Geometry

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Part I

Euclid

I'm writing these notes so that I can understand Euclidean geometry better. We'll build up the axioms from the ground up, so I can understand Euclid's work from the ground up. Thus these notes probably won't be useful for someone trying to understand Euclid themselves, because it's just my ramblings about the subject.

Chapter 1

Book I

Basic Euclidean geometry consists of three objects: Points, Lines (both finite lines with endpoints, an infinite lines with no extremities), and Circles (defined by a point and a radius). Classically, these objects were seen as distinct, but with the power of set theory, it is easier to model lines and circles as sets of points. This has the advantage of making things notationally simple. These is no real logical difference between switching to this notation – any theorem provable in one system is provable in the other. However, we'll avoid from using set theory too much, to avoid making the exposition too austere.

Euclid was the first to pioneer the axiomatic method in mathematics. However, the philosophy behind his proofs was different to ours. At the end of the day, his arguments attack a particular model of the planar geometry find in our world, and he proves things like a physicist, adopted some methods of proof not explicitly stated in his assumptions. This causes problems for us when we try and look at his proofs from a modern day perspective. We will eventually look at other logical systems for geometry, but for now a naive approach will be most useful.

Most of Euclid's proofs concern constructions of certain figures in the planes. Rather than a proof of existence, Euclid literally builds these figures from the ground up. In the early parts of the text these figures will all be defined by a simple curve consisting of straight lines, so that we may describe such a figure by the sequence of points which define the figure. If $X_1, \dots, X_n$ are points, then $X_1 \dots X_n$ will denote the figure obtained by drawing the line $X_1 X_2$, then $X_2 X_3$, and so on, finishing off by drawing $X_n X_1$. Two such figures will be considered equal if we may obtain the points of one from the points of the other by performing a cycle permutation of the points. For instance, a **triangle** is just a sequence of distinct points ABC, and $ABC = BCA = CAB$, and we can abuse the notation, denoting a line between two points A and B as $AB = BA$. The question of whether this is a unique description of such a line is settled by the first axiom of geometry.

Axiom 1. *There is a unique straight line between any pair of points, having those points as endpoints.*

Euclid does not assume that the straight line which exists between the points is unique, but later he uses the fact that a finite line is defined by its endpoints, so we can only assume that he really wants this fact to hold. In order to discuss the lengths of lines, we shall be required to discuss circles at points, and so we introduce the second axiom.

Axiom 2. *A circle may be described with any centre and radius.*

A circle is *defined* by its centre and radius, so the circle which exists by this axiom is unique. Note that circles with a different radii and the same centre may still be equal. Indeed, this happens exactly when the two radii have the same length, a concept we will very shortly discuss.

Euclid defines an **equilateral triangle** as 'a triangle whose three sides are equal', which he really means as saying the *magnitude*, or length, of the sides are equal. In Euclid's synthetic geometry, there do not exist real numbers to assign length to, and as is well known most Greek's did not even believe in irrational numbers. But we shall find that we can get away with much of the theory of magnitude without ever mentioning the concept of a number, which gives a certain sense of satisfaction.

Right now, we only need equality in the length of lines, and we shall discuss a very aggreeable manner in checking equality. If we have two lines AB and AC with a common point, we can check if they have equal length by checking if the circles constructed with centre A and radii AB and AC are equal. This gives us an equivalence relation on the set of lines extending out from A. We shall require that this equivalence relation describes exactly the set of circles with centre A, so that a point C lies on the circle with centre A and radius AB if and only if the length of AC is equal to the length of AB.

Axiom 3. *If C lies on the circle with radius A and radii AB, then AC has the same length as AB.*

In order to generalize equality of length of arbitrary lines, we just make the relation transitive. The relation is already reflexive and symmetric, so this generates an equivalence relation on the set of all lines in the plane. Thus we see that the only basic way to check if two lines AB and CD are equal is to form a sequence of lines beginning at B, and ending at C, which are all equal to one another as lines extending from the same basepoint.

Theorem 1.1. *Any finite line lies on an equilateral triangle.*

Proof. To prove the existence of an equilateral triangle at a line AB, Euclid constructs the circle with radius AB and centre A, and the circle with centre B and radius AB, and considers their point of intersection C. Since C lies on the first circle, AB has the same length as AC, and since C lies on the second circle, CB has the same length as AB. But then the lines AB, BC, and CA describe an equilateral triangle, and so ABC is the triangle required. □

There is only one problem remaining in this proof. There is nothing saying that the two circles given will have a common point of intersection. We could describe an axiom which supplies us with such a point, but this axiom would probably be more general than the theorem itself. Indeed, the existence of a point on the intersection of two circles with the same radius but different centres is equivalent to the theorem we set out to prove. Thus we shall have to settle on the fact that theorem one must be treated as an assumption from our current viewpoint.

