Fractals Avoiding Fractal Configurations

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Method: Discretization of Scales

- Following Fraser, Pramanik, and Keleti, we construct solutions by repeatedly dissecting intervals, ala the construction of the Cantor set.
- If X is the decreasing limit of sets $X_1, X_2, \dots,$ which are unions of intervals, we can discretize the problem so that we only have to avoid a discrete version of the configuration at each dissection.

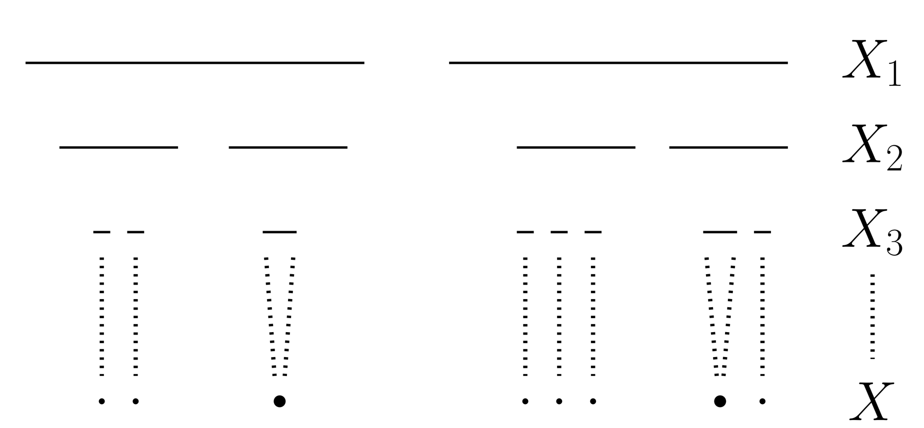

Figure 1: Interval Dissection at Discrete Scales

- **Discrete configuration problem**: If $x_1, \dots, x_d \in X_n$, and $f(x_1, \dots, x_d) = 0$, then some x_i and x_j lie in a common interval in X_n.
- Provided that the discrete configuration problem is satisfied and the length of the intervals forming X_n tends to zero as $n \to \infty$, X avoids all configurations.
- For technical reasons, we consider a slightly different discrete configuration problem where at each scale we consider a partition of X_n into unions of intervals, and the problem then becomes that if $f(x_1, \dots, x_d) = 0$, then some x_i and x_j lie in a common part of the partition of X_n.
- For fractal avoidance, the discrete criterion is obtained if whenever $(x_1, \dots, x_d) \in X_n^d \cap Y$, then $|x_i - x_j| = o(1)$ for some indices i and j.

Our Research Problem: How Large can Sets with a Fixed Irregularity Be?

- The irregularities commonly manifest as avoiding the zero set of a function.
- Largeness is quantified by the Hausdorff dimension of the irregular set.
- Examples of such problems including finding a large set $X \subset \mathbf{R}^3$ such that the angles formed by any three distinct points in X are distinct.
- **Configuration Avoidance**: Find X such that for distinct $x_1, \dots, x_d \in X$, $f(x_1, \dots, x_d) \neq 0$.
- Our new method of finding X more naturally considers a generalization of configuration avoidance.
- **Fractal Avoidance**: Given $Y \subset \mathbf{R}^d$, find X such that $X^d \cap Y \subset \Delta$, where $\Delta = \{x : x_i = x_j \text{ for some } i, j\}$. Generalize configuration avoidance by setting $Y = f^{-1}(0)$.

Main Result:

Theorem.
If the zero set of a function $f : \mathbf{R}^{nd} \to \mathbf{R}$ is α dimensional, then we can find $X \subset \mathbf{R}^d$ with Hausdorff dimension $(nd - \alpha)/(n-1)$ such that $f(x_1, \dots, x_d) \neq 0$ for distinct $x_1, \dots, x_d \in X$.

- Extends results of Pramanik and Fraser (2018) which give the result when f is smooth and nonsingular.
- If $Y \subset \mathbf{R}$ has dimension α, we can find a set $X \subset \mathbf{R}$ of dimension $1 - \alpha$ such that $X + X$, $X - X$, and $X \cdot X$ avoids elements of Y. We hope to extend this result to finding X as a vector space over $\mathbf{Q}$.
- Given a 1 dimensional set Y such that $Y \cap L$ is zero dimensional for each straight line L, and a projection π such that $\pi(Y)$ has non-empty interior, we can find a 1/2 dimensional subset X not containing the vertices of any isoceles triangles.

