Salem Sets Avoiding Rough Configurations

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1 Introduction

Geometric measure theory explores the relationship between the geometry of subsets of Euclidean spaces, and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a particular subset of Euclidean space. In this paper, we study the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a 'rough pattern', in the sense of [3], we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel set $E \subset \mathbf{R}^d$ is it's *Fourier dimension*; given a finite Borel measure μ on $\mathbf{R}^d$, it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$, is the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ |\hat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d \right\} < \infty. \tag{1.1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most classical fractal sets are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem.

Theorem 1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset [0,1]^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n - 1/2}, d \right)$$

such that for any distinct points $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$.

Remark 1. *Theorem 1 is an attempt to strengthen the main result of [3] to give a Fourier dimension bound, albeit under a weaker dimension bound. Unlike in [3], the case of Theorem 1 when $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $[0,\pi]^d - \pi(Z)$ is not necessarily a Salem set, where $\pi(x_1, \ldots, x_n) = x_1$ is projection onto the first coordinate. For instance, in Example 8 of [6] it is shown that there exists a compact set $E \subset [0,1]$ such that $\dim_{\mathbf{M}}(E) < 1$ and $\dim_{\mathbf{F}}([0,1] - E) < 1$. Setting $Z = E \times \{0\} \cup \{0\} \times E$ shows that neither subtracting projections onto the first nor the second coordinate gives the required Fourier dimension bounds.*

Because we are working with *compact* sets avoiding patterns, working in the domain $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, and working in this space has several advantages over the Euclidean case. For a finite measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.2}$$

We can then define the Fourier dimension of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Since $\mathbf{T}^d$ has a natural metric space structure, we can define the Hausdorff dimension of sets on $\mathbf{T}^d$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1.1) is equivalent to the more discrete condition

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.3}$$

A proof is given in [4, Lemma 39]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x) \, d\mu^*(x) = \int_{\mathbf{R}^d} f(x) \, d\mu(x), \tag{1.4}$$

then (1.3), together with the Poisson summation formula, implies $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$. Thus Theorem 1 is clearly equivalent to it's periodic variant, introduced below.

Theorem 2. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset \mathbf{T}^d$ with dimension*
$$\beta = \min \left(\frac{dn - \alpha}{n - 1/2}, d \right)$$
such that for any distinct points $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$.

To construct our set, we rely on a Baire-category argument. Thus we consider a complete metric space $\mathcal{X}$, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in \mathcal{X}$, E is a Salem set of dimension β, and is pattern avoiding, in the sense that the set of pairs (E, μ) which do not satisfy these properties is a set of first category in $\mathcal{X}$. It follows that Theorem 2 holds in a 'generic' sense for elements of $\mathcal{X}$.

Once we have setup the appropriate metric space $\mathcal{X}$, our approach is quite similar to the construction in [3], relying on a random selection procedure, which is now exploited to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach, is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [3]. But the Baire category argument makes our proof much simpler to read, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern.

2 Notation

- Given a metric space Ω, $x \in \Omega$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let

$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

denote the ε-*thickening* of the set E. A subset of Ω is of *first category* in Ω if it is the countable union of closed sets with nonempty interior. We say a property holds *quasi-always*, or a property is *generic* in Ω if the set of points in Ω failing to satisfy that property form a set of first category.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let

$$|x| = \min\{|x + n| : n \in \mathbf{Z}\},$$

and for $x \in \mathbf{T}^d$, we let

$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

The canonical metric on $\mathbf{T}^d$ is then given by $d(x, y) = |x - y|$, for $x, y \in \mathbf{T}^d$.

- Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0, d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as

$$H^\alpha_\delta(E) = \inf\left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } \varepsilon_i \leqslant \delta \text{ for all } i \in \mathbf{N} \right\}.$$

The α dimensional Hausdorff measure of E is equal to

$$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0, d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0, d]$ such that $H^s(E) = 0$.

- Suppose $\mathbf{E} = \mathbf{R}^d$ or $\mathbf{E} = \mathbf{T}^d$, and for a measurable set E, we let $|E|$ denote it's Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon|.$$

- We will make several uses of *Hoeffding's Inequality* to control the deviations of independent families of random variables. The version of Hoeffding's inequality we use states that if $\{X_1, \ldots, X_N\}$ is an independent family of mean-zero random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

$$\mathbf{P}\left(|X_1 + \cdots + X_N| \geqslant t\right) \leqslant 2 \exp\left(\frac{-t^2}{2(A_1^2 + \cdots + A_N^2)} \right).$$

Proofs are given in many probability textbooks, for instance, in Chapter 2 of [5].

- Throughout this paper, we will need to consider a standard mollifier. So we fix a smooth, non-negative function $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$ and

$$\int_{\mathbf{T}^d} \phi(x)\, dx = 1.$$

For each $\varepsilon \in (0,1)$, we can then define $\phi_\varepsilon \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_\varepsilon(x) = \begin{cases} \varepsilon^{-d}\phi(x/\varepsilon) & : |x| < \varepsilon, \\ 0 & : \text{otherwise}. \end{cases}$$

Then the following standard properties hold:

(1) For each $\varepsilon \in (0,1)$, ϕ_ε is a smooth probability density, and $\phi_\varepsilon(x) = 0$ for $|x| \geqslant \varepsilon$.

