Fractals avoiding Fractal Sets

Jacob Denson

University of British Columbia

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Pattern Avoidance Problems

- How large can the Hausdorff dimension of a subset of $\mathbf{R}$ be not containing 3 term arithmetic progressions?
- What is the largest Hausdorff dimension of a subset of $\mathbf{R}^d$ such that the distances between any two points are distinct?
- What is the largest Hausdorff dimension of $\mathbf{R}^d$ such that the angles formed from any three points are irrational?

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

  - If $Z = \{(x, y, z) \in \mathbf{R}^3 : z - x = 2(y - x)\}$, X avoids 3 term arithmetic progressions if and only if $X^3 \cap Z \subset \Delta$.
  - If $Z = \{(x, y, z, w) : d(x, y) = d(z, w)\} \subset \mathbf{R}^{4d}$, then points in X have distinct distances if and only if $X^4 \cap Z \subset \Delta$.
  - If $Z = \{(x, y, z) : \frac{(x-z)\cdot(y-z)}{|x-z||y-z|} = \cos(120^\circ)\}$, X avoids 120° angles if and only if $X^3 \cap Z \subset \Delta$.

The Generic Problem

- **Fractal Avoidance Problem**: Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that $X^n \cap Z \subset \Delta$.
- Mathé (2012): If Z is an algebraic hypersurface specified by a degree r polynomial in nd variables with rational coefficients, then we can find X solving the fractal avoidance problem for Z with dimension d/r. This is independent of n.
- Pramanik and Fraser (2016): If Z is a smooth hypersurface of dimension $nd - d$, we can find X with dimension $d/(n-1)$.

What if the Patterns are Fractally Specified...

Main Result

Theorem

If Z is the countable union of sets with lower Minkowski dimension bounded by α, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathbf{H}}(X) = \min\left(\frac{nd-\alpha}{n-1}, d\right) = \min\left(\frac{codim(Z)}{n-1}, 1\right)$$

- Pramanik and Fraser's Result is a special case. Shows smoothness is only required to get a bound on the Hausdorff dimension.

Low Rank Avoidance

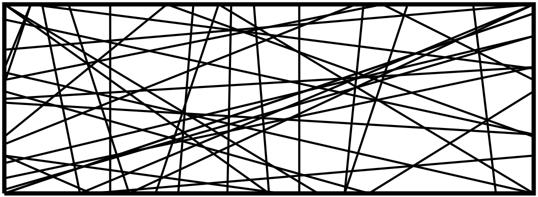
Theorem

If Z is the countable union of sets Z_i for which there exists a linear transformation $T_i : \mathbf{R}^{nd} \to \mathbf{R}^{kd}$, with rational coordinates such that $T_i(Z_i)$ has Minkowski dimension β for each i, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathbf{H}}(X) = \frac{nk - \beta}{2k - 1} = \frac{codim(T(Z))}{2k - 1}$$

- You can think of the hypothesis as saying Z is coverable efficiently by lower dimensional thickened hyperplanes in countably many directions. Result also extends to Z which is efficiently covered by algebraic hypersurfaces.
- Really want to push the $2k - 1$ to $k - 1$, at least for $k \geq 2$. Know this is true for a large number of examples.

Applications



- More robust result of Pramanik and Fraser, showing that we can 'thicken' or 'thin' the zero set of our function with stable effects on the Hausdorff dimension of X.
- Uncountable unions of regular sets are allowed!

Applications

- Given a subset Y of $\mathbf{R}^d$ which is the countable union of sets with Minkowski dimension α, we can find a $\mathbf{Q}$ vector subspace X of $\mathbf{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from Y.
- We can find a full dimensional subset of $\mathbf{R}^d$ avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $y \in \mathbf{Q}$. No dependence on the degree of the polynomial.
- Given a continuous $\gamma : [0,1] \to \mathbf{R}^d$, finding $X \subset [0,1]$ such that $\gamma(x), \gamma(y), \gamma(z)$ do not form an 'arithmetic progression' on the curve, in the sense that

  $$(\gamma(x) - \gamma(y)) - (\gamma(y) - \gamma(z)) = 0$$

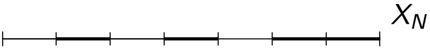
  Pramanik and Fraser can avoid smooth curves. We can now avoid *any* curve given a Hausdorff dimension calculation. Potentially even a Hilbert curve.

The Method

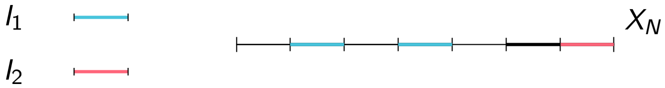
Two key ideas:

- Discretization of Scales by Queueing.
- Random Disection.

