Fractals avoiding Fractal Sets

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Pattern Avoidance Problems

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- How large can the Hausdorff dimension of a $\mathbb{Q}$ vector subspace of $\mathbb{R}^n$ be not containing any points in $\mathbb{Q}^n$.
- What is the largest Hausdorff dimension of a subset of $\mathbb{R}^d$ such that the distances between any two points is irrational?

Pattern Avoidance Problems

- How large can the Hausdorff dimension of a Q vector subspace of R^n be not containing any points in Q^n.
- What is the largest Hausdorff dimension of a subset of R^d such that the distances between any two points is irrational?
- Given any equation f, what is the largest Hausdorff dimension of $X \subset R^d$ such that for any distinct $x_1, \ldots, x_n \in X$, $f(x_1, \ldots, x_n) \neq 0$.

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

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  - For any n, if $Z_n = \{x \in \mathrm{R}^{nd} : \text{there is } a_1, \dots, a_n \text{ s.t. } a_1 x_1 + \dots + a_n x_n \in \mathrm{Q}\}$, then X generates a vector space avoiding the rationals if and only if $X^n \cap Z_n \subset \Delta$ for all n.

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  - If $Z = \{(x,y) : d(x,y) \in \mathrm{Q}\} \subset \mathrm{R}^{2d}$, then points in X have irrational distances if and only if $X^2 \cap Z \subset \Delta$.
  - If $Z = f^{-1}(0)$, X avoids zeroes of f for distinct values if and only if $X^n \cap Z \subset \Delta$.

The Generic Problem

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- Pramanik and Fraser (2016): If Z is a smooth hypersurface of dimension $nd - d$, we can find X with dimension $d/(n-1)$.

Increasing the Difficulty. . .

What if the Patterns are Fractally Specified...

Main Result

Theorem

If Z is the countable union of sets with lower Minkowski dimension bounded by $\alpha \geq d$, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathrm{H}}(X) = \frac{nd - \alpha}{n-1} = \frac{codim(Z)}{n-1}$$

Low Rank Avoidance

Theorem

If we have countably many sets $Z_i \subset \mathrm{R}^{n_i d}$ with linear transformations $T_i : \mathrm{R}^{n_i d} \to \mathrm{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension β_i, we can find X with $X^{n_i} \cap Z_i \subset \Delta$ for each i and

$$\dim_{\mathrm{H}}(X) = \sup_i \left(\frac{n_i k_i - \beta_i}{2k_i - 1} \right) = \sup_i \frac{codim(T_i(Z_i))}{2k_i - 1}$$

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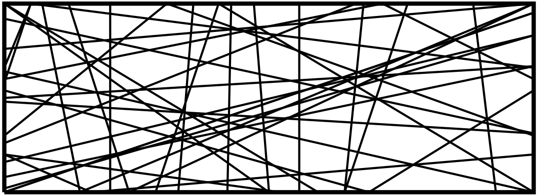
- The hypothesis says Z is coverable efficiently by lower dimensional thickened hyperplanes. Result should also extend when each Z_i is efficiently covered by thickened pencils of low degree algebraic surfaces, i.e. $f(Z)$ has low dimension where f is a polynomial map.
- Want to push the $2k - 1$ to $k - 1$, at least for $k \geq 2$. Know this is true for certain families of examples.

Advantages of Our Method



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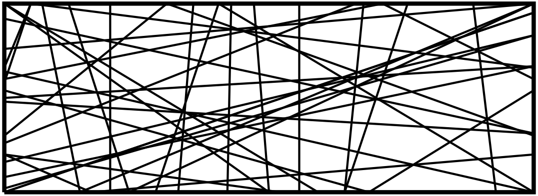

- More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin' Z, we get stable effects on the Hausdorff dimension of X.
- Uncountable unions of regular sets are allowed!

Applications

- Given a subset Y of R^d which is the countable union of sets with Minkowski dimension α, we can find a Q vector subspace X of R^d with Hausdorff dimension $d - \alpha$ disjoint from Y.

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- Given a subset Y of $\mathbb{R}^d$ which is the countable union of sets with Minkowski dimension α, we can find a $\mathbb{Q}$ vector subspace X of $\mathbb{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from Y.
- We can find a full dimensional subset of $\mathbb{R}^d$ avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $x \in X^n, y \in \mathbb{Q}^n$. No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.

