Large Sets Avoiding Rough Patterns

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Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern such as three term arithmetic progressions, or more general patterns such as avoiding points $x_1, \dots, x_n$ such that $f(x_1, \dots, x_n) = 0$ (three term arithmetic progressions are specified by the pattern $x_1 - 2x_2 + x_3 = 0$). Previous work on the subject has considered patterns described by polynomials, or by functions f satisfying certain regularity conditions. We consider the case of 'rough patterns. There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset [0,1]$ is a set with Minkowski dimension α, we construct a set $X \subset [0,1]$ with Hausdorff dimension $1-\alpha$ so that $X + X$ is disjoint from Y. As a second application, given a set Y of dimension close to one, we can construct a subset $X \subset Y$ of dimension 1/2 that avoids isosceles triangles. [**TODO:** I'd like to replace this with a rectifiability statement]

A major question in modern geometric measure theory is whether large Hausdorff dimension guarantees the existence of patterns in sets. There is currently no comprehensive theorem in this setting, and the threshold dimension at which patterns are guaranteed can vary for different patterns, or not even exist at all. For example, the main result of [7] constructs a set $X \subset \mathbf{R}^d$ with full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. On the other hand, Theorem 6.8 of [9] shows that each set $X \subset \mathbf{R}^d$ with Hausdorff dimension exceeding one must contain three colinear points. In this paper, we provide lower bounds for the threshold by solving the pattern avoidance problem; constructing sets with large dimension avoiding patterns.

One natural way to approach the pattern avoidance problem is to find general methods for constructing pattern avoiding sets by exploiting a particular geometric feature of the pattern. In [8], Máthé shows that, given a pattern specified by a countable union of rational coefficient bounded degree polynomials $f_1, f_2, \dots$ in nd variables, one can construct a set $X \subset \mathbf{R}^d$ such that for every collection of distinct points $x_1, \dots, x_n$, and each index k, $f_k(x_1, \dots, x_n) \neq 0$. In [3], Fraser and the second author consider a related problem where the functions f_k are placed with C^1 functions satisfying a mild degeneracy condition.

Rather than avoiding the zeroes of a function, in this paper, we fix $Z \subset \mathbf{R}^{dn}$, and construct sets X such that for each distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. Notice that

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if $Z = \{(x_1, x_2, x_3) \in (\mathbf{R}^d)^3 : (x_1, x_2, x_3) \text{ are colinear}\}$, then $X \subset \mathbf{R}^d$ does not contain three non colinear points precisely when $(x_1, x_2, x_3) \notin Z$ for each distinct $x_1, x_2, x_3 \in X$. Similarily, our problem generalizes the problem statements considered in [8] and [3] if we set $Z = \bigcup_{k=1}^\infty f_k^{-1}(0)$. From this perspective, Máthé constructs sets avoiding a set Z formed from a countable union of algebraic varieties, while Fraser and the second author construct sets avoiding a set Z formed from the countable union of C^1 manifolds. The advantage of this formulation of the problem is it is natural to consider 'rough' sets $Z \subset \mathbf{R}^{dn}$ which are not naturally specified as the zero set of a function. In particular, in this paper we consider Z formed from the countable union of sets, each with lower Minkowski dimension bounded above by some α. In contrast with previous work, no further assumptions are made on Z. The dimension of the avoiding set X we will eventually construct depends only on the codimension $nd - \alpha$ of the set Z, and the number of variables n.

Theorem 1. *Let Z be a countable union of compact sets, each with lower Minkowski dimension at most α, with $d \leqslant \alpha < dn$. Then there exists a set $X \subset [0,1]^d$ with Hausdorff dimension at least $(nd - \alpha)/(n-1)$ such that whenever $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Z$.*

Remark. *When $\alpha < d$, the pattern avoidance problem is trivial, since $X = [0,1]^d - \pi(Z)$ is full dimensional and solves the pattern avoidance problem, where $\pi \colon \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map onto the first coordinate. The case $\alpha = dn$ is trivial as well, since we can set $X = \varnothing$.*

When Z is a countable union of smooth manifolds, Theorem 1 generalizes Theorem 1.1 and 1.2 from [3]. Section 6 is devoted to a comparison of our methods with [3], as well as other general pattern avoidance methods. But our result is more interesting when it can be applied to truly 'rough' sets. One surprising application to our method is obtained by considering a 'rough' set Y in addition to a set Z, and finding a set $X \subset Y$ of large dimension avoiding Z. Previous methods in the literature fundamentally exploit the ability to select points on the entirety of Euclidean space, and so it remains unlikely that we can obtain any result of this form using their methods, unless Y is suitably flat, i.e. a smooth surface. To contrast this, our result even applies for certain sets Y which are of Cantor type. We discuss applications of our result in Section 5.

Theorem 1 is proved using a Cantor-type construction, a common theme in the surrounding literature, described explicitly in Section 3. For a sequence of decreasing lengths $l_n \to 0$, the construction specifies a selection mechanism for a nested family of sets $X_n \to X$, with X_n a union of sidelength l_n cubes and avoiding Z at scales close to l_n. Our *key* contribution to this process is using the probabilistic method to guarantee the existence of efficient selections at each scale, described in Section 2. This selection also assigns mass uniformly on intermediate scales, akin to [3], ensuring that the selection procedure at a single scale is the sole reason for the Hausdorff dimension we calculate in Section 4. Furthermore, the randomization allows us to avoid the complicated queueing techniques in [5] and [3], which has the additional benefit that we can confront the entire behaviour of Z at scales close to l_n simultaneously at the n'th step of the argument.

1 Frequently Used Notation and Terminology

Our argument heavily depends upon discretizing sets into unions of cubes. Throughout our argument, we use the following notation:

(A) A *dyadic scale* is a length l equal to 2^{-k} for some non-negative integer k.

(B) Given a length $l > 0$, we let $\mathcal{B}_l^d$ denote the family of all half open cubes in $\mathbf{R}^d$ with sidelength l and corners on the lattice $(l \cdot \mathbf{Z})^d$. That is,

$$\mathcal{B}_l^d = \{[a_1, a_1 + l] \times \cdots \times [a_d, a_d + l] : a_k \in l \cdot \mathbf{Z}\}.$$

If $E \subset \mathbf{R}^d$, $\mathcal{B}_l^d(E)$ is the family of cubes in $\mathcal{B}_l^d$ intersecting E, i.e.

