfrak{b}$. It is easily verified that $\mathfrak{b}$ converges to x, because if V is any neighbourhood of x, it contains a point x_α, and for $\beta > \alpha$, $W \subset U$, $\mathfrak{b}_{(\beta, W)} \in V$. $\square$

Now suppose we have a class $\mathcal{C}$ consisting of pairs (S, x), where S is a net from some directed set to a specified set X, and x a point in X. Let Ω

consist of all subsets U of X such that, if $x \in U$ and $(S,x) \in \mathcal{C}$, then S is eventually in U. Then Ω is a topology on X, and if $(S,x) \in \mathcal{C}$, $S \to x$ in Ω. Ω is in fact the finest topology in which these nets converge. Nonetheless, in the new topology, even more nets can still converge. It turns out that, under more assumptions, common to convergent nets on any topological space, we can make it so that the $(S,x) \in \mathcal{C}$ are the only nets which converge in our topology Ω.

A convergence class is a class $\mathcal{C}$ of pairs $(\mathfrak{s},x)$, where $\mathfrak{s}$ is a net from some directed set to a fixed set X, and $x \in X$, satisfying the following properties:

1. (Constant Convergence Property) If $\mathfrak{s}$ is a constant net, always valued at $x \in X$, then $(\mathfrak{s},x) \in \mathcal{C}$.

2. (Subnet Convergence Property) If $(\mathfrak{s},x) \in \mathcal{C}$, and $\mathfrak{t}$ is a subnet of $\mathfrak{s}$, then $(\mathfrak{t},x) \in \mathcal{C}$.

3. (Sequence Thinning Property) If $(\mathfrak{s},x) \notin \mathcal{C}$, there's a subnet $\mathfrak{l}$ of $\mathfrak{s}$ such that $(\mathfrak{r},x) \notin \mathcal{C}$ for any subnet $\mathfrak{r}$ of $\mathfrak{l}$.

4. (Iterated Convergence Property) Suppose $(\mathfrak{s},x) \in \mathcal{C}$, and for each $\alpha \in \mathrm{Dom}(S)$, we have a net $\mathfrak{a}^\alpha$ with $(\mathfrak{a}^\alpha, \mathfrak{s}_\alpha) \in \mathcal{C}$. Then $(\mathfrak{a},x) \in \mathcal{C}$, where $\mathfrak{a}$ is the net defined on $\mathrm{Dom}(S) \times \prod_\alpha \mathrm{Dom}(\mathfrak{a}^\alpha)$, ordered by $(\alpha,v) \leqslant (\beta,w)$ if $\alpha \leqslant \beta$ and $v_\alpha \leqslant w_\alpha$ for each α, defined by the equation $\mathfrak{a}_{\alpha,v} = \mathfrak{a}^\alpha_{v_\alpha}$.

These rules are sufficient to generate a unique topology.

Theorem 1.12. *There is a one-to-one correspondence between convergence classes on a set, and topologies on the same set whose convergent nets are exactly those that are members of the convergence class.*

Proof. Consider a convergence class $\mathcal{C}$, with nets with values in a set X, and define

$$\overline{A} = \{x \in X : (\mathfrak{s},x) \in \mathcal{C}, \mathrm{Im}(\mathfrak{s}) \subset A\}$$

We shall verify that this is a closure operator, defining a topology on X.

1. ($A \subset \overline{A}$): For $a \in A$, we define $\mathfrak{s} = (a,a,\dots)$, we find $(\mathfrak{s},a) \in \mathcal{C}$, so $a \in \overline{A}$.

2. If $A \subset B$, $\overline{A} \subset \overline{B}$: Nets taking values in A also only take values in B.

3. ($\overline{\overline{A}} = \overline{A}$): Since $A \subset \overline{A}$, $\overline{A} \subset \overline{\overline{A}}$. Now let $a \in \overline{\overline{A}}$. There is some net $\mathfrak{s}$ with $(\mathfrak{s}, a) \in \mathcal{C}$ only taking values in $\overline{A}$. This means that, for each α in $\mathrm{Dom}(S)$, there is $\mathfrak{a}^\alpha$ with $(\mathfrak{a}^\alpha, \mathfrak{s}_\alpha) \in \mathcal{C}$, only taking values in A. But then there is $(\mathfrak{a}, a) \in \mathcal{C}$, with $\mathfrak{a}_{(\alpha,v)} = \mathfrak{a}^\alpha_{v_\alpha} \in A$, so $a \in \overline{A}$.

