Salem Sets Avoiding Rough Configurations

Jacob Denson

May 28, 2020

1 Introduction

Geometric measure theory explores the relationship between the geometry of a subset of Euclidean space, and regularity properties of the family of Borel measures supported on that set. From the perspective of harmonic analysis, it is often popular to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a set. In this paper, we study the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a 'rough pattern', in the sense of [3], we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel set $E \subset \mathbf{R}^d$ is it's *Fourier dimension*. Given a finite Borel measure μ on $\mathbf{R}^d$, we define it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$, to be the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ |\widehat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d \right\} < \infty. \tag{1.1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most classical fractal sets are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem.

Theorem 1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset [0,1]^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n-1}, d \right)$$

such that for any distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

Remark. *Theorem 1 strengthens the main result of [3]. Unlike in [3], the case of the problem $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $[0,\pi]^d - \pi(Z)$ is not necessarily a Salem set, where $\pi(x_1, \dots, x_n) = x_1$ is projection onto the first coordinate. For instance, $[0,1] - \mathbf{Q}$*

A well-known result in this pattern avoidance setting is that sets with large Fourier dimension satisfy many algebraic relations. More precisely, if integer coefficients $m_1, \dots, m_n \in \mathbf{Z}$

are fixed, and we consider a compact set $X \subset \mathbf{R}$ with $\dim_{\mathbf{F}}(X) > 2/n$, then the sum set $m_1 X + \cdots + m_n X$ contains an open interval. It follows by a slight modification of these coefficients that if $X \subset \mathbf{R}$ and $\dim_{\mathbf{F}}(X) > 2/n$, then there exists $m_1, \ldots, m_n \in \mathbf{Z}$, distinct points $x_1, \ldots, x_n \in X$, and an additional integer $a \in \mathbf{Z}$, such that

$$m_1 x_1 + \cdots + m_n x_n = a. \tag{1.2}$$

It is an interesting to determine how tight this result is. In [2], T.W. Körner constructs a Salem set X with Fourier dimension $1/(n-1)$ such that for non-zero $m \in \mathbf{Z}^n$, and $a \in \mathbf{Z}$, X does not contain distinct points $x_1, \ldots, x_n$ solving (1.2). If, for each nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$, we consider the set

$$Z_{m,a} = \{(x_1, \ldots, x_n) \in [0,1]^n : m_1 x_1 + \cdots + m_n x_n = a\},$$

then $Z_{m,a}$ is a subset of an $n-1$ dimensional hyperplane, and thus can be easily seen to have Minkowski dimension $n-1$. It follows that we can apply Theorem 1 to $Z = \bigcup\{Z_{m,a} : m \neq 0, a \in \mathbf{Z}\}$ to obtain a Salem set $X \subset [0,1]$ with dimension

$$\frac{n-(n-1)}{n-1} = \frac{1}{n-1},$$

such that $(x_1, \ldots, x_n) \notin Z$ for each distinct $x_1, \ldots, x_n \in X$. Thus X avoids solutions to (1.2) for all nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$. Thus we see Theorem 1 generalizes Körner's result, and thus shows the result depends little on the arithmetic properties of the pattern Körner avoids, but rather, depends only on the 'thickness' of the family of tuples $(x_1, \ldots, x_n)$ satisfying the pattern. Since we expect Theorem 1 to be tight for general sets, an improvement to Körner's construction must rely more heavily on the algebraic properties of the pattern involved.

Since we are working with *compact* sets avoiding patterns, working in $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, and working in this space has several advantages over the Euclidean case. For a finite measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.3}$$

We can then define the Fourier dimension of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Similarily, $\mathbf{T}^d$ has a natural quotient metric induced from $\mathbf{R}^d$, so we can consider open balls $B_\varepsilon(x + \mathbf{Z}^d)$, and thus define the Hausdorff dimension of finite Borel measures and sets on $\mathbf{T}^d$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1.1) is equivalent to the more discrete condition

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.4}$$

A proof is given in [4, Lemma 39]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x)\, d\mu^*(x) = \int_{\mathbf{R}^d} f(x + \mathbf{Z}^d)\, d\mu(x),$$

then (1.4) implies $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$. Thus Theorem 1 is clearly equivalent to it's periodic variant, introduced below.

Theorem 2. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset \mathbf{T}^d$ with dimension*

$$\beta = \min\left(\frac{nd-\alpha}{n-1}, d\right)$$

such that for any distinct points $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$. Moreover, for any sequence $\{B(\xi) : \xi \in \mathbf{Z}^d\}$ of positive numbers such that $B(\xi) \to \infty$ as $|\xi| \to \infty$, there is a measure μ supported on X such that for each nonzero $\xi \in \mathbf{Z}^d$,

$$|\widehat{\mu}(\xi)| \leqslant B(\xi)\log(1+|\xi|)^{1/2}|\xi|^{-\beta}.$$

To construct our set, we rely on a Baire-category argument for the purpose of our construction. Thus we consider a complete metric space $\mathcal{X}$, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in \mathcal{X}$, the set E is Salem and pattern avoiding, and for each $\xi \in \mathbf{Z}^d$,

$$|\widehat{\mu}(\xi)| \lesssim B(\xi)\log(1+|\xi|)^{1/2}|\xi|^{-\beta},$$

in the sense that the set of pairs (E, μ) which do not satisfy these properties is a set of first category in $\mathcal{X}$. It follows that Theorem 2 holds in a 'generic' sense for elements of $\mathcal{X}$.