Theorem 1.2. *Given a point A and line BC, to construct a line extending from A with the same length as BC.*

Proof. Construct an equilateral triangle ABD on the line AB. Then construct the circle with centre B and radius BC. Find an intersection point E on the circle which either lies on the line BD, or extends the line, and then construct the circle with centre B and radius BE. Extend the line DA from the extremity A to an intersection point F on the circle. We claim AF has the same length as BC. Indeed, the length of DF is the sum of the length of DA and AF, and the length of DE is the sum of DB and BE. Since the length of DF is equal to DE, since they both lie on the same circle extending from D, and the length of DA is equal to the length of DB, we may subtract to conclude that the length of AF is the same as the length of BE. But BE has the same length of BC, which is all that is required to show AF has the same length as BC. □

Chapter 2

Analytic Geometry

From the perspective of modern axiomatic systems, the axioms of classical Euclidean geometry are a bit of a mess. They not only encompass the theory of lines in planar geometry, but also circles and the interactions between circles and lines. From the perspective of the ancient greeks, this makes sense, because the axioms are meant to describe all the techniques the greeks could interact with in the physical world. The ability to draw circles was provided by a compass; drawing lines was aided by a ruler, which also allowed us to see whether two points were between one another. There was only a single model of the geometry in Euclid's mind – the Euclidean plane $\mathbf{E}^2$. However, when we focus on a single model of the geometry, it is no longer clear what can be *proven* from the axioms (Euclid's axioms are not complete, there are many nonstandard Euclidean planes, including the plane of constructible numbers). In order to apply the completeness theorem to determine the limits of the logical axioms, it is important to be able to classify the various models of Euclidean geometry. In this chapter, we weaken the axioms of geometry to obtain a more general class of models which have a beautiful classification of models. In particular, we study geometries satisfying three axioms

- Any two distinct points X and Y lie together on a unique line XY.

- We say two lines are **parallel** if they are equal to one another, or if they do not intersect. Given a line l and a point X, there is a unique line through X parallel to l.

- There exist three non colinear points in the geometry, and each line has at least two points on it. This axioms is to prevent trivial geometries, provided by a single line.

We call the study of models of this axiom set **affine geometry**.

Example. *Let K be a field, and consider the geometry whose points are elements of K^2, and whose lines are the affine span $v + Kw = \{v + xw : x \in K\}$, for a nonzero $w \in K^2$. It is easy to see that every line through a vector v can be written as $v + Kw$, and for two lines $v + Kw_0$ and $v + Kw_1$, either the two lines intersect only at v, or the two lines are equal and w_0 is a scalar multiple of w_1. This tells us that lines intersecting at two or more common points are equal, and therefore there is a unique line $v + K(w - v)$ between any two points v and w. A line $v_0 + Kw_0$ is parallel to a line $v_1 + Kw_1$ if and only if w_0 is a scalar multiple of w_1, and for any point v_1 not on a line $v_0 + Kw$, the line $v_1 + Kw$ is a line containing v_1 and parallel to $v_0 + Kw$, and this is the unique such line. Finally, the points $(0,1)$, $(1,0)$, and $(0,0)$ are non colinear, so K^2 is a model of affine geometry.*

Our main result will be that the affine planes formed from commutative fields describe 'almost' all possible models of affine geometry. That is, if G is any affine geometry, then there is an isomorphism $f : G \to K^2$ for some field K. To do this, we will define a field structure on some line in G by the natural operations induced by the axioms. This implies that an isomorphism between K^2 and F^2 induces an isomorphism between K and F, so any affine geometry can be *uniquely* coordinatized on some field. The only isomorphisms $f : K^2 \to K^2$ are described by the maps of the form $f(v) = w + Tv$, for some base point $w \in K^2$, and some linear isomorphism $T : K^2 \to K^2$. Thus, in terms of affine geometry, these are the 'only' coordinates one can place on an affine geometry. In order to define this correspondence, we need to add two additional geometric axioms, known as the Desargueian and Pappian theorems. This is essentially a justification of Descartes' program, to understand Euclidean geometry by some coordinatization; provided we prove the theorem in the coordinates of any field, the theorem must be true in every model of Euclidean geometry, and therefore there must exist a synthetic proof of the theorem (a synthetic theorem is one proceeding directly from the axioms of geometry).

Before we begin with the techniques which allow us to prove this theorem, it is useful to see how certain extensions of affine geometry correspond to certain augmentations of the fields we consider. If we add axioms which allow us to consider whether points are 'between' other points, then this corresponds to the geometries K^2 where K has the structure of an *ordered field*.

This shows that over the class of affine geometries, 'Cartesian analytic geometry' suffices to verify any result. Our current axioms do not suffice to uniquely specify all affine geometries; for instance, the above example works if K is a non-commutative division ring. To begin with this construct, consider any affine geometry. We will give a field structure to any line OI in the geometry, with which we may identify the geometry as the vector space $(OI)^2$. We shall define an algebraic structure in which O is the additive identity, and I is the multiplicative identity. Obtaining these definitions requires some nontrivial geometric theorems.

