Theta Functions

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December 11, 2018

In this talk, we discuss a family of modular forms with some important applications to number theory, known as Theta functions. The most basic is the **Jacobi Theta function**, a holomorphic function defined by the q series

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}$$

It arises both as a fundamental solution to the heat equation on the real line, and as a generating function representing certain number theory problems involving squares of integers. It is a special case of the more general **Theta function**

$$\Theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

For each disk in z and proper upper half plane, the series converges uniformly, and so we get a holomorphic function entire in z, and holomorphic for $\tau \in \mathbf{H}$. For convention as is done in the study of these classical functions like the theta series, even though we have used q to denote $e^{2\pi i \tau}$, in this talk we denote q by $e^{\pi i \tau}$. Thus $\theta(\tau) = \sum q^{n^2}$.

Theorem 1. *$\Theta(\cdot|\tau)$ is elliptic in z with period 1 and 'quasiperiod' τ.*

Proof. It is easy to see that $\Theta(z+1|\tau) = \Theta(z|\tau)$, so the function is periodic. A manipulation gives

$$\begin{aligned}\Theta(z+\tau|\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i (n^2+2n)\tau} e^{2\pi i n z} \\ &= e^{-\pi i \tau} \sum_{n=-\infty}^{\infty} e^{\pi i (n+1)^2 \tau} e^{2\pi i n z} = q^{-1} e^{-2\pi i z} \Theta(z|\tau)\end{aligned}$$

So we have a kind of periodicity with an added term. □

Theorem 2. *Θ has modular symmetry in τ.*

Proof. We have $\Theta(z|\tau+2) = \Theta(z|\tau)$. To obtain an interesting modular transformation rule, we apply Poisson summation. Consider the function

$$f(y) = e^{-\pi t (y+x)^2}$$

Then $(\delta_{-x} \circ M_{t^{1/2}})(e^{-\pi y^2}) = f$, and so since $e^{-\pi y^2}$ is it's own Fourier transform,

$$\widehat{f}(\xi) = \frac{e^{-2\pi i \xi x}}{t^{1/2}} e^{-\pi \xi^2/t}$$

Applying Poisson's summation formula, we conclude that

$$e^{-\pi t x^2} \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} e^{-2\pi n t x} = \sum_{n=-\infty}^{\infty} e^{-\pi t (n+x)^2} = \frac{1}{t^{1/2}} \sum_{n=-\infty}^{\infty} e^{2\pi i n x} e^{-\pi n^2/t}$$

We can rearrange this formula to give something interesting about the Theta function. The left hand side is $e^{-\pi t x^2} \Theta(xit|it)$ and the right hand side is $\Theta(x|i/t)/t^{1/2}$. Writing $z = x$ and $\tau = it$, then the equation can be rearranged to read

$$\Theta(z| - 1/\tau) = \sqrt{-i\tau} \cdot e^{\pi i \tau z^2} \cdot \Theta(z\tau|\tau)$$

Both sides of this equation are holomorphic, and we have shown the equation holds for a non isolated set of values z and τ. Because of this, the equation actually holds *everywhere*. □

Corollary 3. $\theta(-1/\tau) = \Theta(0| - 1/\tau) = \sqrt{-i\tau} \cdot \theta(\tau)$

Thus θ has a modular character with respect to the subgroup of Γ generated by the transformations $T : \tau \mapsto \tau + 2$ and $S : z \mapsto -1/\tau$. We shall find that these generate a family of symmetries whose fundamental domain has two, non-equivalent cusps, one at ∞, and the other at 1. We have already given the q series expansion around ∞. Using Θ, we can actually cheat out an expansion for θ near one.

Theorem 4. $\theta(1 - 1/\tau) \sim 2\sqrt{-i\tau} e^{i\pi\tau/4}$ *as* $Im(\tau) \to \infty$.

Proof. We calculate that

$$\theta(1 + 2\tau) = \sum_{n=-\infty}^{\infty} (-1)^{n^2} e^{i\pi n^2 \tau} = \Theta(1/2|\tau)$$

So

$$\begin{aligned}\theta(1 - 1/\tau) &= \Theta(1/2| - 1/\tau) = \sqrt{-i\tau} \cdot e^{\pi i \tau/4} \cdot \Theta(\tau/2|\tau) \\ &= \sqrt{-i\tau} \cdot \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n^2+n+1/4)} = \sqrt{-i\tau} \cdot \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n+1/2)^2}\end{aligned}$$

and $n = 0$ and $n = -1$ give the dominant terms as $\mathrm{Im}(\tau) \to \infty$. □

1 The Modular Character

The transformations T and S generate a subgroup of Γ, which Θ has modular symmetry to.