Once we have setup the appropriate metric space $\mathcal{X}$, our approach is quite similar to the construction in [3], relying on a random selection procedure, which is now exploited to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach, is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [3]. But the Baire category argument makes our proof simpler to read, giving us the 'infinitisimal argument' for free from the discrete case analysis, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern. Moreover, the proof of the Baire category theorem is in some senses, 'hidden' in the queuing method, so the two methods are, aside from small technical differences, equivalent to one another.

2 Notation

- For a positive integer N, we let $[N] = \{1, \ldots, N\}$.
- We let $\mathbf{Z}_0^d = \mathbf{Z}_0^d - \{0\}$.
- Given a metric space Ω, $x \in \Omega$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let

$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

denote the *ε-thickening* of the set E. A subset of Ω is of *first category* in Ω if it is the countable union of closed sets with nonempty interior. We say a property holds *quasi-always*, or a property is *generic* in Ω if the set of points in Ω failing to satisfy that property form a set of first category.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let
$$|x| = \min\{|x+n| : n \in \mathbf{Z}\},$$
and for $x \in \mathbf{T}^d$, we let
$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$
The canonical metric on $\mathbf{T}^d$ is then $d(x,y) = |x-y|$, for $x, y \in \mathbf{T}^d$.

- Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0,d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as
$$H^\alpha_\delta(E) = \inf\left\{\sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } \varepsilon_i \leqslant \delta \text{ for all } i \in \mathbf{N}\right\}.$$
The α dimensional Hausdorff measure of E is equal to
$$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$
The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0,d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0,d]$ such that $H^s(E) = 0$. Frostman's lemma says that if we define the Hausdorff dimension $\dim_{\mathbf{H}}(\mu)$ of a finite Borel measure μ as the supremum of all $s \in [0,d]$ such that
$$\sup\left\{\mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{R}^d, \varepsilon > 0\right\} < \infty, \tag{2.1}$$
then $\dim_{\mathbf{H}}(E)$ is the supremum of $\dim_{\mathbf{H}}(\mu)$, over all Borel probability measures μ supported on E, analogous to the definition of the Fourier dimension of a set E given in the introduction.

- For $\mathbf{E} = \mathbf{R}^d$ or $\mathbf{E} = \mathbf{T}^d$, and for a measurable set E, we let $|E|$ denote it's standard Lebesgue measure. We define the lower and upper Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as
$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon| \quad \text{and} \quad \overline{\dim}_{\mathbf{M}}(E) = \limsup_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon|$$
respectively.

- We will make several uses of *Hoeffding's Inequality* to control the deviations of independent families of random variables. The version of Hoeffding's inequality we use states that if $\{X_1, \ldots, X_N\}$ is an independent family of mean-zero random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,
$$\mathbf{P}\left(|X_1 + \cdots + X_N| \geqslant t\right) \leqslant 2\exp\left(\frac{-t^2}{2(A_1^2 + \cdots + A_N^2)}\right).$$
Proofs are given in many probability textbooks, for instance, in Chapter 2 of [5].

3 Technical Setup

Throughout this paper, we will need to consider a standard mollifier. The next theorem gives the existence of a mollifier with the properties we require; in the remainder of the paper, we fix a single instance of such a mollifier. The proofs are not particular enlightening or novel, and thus for most it will suffice to read the statements of the theorems as a setup for later sections.

Theorem 3. *There exists a smooth probability density $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$, and such that for each $x \in \mathbf{T}^d$*

$$\sum_{k\in\{0,1\}^d} \phi(x + k/2) = 2^d.$$

Proof. Let ψ be a non-negative smooth function on $\mathbf{T}$ such that $\psi(x) = \psi(-x)$ for all $x \in \mathbf{T}$, $\psi(x) = 1$ for $|x| \leqslant 1/10$, $\psi(x) = 0$ for $|x| \geqslant 2/10$, and $0 \leqslant \psi(x) \leqslant 1$ for all $x \in \mathbf{T}$. Then define η to be the non-negative, C^∞ function

$$\eta(x) = \frac{1}{2} - \frac{\psi(x) + \psi(x + 1/2)}{2}.$$

If we define

$$\phi_0(x) = 2(\psi(x) + \eta(x)),$$

then $\phi_0(x) + \phi_0(x + 1/2) = 2$ for all $x \in \mathbf{T}$. Moreover, if $|x| \geqslant 2/5$, then $\psi(x) = 0$, and since this implies $|x + 1/2| \leqslant 1/10$, we find $\eta(x) = 0$. Thus $\phi_0(x) = 0$ for $|x| \geqslant 2/5$. But the condition $\phi_0(x) + \phi_0(x + 1/2) = 2$ implies that ϕ_0 is a probability density function. Thus it suffices to define

$$\phi(x_1, \dots, x_d) = \phi_0(x_1) \dots \phi_0(x_d).$$

□

For each $\varepsilon \in (0, 1)$, we can then define $\phi_\varepsilon \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_\varepsilon(x) = \begin{cases} \varepsilon^{-d}\phi(x/\varepsilon) & : |x| < \varepsilon, \\ 0 & : \text{otherwise.} \end{cases}$$

Then the following properties hold:

