Fractals Avoiding Fractal Sets

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1 A Discrete Building Block

We now develop a discrete technique used to construct solutions to the fractal avoidance problem. It depends very little on the Euclidean structure of the plane. As such, we rephrase the construction as a combinatorial problem on graphs.

Recalling definitions, we say an n *uniform hypergraph* is a collection of *vertices* and *hyperedges*, where a hyperedge is a set of n distinct vertices. An *independent set* is a subset of vertices containing no complete set of vertices in any hyperedge of the graph. A *colouring* is a partition of the vertex set into finitely many independent sets, each of which we call a *colour*. Such a colouring is K *uniform* if each colour class has K elements.

The next lemma is a variant of Turán's theorem on independent sets. For technical reasons, we need an extra restriction on the independant set so it is 'uniformly' chosen over the graph. This is why colorings are introduced.

Lemma 1. *Let G be an n uniform hypergraph with a K uniform coloring. Then there is an independent set W containing elements from all but $|E|/K^n$ colors.*

Proof. Let U be a random vertex set chosen by selecting a vertex of each color uniformly randomly. Every vertex occurs in U with probability $1/K$. For any edge $e = (v_1, \dots, v_n)$, the vertices v_i all have different colors. Thus they have an independent chance of being added to U, and we calculate

$$\mathbf{P}(v_1 \in U, \dots, v_n \in U) = \mathbf{P}(v_1 \in U) \dots \mathbf{P}(v_n \in U) = 1/K^n$$

If we let E' denote the edges $e = (u_1, \dots, u_n)$ with $u_1, \dots, u_n \in U$, then

$$\mathbf{E}|E'| = \sum_{e \in E} \mathbf{P}(e \in E') = \sum_{e \in E} 1/K^n = \frac{|E|}{K^n}$$

This means we may choose a *particular*, nonrandom U for which $|E'| \leqslant |E|/K^n$. If we form a vertex set $W \subset U$ by removing, for each $e \in E'$, a vertex in U adjacent to the edge, then W is an independent set containing all but $|E'| \leqslant |E|/K^n$ colors. $\square$

Corollary. *If $|V| \gtrsim N^a$, $|E| \lesssim N^b$, and $K \gtrsim N^c$, where $b < a + c(n-1)$, then as $N \to \infty$ we can find an independent set containing all but a fraction $o(1)$ of the colors.*

Proof. A simple calculation on the quantities of the previous lemma yields

$$\frac{\#(\text{colors removed})}{\#(\text{all colors})} = \frac{|E|/K^n}{|V|/K} = \frac{|E|}{|V|K^{n-1}} \lesssim \frac{N^b}{N^{a+c(n-1)}}$$

This is $o(1)$ if $b < a + c(n-1)$. □

We now apply these constructions to a problem clearly related to the fractal avoidance problem. It will form our key method to construct fractal avoidance solutions. Given an integer N, we subdivide $\mathbf{R}^d$ into a lattice of sidelength $1/N$ cubes with corners on $\mathbf{Z}^d/N$, the collection of such cubes we will denote by $\mathcal{B}(1/N)$. This grid is used to granularize configuration avoidance.

Theorem 1. *Suppose $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint collections of cubes in $\mathcal{B}(1/N)$, with $|\mathcal{I}_i| \gtrsim N^d$. We assume the lower Minkowski dimension of Y is bounded above by α, and $\beta > d(n-1)/(n-\alpha)$. Then there exists arbitrarily large N and collections of cubes $\mathcal{J}_1, \dots, \mathcal{J}_n \in \mathcal{B}(1/N^\beta)$ with each cube in $\mathcal{J}_1 \times \dots \times \mathcal{J}_n$ disjoint from Y, and as $N \to \infty$, each $\mathcal{J}_i$ contains cubes in all but a fraction $o(1)$ of cubes in $\mathcal{I}_i$.*