Lemma 2.1. *If l is a line intersecting another distinct line r, and r is parallel to another line u, then l intersects u.*

Proof. If l was parallel to u, and l and r shared a common point X, then l and r would both be lines through X parallel to u, hence by uniqueness, $l = r$. □

Corollary 2.2. *Parallellism is an equivalence relation. That is, if l shares no points with r, and r shares no points with u, then l shares no points with u.*

Proof. If l was parallel to r, and r parallel to u, where we assume all three lines are distinct, but l intersected u, then the last lemma would imply l intersected r, which is impossible. □

A **pencil** of parallel lines in affine geometry is an equivalence class of parallel lines. They are a useful family because they allow us to transfer properties between other parallel lines. For instance, given two distinct lines l and r, fix a pencil of parallel lines intersecting both l and r (but parallel to neither). Given $X \in l$, there is a unique line in the pencil through X, and a unique point $Y \in r$ on this line. This generates a bijective map between the points on l and the points on r. More generally, it gives us a projection of the entire plane down onto the line r, but this projection is not a bijection.

Example. *Consider a finite affine plane, with some line l consisting of n points. If r is a line not parallel to l, then the points on r can be put into one to one correspondence with the lines parallel to l. We have seen that any line paralllel* *to l must intersect r at a unique point, and conversely, any point on r generates a line parallel to l. It follows that there are n^2 points in this affine plane, and we call this plane an* **affine plane of order n**.

Now we return to our scenario of defining field operations on a line through two points OI. Given O, fix a line l through O. Fix a point $M \notin L$, and let r be the line through M parallel to l. Given $A, B \in l$, the pencil of lines generated by OM allows us to transport A to a point C on r. Similarily, the pencil of lines generated by BM allows us to transport C back to a point D on l, and we define $D = A + B$. Unfortunately, in general affine geometries this operation will depend on the point M we choose, and this means addition does not in general even have to be associative. When we try to construct addition on two lines, we require a very subtle property. To ensure that addition is independent of the line we choose, we must assume a theorem of Desargues, which specializes our study to the theory of Desarguian planes.

- Suppose we are given three lines A_0A_1, B_0B_1, and C_0C_1, which are either parallel, or intersect in a common point. Then if A_0B_0 is parallel to A_1B_1 and A_0C_0 is parallel to A_1C_1, then B_0C_0 is parallel to B_1C_1.

Once we assume the Desarguian axiom, the proof goes through. The idea of why the operation is well defined corresponds to how translations compose with one another in the affine plane. Given a pencil of lines, and a line l, consider the induced projection $f : G \to l$. Given another pencil of lines, and a line r, we obtain another projection $g : G \to r$. If $f(X) = Y$, and $g(Y) = Z$, consider the pencil of lines generated by XZ. Then the induced projection $h : G \to r$ is equal to $g \circ f$. In other words, projections in Desarguian planes are determined by where they map a single point not on the projected line. This also implies that $f \circ g = g \circ f$, and therefore that addition is a commutative operation. Associativity is also easily verified, as well as the fact that $A + O = O + A = A$. To obtain an additive inverse for each point $A \in l$, we work backwards. First, use the pencil AM to translate O to a point C on the line parallel to l through M, and then use OM to project C down to a point on l, which is $-A$. Thus the addition operation makes the line l into an abelian group.

Example. *Desargues' theorem is easily verified in the case of K^2. Take a parallelogram $ABCD$ in K^2, which is a set of four points, such that the lines of opposite sides of the parallelogram are parallel. This means that $C - D = \alpha(B - A)$, and $C - B = \beta(A - D)$, because the opposite sides of the parallelogram are parallel. This means that*

$$B - A = (B - C) + (C - D) + (D - A) = (1 - \beta)(D - A) + \alpha(B - A)$$

If $\alpha \neq 1$, then we conclude that

$$B - A = \frac{1 - \beta}{1 - \alpha}(D - A)$$

so all four sides of the parallelogram are parallel (we say the parallelogram is degenerate, because then all four points on the parallelogram lie on a single line). Otherwise, we find that $\alpha = 1$, from which we conclude $\beta = 1$.

Now in the case of Desargues' theorem where we consider three parallel lines A_0A_1, B_0B_1, and C_0C_1, the assumptions of the theorem guarantee that $A_0A_1B_1B_0$ and $A_0C_0A_1C_1$ are parallelograms. If $A_0A_1B_1B_0$ is a non-degenerate parallelogram, we conclude that $A_1 - A_0 = B_1 - B_0$ and $A_1 - B_1 = A_0 - B_0$. This means that

$$B_0 - C_0 = B_1 - C_1 + (B_0 - C_0)$$

TODO: FINISH THIS.

To obtain a well defined multiplication operation, we take a line through O distinct to l, project I onto this line by an arbitrary point $M \neq O$, use this projection to project points onto the line, and then project them off to multiply arbitrary points $A, B \in l$. Verification that this axiom requires another property of geometric spaces, known as Pappus' theorem.

- Consider two distinct lines l and r intersecting at a point O, where l contains three points A_0, B_0, C_0, and r contains three points A_1, B_1, and C_1. If A_0B_1 is parallel to B_0C_1, and B_0A_1 is parallel to C_0B_1, then A_0C_1 is parallel to B_0A_1.