1. For each $\varepsilon \in (0, 1)$, ϕ_ε is a smooth probability density, and $\phi_\varepsilon(x) = 0$ for $|x| \geqslant \varepsilon$.
2. For any positive integer N, if $\varepsilon = 1/N$ and $x \in \mathbf{T}^d$,
$$\sum_{k\in[2N]^d} \phi_{1/N}(x + k/2N) = (2N)^d. \tag{3.1}$$
3. For each $\xi \in \mathbf{Z}^d$,
$$\lim_{\varepsilon\to 0} \widehat{\phi}_\varepsilon(\xi) = 1. \tag{3.2}$$
4. For each $T > 0$, and for all $\varepsilon > 0$ and non-zero $k \in \mathbf{Z}^d$,
$$|\widehat{\phi}_\varepsilon(k)| \lesssim_T \varepsilon^{-T}|k|^{-T}. \tag{3.3}$$

Though Theorem 2 works for a very general family of sequences $\{B(\xi)\}$, it will be convenient to make several assumptions on the sequence in our proof. In particular, we can assume that for any n, and for any ξ_1, ξ_2 with $n \leqslant |\xi_1|, |\xi_2| \leqslant 2n$,

$$2^{-1}B(\xi_1) \leqslant B(\xi_2) \leqslant 2B(\xi_1), \tag{3.4}$$

that

$$\sup_{\xi \in \mathbf{Z}^d} B(\xi) \log(1+|\xi|)^{1/2} |\xi|^{-\beta} \leqslant 1, \tag{3.5}$$

and that B is *monotone*, i.e. if $|\xi_1| \leqslant |\xi_2|$, then $B(\xi_1) \leqslant B(\xi_2)$. The next lemma shows (3.4), (3.5), and monotonicity can be assumed without loss of generality.

Lemma 4. *Given any sequence $\{\tilde{B}(\xi)\}$ of positive numbers such that as $|\xi| \to \infty$, $\tilde{B}(\xi) \to \infty$, we can find a sequence $\{B(\xi)\}$ satisfying* (3.4) *and* (3.5) *such that $B(\xi) \to \infty$ as $|\xi| \to \infty$, and such that $B(\xi) \leqslant \tilde{B}(\xi)$ for each $\xi \in \mathbf{Z}^d$.*

Proof. For each nonzero $\xi \in \mathbf{Z}^d$, simultaneously define

$$C_1(\xi) = \min\left\{\tilde{B}(\xi') : \xi' \in \mathbf{Z}^d, 2^{-1}|\xi| \leqslant |\xi'| \leqslant \infty\right\},$$

$$C_2(\xi) = \inf\left\{\frac{|\xi'|^\beta}{\log(1+|\xi'|)^{1/2}} : \xi' \in \mathbf{Z}^d, |\xi'| \geqslant |\xi|\right\},$$

$$C_3(\xi) = \inf\left\{2B(\xi') : \xi' \in \mathbf{Z}^d, |\xi|/2 \leqslant |\xi'| < |\xi|\right\},$$

and

$$B(\xi) = \min(C_1(\xi), C_2(\xi), C_3(\xi)).$$

These quantities are all well defined by an inductive definition on the magnitude of $|\xi|$. Now for each $\xi \in \mathbf{Z}^d$,

$$B(\xi) \leqslant C_1(\xi) \leqslant \tilde{B}(\xi).$$

Clearly C_1 and C_2 are monotone, and one can see C_3 and B are monotone by a simple inductive argument. To verify (3.4), fix n, and consider ξ_1, ξ_2 with $n \leqslant |\xi_1|, |\xi_2| \leqslant 2n$. Without loss of generality, we may assume $|\xi_1| < |\xi_2|$, since the case $|\xi_1| = |\xi_2|$ is obvious. We then find

$$B(\xi_2) \leqslant C_3(\xi_2) \leqslant 2B(\xi_1). \tag{3.6}$$

But the monotonicity of B implies

$$B(\xi_2) \geqslant B(\xi_1). \tag{3.7}$$

A combination of (3.6) and (3.7) shows

$$2^{-1}B(\xi_1) \leqslant B(\xi_2) \leqslant 2B(\xi_1)$$

and

$$2^{-1}B(\xi_2) \leqslant B(\xi_1) \leqslant 2B(\xi_2).$$

Thus the sequence $\{B(\xi)\}$ satisfies (3.4). The fact that $B(\xi) \leqslant C_2(\xi)$ for each ξ implies that the sequence $\{B(\xi)\}$ satisfies (3.5). All that remains is to show that $B(\xi) \to \infty$ as $|\xi| \to \infty$.

It is certainly true that $C_1(\xi), C_2(\xi) \to \infty$. Fix N, and choose M large enough that for $|\xi| \geqslant M$,

$$C_1(\xi), C_2(\xi) \geqslant 10N.$$

To prove that $B(\xi) \geqslant N$ for sufficiently large ξ, it thus suffices to show that $C_3(\xi) \geqslant N$ for sufficiently large ξ. Assume that $C_3(\xi) \leqslant N$ for all ξ. Then for all $|\xi| \geqslant M$, $B(\xi) = C_3(\xi)$. Thus we conclude that for any $|\xi| \geqslant 2M$,

$$C_3(\xi) \geqslant 2C_3(\xi/2). \tag{3.8}$$

Since $C_3(\xi) \geqslant \varepsilon$ for any $|\xi| \geqslant M$, repeated applications of (3.8) show that for each positive integer n, if $|\xi| \geqslant 2^n M$, then $C_3(\xi) \geqslant 2^n \varepsilon$. Taking n large enough that $2^n \varepsilon \geqslant N$ gives a clear contradiction, so for sufficiently large ξ, we must have $B(\xi) \geqslant N$. Since N was arbitrary, this proves our claim. □