(2) For any $\varepsilon \in (0,1)$,

$$\|\widehat{\phi_\varepsilon}\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1. \tag{2.1}$$

(3) For each $\xi \in \mathbf{Z}^d$,

$$\lim_{\varepsilon \to 0} \widehat{\phi_\varepsilon}(\xi) = 1. \tag{2.2}$$

(4) For each $T > 0$, and for all $\varepsilon > 0$ and non-zero $k \in \mathbf{Z}^d$,

$$|\widehat{\phi_\varepsilon}(k)| \lesssim_T \varepsilon^{-T}|k|^{-T}. \tag{2.3}$$

3   A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task. Though in later sections we will specify β as in Theorem 1, in this section we let β be an arbitrary element of $(0, d]$. We proceed as in [2], forming our metric space as a combination of two simpler metric spaces. However, we employ a novel Frechét space construction instead of the Banach space used in [2], which enables us to use softer estimates in our arguments:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

then $(\mathcal{E}, d_H)$ forms a complete metric space.

- We let $M(\beta)$ consist of the class of all finite Borel measures μ on $\mathbf{T}^d$ such that for each $\varepsilon \in (0, \beta/2]$, the quantity

$$\|\mu\|_{M(\beta/2-\varepsilon)} = \sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon}$$

is finite. Then $\|\cdot\|_{M(\beta/2-\varepsilon)}$ is a seminorm on $M(\beta)$, and the collection of all such seminorms for $\varepsilon \in (0, \beta/2]$ gives $M(\beta)$ the structure of a Frechét space. Under this topology, a sequence of probability measures $\{\mu_k\}$ converges to a probability measure μ in $M(\beta)$ if and only if for any $\varepsilon > 0$, $\lim_{k\to\infty} \|\mu_k - \mu\|_{M(\beta/2-\varepsilon)} = 0$.

We now consider a fusion of the metric spaces $\mathcal{E}$ and $M(\beta)$. We let $\mathcal{X}$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(\beta)$, where μ is a probability measure with $\mathrm{supp}(\mu) \subset E$. Then $\mathcal{X}$ is a closed subset of $\mathcal{E} \times M(\beta)$, and thus a complete metrizable space. We remark that for any $\varepsilon > 0$ and $(E, \mu) \in \mathcal{X}$,

$$\lim_{|\xi| \to \infty} |\xi|^{\beta/2 - \varepsilon} |\widehat{\mu}(\xi)| = 0. \tag{3.1}$$

which follows from the fact that $\|\mu\|_{M(\beta/2 - \varepsilon/2)}$ is finite. A density approach following from this remark allows us to work with smooth measures for the remainder of this paper.

Lemma 1. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}$.*

Proof. Consider $(E, \mu) \in \mathcal{X}$. For each $\delta > 0$, consider the convolved measure $\mu_\delta = \mu * \phi_\delta$. Then $\mu_\delta \in C^\infty(\mathbf{T}^d)$. If we set $E_\delta = E \cup \mathrm{supp}(\mu_\delta)$, then we claim

$$\lim_{\delta \to 0} (E_\delta, \mu_\delta) = (E, \mu). \tag{3.2}$$

Since $\mathrm{supp}(\mu_\delta) \subset E_\delta$, we conclude that

$$d_H(E, E_\delta) \leqslant \delta. \tag{3.3}$$

Thus $E_\delta \to E$ as $\delta \to 0$ under the Hausdorff distance. Now fix $\varepsilon_1 \in (0, \beta/2]$ and $\varepsilon > 0$. For each $\xi \in \mathbf{Z}^d$, $|\widehat{\mu_\delta}(\xi)| = |\widehat{\phi_\delta}(\xi)||\widehat{\mu}(\xi)|$, so

$$|\xi|^{\beta/2 - \varepsilon_1} |\mu_\delta(\xi) - \mu(\xi)| = |\xi|^{\beta/2 - \varepsilon_1} |\widehat{\phi_\delta}(\xi) - 1||\widehat{\mu}(\xi)|. \tag{3.4}$$

Since $(E, \mu) \in \mathcal{X}$, we can apply (3.1) to conclude that there exists R such that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2 - \varepsilon_1} |\widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{3.5}$$

Combining (3.4), (3.5), and (2.1), we conclude that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2 - \varepsilon_1} |\mu_\delta(\xi) - \mu(\xi)| \leqslant 2\varepsilon. \tag{3.6}$$

By (2.2), we conclude that there exists $\delta_0 > 0$ such that for $\delta \leqslant \delta_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2 - \varepsilon} |\widehat{\phi_\delta}(\xi) - 1| \leqslant \varepsilon. \tag{3.7}$$

The (L^1, L^∞) bound for the Fourier transform implies that

$$|\widehat{\mu}(\xi)| \leqslant \mu(\mathbf{T}^d) = 1 \tag{3.8}$$

But from (3.7) and (3.8) applied to (3.4), we find that for $\delta \leqslant \delta_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2 - \varepsilon_1} |\mu_\delta(\xi) - \mu(\xi)| \leqslant \varepsilon. \tag{3.9}$$

Putting together (3.6) and (3.9), we find that for $\delta \leqslant \delta_0$,

$$\|\mu_\delta - \mu\|_{M(\beta/2 - \varepsilon_1)} \leqslant 2\varepsilon. \tag{3.10}$$

Since ε and ε_1 were arbitrary, we conclude from (3.10) and (3.3) that $(E_\delta, \mu_\delta) \to (E, \mu)$, completing the proof. $\qquad\square$