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : I \cap E = \varnothing\}.$$

(C) The *lower Minkowski dimension* of a compact set $Z \subset \mathbf{R}^d$ are defined as

$$\underline{\dim}_{\mathbf{M}}(Z) = \liminf_{l \to 0} \frac{\log(\# \mathcal{B}_l^d(Z))}{\log(1/l)}. \tag{1.1}$$

(D) If $0 \leqslant \alpha \leqslant d$ and $\delta > 0$, we define the dyadic Hausdorff content of a set $E \subset \mathbf{R}^d$ as

$$H_\delta^\alpha(E) = \inf\left\{ \sum_{k=1}^m l_k^\alpha : E \subset \bigcup_{k=1}^m I_k \text{ and } I_k \in \mathcal{B}_{l_k}^d, l_k \leqslant \delta \text{ for all } k \right\}.$$

The α dimensional dyadic Hausdorff measure H^α on $\mathbf{R}^d$ is $H^\alpha(E) = \lim_{\delta \to 0} H_\delta^\alpha(E)$. The *Hausdorff dimension* of a set E is $\dim_{\mathbf{H}}(E) = \inf\{\alpha \geqslant 0 : H^\alpha(E) = 0\}$.

(E) Given $I \in \mathcal{B}_l^{dn}$, we can decompose I as $I_1 \times \cdots \times I_n$ for unique cubes $I_1, \dots, I_n \in \mathcal{B}_l^d$. We say I is *strongly non-diagonal* if the cubes $I_1, \dots, I_n$ are distinct. Strongly non-diagonal cubes will play an important role in Section 2, when we solve a discrete version of Theorem 1.

(F) Adopting the terminology of [4], we say a collection of sets $U_1, \dots, U_n$ is a *strong cover* of a set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k. This idea will be useful in Section 3.

(G) A *Frostman measure* of dimension α is a non-zero compactly supported finite Borel measure μ on $\mathbf{R}^d$ such that for every cube I of sidelength l, we have

$$\mu(I) \lesssim l^\alpha. \tag{1.2}$$

Note that a measure μ satisfies (1.2) for every cube I if and only if it satisfies (1.2) for cubes whose sidelengths are dyadic scales.

Frostman's lemma says that

$$\dim_{\mathbf{H}}(E) = \sup\left\{ \alpha : \begin{array}{c} \text{there is an } \alpha \text{ dimensional Frostman} \\ \text{measure supported on } E \end{array} \right\}.$$

2 Avoidance at Discrete Scales

In this section we describe a method for avoiding Z at a single scale. This will be applied in Section 3 at many scales to construct a set X avoiding Z at all scales. The single scale technique is the core part of our construction, and the efficiency with which we can avoid Z at a single scale has direct consequences on the Hausdorff dimension of the set X that we construct.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic scales $l \geqslant s$. In the discrete setting, Z is replaced by a union of cubes in $\mathcal{B}_s^{dn}$, denoted by Z_s. Given a set E, which is a union of cubes in $\mathcal{B}_l^d$, our goal is to construct a union of cubes in $\mathcal{B}_s^d(E)$, denoted F, such that F^n is disjoint from the strongly non-diagonal cubes of Z_s (see Definition (E)).

In order to ensure the final set X obtained in Theorem 1 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of X, it is crucial that F is spread uniformly over E. We achieve this by decomposing E into sub-cubes in $\mathcal{B}_r^d$ for some intermediate scale $r \in (s, l)$, and distributing F evenly among as many of these intermediate sub-cubes as possible. The following lemma shows this is possible provided $|E| r^{-d}$ is much larger than $|Z_s| r^{-dn}$.

Lemma 1. *Fix three dyadic scales $l \geqslant r \geqslant s$. Let E be a union of cubes in $\mathcal{B}_l^d$ and let Z_s be a union of cubes in $\mathcal{B}_s^{dn}$. Then there exists a set $F \subset E$, which is a union of cubes in $\mathcal{B}_s^d$ satisfying the following three properties:*

(A) Avoidance*: For any distinct $J_1, \dots, J_n \in \mathcal{B}_s^d(F)$, we have $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(Z_s)$.*

(B) Non-Concentration*: For each $I \in \mathcal{B}_r^d(E)$, there is at most one $J \in \mathcal{B}_s^d(F)$ with $J \subset I$.*

(C) Large Size*: $\# \mathcal{B}_s^d(F) \geqslant |E| r^{-d} - |Z_s| r^{-dn}$.*

Proof. For each $I \in B_r^d(E)$, let J_I be a random element of $\mathcal{B}_s^d(I)$ chosen independently with uniform probability. Define a random set

$$U = \bigcup \{ J_I : I \in \mathcal{B}_s^d(I) \}. \tag{2.1}$$

By construction, for each $I \in B_s^d(I)$, U contains exactly one subcube $J \in \mathcal{B}_s^d$ with $J \subset I$, so U satisfies Property (B). Our construction also implies $\# \mathcal{B}_s^d(U) = \# \mathcal{B}_r^d(E) = |E| r^{-d}$, so U satisfies Property (C). Unfortunately, U might not satisfy Property (A).

For each cube $J \in \mathcal{B}_s^d(E)$, there is a unique 'parent' cube $I \in \mathcal{B}_r^d(E)$ such that $J \subset I$. Since I contains $(r/s)^d$ elements of $\mathcal{B}_s^d(E)$, for each fixed cube $J \in \mathcal{B}_s^d(E)$,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_I = J) = (s/r)^d. \tag{2.2}$$

Here the probability measure $\mathbf{P}(\cdot)$ is taken with respect to the randomly chosen set U defined in (2.1). Since the cubes J_I are chosen independantly, if $J_1, \dots, J_k$ are distinct cubes in $\mathcal{B}_s^d(E)$, then (2.2) combined with Property (B) shows

$$\mathbf{P}(J_1, \dots, J_k \in U) = \begin{cases} (s/r)^{dk} & : \text{if } J_1, \dots, J_k \text{ have distinct parents} \\ 0 & : \text{otherwise} \end{cases}. \tag{2.3}$$