Remark. *Notice in particular that if we define, for each $\xi \in \mathbf{Z}^d$,*

$$A(\xi) = B(\xi) \log(1 + |\xi|)^{1/2} |\xi|^{-\beta},$$

and set $A(0) = 1$. Then (3.4) implies that for any positive integer n, and any $\xi_1, \xi_2 \in \mathbf{Z}^d$ with $n \leqslant |\xi_1|, |\xi_2| \leqslant 2n$,

$$A(\xi_1) \sim A(\xi_2). \tag{3.9}$$

4 A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task. We proceed as in [2], forming our metric space as a combination of two simpler metric spaces:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

  $$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

  then $(\mathcal{E}, d_H)$ forms a complete metric space.

- We let $M(A)$ consist of the class of all finite Borel measures μ supported on $\mathbf{T}^d$ such that the quantity

  $$\|\mu\|_{M(A)} = \sup_{\xi \in \mathbf{Z}_0^d} \frac{|\widehat{\mu}(\xi)|}{|A(\xi)|},$$

  is finite. Then $\|\cdot\|_A$ is a norm, and $(M(A), \|\cdot\|_A)$ is a Banach space.

We then define $\mathcal{X}$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(A)$, where μ is a probability measure with $\operatorname{supp}(\mu) \subset E$, and

$$\lim_{|\xi| \to \infty} \frac{|\widehat{\mu}(\xi)|}{A(\xi)} = 0.$$

It is easy to see $\mathcal{X}$ is a closed subset of $\mathcal{E} \times M(A)$ under the product topology, and thus if we consider the product metric

$$d_{\mathcal{X}}((E,\mu),(F,\nu)) = \max\left(d_H(E,F), \|\mu-\nu\|_A\right),$$

then $(\mathcal{X}, d_{\mathcal{X}})$ is a complete metric space. The advantage of considering the problem quantitatively is that we can work with smooth measures.

Lemma 5. *The set of all (E,μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}$.*

Proof. We just apply a mollification strategy. Consider $(E,\mu) \in \mathcal{X}$ and $\varepsilon > 0$. For each $\delta > 0$, consider $\mu_\delta = \mu * \phi_\delta$. Then $\mu_\delta \in C^\infty(\mathbf{T}^d)$, and if $\delta \leqslant \varepsilon$, then μ_δ is supported on $\overline{E_\varepsilon}$, so

$$d_H(E, \overline{E_\varepsilon}) \leqslant \varepsilon. \tag{4.1}$$

Since $(E,\mu) \in \mathcal{X}$, there is N depending on μ such that for $|\xi| \geqslant N$,

$$|\widehat{\mu}(\xi)| \leqslant \varepsilon A(k).$$

For $|\xi| \geqslant N$, we thus find that

$$|\widehat{\mu_\delta}(\xi) - \widehat{\mu}(\xi)| = |\widehat{\mu}(k)||\widehat{\phi_\delta}(\xi) - 1| \leqslant 2|\widehat{\mu}(k)| \leqslant 2\varepsilon A(\xi). \tag{4.2}$$

On the other hand, for suitably small $\delta > 0$, (3.2) implies that for $|\xi| \leqslant N$,

$$|\widehat{\phi_\delta}(\xi) - 1| \leqslant \varepsilon A(\xi).$$

But this implies that for $|\xi| \leqslant N$,

$$|\widehat{\mu_\delta}(\xi) - \widehat{\mu}(\xi)| = |\widehat{\mu}(\xi)||\widehat{\phi_\delta}(\xi) - 1| \leqslant \varepsilon A(\xi). \tag{4.3}$$

Thus we conclude that for suitably small δ, $\|\widehat{\mu_\delta} - \widehat{\mu}\|_A \leqslant 2\varepsilon$. But combining (4.1), (4.2), and (4.3), we conclude

$$d_{\mathcal{X}}((E,\mu),(\overline{E_\varepsilon},\mu_\delta)) \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

Our main way of constructing approximations to $(E_0,\mu_0) \in \mathcal{X}$ is to multiply μ_0 by a smooth function f. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which may form parts of a given pattern. As long as μ_0 is fixed, and f is small enough in the $M(A)$ metric, this introduces a neglible amount of error.

Lemma 6. *Consider a smooth finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function f. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(A)} \lesssim_{A,d,\mu_0} \|f\|_{M(A)}.$$

Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu_0}$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| = \left| \sum_{\eta \neq \xi} \widehat{f}(\xi - \eta) \widehat{\mu_0}(\eta) \right|. \tag{4.4}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi - \eta| \leqslant 2|\xi|$, so we can conclude from (3.9) that

$$|\widehat{f}(\xi - \eta)| \lesssim_A \|f\|_{M(A)} \cdot A(\xi). \tag{4.5}$$

Since μ_0 is smooth, for any $T > 0$,

$$|\widehat{\mu_0}(\xi)| \lesssim_{T,\mu_0} |\xi|^{-T} A(\xi). \tag{4.6}$$

Thus we can combine the bounds (4.5) and (4.6), with $T = d + 1$, to conclude that

$$\begin{aligned} \left| \sum_{0 \leqslant |\eta| \leqslant |\xi|/2} \widehat{f}(\eta) \widehat{\mu_0}(\xi - \eta) \right| &\lesssim_{A,\mu_0,d} \left(1 + \sum_{0 < |\eta| \leqslant |\xi|/2} \frac{1}{|\xi|^{d+1}} \right) \|f\|_{M(A)} \cdot A(\xi) \\ &\lesssim_d \|f\|_{M(A)} \cdot A(\xi). \end{aligned} \tag{4.7}$$

