Analysis SEP Problems & Solutions

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Contents

Day 1: Warm Up Question

1. (Fall 2016) For $n \geq 2$ an integer, define

$$F(n) = \max\left\{k \in \mathbb{Z} : 2^k/k \leq n\right\}.$$

Does the infinite series

$$\sum_{n=2}^{\infty} 2^{-F(n)}$$

converge or diverge?

1 Day 1: Basic Analysis

2. (Fall 2017) Let $\{a_n\}$ be a sequence of complex numbers and let

$$c_n = n^{-5} \sum_{k=1}^{n} k^4 a_k.$$

   (a) Prove or Disprove: If $\lim_{n\to\infty} a_n = a$ exists, then $\lim_{n\to\infty} c_n = c$ exists.

   (b) Prove or Disprove: If $\lim_{n\to\infty} c_n = c$ exists, then $\lim_{n\to\infty} a_n = a$ exists.

3. (Fall 2018) For $c_k \in \mathbb{R}$, say that Πc_k converges if $\lim_{K\to\infty} \prod_{k=1}^{K} c_k = C$ exists with $C \neq 0, \infty$.

   (a) Prove that if $0 < a_k < 1$ for all k, or if $-1 < a_k < 0$, for all k, then $\prod(1 + a_k)$ converges if and only if $\sum_k a_k$ converges.

   (b) However, prove that $\prod_{k>1} \left(1 + \frac{(-1)^k}{\sqrt{k}}\right)$ diverges.

4. (Fall 2019) Let f be a continuous function on $\mathbb{R}$ satisfying

$$|f(x)| \leq \frac{1}{1+x^2}.$$

   Define a function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

   (a) Prove that F is continuous and periodic with period 1.

   (b) Prove that if G is continuous and periodic with period one, then

$$\int_0^1 F(x)G(x) = \int_{-\infty}^{\infty} f(x)G(x)\, dx.$$

5. (Fall 2015) Let $a_1, a_2, \ldots$ be a sequence of positive real numbers and assume that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 1.$$

   (a) Show that $\lim_{n\to\infty} a_n n^{-1} = 0$.

   (b) If $b_n = \max(a_1, \ldots, a_n)$, show that $\lim_{n\to\infty} b_n n^{-1} = 0$.

   (c) Show that

$$\lim_{n\to\infty} \frac{a_1^\beta + \cdots + a_n^\beta}{n^\beta} = \begin{cases} 0 & : \beta > 1 \\ \infty & : \beta < 1. \end{cases}.$$

6. (Fall 2021) Let $f \in C^1[0,1]$. Show that for every $\varepsilon > 0$, there exists a polynomial p such that

$$\|f - p\|_{L^\infty[0,1]} + \|f' - p'\|_{L^\infty[0,1]} \leq \varepsilon.$$

Day 2: Warm Up Question

2 Day 2: Basic Analysis

7. (Spring 2017, Spring 2011, and Spring 2007) Show that the sequence of functions

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}, \qquad n = 1, 2, 3, \ldots,$$

is uniformly bounded in $\mathbb{R}$.

Hint: Summation by parts. Break the sum into two parts for $kx \leq 1$ and $kx > 1$ respectively.

8. (Spring 2021)

$$\sum_{k=1}^{\infty} \frac{\cos(\sqrt{k})}{k}$$

converges.

9. (Fall 2015) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(x/n).$$

(a) Show that the series converges pointwise to some function f on $\mathbb{R}$.

(b) Is f continuous on $\mathbb{R}$? Does $f'(x)$ exist for all $x \in \mathbb{R}$?

(c) Does the series converge uniformly on $\mathbb{R}$?

10. (Fall 2019) Show that if $K \subset \mathbb{R}^n$, and every continuous function on K is bounded, then K is compact.

11. (Spring 2015) Prove that the integral

$$f(a) = \int_0^\infty \frac{\sin(x^2 + ax)}{x}\, dx$$

converges for $a \geq 0$, and f is continuous on $[0, \infty)$.

12. (Fall 2017) Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \int_0^n \frac{\sin(sx)}{\sqrt{s}}\, ds.$$

(a) Show that $\{f_n\}$ converges locally uniformly on $(0, \infty)$.

(b) Show that $\{f_n\}$ does *not* converge uniformly on $(0, 1]$.

(c) Does the sequence $\{f_n\}$ converge uniformly on $[1, \infty)$ as $n \to \infty$?

13. (Fall 2021) Does the improper integral

$$\int_2^\infty \frac{x \sin(e^x)}{x + \sin(e^x)}\, dx$$

converge?

