SALEM SETS AVOIDING ROUGH CONFIGURATIONS

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Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension s if it has Hausdorff dimension s, and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that for all $\xi \in \mathbf{R}^d$,

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty.$$

Our goal in these notes is to obtain, for each set $Z \subset \mathbf{R}^{dn}$ with Minkowski dimension s, a Salem set $X \subset \mathbf{R}^d$ with dimension

$$\frac{nd - s}{s},$$

such that for each set of n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We hope that we can rely on the random selection approach of our paper on rough configurations to obtain such a result.

1. Concentration Inequalities Using the Orlicz Norm

Define a convex function $\psi_2 : [0, \infty) \to [0, \infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{ A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \le 1 \}.$$

The family of random variables $\psi_2(L)$ are known as *subgaussian random variables*. Here are some important properties:

- (Gaussian Tails): There exists a universal constant $c > 0$ such that for any random variable X, $\|X\|_{\psi_2(L)} \le A$ if and only if for each $t \ge 0$,
$$\mathbf{P}(|X| \ge t) \le 2 \exp\left(-ct^2/A^2\right).$$
- (Bounded Variables are Subgaussian): For any random X,
$$\|X\|_{\psi_2(L)} \lesssim \|X\|_{L^\infty}.$$
- (Centering) For any random variable X,
$$\|X - \mathbf{E}(X)\|_{\psi_2(L)} \lesssim \|X\|_{\psi_2(L)}.$$
- (Union Bound) If $X_1, \dots, X_N$ are random variables, then
$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \le \|X_1\|_{\psi_2(L)} + \dots + \|X_N\|_{\psi_2(L)}.$$

- (Hoeffding's Inequality): If $X_1, \dots, X_N$ are *independent* random variables, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \lesssim \left(\|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2 \right)^{1/2}.$$

The Orlicz norm is a convenient notation to summarize calculations involving the principle of concentration of measure. Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leq A$ as essentially always lying in the interval $[-A, A]$, very rarely deviating outside this interval.

2. A Family of Cubes

Fix sequences of integers $\{N_m : m \geq 1\}$ and $\{M_m : m \geq 1\}$. We then define two sequences of real numbers $\{l_m : m \geq 0\}$ and $\{r_m : m \geq 0\}$, by defining

$$l_m = \frac{1}{M_1 N_1 \dots M_m N_m} \quad \text{and} \quad r_m = \frac{1}{M_1 N_1 \dots M_m}.$$

For each $m, d \geq 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times \dots \times [N_1]^d \times \dots \times [M_m]^d \times [N_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [N_1]^d \times \dots \times [N_{m-1}]^d \times [M_m]^d.$$

For each string $i \in \Sigma_m^d$, we define a vector $a_i \in (l_m \mathbf{Z})^d$ by setting

$$a_i = i_0 + \sum_{k=1}^m i_{2k-1} \cdot r_k + i_{2k} \cdot l_k$$

Then each string $i \in \Sigma_m^d$ can be identified with the sidelength l_m cube

$$Q_i = \prod_{j=1}^d [a_{ij}, a_{ij} + l_m].$$

centered at a_i. Similarily, for each string $i \in \Pi_m^d$, we define a vector $a \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leq j \leq d$,

$$a_i = i_0 + \left(\sum_{k=1}^{m-1} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k \right) + i_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R_i = \prod_{j=1}^d [a_{ij}, a_{ij} + r_m].$$

We let $\mathcal{Q}_m^d = \{Q_i : i \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R_i : i \in \Pi_m^d\}$. Here are some important properties of this collection of cubes:

- For each m, $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ are covers of $\mathbf{R}^d$.

- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.
- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarly, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{i \in \Sigma_m^d : Q_i \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{i \in \Pi_m^d : R_i \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j, and a third index $k \in \Pi_m^d$, such that $R_k \cap Q_i, R_k \cap Q_j \neq \varnothing$.