We leave it to the reader to verify that all the require properties hold (or you can draw diagrams to convince yourself). The main result is that OI has a natural structure which turns it into a field. If OT is a line through

O distinct to OI, then the pencil generated by IT gives a bijection of OI with OT, and since every point in the plane is uniquely designated by its projection onto OI with respect to the pencil generated by OT, and the projection onto OT with respect to the pencil generated by OI, we conclude that the plane is in one to one correspondence with $OT \times OI$, which we now view as OI^2. Finally, we note that all the lines in the plane are given by $x + OIy$, for $x, y \in OI^2$, and this is left as an exercise.

2.1 Conic Sections

In this section we discuss conic sections, which are planar figures obtained from intersecting a plane with a cone. These may be viewed as two dimensional figures by taking a coordinate system on the plane in question, and we will find that these figures describe all quadratic figures in the plane describable by the equation

$$aX^2 + 2bXY + cY^2 + 2dX + 2eY + f = 0$$

These equations seem complicated, but we will find that we can simplify these equations by applying *affine transformations* to these equations. In projective space, we cannot consider zero sets of arbitrary polynomials in homogenous coordinates, but we can consider the zero sets of *homogenous* polynomials, so we consider any conic as a three dimensional equation adding in the variable Z

$$aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2 = 0$$

We will projectively classify these conics in $\mathbf{RP}^2$.

We can see such a conic as a zero set of a quadratic form in three dimensions, which has a matrix representation

$$(X, Y, Z)^t \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0$$

If we apply a projective transformation to (X, Y, Z) by a matrix M, then this is the same as replacing the interior matrix with

$$M^t \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} M$$

Over the real numbers, it is Sylvester's law of inertia that we can choose a projective transformation M such that the quadratic form is reduced to

$$aX^2 + bY^2 + cZ^2 = 0$$

for some values $a, b, c \in \mathbf{R}$. Since we may always multiply X with a scalar multiple, these polynomials are only affinely distinguished by the sign of their coefficients. We may also multiply the entire equation by a scalar, so we may assume that there are at least as many positive coefficients than negative coefficients. We may also projectively consider a permutation of the variables, like $(X, Y, Z) \mapsto (Z, X, Y)$, so we may assume that $a \geqslant b \geqslant c$. These lead to the following five conics, which can be checked not to be projectively equivalent to one another.

- $X^2 + Y^2 = Z^2$: The standard, *non degenerate conic*.
- $X^2 + Y^2 = 0$: The single point at the origin.
- $X^2 + Y^2 + Z^2 = 0$: No solutions.
- $X^2 = 0$: A projective line
- $X^2 = Z^2$: Two distinct projective lines.
- 0: The entire projective plane.

All of the well known conics, ellipses, circles, hyperbolas, and parabolas, are projectively equivalent to the conic $X^2 + Y^2 = Z^2$, which interestingly enough can be seen as the equation in $\mathbf{R}^3$ for the standard cone. This makes sense, because every plane not passing through the origin in $\mathbf{R}^3$ corresponds to an affine plane embedded in $\mathbf{RP}^2$, and the intersections of this plane with the cone corresponds to all possible situations of what the conic looks like on the affine plane. The projective transformations which maps this plane to itself correspond to all affine transformations of the plane, so the conics which are affinely equivalent may be described as those corresponding affine transformations. If the plane is $Z = 1$, we obtain the canonical embedding of $\mathbf{R}^2$ in $\mathbf{RP}^2$, and the conic $X^2 + Y^2 = Z^2$ becomes the circle $X^2 + Y^2 = 1$. The plane at infinity here is $Z = 0$, so for any $r \neq 0$, the coordinates $Z = r$ induce an affine transformation showing that the circles $X^2 + Y^2 = r^2$ are all affinely equivalent. Similarily, the scalings $X = aX$ and $Y = bY$ induce the equations $a^2X^2 + b^2Y^2 = r^2$, which are the equations for the ellipses whose axis lie on the X and Y axis. If we have an ellipse whose axis do not lie on these axis, a simple rotation of the plane will map this ellipse affinely to one of the forms we understand. Thus circles and ellipse are affinely one and the same.

These may be projectively equivalent, but are not *affinely equivalent*. To distinguish between the projective and affine equations, we distinguish each of the degenerative bullets.

- $X^2 + Y^2 + Z^2 = 0$: All figures obtained by projective transformations from this set have no solutions, and therefore all the figures are trivially affinely equivalent.

- $X^2 + Y^2 = 0$: The projective solution set is just a single point, and the projective transformations can map this point to any other point in the affine plane, or a single point on the line at infinity. These correspond to two solution sets which are not affinely equivalent.

- $X^2 = 0$ has a projective line as the solution set, and every projective transformation can map this line to any other line, so the other affine solutions are all just lines, which are affinely equivalent, or the line at infinity, which will have a trivial solution set in the affine plane. Thust we obtain two solution sets which are not affinely equivalent.