Day 3: Warm Up Question

14. (Fall 2018) Prove that for $1 \leq p \leq 2$ and $0 < b < a$,

$$(a+b)^p + (a-b)^p \geq 2a^p + p(p-1)a^{p-2}b^2.$$

15. (Spring 2018 and Spring 2021) Determine if

$$\sum_{k=1}^{\infty} \frac{\cos(\sqrt{k})}{k}$$

converges.

3 Day 3: Basic Analysis

16. (Spring 2015) Let g be a non-constant differentiable real function on a finite interval $[a, b]$, with $g(a) = g(b) = 0$. Show that there exists $c \in (a, b)$ such that

$$|g'(c)| > \frac{4}{(b-a)^2} \int_a^b |g(t)|\, dt.$$

17. (Fall 2019) If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n - \{0\}$, continuous at 0, and

$$\lim_{x \to 0} \frac{\partial f}{\partial x^i}(x) = 0,$$

for $1 \leq i \leq n$, then f is differentiable at 0.

18. (Spring 2017) For any pair of sequences $\{a_k\}$ and $\{b_n\}$, show that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} \lesssim \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{m=1}^{\infty} b_m^2 \right)^{1/2}.$$

Day 4: Warm Up Problems

4 Day 4: Measure Theory

19. (Spring 2021 and Spring 2016) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $|E| < \infty$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(r) = |E \cap (E + r)|$ is continuous

20. (Spring 2015) Does there exists a Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that

$$\int_a^b f(x)\, dx = \infty$$

for all real numbers $a < b$. Find an example or show that no such function exists.

21. (Fall 2018) Two parts:
   (a) Give an example, with explanation, of each of the following:
   - A sequence of functions on $\mathbb{R}$ that converges to zero in $L^1(\mathbb{R})$, but it does not converge almost anywhere on $\mathbb{R}$ to any function.
   - A sequence of functions in $L^1(\mathbb{R})$ that converges almost everywhere to zero, but it does not converge in measure to any function.

   (b) Prove that a sequence of functions on $\mathbb{R}$ that converges to zero in measure must have a subsequence that converges to zero almost everywhere. Do not quote any theorems that trivialize the problem.

22. (Spring 2017) Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function for which $\|f'\|_\infty < \infty$. Define, for $x > 0$,

$$F(x) = \int_0^\infty f(x + yx)\psi(y)\, dy,$$

where ψ satisfies

$$\int_0^\infty |\psi(y)|\, dy \quad \text{and} \quad \int_0^\infty y \cdot |\psi(y)|\, dy < \infty.$$

Show that $F(x)$ is well defined for all $x \geq 0$, and that F is continuously differentiable.

23. (Fall 2016) Let $f : [0, 1] \to \mathbb{R}$ be continuous with $\min_{0 \leq x \leq 1} f(x) = 0$. Assume that for any $0 \leq a \leq b \leq 1$ we have

$$\int_a^b [f(x) - \min_{a \leq y \leq b} f(y)] dx \leq \frac{|b - a|}{2}.$$

Prove that for any $\lambda \geq 0$, we have

$$|\{x : f(x) > \lambda + 1\}| \leq \frac{1}{2} |\{f(x) > \lambda)\}|.$$

Day 5: Warm Up Problems

24. (Spring 2015) Let $f \in L^2[0,1]$ satisfy $\int_0^1 t^n f(t)\, dt = (n+2)^{-1}$ for $n = 0, 1, \ldots$. Must then $f(t) = t$ for almost every $t \in [0,1]$?

25. (Fall 2021) Let $\{f_n\}$ be a sequence of monotonic functions on $[0,1]$ converging to a function f in measure. Show that f coincides almost everywhere with a monotonic function f_0, and that $f_n(x) \to f_0(x)$ at every point of continuity of f_0.

5 Day 5: Functional Analysis

26. (Fall 2015) Find all $f \in L^2[0, \pi]$ such that

$$\int_0^\pi |f(x) - \sin x|^2 \, dx \le \frac{4\pi}{9}$$

and

$$\int_0^\pi |f(x) - \cos x|^2 \, dx \le \frac{\pi}{9}$$

27. (Spring 2017) Let $l^1(\mathbf{N})$ be the space of summable sequences, i.e.

$$l^1(\mathbf{N}) = \{x : \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

Let $\{a_n\}$ be a sequence with $a_n \ge 0$ for all $n \in \mathbf{N}$ and consider the subset $K \subset l^1(\mathbf{N})$ defined by

$$K = \{x \in l^1(\mathbf{N}) : 0 \le x_n \le a_n \text{ for all } n\}.$$

Show that K is compact if and only if the sequence $\{a_n\}$ itself belongs to $l^1(\mathbf{N})$.