Theorem 5. *T and S generate the group*

$$\Gamma_2(2) = \left\{ z \mapsto \frac{az+b}{cz+d} : a \equiv d \ (mod\ 2), b \equiv c \ (mod\ 2), a \not\equiv b \ (mod\ 2) \right\}$$

which has index three subgroup of Γ.

Proof. One can follow essentially the same proof for Γ to show the first property. If we consider the reduction map $R : SL_2(\mathbf{Z}) \to SL_2(\mathbf{Z}_2)$, then

$$\Gamma_2(2) = R^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

Since $GL_2(\mathbf{Z}_2) = SL_2(\mathbf{Z}_2)$ has 6 elements, $\Gamma_2(2)$ is the inverse of an index 3 subgroup of $SL_2(\mathbf{Z}_2)$. In particular, this means $\Gamma_2(2)$ is also an index 3 subgroup of Γ. □

A geometric consequence is that we have two cusps, and three elliptic points, as well as a fundamental domain $D = \{\tau \in \mathbf{H} : |\mathrm{Re}(\tau)| \leq 1 \ |\tau| \geq 1\}$. In particular, the dimension of $M_k(\Gamma_2(2))$ is bounded above by $k/4 + 1$. We only need the fact that $M_1(\Gamma_2(2))$ is one dimensional, so every modular function with respect to $\Gamma_2(2)$ is constant.

Unfortunately, θ is not really a modular form over $\Gamma_2(2)$, even of half weight, because of the presence of the factor $\sqrt{i}$ in it's functional equation. One can define modular forms with a 'nebentypus coefficient', and prove facts about the dimension of such spaces, which would suffice to classify the behaviour of θ^2, but we prefer to do things in a much more elementary way. Given f and g with $f(-1/\tau) = (-i\tau)^k f(\tau)$ and $g(-1/\tau) = (-i\tau)^k g(\tau)$, we find that $f(-1/\tau)/g(-1/\tau) = f(\tau)/g(\tau)$. If we can prove f/g is holomorphic at cusps, then $f/g \in M_1(\Gamma_2(2))$ must be a constant function, so there is A such that $f(\tau) = Ag(\tau)$. The values at the cusps then dictate the value of A.

2 Sums of Squares

We let $r_2(N)$ denote the number of ordered pairs of integers $n, m \in \mathbf{Z}^2$ such that $n^2 + m^2 = N$. For instance, $r_2(5) = 8$ since

$$5 = 1^2 + 2^2 = (-1)^2 + 2^2 = 1^2 + (-2)^2 = (-1)^2 + (-2)^2$$

and each representation here corresponds to two different ordered sums. Since $\theta(z) = \sum q^{n^2}$, $\theta^2(z) = \sum q^{n^2+m^2}$, so the N'th coefficient in the q series expansion of θ^2 is precisely $r_2(N)$.

Theorem 6. *For all N, we have $r_2(N) = 4[d_1(N) - d_3(N)]$, where $d_1(N)$ denotes the number of divisors of N congruent to one modulo four, and $d_3(N)$ the divisors congruent to three.*

Proof. We reduce the proof to the simple identity

$$\theta(\tau)^2 = 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}}$$

Once this is proved, we have

$$\begin{aligned} 2\sum_{n=-\infty}^{\infty} \frac{1}{q^n + q^{-n}} &= 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \\ &= 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^{4n}} - \frac{q^{3n}}{1-q^{4n}} \end{aligned}$$

Plugging in the identity $(1-q^{4n})^{-1} = \sum q^{4nm}$, then expanding out the right hand side gives

$$\sum \frac{q^{2n}}{1-q^{4n}} = \sum q^{n(4m+1)} = \sum d_1(k) q^k \qquad \sum \frac{q^{3n}}{1-q^{4n}} = \sum q^{n(4m+3)} = \sum d_3(N) q^N$$

Thus provided we are able to prove the initial identity above, we will be done.