On the other hand, we note that (4.5) implies that for all nonzero $\eta \neq \xi$,

$$|\widehat{f}(\xi - \eta)| \lesssim_A \|f\|_{M(A)}, \tag{4.8}$$

Thus applying (4.6) and (4.8), with $T = 3d/2 + 1$ we conclude that

$$\begin{aligned} \left| \sum_{\substack{|\eta| > |\xi|/2 \\ \eta \neq \xi}} \widehat{f}(\xi - \eta) \widehat{\mu_0}(\eta) \right| &\lesssim_{A,d,\mu_0} \sum_{|\eta| > |\xi|/2} \frac{\|f\|_{M(A)}}{|\eta|^{3d/2+1}} \\ &\lesssim_d \frac{\|f\|_{M(A)}}{|\xi|^{d/2+1}} \\ &\lesssim_{A,d} \|f\|_{M(A)} \cdot A(\xi). \end{aligned} \tag{4.9}$$

Combining (4.4), (4.7) and (4.9) completes the proof. □

Thus if $\|f\|_{M(A)}$ is small

The next lemma shows that provided that, for K points $x_1, \ldots, x_K \in \mathbf{T}^d$, that if enough square root cancellation occurs in the exponential sum

$$\frac{1}{K} \sum_{i=1}^{K} e^{2\pi i (x_i \cdot \xi)},$$

then a smooth function f localized to these K points has the appropriate Fourier decay that might be expected by the mass of it's support.

Lemma 7. *Fix $\beta \in [0,d]$, $C > 0$, and $\varepsilon > 0$, and consider K points $x_1, \dots, x_K \in \mathbf{T}^d$ such that if*

$$D(x) = \frac{1}{K}\sum_{i=1}^{K} \delta(x - x_i),$$

then for each $\xi \in \mathbf{Z}^d$ with $0 < |\xi| \leqslant K^{1/\beta+\varepsilon}$,

$$|\widehat{D}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{4.10}$$

Then for any $\delta > 0$, there exists K_0, depending on C, β, d, δ, ε, and the sequence $\{A(\xi)\}$, such that if $K \geqslant K_0$ and $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$, and we define

$$f(x) = \frac{1}{K}\sum_{i=1}^{K} \phi_{\varepsilon_0}(x - x_i),$$

then $\|f\|_{M(A)} \leqslant \delta$.

Proof. Noting that $f = D * \phi_{\varepsilon_0}$, we find that

$$|\widehat{f}| = |\widehat{D}||\widehat{\phi_{\varepsilon_0}}|. \tag{4.11}$$

The standard (L^1, L^∞) bound for the Fourier transform implies that

$$\|\widehat{\phi_{\varepsilon_0}}\|_{L^\infty(\mathbf{Z}^d)} \leqslant \|\phi_{\varepsilon_0}\|_{L^1(\mathbf{T}^d)} = 1. \tag{4.12}$$

Thus for $0 < |\xi| \leqslant K^{1/2\beta}$, we employ (4.10), (4.11) and (4.12) to conclude that

$$|\widehat{f}(\xi)| \lesssim_B \left[CK^{-1/4}\log(K)^{1/2}\right] B(\xi)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2}. \tag{4.13}$$

Since the implicit constant here is absolute, (4.13) implies there exists K_0 depending only on C, the sequence $\{A(\xi)\}$, and δ, such that for $K \geqslant K_0$, and $|\xi| \leqslant K^{1/2\beta}$,

$$|\widehat{f}(\xi)| \leqslant \delta|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2}B(\xi) = \delta A(\xi). \tag{4.14}$$

For $K^{1/2\beta} \leqslant |\xi| \leqslant K^{1/\beta}$, we again combine (4.10), (4.11) and (4.12) to conclude that

$$|\widehat{f}(\xi)| \lesssim_\beta C|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2}. \tag{4.15}$$

Since the constants here also do not depend on K, and as $|\xi| \to \infty$, $B(\xi) \to \infty$, we conclude from (4.15) there exists K_0 depending on β, the sequence $\{A(\xi)\}$, C, and δ, such that for $K \geqslant K_0$ and $K^{1/2\beta} \leqslant |\xi| \leqslant K^{1/\beta}$,

$$|\widehat{f}(\xi)| \leqslant \delta B(\xi)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2} = \delta A(\xi). \tag{4.16}$$

If $K^{1/\beta} \leqslant |\xi| \leqslant K^{1/\beta+\varepsilon}$, we note that (3.3) implies $\widehat{\phi_\varepsilon}(\xi) \lesssim \varepsilon_0^{-\beta/2}|\xi|^{-\beta/2}$, which together with (4.10) and (4.11) implies

$$|\widehat{f}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}\varepsilon^{-\beta/2}|\xi|^{-\beta/2} \lesssim_\beta \left(CK^{-1/2}\varepsilon_0^{-\beta/2}\right)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2}. \tag{4.17}$$