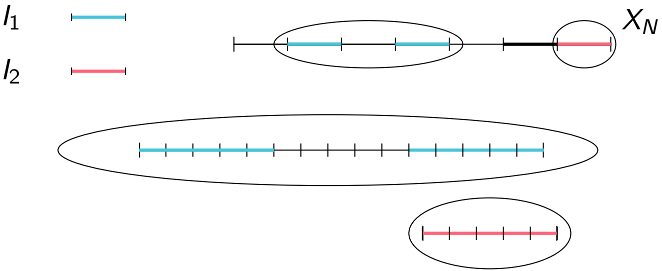
Discretization of Scales



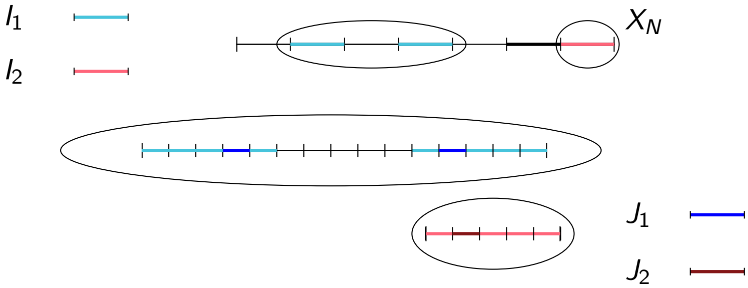
Discretization of Scales



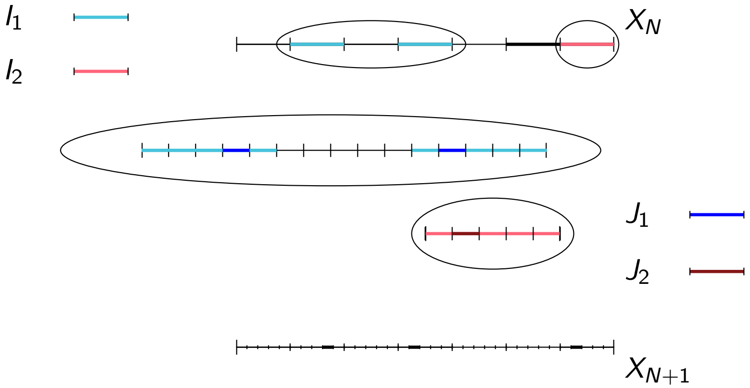
Discretization of Scales



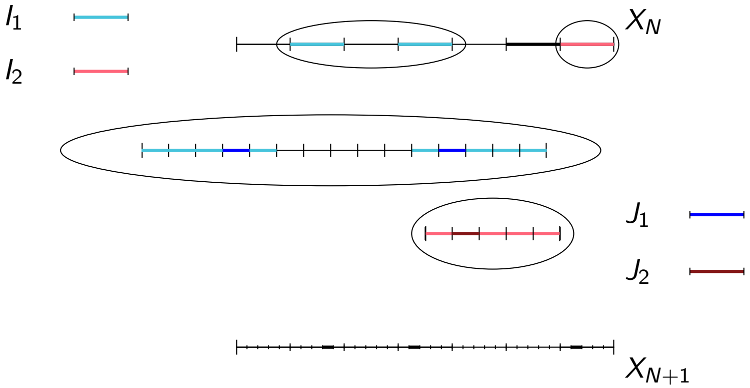
Discretization of Scales



Discretization of Scales



Discretization of Scales

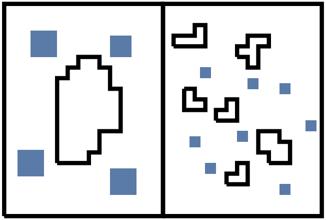


- **Discrete Problem**: Given disjoint unions of length L intervals $I_1, \ldots, I_n$, find J_i for each i containing a part of each interval in I_i such that $J_1 \times \cdots \times J_n$ is disjoint from Z.

Queueing

- Using a queueing procedure, and performing this single scale procedure over all possible choices of sets $I_1, \dots, I_n$ gives a fractal avoiding set X.
- **Reason**: If we consider distinct $x_1, \dots, x_n \in X$, there are intervals $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \cdots \times J_n$ cannot be contained in Z.

Exploiting Randomness



- How do we prove the discrete scale argument?
- Aside from Z's dimension, we have little structural knowledge.
- Random choices of the J_k avoid Z effectively.
- We obtain for all but $o(1)$ of the length L intervals in I_k, J_k contains a length L^β section of each, where $\beta = d(nd - \alpha)/(n-1)$. This ratio gives the Hausdorff dimension bound $(nd - \alpha)/(n-1)$ for X.

So What's Next?

Extension to Hausdorff Dimension

- There is no obvious reason why our techniques should fail when Z has *Hausdorff dimension* α rather than a Minkowski dimension bound.
- We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

Analogies with Hypergraphs

- Partition $I_1, \ldots, I_n$ into length $1/N^\beta$ sections. Form a hypergraph whose vertices are the sections, and add a hyperedge between $K_1 \subset I_1, \ldots, K_n \subset I_n$ if $K_1 \times \cdots \times K_n$ intersects Z.
- Our discrete configuration problem reduces to finding independant sets in hypergraphs.
- Many results in combinatorics have 'sparse analogues' when generalized to hypergraphs and random techniques are used. In this work by thinking of dimension as a continuous analogue of 'sparsity' we have come up with a continuous version of this.
- We are looking to using other methods on hypergraphs to improve the bound when Z has certain structural properties.

You're Interested in Learning More?

Read these to find out more about configuration problems:

- Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.
- Read Máthé (2017) for a construction where f is assumed to be a polynomial of bounded degree.
- Read Pramanik and Fraser (2018) for a general construction where f is smooth and nonsingular. We generalize their work.

Thanks for listening!