We therefore obtain a topology on X. Let $(\mathfrak{s}, x) \in \mathcal{C}$, and suppose $\mathfrak{s}$ does not tend to x in the topology Ω, so there is some open neighbourhood U of x such that, for any α, there is $\beta > \alpha$ with $\mathfrak{s}_\beta \notin U$. The set of such β defines a cofinal set, and thus we gain a subnet $\mathfrak{s}'$, only taking values in U^c, and such that $(\mathfrak{s}', x) \in \mathcal{C}$. But then U^c cannot be closed.

Conversely, suppose $\mathfrak{s}$ converges to x in our new topology, but $(\mathfrak{s}, x) \notin \mathcal{C}$. Then there is a subnet $\mathfrak{l}$ of $\mathfrak{s}$ such that for any subnet $\mathfrak{r}$, $(\mathfrak{r}, x) \notin \mathcal{C}$. For each $\alpha \in \mathrm{Dom}(\mathfrak{l})$, let

$$B_\alpha = \{\beta \in \mathrm{Dom}(\mathfrak{l}) : \beta \geq \alpha\}$$

Then $x \in \overline{\mathfrak{l}(B_\alpha)}$, for each α, since $\mathfrak{l}|_{B_\alpha}$ converges to x, so there is a net $\mathfrak{a}^\alpha$ valued in $\mathfrak{l}(B_\alpha)$ such that $(\mathfrak{a}^\alpha, x) \in \mathcal{C}$. Then the net $\mathfrak{a}_{\alpha,v} = \mathfrak{a}^\alpha_{v_\alpha}$ satisfies $(\mathfrak{a}_{\alpha,v}, x) \in \mathcal{C}$. It turns out that $\mathfrak{a}$ is a subnet of $\mathfrak{l}$. For each (α, v), there is $\beta_{\alpha,v} \in B_\alpha$ such that $\mathfrak{l}_{\beta_{\alpha,v}} = \mathfrak{a}^\alpha_{v_\alpha}$. If $\lambda \leq \gamma$, and $v_\alpha \leqslant w_\alpha$ for all α, then $\beta_{\lambda,v} \preceq \beta_{\gamma,w}$. This contradicts the construction.

It remains to show the class of convergent nets in a topological space form a convergence class, and this is left to the reader to verify. □

1.6 Filters

There is an alternate 'generalized sequence' which some view as more elegant than nets (it is certainly more 'french', if that is your definition of elegance). A **filter** $\mathcal{F}$ is a family of non-empty subsets of a set X, which is closed under finite intersection, and is closed under extension: if $A \subset B$, with $A \in \mathcal{F}$, then $B \in \mathcal{F}$. A filter converges in a topological space X to $x \in X$ if every neighbourhood of x is an element of the filter. It takes a while to get used to the relation between filters and sequences. Think of filters as sets which 'filter' the elements of a sequence. Given some net $\mathfrak{a}$, we have a corresponding filter $\mathcal{F} = \{C : \text{eventually } \mathfrak{a}_\alpha \in C\}$, and $\mathfrak{a}$ converges to x if and only if $\mathcal{F}$ converges to x. Conversely, given a filter $\mathfrak{F}$, define a directed set (x, C), with $x \in C \in \mathfrak{F}$, and let $(x, C) \leqslant (y, D)$ if $C \supset D$. Then we have a natural net $(x, C) \mapsto x$ corresponding to the filter.

- A subsequence of a sequence corresponds to an extension $\mathfrak{G} \supset \mathfrak{F}$ of a filter.

- Switching from viewing a sequence as a sequence in Y rather than X corresponds to switching from a filter $\mathfrak{F}$ on X to a sequence $\mathfrak{F}_Y$ on Y.

- Mapping a sequence x_k to a sequence $f(x_k)$ corresponds to mapping a filter $\mathfrak{F}$ to a filter $f(\mathfrak{F})$.

- A limit point of a filter $\mathcal{F}$ is a point x such that $\mathcal{F} \subset O(x)$.