If $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$, again using the fact that $B(\xi) \to \infty$ as $|\xi| \to \infty$, we conclude from (4.17) that there exists K_0 depending only on δ, C, β, and the sequence $\{A(\xi)\}$, such that if $K \geqslant K_0$, then

$$|\widehat{f}(\xi)| \leqslant \delta B(\xi)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2} = \delta A(\xi). \tag{4.18}$$

If $|\xi| \geqslant K^{1/\beta+\varepsilon}$, we apply (3.3) for $T \geqslant \beta/2$ together with the bound $\varepsilon_0 \geqslant C^{-1}K^{-1/\beta}$ to conclude

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_T \varepsilon_0^{-T}|\xi|^{-T} \\ &\leqslant \left[\frac{C^T K^{T/\beta}|\xi|^{\beta/2-T}}{\log(1+|\xi|)^{1/2}}\right]|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2} \\ &\leqslant \left[\frac{C^T K^{1/2+(\beta/2-T)\varepsilon}}{\log(1+K^{1/\beta+\varepsilon})^{1/2}}\right]|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2}. \end{aligned} \tag{4.19}$$

If we choose $T > \beta/2 + 1/2\varepsilon$, then the value in the brackets in (4.19) decays as $K \to \infty$. In particular, we conclude that there exists a constant K_0 depending on C, δ, β, and ε such that for $K \geqslant K_0$,

$$|\widehat{f}(\xi)| \leqslant \delta B(\xi)|\xi|^{-\beta/2}\log(1+|\xi|)^{1/2} = \delta A(\xi). \tag{4.20}$$

Combining (4.14), (4.16), (4.18), and (4.20) gives $\|f\|_{M(A)} \leqslant \delta$. □

Independence of random variables is sufficient to obtain square root cancellation.

Lemma 8. *Fix a large integer K, let $X_1, \ldots, X_K$ be independant uniformly distributed random variables on $\mathbf{T}^d$. Set*

$$D(x) = \sum_{k=1}^{K} \delta(x - x_k)$$

and

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leqslant K^{1/\beta+1}\}.$$

Then there exists a constant C, depending only on d and β, such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \leqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$, consider the random variable $C(\xi, k) = K^{-1}e^{2\pi i(\xi\cdot X_k)}$. Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} C(\xi, k) = \widehat{D}(\xi). \tag{4.21}$$

We also note that for each ξ and k,

$$|C(\xi, k)| = K^{-1}, \tag{4.22}$$

Moreover, for $\xi \neq 0$,

$$\mathbf{E}(C(\xi, k)) = K^{-1}\int_{\mathbf{T}^d} e^{2\pi i(\xi\cdot x)}\, dx = 0.$$

Since the family of random variables $\{C(\xi,1),\dots,C(\xi,K)\}$ is independent for a fixed non-zero ξ, we can apply Hoeffding's inequality together with (4.21) and (4.22) to conclude that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{D}(\xi)| \geqslant t\right) \leqslant 2e^{-Kt^2/2}. \tag{4.23}$$

A union bound obtained by applying (4.23) over all $|\xi| \leqslant K^{1/\beta+1}$, combined with (4.23), shows that if

$$B = \{\xi \in \mathbf{Z}^d : |\xi| \leqslant K^{1/\beta+1}\},$$

then there exists a constant $C \geqslant 1$ depending only on d and β, such that

$$\mathbf{P}\left(\|\widehat{D}\|_{L^\infty(B)} \geqslant t\right) \leqslant \exp\left(C\log(K) - \frac{5Kt^2}{C}\right). \tag{4.24}$$

But then, setting $t = CK^{-1/2}\log(K)^{1/2}$ in (4.24) completes the proof. □

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}$ to be as low as possible. For each $(E,\mu) \in \mathcal{X}$, the condition that $\mu \in M(A)$ implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus we might expect that for quasi-all $(E,\mu) \in M(A)$, the set E has both Hausdorff dimension and Fourier dimension equal to β. This turns out to be the correct assumption.

Lemma 9. *For quasi-all $(E,\mu) \in \mathcal{X}$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since the case $\beta = d$ is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that quasi-all $\mu \in M(A)$ have Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha,\delta,s) = \{(E,\mu) \in \mathcal{G} : H^\alpha_\delta(E) < s\}$. Then $A(\alpha,\delta,s)$ is an open set, and

$$\bigcap_{n=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty} A(\beta+1/n, 1/m, 1/k)$$

is precisely the family of $(E,\mu) \in \mathcal{X}$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha,\delta,s)$ is dense in $\mathcal{X}$ for $\alpha \in (\beta,d)$ and $\delta, s > 0$. Fix $(E,\mu_0) \in \mathcal{X}$, $\alpha \in (\beta,d)$, $\delta > 0$, and $s > 0$, and consider $\varepsilon > 0$. We aim to show that for each $(E_0,\mu_0) \in \mathcal{X}$, there is $(E,\mu) \in A(\alpha,\delta,s)$ such that $d_{\mathcal{X}}((E_0,\mu_0),(E,\mu)) \leqslant \varepsilon$. Without loss of generality, in light of Lemma 5, we may assume without loss of generality that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a large integer K. Lemma 8 shows that there exists a constant C depending only on d and β and K points $x_1,\dots,x_K \in \mathbf{T}^d$ such that if

$$D(x) = \sum_{k=1}^{K} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{4.25}$$