Proof. If $\mathcal{K} \subset \mathcal{B}(1/N^\beta)^n$ is the collection of all cubes in a sidelength $1/N^\beta$ lattice intersecting Y, then $|\mathcal{K}| \lesssim N^{\alpha\beta}$. We then let $\mathcal{I}'_i$ be all cubes in $\mathcal{B}(1/N^\beta)$ contained in $\mathcal{I}_i$. Considering these cubes as vertices gives us an n uniform hypergraph G with a hyperedge between $I_1 \in \mathcal{I}'_1, \dots, I_n \in \mathcal{I}'_n$ if $I_1 \times \dots \times I_n \in \mathcal{K}$. We say two cubes in G are the same color if they are contained in a common cube in $\mathcal{I}_i$.

Using the fact that a sidelength $1/N$ cube contains $N^{d(\beta-1)}$ sidelength $1/N^\beta$ cubes, we conclude that G has $\sum |\mathcal{I}_i| = N^{d(\beta-1)} \sum |\mathcal{I}_i| \gtrsim N^{d\beta}$ vertices. The number of edges in G is bounded by $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Finally, the coloring is $N^{d(\beta-1)}$ uniform. Thus in the terminology of the previous corollary, $a = d\beta$, $b = \alpha\beta$, and $c = d(\beta-1)$, and the inequality in the hypothesis of this theorem is then equivalent to the inequality in the hypothesis of the corollary. Applying the corollary gives the required result. □

The value $d(n-1)/(n-\alpha)$ in the theorem is directly related to the dimension $(n-\alpha)/(n-1)$ we obtain in our main result. Any improvement on this bound for special cases of the fractal avoidance problem immediately leads to improvements on the Hausdorff dimension of the set constructed. The fact that our hypergraph result is tight indicates that for the general fractal avoidance problem, our construction gives tight bounds.

2 A Fractal Avoiding Set

Our solutions X to fractal avoidance problems will be obtained by breaking the problem down into a sequence of discrete configuration problems. The central idea was first used in [3]. We construct X as a limit $\lim X_N$, where X_N is a disjoint union of sidelength L_N cubes, and X_{N+1} is obtained from X_N by subdividing the cubes into length R_N cubes, then further subdividing these cubes into cubes of smaller sidelength L_{N+1}, and removing a portion of them.

At each step N, we consider a disjoint collection of sidelength R_N cubes $\mathcal{I}_1(N), \dots, \mathcal{I}_n(N) \subset \mathcal{B}(R_N)$, each cube contained in X_N. The main result of the previous section allows us to find a collection of sidelength $L_{N+1} = R_N^{\beta_N}$ cubes $\mathcal{J}_i(N) \subset \mathcal{I}_i(N)$ with all cubes in $\mathcal{J}_1(N) \times \cdots \times \mathcal{J}_n(N)$ disjoint from Y, and where β_N converges to $\beta = d(n-1)/(n-\alpha)$ from above. We then form X_{N+1} from X_N by removing the parts of cubes in $\mathcal{I}_i(N)$ which are not contained in the cubes in $\mathcal{J}_i(N)$. Once parameters are fixed, and an initial set X_0 is chosen, which we might as well assume to be $[0,1]^d$, we obtain a sequence $X_0, X_1, \dots$ converging to a set X. A simple constraint detailed below is all that is required to ensure that X is a solution to the fractal avoidance problem.

Lemma 2. *Suppose that for any choice of distinct $x_1, \dots, x_n \in X$, there exists N such that each x_i is contained in a cube in $\mathcal{I}_i(N)$. Then $X^d \cap Y \subset \Delta$.*

Proof. For then $x_1, \dots, x_n \in X_{N+1}$, so $x_1 \in \mathcal{J}_1(N), \dots, x_n \in \mathcal{J}_n(N)$, and so the tuple $(x_1, \dots, x_n)$ is contained in a cube in $\mathcal{J}_1(N) \times \cdots \times \mathcal{J}_n(N)$, which is disjoint from Y. Taking contrapositives of this argument shows that if $y \in X^d \cap Y$, then there must be some i and j for which $y_i = y_j$, so $y \in \Delta$. $\square$