Let $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(Z_s)$ be a strongly non-diagonal cube. Since the cubes $J_1, \dots, J_n$ are distinct, we can apply (2.3) to conclude

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1, \dots, J_k \in U) \leqslant (s/r)^{dn}. \tag{2.4}$$

Define a random collection of cubes

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(Z_s) \colon U \in U^n,\ K \text{ strongly non-diagonal}\}. \tag{2.5}$$

By (2.4) and linearity of expectation,

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_K \mathbf{P}(K \subset U^n) \leqslant (\#\,\mathcal{B}_s^{dn}(Z_s))(s/r)^{dn} = |Z_s| r^{-dn}, \tag{2.6}$$

where K ranges over strongly non-diagonal cubes in $\mathcal{B}_s^{dn}(Z_s)$. In particular, (2.6) implies that there exists at least one (non-random) set U_0 such that

$$\#\mathcal{K}(U_0) \leqslant |Z_s| r^{-dn}. \tag{2.7}$$

The set F is now obtained by removing a small number of cubes from U_0.

We now define

$$F = U_0 - \{\pi(K) \colon K \in \mathcal{K}(U_0),\ K \text{ strongly non-diagonal}\}, \tag{2.8}$$

where $\pi \colon \mathcal{B}_s^{dn} \to \mathcal{B}_s^{d}$ projects a set $J_1 \times \cdots \times J_n$ to J_1. Since $F \subset U_0$, and U_0 satisfies Property (B), F also satisfies Property (B). Property (C) follows from (2.7). Finally, given any strongly non-diagonal cube $J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(Z_s)$, either $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(U_0^n)$, or $J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(U_0^n)$. If the former occurs then $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(F^n)$ since $F \subset U_0$, while if the latter occur then (2.8) implies $J_1 \notin \mathcal{B}_s^{d}(F)$. In either case, we have $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(F^n)$, so F satisfies Property (A). □

Remark. *While the existence of the set F in Lemma 1 was obtained by probabilistic techniques, it's existence is a purely deterministic statement. One can find a candidate F constructively by checking every possible choice of U (there are only finitely many) to find one particular choice U_0 which satisfies* (2.7)*, and then defining F by* (2.8)*. Thus the set we obtain in Theorem 1 exists by purely constructive means.*

Our inability to select almost every cube in Lemma 1 means that repeated applications of the result will lead to a loss in Hausdorff dimension. In fact, in the worst case, applying the lemma causes us to lose as much Hausdorff dimension as is permitted by Theorem 1. To see why, note that to ensure F is non-empty, our parameters to Lemma 1 must satisfy

$$r^{d(n-1)} > |Z_s||E|^{-1}. \tag{2.9}$$

If Z has dimension α, we will later determine that the scale s discretization Z_s satisfies $|Z_s| \leqslant s^{dn-\alpha-\varepsilon}$, for some small positive ε converging to zero as $s \to 0$. In the worst case, however, it is certainly possible that

$$|Z_s| \geqslant s^{dn-\alpha-\varepsilon}. \tag{2.10}$$

In this situation, we therefore must have

$$r \geqslant s^{(dn-\alpha)/d(n-1)}, \tag{2.11}$$

Otherwise, if we assume $r \leqslant s^{(dn-\alpha)/d(n-1)}$, then (2.10) shows

$$r^{d(n-1)} \leqslant s^{dn-\alpha} \leqslant |Z_s| \leqslant |Z_s||E|^{-1}, \tag{2.12}$$

so (2.9) is not satisfied. But then (2.11) combined with Property (B) of Lemma 1 shows

$$\frac{\log \# \mathcal{B}_s^d(F)}{\log(1/s)} \leqslant \frac{\log \# \mathcal{B}_r^d(E)}{\log(1/s)} \leqslant \frac{d \log(1/r) - \log |E|^{-1}}{\log(1/s)} \leqslant \frac{dn-\alpha}{n-1} \tag{2.13}$$

If $F = \bigcap F_k$, where $\{F_k\}$ are infinitely many sets obtained at a sequence of scales s_k converging to zero as $k \to \infty$, then (2.13) can be easily used to show

$$\dim_{\mathbf{H}}(F) \leqslant (dn-\alpha)/(n-1). \tag{2.14}$$

Comparing (2.14) to the Hausdorff dimension of the set X obtained in Theorem 1, we see we must be very careful to ensure applications of the discrete lemma infinitely many times are the only place in our proof where dimension is lost.

In our construction of the set X in Theorem 1, Lemma 1 will be applied with r slightly greater than $s^{(dn-\alpha)/d(n-1)-\varepsilon}$, which gives an epsilon of room to escape the problems of the last paragraph. When combined with the large difference between s and l, we find $\# \mathcal{B}_s^d(F)$ is large when compared to $\# \mathcal{B}_r^d(E)$. The following corollary repackages Lemma 1 to work in this scenario.

Corollary 1. *Let $\alpha \in (0, dn)$, let $\varepsilon \in (0, (dn-\alpha)/d(n-1))$, and let $s \leqslant l$ be two dyadic scales. Let E be a nonempty union of cubes in $\mathcal{B}_l^d$, and Z_s a union of cubes in $\mathcal{B}_s^{dn}$. Suppose that $|Z_s| \leqslant s^{dn-\alpha-\varepsilon}$ and $s^{(dn-\alpha)/d(n-1)-\varepsilon} \leqslant l$. Then there is a dyadic number $r \sim s^{(dn-\alpha)/d(n-1)-\varepsilon}$ with $s \leqslant r \leqslant l$ and a set $F \subset E$ which is a union of cubes in $\mathcal{B}_s^d(E)$ satisfying the following three properties*

(A) Avoidance*: For each distinct $J_1, \dots, J_n \in \mathcal{B}_s^d(F)$, $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(Z_s)$.*

(B) Non-Concentration*: For each $I \in \mathcal{B}_r^d(E)$, there is at most one $J \in \mathcal{B}_s^d(F)$ with $J \subset I$.*

(C) Large Size*: $\# \mathcal{B}_s^d(F) \geqslant (1 - s^{\varepsilon d(n-1)} l^{-d}) \# \mathcal{B}_r^d(E)$.*