Remark 2. *We note that if (E_0, μ_0) are such that $\text{supp}(\mu_0) = E_0$, then in the proof above, one may let E_δ be equal to $\text{supp}(\mu_\delta)$, since then $d_H(E_0, E_\delta) \leqslant \delta$. It then follows that such pairs (E_0, μ_0) are approximated by pairs (E, μ), where μ is smooth and $\text{supp}(\mu) = E$.*

The reader might not be wondering why we don't work with the smaller space $\tilde{\mathcal{X}} \subset \mathcal{X}$ of pairs $(E, \mu) \in \mathcal{X}$ with $\text{supp}(\mu) = E$. The reason is that $\tilde{\mathcal{X}}$ is not a closed subset of $\mathcal{E} \times M(\beta)$, and so is not a complete metrizable space under the product topology. However, as a consolation, we note that one can prove that quasi-all elements of $\mathcal{X}$ belong to $\tilde{\mathcal{X}}$, so that one can think of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ as being equal 'generically'.

Lemma 2. *For quasi-all $(E, \mu) \in \mathcal{X}$, $\text{supp}(\mu) = E$.*

Proof. For each closed cube $I \subset \mathbf{T}^d$, let

$$A(I) = \{(E, \mu) \in \mathbf{T}^d : (E \cap I) = \varnothing \text{ or } \mu(I) > 0\}.$$

Then $A(I)$ is an open set. If $\{I_k\}$ is a sequence enumerating all cubes with rational corners in $\mathbf{T}^d$, then

$$\bigcap_{k=1}^{\infty} A(I_k)$$

is the collection of $(E, \mu) \in \mathcal{X}$ with $\text{supp}(\mu) = E$. Thus it suffices to show that $A(I)$ is dense in $\mathcal{X}$ for each closed cube I.

Consider $(E_0, \mu_0) \in \mathcal{X} - A(I)$, $\varepsilon_1 \in (0, \beta/2]$, and $\varepsilon > 0$. Our goal is to find $(E, \mu) \in A(I)$ with $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu_0 - \mu\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. Without loss of generality by Lemma 1 we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$. Because $(E_0, \mu_0) \in \mathcal{X} - A(I)$, we know $E \cap I \neq \varnothing$ and $\mu(I) = 0$. Find a smooth probability measure ν supported on $E_\varepsilon \cap I$ and, for $t \in (0, 1)$, define $\mu_t = (1 - t)\mu_0 + t\nu$. Then $\text{supp}(\mu_t) \subset E_\varepsilon$, so if we let $E = \text{supp}(\nu) \cup \text{supp}(\mu)$, then $d_H(E, E_0) \leqslant \varepsilon$. Clearly $(E, \mu_t) \in A(I)$ for $t > 0$. And

$$\|\mu_t - \mu_0\|_{M(\beta/2 - \varepsilon)} \leqslant t \left(\|\mu_0\|_{M(\beta/2 - \varepsilon)} + \|\nu\|_{M(\beta/2 - \varepsilon)} \right),$$

so taking $t \leqslant \varepsilon(\|\mu\|_{M(\beta/2 - \varepsilon)} + \|\nu\|_{M(\beta/2 - \varepsilon)})^{-1}$ shows $\|\mu_t - \mu\|_{M(\beta/2 - \varepsilon)} \leqslant \varepsilon$. Since ε was arbitrary, we conclude $A(I)$ is dense in $\mathcal{X}$. $\square$

Combining Lemma 2 with Remark 2 gives the following simple corollary.

Corollary 1. *The family of (E, μ) with $\text{supp}(\mu) = E$ and $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}$.*

Our main way of constructing approximations to $(E_0, \mu_0) \in \mathcal{X}$ is to multiply μ_0 by a smooth function f. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which may contribute to the formation of a particular pattern from the set Z we are trying to avoid. As long as f is small enough in the topology of $M(\beta)$, this introduces a neglible amount of error.

Lemma 3. *Consider a smooth finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function f. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon)} \lesssim_{d, \mu_0} \|f\|_{M(\beta/2 - \varepsilon)}.$$

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Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu_0}$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\xi|^{\beta/2-\varepsilon}|\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| = |\xi|^{\beta/2-\varepsilon}\left|\sum_{\eta \neq \xi} \widehat{f}(\xi - \eta)\widehat{\mu_0}(\eta)\right|. \tag{3.11}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi - \eta| \leqslant 2|\xi|$, so

$$|\xi|^{\beta/2-\varepsilon}|\widehat{f}(\xi-\eta)| \leqslant \|f\|_{M(\beta/2-\varepsilon)}|\xi|^{\beta/2-\varepsilon}|\xi-\eta|^{\varepsilon-\beta/2} \leqslant 2^{\beta/2-\varepsilon}\|f\|_{M(\beta/2-\varepsilon)} \lesssim_d \|f\|_{M(\beta/2-\varepsilon)}. \tag{3.12}$$

Since μ_0 is smooth, for any $T \geqslant 0$ and $\xi \in \mathbf{Z}^d$,

$$|\xi|^T|\widehat{\mu_0}(\xi)| \lesssim_{T,\mu_0} 1. \tag{3.13}$$

Thus we can combine the bounds (3.12) and (3.13), with $T = d + 1$, to conclude that

$$|\xi|^{\beta/2-\varepsilon}\left|\sum_{0 \leqslant |\eta| \leqslant |\xi|/2} \widehat{f}(\eta)\widehat{\mu_0}(\xi - \eta)\right| \lesssim_{\mu_0,d} \left(1 + \sum_{0 < |\eta| \leqslant |\xi|/2} \frac{1}{|\xi|^{d+1}}\right)\|f\|_{M(\beta/2-\varepsilon)} \tag{3.14}$$

$$\lesssim_d \|f\|_{M(\beta/2-\varepsilon)}.$$

On the other hand, we note that (3.12) implies that for all $\eta \neq \xi$,

$$|\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2-\varepsilon)}, \tag{3.15}$$

