Radial Multipliers

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Chapter 1

Heo, Nazarov, and Seeger

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's paper 2011 *Radial Fourier Multipliers in High Dimensions* [1]. Recall that if $m \in L^\infty(\mathbf{R}^d)$ is the symbol of a Fourier multiplier operator T_m, then we let $\|m\|_{M^p(\mathbf{R}^d)}$ denote the operator norm of T_m from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. The goal of this paper is to show that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, $1 < p < (2d-2)/(d+1)$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero,

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}.$$

where the implicit constant depends on p and η. Since

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\mathrm{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}$$

we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family $\{\mathrm{Dil}_t \eta\}$.

In Garrigós and Seeger's 2007 paper *Characterizations of Hankel Multipliers*, it is proved that for $1 < p < 2d/(d+1)$, if $\eta \in \mathcal{S}(\mathbf{R}^d)$ is a nonzero, radial Schwartz function, then

$$\|m\|_{M_{\mathrm{rad}}^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

where $M_{\mathrm{rad}}^p(\mathbf{R}^d)$ is the operator norm of T_m from $L_{\mathrm{rad}}^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. Thus, at least in the range $1 < p < (2d-2)/(d+1)$, boundedness of T_m on radial functions is equivalent to boundedness on all functions.

1.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a convolution inequality, which also proves local smoothing results for the wave equation.

Let σ_r be the surface measure on the radius r sphere centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial Schwartz function $\psi \in \mathcal{S}(\mathbf{R}^d)$. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $f_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$. Our goal is to prove the following inequality.

Lemma 1.1. *For any $a : \mathbf{R}^d \times [1, \infty) \to \mathbf{C}$, and $1 \leqslant p < (2d-2)/(d+1)$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.$$

where the implicit constant depends on p, d, and ψ.

Why is Lemma 1.1 useful? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier given by some function $\tilde{m} : [1, \infty) \to \mathbf{C}$, and we set $a_r(x) = \tilde{m}(r)f(x)$ for some $f : \mathbf{R}^d \to \mathbf{C}$. Then it is simple to check that

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr = K * \psi * f$$

where $K(x) = |x|^{1-d} m(x)$. In this setting, Lemma 1.1 says that

$$\|K * \psi * f\|_{L^p(\mathbf{R}^d)} \lesssim \|m\|_{L^p(\mathbf{R}^d)} \|f\|_{L^p(\mathbf{R}^d)},$$

which is clearly related to the convolution bound we want to show if $\psi = \hat{\eta}$, provided that we are dealing with a multiplier m that is supported away from the origin.

To understand Lemma 1.1 it suffices to prove the following discretized estimate.

Theorem 1.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1, \infty)$, which is discretized in the sense that $|(x_1, r_1) - (x_2, r_2)| \geqslant 1$ for each distinct pair (x_1, r_1) and (x_2, r_2) in $\mathcal{E}$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < (2d - 2)/(d + 1)$,*

$$\left\| \sum_{(x,r) \in \mathcal{E}} a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r) \in \mathcal{E}} |a_r(x)|^p r^{p-1} \right)^{1/d},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independant of $\mathcal{E}$.

Proof of Lemma 1.1 from Lemma 1.2. For any $a : \mathbf{R}^d \times [1, \infty) \to \mathbf{C}$,

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} = \int_{[0,1)^d} \int_0^1 \sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx}) \, dr \, dx$$

Minkowski's inequality thus implies that

$$\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} \|_{L^p(\mathbf{R}^d)} \leqslant \int_{[0,1)^d} \int_0^1 \| \sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx}) \|_{L^p(\mathbf{R}^d)} \, dr \, dx$$

$$\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} |a_r(x)|^p r^{p-1} \right)^{1/p} dr \, dx$$

$$\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} |a_r(x)|^p r^{p-1} \, dr \, dx \right)^{1/p}$$

$$= \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} \, dr \, dx \right)^{1/p}. \qquad \square$$

Lemma 1.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r) \in \mathcal{E}} a_r(x) f_{xr}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x, r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 1.2 is implied by the following Lemma.

