

Fractals avoiding Fractal Sets

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Pattern Avoidance Problems

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- ▶ How large can the Hausdorff dimension of a \mathbf{Q} vector subspace of \mathbf{R}^n be not containing any points in \mathbf{Q}^n .
- ▶ What is the largest Hausdorff dimension of a subset of \mathbf{R}^d such that the distances between any two points is irrational?
- ▶ Given any equation f , what is the largest Hausdorff dimension of $X \subset \mathbf{R}^d$ such that for any distinct $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$.

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z , except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

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- For any n , if $Z_n = \{x \in \mathbf{R}^{nd} : \text{there is } a_1, \dots, a_n \text{ s.t. } a_1 x_1 + \dots + a_n x_n \in \mathbf{Q}\}$, then X generates a vector space avoiding the rationals if and only if $X^n \cap Z_n \subset \Delta$ for all n .

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- If $Z = \{(x, y) : d(x, y) \in \mathbf{Q}\} \subset \mathbf{R}^{2d}$, then points in X have irrational distances if and only if $X^2 \cap Z \subset \Delta$.
- If $Z = f^{-1}(0)$, X avoids zeroes of f for distinct values if and only if $X^n \cap Z \subset \Delta$.

The Generic Problem

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- ▶ Pramanik and Fraser (2016): If Z is a smooth hypersurface of dimension $nd - d$, we can find X with dimension $d/(n - 1)$.

Increasing the Difficulty...

*What if the Patterns are
Fractally Specified...*

Main Result

Theorem

If Z is the countable union of sets with lower Minkowski dimension bounded by $\alpha \geq d$, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathbf{H}}(X) = \frac{nd - \alpha}{n - 1} = \frac{\text{codim}(Z)}{n - 1}$$

Low Rank Avoidance

Theorem

If we have countably many sets $Z_i \subset \mathbf{R}^{n_i d}$ with linear transformations $T_i : \mathbf{R}^{n_i d} \rightarrow \mathbf{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension β_i , we can find X with $X^{n_i} \cap Z_i \subset \Delta$ for each i and

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- ▶ The hypothesis says Z is coverable efficiently by lower dimensional thickened hyperplanes. Result should also extend when each Z_i is efficiently covered by thickened pencils of low degree algebraic surfaces, i.e. $f(Z)$ has low dimension where f is a polynomial map.

Low Rank Avoidance

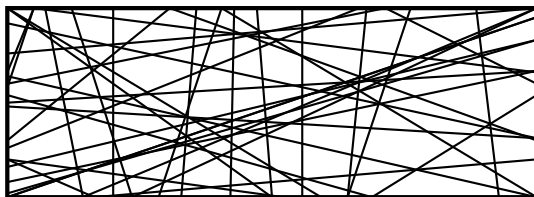
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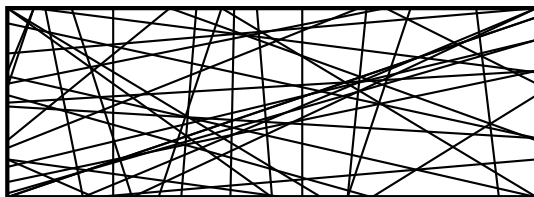
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- ▶ Want to push the $2k - 1$ to $k - 1$, at least for $k \geq 2$. Know this is true for certain families of examples.

Advantages of Our Method



- More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin' Z , we get stable effects on the Hausdorff dimension of X .

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- ▶ More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin' Z , we get stable effects on the Hausdorff dimension of X .
- ▶ Uncountable unions of regular sets are allowed!

Applications

- ▶ Given a subset Y of \mathbf{R}^d which is the countable union of sets with Minkowski dimension α , we can find a \mathbf{Q} vector subspace X of \mathbf{R}^d with Hausdorff dimension $d - \alpha$ disjoint from Y .

Applications

- ▶ Given a subset Y of \mathbf{R}^d which is the countable union of sets with Minkowski dimension α , we can find a \mathbf{Q} vector subspace X of \mathbf{R}^d with Hausdorff dimension $d - \alpha$ disjoint from Y .
- ▶ We can find a full dimensional subset of \mathbf{R}^d avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $x \in X^n, y \in \mathbf{Q}^n$. No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.

Two key ideas:

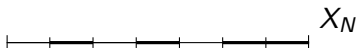
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- ▶ Random Dissection.