Thus applying (3.13) and (3.15), with $T = 3d/2$, we conclude that

$$|\xi|^{\beta/2-\varepsilon}\left|\sum_{\substack{|\eta|>|\xi|/2 \\ \eta \neq \xi}} \widehat{f}(\xi - \eta)\widehat{\mu_0}(\eta)\right| \lesssim_{d,\mu_0} |\xi|^{\beta/2-\varepsilon} \sum_{|\eta|>|\xi|/2} \frac{\|f\|_{M(\beta/2-\varepsilon)}}{|\eta|^{3d/2}} \lesssim_d \|f\|_{M(\beta/2-\varepsilon)}. \tag{3.16}$$

Combining (3.11), (3.14) and (3.16) completes the proof. $\qquad\square$

Remark 3. *In particular, we note that this lemma implies that $\mu(\mathbf{T}^d) \geqslant 1 - O_{d,\mu_0}(\|f\|_{M(0)})$.*

The next lemma shows that provided that, for K points $x_1, \ldots, x_K \in \mathbf{T}^d$, that there is enough square root cancellation in the exponential sum

$$\frac{1}{K}\sum_{i=1}^{K} e^{2\pi i(x_i \cdot \xi)},$$

then a smooth function f localized to these K points has the appropriate Fourier decay that might be expected by the mass of it's support.

Lemma 4. *Fix $C > 0$, $\varepsilon_1, \varepsilon_2 > 0$, and $\varepsilon > 0$. Consider K points $x_1, \ldots, x_K \in \mathbf{T}^d$ such that for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant K^{1/\beta+\varepsilon_2}$,*

$$\left|\frac{1}{K}\sum_{i=1}^{K} e^{2\pi i(x_i \cdot \xi)}\right| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{3.17}$$

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Then there exists a large integer $K_0(C, \beta, \varepsilon_1, \varepsilon_2, \varepsilon)$, such that if $K \geqslant K_0$, $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$, and we define

$$f(x) = \frac{1}{K}\sum_{i=1}^{K}\phi_{\varepsilon_0}(x - x_i),$$

then $\|f\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$.

Proof. Set

$$D(x) = \frac{1}{K}\sum_{i=1}^{K}\delta(x - x_i),$$

then (3.17) is equivalent to the property that for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant K^{1/\beta + \varepsilon_2}$,

$$|\widehat{D}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{3.18}$$

Noting that $f = D * \phi_{\varepsilon_0}$, we conclude that

$$|\widehat{f}| = |\widehat{D}||\widehat{\phi_{\varepsilon_0}}|. \tag{3.19}$$

For $0 < |\xi| \leqslant K^{1/\beta}$, we combine (3.18), (3.19) and (2.1) to conclude that

$$|\widehat{f}(\xi)| \leqslant \left[CK^{-\varepsilon_1/\beta}\log(K)^{1/2}\right]|\xi|^{\varepsilon_1 - \beta/2}. \tag{3.20}$$

As $K \to \infty$, $K^{-\varepsilon_1/\beta}\log(K)^{1/2} \to 0$, so we conclude from (3.20) there exists a large integer $K_1(C, \beta, \varepsilon_1, \varepsilon)$ such that for $K \geqslant K_1(C, \beta, \varepsilon_1, \varepsilon)$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon|\xi|^{\varepsilon_1 - \beta/2}. \tag{3.21}$$

If $K^{1/\beta} \leqslant |\xi| \leqslant K^{1/\beta + \varepsilon_2}$, we note that (2.3) implies $\widehat{\phi_{\varepsilon_0}}(\xi) \lesssim_d \varepsilon_0^{-\beta/2}|\xi|^{-\beta/2}$, which together with (3.18), (3.19), and the bound $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$, imply

$$\begin{aligned}
|\widehat{f}(\xi)| &\lesssim_d \left(CK^{-1/2}\varepsilon_0^{-\beta/2}K^{-\varepsilon_1/\beta}\log(K)^{1/2}\right)|\xi|^{\varepsilon_1 - \beta/2} \\
&\leqslant \left(C^{1+\beta/2}K^{-\varepsilon_1/\beta}\log(K)^{1/2}\right)|\xi|^{\varepsilon_1 - \beta/2}.
\end{aligned} \tag{3.22}$$

Again, we find that as $K \to \infty$, $K^{-\varepsilon_1/\beta}\log(K)^{1/2} \to 0$, so we conclude from (3.22) that there exists $K_2(C, \beta, \varepsilon_1, \varepsilon)$ such that if $K \geqslant K_2(C, \beta, \varepsilon_1, \varepsilon)$, then

$$|\widehat{f}(\xi)| \leqslant \varepsilon|\xi|^{\varepsilon_1 - \beta/2}. \tag{3.23}$$

If $|\xi| \geqslant K^{1/\beta + \varepsilon_2}$, we apply (2.3) for $T \geqslant \beta/2$ together with the bound $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$ to conclude

$$\begin{aligned}
|\widehat{f}(\xi)| &\lesssim_T \varepsilon_0^{-T}|\xi|^{-T} \\
&\leqslant \left[C^T K^{T/\beta}|\xi|^{\beta/2 - T}\right]|\xi|^{-\beta/2} \\
&\leqslant \left[C^T K^{1/2 + (\beta/2 - T)\varepsilon_2}\right]|\xi|^{-\beta/2}.
\end{aligned} \tag{3.24}$$