- $X^2 = Z^2$: The solution set corresponds to the intersection of two projective lines, and every projective transformation will map this solution set to the intersection of two projective lines. If these lines intersect at infinity, then the affine solution set will correspond to two parallel lines, which are all affinely equivalent, or the affine solution set will correspond to two intersecting liens, which are also all affinely equivalent. Thus we obtain two affine solution sets here.

- 0: Whose solution set is the entire plane $\mathbf{RP}^2$, and whose affine solution set will always be $\mathbf{R}^2$.

- $X^2 + Y^2 = Z^2$: Whose solution set is a circle with radius one. The projective transformations take this conic into a conic of the form $aX^2 + 2bXY + cY^2 + 2dX + 2eY + f$, where the matrix

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

is invertible. We can actually classify these conics affinely by looking at the discriminant $b^2 - ac$. It is clear that the only conics affinely equivalent to $X^2 + Y^2 = Z^2$ are the ellipses,

It is interesting that, in the projective case, two polynomials are projectively equivalent if and only if their solution sets are projectively equivalent, whereas this does not hold in the affine case – two solution sets can be affinely equivalent, while their polynomials cannot be affinely transformed onto one another.

Chapter 3

Projective Geometry

Projective geometry can be viewed as a slight modification to affine geometry. Rather than distinguishing between parallel and intersecting lines, in projective geometry we deduce theorems based on the assumption that all lines intersect. Originally studied as a 'paradoxical' system used to prove the parallel postulate by contradiction, projective geometry is now seen as a perfectly sound geometric system, which is often more elegant than classical Euclidean geometry. Projective geometry proceeds in studying geometries satisfying two geometric axioms:

- Every distinct pair of lines has a unique intersection.
- Every distinct pair of points is connected by a unique line.

as well as a 'regularity' axiom, to protect against degeneracy:

- Every line has 3 or more points, and there is more than one line.

The most visual way to see a model of projective geometry is as a model of affine geometry, with added 'points at infinity'. In particular, if we take a model of projective geometry, and remove any line from the geometry (as well as points on that line), then the resulting geometry will be a model of affine geometry. Conversely, there is a unique way of extending any affine geometry to obtain a projective geometry by adding a 'line at infinity'.

Example. *We can obtain a projective geometry from any affine geometry by adding a single point for each set of parallel lines, causing these parallel lines to intersect. More specifically, in any affine geometry G, for each line l we consider* *the equivalence class $[l]$ of lines parallel to l (the pencil of lines generated by l). We obtain a projective geometry G_∞ by adding a new point $p_{[l]}$ for each pencil of lines, extending each line l so that $p_{[l]}$ lies on l, and adding a new line at infinity l_∞, which consists of all points $p_{[l]}$ not found in the original geometry.*

Example. *There is a particular way to coordinatize the projective plane K^2_∞, where K is some field. We consider the projective geometry $\mathbf{P}K^3$, also denoted $K\mathbf{P}^2$, whose points are lines through the origin in K^3, and a line is a plane in the origin in K^3. This is a projective geometry, because two planes through the origin in three dimensions always intersect in a line, and two lines through the origin generate a unique plane through the origin. This space has a* **homogenous coordinate system** *obtained by the correspondence $K^3 - \{0\} \to \mathbf{P}K^3$, taking a vector v to its span Kv. The fibres of this correspondence are written as $[x : y : z]$, and are essentially the smallest equivalence classes such that $[\lambda x : \lambda y : \lambda z] = [x : y : z]$ for all nonzero $\lambda \in K$. It is often very useful to introduce a homogenous coordinate system to a projective geometry, because then we can understand the space through algebraic operations. For instance, a line in $\mathbf{P}K^3$ can be described as the solution set to the equation $aX + bY + cZ = 0$ (which is well defined because the polynomial is homogenous), for some $a, b, c \in K$, which is obvious if we think of a line in $\mathbf{P}K^3$ as a plane through the origin. We embed K^2 in $K\mathbf{P}^2$ by mapping (x, y) to $[x : y : 1]$. In this case, the 'line at infinity' in $K\mathbf{P}^2$ can be thought of as the points $[x : y : 0]$, which are not in the image of this correspondence. In particular, a line in K^2, which can be identified as the solution set of the equation $aX + bY = c$, consists of all points which are in the plane $aX + bY = cZ$, and if $Z = 0$, then the added point at infinity of this line is just the solution the line $aX + bY = 0$ through the origin. We switch between the notations $K\mathbf{P}^2$ and $\mathbf{P}K^3$ depending on whether we want to view the object as 'two dimensional' or 'one dimensional'. In general, for any vector space V, $\mathbf{P}V$ is the space of lines through the origin in V. It is a more natural functor from the point of view of the category of vector spaces, but less intuitive geometrically, because in the general case we have no natural embedding of some 'affine space' W in $\mathbf{P}V$.*

If we remove a line from the projective geometry to obtain an affine geometry, and then add a line at infinity back in, we obtain a geometry isomorphic to the original geometry, so the two axiom systems are essentially in one to one correspondence. In particular, any projective geometry satisfying Desargues' theorem and Pappus' theorem must be isomorphic to $\mathbf{P}K^2$ for some field K, because if we remove a line at infinity we obtain a geometry isomorphic to K^2. Thus systems of homogenous coordinates over an arbitrary field as the 'correct' way to introduce analytic geometry to the study of projective geometry, and we will find ourselves focusing on projective geometries which have homogenous coordinates more and more over the course of our study of projective geometry.