Let us define

$$f(\tau) = 2\sum \frac{1}{q^n + q^{-n}} = \sum_{n=-\infty}^{\infty} \frac{1}{\cos(n\pi\tau)}$$

We will prove that f has the same symmetry as θ^2, and also has the same asymptotics near cusps. It is obvious that $f(\tau+2) = f(\tau)$. We require two trigonometric identities:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi n t)} = \frac{1}{t}\sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi n/t)}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\cosh(\pi n/t)} = t\sum_{n=-\infty}^{\infty} \frac{1}{\cos(\pi(n+1/2)t)}$$

The first says exactly that $f(-1/\tau) = -i\tau f(\tau)$. The second ca be analytically continued to say

$$f(1-1/\tau) = -i\tau \sum_{n=-\infty}^{\infty} \frac{1}{\cos(\pi(n+1/2)\tau)}$$

This gives $f(1-1/\tau) \sim -4i\tau \cdot e^{\pi i \tau/2}$ as $\mathrm{Im}(\tau) \to \infty$. And $f(i\infty) = 1$. Thus $f(\tau)/\theta^2(\tau)$ is holomorphic at cusp points, and since $f(i\infty)/\theta^2(i\infty) = 1$, we conclude $f(\tau) = \theta^2(\tau)$ for all τ. This completes the proof. $\square$

Corollary 7. *An integer N is the sum of two perfect squares if and only if every prime p congruent to three modulo four divides into N an even number of times.*

Proof. Let p be a prime with $p \equiv 1$ modulo 4. Then $r_2(p^n) = 4(n+1)$. This is because p has precisely $n+1$ divisors equal to 1 modulo 4, and zero equal to 3 modulo 4. In particular, $r_2(p) = 8$, and thus there is a *unique* choice of perfect squares n^2 and m^2 such that $n^2 + m^2 = p$. On the other hand, if $p \equiv 3$ modulo 4, then $d_1(p^n) = \lfloor (n+1)/2 \rfloor$, since only odd powers are congruent to one modulo 4, and $d_3(p^n) = \lceil (n+1)/2 \rceil$. Now these two numbers are equal if and only if n is odd, so p^n can be written as a sum of squares if and only if n is odd. Now if N and M are relatively prime, a divisor of NM can be split uniquely into a divisor of N and M and so

$$\begin{aligned}
r_2(NM) &= 4[d_1(NM) - d_3(NM)] \\
&= 4(d_1(N)d_1(M) + d_3(N)d_3(M)) - 4(d_1(N)d_3(M) + d_3(N)d_1(M)) \\
&= r_1(N)r_2(M)/4
\end{aligned}$$

Thus $r_2(NM) = 0$ if and only if $r_1(N) = 0$ or $r_2(M) = 0$. ☐

3 Forbidden Eisenstein Series

We couldn't define the weight two Eisenstein series properly since the series

$$\sum_{n,m \in \mathbf{Z}} \frac{1}{(n\tau + m)^2}$$

does not converge absolutely. Nonetheless, we shall find it *is* useful to define a 'forbidden' weight two Eisenstein series so we can attack the four squares problem. We consider the two functions

$$F(\tau) = \sum_m \sum_n \frac{1}{(m\tau + n)^2} \qquad \tilde{F}(\tau) = \sum_n \sum_m \frac{1}{(m\tau + n)^2}$$

where the order of m and n is *integral* in both series is integral. The modular identity $F(-1/\tau) = \tau^2 F(\tau)$ does not hold, since we cannot rearrange sums, but the standard manipulations for Eisenstein series gives $F(-1/\tau) = \tau^2 \tilde{F}(\tau)$. Using the Dedekind eta function $\eta(\tau) = q^{1/12} \prod (1 - q^{2n})$, which has the modular symmetry $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, we can give a modular symmetry for F itself.

Theorem 8. $F(-1/\tau) = \tau^2 F(\tau) - 2\pi i \tau$.

Proof. The same manipulations as for normal Eisenstein series give that $F(-1/\tau) = \tau^2 \tilde{F}(\tau)$, albeit this time we cannot interchange summation to give $\tilde{F}(\tau) = F(\tau)$. Now if we take the logarithmic derivative of η, we get

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{\pi i}{12} - 2\pi i \sum \frac{1}{1 - q^{2n}} = \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \sigma_1(n) q^{2n}$$

But the standard expansion of Eisenstein series in terms of the divisor function give that $F(\tau) = \pi^2/3 - 8\pi^2 \sum \sigma_1(n) q^{2n}$, so we have shown that $\eta'(\tau)/\eta(\tau) =$

$(i/4\pi)F(\tau)$. Taking the logarithmic derivative of both sides of the functional equation gives

$$\frac{\eta'(-1/\tau)}{\eta(-1/\tau)} = \frac{i}{4\pi\tau^2}F(-1/\tau) = \frac{i\tilde{F}(\tau)}{4\pi}$$

$$\frac{(\sqrt{-i\tau}\eta(\tau))'}{\sqrt{-i\tau}\eta(\tau)} = \frac{1}{2\tau} + \frac{iF(\tau)}{4\pi}$$

Putting these two equations together gives the required result. □

4 Four Squares

We now prove Lagrange's theorem that every positive integer is the sum of four squares. Moreover, to do this we will define an exact formula for the number of ways $r_4(N)$ that we can do this for each N.