Lemma 1.3. *For any $1 \leqslant p < (2d - 2)/(d + 1)$ and $k \geqslant 1$,*

$$\| \sum_{(x,r) \in \mathcal{E}_k} f_{xr} \|_{L^p(\mathbf{R}^d)} \lesssim 2^{k(d-1)} \#(\mathcal{E}_k)^{1/p} = 2^k \cdot (2^{k(d-p-1)} \#(\mathcal{E}_k))^{1/p}.$$

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Proof of Lemma 1.2 from Lemma 1.3. Applying a dyadic interpolation result (Lemma 2.2 of the paper), Lemma 1.3 implies that

$$\|\sum_{(x,r)\in\mathcal{E}} f_{xr}\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum 2^{kp}2^{k(d-p-1)}\#(\mathcal{E}_k)\right)^{1/p} = \left(\sum 2^{k(d-1)}\#(\mathcal{E}_k)\right)^{1/p}$$

This is a restricted strong type bound for Lemma 1.2, which we can then interpolate. □

If ψ is compactly supported, and r is sufficiently large depending on the size of this support, then f_{xr} is supported on an annulus with centre x, radius r, and thickness $O(1)$. Thus $\|f_{xr}\|_{L^p(\mathbf{R}^d)} \sim r^{(d-1)/p}$, which implies that

$$\|\sum_{(x,r)\in\mathcal{E}_k} f_{xr}\|_{L^p(\mathbf{R}^d)} \gtrsim 2^{k(d-1)/p}\#(\mathcal{E}_k)^{1/p}$$

so this bound can only be true if $p \geqslant 1$, and becomes tight when $p = 1$, where we actually have

$$\|\sum_{(x,r)\in\mathcal{E}_k} f_{xr}\|_{L^1(\mathbf{R}^d)} \sim 2^{k(d-1)}\#(\mathcal{E}_k)$$

because there can be no constructive interference in the L^1 norm. Understanding the sum in Lemma 1.3 for $1 < p < (2d-2)/(d+1)$ will require an understanding of the interference patterns of annuli with comparable radius. We will use almost orthogonality principles to understand these interference patterns.

Lemma 1.4. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ with $|x_1 - x_2| \geqslant 1$ or $x_1 = x_2$, and $r_1, r_2 > 1$,*

$$|\langle f_{x_1 r_1}, f_{x_2 r_2}\rangle| \lesssim_N (r_1 r_2)^{(d-1)/2}(1+|r_1-r_2|+|x_1-x_2|)^{-(d-1)/2}\sum_{\pm,\pm}(1+||x_1-x_2|\pm r_1\pm r_2|)^{-N}.$$

Remark. Lemma 1.4 implies that $\langle f_{x_1 r_1}, f_{x_2 r_2}\rangle$ is very small, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition:

- $r_1 + r_2 - |x_1 - x_2| \approx 0$, which holds when the two annuli are 'approximately' externally tangent to one another.

- $r_1 - r_2 + |x_1 - x_2| \approx 0$ or $r_1 - r_2 - |x_1 - x_2| \approx$, which holds when the two annuli are 'approximately' internally tangent to one another.

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Heo, Nazarov, and Seeger do not exploit the tangency situations, though utilizing the tangencies seems important to improve the results they obtain.

Proof. We write

$$
\begin{aligned}
\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle &= \left\langle \hat{f}_{x_1 r_1}, \hat{f}_{x_2 r_2} \right\rangle \\
&= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbf{R}^d} \hat{\sigma}(r_1 \xi) \overline{\hat{\sigma}(r_2 \xi)} |\hat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}
$$

We have $\hat{\sigma}(\xi) = C_d |\xi|^{1-d/2} J_{d/2-1}(2\pi |\xi|)$ for some constant c_d, where $J_{d/2-1}$ is the Bessel function of order $d/2 - 1$. If we write $a(|\xi|) = |\hat{\psi}(\xi)|^2$ for each $\xi \in \mathbf{R}^d$ (which is possible because ψ is radial), then we find

$$
\begin{aligned}
&\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle \\
&= C_d (r_1 r_2)^{d/2} \\
&\quad \int_{\mathbf{R}^d} |\xi|^{2-d} J_{d/2-1}(2\pi r_1 |\xi|) J_{d/2-1}(2\pi r_2 |\xi|) |\hat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= C_d (r_1 r_2)^{d/2} |x_2 - x_1|^{1-d/2} \\
&\quad \int_0^\infty s^{d/2+1} \cdot a(s) \cdot J_{d/2-1}(2\pi r_1 s) \cdot J_{d/2-1}(2\pi r_2 s) \cdot J_{d/2-1}(2\pi s |x_2 - x_1|) \, ds.
\end{aligned}
$$