Discretization of Scales



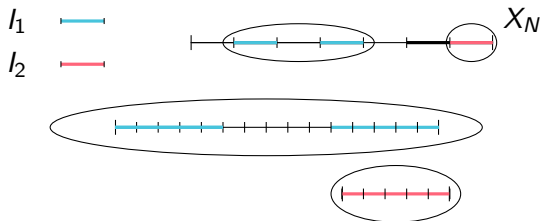
- Construct X as a Cantor set by limits of interval sets X_N .

Discretization of Scales



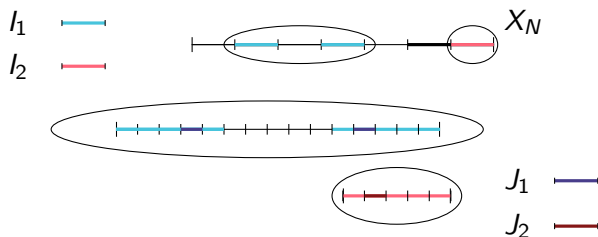
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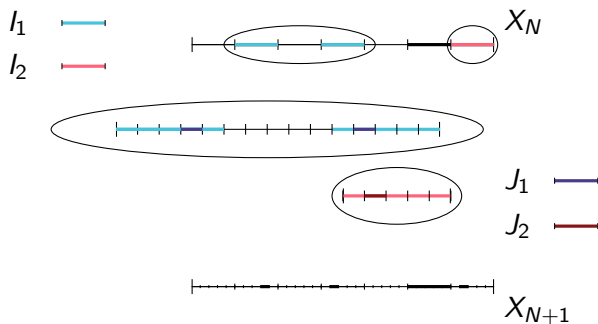
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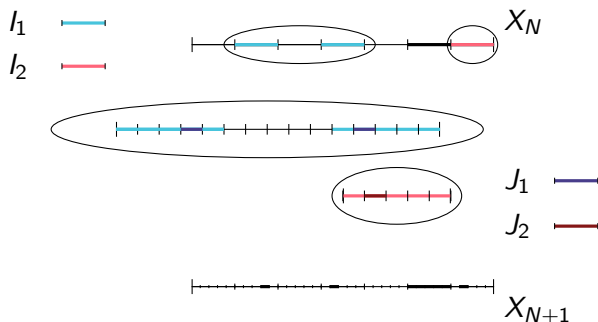
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Discretization of Scales



- ▶ Construct X as a Cantor set by limits of interval sets X_N .
- ▶ **Discrete Problem:** Given disjoint unions of length L intervals l_1, \dots, l_n , find J_i for each i containing a part of each interval in l_i such that $J_1 \times \dots \times J_n$ is disjoint from Z .

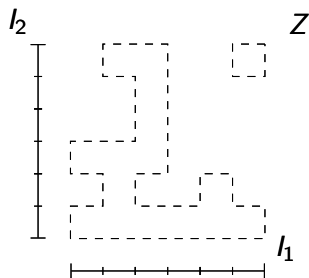
Queueing

- ▶ Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers I_1, \dots, I_n of the set X gives a fractal avoiding set X .

Queueing

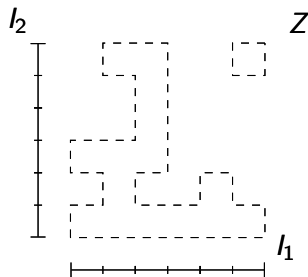
- ▶ Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers I_1, \dots, I_n of the set X gives a fractal avoiding set X .
- ▶ **Reason:** If we consider distinct $x_1, \dots, x_n \in X$, there are intervals I_1, \dots, I_n with $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \dots \times J_n$ cannot be contained in Z .

Exploiting Randomness



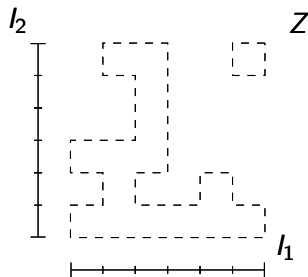
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Exploiting Randomness



- ▶ How do we prove the discrete scale argument?
- ▶ Aside from Z 's dimension, we have little structural knowledge.
- ▶ Random choices of the J_k avoid Z effectively.
- ▶ We essentially obtain for all but a fraction $o(1)$ of the length L intervals in I_k , J_k contains a length L^β section of each, where $\beta = d(n-1)/(nd - \alpha)$. This ratio gives the Hausdorff dimension bound $(nd - \alpha)/(n - 1)$ for X .

So What's Next?