Example. *The family of neighbourhoods of x is a filter, and converges to x.*

Example. *The set of cofinite sets in a space X form a filter. The cofinite filter on the set of positive integers is known as the Fréchet filter.*

Lemma 1.13. *If $\mathcal{A}$ is a family of non-void sets closed under finite intersection, then there is a smallest filter $F(\mathcal{A})$ which contains $\mathcal{A}$.*

Proof. Let $F(\mathcal{A}) = \{B : (\exists A \in \mathcal{A} : B \supset A)\}$. Then $F(\mathcal{A})$ is a family of nonvoid sets, and if $B \supset A$, $C \supset A'$, $B \cap C \supset A \cap A'$, so $F(\mathcal{A})$ is closed under finite intersections. It is certainly obvious that $F(\mathcal{A})$ is closed under intersection, so that $F(\mathcal{A})$ is a filter. □

A family of non-void sets B which is closed under finite extension is called a base for the filter $F(B)$. $B \subset \mathcal{F}$ is a base if and only if every element U of $\mathcal{F}$, there is $V \in B$ for which $V \subset U$.

Corollary 1.14. *A family $\{\mathcal{F}_\alpha\}$ of filters has a least upper bound if and only if for any finite index $\mathcal{F}_{\alpha_1}, \dots, \mathcal{F}_{\alpha_k}$, with $U_k \in \mathcal{F}_{\alpha_k}$, $\bigcap_{k=1}^n U_k$ is non-empty.*

Lemma 1.15. *x is a limit point of Y if and only if A is contained in a filter which converges to x.*

Proof. Let $\mathcal{F}$ be a filter which converges to x, an contains A. If U is a neighbourhood of x, then $U \cap A \in \mathcal{F}$ is non-empty. Thus x is a limit point of A. Conversely, let x be a limit point of A. Then, if $N(x)$ is the set of neighbourhoods of x, then $F(N(x) \cup A)$ is a filter converging to x, containing A. □

Example. *A family $\mathcal{U}$ of open sets is a neighbourhood basis of x if and only if $F(\mathcal{U})$ converges to x.*

Theorem 1.16. *U is open if and only if U is in every filter converging to a point in U.*

Proof. If x is a point, then the set $N(x)$ of all neighbourhoods of x is a filter converging to U, so $U \in N(x)$. We then apply the thoerem that a set is open if and only if it is a neighbourhood of all its containing points. The converse is true by definition. □

1.6.1 Mapping Filters

If $f : X \to Y$ is a surjective mapping, with $\mathcal{F}$ a filter on X, then the family

$$f(\mathcal{F}) = \{f(C) : C \in \mathcal{F}\}$$

is also a filter, for $f(C)$ is non-empty for each $C \neq \emptyset$, and

$$\emptyset \neq f(C \cap D) \subset f(C) \cap f(D)$$

The mapping also maps bases onto bases.

Theorem 1.17. *f is continuous if and only if $f(\mathcal{F})$ converges to $f(x)$ whenever $\mathcal{F}$ converges to x.*

Proof. If $\mathcal{F}$ contains every neighbourhood of a point x, then $f(\mathcal{F})$ contains a point $f(x)$, because if U is an open neighbourhood of $f(x)$, $f^{-1}(U)$ is an open neighbourhood of x, and $f(f^{-1}(U)) = U$. Conversely, let $f(\mathcal{F})$ converge to $f(x)$ whenever $\mathcal{F}$ converges to x. If $O(x)$ is the set of neighbourhoods of x, then $O(x)$ is a filter converging to x, so $f(O(x))$ converges to $f(x)$, so given any open neighbourhood U of $f(x)$, $U = f(V)$ for some open neighbourhood V of x, which implies continuity. □

Example. *If x_n is a sequence, then the filter corresponding to x_n is the image of the Fréchet filter under the map $n \mapsto x_n$. x_n converges to x if and only if its corresponding filter converges.*