3.1 Duality

Unlike affine geometry, projective geometry has an incredibly rich duality theory, because the axioms defining projective geometry are symmetric with respect to points and lines. If we take any model of projective geometry, and exchange the specification of points and lines, we end up with another projective geometry. Thus if we take any theorem of projective geometry and exchange the definition of points and lines, we obtain another theorem of projective geometry. We will see this duality theory appear again and again in our study of projective systems.

In another direction, note that we have a duality theory between the set of lines in K^3 and the set of planes in K^3, because every plane in K^3 is defined to be the solution set of the equation $f(x) = 0$, where $f : K^3 \to K$ is some nonzero linear functional over K. Two nonzero linear functionals determine the same plane if and only if they are scalar multiples of one another, so we have a nice parameterization of the planes in K^3 with $\mathbf{P}(K^3)^*$. We can introduce homogenous coordinates to $\mathbf{P}(K^3)^*$ by defining $[x : y : z] \in K\mathbf{P}^2$ to correspond to the functional $(x', y', z') \mapsto xx' + yy' + zz'$, and this gives an isomorphism of $K\mathbf{P}^2$ with $\mathbf{P}(K^3)^*$.

In particular, the isomorphism of $\mathbf{RP}^2$ with $\mathbf{P}(\mathbf{R}^3)^*$ on $\mathbf{R}^3$ preserves the Euclidean inner product on the space. Thus we can think of a plane in $\mathbf{R}^3$ as being parameterized by the unique line orthogonal to it. In particular, given any two distinct vectors X and Y in $\mathbf{R}^3$ which are not scalar multiples of one another, the cross product $X \times Y$ is a vector orthogonal to both X and Y. This operation is bilinear, and therefore descends to a map on $\mathbf{P}K^2$. Because of the orthogonality properties of the cross product, we conclude that $X \times Y$ describes the plane connecting X to Y. Conversely, if X and Y are the homogenous coordinates of some plane, then $X \times Y$ describes the unique line orthogonal to X and Y, and therefore $X \times Y$ can be seen as the homogenous coordinates of the unique point in the intersection of X and Y. More generally, we can define the cross product on K^3, and we find the same result holds with respect to the abstract bilinear map $\langle x,y\rangle = x_1y_1 + x_2y_2 + x_3y_3$ on the space.

Example. *Given two lines l and l_∞, and an additional point X, we often want to calculate the unique line through X passing through the point on the intersection of l and l_∞. If l_∞ is viewed as the line at infinity, this means exactly that we want to find the line through X parallel to l. Since $l \times l_\infty$ gives the homogenous coordinates of the point we want X to pass through, we can find this value as $X \times (l \times l_\infty)$.*

3.2 Projective Transformations

The interesting transformations on projective space are those preserving colinearity. They are know as **perspectivities**, or **homographies**. Since linear transformations in $T : K^3 \to K^3$ map lines to lines, they induce maps from $\mathbf{P}K^2$ to $\mathbf{P}K^2$, and since T maps planes to planes, it preserves colinearity. In coordinates, we can describe these transformations by 3 by 3 invertible matrices

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in GL_3(K)$$

The fundamental theorem of projective geometry says that the only transformations on projective spaces are those induced by **semilinear maps** $T : K^3 \to K^3$, which are maps satisfying

$$T(x+y) = Tx + Ty \quad T(cx) = \theta(c)Tx$$

for some particular field automorphism $\theta : K \to K$. They form the **projective group** $P_2(K)$. For now, lets explore the subgroup of homographys obtained by linear maps on K^3, which is the **projective linear group** $PL_2(K)$, and can be described as $GL_3(K)$ modulo the center $Z(K)$ (which in most cases is just the set of scalar multiples of the identity). First, we note the following 'rigid' properties of these maps.

Theorem 3.1. *Given four points $A,B,C,D \in K\mathbf{P}^2$, no three of which are colinear, and four other points $A',B',C',D' \in K\mathbf{P}^2$, no three of which are colinear, there is a unique transformation in $PL_3(K)$ mapping A to A', B to B', C to C', and D to D'.*

Proof. First, consider the special case where $A = [1:0:0]$, $B = [0:1:0]$, $C = [0:0:1]$, and $D = [1:1:1]$. If A' has homogenous coordinates $[A_1 : A_2 : A_3]$, B' has homogenous coordinates $[B_1 : B_2 : B_3]$, and so on and so forth, we essentially must find an invertible matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

such that

$$M[1:0:0] = [a:d:g] = [A_1 : A_2 : A_3]$$
$$M[0:1:0] = [b:e:h] = [B_1 : B_2 : B_3]$$
$$M[0:0:1] = [c:f:i] = [C_1 : C_2 : C_3]$$
$$M[1:1:1] = [a+b+c : d+e+f : g+h+i] = [D_1 : D_2 : D_3]$$

