# Sets, Patterns, and Fourier Decay

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November 12, 2021

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- A set has *Fourier Dimension* $s$ if $|\widehat{\mu}(k)| \lesssim |k|^{-s/2}$ for *all* $n$.
- $\dim_{\mathbb{F}}(E) \leq \dim_{\mathbb{H}}(E) \leq \dim_{\mathbb{M}}(E)$.

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  - Thus $\text{supp}(\nu) \subset E + \dots + E$ contains an interval.
- If $\dim_{\mathbb{F}}(E) > 2/n$, then there are $m_1, \dots, m_n$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$.

# Independent Sets

- (Rudin, 1960): There exists $E \subset \mathbb{T}$ and a finite Borel measure $\mu$ with $\mathrm{supp}(\mu) \subset E$ such that $E$ is independent but $|\widehat{\mu}(k)| \to 0$ as $|k| \to \infty$.

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  - Can we improve this?

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- (Schmerkin, 2017): There is $E \subset \mathbb{T}$ avoiding arithmetic progressions with $\dim_{\mathbb{F}}(E) = 1$.
- (Liang and Pramanik, 2019): Generalized Schmerkin's construction to all translation-invariant patterns.

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- (Henriot and Łaba and Pramanik, 2016): For certain linear maps $A_1, \ldots, A_n$ and polynomials $Q$, there is $\varepsilon > 0$ such that if $E \subset \mathbb{T}$ and $\dim_{\mathbb{F}}(E) \geq 1 - \varepsilon$, $E$ contains a family of points of the form

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- (Kuca, Orponen, Sahlsten, Preprint 2021): If $E \subset \mathbb{T}^2$ and $\dim_{\mathbb{H}}(E) \geq 2 - \varepsilon$, then $E$ contains solutions to $y_2 - x_2 = (y_1 - x_1)^2$ for distinct $x, y \in E$.

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| **Author** | **Property of $f$** | $\dim_{\mathbb{H}}(X)$ |
|---|---|---|
| Mathé (2017) | A degree $r$ polynomial | $d/r$ |
| Fraser Pramanik (2018) | $f$ is $C^1$ | $m/(n-1)$ |
| D. Pramanik Zahl (2020) | $f$ Lipschitz | $m/(n-1)$ |
| D. (2020) | $f = g \circ \pi$ where the linear map $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$ is is surjective | $1/(m-1)$ |

- Can we modify these constructions to obtain Salem sets?

# Main Result

## Theorem

*Suppose* $f(x_1, \ldots, x_{n-1})$ *is* $C^\infty$, *and for each* $1 \le i \le n-1$,

$$D_{x_k} f = \left( \frac{\partial f_i}{\partial x_{kj}} \right)$$

*is invertible. Then there exists* $E \subset \mathbb{T}^d$ *with*

$$dim_{\mathbb{F}}(E) = \frac{d}{n - 3/4}$$

*avoiding solutions to the equation* $x_n = f(x_1, \ldots, x_{n-1})$.

- (Fraser and Pramanik, 2016) obtains a set $E \subset \mathbb{R}$ with

$$\dim_{\mathbb{H}}(E) = \frac{d}{n-1}.$$