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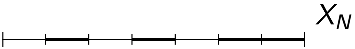
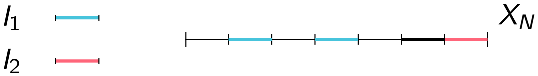
- Random Disection.

Discretization of Scales



- Construct X as a Cantor set by limits of interval sets X_N.

Discretization of Scales



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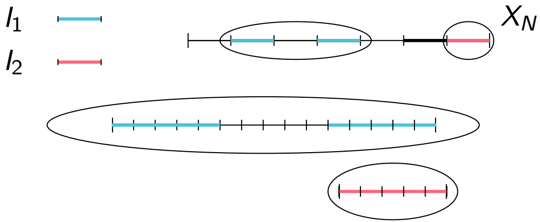

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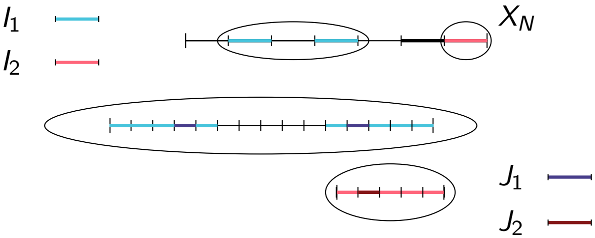

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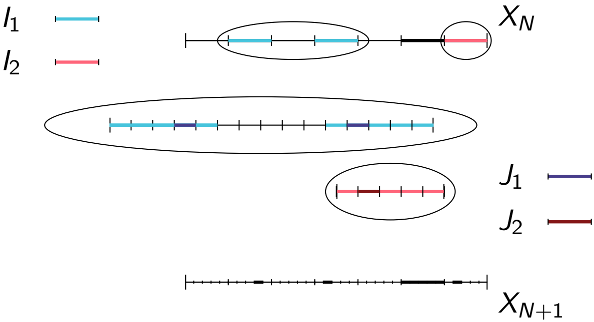

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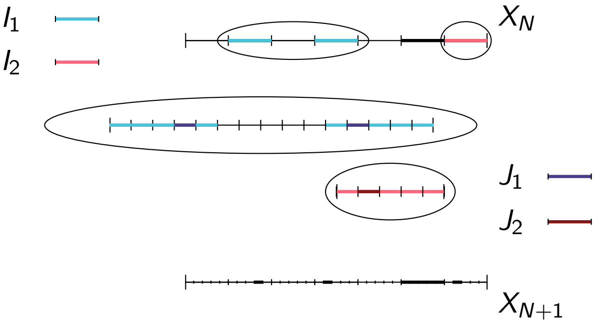

- Construct X as a Cantor set by limits of interval sets X_N.
- **Discrete Problem**: Given disjoint unions of length L intervals $I_1, \ldots, I_n$, find J_i for each i containing a part of each interval in I_i such that $J_1 \times \cdots \times J_n$ is disjoint from Z.

Queueing

- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set X gives a fractal avoiding set X.

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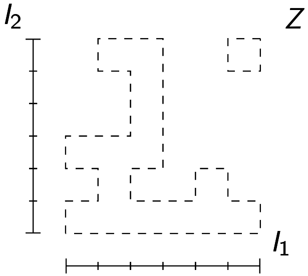
- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set X gives a fractal avoiding set X.
- **Reason**: If we consider distinct $x_1, \dots, x_n \in X$, there are intervals $I_1, \dots, I_n$ with $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \cdots \times J_n$ cannot be contained in Z.

Exploiting Randomness



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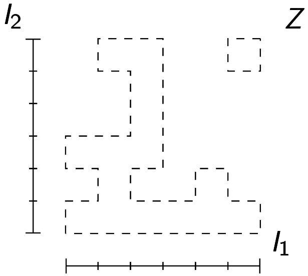

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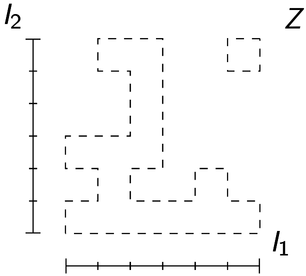

- How do we prove the discrete scale argument?
- Aside from Z's dimension, we have little structural knowledge.
- Random choices of the J_k avoid Z effectively.
- We essentially obtain for all but a fraction $o(1)$ of the length L intervals in I_k, J_k contains a length L^β section of each, where $\beta = d(n-1)/(nd-\alpha)$. This ratio gives the Hausdorff dimension bound $(nd-\alpha)/(n-1)$ for X.

