

Cartesian Products Avoiding Patterns

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Abstract

We construct subsets of $[0, 1]^d$ with large Hausdorff dimension whose Cartesian product avoids countably many sets with low Minkowski dimension. This generalizes the pattern avoidance problem often used in the literature. We use the result to construct high dimensional sets whose sum set avoids a given set, as well as construct dimension $1/2$ sets avoiding isosceles triangles, which are restricted to lie on an arbitrary set with fractional dimension close to that of a line. General pattern avoidance methods in the literature are completely unable to perform anything of this sort, which makes the latter result particularly surprising.

Can we construct high dimensional subsets of \mathbf{R}^d avoiding patterns? For instance, can we find large sets containing no colinear triple of points? What about a set not containing any three vertices forming an isosceles triangle? If we can specify the pattern as the zero set of a function $f : (\mathbf{R}^d)^n \rightarrow \mathbf{R}$, results in the literature give general methods for finding sets $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that for any distinct points $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$. [2] gives good results when f is C^1 and n is small, and [5] is successful when f is a low degree polynomial. Rather than avoiding the zeroes of a function, in this paper, we fix a set $Z \subset (\mathbf{R}^d)^n$, and construct sets X such that for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. Surprisingly, a fractional dimension bound on Z is all that is needed to find a high dimensional set X in this setting.

Theorem 1. *Suppose $Z \subset (\mathbf{R}^d)^n$ is the countable union of compact sets, each with lower Minkowski dimension at most α . Then there exists $X \subset [0, 1]^d$ with*

$$\dim_{\mathbf{H}}(X) = \min \left(\frac{nd - \alpha}{n - 1}, d \right)$$

such that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Z$.

Our novel formulation of the pattern avoidance problem makes certain geometric features of Z much more explicit than when we can express Z as the zero set of the function. In particular, exploiting the fractional dimension of Z is completely non-obvious from the functional perspective. Moreover, the relation of the hypothesis of previous results to their conclusion when viewed as statements about the geometric set Z .

Despite the generality of our method, we are still able to recover Theorems 1.1 and 1.2 of [2] as special cases when Z is formed from a countable collection of smooth manifolds. Meanwhile, our proof is less technical than their approach. We compare our methods with [2], as well as other generic pattern avoidance methods, in Section 6.

Because our result is the only general avoidance method which can avoid singular sets Z , our method can be used to consider problems that other methods cannot touch at all. Most interesting of these results is a construction of a large ‘restricted’ set avoiding configurations. In the restricted scenario, in addition to Z , we are given an arbitrary set Y , and we must construct a high dimensional X avoiding Z which is *restricted* to be a subset of Y . If Y has non-empty interior, and Z is smooth, we can apply the results of [2] and [5] in this setting. But our result even applies for certain Y with non-empty interior, or even a totally disconnected Cantor-like set. We discuss the applications of our method in Section 5.

The key idea to avoiding low dimension configurations is a random mass selection strategy. This is the main technique we use to approach a discrete variant of Theorem 1 in Section 2. The size of an optimal solution to the discrete problem is tricky to calculate exactly, but we can compute the expected size of a random selection, which is optimal enough in expectation, and is likely tight for general inputs to the problem. By overlaying the solution to the discretized problem at a sequence of scales, in Section 3 we are able to obtain the required set X via a Cantor-type construction.

An important property of our discrete strategy is that it assigns mass ‘uniformly’ at each iteration. Exploiting this fact, in Section 4 we are able to show the set X has the required Hausdorff dimension regardless of how fast our sequence of scales decay. The uniform mass assignment technique occurs implicitly in at least one other Hausdorff dimension calculation, for example, in [2]. But we do not believe the uniform strategy has been explicitly identified in the literature as a method to maintain fractional dimension despite a rapid decay of scales used in the construction of a set.

Remark. *The difficult setting of Theorem 1 occurs when $\alpha \geq d$. If $\alpha < d$,*

$$X = \{x \in [0, 1]^d : x \neq z_k \text{ for all } (z_1, \dots, z_n) \in Z \text{ and } 1 \leq k \leq d\}$$

gives a set with full Hausdorff dimension satisfying the properties of the theorem. In our proof, we will assume $d \leq \alpha < dn$, and so must find a set X with $\dim_{\mathbf{H}}(X) = (dn - \alpha)/(n - 1)$.

1 Frequently Used Notation and Terminology

Our argument heavily depends upon discretizing sets into unions of cubes. Throughout our argument, we use the following notation:

- (A) A *dyadic scale* is a length l equal to 2^{-k} for some non-negative integer k .

- (B) Given a length l , \mathcal{B}_l^d denotes the family of all half open cubes in \mathbf{R}^d with side length l and corners on the lattice $(l \cdot \mathbf{Z})^d$. That is,

$$\mathcal{B}_l^d = \{[a_1, a_1 + l) \times \cdots \times [a_d, a_d + l) : a_i \in l \cdot \mathbf{Z}\}.$$

If $E \subset \mathbf{R}^d$, $\mathcal{B}_l^d(E)$ is the family of cubes in \mathcal{B}_l^d intersecting E , i.e.

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : I \cap E \neq \emptyset\}.$$

In section 2, we solve a discrete version of our Theorem 1. Important to this approach is a discretization of the diagonal:

- (C) Given $I \in \mathcal{B}_l^{dn}$, we can decompose I as $I_1 \times \cdots \times I_n$ for unique cubes $I_1, \dots, I_n \in \mathcal{B}_l^d$. We say I is *weakly non-diagonal* if the cubes I_1, \dots, I_n are distinct.