For any $N > 0$, we have an asymptotic expansion

$$
J_{d/2-1}(t) = t^{-1/2} \left(\sum_{n=0}^{N-1} (a_{n,+} e^{it} + a_{n,-} e^{-it}) t^{-n} + O_N(t^{-N}) \right).
$$

Thus we find

$$\int_0^\infty s^{1+d/2} \cdot a(s) \cdot J_{d/2-1}(2\pi r_1 s) \cdot J_{d/2-1}(2\pi r_2 s) \cdot J_{d/2-1}(2\pi s|x_2 - x_1|) \, ds$$

$$= \sum_{n_1,n_2,n_3=0}^{N-1} \sum_{\tau_1,\tau_2,\tau_3 \in \{\pm 1\}} c_{d,n,\tau} \cdot r_1^{-1/2-n_1} r_2^{-1/2-n_2} |x_2 - x_1|^{-1/2-n_3}$$

$$\int_0^\infty s^{(d-1)/2-n_1-n_2-n_3} \cdot a(s) \cdot e^{2\pi i(\tau_1 r_1 + \tau_2 r_2 + \tau_3|x_2-x_1|)s} \, ds$$

$$+ O_{N,d}\left((r_1 + r_2 + |x_2 - x_1|)^{-N} \int_0^\infty s^{(d-1)/2-N} \cdot a(s)\right).$$

If we assume that $|a(s)| \lesssim s^{10N}$ for small $s > 0$, then

$$\left| \int_0^\infty s^{(d-1)/2-n_1-n_2-n_3} \cdot a(s) \cdot e^{2\pi i(\tau_1 r_1 + \tau_2 r_2 + \tau_3(x_2-x_1))s} \, ds \right|$$

$$\lesssim_N (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3(x_2 - x_1)|)^{-5N}.$$

Thus for $r_1, r_2 > 1$,

$$\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle \lesssim_{N,d} (r_1 r_2)^{(d-1)/2} |x_2 - x_1|^{-(d-1)/2} \sum_{\pm,\pm} (1 + (|x_2 - x_1| \pm r_1 \pm r_2))^{-N}$$

$$+ (r_1 r_2)^{d/2} |x_2 - x_1|^{1-d/2} (r_1 + r_2 + |x_2 - x_1|)^{-5N}$$

$$\lesssim (r_1 r_2)^{(d-1)/2} |x_2 - x_1|^{-(d-1)/2} \sum_{\pm,\pm} (1 + (|x_2 - x_1| \pm r_1 \pm r_2))^{-N}.$$

If $|r_2 - r_1| \gtrsim |x_2 - x_1|$, then $f_{x_1 r_1}$ and $f_{x_2 r_2}$ have disjoint support, which implies $\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle = 0$. Thus we conclude that

$$\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle \lesssim_{N,d} (r_1 r_2)^{(d-1)/2} (1 + |r_2 - r_1| + |x_2 - x_1|)^{-(d-1)/2}$$

$$\cdot \sum_{\pm,\pm} (1 + (|x_2 - x_1| \pm r_1 \pm r_2))^{-N}. \qquad \square$$

Lemma 1.4 implies that

$$|\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle| \lesssim \left(\frac{r_1 r_2}{(1 + |x_1 - x_2|)(1 + |r_1 - r_2|)} \right)^{(d-1)/2}.$$

The exponent $(d-1)/2$ is too weak to apply almost orthogonality directly.

Bibliography

[1] Andreas Seeger Yaryong Heo, Fëdor Nazrov. Radial fourier multipliers in high dimensions. 2011.