Outline of Proposed Research

The question of the L^p boundedness of translation-invariant operators on $\mathbb{R}^n$ has proved central to the development of modern harmonic analysis. Indeed, answers to these questions underpin any subtle understanding of the Fourier transform, since with essentially any such operator T, we can associate a function $m : \mathbb{R}^n \to \mathbb{C}$, known as the *symbol* of T, such that for any function f, the Fourier transform of Tf obeys the relation $\widehat{Tf} = m\widehat{f}$; thus translation invariant operators are also called *Fourier multiplier operators*. Initial questions about Fourier multipliers emerged from classical questions concerning the convergence properties of Fourier series, and in the study of the classical equations of physics, like the heat and wave equation. Such operators often have rotational symmetry, so it is natural to restrict our attention to multiplier operators which are also rotation-invariant. These operators are called *radial Fourier multipliers*, since the associated symbol is then a radial function. This research project proposes the study of necessary and sufficient conditions to guarantee L^p boundedness of radial multiplier operators, stimulated by recent developments which indicate lines of attack for three related problems in the field.

The general study of the boundedness of Fourier multipliers was intiated in the 1960s. It was quickly realized that the most fundamental estimates for a translation-invariant operator T were L^p estimates of the form $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ in the range $1 \leq p \leq 2$. For $p = 1$ and $p = 2$, mathematicians found simple necessary and sufficient conditions to ensure boundedness [7]. But the problem of finding necessary and sufficient conditions for boundedness in the range $1 < p < 2$ proved impenetrable. Indeed, many interesting problems about the boundedness of *specific* Fourier multiplier operators for values of p in the range, such as the Bochner-Riesz conjecture, remain largely unsolved today.

Thus it came as a surprise when recent results indicated necessary and sufficient conditions for *radial* Fourier multiplier operators to be bounded for values of p in this range. First came the result of [3], which gave a simple sufficient and necessary condition for bounds of the form $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ to hold for a particular radial multiplier operator T, uniformly over *radial* functions f, precisely in the range $1 < p < 2 - 2/(n+1)$. It is natural to conjecture that the same criterion, applied to the same range of p, gives the bound $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ for *general* functions f. In this outline we call this statement the *radial multiplier conjecture*. We now know, by the results of [6] and [2], that the radial multiplier conjecture is true when $n > 4$ and $1 < p < 2 - 4/(n+1)$, and when $n = 4$ and $1 < p < 2 - 3.79/(n+1)$. We also know [2] the criterion in the conjecture is sufficient to obtain a *restricted weak type* bound $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^n)}$ when $n = 3$ and $1 < p < 2 - 3.66/(n+1)$. But the radial multiplier conjecture has not yet been completely resolved in any dimension n, we do not have any strong type L^p bounds when $n = 3$, and we don't have any bounds whatsoever when $n = 2$.

If fully proved, the radial multiplier conjecture would imply the Bochner-Riesz conjecture, and thus the Kakeya and restriction conjectures as a result. All three consequences are major unsolved problems in harmonic analysis, so a complete resolution of the conjecture is far beyond the scope of current research techniques. On the other hand, the Bochner Riesz conjecture is completely resolved when $n = 2$, while in contrast, no results related to the radial multiplier conjecture are known in this dimension at all. And in any dimension $n > 2$, the range under which the Bochner-Riesz multiplier is known to hold [4] is strictly larger than the range under which the radial multiplier conjecture is known to hold, even for the restricted weak-type bounds obtained in [2]. Thus it still seems within hope that the techniques recently applied to improve results for Bochner-Riesz problem, such as broad-narrow analysis [1], the polynomial Wolff axioms [8], and methods of incidence geometry and polynomial partitioning [11] can be applied to give improvements to current results characterizing the boundedness of general radial Fourier multipliers.

Our hopes are further emboldened when we consult the proofs in [6] and [2], which reduce the radial multiplier conjecture to the study of upper bounds of quantities of the form $\|\sum_{(y,r)\in\mathscr{E}} F_{y,r}\|_{L^p(\mathbb{R}^n)}$, where $\mathscr{E} \subset \mathbb{R}^n \times (0,\infty)$ is a finite collection of pairs, and $F_{y,r}$ is an oscillating function supported on a $O(1)$ neighborhood of a sphere of radius r centered at a point y. The L^p norm of this sum is closely related to the study of the tangential intersections of these spheres, a problem successfully studied in more combinatorial settings using incidence geometry and polynomial partitioning methods [12], which provides further estimates that these methods might yield further estimates on the radial multiplier conjecture.