In section 3, we discretize the set Z in the hypothesis of Theorem 1 into a union of boxes in order to construct our solution X . Our argument is aided by the following definitions:

- (D) Adopting the terminology of [3], we say a collection of sets U_1, U_2, \dots is a *strong cover* of some set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k .
- (E) The *lower Minkowski dimension* of a compact set $Z \subset \mathbf{R}^d$ is

$$\underline{\dim}_{\mathbf{M}}(Z) = \liminf_{l \rightarrow 0} \frac{\log(\#(\mathcal{B}_l^d(Z)))}{\log(1/l)}. \quad (1)$$

This means if $\mathcal{B}_l^d(E) \leq \alpha$, there is a sequence of lengths $l_1, l_2, \dots \rightarrow 0$ and positive $\delta_1, \delta_2, \dots \rightarrow 0$ such that $\#(\mathcal{B}_{l_k}^d(Z)) \leq (1/l_k)^{\alpha + \delta_k}$.

Once X has been constructed, in Section 4 we show that it has the Hausdorff dimension guaranteed by Theorem 1. Here we use a Frostman's lemma type approach, summarized by the following definitions:

- (F) The α dimensional Hausdorff measure H^α on \mathbf{R}^d is defined to be $H^\alpha(E) = \lim_{\delta \rightarrow 0} H_\delta^\alpha(E)$, where

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{k=1}^m l_k^\alpha : E \subset \bigcup_{k=1}^m I_k, \begin{array}{l} \text{each } I_k \text{ is a dyadic cube} \\ \text{of length } l_k \leq \delta \end{array} \right\}. \quad (2)$$

The *Hausdorff dimension* of a set E is

$$\dim_{\mathbf{H}}(E) = \inf\{\alpha \geq 0 : H^\alpha(E) = 0\}$$

- (G) A *Frostman measure* of dimension α is a non-zero compactly supported finite Borel measure μ on \mathbf{R}^d such that for any dyadic scale l , and $I \in \mathcal{B}_l^d$, $\mu(I) \lesssim l^\alpha$. *Frostman's lemma* shows that

$$\dim_{\mathbf{H}}(E) = \sup \left\{ \alpha : \begin{array}{l} \text{there is an } \alpha \text{ dimensional Frostman} \\ \text{measure supported on } E \end{array} \right\}, \quad (3)$$

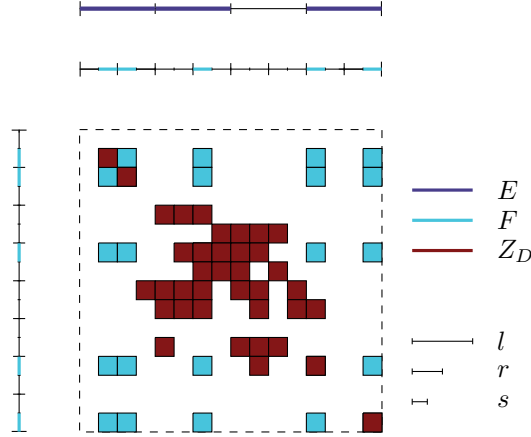


Figure 1: An example choice of F satisfying the conclusions of Lemma 1 where $d = 1$ and $n = 2$. F satisfies the non-concentration and avoidance property, as well as containing an interval from all but 3 of the intervals in $\mathcal{B}_r^d(E)$.

2 Avoidance at Discrete Scales

We avoid Z via a multi-scale construction at an infinite sequence of scales. At each scale, we solve a discretized version of the problem. Combining these solutions then solves the original problem. This section describes the discretized avoidance technique. This is the *core* part of our construction, and the Hausdorff dimension we achieve is a direct result of our success in the discrete setting.

Fix two dyadic scales l and s , with $l > s$. In the discrete setting, we replace Z by a union of side length s cubes in \mathbf{R}^{dn} , denoted Z_D . Our goal is to take a set E , which is a union of side length l cubes, and find a union of side length s cubes $F \subset E$ such that F^n is disjoint from the weakly non-diagonal (see definition C) subcubes of Z_D .

In order to ensure that X has large Hausdorff dimension, it is crucial that the mass of F is spread uniformly over E in the discrete setting. We can achieve this by trying to include a equal portion of mass in each side length r sub cube of E , for some intermediary dyadic scale r between l and s . The next lemma shows that we can select a equal portion of mass from *almost* all the side length r cubes, provided that $|Z_D| \ll |E|r^{d(n-1)}$.

Lemma 1. *Fix three dyadic lengths $l > r > s$. Let E be a union of cubes in \mathcal{B}_l^d , and Z_D a union of cubes in \mathcal{B}_s^{dn} . Then there exists $F \subset E$, which is a union of cubes in \mathcal{B}_s^d , such that*

- *Avoidance: For any distinct $I_1, \dots, I_n \in \mathcal{B}_s^d(F)$, $I_1 \times \dots \times I_n \notin \mathcal{B}_s^{dn}(Z_D)$.*
- *Non-Concentration: F contains at most one sidelength s subcube of each cube in $\mathcal{B}_r^d(E)$.*

- Breadth: For all but $|Z_D|r^{-dn}$ of the cubes in $\mathcal{B}_r^d(E)$, F contains a side-length s subcube.

Proof. Form a random set U by selecting a side length s cube from each side length r cube uniformly at random. More precisely, set

$$U = \bigcup \{J_I : I \in \mathcal{B}_r^d(E)\},$$

where J_I is an element selected uniformly randomly from $\mathcal{B}_s^d(I)$. Then U certainly satisfies the non-concentration and breadth properties, but U does not satisfy the avoidance property. We will show that with non-zero probability, we can obtain the avoidance property by removing at most $|Z_D|r^{-dn}$ cubes from U , so that the breadth property remains satisfied.