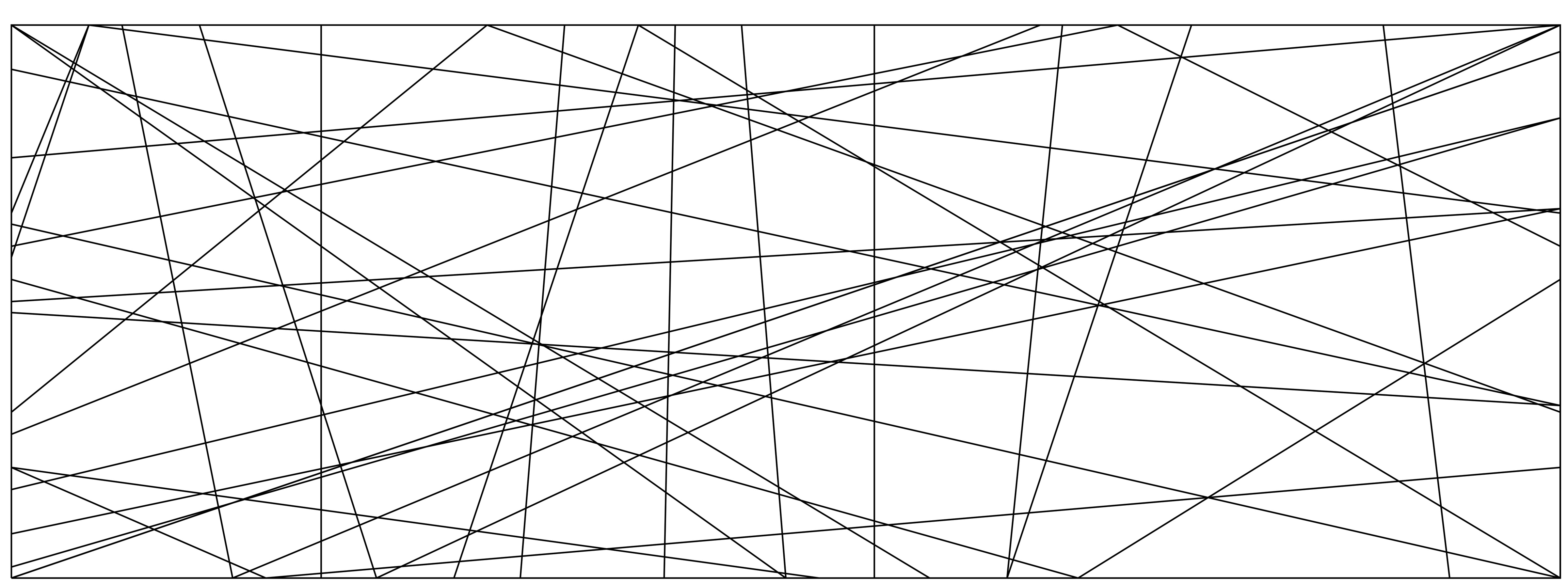

Figure 2: Our method can now find a full dimensional X avoiding Y formed by uncountably many lines that aren't too 'bushy'.

Method: Random Selection

- The dimension of Y gives us very little structural information about Y, so it behaves like a random distribution of mass.
- To combat this, we choose random interval dissections to form X at each discrete scale, pruning intersections with Y.
- If Y concentrates at a particular location, the random choice of X can stay away from this location. On the other hand, if Y is spread out rather uniformly, we can spread out X uniformly while still avoiding the elements of Y.

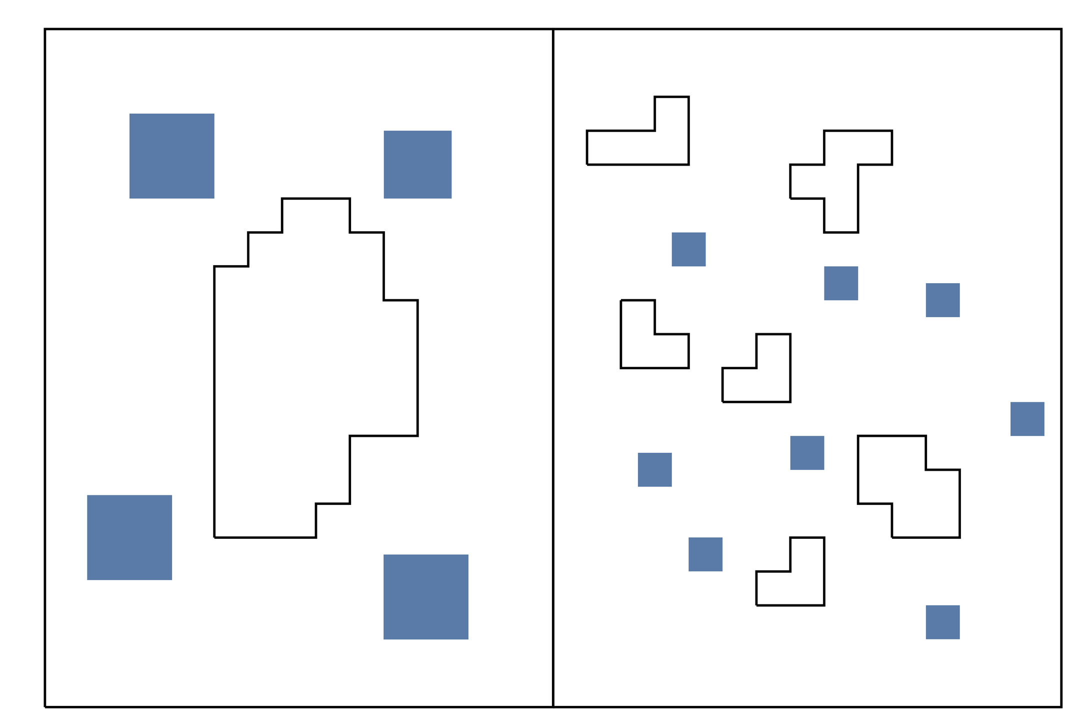

Figure 3: Random choices of X avoid Y.

- Our method currently works when the dimension of Y is quantified as the box counting dimension.
- We are currently working on using hyperdyadic scaling to extend the result where Y is quantified by it's Hausdorff dimension.
- The construction parallels a random construction of an independant set in a hypergraph, similar to Turan's theorem.
- We are also currently looking at using other techniques on hypergraphs to improve the dimension of X when Y has more structure.