Proof. Let r be the smallest dyadic number that is greater than $s^{(dn-\alpha)/d(n-1)-\varepsilon}$. Since $s^{(dn-\alpha)/d(n-1)-\varepsilon} \leqslant l$, we find $r \leqslant l$. Because $0 \leqslant (dn-\alpha)/d(n-1) - \varepsilon \leqslant 1$, we find $r \geqslant s$. Applying Lemma 1, we obtain a set $F \subset E$ satisfying Properties (A) and (B), and we calculate

$$\frac{\# \mathcal{B}_s^d(F)}{\# \mathcal{B}_r^d(E)} \geqslant \frac{|E| r^{-d} - s^{dn-\gamma} r^{-dn}}{|E| r^{-d}} = 1 - \frac{s^{dn-\gamma} r^{-d(n-1)}}{|E|} \geqslant 1 - \frac{s^{\varepsilon d(n-1)}}{|E|} \geqslant 1 - s^{\varepsilon d(n-1)} l^{-d},$$

so Property (C) is satisfied. □

Remark. *Lemma 1 is the core of our avoidance technique. The remaining argument is fairly modular. If, for a special case of Z, one can improve Lemma 1 using a different proof technique so that a better bound can be placed on the number of cubes discarded, then the remaining parts of our paper can be applied near verbatim to yield a set X with larger Hausdorff dimension to Theorem 1. The dimension your technique yields can be found by an analogous calculation to* (2.14).

3 Fractal Discretization

In this section we will construct the set X by applying Lemma 1 at many scales. Since Z is a countable union of compact sets with Minkowski dimension at most α, there exists a strong cover (see Definition (F)) of Z by cubes restricted to a sequence of dyadic scales l_k. We will select this strong cover so that the scales l_k converge to 0 very quickly.

Lemma 2. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Let $\{\varepsilon_k\}$ be a sequence of positive numbers and let $\{f_k\}$ be a sequence of functions from $(0,\infty)$ to $(0,\infty)$. Then there exists a sequence of dyadic lengths $\{l_k\}$ and compact sets $\{Z_k\}$ such that*

(A) *For each index $k \geqslant 2$, $l_k \leqslant f_{k-1}(l_{k-1})$.*

(B) *For each index k, Z_k is a union of cubes in $\mathcal{B}^{dn}_{l_k}$.*

(C) *Z is strongly covered by the sets $\{Z_k\}$.*

(D) *For each index k, $\#\,\mathcal{B}^{dn}_{l_k}(Z_k) \leqslant 1/l_k^{\alpha+\varepsilon_k}$.*

Proof. Let Z be the union of sets $\{Y_k\}$ with $\underline{\dim}_{\mathbf{M}}(Y_k) \leqslant \alpha$ for each k. Let $m_1, m_2, \dots$ be a sequence of integers that repeats each integer infinitely often. For each positive integer k, since $\underline{\dim}_{\mathbf{M}}(Y_{m_k}) \leqslant \alpha$, (1.1) implies that there exists a sequence of lengths l tending to 0 that satisfy $\#\,\mathcal{B}^{dn}_{l}(Y_{m_k}) \leqslant 1/l^{\alpha+(\varepsilon_k/2)}$. Replacing l with a dyadic scale at most twice the size of l, there are infinitely many *dyadic* scales l with

$$\#\,\mathcal{B}^{dn}_{l}(Y_{m_k}) \leqslant \frac{1}{(l/2)^{\alpha+\varepsilon_k}} \leqslant \frac{2^{dn}}{l^{\alpha+(\varepsilon_k/2)}} = \frac{\left(2^{dn} l^{\varepsilon_k/2}\right)}{l^{\alpha+\varepsilon_k}}. \tag{3.1}$$

In particular, we may select a dyadic length l such that $2^{dn} l^{\varepsilon_k/2} \leqslant 1$, and, if $k \geqslant 2$, also satisfying $l \leqslant f_{k-1}(l_{k-1})$. The first constraint together with (3.1) implies $\#\,\mathcal{B}^{dn}_{l}(Y_{m_k}) \leqslant 1/l^{\alpha+\varepsilon_k}$. We then set $l_k = l$, and define Z_k to be the union of all cubes in $\mathcal{B}^{dn}_{l_k}(Y_{m_k})$. □

We now construct X by avoiding the various discretizations of Z at each scale. The aim is to find a nested decreasing family of discretized sets $\{X_k\}$ with $X = \bigcap X_k$. One condition guaranteeing that X avoids Z is that X_k^n is disjoint from *strongly non-diagonal* cubes in Z_k.

Lemma 3. *Let $Z \subset \mathbf{R}^{dn}$ and let $\{l_k\}$ be a sequence of lengths converging to zero. For each index k, let Z_k be a union of cubes in $\mathcal{B}^{dn}_{l_k}$, and suppose the sets $\{Z_k\}$ strongly cover Z. For each index k, let X_k be a union of cubes in $\mathcal{B}^{d}_{l_k}$. Suppose that for each k, X_k^n avoids strongly non-diagonal cubes in Z_k. If $X = \bigcap X_k$, then $(x_1, \dots, x_n) \notin Z$ for each distinct $x_1, \dots, x_n \in X$.*

Proof. Let $z \in Z$ be a point with distinct coordinates $z_1, \dots, z_n$. Set

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance between Δ and z. Since $\{Z_k\}$ strongly covers Z, there is a subsequence $\{k_m\}$ such that $z \in Z_{k_m}$ for each index m. For suitably large m, the sidelength l_k cube I in Z_{k_m} containing z is disjoint from Δ. But this means I is strongly non-diagonal, and so $z \notin X^n_{k_m}$. In particular, z is not an element of X^n. □

We are now ready to construct the set X in Theorem 1. Let $l_0 = 1$ and $X_0 = [0,1]^d$. For each $k \geqslant 1$, define

$$\varepsilon_k = \frac{dn - \alpha}{4d(n-1)k}, \tag{3.2}$$

and

$$f_k(x) = \min\left(x^{k^2}, (x^d 4^{-k-1})^{\frac{1}{\varepsilon_{k+1} d(n-1)}}\right). \tag{3.3}$$

Apply Lemma 2 to Z with this choice of $\{\varepsilon_k\}$ and $\{f_k\}$; let $\{l_k\}$ be the resulting sequence of length scales and let $\{Z_k\}$ be the resulting strong cover of Z. Observe that for each $k \geqslant 1$,

$$l_{k+1} \leqslant \min\left(l_k^{k^2},\ (l_k^d 4^{-k-1})^{\frac{1}{\varepsilon_{k+1} d(n-1)}}\right). \tag{3.4}$$

This rapid decay is sufficient to obtain the Hausdorff dimension we desire from X in the next section.