Trying to reduce this to a problem of linear algebra, we have to try and find four constants $\alpha, \beta, \lambda, \gamma$ such that

$$\alpha M(1,0,0) = (A_1, A_2, A_3) \quad \beta M(0,1,0) = (B_1, B_2, B_3)$$

$$\lambda M(0,0,1) = (C_1, C_2, C_3) \quad \gamma M(1,1,1) = (D_1, D_2, D_3)$$

Solving the first three constraints is equivalent to solving the matrix equation

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

and the matrix on the right is invertible because these vectors are not colinear, hence we can uniquely find the matrix of coefficients on the left for any particular nonzero values of α, β, and λ. Since A, B and C are not colinear, we may write $(D_1, D_2, D_3) = a(A_1, A_2, A_3) + b(B_1, B_2, B_3) + c(C_1, C_2, C_3)$ for a unique pair of coefficients a, b and c, which all must be nonzero, because (D_1, D_2, D_3) is not colinear with any of the other two points. Now we need only to find values α, β, and λ and γ such that

$$\gamma M(1,1,1) = \gamma\alpha^{-1}(A_1, A_2, A_3) + \gamma\beta^{-1}(B_1, B_2, B_3) + \gamma\lambda^{-1}(C_1, C_2, C_3)$$

is equal to (D_1, D_2, D_3), and we set $\gamma\alpha^{-1} = a$, $\gamma\beta^{-1} = b$, and $\gamma\lambda^{-1} = c$. If we fix $\gamma = 1$, the values of α, β, and γ are uniquely determined. □

Projective linear transformations which fix the line at infinity in $K\mathbf{P}^2$ can model all sorts of interesting transformations on Euclidean space. First, note that since the line at infinity may be identified with the set of points with homogenous coordinates $[x : y : 0]$, a matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

preserves the line at infinity if and only if $g = h = 0$. It then follows that in order to be invertible, i must be nonzero, and by normalizing, we determine that we can describe such maps as

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$$

restricting this map to points in K^2, we find that it takes the form

$$X \mapsto \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} [X : 1] = \begin{pmatrix} a & b \\ d & e \end{pmatrix} X + \begin{pmatrix} c \\ f \end{pmatrix}$$

so projective linear transformations which fix the line at infinity are exactly the affine linear transformations, consisting of all skews of the plane combined with a translation.

3.3 The Projective Line and Cross Ratios

The space $K\mathbf{P}^1 = \mathbf{P}K^2$ is known as the **projective line**, and can be seen as a one dimensional projective geometry. Indeed, it contains the affine line K where $x \in K$ is viewed as $[x : 1]$, as well as a unique point at infinity $[1 : 0]$, which we also denote as ∞. In Euclidean geometry, we understand the plane by fixing an arbitrary line and studying the properties of the line invariant under affine transformations. We shall try to do the same in $K\mathbf{P}^1$. Given a line of the form $\{[\alpha v + \beta w] : \alpha, \beta \in K\}$ in $\mathbf{P}K^3$, we can coordinatize the line once we fix v and w by the map $[\alpha v + \beta w] \mapsto [\alpha : \beta]$, which are the homogenous coordinates in $K\mathbf{P}^1$. This is equivalent to viewing w as the origin of the projective line, v as the unique point at infinity, and then fixing a scale so that $[v+w]$ is expressed as 1 in the coordinates. Given any other spanning set v', w', there is a change of basis matrix T mapping v to v' and w to w', which by linearity induces a transformation on $K\mathbf{P}^1$. The family of all such transformations is the one dimensional projective linear group, denoted $PL_1(K)$, which can be described as those maps induced by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

And also are of the form

$$x \mapsto \frac{ax+b}{cx+d} = [ax+b : cx+d]$$

By applying similar techniques to the theory of projective linear transformations in $K\mathbf{P}^2$, we find that a projective linear transformation is uniquely specified by where it maps 3 distinct points. In particular, a projective transformation preserving 0, 1 and ∞ must be the identity, and any transformation preserving 0 and ∞ must be of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for nonzero α, β, and therefore after normalizing by setting $\beta = 1$, we see that the transformation is just $x \mapsto \alpha x$.

Lemma 3.2. *If $l = span(v, w)$ and $l' = span(v', w')$ are two lines in $\mathbf{P}K^2$, and O is a point not on either of the two lines, consider the projection $X \mapsto (X \times O) \times l'$. If $v \mapsto v'$ and $w \mapsto w'$, then the induced function on homogenous coefficients induced by the specification (v, w) and (v', w') is just multiplication by some scalar in K.*

Proof. It is clear that the map is linear in X, because the cross product is bilinear, hence the induced transformation on homogenous cooeficients must be in $PL_1(K)$. Switching to the coefficients, we see that $v = v' = \infty$ and $w = w' = 0$, so this transformation must preserve these two points, and is therefore just given by multiplication by a scalar. □

If we consider a projective transformation on $K\mathbf{P}^2$ mapping a line l to a line l', then in arbitrary projective coordinates on l and l', the induced transformation will be a projective map on $K\mathbf{P}^1$, and it is therefore of interest to analyze projective transformations on the projective line, and the invariants of these transformations.