If we choose $T > \beta/2 + 1/2\varepsilon_2$, then as $K \to \infty$, $K^{1/2 + (\beta/2 - T)\varepsilon_1} \to 0$. Thus we conclude from (3.24) that there exists a large integer $K_3(C, \beta, \varepsilon_2, \varepsilon)$ such that for $K \geqslant K_3(C, \beta, \varepsilon_2, \varepsilon)$ and $|\xi| \geqslant K^{1/\beta + \varepsilon_2}$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon|\xi|^{-\beta/2}. \tag{3.25}$$

All that remains is to combine (3.21), (3.23), and (3.25), defining $K_0 = \max(K_1, K_2, K_3)$. $\square$

Lemma 5. *Consider a smooth finite measure μ_0 on $\mathbf{T}^d$. Fix $C > 0$, $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$, and $\varepsilon > 0$. Consider K points $x_1, \ldots, x_K \in \mathbf{T}^d$ such that if*

$$D(x) = \frac{1}{K} \sum_{i=1}^{K} \delta(x - x_i),$$

then for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leq K^{1/\beta + \varepsilon_2}$,

$$|\widehat{D}(\xi)| \leq C K^{-1/2} \log(K)^{1/2}. \tag{3.26}$$

Then there exists a large integer $K_0(C, d, \beta, \mu_0, \varepsilon_1, \varepsilon_2, \varepsilon)$, such that if $K \geq K_0$ and $\varepsilon_0 \geq C^{-1} K^{-1/\beta}$, and we define

$$f(x) = \frac{1}{K} \sum_{i=1}^{K} \phi_{\varepsilon_0}(x - x_i),$$

and a smooth probability measure

$$\mu = \frac{f \mu_0}{(f \mu_0)(\mathbf{T}^d)},$$

then $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leq \varepsilon$.

Proof. It suffices to combine Lemmas 3 and 4 to show that there exists $K_0(C, d, \beta, \mu_0, \varepsilon_1, \varepsilon_2, \varepsilon)$ such that

$$\|f \mu_0 - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leq \varepsilon/10, \tag{3.27}$$

and

$$\|f \mu_0 - \mu_0\|_{M(0)} \leq \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2 - \varepsilon)}}\right). \tag{3.28}$$

As mentioned in Remark 3, (3.28) implies that

$$1 - \min\left(1/2, (\varepsilon/4)\|\mu\|_{M(\beta/2 - \varepsilon)}\right) \leq (f\mu_0)(\mathbf{T}^d) \leq 1. \tag{3.29}$$

But now (3.27) and (3.29) show that

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leq \|f\mu_0 - \mu_0\|_{M(\beta/2 - \varepsilon)} + \|\mu - f\mu_0\|_{M(\beta/2 - \varepsilon)}$$

$$\leq (\varepsilon/2) + \left(1 - \frac{1}{(f\mu_0)(\mathbf{T}^d)}\right)\|\mu_0\|_{M(\beta/2 - \varepsilon)} \leq \varepsilon. \qquad \square$$

Independence of random variables is sufficient to obtain square root cancellation.

Lemma 6. *Fix a large integer K, let $X_1, \ldots, X_K$ be independant uniformly distributed random variables on $\mathbf{T}^d$. Set*

$$D(x) = \sum_{k=1}^{K} \delta(x - x_k)$$

and

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leq K^{1/\beta + 1}\}.$$

Then there exists a constant $C(\beta, d)$, such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \leq C K^{-1/2} \log(K)^{1/2}\right) \leq 1/10.$$

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$, consider the random variable $Y(\xi, k) = K^{-1} e^{2\pi i(\xi \cdot X_k)}$. Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} Y(\xi, k) = \widehat{D}(\xi). \tag{3.30}$$

We also note that for each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$,

$$|Y(\xi, k)| = K^{-1}, \tag{3.31}$$

Moreover, for $\xi \neq 0$,

$$\mathbf{E}(Y(\xi, k)) = K^{-1} \int_{\mathbf{T}^d} e^{2\pi i(\xi \cdot x)} \, dx = 0. \tag{3.32}$$

Since the family of random variables $\{Y(\xi, 1), \ldots, Y(\xi, K)\}$ is independent for a fixed non-zero ξ, we can apply Hoeffding's inequality together with (3.30) and (3.31) to conclude that for all $t \geq 0$,

$$\mathbf{P}\left(|\widehat{D}(\xi)| \geq t\right) \leq 2e^{-Kt^2/2}. \tag{3.33}$$

A union bound obtained by applying (3.33) over all $|\xi| \leq K^{1/\beta+1}$ shows that if

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leq K^{1/\beta+1}\},$$

then there exists a constant $C(d, \beta) \geq 1$ such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \geq t\right) \leq \exp\left(C \log(K) - \frac{5Kt^2}{C}\right). \tag{3.34}$$

But then, setting $t = CK^{-1/2} \log(K)^{1/2}$ in (3.34) completes the proof. $\qquad\square$

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}$ to be as low as possible. For each $(E, \mu) \in \mathcal{X}$, the condition that $\mu \in M(\beta)$ implies that $\dim_{\mathbf{F}}(\mu) \geq \beta$, so $\dim_{\mathbf{F}}(E) \geq \beta$. Thus it is natural to expect that for quasi-all $(E, \mu) \in M(\beta)$, the set E has both Hausdorff dimension and Fourier dimension equal to β.