For any $J \in \mathcal{B}_s^d(E)$, there is a unique $I \in \mathcal{B}_r^d(E)$ such that $J \subset I$, so

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_I = J) = (s/r)^d.$$

Since any two elements of $\mathcal{B}_s^d(U)$ lie in distinct cubes of \mathcal{B}_r^d , the only way that a weakly non-diagonal cube $K = J_1 \times \cdots \times J_n$ in $\mathcal{B}_s^{dn}(Z_D)$ is a subset of U^n is if J_1, \dots, J_n all lie in separate cubes of \mathcal{B}_r^d . In this case, the events $\{J_k \subset U\}$ are independent of one another, and so

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1 \subset U) \cdots \mathbf{P}(J_n \subset U) = (s/r)^{dn}.$$

If $\mathcal{K}(U)$ denotes the family of all weakly non-diagonal cubes $K \in \mathcal{B}_s^{dn}(Z_D)$ contained in U^n , then, letting K range over the weakly non-diagonal cubes of $\mathcal{B}_s^{dn}(Z_D)$, we find

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_K \mathbf{P}(K \subset U^n) \leq |\mathcal{B}_s^{dn}(Z_D)|(s/r)^{dn} = |Z_D|r^{-dn}.$$

In particular, there is at least one outcome U_0 for U we can choose for which

$$\#\mathcal{K}(U_0) \leq \mathbf{E}(\mathcal{K}(U)) = |\mathcal{B}_s^{dn}(Z_D)|(s/r)^{dn} = |Z_D|r^{-dn}.$$

Thus we have selected a section of mass with very few intersections with Z_0 .

We now define $F = U_0 - \{J_1 : K = J_1 \times \cdots \times J_n \in \mathcal{K}(U_0)\}$. As a subset of U_0 , F inherits the non-concentration property. We have removed at most $|Z_D|r^{-dn}$ cubes from U_0 , so F satisfies the breadth property. Finally, since we have removed a single side from every weakly non-diagonal cube in U_0^n intersecting Z_D , F satisfies the avoidance property. So our construction is complete. \square

Remark. The existence of U_0 was justified by a randomized selection process. Nonetheless, its existence can be made constructive: We simply iterate through all possible outcomes of U and pick one minimizing the cardinality of $\mathcal{K}(U)$. As a result, the set X in our theorem is obtained by explicit, constructive means.

If the set Z has dimension α , we will later show its discretization Z_D will satisfy bounds of the form $|Z_D| \leq s^{dn-\gamma}$, with γ converging to α as $s \rightarrow 0$. For

convenience, we will also set r to be the closest dyadic scale to s^λ , for some $\lambda \in (0, 1)$. The size of λ is directly related to the Hausdorff dimension of the set X we construct, the next corollary shows precisely how large we can set it to be so that F contains subcubes from a constant fraction of the cubes in $\mathcal{B}_r^d(E)$. The error term $3m \log_s |E|$ will be made insignificant by the rapid decay of the values s used in our construction.

Corollary 1. *Fix two dyadic scales $l > r$. Let E be a union of cubes in \mathcal{B}_l^d , and Z_D a union of cubes in \mathcal{B}_s^{dn} . Also, consider three parameters $\lambda \in (0, 1]$, $\gamma \in [d, dn]$, and $m > 0$. Suppose r is the closest dyadic scale to s^λ , i.e. $r = 2^{-\lfloor \lambda \log_2(1/s) \rfloor}$, $|E| \leq 1/2$, and $|Z_D| \leq s^{dn-\gamma}$. If*

$$0 < \lambda \leq \frac{dn - \gamma}{d(n-1)} - 3m \log_s |E|, \quad (4)$$

then we can find F satisfying the avoidance, and non-concentration property, containing intervals from all but a fraction $1/2^m$ of the cubes in $\mathcal{B}_r^d(E)$.

Proof. Rearranging (4) gives

$$dn - \gamma - \lambda d(n-1) \geq 3d(n-1)m \log_s |E|.$$

Since r is within a factor of two from s^λ , we consider the set E obtained from Lemma 1, and we calculate that

$$\begin{aligned} \frac{|\{I \in \mathcal{B}_r^d(E) : \mathcal{B}_s^d(I) \cap \mathcal{B}_s^d(F) = \emptyset\}|}{|\mathcal{B}_r^d(E)|} &\leq \frac{|Z_D| r^{-dn}}{|E| r^{-d}} \leq \frac{s^{dn-\gamma} r^{-dn}}{|E| r^{-d}} \\ &\leq s^{dn-\gamma} r^{-d(n-1)} |E|^{-1} \leq (s^{dn-\gamma}) (s/2)^{-\lambda d(n-1)} |E|^{-1} \\ &\leq 2^{\lambda d(n-1)} s^{dn-\gamma-\lambda d(n-1)} |E|^{-1} \leq 2^{\lambda d(n-1)} |E|^{3d(n-1)m-1} \\ &\leq 2^{d(n-1)-(3d(n-1)m-1)} \leq 1/2^m. \end{aligned}$$

The final inequality is true because

$$[d(n-1) - (3d(n-1)m - 1)] = 1 - d(n-1)(3m-1) \leq 1 - 2m \leq -m. \quad \square$$

Remark. *We reemphasize that the discrete method is the core of our avoidance technique. The remaining argument is modular. Indeed, we based the remaining parts of our paper on the construction method of [2]. If for a special case of Z , one can improve the lemma so that fewer cubes are discarded, then the remaining parts of our paper can likely be applied near verbatim to yield a set X with a larger Hausdorff dimension. For instance, if Z is a degree m algebraic hypersurface, and $Z_D = \mathcal{B}_l^{dn}(Z)$, then a variation on the technique used in Proposition 2.2 of [5] enables one to obtain a version of Corollary 1 with $\lambda \approx 1/m$. Following through the remainder of our proof replicates Theorem 2.3 of [5].*

3 Fractal Discretization

Now we apply the discrete result at many scales. The fact that Z is the countable union of compact sets with Minkowski dimension α implies that we can find an efficient *strong cover* (see definition D) of Z by cubes restricted to lie at a sequence of dyadic scales l_k converging to zero arbitrarily fast.