1.6.2 Germs

For completeness, we now mentioned germs with respect to a filter, a simple application of filters which allows us to identify functions which are locally equal. Let $\mathcal{F}$ be a filter on a set X. Define an equivalence relationship on 2^X by saying $A \sim B$ if there is $V \in \mathfrak{F}$ such that $A \cap V = B \cap V$. The equivalence classes formed from this equivalence relation is known as the set of **germs** with respect to $\mathcal{F}$. The operations of intersection and union are well defined on germs, because if $A \cap V = B \cap V$, and $C \cap V = D \cap V$, then

$$(A \cap C) \cap V = (A \cap V) \cap (C \cap V) = (B \cap V) \cap (D \cap V) = (B \cap D) \cap V$$
$$(A \cup C) \cap V = (A \cap V) \cup (C \cap V) = (B \cap V) \cup (D \cap V) = (B \cup D) \cap V$$

More generally, define an equivalence relation on Y^X by letting $f \sim g$ if f agrees with g on an element V of $\mathfrak{F}$. Then we obtain germs of mappings with respect to the filter $\mathfrak{F}$, and composition from the range of the function to a new range is well defined. That is, if $f \sim g$, then $h \circ f \sim h \circ g$. A simple generalization of this is that every law of composition descends to the class of germs, so that we put germs on groups, rings, and algebras.

Example. *The primary example of a germ is obtained when we consider the filter $O(x)$. Then two functions equal one another as germs with respect to $O(x)$ if and only if they are locally equal at x. These germs occur again and again in differential geometry, algebraic geometry, and many other areas of analysis.*

1.7 saopwkdpoa

Surely this specification must involve the circles we were discussing as our fundamental example. Of course, these are not the only open sets, since the union of two balls is not necessarily an open ball, but we may use these as the fundamental sets from which all open sets are constructed.

The intuitive topology on $\mathbf{R}$ can be generalized to any linearly ordered set. In most cases, we may take the same open intervals. One case is more complicated; if the set contains a minimum or maximum element, then these elements are contained in no open intervals. We fix this by allowing in combination with open sets half closed rays $[-\infty, x)$ and $(x, \infty]$ into the definition of openess. We call the topology defined the order topology on the ordered set.

If X is a topological space, and Z is a subset, we can see intuitively how Z may inherit the notion of space from X. We take open sets in Z to be the intersection of open sets in Z with Z, and we call this the subspace topology.

As examples, the topology $\mathbf{Z}$ inherits from the order topology on $\mathbf{R}$ the discrete topology (which is why we think of integers as being separated on the real line). Conversely, the set $\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$ inherits a completely different topology (zero is no longer open on its own).

Most of the time, when specifying a set, it is difficult to specify precisely the set of all open sets that define a topology. Since, when $\{C_i\}$ is a indexed set of topology, $\bigcap_{i \in I} C_i$ is also topology, we may, when given a subcollection of the power set of a set, generate a topology on that subset by taking the smallest topology which contains the subcollection. Special collections of subs ets $\mathcal{C}$ with the following properties are of increased importance:

1. Every element in the space is contained in one of the subsets
2. If x is contained in two sets C_1 and C_2 in $\mathcal{C}$, there is a third set C_3 in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Then the topology generated has the property that a set A is open if and only if, for any element x in A, there is a set C in $\mathcal{C}$ containing x such that $C \subseteq A$. We call such a collection a basis for the topology. Specific examples include open intervals in $\mathbf{R}$. We say $\mathcal{C}$ covers the topology it generates.

Unlike in linear algebra, a basis for a topology is not unique up to bijection. We cannot even always find a minimal basis for all topologies in terms of containment (consider open balls in $\mathbf{R}^n$). We can only promise that there is a basis with minimum cardinality, which exists due to the well ordering property of cardinal numbers.

Theorem 1.18. *Let X be a topological space, and Y a subset with the subspace topology. Then a subset A is closed in Y if and only if $A = B \cap Y$ for a closed set B in X.*

Proof. Suppose $A = B \cap Y$ as in the theorem's statement. Then $X - B$ is open in X, so $(X - B) \cap Y$ is open in Y, and this set is just $Y - B$. But this $B \cap Y$, which is the complement of $Y - B$, is closed in Y.

Conversely, if A is closed in Y, $Y - A$ is open in Y, hence $Y - A = V \cap Y$ for some open set V in X. Since $V^c \cap Y = A$, we see that A is the intersection of Y with a closed set in X. $\square$

Chapter 2

Distances

In topology, we escrew the concept of an exact distance – we care only about the distance that lies infinitely small between different objects. Nonetheless, distances are more consistant with out intuitions about space, especially in the context of the euclidean plane and analysis, so it is useful to be able to construct topological space using an abstract definition of distance.