Lemma 7. *For quasi-all $(E, \mu) \in \mathcal{X}$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since the case $\beta = d$ is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that quasi-all $\mu \in M(\beta)$ have Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{G} : H_\delta^\alpha(E) < s\}$. Then $A(\alpha, \delta, s)$ is an open subset of $\mathcal{X}$, and

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(\beta + 1/n, 1/m, 1/k)$$

is precisely the family of $(E, \mu) \in \mathcal{X}$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha, \delta, s)$ is dense in $\mathcal{X}$ for $\alpha \in (\beta, d)$ and $\delta, s > 0$. Fix $(E_0, \mu_0) \in \mathcal{X}$,

$\alpha \in (\beta, d)$, $\delta > 0$, $s > 0$, and $\varepsilon_1 > 0$. We aim to show that for each $\varepsilon > 0$, there exists $(E, \mu) \in A(\alpha, \delta, s)$ such that $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. Without loss of generality, in light of Lemma 1, we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a large integer K. Lemma 6 shows that there exists a constant $C(\beta, d)$, as well as K points $x_1, \ldots, x_K \in \mathbf{T}^d$ such that if

$$D(x) = \sum_{k=1}^{K} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta + 1}$,

$$|\widehat{D}(\xi)| \leqslant C K^{-1/2} \log(K)^{1/2}. \tag{3.35}$$

Using (3.35) to apply Lemma 5, we conclude that there exists $K_1(d, \beta, \mu_0, \varepsilon_1, \varepsilon)$, such that if $K \geqslant K_1$, if $\varepsilon_0 = K^{-1/\beta}$, if

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k=1}^{K} \phi_{\varepsilon_0}(x - x_k) \right) \mu_0(x),$$

and if

$$\mu = \mu_1 / \mu_1(\mathbf{T}^d),$$

then

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon. \tag{3.36}$$

Note that μ is supported on K balls of radius ε_0. Thus if we assume $K \geqslant \delta^{-1/\beta}$, so that $\varepsilon_0 \leqslant \delta$, we find that

$$H_\delta^\alpha(\text{supp}(\mu)) \leqslant K \varepsilon_0^\alpha = K^{1 - \alpha/\beta}. \tag{3.37}$$

Since $\alpha > \beta$, (3.37) implies that there is $K_2(\alpha, \beta)$ such that for $K \geqslant K_2$,

$$H_\delta^\alpha(\text{supp}(\mu)) \leqslant s. \tag{3.38}$$

Now let

$$E = \text{supp}(\mu) \cup \{y_1, \ldots, y_N\},$$

where $\{y_1, \ldots, y_N\} \subset E_0$ is a ε net of E_0. Set $K_0(d, \alpha, \beta, \delta, \mu_0, \varepsilon_1, \varepsilon) = \max(K_1, K_2, \delta^{-1/\beta})$, and suppose $K \geqslant K_0$. Equation (3.38) implies that $H_\delta^\alpha(E) \leqslant s$, so $(E, \mu) \in A(\alpha, \delta, s)$. Since $\text{supp}(\mu) \subset E$,

$$d_H(E, E_0) \leqslant \varepsilon. \tag{3.39}$$

Recalling (3.36), we see that we have proved what was required. $\square$

All that now remains is to show that quasi-all elements of $\mathcal{X}$ avoid the given set Z; just as with the proof above, the advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

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4   Random Avoiding Sets

In the last section, our results held for an arbitrary $\beta \in (0, d]$. But in this section, we assume

$$\beta = \frac{dn - \alpha}{n - 1/2},$$

which will enable us to generically avoid the pattern Z.

Lemma 8. *For quasi-all $(E, \mu) \in \mathcal{X}$, for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.*

Proof. The set $Z \subset \mathbf{R}^{dn}$ is the countable union of sets with lower Minkowski dimension at most α. For a closed set $W \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, and $s > 0$, consider the set

$$B(W, s) = \left\{ (E, \mu) \in \mathcal{X} : \begin{array}{c} \text{for all } x_1, \ldots, x_n \in E \text{ such that} \\ |x_i - x_j| \geqslant s \text{ for } i \neq j, \ (x_1, \ldots, x_n) \notin W \end{array} \right\}.$$

Then $B(W, s)$ is an open subset of $\mathcal{X}$. If Z is a countable union of closed sets $\{Z_k\}$ with lower Minkowski at most α, then clearly the set

$$\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} B(Z_k, 1/n)$$

consists of the family of sets (E, μ) such that for distinct $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$. Thus it suffices to show that $B(W, s)$ is dense in $\mathcal{X}$ for any $s > 0$, and any closed set W with lower Minkowski dimension at most α. So we fix a set $W \subset \mathbf{T}^{dn}$ and a pair $(E_0, \mu_0) \in \mathcal{X}$. Our goal is to show that for any $\varepsilon_1 > 0$ and $\varepsilon > 0$, we can find $(E, \mu) \in B(W, s)$ with $d_H(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. We may assume by Corollary 1 that $\text{supp}(\mu) = E$ and $\mu \in C^{\infty}(\mathbf{T}^d)$.