3. A Family of Mollifiers

We now consider a family of mollifiers, which we will use to smooth out the Fourier transform of the measures we study. Begin by choosing a non-negative C^∞ function ψ supported on $[-1,1]^d$ such that

$$\int_{\mathbf{R}^d} \psi(x)\, dx = 1, \tag{1}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{2}$$

Since ψ is C^∞ and compactly supported, then for each $t \in [0,\infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \tag{3}$$

Now we rescale the mollifier. For each $m > 0$, we let

$$\psi_m(x) = l_m^{-d} \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \tag{4}$$

Equation (2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \tag{5}$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \xi), \tag{6}$$

In particular, (6) implies that for each $t \geq 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)||\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)||\xi|^t. \tag{7}$$

This implies that, uniformly in m, $\widehat{\psi_m}$ decays sharply outside of the box $[-l_m^{-1}, l_m^{-1}]^d$, a manifestation of the Heisenberg uncertainty principle.

4. Discrete Lemma

We now consider a discrete form of the Fourier bound argument, which we can apply iteratively to obtain a Salem set

Lemma 1. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (n-s)/4)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth probability measure compactly supported on T, together with a constant $C > 0$ such that for each $m \in \mathbf{Z}^d$,*

$$|\widehat{\mu_T}(m)| \leq C|m|^{-s/2}.$$

Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}. \tag{8}$$

Then there exists constants $C_1(\mu_T, n, d, s)$ and $C_2(\mu_T, n, d, s)$ such that if

$$N_{m+1} \geq C_1(T, \mu_T, n, d, s, \varepsilon), \tag{9}$$

and

$$\frac{N_{m+1}^{\frac{dn-s-2\varepsilon}{s+\varepsilon}}}{C_2(T, \mu_T, n, d, s)} \leq M_{m+1} \leq \frac{N_{m+1}^{\frac{dn-s-2\varepsilon}{s+\varepsilon}}}{C_2(T, \mu_T, n, d, s)} + 1, \tag{10}$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

- *For any strongly non-diagonal cube*
$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$
There exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.
- *For any $m \in \mathbf{Z}^d$,*
$$|\widehat{\mu}(m)| \leq (C + l_{m+1}^{-1})|m|^{-s/2}.$$

Proof. For each $i \in \Pi_{m+1}^d$, let j_i be a random integer vector chosen from $[N_{m+1}]^d$, such that the family $\{j_i : i \in \Pi_{m+1}^d\}$ is independent. We define a measure ν_S by setting, for each $x \in \mathbf{R}^d$,

$$d\nu_S(x) = r_{m+1}^d \sum_{i \in \Pi_{m+1}^d} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

We then normalize, defining

$$\mu_S = \frac{\nu_S}{\nu_S(\mathbf{R}^d)}.$$

If we set

$$S = \bigcup \{ Q \in \mathcal{Q}^d_{m+1} : \mu_S(Q) > 0 \},$$

then by definition, S is a $\mathcal{Q}_{m+1}$ discretized set, μ_S is supported on S, and $S \subset T$. It now suffices to show that with nonzero probability, $\mathcal{Q}_m(S^n)$ is disjoint from the strongly non diagonal cubes in $\mathcal{Q}_m(B)$, and μ_S has the required Fourier decay.

In our calculations, it will be helpful to decompose the measure ν_S. For each $i \in \Pi_{m+1}(T)$, define a random measure ν_i by setting

$$d\nu_i(x) = r^d_{m+1} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

Then $\nu_S = \sum_{i \in \Pi_{m+1}(T)} \nu_i$.