Definition. A **Metric** on a set X is a function $d : X \times X \to \mathbf{R}$ such that for any $x, y, z \in X$

1. (Nondegeneracy) $d(x,y) \geqslant 0$, and $d(x,y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x,y) = d(y,x)$.
3. (The Triangle Inequality) $d(x,z) \leqslant d(x,y) + d(y,z)$.

Given a subset A of X, we define $d(x,A) = \inf \{d(x,a) : a \in A\}$, and the **diameter** of A to be $\text{diam}(A) = \sup\{d(x,y) : x, y \in A\}$. A set is **bounded** if its diameter is finite. The **open ball** of radius r and center x is $B_r(x) = \{y \in X : d(x,y) < r\}$, and the **closed ball** $D_r(x) = \{y \in X : d(x,y) \leqslant r\}$. If we take the set of all open balls to be the basis of a topological space, we obtain a topological structure – a **metric space**.

Example. *The canonical metric on* $\mathbf{R}^n$ *is the euclidean metric*

$$d(x,y) = \sum_{i=1}^{n} (x_i - y_i)^2$$

But we may consider many different metrics on $\mathbf{R}^n$.

$$d(x,y) = \sum_{k=1}^{n} |y_i - x_i|$$

The open balls of this metric are shaped like diamonds; the metric

$$d(x,y) = \max_{i=1,\dots,n} |y_i - x_i|$$

induces balls shaped like squares whose sides are oriented to the axes. The topologies induced by all three metrics are the same.

Example. *On any set* X, *take* $d(x,y) = \delta_{x,y}$. *The topology induced is discrete.*

Theorem 2.1. *Every metric space is normal*

Proof. Let A and B be two closed, disjoint sets in a metric space. Consider $U = \{x : d(x,A) < d(x,B)\}$, and $V = \{x : d(x,B) < d(x,A)\}$. Then U and V are disjoint open sets (since $x \mapsto d(x,A)$ is continuous), with U containing A and V containing B. □

Theorem 2.2. *A subset of a complete metric space is precompact if and only if it is totally bounded.*

Proof. Let X be a totally bounded metric space. Let $\{x_i\}$ be a sequence in X. Choose a finite covering of X by balls of radius 1. Select a subsequence $x_{1,i}$ of x_i which lies in some specific ball B_1. Cover B_1 by a finite covering of radius $1/2$, and take a subsequence $x_{2,i}$ of $x_{1,i}$. Proceed inductively, considering subsequences $x_{n,i}$. Then $x_{i,i}$ is a cauchy sequence in X, and we have shown X is precompact. □

Theorem 2.3. *Let* X *be a compact metric space, and let* $\{U_i\}$ *be an open cover. Then the cover admits a* **Lebesgue number** δ *such that if* $diam(Y) < \delta$, *then* $Y \subset U_i$.

Proof. Let $U_1, \ldots, U_n$ be a finite subcover of X, and let $C_i = X - U_i$. Define

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

Then f is continuous, and is always positive. Since f is defined on a compact set, it attains a minimum value $\delta > 0$. Suppose Y is a set with $\mathrm{diam}(Y) < \delta$, containing points $y_1, \ldots y_n$ in each C_i. But then

$$f(y_1) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) = \frac{(n-1)}{n} \delta < \delta$$

A contradiction which proves the claim. $\square$

Chapter 3

Constructions

Here we get to the visually interesting part of topology, providing methods to mold and curve the topological structures of your choosing, gluing, stretching, and all other kinds of fun stuff. It will explain how we get from a plane to a torus, or from $\mathbf{R}^2$ to S^2. The construction we will make can be shown in a very general manner.

Definition. Let X be a topological space, and $\sim$ an equivalence relation on X. From this equivalence relation, we form the set $X/\sim$, and consider the projection mapping π from X to $X/\sim$. The quotient topology on $X/\sim$ is the coarsest topology that makes π continuous, and makes $X/\sim$ the quotient space of X and $\sim$. A set A is open in $X/\sim$ if and only if $\pi^{-1}(A)$ is open in X.