28. (Spring 2018, Spring 2021) Let K be a continuous function on $[0, 1] \times [0, 1]$. Suppose that g is a continuous function on $[0, 1]$. Show that there exists a unique continuous function f on $[0, 1]$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x, y)dy$$

29. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.

   (a) Prove that there exists a constant A such that

   $$\|f * \phi\|_q \le A\|f\|_p$$

   for $1 \le p \le q \le \infty$.

   (b) Show by example that such general inequality cannot hold for $p > q$.

30. (Fall 2016) Give an example of a non-empty closed subset of $L^2([0, 1])$ that does not contain a vector of smallest norm. Prove your assertion.

Day 6: Warm Up Problems

6 Day 6: Functional Analysis

31. (Fall 2019) Show that there is no sequence $\{a_n\}$ of positive numbers such that $\sum a_n|c_n| < \infty$ if and only if $\{c_n\}$ is a bounded sequence. Hint: Suppose that there exists a sequence and consider the map $T : l^\infty \to l^1$ given by $Tf(n) = a_n f(n)$. The set of f such that $f(n) = 0$ for all but finitely many n is dense in l^1 but not in l^∞.

32. (Fall 2020)

    Suppose that X, Y and Z are Banach spaces, and $T : X \times Y \to Z$ is a mapping such that:

    (a) For each $x \in X$, the map $y \mapsto T(x, y)$ is a bounded linear map $Y \to Z$.

    (b) For each $y \in Y$, the map $x \mapsto T(x, y)$ is a bounded linear map $X \to Z$.

    Prove there exists a constant C such that

$$\|T(x, y)\|_Z \leq C \|x\|_X \|y\|_Y$$

33. (Fall 2015) For $p \in (1, \infty)$, and for $f \in L^p(\mathbb{R})$ define

$$Tf(x) = \int_0^1 f(x + y)\, dy.$$

    (a) Show that $\|Tf\|_p \leq \|f\|_p$ and equality holds if and only if $f = 0$ almost everywhere.

    (b) (Fall 2015) Prove that the map $f \mapsto Tf - f$ does not map $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$.

34. (Fall 2014)

    (a) For any $n \geq 1$ an integer, there exists two positive measures μ_1^n, μ_2^n supported on $[0, 1]$ such that for any polynomial $P(x)$ with $\deg P(x) \leq n$ it holds:

$$P'(0) = \int_0^1 P(x) d\mu_1^n(x) - \int_0^1 P(x) d\mu_2^n(x).$$

    (b) Does there exist two finite positive measures μ_1, μ_2 supported on $[0, 1]$ such that for any polynomial $P(x)$, it holds

$$P'(0) = \int_0^1 P(x) d\mu_1(x) - \int_0^1 P(x) d\mu_2(x)?$$

Day 7: Warm Up Problems

7 Day 7: Baire Category

35. (Spring 2020) A **Hamel basis** for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. (Hint: otherwise the Banach space would be first category in itself.)

36. (Fall 2017) Let f_n be a sequence of real functions on $\mathbb{R}$ such that each f_n' is continuous on $\mathbb{R}$. Suppose that as $n \to \infty$, f_n converges to a function $f : \mathbb{R} \to \mathbb{R}$ pointwise, and f_n' converges to a function g pointwise.

    Prove that there exists a non-empty interval (a, b) and a constant $L < \infty$ such that

    $$|f(x) - f(y)| \leq L|x - y|.$$

    Hint: Consider the sets $K_c = \{x : \sup_n |f_n'(x)| \leq c\}$.

37. (Fall 2016) Show that there is a continuous real valued function on $[0, 1]$ that is not monotone on any open interval $(a, b) \subset [0, 1]$.

38. (Spring 2014) Does there exist a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that $f_n \to \chi_{\mathbb{Q}}$ pointwise?

39. (Fall 2014) Let X, Y be Banach spaces and $\{T_{j,k} : j, k \in \mathbb{N}\}$ be a set of bounded linear transformations $X \to Y$. Suppose for each k, there exists $x \in X$ such that $\sup\{\|T_{j,k}x\| : j \in \mathbb{N}\} = \infty$. Then there is an $x \in X$ such that $\sup\{\|T_{j,k}x\| : j \in \mathbb{N}\} = \infty$ for all k.

Day 8: Warm Up Problems

8 Day 8: Intro to Distribution

40. (Fall 2020) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |x|$. Find $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$, where the derivative is taken in the sense of distributions.