Lemma 2. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α , and consider any decreasing sequence ε_k converging to zero with $\alpha + \varepsilon_k \leq dn$. Then there is a decreasing sequence of dyadic scales l_1, l_2, \dots , and compact sets Z_k , which is a union of cubes in $\mathcal{B}_{l_k}^{dn}$, such that Z is strongly covered by the sets Z_k and $\#(\mathcal{B}_{l_k}^{dn}(Z_k)) \leq 1/l_k^{\alpha+\varepsilon_k}$.*

Proof. Let Z be the union of sets Y_i with $\underline{\dim}_{\mathbf{M}}(Y_i) \leq \alpha$ for each i . Consider any sequence of integers m_1, m_2, \dots which repeats each integer infinitely often. Given k , since $\underline{\dim}_{\mathbf{M}}(Y_{m_k}) \leq \alpha$, equation (1) implies there are infinitely many lengths l with $\#(\mathcal{B}_l^{dn}(Y_{m_k})) \leq 1/l^{\alpha+(\varepsilon_k/2)}$. Replacing l with a dyadic number at most twice the size of l , there are infinitely many *dyadic* lengths l with

$$\#(\mathcal{B}_l^{dn}(Y_{m_k})) \leq 1/(l/2)^{\alpha+\varepsilon_k} \leq 2^{dn}/l^{\alpha+(\varepsilon_k/2)} = (2^{dn}l^{(\varepsilon_k/2)})/l^{\alpha+\varepsilon_k}.$$

In particular, we may fix a length l_k smaller than l_1, \dots, l_{k-1} , which in addition satisfies the inequality $2^{dn}l_k^{(\varepsilon_k/2)} \leq 1$ so that $\#(\mathcal{B}_{l_k}^{dn}(Y_{m_k})) \leq 1/l_k^{\alpha+\varepsilon_k}$. Then the union of the cubes in $\mathcal{B}_{l_k}^{dn}(Y_{m_k})$ forms the set Z_k . \square

Remark. *In the proof, we are free to make l_k arbitrarily small in relation to the previous parameters l_1, \dots, l_{k-1} we have chosen. For instance, in Section 4, we will assume that $l_{k+1} \leq l_k^2$, and the argument above can be easily modified to incorporate this inequality. We will also find that setting $\varepsilon_k = c \cdot k^{-1}$ suffices to give the results we need, where c is a sufficiently small constant such that $\alpha + c \leq dn$.*

We can now construct X by avoiding the various discretizations of Z at each scale. The aim is to find a nested decreasing family of discretized sets X_k with $X = \lim X_k$. One condition guaranteeing that X avoids Z is that X_k^n is disjoint from *weakly non-diagonal* cubes (see definition D) in Z_k .

Lemma 3. *Suppose Z_k is a strong cover of Z , where Z_k is a union of cubes in $\mathcal{B}_{l_k}^{dn}$ with $l_k \rightarrow 0$, and X_k is a nested decreasing family of sets such that for each k , X_k^n is disjoint from weakly non-diagonal cubes in Z_k . If $X = \lim X_k$, then $(x_1, \dots, x_n) \notin Z$ for any distinct $x_1, \dots, x_n \in X$.*

Proof. Pick $z \in Z$ with distinct coordinates z_1, \dots, z_n . Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance. Since Z_k is a strong cover of Z , the point z is covered by cubes in infinitely many of collections Z_{k_m} . For suitably large m , the sidelength l_k cube I in Z_{k_m} containing z is disjoint from Δ . But this means that I is weakly non-diagonal, and so $z \notin X_m^d$. In particular, z is not an element of X^n . \square

We now quite simply iterate our discrete scale argument to construct the set X . First, we set $X_0 = [0, 1/2]^d$, so that $|X_0| \leq 1/2$, enabling us to apply Corollary 1. To get X_{k+1} from X_k , we apply Corollary 1, setting $E = X_k$, $Z_D = Z_{k+1}$, $l = l_k$, $s = l_{k+1}$, $\gamma = \alpha + \varepsilon_k = \alpha + c \cdot k^{-1}$, and r the closest dyadic scale to $l_{k+1}^{\beta_{k+1}}$, where

$$\beta_{k+1} = \frac{dn - \alpha}{d(n-1)} - \left(\frac{\varepsilon_{k+1}}{d(n-1)} + 6(k+1) \log_{L_{k+1}} |X_k| \right). \quad (5)$$

We can now apply Corollary 1 to construct a set F with F^n avoiding weakly non-diagonal cubes in Z_{k+1} , and containing a $\mathcal{B}_{l_{k+1}}^d$ subcube from all but a fraction $1/2^{2k+2}$ of the $\mathcal{B}_{r_{k+1}}^d$ cubes in I . We set $X_{k+1} = F$. Repeatedly doing this builds an infinite sequence of the X_k . Since X_k^n avoids Z_k , for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin X$.

4 Dimension Bounds

All that remains in our argument is showing X has the right Hausdorff dimension. First, we begin with a rough outline of our proof strategy. At the discrete scale l_k , X looks like a $d\beta_k$ dimensional set. If the lengths l_k rapidly converge to zero, then we can ensure $\beta_k \rightarrow \beta$, where

$$\beta = \frac{dn - \alpha}{d(n-1)}.$$

Thus X looks $d\beta = (dn - \alpha)/(n-1)$ dimensional at the discrete scales l_k , which is the Hausdorff dimension we want. To obtain the complete dimension bound, it then suffices to interpolate to get a $d\beta$ dimensional behavior at all intermediary scales. We won't be penalized here by making the gaps between discrete scales too large, because the uniform way that we have selected cubes in consecutive scales implies that between the scales l_k and l_{k+1}^β , X behaves like a full dimensional set. This is the uniform mass assignment technique mentioned in the introduction.

Lemma 4. $\beta_k = \beta - O(1/k)$.