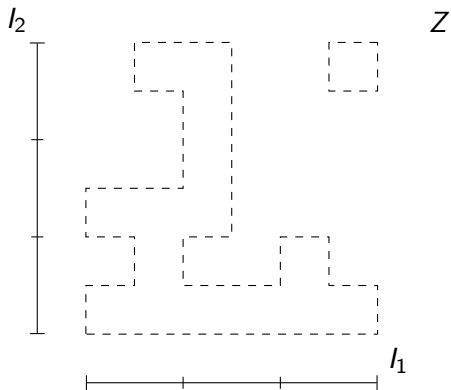
Extension to Hausdorff Dimension

- ▶ There is no obvious reason why our techniques should fail when Z has *Hausdorff dimension* α rather than a Minkowski dimension bound.

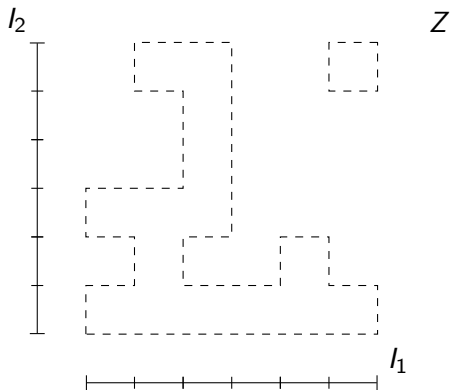
Extension to Hausdorff Dimension

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- ▶ We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

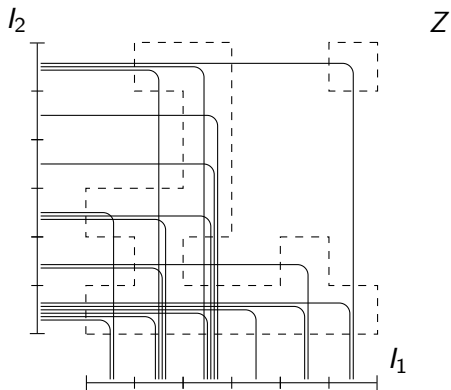
Analogies with Hypergraphs



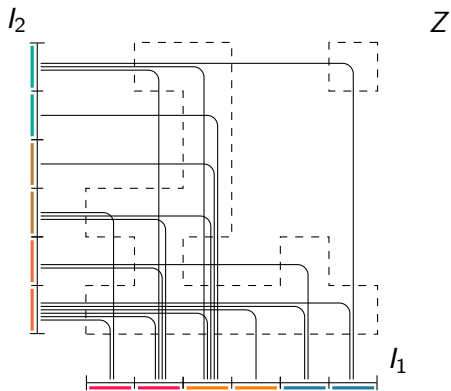
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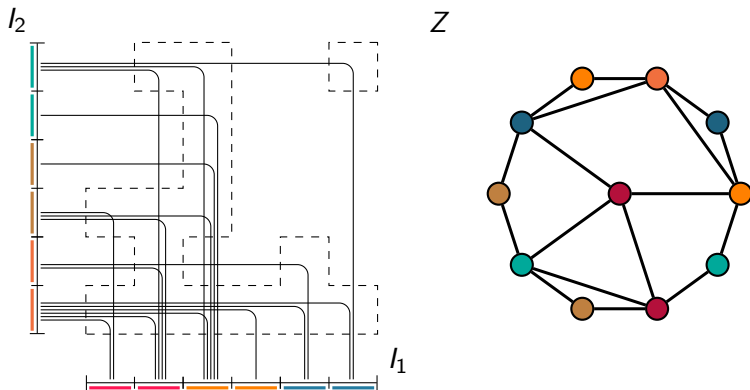
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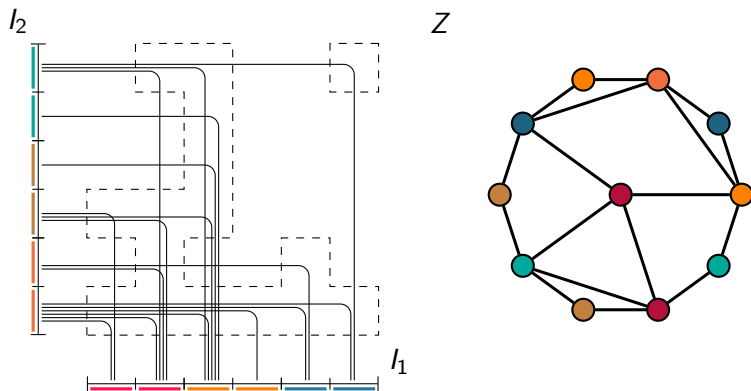


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Analogies with Hypergraphs



- ▶ The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- ▶ We are looking to using other methods on hypergraphs to improve the bound when Z has certain structural properties (like for the low rank result).

Interested in Learning More?

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