For each index $k \geqslant 1$, define $l = l_k$, $s = l_{k+1}$. Observe that X_k is a non-empty union of cubes in $\mathcal{B}_l^d$; that Z_{k+1} is a union of cubes in $\mathcal{B}_s^{dn}$; and that $|Z_{k+1}| \leqslant s^{dn-\alpha-\varepsilon_{k+1}}$. Furthermore, (3.3) implies that

$$s^{\varepsilon_{k+1} d(n-1)} l^{-d} \leqslant 4^{-k-1}. \tag{3.5}$$

Apply Corollary 1 to the sets X_k and Z_{k+1} with the choices of l and s described above and with $\varepsilon = \varepsilon_{k+1}$. We obtain a dyadic number $r \sim s^{(dn-\alpha)/d(n-1)-\varepsilon}$ with $s \leqslant r \leqslant l$ and a set $F \subset X_k$ which is a union of cubes in $\mathcal{B}_s^d(E)$ satisfying Properties (A), (B), and (C) from the corollary. Define $X_{k+1} = F$. Note that by (3.5) and Property (C), we have

$$\frac{\#\,\mathcal{B}_s^d(X_{k+1})}{\#\,\mathcal{B}_r^d(X_k)} \geqslant 1 - 4^{-k-1}. \tag{3.6}$$

Property (A) implies X_{k+1} avoids strongly non-diagonal cubes in Z_{k+1}. Define $X = \bigcap X_k$. By Lemma 3, $(x_1, \dots, x_n) \notin Z$ for each distinct $x_1, \dots, x_n \in X$.

4 Dimension Bounds

To complete the proof of Theorem 1, we must show that X has the claimed Hausdorff dimension. First, we begin with a rough outline of our proof strategy. Recall that from the previous section, we have a sequence of length scales $\{l_k\}$. For each index k, define r_k to be the smallest dyadic number larger than $s^{(dn-\alpha)/d(n-1)-\varepsilon}$.

At the discrete scale l_k, X looks like a set with dimension $\beta - 2d\cdot\varepsilon_{k+1} = \beta - O(1/k)$, where $\beta = (dn-\alpha)/(n-1)$. As $k \to \infty$, $\beta - O(1/k) \to \beta$, so X looks β dimensional at the discrete scales, which is the Hausdorff dimension we want. To obtain the complete dimension bound, it then suffices to interpolate to get a β dimensional behaviour at all intermediate scales. In this construction, as in [3], these intermediate scales posed a significant difficulty. We avoid this difficulty because of the uniform way that we have selected cubes in consecutive scales. This means that between the scales l_k and r_{k+1}, X behaves like a full dimensional set.

The most convenient way to examine the dimension of X at various scales is to use Frostman's lemma. We construct a non-zero Borel measure μ supported on X such that for all $\varepsilon > 0$, for all lengths l, and for all $I \in \mathcal{B}_l^d$, we have $\mu(I) \lesssim_\varepsilon l^{\beta-\varepsilon}$. For each $\varepsilon > 0$, the measure μ is a Frostman measure (see Definition (G)) with dimension at least $\beta - \varepsilon$. This implies that $\dim_{\mathbf{H}}(X) \geqslant \beta$. The advantage of this approach is that once a natural choice of μ is fixed, it is easy to understand the behaviour of X at a scale l by looking at the behaviour of μ restricted to cubes at the scale l.

To construct μ, we take a sequence of measures μ_k, supported on X_k, and then take a weak limit. We initialize this construction by setting μ_0 to be the uniform probability measure on $X_0 = [0,1]^d$. We then define μ_{k+1} supported on X_{k+1}, by modifying the distribution μ_k. First, we throw away the mass of the cubes $I \in \mathcal{B}_{l_k}^d(X_k)$ for which half of the elements of $\mathcal{B}_{r_{k+1}}^d(I)$ fail to intersect X_{k+1}. For the cubes I with more than half of the cubes in $\mathcal{B}_{r_{k+1}}^d(I)$ intersecting X_{k+1}, we distribute the mass $\mu_k(I)$ uniformly over the subcubes of I in X_{k+1}, giving the distribution of μ_{k+1}.

Observe that for each dyadic scale $l > 0$, there is a number N so that for each cube $I \in \mathcal{B}_l^d$, the sequence $\{\mu_k(I)\}$ is monotone decreasing for all $k \geqslant N$. This implies that the sequence of measures $\{\mu_k\}$ converge weakly to a measure μ.

Observe that for each $k \geqslant 1$ and each $I \in \mathcal{B}_{l_k}^d$, we have $\mu(I) \leqslant \mu_k(I)$. This monotonicity will be useful for passing from bounds on the discrete measures to bounds on the final measure. We will now show that the measure μ is a Frostman measure of dimension $\beta = (dn - \alpha)/(n-1)$.

Lemma 4. *If $I \in \mathcal{B}_{l_k}^d$, then*

$$\mu(I) \leqslant \mu_k(I) \leqslant 2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1}\right]^d. \tag{4.1}$$

Proof. Consider $I \in \mathcal{B}_{l_{k+1}}^d$, $J \in \mathcal{B}_{l_k}^d$. If $\mu_k(I) > 0$, J contains an element of $\mathcal{B}_{l_k}^d$ in at least half of the cubes in $\mathcal{B}_{r_k}^d(J)$. Thus the mass of J distributes itself evenly over at least $2^{-1}(l_{k-1}/r_k)^d$ cubes, which gives that $\mu_k(I) \leqslant 2(r_k/l_{k-1})^d \mu_{k-1}(J)$. Expanding this recursive inequality completes the proof, using that μ_0 has total mass one as a base case. □