Using (4.25) to apply Lemma 7, and then applying Lemma 6, we conclude that there exists K_0 depending on β, d, ε, and the sequence $\{A(\xi)\}$, such that if $K \geqslant K_0$, if $\varepsilon_0 = K^{-1/\beta}$, and if

$$\mu(x) = \left(\sum_{k=1}^{K} \phi_{\varepsilon_0}(x - x_k) \right) \mu_0(x),$$

then

$$\|\mu - \mu_0\|_{M(A)} \leqslant \varepsilon. \tag{4.26}$$

Since μ is supported on K balls of radius ε_0, and for K suitably large, $\varepsilon_0 \leqslant \delta$, we find that

$$H^{\alpha}_{\delta}(\mathrm{supp}(\mu)) \leqslant K\varepsilon_0^{\alpha} = K^{1-\alpha/\beta}. \tag{4.27}$$

Since $\alpha > \beta$, (4.27) implies that for suitably large K,

$$H^{\alpha}_{\delta}(\mathrm{supp}(\mu)) \leqslant s. \tag{4.28}$$

Now let

$$E = \mathrm{supp}(\mu) \cup \{y_1, \ldots, y_N\},$$

where $\{y_1, \ldots, y_N\} \subset E_0$ is a ε net of E_0. Equation (4.28) implies that $H^{\alpha}_{\delta}(E) \leqslant s$ for sufficiently large K, so $(E, \mu) \in A(\alpha, \delta, s)$. Since $\mathrm{supp}(\mu) \subset E$,

$$d_H(E, E_0) \leqslant \varepsilon. \tag{4.29}$$

Combining (4.26) with (4.29) shows $d_{\mathcal{X}}((E_0, \mu_0), (E, \mu)) \leqslant \varepsilon$, which completes the proof. □

All that now remains is to show that quasi-all elements of $\mathcal{X}$ avoid the given set Z; just as with the proof above, the advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

5 Random Avoiding Sets

Lemma 10. *For quasi-all (E, μ), for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.*

Proof. Let us fix a set $W \subset \mathbf{R}^{nd}$, and $\varepsilon_0 > 0$ such that

$$|W_{\varepsilon_0}| \leqslant \varepsilon_0^{nd-\alpha}. \tag{5.1}$$

Fix a large integer K, and consider a family

$$\{X_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant K\}$$

of independant, uniformly distributed random variables on $\mathbf{T}^d$. Set $E_i = \{X_{i1}, \ldots, X_{iK}\}$. We wish to determine the density of a uniformly distributed $X \in \mathbf{T}^d$, subject to the condition that $X \in (\mathbf{T} \times E_2 \times \cdots \times E_n) \cap W_{\varepsilon_0}$. If $E_2, \ldots, E_n$ are fixed, non-random sets, then the distribution of X is constant on $(E_2 \times \cdots \times E_n) \cap W_{\varepsilon_0}$.

Fix a large integer K, and let $X_1, \dots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $i_1, \dots, i_n \in \{1, \dots, K\}$, the random vector $X_i = (X_{i_1}, \dots, X_{i_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (5.7) implies

$$\mathbf{P}(d(X_i, W) \leqslant \varepsilon_0) \leqslant |W_{\varepsilon_0}| \leqslant \varepsilon_0^{nd-\alpha}. \tag{5.2}$$

If M denotes the number of indices i such that $d(X_i, W) \leqslant \varepsilon_0$, then by linearity of expectation we conclude from (5.8) that

$$\mathbf{E}(M) \leqslant K^n \varepsilon_0^{nd-\alpha}. \tag{5.3}$$

Applying Markov's inequality to (5.9), we conclude that

$$\mathbf{P}(M \geqslant 10K^n \varepsilon_0^{nd-\alpha}) \leqslant 1/10. \tag{5.4}$$

Taking a union bound to (5.10) and the result of Lemma 8, we conclude that there exists K points $x_1, \dots, x_K \in \mathbf{T}^d$ and a constant C depending on d and β such that if

$$D_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - x_k)$$

then for $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D_0}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}, \tag{5.5}$$

and if S is the set of indices $k_1 \in \{1, \dots, K\}$ such that we can find distinct $k_1, k_2, \dots, k_n \in \{1, \dots, K\}$ such that $d((x_{k_1}, \dots, x_{k_n}), W) \leqslant \varepsilon_0$, then

$$\#(S) \leqslant 10K^n \varepsilon_0^{nd-\alpha}. \tag{5.6}$$

In particular, if we set $C_1 = 10^{1/(nd-\alpha)}$, and we set $\varepsilon_0 = C_1^{-1} K^{-1/\beta}$ (TODO: WHERE $\beta = (nd - \alpha)/(n - 1/2)$), then

$$\#(S) \leqslant K^{1/2}.$$

In particular, this means that if we set

$$D_1(x) = \sum_{k \notin S} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D_1}(\xi)| \leqslant (C + 1)K^{-1/2} \log(K).$$

Applying Lemmas 6 and 7, we conclude that there is K_0 such that if

$$\mu(x) = \left(\sum_{k \notin S} \phi_{0.5\varepsilon_0}(x - x_k) \right) \mu_0(x)$$

and $K \geqslant K_0$, then

$$\|\mu - \mu_0\|_{M(A)} \leqslant \varepsilon.$$

Notice that $\mathrm{supp}(\mu) = \mathrm{supp}(D_1)_{0.5\varepsilon_0}$. For any distinct $x_1, \ldots, x_n \in \mathrm{supp}(D_1)$,

$$d((x_1, \ldots, x_n), W) \geqslant \varepsilon_0.$$

Thus if $x_1', \ldots, x_n' \in \mathrm{supp}(\mu)$ are s-separated, and $\varepsilon_0 \leqslant s$, then

$$d((x_1', \ldots, x_n'), W) \geqslant 0.5\varepsilon_0.$$

Thus □

This gives the result for $\beta = (nd - \alpha)/(n - 1/2)$. Let's see if we can improve this to $(nd - \alpha)/(n - 1)$.