If we have a surjective map $f : X \to Y$, where X is some topological space, and Y is any set, then we may construct a topology on Y analogous to the quotient topology. First we consider the fibers of X relative to the mapping f, and identify the quotient topology on this set. We obtain a bijective mapping $\overline{f}$. The quotient topology on Y is the topology which makes $\overline{f}$ a homeomorphism. Since, in the context of topology, homeomorphisms preserve all important properties, we may as well consider this definition no different from the definition in terms of equivalence relations.

Chapter 4

Algebraic Topology

To verify that two topological spaces are homeomorphic, we need only find a single homeomorphism that connects the two spaces. On the contrary, to verify that two topological spaces are not homeomorphic, we need to somehow show that every function from one space to the other is not a homeomorphism, a computationally intractable problem. One trick we can use to separate topological spaces is to find fundamental topological properties which distinguish two topological spaces. Connectedness, Compactness, and Hausdorffiness are all preserved by homeomorphism, as does the topological properties of subspaces. Nonetheless, sometimes these properties are not enough to distinguish two spaces. This chapter shows a deep technique which is often useful for characterizing spaces.

Consider two functions f and g between topological spaces X and Y. Though f might not be equal to g, they may be in some sense topologically equal – we may be able to deform one to the other in a continuous fashion. This is a homotopy.

Definition. Let $f, g : X \to Y$ be two continuous functions. Define a topology on $Hom(X,Y)$ as a subspace of Y^X, which can be viewed as the product topology of Y with itself Y times. Then f and g are homotopic if there exists a path in $Hom(X,Y)$ which connects f to g. Alternatively, these two functions are homotopic if there exists a continuous function $F : [0,1] \times X \to Y$ such that for all x, $F(0,x) = f(x)$, and $F(1,x) = g(x)$.

The fact that homotopy is an equivalence relation will allow us to distill functions between spaces to their fundamental properties. We need to specialize our definition for it to be more of more use to us.

Definition. Two paths in X are path homotopic if they have the same start and end point, and are homotopic to each other.

Let f and g be two paths in X, where the end point of f is the start point of g. Then we may compose the two paths to form a new path $f * g$, defined by

$$(f * g)(x) = f(2x) : x \in [0, 1/2] g(2x - 1) : x \in [1/2, 1]$$

By the pasting lemma, this function is a path which connects the start point of f to the end point of g. Unfortunately, concatenation is not associative, we do not have that $f*(g*h) = (f*g)*h$. These paths are homotopic to each other, however, and moving to path homotopy classes makes the definition much simpler.

Theorem 4.1. *Let f be path homotopic to f', and g path homotopic to g'. Then $f * g$ is homotopic to $f' * g'$.*

Proof. Let F be the path homotopy from f to f', and G the path homotopy from g to g'. Define a homotopy H between $f * g$ and $f' * g'$ by

$$H(\cdot, y) = F(\cdot, y) * G(\cdot, y)$$

More specifically

$$H(x, y) = \begin{cases} F(2x, y) & \text{if } x \in [0, 1/2] \\ G(2x - 1, y) & \text{if } x \in [1/2, 1] \end{cases}$$

The pasting lemma guarentees this function is a new homotopy. □

We now consider homotopy classes of paths, so when we talk about a path f, we are really talking about all paths homotopic to f.

Theorem 4.2. $[f] * ([g] * [h]) = ([f] * [g]) * [h]$.

Chapter 5

Compactification

In the following, we consider only locally connected, locally compact, connected Hausdorff spaces.

Definition. An **end** of a X is a map ε defined on compact subsets of X, such that $\varepsilon(C)$ is a connected component of $X - C$ for each compact C, and $C \subset D$ implies $\varepsilon(D) \subset \varepsilon(C)$. Denote the set of all ends on X by $\mathcal{E}(X)$.

Definition. The end compactification of a space X is the space $\mathbf{X} = X \cup \mathcal{E}(X)$, where a set is open if it is open in X, or if it is of the form $U_{\varepsilon(C)} := \varepsilon(C) \cup \{\varepsilon' \in \mathcal{E}(X) : \varepsilon'(C) = \varepsilon(C)\}$, where C is compact.

