Radial Multipliers

Jacob Denson

February 15, 2021

Table Of Contents

Chapter 1

Heo, Nazarov, and Seeger

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's paper 2011 *Radial Fourier Multipliers in High Dimensions* [1]. Recall that if $m \in L^\infty(\mathbf{R}^d)$ is the symbol of a Fourier multiplier operator T_m, then we let $\|m\|_{M^p(\mathbf{R}^d)}$ denote the operator norm of T_m from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. The goal of this paper is to show that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, $1 < p < (2d-2)/(d+1)$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero,

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}.$$

where the implicit constant depends on p and η. Since

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\mathrm{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}$$

we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family $\{\mathrm{Dil}_t \eta\}$.

In Garrigós and Seeger's 2007 paper *Characterizations of Hankel Multipliers*, it is proved that for $1 < p < 2d/(d+1)$, if $\eta \in \mathcal{S}(\mathbf{R}^d)$ is a nonzero, radial Schwartz function, then

$$\|m\|_{M^p_{\mathrm{rad}}(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

where $M^p_{\mathrm{rad}}(\mathbf{R}^d)$ is the operator norm of T_m from $L^p_{\mathrm{rad}}(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. Thus, at least in the range $1 < p < (2d-2)/(d+1)$, boundedness of T_m on radial functions is equivalent to boundedness on all functions.

1.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a convolution inequality, which also proves local smoothing results for the wave equation.

Let σ_r be the surface measure on the radius r sphere centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial Schwartz function $\psi \in \mathcal{S}(\mathbf{R}^d)$. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $f_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$. Our goal is to prove the following inequality.

Lemma 1.1. *For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, and $1 \leqslant p < (2d-2)/(d+1)$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.$$

where the implicit constant depends on p, d, and ψ.

Why is Lemma 1.1 useful? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier given by some function $\tilde{m} : [1,\infty) \to \mathbf{C}$, and we set $a_r(x) = \tilde{m}(r) f(x)$ for some $f : \mathbf{R}^d \to \mathbf{C}$. Then it is simple to check that

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr = K * \psi * f$$

where $K(x) = |x|^{1-d} m(x)$. In this setting, Lemma 1.1 says that

$$\|K * \psi * f\|_{L^p(\mathbf{R}^d)} \lesssim \|m\|_{L^p(\mathbf{R}^d)} \|f\|_{L^p(\mathbf{R}^d)},$$

which is clearly related to the convolution bound we want to show if $\psi = \widehat{\eta}$, provided that we are dealing with a multiplier m that is supported away from the origin.

To understand Lemma 1.1 it suffices to prove the following discretized estimate.

Theorem 1.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1,\infty)$, which is* discretized *in the sense that $|(x_1,r_1)-(x_2,r_2)| \geqslant 1$ for each distinct pair (x_1,r_1) and (x_2,r_2) in $\mathcal{E}$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < (2d-2)/(d+1)$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r)\in\mathcal{E}} |a_r(x)|^p r^{p-1} \right)^{1/d},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independant of $\mathcal{E}$.

Proof of Lemma 1.1 from Lemma 1.2. For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$,

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} = \int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx})\, dr\, dx$$

Minkowski's inequality thus implies that

$$\begin{aligned}
\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} \|_{L^p(\mathbf{R}^d)} &\leqslant \int_{[0,1)^d} \int_0^1 \| \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx}) \|_{L^p(\mathbf{R}^d)}\, dr\, dx \\
&\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} |a_r(x)|^p r^{p-1} \right)^{1/p} dr\, dx \\
&\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} |a_r(x)|^p r^{p-1}\, dr\, dx \right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.
\end{aligned}$$

□

Lemma 1.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r)\in\mathcal{E}} a_r(x) f_{xr}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 1.2 is implied by the following Lemma.