Let

$$\beta_0 = \frac{dn - \alpha}{n - 1/2} - \varepsilon_1$$

Since W has lower Minkowski dimension at most α, we can find arbitrarily small $\varepsilon_0 > 0$ such that

$$|W_{\varepsilon_0}| \leqslant \varepsilon_0^{dn - \alpha - (n - 1/2)\varepsilon_1}. \tag{4.1}$$

Now fix a large integer K such that

$$(1/10)K^{-1/\beta_0} \leqslant \varepsilon_0 \leqslant K^{-1/\beta_0}.$$

As $\varepsilon_0 \to 0$, $K \to \infty$, so we may take K as large as we like by taking ε_0 arbitrarily small. Let $X_1, \ldots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $k_1, \ldots, k_n \in \{1, \ldots, K\}$, the random vector $X_k = (X_{k_1}, \ldots, X_{k_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (4.1) implies

$$\mathbf{P}(d(X_k, W) \leqslant \varepsilon_0) \leqslant |W_{\varepsilon_0}| \leqslant \varepsilon_0^{dn - \alpha - (n - 1/2)\varepsilon_1}. \tag{4.2}$$

If M_0 denotes the number of indices i such that $d(X_i, W) \leqslant \varepsilon_0$, then by linearity of expectation we conclude from (4.2) that

$$\mathbf{E}(M_0) \leqslant K^n \varepsilon_0^{dn - \alpha - (n - 1/2)\varepsilon_1}. \tag{4.3}$$

Applying Markov's inequality to (4.3), we conclude that

$$\mathbf{P}(M_0 \geqslant 10 K^n \varepsilon_0^{dn - \alpha - (n - 1/2)\varepsilon_1}) \leqslant 1/10. \tag{4.4}$$

Taking a union bound to (4.4) and the result of Lemma 6, we conclude that there exists K points $x_1, \ldots, x_K \in \mathbf{T}^d$ and a constant $C(d, \beta)$ such that the following two statements hold:

(1) Let S be the set of indices $k_1 \in \{1, \ldots, K\}$ with the property that we can find distinct $k_1, k_2, \ldots, k_n \in \{1, \ldots, K\}$ such that if $x = (x_{k_1}, \ldots, x_{k_n})$, then $d(x, W) \leqslant \varepsilon_0$. Then

$$\#(S) \leqslant 10 K^n \varepsilon_0^{dn - \alpha - (n - 1/2)\varepsilon_1}. \tag{4.5}$$

(2) If we define

$$D_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - x_k)$$

then for $|\xi| \leqslant K^{1/\beta_0 + 1}$,

$$|\widehat{D_0}(\xi)| \leqslant C K^{-1/2} \log(K)^{1/2}. \tag{4.6}$$

Since $\varepsilon_0 \leqslant K^{-1/\beta_0}$, (4.6) implies that

$$\#(S) \leqslant 10 K^{1/2}. \tag{4.7}$$

In particular, (4.6) and (4.7) imply that if

$$D_1(x) = \sum_{k \notin S} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta_0 + 1}$,

$$|\widehat{D_1}(\xi)| \leqslant (C + 10) K^{-1/2} \log(K). \tag{4.8}$$

We define

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k \notin S} \phi_{(d^{1/2}/2)\varepsilon}(x - x_k) \right) \mu_0(x)$$

and then define

$$\mu(x) = \mu_1 / \mu_1(\mathbf{T}^d). \tag{4.9}$$

Since $\varepsilon_0 \geqslant (1/10) K^{-1/\beta_0}$, we can apply Lemma 5 in conjunction with (4.8) for any $\delta > 0$ to find an integer $K_0(d, \beta, \mu_0, \delta, \varepsilon_1)$, such that if $K \geqslant K_0$, if we define

$$\mu_1(x) = \left(\sum_{k \notin S} \phi_{(d^{-1/2}/2)\varepsilon_0}(x - x_k) \right) \mu_0(x),$$

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and if we then define $\mu = \mu_1 / \mu_1(\mathbf{T}^d)$, then

$$\|\mu - \mu_0\|_{M(\beta_0/2 - \varepsilon_1/2)} = \|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \delta. \tag{4.10}$$

The topology induced by the norm $\| \cdot \|_{M(\beta/2 - \varepsilon_1)}$ is stronger than the weak $*$ topology on $M(\beta)$. We note that if $\{\mu_k\}$ is a sequence of measures converging weakly to a measure μ, with $\mathrm{supp}(\mu_k) \subset \mathrm{supp}(\mu)$ for each k, then $\mathrm{supp}(\mu_k)$ converges in the Hausdorff metric to $\mathrm{supp}(\mu)$. Thus if δ is suitably small relative to ε, then we conclude $d_H(\mathrm{supp}(\mu), E_0) \leqslant \varepsilon$. If we also choose $\delta \leqslant \varepsilon$ then from (4.10) we also conclude that $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. Thus all that remains is to show that if $E = \mathrm{supp}(\mu_k)$, then for sufficiently large K, $(E, \mu) \in B(W, s)$. If fact, we will show this is the case for $\varepsilon_0 \leqslant s$.