Corollary 2. *The measure μ is non-trivial, i.e. $\mu \neq 0$.*

Proof. To prove the result, it suffices to show that the total mass of μ_k is bounded below, independantly of k. At each stage k,

$$\# \mathcal{B}_{l_k}^d(X_k) \leqslant \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1}\right]^d.$$

Since only a fraction $1/4^{k+1}$ of the cubes in $\mathcal{B}_{r_k}^d(X_k)$ do not contain a cube in X_{k+1} (this is a consequence of (3.6)), it is only for at most a fraction $1/(2 \cdot 4^k)$ of the cubes in $\mathcal{B}_{r_k}^d(X_k)$ cubes that X_{k+1} fails to contain more than half of the subcubes. But this means that

$$\mu_k[0,1]^d - \mu_{k+1}[0,1]^d \leqslant \left(\frac{1}{2 \cdot 4^k}\left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1}\right]^d\right)\left(2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1}\right]^d\right) \leqslant \frac{1}{2^{k+1}}.$$

Thus

$$\mu_m[0,1]^d = \mu_0[0,1]^d - \left(\sum_{k=0}^{m-1} \mu_k[0,1]^d - \mu_{k+1}[0,1]^d\right) \geqslant 1 - \sum_{k=0}^{m-1} 1/2^{k+1} \geqslant 1/2.$$

This implies $\mu[0,1]^d \geqslant 1/2$, and so in particular, $\mu \neq 0$. $\square$

Treating all parameters in (4.1) which depend on indices smaller than k as essentially constant, we 'conclude' that $\mu_k(I) \lesssim r_k^d \lesssim l_k^{\beta - 2d\cdot\varepsilon_{k+1}}$. The bound $l_{k+1} \leqslant l_k^{k^2}$ from (3.4) implies l_k decays very rapidly, which enables us to ignore quantities depending on previous indices, and obtain a true inequality.

Corollary 3. *For all $I \in \mathcal{B}_{l_k}^d$, $\mu(I) \leqslant \mu_k(I) \lesssim l_k^{\beta - O(1/k)}$.*

Proof. Given $\varepsilon > 0$, (4.1) and the inequality $l_k \leqslant l_{k-1}^{(k-1)^2}$ imply

$$\begin{aligned}
\mu_k(I) &= 2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1} \right]^d \leqslant \left(\frac{2^k}{l_{k-1}^d \dots l_1^d} \right) l_k^{\beta - O(1/k)} \\
&= \left(\frac{2^k l_k^{2d/k}}{l_{k-1}^{d(k-1)}} \right) l_k^{\beta - O(1/k) - 2d/k} \leqslant \left(2^k l_{k-1}^{(2d/k)(k-1)^2 - d(k-1)} \right) l_k^{\beta - O(1/k)} = o(l_k^{\beta - O(1/k)}).
\end{aligned}$$

$\square$

Corollary 3 gives the cleanest expression of the β dimensional behaviour of μ at discrete scales we will need. To obtain a Frostman measure bound at *all* scales, we need to apply a covering argument. This is where the uniform mass assignment technique comes into play. Because μ behaves like a full dimensional set between the scales l_k and r_{k+1}, we won't be penalized for making the gap between l_k and r_{k+1} arbitrarily large. This is essential to our argument, because l_k decays faster than 2^{-k^m} for each $m > 0$.

Lemma 5. *If l is dyadic and $I \in \mathcal{B}_l^d$, then $\mu(I) \lesssim l^{\beta - \varepsilon}$ for all $\varepsilon > 0$.*

Proof. We begin by assume $l \leqslant l_k$ for some k, and prove $\mu(I) \lesssim l^{\beta - O(1/k)}$. To do this, we apply a covering argument, which breaks into cases depending on the size of l in proportion to the scales l_k and r_k:

- If $r_{k+1} \leqslant l \leqslant l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}_{r_{k+1}}^d$. Because of the non-concentration and breadth properties of our construction, we therefore know that the mass of each cube in $\mathcal{B}_{r_{k+1}}^d$ is bounded by at most $2(r_{k+1}/l_{k+1})^d$ times the mass of a cube in $\mathcal{B}_{l_k}^d$. Thus

$$\mu(I) \lesssim (l/r_{k+1})^d (2(r_{k+1}/l_k)^d) l_k^{\beta - O(1/k)} \leqslant 2l^d / l_k^{d - \beta + O(1/k)} \leqslant 2l^{\beta - O(1/k)},$$

  where we used the fact that $d - \beta + O(1/k) \geqslant 0$, so $l_k^{d - \beta + O(1/k)} \geqslant l^{d - \beta + O(1/k)}$.

- If $l_{k+1} \leqslant l \leqslant r_{k+1}$, we can cover I by a single cube in $\mathcal{B}_{r_{k+1}}^d$. Each cube in $\mathcal{B}_{r_{k+1}}^d$ contains at most one cube of $\mathcal{B}_{l_{k+1}}^d$ also contained in X_{k+1}, so $\mu(I) \lesssim l_{k+1}^{\beta - O(1/k)} \leqslant l^{\beta - O(1/k)}$.

- If $l \leqslant l_{k+1}$, there certainly exists m such that $l_{m+1} \leqslant l \leqslant l_m$, and one of the previous cases yields that $\mu(I) \lesssim l^{\beta - O(1/m)} \leqslant l^{\beta - O(1/k)}$.

If $l \geqslant l_k$, then $\mu(I) \leqslant 1 \lesssim_k l_k^{\beta - O(1/k)} \leqslant l^{\beta - O(1/k)}$, so we obtain $\mu(I) \lesssim_k l^{\beta - O(1/k)}$ for arbitrary $I \in \mathcal{B}_l^d$. Since k is arbitrary, this proves the claim. $\square$

Applying Frostman's lemma gives $\dim_{\mathbf{H}}(X) \geqslant \beta$, concluding the proof of Theorem 1.

5 Applications

As discussed in the introduction, Theorem 1 generalizes Theorems 1.1 and 1.2 from [3]. In this section, we present two applications of Theorem 1 in settings where previous methods cannot obtain any results.