When $n = 3$, [2] is only able to obtain bounds on the L^p sums in the last paragraph when $\mathscr{E}$ is a Cartesian product of two subsets of $(0,\infty)$ and $\mathbb{R}^n$. This is why only restricted weak-type bounds have been obtained in this dimension. It is therefore an interesting question whether different techniques enable one to extend the L^p bounds of these sums when the set $\mathscr{E}$ is *not* a Cartesian product, which would allow us to upgrade the result of [2] in $n = 3$ to give strong L^p bounds. This question also has independent interest, because it would imply new results for the 'endpoint' local smoothing conjecture, which concerns the regularity of solutions to the wave equation in $\mathbb{R}^n$. Incidence geometry has been recently applied to yield results on the 'non-endpoint' local smoothing conjecture [5], which again suggests these techniques might be applied to yield the estimates needed to upgrade the result of [2] to give strong L^p-type bounds.

A third line of questioning about the radial multiplier conjecture is obtained by studying natural analogues of Fourier multiplier operators on Riemannian manifolds. Using functional calculi, for any function $m : [0,\infty) \to \mathbf{C}$, we can associate an operator $m(\sqrt{-\Delta})$ on a compact Riemannian manifold M, where Δ is the Laplace-Beltrami on M. Just like multiplier operators on $\mathbb{R}^n$ are crucial to an understanding of the Fourier transform, the operators $m(\sqrt{-\Delta})$ are crucial to understand the behaviour of eigenfunctions of the Laplace-Beltrami operator on M.

The direct analogue of the boundedness problems for multipliers is trivial in this setting, since any compactly supported function m will induce an operator $T = m(\sqrt{-\Delta})$ satisfying estimates of the form $\|Tf\|_{L^p(M)} \lesssim \|f\|_{L^p(M)}$ for all $1 \leq p \leq 2$. The correct formulation of the problem is instead to consider the alternate bound $\sup_{R>0} \|m(\sqrt{-\Delta}/R)f\|_{L^p(M)} \lesssim \|f\|_{L^p(M)}$. In fact, a transference principle of Mitjagin [10] implies that if the latter bound holds for a multiplier $m(\sqrt{-\Delta})$, then a bound of the form $\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ holds where T is the Fourier multiplier operator associated with the symbol $m(|\xi|)$. Thus boundedness in this alternate sense is at least as hard on a compact manifold M as it is in $\mathbb{R}^n$. The research project here is to try and extend the radial multiplier conjecture to this setting.

For general compact manifolds difficulties arise in generalizing the conjecture, connected to the fact that analogues of the Kakeya / Nikodym conjecture are false in this general setting [9]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [3] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [6] or even [2].

In conclusion, the results of [6] and [2] indicate three lines of questioning about radial Fourier multiplier operators, which current research techniques place us in reach of resolving. The first question is whether we can extend the range of exponents upon which the conjecture of [3] is true, at least in the case $n = 2$ where Bochner-Riesz has been solved. The second is whether we can use more sophisticated arguments to prove the L^p sum bounds obtained in [2] when $n = 3$ when the sums are no longer cartesian products, thus obtaining strong L^p characterizations in this settiong, as well as new results about the endpoint local smoothing conjecture. The third question is whether we can generalize these bounds obtained in these two papers to study radial Fourier multipliers on the sphere.

References

[1] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[2] Laura Cladek. Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Anal. PDE*, 11(2):467–498, 2018.

[3] Gustavo Garrigós and Andreas Seeger. Characterizations of Hankel multipliers. *Math. Ann.*, 342(1):31–68, 2008.

[4] Shaoming Guo, Changkeun Oh, Wu Shukun Wang, Hong, and Ruixiang Zhang. The Bochner-Riesz problem: An old approach revisited. 2021.

[5] Larry Guth, Hong Wang, and Ruixiang Zhang. A sharp square function estimate for the cone in $\mathbb{R}^3$. *Ann. of Math.*, 192(2):551–581, 2020.

[6] Yaryong Heo, Fdor Nazarov, and Andreas Seeger. Radial Fourier multipliers in high dimensions. *Acta Math.*, 206(1):55–92, 2011.

[7] Lars Hörmander. Estimates for translation invariant operators in lp spaces. *Acta Math.*, 104(1-2):93–140, 1960.

[8] Nets Hawk Katz and Keith M. Rogers. On the polynomial Wolff axioms. *Geom. Funct. Anal.*, 28(6):1706–1716, 2018.

[9] William P. Minicozzi and Christopher D. Sogge. Negative results for Nikodym maximal functions and related oscillatory integrals in curved spaces. *Math. Res. Lett.*, 4(2):221–237, 2000.

[10] Boris S. Mitjagin. Divergenz von spektralentwicklungen in L_p-räumen. *Internat. Ser. Numer. Math.*, 25:521–530, 1974.

[11] Joshua Zahl. On the Wolff circular maximal function. *Illinois J. Math.*, 56(4):1281–1295, 2012.

[12] Joshua Zahl. Sphere tangencies, line incidences, and Lie's line-sphere correspondence. *Math. Proc. Cambridge Philos. Soc.*, pages 1–21, 2021.