Graph Theory

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March 1, 2021

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Chapter 1

Basic Graph Theory

Many situations can be formalized by diagrams involving points connected between one another, in such a way that the particular way they are connected is immaterial. Graph theory is the mathematical study of such situations.

1.1 Bipartite Graphs

A graph G is *bipartite* if it's vertices can be divided into two disjoint sets V_1 and V_2 such that every edge in G has one end in V_1 and one end in V_2. Given such a decomposition, we say elements of V_1 are 'left hand vertices', and elements of V_2 are 'right hand vertices'.

Theorem 1.1. *A graph G is bipartite if and only if every cycle in G has even length.*

Proof. One direction is clear, and so it suffices to show that if every cycle in G has even length has even length, then G is bipartite. We proceed by induction, the base case being the trivial graph with no vertices, which is obviously bipartite. Without loss of generality, we may assume that G is connected. For the inductive case, deleting some vertex v to form a graph G' with connected components $H_1 \cup \cdots \cup H_n$. By induction, each graph H_i is bipartite, so we can write the vertices of H_i as a disjoint union $V_{i1} \cup V_{i2}$ which gives the graph a bipartite structure. We claim that for each $i \in \{1, \dots, n\}$, there exists $j_i \in \{1, 2\}$ such that v is only connected to vertices in V_{ij_i}, because otherwise we could find an odd length cycle containing v.

If we write $V_1 = V_{1j_1} \cup \cdots \cup V_{1j_n}$ and $V_2 = V_{1j_1}^c \cup \cdots \cup V_{1j_n}^c \cup \{v\}$, then this gives a bipartite structure on G. $\square$

Theorem 1.2. *Let G be a bipartite graph with vertex set $V_1 \cup V_2$, where V_2 is nonempty, and edge set E. Then G has a bipartite subgraph G' with vertex set $V_1' \cup V_2'$ and edge set E', where $\#(E') \geqslant \#(E)/2$, and $\deg_{G'}(v_2) \geqslant \#(E)/2\#(V_2)$ for each $v_2 \in V_2'$.*

Proof. For any integer N,

$$\sum_{\deg(v_2)>N} \deg(v_2) \geqslant \#(E) - N\#(V_2).$$

Thus if $N = \#(E)/2\#(V_2)$,

$$\sum_{\deg(v_2)>N} \deg(v_2) \geqslant \#(E)/2,$$

so we can set $V_1' = V_1$, and $V_2' = \{v_2 \in V_2 : \deg(v_2) > \#(E)/2\#(V_2)\}$, and then let E' be the set of all vertices from V_1' to V_2'. $\square$

Theorem 1.3. *Let G be a bipartite graph with vertex set $V_1 \cup V_2$. Then G contains $\#(E)^2/\#(V_2)$ length two paths with both endpoints in V_1. G also contains $\#(E)^4/\#(V_1)\#(V_2)$ length four cycles.*

Proof. s $\square$