Theorem 2 ((Sum-sets avoiding specified sets))**.** *Let $Y \subset \mathbf{R}^d$ be a countable union of sets of Minkowski dimension at most α. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $1 - \alpha$ such that $X + X$ is disjoint from Y.*

Proof. Define $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x, y) : x + y \in Y\} \quad \text{and} \quad Z_2 = \{(x, y) : y \in Y/2\}.$$

Since Y is a countable union of sets of Minkowski dimension at most α, Z is a countable union of sets with lower Minkowski dimension at most $1 + \alpha$. Applying Theorem 1 with $d = 1$, $n = 2$, giving a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $1 - \alpha$ avoiding Z. Since X avoids Z_1, whenever $x, y \in X$ are distinct, $x + y \notin Y$. Since X avoids Z_2, $X \cap (Y/2) = \varnothing$, and thus for each $x \in X$, $x + x \notin Y$. Thus $X + X$ is disjoint from Y. □

Remark. *One weakness of our result is that as the number of variables n increases, the dimension of X tends to zero. If we try and make the n-fold sum $X + \cdots + X$ disjoint from Y, current techniques only yield a set of dimension $(1 - \alpha)/(n - 1)$. We have ideas on how to improve our main result when Z is 'flat', in addition to being low dimension, which will enable us to remove the dependence of $\dim_{\mathbf{H}}(X)$ on n. In particular, we expect to be able to construct a set X of dimension $1 - \alpha$, such that X is disjoint from Y, and X is closed under addition, and multiplication by rational numbers. In particular, given a $\mathbf{Q}$ subspace V of $\mathbf{R}^d$ with dimension α, we can always find a 'complementary' $\mathbf{Q}$ vector space W with complementary fractional dimension $d - \alpha$ such that $V \cap W = (0)$.*

In [3], Fraser and the second author show that if γ is a C^2 curve with non-vanishing curvature, then there exists a set $E \subset \gamma$ of Hausdorff dimension 1/2 that does not contain isoceles triangles. Our method extends this result from the case of curves to more general plane sets.

Theorem 3 (Restricted sets avoiding isoceles triangles)**.** *Let $Y \subset \mathbf{R}^2$ and let $\pi : \mathbf{R}^2 \to \mathbf{R}$ be an orthogonal projection such that $\pi(Y)$ has non-empty interior. Let d be an arbitrary metric on $\mathbf{R}^2$. Suppose that*

$$Z_0 = \{(y_1, y_2, y_3) \in Y^3 : d(y_1, y_2) = d(y_1, y_3)\}$$

is the countable union of sets with lower Minkowski dimension at most $2 + \varepsilon$, for $\varepsilon \geqslant 0$. Then there exists a set $X \subset Y$ with dimension $1/2 - O(\varepsilon)$ so that no triple of points $(x_1, x_2, x_3) \in X^3$ form the vertices of an isoceles triangle.

Proof. Without loss of generality, by translation and rescaling, assuming $\pi(Y)$ contains $[0, 1]$. Form the set

$$Z = \pi(Z_0) = \{(\pi(y_1), \pi(y_2), \pi(y_3)) : y \in Z_0\}$$

Then Z is the projection of a $2 + t$ dimensional set, and therefore has dimension at most $2 + t$. Applying Theorem 1 with $d = 1$ and $n = 3$, we construct a Hausdorff dimension $1/2 - \varepsilon/2$ set $X_0 \subset [0, 1]$ such that for each distinct $x_1, x_2, x_3 \in X_0$, $(x_1, x_2, x_3) \notin Z$. Thus if we form a set X by picking, from each $x \in X_0$, a single element of $\pi^{-1}(x)$, then X avoids isoceles triangles, and has Hausdorff dimension at least as large as X_0. $\square$

To see that Theorem 3 indeed generalizes the result of Fraser and the second author, observe that if d is the Euclidean metric, then for every pair of points $x, y \in \mathbf{R}^2$, the set

$$\{z \in \mathbf{R}^2 : d(x, z) = d(y, z)\}$$

is the perpendicular bisector B_{xy} of x and y. If γ is a compact portion of a smooth curve with non-vanishing curvature, then the number of points in $\gamma \cap B_{xy}$ is bounded independantly of x and y. Thus Z_0 in the statement of Theorem 3 has Minkowski dimension at most 2.

Results about slice of measures, e.g. those detailed in Chapter 6 of [9] show that for each one dimensional set Y, for almost every line L, $L \cap Y$ consists of a finite collection of points. This suggests that if Y is any set with fractional dimension one, then Z_0 has dimension at most 2. We are unsure if this is true for every set with dimension one, but we provide two separate results which suggest that the result is true for a generic set. The first result shows that for an infinite family of Cantor-type sets with dimension $1 + \varepsilon$, Z_0 has dimension at most $3 - \varepsilon$. Thus Theorem 3 can be applied in settings where Y is incredibly 'rough', i.e. totally disconnected.

The probabilistic model we study is obtained by considering a nested family of random discretized sets $C_0, C_1, \dots$ with C_k a union of sidelength $1/2^k$ squares. First, we fix $p \in [0, 1]$. Then we set $C_0 = [0, 1]^2$. To construct C_{k+1}, we split C_k into sidelength $1/2^{k+1}$ squares, and keep each square in C_{k+1} with probability p. Then every sidelength $1/2^k$ cube is contained in C_k with probability p^k, and so C_k contains, on average, $(4p)^k$ sidelength $1/2^k$ cubes. We now show that if $p > 1/4$, C almost surely has Minkowski dimension $2 - \log_2(1/p)$.

Lemma 6. *If $p > 1/4$, then almost surely, $C = \varnothing$, or has Minkowski dimension $2 - \log_2(1/p)$, and this second case occurs with non-zero probability.*

Proof. Let $p > 1/4$. For each k, let Z_k denote the number of sidelength $1/2^k$ cubes in $[0, 1]^2$. Then $\mathbf{E}[Z_{k+1}|Z_k] = (4p)Z_k$, so by induction we can verify that $\mathbf{E}[Z_k] = (4p)^k$. The sequence Z_k is a branching process, where each cube can produce between zero and four subcubes. The Kesten-Stigum theorem (the main result of [6]) implies that almost surely, there exists a random constant A_0 such that $Z_k \sim A_0(4p)^k$, $\mathbf{E}(A_0) = 1$, and $A_0 = 0$ if and only if it is eventually true that $Z_k = 0$ for sufficiently large k, so $C_k = \varnothing$ eventually. Whenever A_0 is non-zero, we find

$$\dim_{\mathbf{M}}(C) = \lim_{k \to \infty} \frac{\log Z_k}{k \log 2} = \lim_{k \to \infty} \frac{\log(Z_k/A_0(4p)^k) + \log A_0(4p)^k}{k \log 2} = 2 - \log_2(1/p)$$

Since $\mathbf{E}(A_0) \neq 0$, A_0 is non-zero with positive probability. $\square$

We now show similar theory shows that the resulting set Z_0 associated with C almost surely has Minkowski dimension $3 - \alpha$. To do this, we must first prove a lemma about branching processes.