First, let's show that ν_S only needs to be slightly perturbed to become a probability measure. Note that if $j_0, j_1 \in [N_{m+1}]^d$, then

$$|a_{ij_0} - a_{ij_1}| = |j_0 - j_1| \cdot l_{m+1} \lesssim_d N_{m+1} l_{m+1} = r_{m+1},$$

which, together with (4), implies

$$\begin{aligned} &\left| r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_0}) \mu_T(x) - r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_1}) \mu_T(x) \right| \\ &\quad \leq r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x) \left| \mu_T(x + a_{ij_0}) - \mu_T(x + a_{ij_1}) \right| \\ &\quad \lesssim_d r^{d+1}_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x) \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} = r^{d+1}_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \tag{11}$$

Thus (11) implies that for each i,

$$\|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \lesssim_d r^{d+1}_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \tag{12}$$

Furthermore, (5) implies

$$\begin{aligned} \sum_{i \in \Pi^d_{m+1}} \mathbf{E}(\nu_i(\mathbf{R}^d)) &= r^d_{m+1} \sum_{(i,j) \in \Sigma^d_{m+1}} \mathbf{P}(j_i = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij}) \mu_T(x)\, dx \\ &= \frac{r^d_{m+1}}{N^d_{m+1}} \int_{\mathbf{R}^d} \left(\sum_{(i,j) \in \Sigma^d_{m+1}} \psi_{m+1}(x - a_{ij}) \right) \mu_T(x)\, dx \\ &= \frac{r^d_{m+1} l^{-d}_{m+1}}{N^d_{m+1}} = 1. \end{aligned} \tag{13}$$

For all but $O_d(r_{m+1}^{-d})$ indices i, $\nu_i = 0$ almost surely. Thus we can apply the triangle inequality together with (12) and (13) to conclude that

$$\begin{aligned}\|\nu_S(\mathbf{R}^d) - 1\|_{L^\infty} &= \| \textstyle\sum_{i\in\Pi_{m+1}^d} \left[\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\right] \|_{L^\infty} \\ &\le \textstyle\sum_{i\in\Pi_{m+1}^d} \|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \\ &\lesssim_d r_{m+1}^{-d} r_{m+1}^{d+1} \|\nabla\mu\|_{L^\infty(\mathbf{R}^d)} \\ &= r_{m+1}\|\nabla\mu\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{14}$$

Thus if (9) and (10) hold for appropriately chosen constants, we can apply (14) to conclude $\|\nu_S(\mathbf{R}^d) - 1\|_{L^\infty} \le 1/2$, or, in other words, that it is almost surely true that

$$1/2 \le \nu_S(\mathbf{R}^d) \le 3/2. \tag{15}$$

This means normalizing this measure to a probability measure is neglible to the asymptotics we consider.

For each $i \in \Pi_{m+1}^d$, let

$$S_i = \bigcup\{Q \in \mathcal{Q}_{m+1}^d : \nu_i(Q) > 0\}.$$

Then $S = \bigcup_{i\in\Pi_{m+1}^d} S_i$. Because j_i is selected uniformly from $[N_{m+1}]^d$ for each i, for any $j \in [N_{m+1}]^d$,

$$\mathbf{P}(j_i = j) = N_{m+1}^{-d}. \tag{16}$$

Since ψ_{m+1} is supported on $[-l_{m+1}, l_{m+1}]^d$,

$$S_i \subset \bigcup\{R_{i_0} : R_{i_0} \cap R_i \neq \varnothing\}.$$

For any cube $Q_{ij} \in \Sigma_{m+1}^d$, there are $O_d(1)$ pairs (i_0, j_0) such that $Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing$, and so a union bound together with (16) gives

$$\mathbf{P}(Q_{ij} \in \mathcal{Q}_{m+1}(S)) \le \textstyle\sum_{Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing} \mathbf{P}(j_{i'} = j') \lesssim_d N_{m+1}^{-d}. \tag{17}$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_{i_1 j_1} \times \dots \times Q_{i_n j_n} \in \mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_{i_k j_k} \in S\}$ are independent from one another, which together with (17) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_{i_1 j_1} \in S) \dots \mathbf{P}(Q_{i_n j_n} \in S) \lesssim_{d,n} N_{m+1}^{-dn}. \tag{18}$$