Lemma 1.3. *For any $1 \leqslant p < (2d-2)/(d+1)$ and $k \geqslant 1$,*

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \|_{L^p(\mathbf{R}^d)} \lesssim 2^{k(d-1)} \#(\mathcal{E}_k)^{1/p} = 2^k \cdot (2^{k(d-p-1)} \#(\mathcal{E}_k))^{1/p}.$$

Proof of Lemma 1.2 from Lemma 1.3. Applying a dyadic interpolation result (Lemma 2.2 of the paper), Lemma 1.3 implies that

$$\| \sum_{(x,r)\in\mathcal{E}} f_{xr} \|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum 2^{kp} 2^{k(d-p-1)} \#(\mathcal{E}_k) \right)^{1/p} = \left(\sum 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}$$

This is a restricted strong type bound for Lemma 1.2, which we can then interpolate. $\square$

If ψ is compactly supported, and r is sufficiently large depending on the size of this support, then f_{xr} is supported on an annulus with centre x, radius r, and thickness $O(1)$. Thus $\|f_{xr}\|_{L^p(\mathbf{R}^d)} \sim r^{(d-1)/p}$, which implies that

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \|_{L^p(\mathbf{R}^d)} \gtrsim 2^{k(d-1)/p} \#(\mathcal{E}_k)^{1/p}$$

so this bound can only be true if $p \geqslant 1$, and becomes tight when $p = 1$, where we actually have

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \|_{L^1(\mathbf{R}^d)} \sim 2^{k(d-1)} \#(\mathcal{E}_k)$$

because there can be no constructive interference in the L^1 norm. Understanding the sum in Lemma 1.3 for $1 < p < (2d-2)/(d+1)$ will require an understanding of the interference patterns of annuli with comparable radius. We will use almost orthogonality principles to understand these interference patterns.

Lemma 1.4. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ with $|x_1 - x_2| \geqslant 1$ or $x_1 = x_2$, and $r_1, r_2 > 1$,*

$$|\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

Remark. Lemma 1.4 implies that $\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle$ is very small, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition:

- $r_1 + r_2 - |x_1 - x_2| \approx 0$, which holds when the two annuli are 'approximately' externally tangent to one another.

- $r_1 - r_2 + |x_1 - x_2| \approx 0$ or $r_1 - r_2 - |x_1 - x_2| \approx$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency situations, though utilizing the tangencies seems important to improve the results they obtain.

Proof. We write

$$\begin{aligned}\langle f_{x_1r_1}, f_{x_2r_2}\rangle &= \left\langle \widehat{f}_{x_1r_1}, \widehat{f}_{x_2r_2}\right\rangle \\ &= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1)\cdot \xi}\, d\xi \\ &= (r_1r_2)^{d-1} \int_{\mathbf{R}^d} \widehat{\sigma}(r_1\xi)\overline{\widehat{\sigma}(r_2\xi)}|\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2-x_1)\cdot\xi}\, d\xi.\end{aligned}$$

We have $\widehat{\sigma}(\xi) = C_d|\xi|^{1-d/2}J_{d/2-1}(2\pi|\xi|)$ for some constant c_d, where $J_{d/2-1}$ is the Bessel function of order $d/2-1$. If we write $a(|\xi|) = |\widehat{\psi}(\xi)|^2$ for each $\xi \in \mathbf{R}^d$ (which is possible because ψ is radial), then we find

$$\begin{aligned}&\langle f_{x_1r_1}, f_{x_2r_2}\rangle \\ &\quad = C_d(r_1r_2)^{d/2} \\ &\qquad \int_{\mathbf{R}^d} |\xi|^{2-d} J_{d/2-1}(2\pi r_1|\xi|)J_{d/2-1}(2\pi r_2|\xi|)|\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2-x_1)\cdot\xi}\, d\xi \\ &\quad = C_d(r_1r_2)^{d/2}|x_2-x_1|^{1-d/2} \\ &\qquad \int_0^\infty s^{d/2+1}\cdot a(s)\cdot J_{d/2-1}(2\pi r_1 s)\cdot J_{d/2-1}(2\pi r_2 s)\cdot J_{d/2-1}(2\pi s|x_2-x_1|)\, ds.\end{aligned}$$