Consider n points $y_1, \ldots, y_n \in \mathrm{supp}(\mu)$, with $|y_i - y_j| \geqslant \varepsilon_0$ for any two indices $i \neq j$. We can therefore find distinct indices $k_1, \ldots, k_n \in \{1, \ldots, K\}$ such that for each $i \in \{1, \ldots, n\}$,

$$|x_{k_i} - y_i| \leqslant (d^{-1/2}/2) \cdot \varepsilon_0. \tag{4.11}$$

If we set $x = (x_{k_1}, \ldots, x_{k_n})$ and $y = (y_1, \ldots, y_n)$, then (4.11) implies

$$|x - y| \leqslant (\varepsilon_0/2). \tag{4.12}$$

Since $i_1 \notin S$, $d(x, W) \geqslant \varepsilon_0$, which combined with (4.12) implies

$$d(y, W) \geqslant d(x, W) - |x - y| \geqslant \varepsilon_0/2. \tag{4.13}$$

Thus in particular we conclude $y \notin W$, which shows $(E, \mu) \in B(W, s)$ if $\varepsilon_0 \leqslant s$. $\qquad\square$

Concluding Remarks. *Let us end this Section by discussing where the loss in Theorem 1 occurs in our proof, as compared to [3].*

The reason for the loss of dimension as compared to the result of [3] is that if we use a (L^1, L^∞) bound for the Fourier transform we are forced to choose parameters such that $\#(S) \leqslant K^{1/2}$. However, if we were able to justify that square root cancellation occured in the sum

$$\sum_{k \in S} e^{2\pi i (\xi \cdot x_k)},$$

then we would only be forced to pick $\#(S) \leqslant K$, which would lead to a dimension bound of the form

$$\frac{dn - \alpha}{n - 1}.$$

5 Techniques for Avoiding Hyperplanes

Let $y = f(x)$ be a curve in $\mathbf{T}^2$ defining a curve S, where f is an analytic function (except perhaps at finitely many points?). Given $\varepsilon > 0$, we want to determine the differentiability of the map

$$A(x) = H^1(S_\varepsilon \cap \{x \times \mathbf{T}\}).$$

We wish to show A is a smooth function. The tangent to S at a point $(x, f(x))$ is given by $(1, f'(x))$, and so the unit normal vector is

$$N(x) = \frac{(f'(x), -1)}{\sqrt{1 + f'(x)^2}}.$$

Suppose that $x_0 \in \mathbf{T}$ is fixed, and let $x \in \mathbf{T}$ and $|\delta| \leqslant \varepsilon$ be given such that

$$(x_0, y_0) = (x, f(x)) + \delta N(x) = \left(x + \frac{\delta f'(x)}{\sqrt{1 + f'(x)^2}}, f(x) - \frac{\delta}{\sqrt{1 + f'(x)^2}} \right)$$

Thus

$$x_0 = x + \frac{\delta f'(x)}{\sqrt{1 + f'(x)^2}}$$

and

$$y_0 = f(x) - \frac{\delta}{\sqrt{1 + f'(x)^2}}.$$

Now the first equation tells us that

$$\delta = -(x - x_0) \frac{\sqrt{1 + f'(x)^2}}{f'(x)}.$$

Thus if we define

$$g(x, x_0) = \begin{cases} f(x) + \frac{x - x_0}{f'(x)} & : f'(x) \neq 0 \\ BLAH & : BLAH, \end{cases}$$

then $y_0 = g(x, x_0)$. Thus we ask ourselves what is the value of

$$A(x_0) = \max \left\{ f(x) + \frac{x - x_0}{f'(x)} : |x - x_0| \leqslant \frac{\varepsilon |f'(x)|}{\sqrt{1 + f'(x)^2}} \right\}.$$

Now $g(x, x_0)$ is a smooth function except where $f'(x) = 0$. In particular, if $f'(x_0) \neq 0$, then the constraint region defining $A(x_0)$ is a finite union of closed intervals. And $f'(x_0) = 0$ only at finitely many points, and if we make ε small enough we can make $A(x_0) = f(x_0) + \varepsilon$ at these points.

so this causes us no problems since we only care about whether A is differentiable except at finitely many points. To analyze $A(x_0)$ when $f'(x_0) = 0$, we note that a solution that gives the maximum either satisfies

$$f'(x)(f'(x)^2 + 1) + (x - x_0)f''(x) = 0$$

or

$$x - x_0 = \frac{\varepsilon f'(x)}{\sqrt{1 + f'(x)^2}}$$

or

$$x - x_0 = \frac{-\varepsilon f'(x)}{\sqrt{1 + f'(x)^2}}.$$

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If ε is small enough, then the implicit function theorem implies that the second and third equations have finitely many solutions for each x_0, which are locally smoothly parameterized. Since $f'(x_0) \neq 0$, the first equation does not even have any solutions if $\varepsilon \lesssim 1$. Thus we conclude that if ε is small enough, there exists a function $x(x_0)$ which is smooth, except at finitely many points, such that

$$g(x_0) = f(x) + \frac{x - x_0}{f'(x)}.$$

Thus at any x_0 where x is smooth, we conclude

$$g'(x_0) = f'(x) \cdot x' + \frac{x' - 1}{f'(x)} - \frac{x - x_0}{f'(x)^2} f''(x) x'.$$

References

[1] T.W. Körner, *Measures on Independent Sets, A Quantitative Version of Rudin's Theorem.*

[2] T.W. Körner, *Fourier transforms of measures and algebraic relations on their supports.*

[3] Jacob Denson, Malabika Pramanik, Joshua Zahl, *Large sets avoiding rough patterns.*

[4] Jacob Denson, *Cartesian products avoiding patterns.*

[5] Roman Vershynin, *High dimensional probability*, Cambridge Series in Statistical and Probabilistic Mathematics, 2018.

[6] Fredrik Ekström, Tomas Persson Jörg Schmeling, *On the Fourier dimension and a modification*, 2015.