Taking expectations over all cubes in B, and applying (8) and (18) gives

$$\begin{aligned}\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\lesssim_{d,n} \#(\mathcal{Q}_{m+1}(B)) \cdot N_{m+1}^{-dn} \\ &\le l_{m+1}^{-(s+\varepsilon)} N_{m+1}^{-dn} \\ &= \frac{M_{m+1}^{s+\varepsilon} l_m^{-(s+\varepsilon)}}{N_{m+1}^{dn-s-\varepsilon}}.\end{aligned} \tag{19}$$

If (10) holds, for an appropriately chosen constant depending only on l_m, d, n, and s, we can apply Markov's inequality together with (19) to conclude

$$\begin{aligned}\mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \varnothing) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geq 1)\\ &\leq \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n)))\\ &\leq 1/N_{m+1}^{\varepsilon} \leq 1/10.\end{aligned} \tag{20}$$

Thus $\mathcal{Q}_{k+1}(S^n)$ is disjoint from $\mathcal{Q}_{k+1}(B)$ with high probability.

Now we analyze the Fourier transform of the measure ν. For each $i \in \Pi_{m+1}^d$, and $m \in \mathbf{Z}$, define $X_{im} = \widehat{\nu_i}(m) - \widehat{\mathbf{E}(\nu_i)}(m)$. Applying (2) gives

$$\begin{aligned}\sum_{i \in \Pi_{m+1}^d} \widehat{\mathbf{E}(\nu_i)}(m) &= \sum_{i \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [N_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} \psi_{m+1}(x - a_{ij}) d\mu_T(x)\\ &= \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} d\mu_T(x) = \widehat{\mu_T}(m).\end{aligned} \tag{21}$$

For each i and m, the standard (L^1, L^∞) bound on the Fourier transform, combined with (12), shows

$$\begin{aligned}\|X_{im}\|_{\psi_2(L)} &\leq \|X_{im}\|_{L^\infty}\\ &\leq \|\nu_i(\mathbf{R}^d)\|_{L^\infty} + \mathbf{E}(\nu_i)(\mathbf{R}^d)\\ &\lesssim_d \mathbf{E}(\nu_i)(\mathbf{R}^d) + r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{22}$$

For a fixed m, the family of random variables $\{X_{im}\}$ are independent. Furthermore, $\sum X_{im} = \widehat{\nu}(m) - \widehat{\mathbf{E}(\nu)}(m)$, and furthermore, (5) and (16) imply that

$$\begin{aligned}\mathbf{E}(\widehat{\nu_S}(m)) &= \frac{r_{m+1}^d}{N_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} \left(\sum_{(i,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij}) \right) d\mu_T(x)\\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{N_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} d\mu_T(x)\\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{N_{m+1}^d} \widehat{\mu_T}(m) = \widehat{\mu_T}(m).\end{aligned} \tag{23}$$

Hoeffding's inequality, together with (22) and (23), imply that

$$\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \lesssim_d \left(\sum \mathbf{E}(\nu_i)(\mathbf{R}^d)^2\right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}.$$

Now taking in absolute values into the definition of ν_i gives

$$\begin{aligned}\mathbf{E}(\nu_i)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j \in [N_{m+1}]^d} \int \psi_{m+1}(x - a_{ij}) d\mu_T(x)\\ &\leq r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned}$$

Applying (12) shows

$$\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \lesssim_d [\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}] r_{m+1}^{d/2}. \tag{24}$$

We can then apply a union bound over $D = \{m \in \mathbf{Z}^d : |m| \leq 10 l_{m+1}^{-1}\}$ together with (24) to conclude that there exists a constant $c(\mu_T, d)$ such that

$$\mathbf{P}\left(\|\widehat{\nu} - \widehat{\mu_T}\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1})\right) \lesssim_d l_{m+1}^{-d} \exp\left(-c(\mu_T, d)\log(M_{m+1})^2\right). \tag{25}$$

If (10) holds, for an appropriately chosen constant depending only on l_m, d, n, and s, then (25) implies

$$\mathbf{P}\left(\|\widehat{\nu} - \widehat{\mu_T}\|_{L^\infty(D)}\right) \leq 1/10.$$

Thus $\widehat{\nu}$ and $\widehat{\mu_T}$ are highly like to differ only by a miniscule amount over small frequencies.