For any $N > 0$, we have an asymptotic expansion

$$J_{d/2-1}(t) = t^{-1/2}\left(\sum_{n=0}^{N-1}(a_{n,+}e^{it} + a_{n,-}e^{-it})t^{-n} + O_N(t^{-N})\right).$$

Thus we find

$$\begin{aligned}
&\int_0^\infty s^{1+d/2}\cdot a(s)\cdot J_{d/2-1}(2\pi r_1 s)\cdot J_{d/2-1}(2\pi r_2 s)\cdot J_{d/2-1}(2\pi s|x_2-x_1|)\,ds\\
&\quad=\sum_{n_1,n_2,n_3=0}^{N-1}\ \sum_{\tau_1,\tau_2,\tau_3\in\{\pm1\}} c_{d,n,\tau}\cdot r_1^{-1/2-n_1}r_2^{-1/2-n_2}|x_2-x_1|^{-1/2-n_3}\\
&\qquad\qquad\int_0^\infty s^{(d-1)/2-n_1-n_2-n_3}\cdot a(s)\cdot e^{2\pi i(\tau_1 r_1+\tau_2 r_2+\tau_3|x_2-x_1|)s}\,ds\\
&\qquad\qquad+O_{N,d}\left((r_1+r_2+|x_2-x_1|)^{-N}\int_0^\infty s^{(d-1)/2-N}\cdot a(s)\right).
\end{aligned}$$

If we assume that $|a(s)|\lesssim s^{10N}$ for small $s>0$, then

$$\begin{aligned}
&\left|\int_0^\infty s^{(d-1)/2-n_1-n_2-n_3}\cdot a(s)\cdot e^{2\pi i(\tau_1 r_1+\tau_2 r_2+\tau_3(x_2-x_1))s}\,ds\right|\\
&\qquad\lesssim_N (1+|\tau_1 r_1+\tau_2 r_2+\tau_3(x_2-x_1)|)^{-5N}.
\end{aligned}$$

Thus for $r_1,r_2>1$,

$$\begin{aligned}
\langle f_{x_1r_1},f_{x_2r_2}\rangle &\lesssim_{N,d} (r_1r_2)^{(d-1)/2}|x_2-x_1|^{-(d-1)/2}\sum_{\pm,\pm}(1+(|x_2-x_1|\pm r_1\pm r_2))^{-N}\\
&\quad+(r_1r_2)^{d/2}|x_2-x_1|^{1-d/2}(r_1+r_2+|x_2-x_1|)^{-5N}\\
&\lesssim (r_1r_2)^{(d-1)/2}|x_2-x_1|^{-(d-1)/2}\sum_{\pm,\pm}(1+(|x_2-x_1|\pm r_1\pm r_2))^{-N}.
\end{aligned}$$

If $|r_2-r_1|\gtrsim|x_2-x_1|$, then $f_{x_1r_1}$ and $f_{x_2r_2}$ have disjoint support, which implies $\langle f_{x_1r_1},f_{x_2r_2}\rangle=0$. Thus we conclude that

$$\begin{aligned}
\langle f_{x_1r_1},f_{x_2r_2}\rangle &\lesssim_{N,d} (r_1r_2)^{(d-1)/2}(1+|r_2-r_1|+|x_2-x_1|)^{-(d-1)/2}\\
&\quad\cdot\sum_{\pm,\pm}(1+(|x_2-x_1|\pm r_1\pm r_2))^{-N}.
\end{aligned}$$

□

Lemma 1.4 implies that

$$|\langle f_{x_1r_1},f_{x_2r_2}\rangle|\lesssim\left(\frac{r_1r_2}{(1+|x_1-x_2|)(1+|r_1-r_2|)}\right)^{(d-1)/2}.$$

The exponent $(d-1)/2$ is too weak to apply almost orthogonality directly.

Bibliography

[1] Andreas Seeger Yaryong Heo, Fëdor Nazrov. Radial fourier multipliers in high dimensions. 2011.