Proof. We must show

$$\beta - \beta_k = \frac{\varepsilon_{k+1}}{d(n-1)} + 6(k+1) \log_{l_{k+1}} |X_k| = O(1/k).$$

Since $\varepsilon_k = c \cdot k^{-1}$, the first term is easily seen to be $O(1/k)$. On the other hand, we need the lengths to tend to zero rapidly to make the other error term decay to zero. The remark after Lemma 2 infers we can assume $l_{k+1} \leq l_k^{k^2}$, so we find

$$(k+1) \log_{l_{k+1}} |X_k| \leq \frac{(k+1) \log l_k}{\log l_{k+1}} \leq \frac{(k+1) \log l_k}{k^2 \log l_k} = \frac{k+1}{k^2} = O(1/k).$$

Thus both components of the error term are $O(1/k)$. \square

The most convenient way to understand the dimension of X at various scales is to use a Frostman type approach. We construct a finite Borel measure μ supported on X which is a Frostman measure (see definition G) of dimension $d\beta - \varepsilon$ for each $\varepsilon > 0$. Combined with Frostman's lemma, this implies that $\dim_{\mathbf{H}}(X) \geq d\beta$. The advantage of this approach is that once a natural choice of μ is fixed, the behaviour of X at a scale l can be understood by looking at the behaviour of μ restricted to cubes at the scale l .

To construct μ , we take a sequence of measures μ_k , supported on X_k , and then take a weak limit. We initialize this construction by setting μ_0 to be the uniform probability measure on $X_0 = [0, 1/2]^d$. We then define μ_{k+1} , supported on X_{k+1} , by modifying the distribution of μ_k . First, we throw away the mass of the cubes in $\mathcal{B}_{l_k}^d(X_k)$ for which more than half of the cubes in $\mathcal{B}_{r_{k+1}}^d(I)$ are disjoint from X_{k+1} . For the cubes I with more than half of the cubes $\mathcal{B}_{r_{k+1}}^d(I)$ containing a part of X_{k+1} , we distribute the mass $\mu_k(I)$ uniformly over the subcubes of I in X_{k+1} , giving the distribution of μ_{k+1} .

A glance at the cumulative distribution functions of the μ_k shows these measures converge weakly to a function μ . For any $I \in \mathcal{B}_{l_k}^d$, we find $\mu(I) \leq \mu_k(I)$, which will be useful for passing from bounds on the discrete measures to bounds on the final measure.

Lemma 5. *If $I \in \mathcal{B}_{l_k}^d$, then*

$$\mu(I) \leq \mu_k(I) \leq 2^k \left[\frac{r_k r_{k-1} \dots r_1}{l_{k-1} \dots l_1} \right]^d. \quad (6)$$

Proof. Consider $J \in \mathcal{B}_{l_{k-1}}^d$ with $I \subset J$. If $\mu_k(I) > 0$, more than half of the elements of $\mathcal{B}_{r_k}^d(J)$ contain a single sidelength l_k subcube in X_k . Since the mass $\mu_{k-1}(J)$ distributes itself evenly over all such cubes, of which there must be more than $2^{-1}(l_{k-1}/r_k)^d$, we find

$$\mu_k(I) \leq 2(r_k/l_{k-1})^d \mu_{k-1}(J)$$

Expanding this inequality recursively completes the proof, using the fact that μ_0 has total mass one as a base case. \square

Corollary 2. *The measure μ is non-zero.*

Proof. The nonconcentration property at each discrete scales implies that for each k ,

$$\#(\mathcal{B}_{l_k}^d(X_k)) \leq \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} \right]^d.$$

Since only a fraction $1/2^{2k+2}$ of the cubes in $\mathcal{B}_{r_k}^d(X_k)$ do not contain an cube in X_{k+1} , it is only for at most a fraction $1/2^{2k+1}$ of the cubes in $\mathcal{B}_{r_k}^d(X_k)$ cubes that X_{k+1} fails to contain more than half of the subcubes. But this means that

$$\mu_k[0, 1]^d - \mu_{k+1}[0, 1]^d \leq \left(\frac{1}{2^{2k+1}} \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} \right]^d \right) \left(2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1} \right]^d \right) \leq 1/2^{k+1}.$$

Thus for any m ,

$$\mu_m[0, 1]^d = \mu_0[0, 1]^d - \left(\sum_{k=0}^{m-1} \mu_k[0, 1]^d - \mu_{k+1}[0, 1]^d \right) \geq 1 - \sum_{k=0}^{m-1} 1/2^{k+1} \geq 1/2.$$

Since m was arbitrary, this implies $\mu[0, 1]^d \geq 1/2$, so in particular, $\mu \neq 0$. \square

Treating all parameters in equation (6) which depend on indices smaller than k as essentially constant, we ‘conclude’ that $\mu_k(I) \lesssim r_k^d \lesssim l_k^{\beta d - O(1/k)}$. The equation $l_{k+1} \leq l_k^{k^2}$ implies l_k decays very rapidly, which enables us to rigorously obtain this inequality.

Corollary 3. *For all $I \in \mathcal{B}_{l_k}^d$, $\mu(I) \leq \mu_k(I) \lesssim l_k^{d\beta - O(1/k)}$.*

Proof. Given $\varepsilon > 0$, equation (6) and the inequality $l_k \leq l_{k-1}^{(k-1)^2}$ imply

$$\begin{aligned} \mu_k(I) &\leq 2^k \left[\frac{r_k \cdots r_1}{l_{k-1} \cdots l_1} \right]^d \leq \left(\frac{2^k}{l_{k-1}^d \cdots l_1^d} \right) l_k^{d\beta_k} \\ &\leq \left(2^k l_k^{2d/k} / l_{k-1}^{d(k-1)} \right) l_k^{d\beta_k - 2d/k} \\ &\leq \left(2^k l_{k-1}^{(2d/k)(k-1)^2 - d(k-1)} \right) l_k^{d\beta_k - 2d/k} = o(l_k^{d\beta_k - 2d/k}). \end{aligned} \quad \square$$

Corollary 3 gives the cleanest expression of the $d\beta$ dimensional behavior of μ at discrete scales we will need. To obtain a Frostman measure bound at *all* scales, we need to apply a covering argument. This is where the uniform mass assignment technique comes into play. Because μ behaves like a full dimensional Frostman measure between the discrete scales r_{k+1} and l_k , the argument below shows the gap between l_k and r_{k+1} can be arbitrary. This is essential to our argument, because l_k decays faster than 2^{-k^m} for any $m > 0$.