The first invariant of projective geometry is incredibly important. For notational convinience, given $v, w \in K^2$, let $[v, w] \in K$ denote the determinant of the matrix obtained by stacking the column vectors v and w into a matrix. That is, $[v, w] = v_1 w_2 - v_2 w_1$. For four points $A, B, C, D \in K^2$, we define the cross ratio to be

$$(A, B; C, D) = ([A, C][B, D] : [A, D][B, C]) = \frac{[A, C][B, D]}{[A, D][B, C]}$$

The cross ratio is invariant under any linear transformation $T \in GL_2(K)$, because

$$\begin{aligned}(TA, TB; TC, TD) &= ([TA, TC][TB, TD] : [TA, TD][TB, TC]) \\ &= (\det(T)^2[A, C][B, D] : \det(T)^2[A, D][B, D]) = (A, B; C, D)\end{aligned}$$

This also implies the cross ratio descends to homogenous coordinates. For any nonzero $\lambda \in K$, $(\lambda A, \lambda B; \lambda C, \lambda D) = (A, B; C, D)$, and therefore the cross ratio can be considered for points on the projective line. It is invariant under the action of $PL_1(K)$. This implies, in particular, that for any four points A, B, C, D on a line l in $K\mathbf{P}^2$, the cross ratio is defined irrespective of the particular homogenous coordinate system, and is also invariant under the action of $PL_2(K)$. What's more, if O is not on the line l, then we may calculate the cross ratio in $K\mathbf{P}^2$ as

$$(A, B; C, D) = \frac{[O, A, C][O, B, D]}{[O, A, D][O, B, C]}$$

where $[A, B, C]$ is the determinant of the three by three stacked matrix, which is invariant under $PL_2(K)$. This means that we may assume $O = [0 : 0 : 1]$, $A = [1 : 0 : 0]$, and $B = [0 : 1 : 0]$, in which case it follows that $C = [C_1 : C_2 : 0]$ and $D = [D_1 : D_2 : 0]$ for some values C_1, C_2, D_1, D_2, and then the theorem is obvious.

The cross ratio is not a commutative operation, not even among the two 'pairs' upon which the ratio is well defined. We find that swapping one of these pairs corresponds to inverting the cross ratio

$$(b, a; c, d) = (a, b; d, c) = \frac{1}{(a, b; c, d)}$$

conversely, swapping two elements on either side corresponds to reflecting the ratio about the line $X = 1/2$.

$$(d,b;c,a) = (a,c;b,d) = 1 - (a,b;c,d)$$

If $(a,b;c,d) = \lambda$, then the six possible values of the cross ratio obtained by permuting points in the ratio are obtained by composing these two operations, and therefore consist of

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}$$

and the transformations formed by reflection at $1/2$ and inversion form the anharmonic group, which is isomorphic to the Klein four group. The values of the cross ratio become degenerate when $\lambda = 1-\lambda$, or $\lambda = \lambda^{-1}$, that is, at the values $\lambda = 1/2$, $\lambda = -1$, $\lambda = 1$, and $\lambda = \infty$. These break down into the orbit sets $\{-1, 2, 1/2\}$ and $\{1, 0, \infty\}$. In the latter case, $(A,B;C,D) = 0$ implies that one of the points A and B is equal to one of the other points C, D, so assuming our points A, B, C, D are distinct, the only degenerate case up to orbits is $(A,B;C,D) = -1$. In this case, we say (A,B) and (C,D) are in **harmonic position**.

Given three distinct points A, B, and C on a projective line, there is a unique point D such that (A,B) and (C,D) are in Harmonic position. This is clear because for any fixed A, B and C, the map $D \mapsto (A,B;C,D)$ is a projective transformation. given by the linear operator

$$D \mapsto ([A,C](B_1 D_2 - D_1 B_2), [B,C] A_1 D_2 - A_2 D_1) = \begin{pmatrix} -[A,C]B_2 & [A,C]B_1 \\ -[B,C]A_2 & [B,C]A_1 \end{pmatrix} D$$

and the determinant of the operator is $[A,C][B,C][B,A]$, which is nozero provided A, B and C are distinct points. One way to construct D is by fixing an auxillary point O off of the line l containing A, B, and C, choosing another auxilary point P on $O \times C$, and then considering the point $A' = (O \times A) \times (P \times B)$, $B' = (O \times B) \times (P \times A)$, and $D = (A' \times B') \times l$. If we let $C' = (A' \times B') \times (O \times C')$, then we find $(A,B;C,D) = (A',B';C',D')$, and also $(A,B;C,D) = (B',A' : C',D')$, so it follows that since the opposite pairs of points are distinct, that $(A,B;C,D) = -1$. If $A = B$, then $(A,B;C,D) = 1$, so this construction is impossible.