41. (Spring 2017)
    (a) Let $f \in L^1(\mathbb{R})$ and consider the sequence of distributions $T_n(x) = \sin(nx^2)f(x)$. Show that $\lim_{n\to\infty} T_n = 0$ in the sense of distributions.
    (b) Find a distribution $T \in \mathcal{D}'(\mathbb{R})$ such that $T_n = \sin(nx^2)T$ does not converge to 0 in the sense of distributions as $n \to \infty$.

42. (Spring 2015)
    (a) If $f \in C[0, 1]$, and the distributional derivative f' of f on $(0, 1)$ is in $L^1((0, 1))$, prove that

    $$f(1) - f(0) = \int_0^1 f'(x)\, dx.$$

    (b) Let $p \in [1, \infty)$ and let $F \subset C[0, 1]$ be such that for each $f \in F$ we have $\|f\|_{L^1[0,1]} \le 1$ and $\|f'\|_{L^p[0,1]} \le 1$, where f' is the distributional derivative of f. Prove that F is precompact in $C[0, 1]$, or find a counter-example.

43. (Fall 2021, Spring 2016) Prove or disprove:
    (a) There exists a distribution $u \in \mathcal{D}(\mathbb{R})$ so that the restriction to $(0, \infty)$ is given by

    $$\langle u, f\rangle = \int_0^\infty e^{1/x^2} f(x)dx$$

    for all $f \in C^\infty(\mathbb{R})$ which are compactly supported in $(0, \infty)$.
    (b) There exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by

    $$\langle u, f\rangle = \int_0^\infty x^{-2} e^{i/x^2} f(x)dx$$

    for all $f \in C^\infty$ which are compactly supported in $(0, \infty)$.

44. (Fall 2017) A distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is said to be *nonnegative* if $\langle T, \phi\rangle \ge 0$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(x) \ge 0$ for all $x \in \mathbb{R}^n$.
    (a) Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and let T_f be the distribution defined by f. Show that $T_f \ge 0$ if and only if $f \ge 0$ for almost all $x \in \mathbb{R}^n$.
    (b) Show that if $T_n \to T$ in the sense of distributions, and if $T_n \ge 0$ for all n, then $T \ge 0$.
    (c) Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^2 function with $\Phi'' \ge 0$ in $\mathbb{R}$, and let $f \in C^2(\mathbb{R}^n)$ have compact support. Show that $\Delta(\Phi(f(x)) \ge \Phi'(f(x))\Delta f(x)$.
    (d) Suppose $f \in C^2(\mathbb{R}^n)$ has compact support. Show that $\Delta|f| \ge \text{sign}(f(x))\Delta f(x)$ holds in the sense of distributions. (Hint use (c) with $\Phi(t) = \sqrt{\varepsilon + t^2}$).

45. (Spring 2020)
    (a) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) = \{0\}$. If $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$, does it follow that $f\Lambda = 0$ as a distribution?
    (b) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\text{supp}(\Lambda) \subseteq K$, where $K = \{x \in \mathbb{R}^n : |x| \le 1\}$. If $f \in C^\infty(\mathbb{R}^n)$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

Day 9: Warm Up Problems

9 Day 9: Fourier Analysis + Distribution Theory

46. (Spring 2017) Let $f \in L^1(\mathbb{R}^n)$ be a function all of whose distributional derivatives $D^\alpha f$ of order $|\alpha| = m$ also belong to $L^1(\mathbb{R}^n)$. Show that if $m > n$, then $f \in C(\mathbb{R}^n)$.

47. (Fall 2019) Let $s \in \mathbb{R}$, and let $H^s(\mathbb{R})$ be the Sobolev space on $\mathbb{R}$ with the norm

$$\|u\|_{(s)} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

where $\widehat{u}$ is the Fourier transform of u. Let $r < s < t$ be real numbers. Prove that for every $\varepsilon > 0$ there is $C > 0$ such that

$$\|u\|_{(s)} \le \varepsilon \|u\|_{(t)} + C\|u\|_{(r)}$$

for every $u \in H^t(\mathbb{R})$.

48. (Fall 2019) Let $f \in L^2(\mathbb{R})$. Define

$$g(x) = \int_{-\infty}^{\infty} f(x - y) f(y) \, dy$$

Show that there exists a function $h \in L^1(\mathbb{R})$ such that

$$g(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} h(x) \, dx,$$

i.e. g is a Fourier transform of a function in $L^1(\mathbb{R})$. Hint: The following formal argument may be helpful:

$$\widehat{g}(x) = \widehat{f * f}(x) = \widehat{f}(x)^2,$$

where $*$ denotes convolution, and $\widehat{\cdot}$ denotes the Fourier transform.