Theorem 9. *For all N, $r_4(N) = 8\sigma_1^*(N)$, where $\sigma_1^*(N)$ denotes the number of divisors of N not divisible by 4.*

Proof. Like with two squares, it shall suffice to prove an identity for $\theta^4(\tau)$. To do this, we must dive into the 'forbidden' Eisenstein series of weight two. We consider

$$E_2^*(\tau) = \sum_m \sum_n \frac{1}{(m\tau/2+n)^2} - \sum_m \sum_n \frac{1}{(m\tau+n/2)^2} = F(\tau/2) - 4F(2\tau)$$

A manipulation of the Eisenstein series gives that

$$E_2^*(\tau) = \left(\frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k)q^k\right) - 4\left(\frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k)q^{4k}\right)$$

The fact that $\sigma_1^*(N) = \sigma_1(N)$ if N is not divisible by four, and $\sigma_1^*(N) = \sigma_1(N) - 4\sigma_1(N/4)$ if N is divisible by four, shows

$$E_2^*(\tau) = -\pi^2 - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1^*(k)q^k$$

It therefore suffices to prove that $\theta^4(\tau) = -E_2^*(\tau)/\pi^2$. To do this, we need only verify that E_2^* satisfies the same transformational properties as θ^4, and has the same asymptotic properties near cusp points.

It is obvious that $E_2^*(\tau+2) = E_2^*(\tau)$. We then calculate that

$$\begin{aligned} E_2^*(-1/\tau) &= F(-1/2\tau) - 4F(-2/\tau) \\ &= [4\tau^2 - 4\pi i\tau] - 4[(\tau/2)^2 F(\tau/2) - \pi i\tau] \\ &= 4\tau^2 F(2\tau) - 4(\tau^2/4)F(\tau/2) \\ &= -\tau^2(F(\tau/2) - 4F(2\tau)) = -\tau^2 E_2^*(\tau) \end{aligned}$$

We have already calculated the q series expansion for E_2^*. To calculate the q series expansion around 1, we calculate that

$$F(1/2-1/2\tau) = F\left(\frac{\tau-1}{2\tau}\right) = \left(\frac{2\tau}{\tau-1}\right)^2 F\left(\frac{2\tau}{1-\tau}\right) - \frac{4\pi i\tau}{1-\tau}$$

$$\begin{aligned} F\left(\frac{2\tau}{1-\tau}\right) &= F\left(-2+\frac{2}{1-\tau}\right) \\ &= F\left(\frac{2}{1-\tau}\right) = \left(\frac{1-\tau}{2}\right)^2 F\left(\frac{\tau-1}{2}\right) - 2\pi i\left(\frac{\tau-1}{2}\right) \end{aligned}$$

Thus

$$F(1/2-1/2\tau) = \tau^2 F\left(\frac{\tau-1}{2}\right) - \frac{4\pi i\tau}{1-\tau} - 2\pi i\frac{4\tau^2}{(\tau-1)^2}\left(\frac{\tau-1}{2}\right)$$

But $F(2-2/\tau) = F(-2/\tau) = (\tau^2/4)F(\tau/2) - 2\pi i\tau/2$, and so

$$\begin{aligned} E_2^*(1-1/\tau) &= F(1/2-1/2\tau) - 4F(2-2/\tau) \\ &= \tau^2\left(F\left(\frac{\tau-1}{2}\right) - F(\tau/2)\right) - 2\pi i\left(\frac{2\tau}{1-\tau} + \frac{2\tau^2}{\tau-1}\right) + 4\pi i\tau \\ &= \tau^2\left(F\left(\frac{\tau-1}{2}\right) - F(\tau/2)\right) \end{aligned}$$

Thus we conclude that $|E_2^*(1-1/\tau)| \lesssim |\tau^2 e^{\pi i\tau}|$ as $\text{Im}(\tau) \to \infty$, since we have $F(\tau) = \pi^2/3 + O(e^{2\pi i\tau})$ as $\text{Im}(\tau) \to \infty$. But this means $E_2^*(\tau)/\theta^4(\tau)$ is bounded as we approach cusp points, and therefore holomorphic there. Since we have $E_2^*(i\infty)/\theta^4(\tau) = -\pi^2$, this completes the proof. □

Since it is obvious that $\sigma_1^*(N) \geq 1$ for all $N \geq 1$, this shows that it is possible to write *every* integer as a sum of four squares.

Remark. We can continue this pattern. In the case of sums of eight squares, it is actually much simpler than in the two cases to come up with a formula for the number of representations, because we actually do have a proper modular form of weight 2.