Lemma 6. *If $l \leq l_k$ is dyadic and $I \in \mathcal{B}_l^d$, then $\mu(I) \lesssim l^{d\beta - O(1/k)}$.*

Proof. We use a covering argument, which breaks into cases depending on the size of l in proportion to l_k and r_k :

- If $r_{k+1} \leq l \leq l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}_{r_{k+1}}^d$. Because of the non-concentration and breadth properties of our construction, we therefore know that the mass of each cube in $\mathcal{B}_{r_{k+1}}^d$ is bounded by at most $2(r_{k+1}/l_{k+1})^d$ times the mass of a $\mathcal{B}_{l_k}^d$ cube. Thus

$$\mu(I) \lesssim (l/r_{k+1})^d (2(r_{k+1}/l_k)^d) l_k^{d\beta - O(1/k)} \leq 2l^d / l_k^{d + O(1/k) - d\beta} \leq 2l^{d\beta - O(1/k)}.$$

- If $l_{k+1} \leq l \leq r_{k+1}$, we can cover I by a single cube in $\mathcal{B}_{r_{k+1}}^d$. Each cube in $\mathcal{B}_{r_{k+1}}^d(X)$ contains at most one cube in $\mathcal{B}_{l_{k+1}}^d(X)$, so

$$\mu(I) \lesssim l_{k+1}^{d\beta - O(1/k)} \leq l^{d\beta - O(1/k)}.$$

- If $l \leq l_{k+1}$, there certainly exists M such that $l_{M+1} \leq l \leq l_M$, and one of the previous cases yields that

$$\mu(I) \lesssim 2l^{d\beta - O(1/M)} \leq 2l^{d\beta - O(1/k)}. \quad \square$$

To prove μ is a Frostman measure with dimension $d\beta - \varepsilon$ for all $\varepsilon > 0$, we need to prove $\mu(I) \lesssim l^{d\beta - O(1/k)}$ for an *arbitrary* dyadic cube with sidelength l , not just one with $l \leq l_k$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. If $l \geq l_k$, then $\mu(I)/l^{d\beta - O(1/k)} \leq 1/l_k^{d\beta - O(1/k)} \lesssim_k 1$, so $\mu(I) \lesssim_k l^{d\beta - O(1/k)}$ holds automatically for all sufficiently large cubes. Thus we know $\dim_{\mathbf{H}}(X) \geq d\beta = (dn - \alpha)/(n - 1)$. It is simple to show that the dimension of X is precisely $d\beta$. Thus the following lemma concludes the proof of Theorem 1.

Lemma 7. $\dim_{\mathbf{H}}(X) = (dn - \alpha)/(n - 1)$.

Proof. By the non-concentration property at discrete scales, X_k is covered by at most

$$\left[\frac{l_{k-1} \cdots l_1}{r_k \cdots r_1} \right]^d$$

side length l_k cubes. It follows that if $\gamma > \beta_k$, then

$$H_{l_k}^{d\gamma}(X) \leq \left[\frac{l_{k-1} \cdots l_1}{r_k \cdots r_1} l_k^\gamma \right]^d \lesssim \left[\frac{l_{k-1} \cdots l_1}{r_{k-1} \cdots r_1} l_k^{\gamma - \beta_k} \right]^d \leq l_k^{d(\gamma - \beta_k)}.$$

Since $l_k \rightarrow 0$ as $k \rightarrow \infty$, $H^{d\gamma}(X) = 0$. Since γ was arbitrary, $\dim_{\mathbf{H}}(X) \leq d\beta_k$, and since k was arbitrary, $\dim_{\mathbf{H}}(X) \leq d\beta$. \square

5 Applications

Our result already generalizes methods with interesting applications. But the most novel applications of our method occur when the configurations truly form a set of fractional dimension.

Theorem 2 (Sum-Sets Avoiding Fractals). *If $Y \subset \mathbf{R}^d$ is a countable union of sets with lower Minkowski dimension upper at most α , then there exists a set X with Hausdorff dimension $d - \alpha$ such that $X + X$ is disjoint from Y .*

Proof. Consider

$$Z = \{(x, y) : x + y \in Y\} \cup \{(x, y) : y \in Y/2\} = Z_1 \cup Z_2$$

Since Y is the countable union of sets with lower Minkowski dimension upper at most α , Z is the countable union of sets with lower Minkowski dimension at most $1 + \alpha$. Applying Theorem 1 with $n = 2$ gives a set X of dimension $1 - \alpha$ avoiding Z . In particular, avoiding Z_1 implies that for x, y distinct, $x + y \notin Y$, and avoiding Z_2 implies that $x \notin Y/2$ for any x , so $x + x = 2x \notin Y$. \square

Remark. One problem with our result is that as the number of variables n increases, the dimension of X tends to zero. If we try and make the n -fold sum $X + \dots + X$ be disjoint from Y , Theorem 1 only yield a dimension $(d - \alpha)/(n - 1)$ set. We have ideas on how to improve our main result when Z is ‘flat’, in addition to being low dimension, which will enable us to remove the dependence of $\dim_{\mathbf{H}}(X)$ on n , which we plan to publish in a later paper. This will enable us to still obtain us to consider sums of arbitrary length. In particular, we expect to be able to construct a set X disjoint from Y with the same dimension $d - \alpha$, such that X is closed under addition, and multiplication by rational numbers. In particular, given a \mathbf{Q} subspace V of \mathbf{R}^d with dimension α , we can always find a ‘complementary’ \mathbf{Q} vector space W with complementary fractional dimension $d - \alpha$ such that $V \cap W = \{0\}$.