Lemma 5.1. *The end compactification is Hausdorff.*

Proof. If $\varepsilon, \varepsilon \in \mathcal{E}(X)$ are two unequal ends, then there is some compact set C for which $\varepsilon(C) \neq \varepsilon'(C)$. But then $U_{\varepsilon(C)}$ and $U_{\varepsilon'(C)}$ are disjoint. If $x, y \in X$, then they can surely be separated in the end compactification because X is Hausdorff. If x is a point in X, and ε is an end, then because X is locally compact, x possesses a precompact neighbourhood V of x, and then $U_{\varepsilon(\overline{V})}$ is disjoint from V. $\square$

Lemma 5.2. $\mathbf{R}$ *has two ends, the 'left' and 'right' ends.*

Proof. $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n,n]$, and each $[-n,n]$ is compact. We contend every end ε on $\mathbf{R}$ is defined by its action on $[-n,n]$. If C is any compact set, then C is contained in an interval of the form $[-n,n]$. Clearly, $\varepsilon(C)$ must be the unique connected extension of $\varepsilon([-n,n])$, since $\varepsilon(C) \supset \varepsilon([-n,n])$. In fact, ε is defined solely by its action on $[-1,1]$, since $[-1,1] \subset [-2,2] \subset \ldots$. Since the two choices $\varepsilon([-x,x]) = (-\infty,x)$ and $\varepsilon([-x,x]) = (x,\infty)$ constitute ends, the space has two ends. □

In general, if a space X can be written as $C_1 \subset C_2 \subset \cdots \to X$, where each C_i is compact, then all ends are defined by their action on C_1. We shall call such a space **hemicompact**. Not all choices of components of $X - C_1$ will work, however.

Lemma 5.3. *The end compactification of a hemicompact space is compact.*

Proof. s □

Chapter 6

Compactification

In the following, we consider only locally connected, locally compact, connected Hausdorff spaces.

Definition. An **end** of a X is a map ε defined on compact subsets of X, such that $\varepsilon(C)$ is a connected component of $X - C$ for each compact C, and $C \subset D$ implies $\varepsilon(D) \subset \varepsilon(C)$. Denote the set of all ends on X by $\mathcal{E}(X)$.

Definition. The end compactification of a space X is the space $\mathbf{X} = X \cup \mathcal{E}(X)$, where a set is open if it is open in X, or if it is of the form $U_{\varepsilon(C)} := \varepsilon(C) \cup \{\varepsilon' \in \mathcal{E}(X) : \varepsilon'(C) = \varepsilon(C)\}$, where C is compact.

Lemma 6.1. *The end compactification is Hausdorff.*

Proof. If $\varepsilon, \varepsilon \in \mathcal{E}(X)$ are two unequal ends, then there is some compact set C for which $\varepsilon(C) \neq \varepsilon'(C)$. But then $U_{\varepsilon(C)}$ and $U_{\varepsilon'(C)}$ are disjoint. If $x, y \in X$, then they can surely be separated in the end compactification because X is Hausdorff. If x is a point in X, and ε is an end, then because X is locally compact, x possesses a precompact neighbourhood V of x, and then $U_{\varepsilon(\overline{V})}$ is disjoint from V. $\square$

Lemma 6.2. $\mathbf{R}$ *has two ends, the 'left' and 'right' ends.*

Proof. $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n,n]$, and each $[-n,n]$ is compact. We contend every end ε on $\mathbf{R}$ is defined by its action on $[-n,n]$. If C is any compact set, then C is contained in an interval of the form $[-n,n]$. Clearly, $\varepsilon(C)$ must be the unique connected extension of $\varepsilon([-n,n])$, since $\varepsilon(C) \supset \varepsilon([-n,n])$. In fact, ε is defined solely by its action on $[-1,1]$, since $[-1,1] \subset [-2,2] \subset \ldots$. Since the two choices $\varepsilon([-x,x]) = (-\infty, x)$ and $\varepsilon([-x,x]) = (x, \infty)$ constitute ends, the space has two ends. $\square$

In general, if a space X can be written as $C_1 \subset C_2 \subset \cdots \to X$, where each C_i is compact, then all ends are defined by their action on C_1. We shall call such a space **hemicompact**. Not all choices of components of $X - C_1$ will work, however.

Lemma 6.3. *The end compactification of a hemicompact space is compact.*

Proof. s $\square$