Large Salem Sets Avoiding Nonlinear Configurations 2

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Our goal is to improve the result of the last paper. Most of the work has been done in that setting in the precise range needed, so we can skip all but the final analysis of an expectation in this paper. Let us reintroduce notation:

- We consider a family of axis-aligned cubes $Q_1, \dots, Q_n \subset [0,1]^d$, each with common sidelength $s > 0$, such that $d(Q_i, Q_j) \geqslant 10s$ for $i \neq j$.
- We consider a family of density functions $\psi_1, \dots, \psi_n \in C^\infty(\mathbb{T}^d)$ supported on $2Q_i$. We fix a large integer $M > 0$, and consider a family of independent random variables
$$\{X_i(k) : 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant M\}$$
where $X_i(k)$ is chosen with respect to the probability density function ψ_i.
- Let $r = M^{-1/\lambda}$, and consider the random set I of all indices $k_n \in \{1, \dots, M\}$ such that there exists indices $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$ such that
$$|X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leqslant r.$$
- Finally, define
$$H(\xi) = \sum_{k \in I} e^{2\pi i \xi \cdot X_n(k)}.$$

Our goal is to show that for any $\delta > 0$, there is $M_0 > 0$ such that for $M \geqslant M_0$, and any $\xi \in \mathbb{Z}^d$,
$$\mathbb{E}[H(\xi)] \leqslant \delta M |\xi|^{-\lambda/2} + O_\delta(M^{1/2}).$$
In the previous paper, we were able to establish this result with $\lambda = d/(n - 3/4)$, but we expect that one can establish this result with $\lambda = d/(n-1)$. This was established in the case $n = 2$, where $\lambda = d$. So the case of interest is where $n \geqslant 3$.

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1 Review of Last Paper Proof

We start by writing

$$\mathbb{E}[H(\xi)] = M \int \psi_n(x_n) p_M(x_n) e^{2\pi i \xi \cdot x_n}\, dx_n,$$

where $p_M(x_n)$ denotes the probability that there exists $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$ such that

$$|x_n - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leqslant r.$$

In that paper, it was shown using an inclusion exclusion argument that if $E_{x_n} = f^{-1}(x_n)$, then

$$P_M(x_n) = M^{n-1} \int_{E_{x_n}} \psi_1(x_1) \cdots \psi_{n-1}(x_{n-1})\, dx_1 \dots\, dx_{n-1} + O(M^{2(n-1)-2d/\lambda}).$$

The error here is $O(M^{-1/2})$ provided that $\lambda \leqslant d/(n-3/4)$. Thus in this situation if we write $\psi = \psi_1 \otimes \cdots \otimes \psi_n$ then

$$\mathbb{E}[H(\xi)] = \left(M \int \int_{E_{x_n}} \psi(x) e^{2\pi i \xi \cdot x_n} \right) + O(M^{1/2}).$$

The integral here can be converted using the coarea formula into an oscillatory integral, which yields the $\delta|\xi|^{-\lambda/2}$ term required.

2 Does Smoothness Help?

The function ψ_n is smooth. Thus if we could show that

$$\|p_M\|_{W^{s/2,1}(\mathbb{T}^d)} \lesssim M^{-1/2}$$

then the fact that $\mathbb{E}[H(\xi)]$ is M times the Fourier transform of $\psi_n \cdot P_M$ would give the required result. But heuristically, it doesn't seem like the function should be that smooth, though I should redo the calculation just to be sure.

3 Counting Solutions

The reason the argument in the first section worked was that with high probability, the number of solutions to

$$|x_n - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leqslant r$$

was equal to M^{n-1} times the surface area of the hyperplane E_{x_n}. Could we do something more robust to get around the inclusion-exclusion argument?

One potential idea, instead of drawing $X_1, \dots, X_{n-1}$ from a continuous distribution, is to draw the points from a discrete distribution, e.g. uniformly distributed on some rational points with a fixed denominator in the cube Q_i? If f is an integer valued polynomial, then perhaps the circle method might then be of some use since then we might get much better bounds on the number of solutions to the equation? We could also use probabilistic decoupling to remove the cubes $\{Q_i\}$ from the equation if needed, provided the concentration argument goes through.