Finally, it suffices to analyze the values of $\widehat{\nu_S}(m)$ when $|m| \geq 10 l_{m+1}^{-1}$. If we define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{i \in \Pi_{m+1}^d \\ d(a_i, T) \leq 2 r_{m+1}^{-1}}} \delta_{a_{i j_i}},$$

then $\nu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\nu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup |\widehat{\mu_T}(\xi)||\xi|^t < \infty.$$

In light of (6), if we define, for each $t > 0$,

$$B(t) = \sup |\widehat{\psi}(\xi)||\xi|^t,$$

then

$$\sup |\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t} B(t).$$

It now suffices to bound

$$\sup_{|\eta| \geq 10 l_{m+1}^{-1}} |\eta|^{s/2} \int \widehat{\mu_T}(\eta - \xi)\widehat{\alpha}(\xi)\widehat{\psi_{m+1}}(\xi)\, d\xi.$$

The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, we have $\alpha(\mathbf{R}^d) \leq 2^d$, so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leq 2^d$, so it suffices to bound

$$\int |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi.$$

If $|\xi| \leq |\eta|/2$, $|\eta - \xi| \geq |\xi|/2$, so for all $t > 0$, and combined with the fact that $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leq 1$, we find

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leq \frac{A(t) 2^{t-d}}{|\eta|^{t-d}}.$$

If we set $t = d + 1 + s/2$ and apply (10) for an appropriate chosen constant depending only on d and μ_T, we conclude

$$\int_{0\le|\xi|\le|\eta|/2} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi_{m+1}}(\xi)|\,d\xi \le \frac{A(d+1+s/2)2^{1+s/2}|\eta|^{-1}}{|\eta|^{s/2}} \le \frac{|\eta|^{-s/2}}{10\cdot 2^d}.$$

Conversely, if $|\xi| \ge 2|\eta|$, then $|\eta-\xi| \ge |\xi|/2$, so for each $t > 0$,

$$\begin{aligned}\int_{|\xi|\ge 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi}_{m+1}(\xi)|\,d\xi &\le \int_{|\xi|\ge 2|\eta|} \frac{A(t)2^t}{|\xi|^t}\\ &\lesssim_d \int_{2|\eta|}^{\infty} r^{d-t}A(t)2^t.\end{aligned}$$

Provided $t > d+1$, this integral is finite, and is

$$\lesssim_{d,t} A(t)2^{d+1}|\eta|^{d+1-t}.$$

Setting $t = d+2+s/2$, and applying (10), we conclude

$$\int_{|\xi|\ge 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi}_{m+1}(\xi)|\,d\xi \le \frac{|\eta|^{-s/2}}{10\cdot 2^d}.$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu_T}\|_{L^\infty(\mathbf{R}^d)} \le 1$ to conclude that

$$\int_{|\eta|/2\le|\xi|\le 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi_{m+1}}(\xi)|\,d\xi \le \frac{2^{d+t}B(t)}{|\eta|^{t-d}}.$$

If we set $t = d + s/2 + 1$ and apply (10), we conclude

$$\int_{|\eta|/2\le|\xi|\le 2|\eta} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi_{m+1}}(\xi)|\,d\xi \le \frac{|\xi|^{-s/2}}{10\cdot 2^d}.$$

Summing up the three bounds, we conclude that if $|\eta| \ge 10l_{m+1}^{-1}$, then

$$|\widehat{\nu_S}(\eta)| \le |\eta|^{-s/2}.$$

Thus, almost surely, $\widehat{\nu_S}(\eta)$ has fast Fourier decay for $|\eta| \ge 10l_{m+1}^{-1}$.

Let us put all our calculations together. $\square$