49. (Fall 2015) Let f be a tempered distribution on $\mathbb{R}$ with Fourier transform

$$\widehat{f}(\xi) = 1 + \xi^{12} + \sin \xi + \operatorname{sign}(\xi).$$

Find f and f' (specify the definition of the Fourier transform you are using).

50. (Fall 2015) Recall that $H^s(\mathbb{R}^n)$ is the Sobolev space consisting of all tempered distributions g on $\mathbb{R}^n$ for which the Fourier transform $\widehat{g}$ of g is locally integrable and satisfies

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{g}(\xi)|^2 \, d\xi < \infty.$$

Let u be a Schwartz function on $\mathbb{R}^n$ and for $a \in \mathbb{C}$, let

$$f_a(x) = |x|^a u(x).$$

Show that if $\operatorname{Re}(a) > -n/2$ and $s \in [0, \operatorname{Re}(a) + n/2)$, then $f_a \in H^s(\mathbb{R}^n)$.

Day 10: Warm Up Problems

10 Day 10: Bonus Questions

51. (Fall 2017) Let $a_1, \ldots, a_n > 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{1 + \sum |x_i|^{\alpha_i}}.$$

Determine for each $p > 0$ whether

$$\int |f(x)|^p \, dx < \infty.$$

52. Let $f \in L^1(\mathbb{R})$. Let

$$G(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t^2} f(t) \, dt.$$

Prove that G is a continuous function and that $\lim_{\lambda \to \infty} G(\lambda) = 0$.

53. Let (X, μ) be a σ-finite measure space. Let $\{f_n\}$ be a sequence of measurable functions and assume that $f_n \to f$ almost everywhere. Prove that there exists measurable $A_1, A_2, \cdots \subset X$ such that $\mu(X - \bigcup_i A_i) = 0$, and such that $f_n|_{A_i} \to f|_{A_i}$ uniformly for each i.

54. (Spring 2017) Given for each function $f \in C^0(\mathbb{R}^2)$ we define for each $y \in \mathbb{R}$ a function $f_y \in C[0,1]$ by $f_y(x) = f(x, y)$. Assume that for each fixed y, the distributional derivative of $f_y \in \mathcal{D}'(\mathbb{R})$ defines a function $a_y \in L^p(\mathbb{R})$. Assume further that

$$\|a_y\|_p \leq C < \infty$$

for some constant C independent of y. Show that the distributional derivative $\partial_x f \in \mathcal{D}'(\mathbb{R}^2)$ is in $L^p_{\text{loc}}(\mathbb{R}^2)$, provided $1 < p \leq \infty$.

55. (Spring 2020) Let $E \subset [0, 1]$ be a measurable set with positive Lebesgue measure. Moreover, it satisfies the following property: As long as x and x belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

56. (a) Does $p_N = \prod_{n=2}^{N} (1 + (-1)^n/n)$ tend to a nonzero limit as $N \to \infty$.
    (b) Does $q_N = \prod_{n=2}^{N} (1 + (-1)^n/\sqrt{n})$ tend to a nonzero limit as $N \to \infty$.

11 Day 11: Bonus Questions

57. (Fall 2015) Let $\chi \in C^\infty(\mathbb{R})$ have a compact support and define

$$f_n(x) = n^2 \chi'(nx).$$

   (a) Does f_n converge in the sense of distributions as $n \to \infty$? If so, what is the limit?

   (b) Let $p \in [1, \infty)$ and $g \in L^p(\mathbb{R})$ be such that the distributional derivative of g also lies in $L^p(\mathbb{R})$. Does $f_n * g$ converge in $L^p(\mathbb{R})$ as $n \to \infty$? If so, what is the limit?

58. Let

$$s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}.$$

Prove that $s_N(x)$ converges to a limit $s(x)$ on $[0, 1]$, and that there is a constant $C > 0$ so that for all $N \geq 1$ the inequality

$$\sup_{x \in [0,1]} |s_N(x) - s(x)| \leq C N^{-2/3}$$

holds.

59. (Fall 2018) Prove that in an infinite dimensional Banach space,

   (a) every norm bounded set is weakly bounded,

   (b) every norm closed set is weakly closed

   (c) a norm bounded set has empty interior in the weak topology

60. (Spring 2016) Let $1 < p < \infty$, and let $\chi_{[1-\frac{1}{n},1]}$ denote the characteristic function of $[1 - \frac{1}{n}, 1]$. For which $\alpha \in \mathbb{R}$ does the sequences $n^\alpha \chi_{[1-\frac{1}{n},1]}$ converge weakly to 0 in $L^p(\mathbb{R})$?

