Marstrand Projection Theorem Via Marstrand Projection Theorem

Jacob Denson*

April 4, 2022

Abstract

TODO

Recall the classic Marstrand Projection Theorem.

Theorem 0.1. *Suppose $E \subset \mathbb{R}^n$ has Hausdorff dimension s. If $s < m$, then for almost every $\pi \in G(n,m)$, $\dim_{\mathbb{H}}(E) = s$, and if $s \geqslant m$, $\dim_{\mathbb{H}}(E) = m$.*

The goal of this paper is to discuss the connection between Marstrand's projection theorem, and the following result from metric geometry.

Theorem 0.2. *Fix $0 < \delta < 1$, let X be a set of N points in $\mathbb{R}^n$, and suppose $m > 8\ln(N)/\delta^2$. Then with probability greater than or equal to $1 - 2\exp(-c\delta^2 m)$, a random projection $\pi \in G(n,m)$ will satisfy*

$$(1-\delta)(m/n)^{1/2}|x-y| \leqslant |\pi(x) - \pi(y)| \leqslant (1+\delta)(m/n)^{1/2}|x-y|,$$

i.e. $(n/m)^{1/2}\pi$, restricted as a map from X to $\mathbb{R}^m$, will be an approximate isometry.

Let us recall some notation, introduced by Katz and Tao, and modified by Hera, Schmerkin, and Yavicoli. Fix some small quantity $\varepsilon_0 \ll 1$:

- A *hyper-dyadic* number will be a number of the form $2^{-\lfloor (1+\varepsilon_0)^k \rfloor}$ for some $k \geqslant 0$. A *hyper-dyadic cube* is a cube with hyper-dyadic sidelengths. We note that for any N, there are $O_{\varepsilon_0}(\log N)$ hyper-dyadic numbers between δ and δ^N for any $N > 0$, which is much less than the $O_{\varepsilon_0}(N \log(1/\delta))$ many dyadic numbers between δ and δ^N, which depends on δ.

- A family of sets $\{X_\alpha\}$ *strongly covers* a set X if each point in X is contained in infinitely many of the sets $\{X_\alpha\}$.

- A set E is δ *discretized* if it is the union of δ balls.

*University of Madison Wisconsin, Madison, WI, jcdenson@wisc.edu

- A set $E \subset \mathbb{R}^n$ is a (δ, s) set if E is a δ discretized subset of $B(0,2)$, and for all $\delta \leqslant r \leqslant 2$,

$$|E \cap B(x,r)| \lesssim \delta^{n-\varepsilon}(r/\delta)^s.$$

- $|E| \gtrsim \delta^{n-s}$.

A result of Katz and Tao gives the following.

Theorem 0.3. *Suppose $0 < s < n$, and let E be a compact subset of $\mathbb{R}^n$. If $\dim_{\mathbb{H}}(E) \leqslant s$, we can find a (δ, s) set X_δ for each hyperdyadic number δ such that $\{X_\delta\}$ strongly covers E. Conversely, if $C > 0$ is sufficiently large, we can find a family $\{X_\delta\}$, where X_δ is a (δ, s) set for each δ, with implicit constants bounded uniformly in δ, then $\dim_{\mathbb{H}}(E) \leqslant s$.*

Proof. Suppose the latter constraint. Since X_δ is a (δ, s) set, it is δ discretized. It is therefore the union of a family of radius δ balls $\{B_i\}$. Applying the Vitali covering lemma, we may find a disjoint subfamily of balls $S = \{B_{j_i}\}$ such that $X_\delta \subset \bigcup 5B_{j_i}$. Thus

$$\#(S)\delta^n \lesssim |X_\delta| = |X_\delta \cap B(0,2)| \lesssim \delta^{n-s},$$

so $\#(S) \lesssim \delta^{-s}$. But this means that X_δ is covered by $O(\delta^{-s})$ balls of radius 5δ, so

$$H^{s+\varepsilon}_{5\delta}(X_\delta) \lesssim \delta^{-s}(5\delta)^{s+\varepsilon} \lesssim \delta^\varepsilon.$$

Since E is compact, and strongly covered by the sets $\{X_\delta\}$, for any hyperdyadic $\delta_1 > 0$, there exists δ_2 such that

$$E \subset \bigcup_{\delta_2 \leqslant \delta \leqslant \delta_1} X_\delta.$$

But this means that

$$H^{s+\varepsilon}_{5\delta_1}(E) \leqslant \sum_{\delta_2 \leqslant \delta \leqslant \delta_1} H^{s+\varepsilon}_{5\delta_1}(X_\delta) \lesssim \sum_{\delta_2 \leqslant \delta \leqslant \delta_1}$$

in particular, δ discretized, so is the union of a family of balls $\{B_i\}$, where B_i has radius $r_i \approx \delta$. Applying Vitali's covering lemma, we may find a disjoint subset $\{B_{i_j}\}$ such that X_δ is covered by the family of balls $\{5B_{i_j}\}$. If we let X'_δ denote the union of balls $\{5B_{i_j}\}$, then X'_δ is still a $(\delta, s - C\varepsilon_0)$ set, since it is certainly δ discretized, and

$$|X'_\delta \cap B(x,r)|$$

Thus

$$|X_\delta| \gtrsim_d \sum r^d_{i_j}$$

Suppose the latter constraint. Since X_δ is a $(\delta, s - C\varepsilon_0)$ set, for any $x \in \mathbb{R}^d$,

$$|E \cap B(x,1)| \lesssim_{x,\varepsilon_0} \delta^{n-s+(C-C_1)\varepsilon_0}.$$

Since E is covered by $O_d(C_0^d)$ balls of radius one independantly, it follows that

$$|E| \lesssim_{C_0,\varepsilon_0,d} \delta^{n-s+(C-C_1)\varepsilon_0}$$

it satisfies the bound $|X_\delta| \lessapprox \delta^{n-s+C\varepsilon}$
it is a union of balls $\{B(x_i, r_i)$, where $r_i \approx \delta$. But then $N(X_\delta, \varepsilon/2)$
Thus $r_i \lesssim_\varepsilon \delta^{-O(\varepsilon)}\delta$ ∎