Proof. The set $Z \subset \mathbf{R}^{nd}$ is the countable union of sets with Minkowski dimension at most α. Consider n open sets $U_1, \ldots, U_n$ with $d(U_i, U_j) \geqslant 1/N$ for $i \neq j$. For a closed set W with Minkowski dimension at most α, consider the set

$$B(W, U_1, \ldots, U_n) = \{(E, \mu) \in \mathcal{X} : \text{for } x_1 \in E \cap U_1, \ldots, x_n \in E \cap U_n,\ (x_1, \ldots, x_n) \notin W\}.$$

It thus suffices to show that $B(W, s)$ is dense in $\mathcal{X}$ for any W and s. Thus fix $(E_0, \mu_0) \in \mathcal{X}$ and $\varepsilon > 0$. We will construct $(E, \mu) \in \mathcal{X}$ with $(E, \mu) \in B(W, s)$ and $d_{\mathcal{X}}((E_0, \mu_0), (E, \mu)) \leqslant \varepsilon$.

If we fix a set $W \subset \mathbf{R}^{nd}$ with Minkowski dimension at most α, then there exists arbitrarily small $\varepsilon_0 > 0$ such that

$$|W_{\varepsilon_0}| \leqslant \varepsilon_0^{nd-\alpha}. \tag{5.7}$$

Fix a large integer K, and let $X_1, \ldots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $i_1, \ldots, i_n \in \{1, \ldots, K\}$, the random vector $X_i = (X_{i_1}, \ldots, X_{i_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (5.7) implies

$$\mathbf{P}(d(X_i, W) \leqslant \varepsilon_0) \leqslant |W_{\varepsilon_0}| \leqslant \varepsilon_0^{nd-\alpha}. \tag{5.8}$$

If M denotes the number of indices i such that $d(X_i, W) \leqslant \varepsilon_0$, then by linearity of expectation we conclude from (5.8) that

$$\mathbf{E}(M) \leqslant K^n \varepsilon_0^{nd-\alpha}. \tag{5.9}$$

Applying Markov's inequality to (5.9), we conclude that

$$\mathbf{P}(M \geqslant 10K^n \varepsilon_0^{nd-\alpha}) \leqslant 1/10. \tag{5.10}$$

Taking a union bound to (5.10) and the result of Lemma 8, we conclude that there exists K points $x_1, \ldots, x_K \in \mathbf{T}^d$ and a constant C depending on d and β such that if

$$D_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta(x - x_k)$$

then for $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D_0}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}, \tag{5.11}$$

and if S is the set of indices $k_1 \in \{1, \dots, K\}$ such that we can find distinct $k_1, k_2, \dots, k_n \in \{1, \dots, K\}$ such that $d((x_{k_1}, \dots, x_{k_n}), W) \leqslant \varepsilon_0$, then

$$\#(S) \leqslant 10K^n \varepsilon_0^{nd-\alpha}. \tag{5.12}$$

In particular, if we set $C_1 = 10^{1/(nd-\alpha)}$, and we set $\varepsilon_0 = C_1^{-1} K^{-1/\beta}$ (TODO: WHERE $\beta = (nd - \alpha)/(n - 1/2)$), then

$$\#(S) \leqslant K^{1/2}.$$

In particular, this means that if we set

$$D_1(x) = \sum_{k \notin S} \delta(x - x_k),$$

then for each $|\xi| \leqslant K^{1/\beta+1}$,

$$|\widehat{D_1}(\xi)| \leqslant (C+1)K^{-1/2} \log(K).$$

Applying Lemmas 6 and 7, we conclude that there is K_0 such that if

$$\mu(x) = \left(\sum_{k \notin S} \phi_{0.5\varepsilon_0}(x - x_k) \right) \mu_0(x)$$

and $K \geqslant K_0$, then

$$\|\mu - \mu_0\|_{M(A)} \leqslant \varepsilon.$$

Notice that $\operatorname{supp}(\mu) = \operatorname{supp}(D_1)_{0.5\varepsilon_0}$. For any distinct $x_1, \dots, x_n \in \operatorname{supp}(D_1)$,

$$d((x_1, \dots, x_n), W) \geqslant \varepsilon_0.$$

Thus if $x_1', \dots, x_n' \in \operatorname{supp}(\mu)$ are s-separated, and $\varepsilon_0 \leqslant s$, then

$$d((x_1', \dots, x_n'), W) \geqslant 0.5\varepsilon_0.$$

Thus □

References

[1] T.W. Körner, *Measures on Independent Sets, A Quantitative Version of Rudin's Theorem.*

[2] T.W. Körner, *Fourier Transforms of Measures and Algebraic Relations on Their Supports.*

[3] Jacob Denson, Malabika Pramanik, Joshua Zahl, *Large Sets Avoiding Rough Patterns.*

[4] Jacob Denson, *Cartesian Products Avoiding Patterns.*

[5] Roman Vershynin, *High Dimensional Probability*, Cambridge Series in Statistical and Probabilistic Mathematics, 2018.