In [2], Hausdorff dimension $1/2$ subsets of smooth curves with non-vanishing curvature are constructed avoiding isosceles triangles. Our method improves this to find subsets of sets avoiding isosceles triangles, with the curvature condition replaced with a hypothesis more fitting geometric measure theory. We are unaware of methods in the literature which enable one to construct sets avoiding configurations which are restricted to lie in a set of fractional dimension, which makes this result particularly interesting.

Theorem 3 (Restricted Isosceles Triangle Avoiding Sets). *Suppose we are given $Y \subset \mathbf{R}^2$ together with an orthogonal projection $\pi : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that $\pi(Y)$ has non-empty interior. Let d be an arbitrary metric on \mathbf{R}^2 . Provided that the set*

$$Z_0 = \{(y_1, y_2, y_3) \in Y^3 : d(y_1, y_2) = d(y_1, y_3)\}$$

is the countable union of sets with lower Minkowski dimension bounded by $2 + \varepsilon$, there exists a set with dimension $1/2 - O(\varepsilon)$ subset $X \subset Y$ with no triple $(x_1, x_2, x_3) \in X^3$ forming the vertices of an isosceles triangle.

Proof. Without loss of generality, by translation and rescaling, assume $\pi(Y)$ contains $[0, 1]$. Form the set

$$Z = \pi(Z_0) = \{(\pi(y_1), \pi(y_2), \pi(y_3)) : y \in Z_0\},$$

Then Z is the projection of a $2 + \varepsilon$ dimensional set, and therefore has dimension at most as large as $2 + \varepsilon$. Applying Theorem 1 with $d = 1$ and $n = 3$, we construct a Hausdorff dimension $1/2 - O(\varepsilon)$ set $X_0 \subset [0, 1]$ such that for any distinct $x_1, x_2, x_3 \in X_0$, $(x_1, x_2, x_3) \notin Z$. Thus if we form a set X by picking, from each $x \in X_0$, a single element of $\pi^{-1}(x)$, then X avoids isosceles triangles, and has Hausdorff dimension at least as large as X_0 . \square

Remark. *The existence of a projection as in this theorem is guaranteed if Y is a rectifiable set, which makes it not too rigid of an assumption.*

Let d be the Euclidean metric. For any fixed points P and Q , the points R with $d(P, R) = d(Q, R)$ form a line L_{PQ} bisecting the plane between P and

Q . Understanding the dimension of $L_{PQ} \cap Y$ for $P, Q \in Y$ is therefore key to prove that the set Z_0 in the hypothesis of the theorem has small dimension. If Y is a compact portion of a smooth curve with non-vanishing curvature, then $Y \cap L_{PQ}$ consists of finitely many points, bounded independently of any choice of P, Q in the plane. This provides an alternate viewpoint to the result in [2].

Results about slices of measures, e.g. in Chapter 6 of [6] indicate that for any one dimensional set Y , for almost every line L , $L \cap Y$ is a finite collection of points. This suggests that if Y is a generic set with fractional dimension one, then Z_0 has dimension at most 2, leading to a general result finding dimension $1/2$ isosceles-avoiding subsets X of ‘projectable’ sets Y . We do not know this is true, but it is almost surely true for a random family of Cantor sets.

Example. Consider a random ‘Cantor dust’ C obtained as a limit of a decreasing family of discrete sets C_0, C_1, C_2, \dots , where C_k is a union of sidelength $1/2^k$ squares. We start by setting $C_0 = [0, 1]^2$. To construct C_{k+1} , we split C_k into sidelength $1/2^{k+1}$ squares, and then keep each square in C_{k+1} with probability p . In particular, then the probability that any sidelength $1/2^k$ square lies in C_k is p^k . On average, C_k contains $(4p)^k$ sidelength $1/2^k$ squares. If, for each dyadic square $I \in \mathcal{B}_{1/2^k}^2$, we let $X_I = \mathbf{I}(I \in C_k)$, then

$$X_k = \sum_{I \in \mathcal{B}_{2^{-k}}^2} X_I$$

denotes the number of cubes in C_k . Since the X_I are independent, we may apply Hoeffding’s inequality to conclude that

$$\mathbf{P}(|X_k - (4p)^k| \geq (4p)^k/2) = \exp(-O((8p^2)^k))$$

If $p \geq 1/2$, the term on the right hand side is certainly summable over positive integers k , which shows by the Borel Cantelli lemma that almost surely, we have $X_k \sim (4p)^k$ for sufficiently large k . This implies that the Minkowski dimension of X is equal to $2 - \log_2(1/p)$ almost surely. We are most interested when p is equal to, or slightly larger than $1/2$, so C is almost surely close to one dimensional.

Consider an arbitrary line L in the plane, such that the $1/2^k$ thickened line $L_{1/2^k}$ intersects M sidelength $1/2^k$ dyadic squares I_1, \dots, I_M in $[0, 1]^2$. We know that $M \lesssim 2^k$, and on average, intersects $p^k M \lesssim (2p)^k$ dyadic cubes in C_k . We let $Y = \sum_{m=1}^M X_{I_m}$ count the number of dyadic cubes in C_k intersecting $L_{1/2^k}$. We wish to show $Y \lesssim 2^{k(1+\varepsilon)} p^k$ with overwhelming probability, so we may assume without loss of generality that $M \geq 2^{k(1+\varepsilon)} p^k$. Hoeffding’s inequality then implies that

$$\mathbf{P}(Y \geq 2^{k(1+\varepsilon)} p^k) \leq \mathbf{P}(Y \geq p^k M + 2^{k(1+\varepsilon)} p^k) \leq \exp\left(-2p^k 2^{k(1+\varepsilon)}\right)$$