Conclusion

So What's Next?

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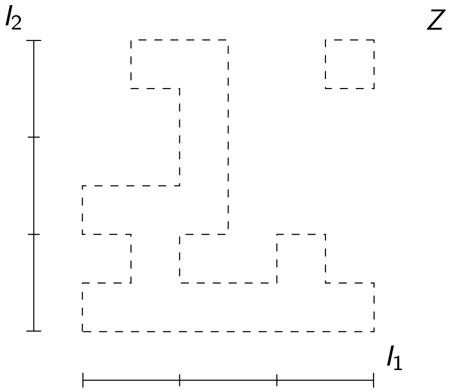
- We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

Analogies with Hypergraphs



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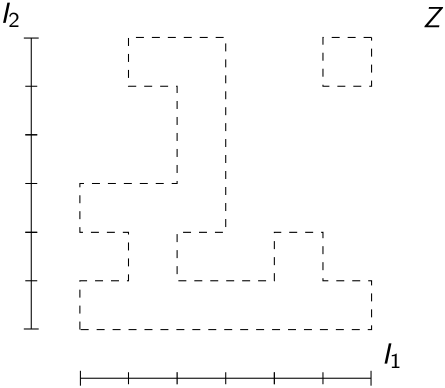

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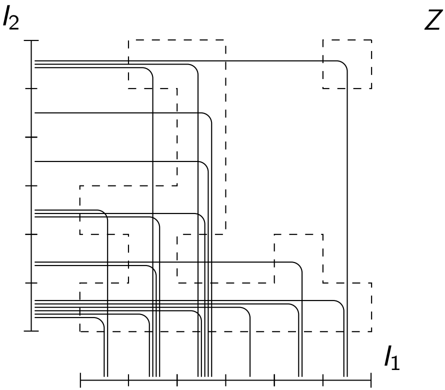

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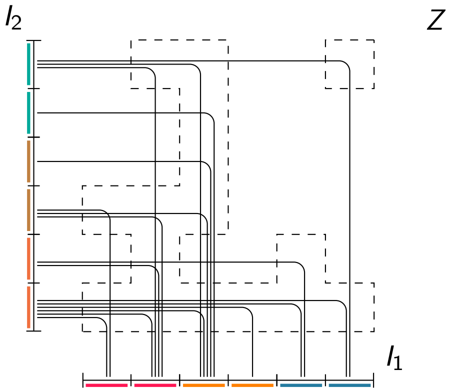

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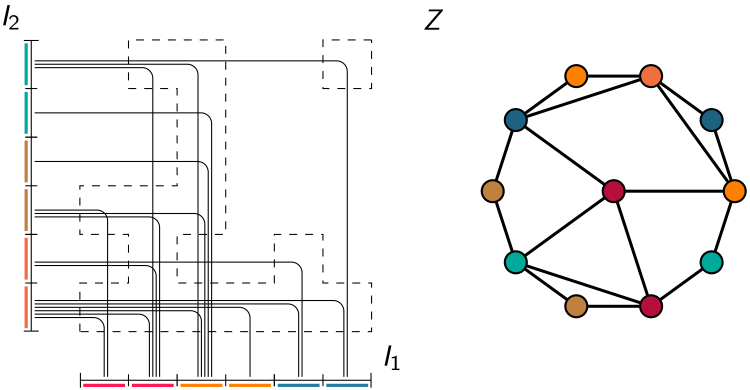

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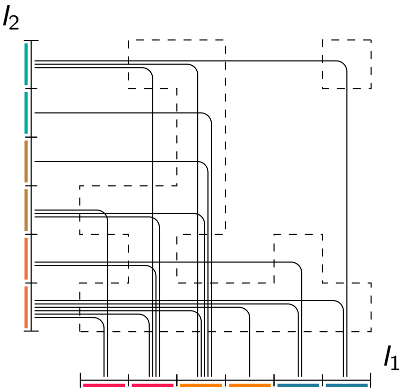

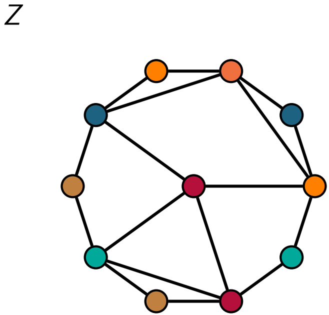

- The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- We are looking to using other methods on hypergraphs to improve the bound when Z has certain structural properties (like for the low rank result).

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Thanks for listening!