We achieve the constraint in the lemma by dynamically choosing parameters subject to a queueing process. The queue will consist of an ordered sequence of tuples $(I_1, \dots, I_n)$, where $I_1, \dots, I_n$ are disjoint cubes. At stage N of the construction, we take off the front tuple $(I_1, \dots, I_n)$ from the queue, and set $\mathcal{I}_i(N)$ to be the set of all cubes in $\mathcal{B}(R_N)$ which are a subset of both I_i and X_N. We then subdivide X_N using these parameters to form the set X_{N+1} as a union of length $L_{N+1} = R_N^{\beta}$ intervals. After this, for *any* ordered choice of distinct intervals $I_1, \dots, I_n \in \mathcal{B}(L_{N+1})$, with each interval I_i a subset of X_{N+1}, we add the tuple $(I_1, \dots, I_n)$ to the end of the queue.

Provided that $L_N \to 0$, for any distinct choice of $x_1, \dots, x_n \in X$, there exists N and L_N such that $|x_i - x_j| \geqslant 2L_N$ for all $i \neq j$. Thus at stage N, a tuple $(I_1, \dots, I_n)$ is added to the end of the queue with $x_i \in I_i$, and at a *much much* later stage M of the construction, this tuple is popped off the front of the queue, and so each x_i is contained in a cube in $\mathcal{I}_i(M)$. Thus we conclude that X is a solution to the fractal avoidance problem.

3 Dimension Bounds

To complete the proof, it suffices to choose the parameters R_N and β_N which lead to the correct Hausdorff dimension bound on X. The actual choice of β_N doesn't matter, only that it is an increasing sequence converging to β in the limit. We also fix a decreasing sequence λ_N such that $\lambda_N \beta_N > d$, to be used later on in our argument. Since β_N converges to β from above, we can let λ_N

tend to $\lambda = (dn-\alpha)/(n-1)$ from below. The fact that the dissection of X_{N+1} for X_N occurs uniformly over the will aid us in annihilating the superexponentially increasing constants which inherently occur from the exponentially decreasing values of L_N we are forced to choose.

We rely on the mass distribution principle to construct a probability measure μ supported on X. This enables us to calculate the Hausdorff dimension of X using Frostman's lemma. We begin by putting the uniform probability measure μ_0 on $X_0 = [0,1]^d$. Then, at each stage of the construction, we construct μ_{N+1} from μ_N by taking the mass on a certain sidelength L_N cube in X_N, and uniformly distributing it's mass over the sidelength L_{N+1} cubes in $I \cap X_{N+1}$. Using the weak compactness of the unit ball in $L^1(\mathbf{R}^d)^*$, we obtain a weak limit $\mu = \lim \mu_n$. The fact that μ_n is supported on X_n for each n implies μ is supported on X.

It is intuitive that the mass on μ will be distributed more thinly at each stage the fatter the cubes that are kept. Quantifying this precisely allows us to apply Frostman's lemma. More precisely, we will prove that for each length L interval I, $\mu(I) \lesssim_N L^{\lambda_N}$. Thus Frostman's lemma guarantees that $\dim_{\mathbf{H}}(X) \geqslant \lambda_N$, and taking $\lambda_N \to \lambda$ will complete the proof.

Lemma 3. *For R_N sufficiently large, and $I \in \mathcal{B}(L_{N+1})$,*

$$\mu(I) \leqslant 2^N R_0^{d-\beta_0} \dots R_N^{d-\beta_N} R_N^d$$

Proof. If I is not a cube in X_{N+1}, then $\mu(I) = \mu_{N+1}(I) = 0$, so the inequality is obviously true. Otherwise, we can find a cube $J \in \mathcal{B}(L_N)$ in $I \cap X_N$. J contains $(L_N/R_N)^d$ sidelength R_N cubes. Our main discrete result implies that X_{N+1} contains a sidelength L_{N+1} cube in all but a fraction $o(1)$ of these cubes,. In particular, if we choose R_N sufficiently large, then we know that we keep a sidelength L_N portion of at least half of these cubes. Thus