There are $O(4^k)$ lines which when thickened by $1/2^k$ are such that all lines are contained in one of the $1/2^k$ thickened line. Thus we can bound the probability

that any line intersects more than $2^{k(1+\varepsilon)}p^k$ by

$$4^k \exp(-2p^k 2^{k(1+\varepsilon)})$$

If $p \geq 1/2$, this value is summable in k , implying by the Borel Cantelli lemma that almost surely, it is eventually true that the $1/2^k$ thickened lines intersect fewer than $2^{k(1+\varepsilon)}p^k$ sidelength $1/2^k$ dyadic cubes. This implies that if

$$Z_0 = \{(y_1, y_2, y_3) \in C^3 : d(y_1, y_2) = d(y_1, y_3)\}$$

Then almost surely, we may for sufficiently large k , cover Z_0 by $32^{k+\varepsilon}p^{3k}$ dyadic cubes. This implies that the Minkowski dimension of Z_0 is bounded by $5 - 3\log_2(1/p)$. In particular, if $p = 1/2$, this shows Z_0 is almost surely two dimensional.

6 Relation to Literature, and Future Work

Our result is part of a growing body of work finding general methods to find sets avoiding patterns. The main focus of this section is comparing our method to the two other major results in the literature. [2] constructs sets with dimension $k/(n-1)$ avoiding the zero sets of rank k C^1 functions. In [5], sets of dimension d/l are constructed avoiding a degree l algebraic hypersurface specified by a polynomial with rational coefficients.

We can view our result as a robust version of Pramanik and Fraser's result. Indeed, if we try and avoid the zero set of a C^1 rank k function, then we are really avoiding a dimension $dn - k$ dimensional manifold. Our method gives a dimension

$$\frac{dn - (dn - k)}{n - 1} = \frac{k}{n - 1}$$

set, which is exactly the result obtained in [2].

That our result generalizes [2] should be expected because the technical skeleton of our construction is heavily modeled after their construction technique. Their result also reduces the problem to a discrete avoidance problem. But they *deterministically* select a particular side length S cube in every side length R cube. For arbitrary Z , this selection procedure can easily be exploited for a particularly nasty Z , so their method must rely on smoothness in order to ensure some cubes are selected at each stage. Our discrete avoidance technique was motivated by other combinatorial optimization problems, where adding a random quantity prevents inefficient selections from being made in expectation. This allows us to rely purely on combinatorial calculations, rather than employing smoothness, and greatly increases the applicability of the sets Z we can apply our method to. Furthermore, it shows that the underlying problem is robust to changes in dimension; slightly 'thickening' Z only slightly perturbs the dimension of X .

One useful technique in [2], and its predecessor [4], is the use of a Cantor set construction 'with memory'; a queue in the construction algorithm for their sets

allows storage of particular discrete versions of the problem to be stored, and then retrieved at a much later stage of the construction process. This enables them to ‘separate’ variables in the discrete version of the problem, i.e. instead of forming a single set F from a set E , they from n sets F_1, \dots, F_n from disjoint sets E_1, \dots, E_n . The fact that our result is more general, yet does not rely on this technique is an interesting anomaly. An obvious advantage is that the description of the technique is much more simple. But an additional advantage is that we can attack ‘one scale’ of the problem at a time, rather than having to rely on stored memory from a vast number of steps before the current one. We believe that we can exploit the single scale approach to the problem to generalize our theorem to a much wider family of ‘dimension α ’ sets Z , which we plan to discuss in a later paper.

As a generalization of the result in [2], our result has the same issues when compared to the result of [5]. When the parameter n is large, the dimension of our result suffers greatly, as with the n fold sum application in the last section. Furthermore, our result can’t even beat trivial results if Z is almost full dimensional, as the next example shows.

Example. *Consider an α dimensional set of angles Y , and try and find $X \subset \mathbf{R}^2$ such that the angle formed from any three points in X avoids Y . If we form the set*

$$Z = \left\{ (x, y, z) : \text{There is } \theta \in Y \text{ such that } \frac{(x - y) \cdot (x - z)}{|x - y||x - z|} = \cos \theta \right\}$$

Then we can find X avoiding Z . But one calculates that Z has dimension $3d + \alpha - 1$, which means X has dimension $(1 - \alpha)/2$. Provided the set of angles does not contain π , the trivial example of a straight line beats our result.

Nonetheless, we still believe our method is a useful inspiration for new techniques in the ‘high dimensional’ setting. Most prior literature studies sets Z only implicitly as zero sets of some regular function f . The features of the function f imply geometric features of Z , which are exploited by these results. But some geometric features are not obvious from the functional perspective; in particular, the fractional dimension of the zero set of f is not an obvious property to study. We believe obtaining methods by looking at the explicit geometric structure of Z should lead to new techniques in the field, and we already have several ideas in mind when Z has geometric structure in addition to a dimension bound, which we plan to publish in a later paper.

We can compare our randomized selection technique to a discrete phenomenon that has been recently noticed, for instance in [1]. There, certain combinatorial problems can be rephrased as abstract problems on hypergraphs, and one can then generalize the solutions of these problems using some strategy to improve the result where the hypergraph is sparse. Our result is a continuous analogue of this phenomenon, where sparsity is represented by the dimension of the set Z we are trying to avoid. One can even view Lemma 1 as a solution to a problem about independent sets in hypergraphs. In particular, we

can form a hypergraph by taking the cubes $\mathcal{B}_s^d(E)$ as vertices, and adding an edge (I_1, \dots, I_n) between n distinct cubes $I_k \in \mathcal{B}_s^d(E)$ if $I_1 \times \dots \times I_n$ intersects Z_D . An independent set of cubes in this hypergraph corresponds precisely to a set F with F^n disjoint except on a discretization of the diagonal. And so Lemma 1 really just finds a ‘uniformly chosen’ independent set in a sparse graph. Thus we really just applied the discrete phenomenon at many scales to obtain a continuous version of the phenomenon.

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