61. (Spring 2018) Let x_n be a sequence in a Hilbert space H. Suppose that x_n converges to x weakly. Prove that there is a subsequence x_{n_k} such that

$$\frac{1}{N} \sum_{k=1}^{N} x_{n_k}$$

converges to x (in norm) as $N \to \infty$.

62. (Fall 2015) Let $E \subset \mathbb{R}$ be a measurable set, such that $E + r = E$ for all $r \in \mathbf{Q}$. Show that $|E| = 0$ or $|E^c| = 0$.

63. (Spring 2015) Let $\{r_n\} \in [0, 1]$ be an arbitrary sequence, and define the function

$$f(x) = \sum_{r_n < x} \frac{1}{2^n}$$

Show that f is Borel measurable, find all it's points of discontinuity, and find $\int_0^1 f(x)\, dx$.

12 Day 12: Bonus Questions

64. (Fall 2013) Let $E = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}, x_1 - x_2 \in \mathbb{Q}\}$. Is it possible to find to Lebesgue measurable sets $A_1, A_2 \subset \mathbb{R}$ such that $|A_1|, |A_2| > 0$, and $A_1 \times A_2 \subset E^c$?

65. (Fall 2015) Let (X, μ) be a measure space, and let $f : X \to \mathbb{R}$ be measurable. Then if $1 \le p < r < q < \infty$ and there is $C < \infty$ such that
$$\mu(\{x : |f(x)| > \lambda\}) \le \frac{C}{\lambda^p + \lambda^q}$$
for every $\lambda > 0$. Then $f \in L^r(\mu)$.

66. (Spring 2017) Let $E \subset \mathbb{R}^n$ be a set of finite, positive measure, and let $\{t_k\}$ be a sequence with $\{t_k\} > 0$ and $\lim_k t_k = 0$. Define, for $f \in L^p(\mathbb{R}^n)$,
$$Mf(x) = \sup_k \fint_{t_k E} |f(x - y)| \, dy.$$
Suppose furthermore that there is $C > 0$ such that
$$|\{x : Mf(x) > \lambda\}| \le C\lambda^{-p}\|f\|_p^p.$$
Show that for every $f \in L^p(\mathbb{R}^n)$,
$$\lim_k \fint_{t_k E} f(x - y) \, dy = f(x).$$
for almost every $x \in \mathbb{R}^d$.

67. (Spring 2020) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f almost everywhere on $[0, 1]$ and such that $\|f_n\|_{L^2([0,1])} \le 1$ for all n. Show that
$$\lim_{n\to\infty} \|f_n - f\|_{L^1([0,1])} = 0.$$

68. (Spring 2020) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that
$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = 0.$$

69. (Fall 2018) Let $1 < p \le \infty$. Let $(X, \mathcal{M}, \mu)$ be a finite measure speace. Let $\{f_n\}$ be a sequence of measurable functions converging μ-a.e. to the function f. Assume further that $\|f_n\|_p \le 1$ for all n. Prove that $f_n \to f$ as $n \to \infty$ in L^r for all $1 \le r < p$.

70. (Fall 2017) Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0, 1)$, i.e. that there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \le A|x - y|^\beta$. Consider the function g defined by
$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^\alpha} \, dy,$$
where $\alpha \in (0, \beta)$.
    (a) Prove that g is a continuous function at zero.
    (b) Prove that g is differentiable at zero. (Hint: Try the dominated convergence theorem).

13 Day 13: Interchanging limits and integrals + Arzela-Ascoli Theorem

13.1 Limits and Integrals (DCT and friends)

71. (Rice, Winter 2011) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on $[0,1]$ such that $\|f(x)\| \le 1$ for all $n \ge 1$ and all $0 < x \le 1$, and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for each $0 \le x \le 1$. Prove that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{\sqrt{|x-1/n|}} = \int_0^1 \frac{f(x)}{\sqrt{x}}\, dx$$

72. (Rice, Spring 2005) Compute
   (a) $\lim_{n\to\infty} \int_0^\infty \frac{x^{n-2}}{1+x^n}$.
   (b) $\lim_{n\to\infty} n\int_0^\infty \frac{\sin y}{y(1+n^2y)}\, dy$ (Hint: Substitute $x = ny$).

73. (Spring 2017) Let $f : [0,\infty) \to \mathbb{R}$ be a continuously differentiable function for which $\|f'\|_\infty < \infty$. Define, for $x > 0$,

$$F(x) = \int_0^\infty f(x + yx)\psi(y)\, dy,$$

where ψ satisfies

$$\int_0^\infty |\psi(y)|\, dy \quad \text{and} \quad \int_0^\infty y \cdot |\psi(y)|\, dy < \infty.$$

Show that $F(x)$ is well defined for all $x \ge 0$, and that F is continuously differentiable.

