Ordinary Differential Equations

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Chapter 1

Vector Fields on $\mathbf{RP}^n$

On $\mathbf{R}^n$, local flows related to a smooth vector field v may fail to extend to global flows because solutions approach ∞ in finite time. However, often we may be able to embed $\mathbf{R}^n$ in a compact manifold K, and extend v to a smooth vector field on K. Since K is compact, all local flows extend to global flows, and thus we can consider a global flow on $\mathbf{R}^n$ which 'passes through ∞' in finite time.

For instance, recall that the space $\mathbf{RP}^n$ is the compact quotient space of $\mathbf{R}^{n+1} - \{0\}$ by the group action of $\mathbf{R}^\times$ by scaling, so that x is identified with λx for any $\lambda \neq 0$. The quotient structure gives it a natural topological structure, which can also be identified with the topology which makes the projection maps on each of the coordinate systems

$$x_i : [x] \mapsto (x^1/x^i, \dots, \widehat{x^i/x^i}, \dots, x^{n+1}/x^i)$$

defined on $U_i = \{[x] : x_i \neq 0\}$, homeomorphisms. It is a smooth manifold if we consider the x_i as diffeomorphisms.

Example. *The classic example of a vector field which cannot be extended to a global flow is $v(x) = x^2$ on $\mathbf{R}$, which has a flow*

$$\varphi_t(x) = \frac{x}{1 - xt}$$

Which has a singularity at $t = x^{-1}$. Note, however, that if we write this map in projective coordinates, then we find $\varphi_t[x : y] = [x : y - tx]$. In this formulation, it is easy to see that each map φ_t can be extended uniquely to a smooth map *from* $\mathbf{RP}^1$ *to* $\mathbf{RP}^1$*, and the group equation still holds.*

$$\varphi_{t+s}[x:y] = [x:y-x(t+s)] = [x:(y-sx)-tx] = \varphi_t(\varphi_s[x:y])$$

An alternate way to see this is to let $y = 1/x$ *denote the inverse coordinate system on projective space. We then calculate that for* $y \neq 0, \infty$*, that*

$$v(y) = x^2\partial_x(y) = -x^2y^2 = -\partial_y$$

and v *can be uniquely extended to a smooth vector field on* $\mathbf{RP}^1$ *by defining* $v(\infty) = \partial_y$*, and therefore generates a global flow on* $\mathbf{RP}^1$ *because* $\mathbf{RP}^1$ *is compact. This technique is not general, however. If we consider the vector field* $v(x) = x^3\partial_x$*, then we find that* $v(y) = -y^{-1}\partial_y$*, which cannot be extended to a smooth vector field at* $y = 0$*. This is because solutions approach infinity 'too fast' – we find the flows take the form*

$$\varphi_t(y) = \sqrt{y^2 - 2t}$$

And these solutions approach $y = 0$ *with infinite slope.*

Sometimes the geometry of projective space provides an enlightening viewpoint on a particular differential equation.

Example. *Consider the differential equation* $\ddot{u} + \alpha u = 0$*, as* α *ranges over* $\mathbf{R}$*. This corresponds to the two dimensional first order system specified by the vector field* $v(u,w) = (w, -\alpha u)$*. This means that on the integral curves defined by this vector field,*

$$-\alpha u du = w dw$$

so the integral curves lie on the level curves to $w^2 + \alpha u^2$*. For* $\alpha > 0$*, this value is always positive, and defines an ellipse. Since* v *does not vanish on any ellipse of a positive radius, we see these ellipses must describe the integral curves. For* $\alpha < 0$*, the level curves of* $w^2 + \alpha u^2$ *describe hyperbolas not passing through the origin, so these hyperbolas are the integral curves. For* $\alpha = 0$*, the integral curves are easily seen to be the lines parallel to the* x *axis. Switching to the coordinates* $x = u/w$*,* $y = 1/w$*, we find that for* $y \neq 0$*,*

$$\begin{aligned} v(x,y) &= (w\partial_u(x) - \alpha u\partial_w(x), w\partial_u(y) - \alpha u\partial_w(y)) \\ &= (1 + \alpha u^2/w^2, \alpha u/w^2) = (1 + \alpha x^2, \alpha xy) \end{aligned}$$

This function can be extended to a smooth vector field on the whole of $\mathbf{RP}^2$ *by defining* $v(x,0) = (1 + \alpha x^2, 0)$*.*

Bibliography