$$\mu(I) = \mu_{N+1}(I) \leqslant \frac{\mu_N(J)}{(L_N/R_N)^d/2} = 2\mu_N(J)(R_N/L_N)^d = 2\mu(J)(R_N/L_N)^d$$

completing the calculation. Applying this lemma iteratively, we conclude that

$$\mu(I) \leqslant 2^N (R_0/L_0)^d (R_1/L_1)^d \dots (R_N/L_N)^d = 2^N R_0^{d-\beta_0} \dots R_{N-1}^{d-\beta_{N-1}} R_N^d$$

completing the calculation. □

Corollary. *If R_N is even larger, we can force $\mu(I) \leqslant L_N^{\lambda_N}$ for $I \in \mathcal{B}(L_N)$.*

Proof. We write the inequality in the last problem as

$$\mu(I) \leqslant [2^N R_0^{d-\beta_0} \dots R_{N-1}^{d-\beta_{N-1}} R_N^{d-\lambda_N \beta_N}] L_{N+1}^{\lambda_N}$$

Since $\lambda_N \beta_N > d$, the quantity in the square brackets is $o(1)$ as $R_N \to \infty$. Thus for sufficiently large R_N, we conclude that $\mu(I) \leqslant L_N^{\lambda_N}$. □

This is almost the required inequality, except we have only proven it for intervals at particular scales. To obtain a general inequality, we use the fact that our construction is obtained uniformly across all intervals.

Theorem 2. *If R_N is chosen large enough that the previous inequalities hold, then we have $\mu(I) \leqslant 2^{d+1} L^{\lambda_N}$ for all intervals I with sidelength $L \leqslant L_N$.*

Proof. We break our analysis into three cases, depending on the size of L:

- If $R_N \leqslant L \leqslant L_N$, we can cover I by at most $2^d (L/R_N)^d$ cubes in $\mathcal{B}(R_N)$. For each such cube, we know that the mass on each sidelength R_N cube is at most $2(R_N/L_N)^d$ times the mass on an element of $\mathcal{B}(L_N)$. Thus we obtain a bound

$$\mu(I) \leqslant [2^d (L/R_N)^d][2(R_N/L_N)^d][L_N^{\lambda_N}] \leqslant \frac{2^{d+1} L^d}{L_N^{d-\lambda_N}} \leqslant 2^{d+1} L^{\lambda_N}$$

  which gives the required calculation.

- If $L_{N+1} \leqslant L \leqslant R_N$, we can cover L by at most 2^d cubes in $\mathcal{B}(R_N)$. Each cube in $\mathcal{B}(R_N)$ contains at most one cube in $\mathcal{B}(L_{N+1})$ which is also contained in X_{N+1}, so the bound in the last corollary gives that $\mu(I) \leqslant 2^d L_{N+1}^{\lambda_N} \leqslant 2^d L^{\lambda_N}$.

- If $L \leqslant L_{N+1}$, there certainly exists M such that $L_{M+1} \leqslant L \leqslant L_M$, and one of the previous cases yields that $\mu(I) \leqslant 2^{d+1} L^{\alpha_M} \leqslant 2^{d+1} L^{\alpha_N}$.

This covers all possible situations, completing the proof. $\square$

To employ Frostman's lemma, we need the result $\mu(I) \lesssim L^{\lambda_N}$ for an *arbitrary* interval, not just one with $L \leqslant L_N$. But this is no trouble; it is only the behaviour of the measure on arbitrarily small scales that matters. This is because if $L \geqslant L_N$, then $\mu(I)/L^{\lambda_N} \leqslant 1/L_N^{\lambda_N} \lesssim_N 1$, so $\mu(I) \lesssim_N L^{\lambda_N}$. Thus all problems with the Hausdorff dimension argument are complete, and we have proven that there is a choice of parameters which constructs a set X with Hausdorff dimension no less than $(nd - \alpha)/(n-1)$.

References

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