13.2 Arzela-Ascoli

The key part of the Arzela-Ascoli theorem to know for the qual is the following: If $\{f_n\} \subset C[0,1]$ is a sequence which is uniformly bounded and equicontinuous, then $\{f_n\}$ has a uniformly convergent subsequence.

(Note that we can replace $[0,1]$ by any compact subset of $\mathbb{R}^d$. Also, there is a converse to the theorem, but I haven't seen it used in any qual problems.)

By *uniformly bounded*, we mean that $|f_n(x)| \le C$ for all $x \in [0,1]$, $n \in \mathbb{N}$.

By *equicontinuous*, we mean that for all $\epsilon > 0$, there exists δ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $n \in \mathbb{N}$.

74. (From a UBC Math 321 Midterm) Let $\{f_n\}$ be a sequence of functions in $C[a,b]$ with no uniformly convergent subsequence. Define

$$F_n(x) = \int_a^x \sin(f_n(t))\, dt.$$

Does $\{F_n\}$ has a uniformly convergent subsequence.

75. (Fall 2004) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions whose derivatives f_n' in the sense of distributions belong to $L^2(0,1)$. The functions also satisfy $f_n(0) = 0$.

(a) Assume that

$$\lim_{n\to\infty}\int_0^1 f_n'(x)g(x)dx$$

exists for all $g \in L^2(0,1)$. Show that the f_n converge uniformly as $n \to \infty$.

(b) Assume that

$$\lim_{n\to\infty}\int_0^1 f_n'(x)g(x)dx$$

exists for all $g \in C([0,1])$. Do we still have the f_n converge uniformly?

76. (Spring 2014)

Consider the following operator

$$Af = \frac{1}{x\sqrt{1+|\log x|}}\int_0^x f(t)dt$$

Is A bounded as an operator from $L^2[0,1]$ to $L^2[0,1]$. Is it compact?

77. (Problem 36 from the 2017 SEP) Consider the Hilbert space $L^2([0,1])$ with inner product $(f,g) := \int_0^1 f(t)\bar{g}(t)dt$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system of functions in $L^2([0,1])$.

(a) Suppose that $e_n \in L^2([0,1])$ for all $n \in \mathbb{N}$. Show that

$$\sup_n \max_{x\in[0,1]} |e_n'(x)| = \infty.$$

(b) Suppose that e_n is complete, which means $(g, e_n) = 0$ for all n implies $g = 0$ almost everywhere. Prove

$$\sum_{n=1}^{\infty} |e_n(x)|^2 = \infty, \quad \text{almost everywhere.}$$

14 Day 14: Misc. Topics

78. (Rice, Winter 2008) Is it possible to construct a measurable set $E \subset \mathbb{R}$ of positive measure such that for any pair $a < b$, $|E \cap [a, b]| \leq 0.5(b - a)$?

79. (Spring 2010) For $\lambda > 0$, set
$$F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} dx$$
Prove there exists constants A and $C > 0$, such that $F(\lambda) = \frac{A}{\lambda^{\frac{1}{4}}} + E(\lambda)$ where $|E(\lambda)| \leq \frac{C}{\lambda^{\frac{1}{2}}}$.

80. (Fall 2010) Let $I = [0, 1]$ and define for $f \in L^2(I)$ the Fourier coefficients as $\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}dt$ for any $k \in \mathbb{Z}$.
    (a) Let $\mathcal{G}$ be the set of all $L^2(I)$ functions with the property that $|\hat{f}(0)| \leq 1$ and $|\hat{f}(k)| \leq |k|^{-3/5}$ for any $k \in \mathbb{Z}$, $k \neq 0$. Prove that $\mathcal{G}$ is a compact subset of $L^2(I)$.
    (b) Let $\mathcal{E}$ be the set of all $L^2(I)$ functions with the property that $\sum_k |\hat{f}(k)|^{5/3} \leq 2016^{-2016}$. Is $\mathcal{E}$ a compact subset of $L^2(I)$?

81. (Fall 2011) Let $\ell^2(\mathbb{N})$ denote the Hilbert space of square summable sequences with inner product $(x, y) = \sum_{n=1}^{\infty} x_n y_n$, where $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$.
    (a) What are the necessary and sufficient conditions on $\lambda_n > 0$ for the set
    $$S = \big\{(x_1, x_2 \cdots) \in \ell^2(\mathbb{N}) : |x_n| \leq \lambda_n, \forall n\big\}$$
    to be compact in $\ell^2(\mathbb{N})$?
    (b) What are the necessary and sufficient conditions on $\mu_n > 0$ for the set
    $$\left\{(x_1, x_2 \cdots) \in \ell^2(\mathbb{N}) : \sum_n \frac{|x_n|^2}{\mu_n^2} \leq 1\right\}$$
    to be compact in $\ell^2(\mathbb{N})$?

82. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, let $E = \{x \in \mathbb{R} : f \text{ is not differentiable at } x\}$. Show that E is at most countable.

83. (Fall 2015) Identify all $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} \sin(2\pi n\alpha)$ exists.

15 Day 15: Spring 2021 Final Qualifying Exam

84. (3) For a Lebesgue measurable subset E of $\mathbb{R}$, denote $\mathbf{1}_E$ the indicator function of E (i.e. $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \in E^c$).

    Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that

    $$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \mathbf{1}_{E_n}(x)| dx = 0.$$

    Prove that f is the indicator function of a measurable set.

85. (Spring 2021) Let f be a C^1 function on $[0, \infty)$. Suppose that

    $$\int_0^\infty t|f'(t)|^2 \, dt < \infty$$

    and

    $$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) \, dt = L.$$

    Show that $f(t) \to L$ as $t \to \infty$.

86. (6) Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Holder condition with exponent $\beta \in (0, 1)$, i.e., there exists a constant $A < \infty$ such that $\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq A|x - y|^\beta$. Consider the function g defined by

    $$g(x) = \int_{-\infty}^\infty \frac{f(y)}{|x - y|^\alpha} dy$$

    where $\alpha \in (0, \beta)$.

    (a) Prove that g is a continuous function at 0.

    (b) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem).

87. (7R) Let $f_n \to f$ weakly in $L^2(\mathbb{R})$ and $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$. Show that $f_n \to f$ strongly in $L^2(\mathbb{R})$.

88. (8R)

    (a) Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \to H_2$ be a continuous linear operator. Give a precise definition of the adjoint operator T^*.

    (b) Let $(a, b) \subset \mathbb{R}$ be a (possibly infinite) open interval. If $f \in L^2(a, b)$, explain what it means that the distributional derivative f' is also in $L^2(a, b)$.

    (c) Let $\mathbb{R}_+$ denote the positive real axis $[0, \infty)$. Let $H^1(\mathbb{R})$ (respectively $H^1(\mathbb{R}_+)$) be the space of real-valued functions $f \in L^2(\mathbb{R})$ (respectively $f \in L^2(\mathbb{R}_+)$ such tha thte distributional derivative f' is also in $L^2(\mathbb{R})$ (respectively $L^2(\mathbb{R}_+)$). Then $H^1(\mathbb{R})$ and $H^1(\mathbb{R}_+)$ are Hilbert spaces with inner product given by

    $$\langle f, g\rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)dx + \int_{\mathbb{R}} f'(x)g'(x)dx,$$
    $$\langle f, g\rangle_{H^1(\mathbb{R}_+)} = \int_{\mathbb{R}_+} f(x)g(x)dx + \int_{\mathbb{R}_+} f'(x)g'(x)dx$$

    Let $T : H^1(\mathbb{R}) \to H^1(\mathbb{R}_+)$ be the mapping given by the restriction. Compute exactly the adjoint operator T^*.

89. (9R) A real valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there exists a constant C so that for every $\varepsilon > 0$, there is a bounded function $\varphi \in C^{\infty}(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} |f(x)-\varphi(x)| \leq C\varepsilon^{1/2} \quad \text{and} \quad \sup_{x\in\mathbb{R}} \varepsilon^{1/2}|\varphi'(x)| \leq C.$$

16 Questions that need solutions

90. (Fall 2017) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function that has continuous partial derivatives in $\mathbb{R}^2$. Define $\chi : \mathbb{R}^3 \to \{0,1\}$ by $\chi(x) = 1$ if $x_3 > g(x_1, x_2)$, and $\chi(x) = 0$ otherwise. Compute the derivatives $\partial\chi/\partial x^i$ for $i = 1, 2, 3$.

91. Let $\alpha \in (0,1)$, and for $f \in C[0,1]$, and $x \in [0,1]$, define

$$(T_\alpha f)(x) = \int_0^1 \sin(x+y)|x-y|^{-\alpha} f(y)\, dy.$$

   (a) Prove that T_α extends to a bounded operator on $L^2[0,1]$.

   (b) For which $\alpha \in (0,1)$ is $T_\alpha : L^2[0,1] \to L^2[0,1]$ a compact operator?