Lemma 7. *Let $Z_0, Z_1, Z_2, \dots$ be a supercritical branching process generated by an offspring law $p_0, \dots, p_N$, with $\sum_{k=0}^N kp_k = E > 1$. Then there exists a random constant A, and a universal constant c such that*

$$\mathbf{P}(Z_k \geqslant A(p_0 + E)^k) \lesssim \exp(-cE^k)$$

Proof. Without loss of generality, we may assume the existence of a grid of independant, identically distributed discrete random variables X_{ij} with $\mathbf{P}(X_{ij} = k) = p_k$ such that

$$Z_{k+1} = \sum_{j=1}^{Z_k} X_{ij}.$$

Now consider the branching process $\{\tilde{Z}_k\}$ defined by setting $\tilde{Z}_0 = 1$, and

$$\tilde{Z}_{k+1} = \sum_{j=1}^{\tilde{Z}_k} \max(X_{ij}, 1).$$

Then $Z_k \leqslant \tilde{Z}_k$, and $\mathbf{E}(\tilde{Z}_{k+1}|\tilde{Z}_k) = p_0 + \sum_{k=1}^N kp_k = p_0 + E$. Applying Theorem 5 of [1] implies that if A_0 is the random constant obtained as in Lemma 6, then there exists a small constant c such that

$$\begin{aligned}\mathbf{P}(Z_k \geqslant (A_0 + 1)(p_0 + E)^k) &\leqslant \mathbf{P}(\tilde{Z}_k \geqslant (A_0 + 1)(p_0 + E)^k) \\ &\lesssim \exp(-c(p_0 + E)^k) \leqslant \exp(-cE^k).\end{aligned}$$ □

We are now able to obtain bounds on the intersection of C with lines in the plane by proceeding along similar lines to the previous argument, together with a chaining argument which enables one to place probability bounds on the intersection properties of C with the infinite set of lines in the plane.

Lemma 8. *If $p > 1/2$, then almost surely, Z_0 associated with C has lower Minkowski dimension at most $3 - O(\log_2(1/p))$.*

Proof. Fix N, and for each index k let $\delta_k = 1/2^{Nk}$. Given a line L, let I be a sidelength δ_k square I intersecting the δ_k thickened line L_{δ_k}. Then L passes from one side of the box to the other, and so if we split I into 4^N sidelength δ_{k+1} squares, then $L_{\delta_{k+1}}$ intersects at most $10 \cdot 2^N$ of these boxes. Conditioned on I being contained in C_k, each of these subboxes occurs in C_{k+N} with probability p^N. We let $Z_k(L)$ denote the number of sidelength $1/2^{Nk}$ boxes in C_{Nk} intersecting L_{δ_k}. Let $N > 3$, so that $10 \cdot 2^N < 4^N$. If we let $\tilde{Z}_k(L)$ denote the number of sidelength $1/2^{Nk}$ boxes in C_{Nk}, but augmented so exactly $10 \cdot 2^N$ subboxes are added at each stage, then $\tilde{Z}_k(L)$ is a branching process with $Z_k(L) \leqslant \tilde{Z}_k(L)$ and $\mathbf{E}[\tilde{Z}_{k+1}(L)|\tilde{Z}_k(L)] = 10 \cdot (2p)^N \tilde{Z}_k(L)$. The offspring law for this branching process is such that a cube produces no subcubes with probability $(1 - p^N)^{10 \cdot 2^N}$, which converges to zero as $N \to \infty$ since $p > 1/2$. Applying Lemma 7, we conclude that there is a random constant A and a small constant c such that

$$\mathbf{P}(Z_k(L) \geqslant A(o(1) + M)^k) \lesssim \exp(-c(2p)^{kN})$$

Now the main result of BIGGINS AND BINGHAM show there are two universal constants c and $\lambda > 1$ such that

$$\mathbf{P}(A \geqslant t) \leqslant \exp(-ct^{\lambda})$$

In particular, $\mathbf{P}(A \geqslant k) \leqslant \exp(-ck^{\lambda})$. Putting these two inequalities together gives

$$\mathbf{P}(Z_k(L) \geqslant k(o(1) + 10(2p)^N)^k) \lesssim \exp(-c(2p)^{kN}) + \exp(-ck^{\lambda}) \lesssim \exp(-ck^{\lambda})$$

For each k, we now find lines $L_{k,1}, \dots, L_{k,M}$ such that for any line L, there exists i such that $[0,1]^d \cap L_{\delta_k/2} \subset (L_{k,i})_{\delta_k}$. If we do this efficiently, if B is a sufficiently large universal constant, we can guarantee that $M \leqslant B \cdot 4^{kN}$. Applying a union bound, we find that

$$\mathbf{P}\left(\text{there is } i \text{ such that } Z_k(L_{k,i}) \geqslant k(o(1) + 10(2p)^N)^k\right) \lesssim B \cdot 4^{kN} \exp(-ck^{\lambda})$$

Since $\lambda > 1$, we have

$$\sum_{k=1}^{\infty} \mathbf{P}\left(\text{there is } i \text{ such that } Z_k(L_{k,i}) \geqslant k(o(1) + 10(2p)^N)^k\right) < \infty$$

Applying Borel-Cantelli, we conclude that almost surely, it is eventually true for sufficiently large k that $Z_k(L_{k,i}) \leqslant k(o(1) + 10(2p)^N)^k$ for all i. In particular, the number of cubes intersecting $L_{\delta_k/2}$ for any line L is bounded by $k(o(1) + 10(2p)^N)^k = O((100 \cdot (2p)^N)^k)$. Taking $N \to \infty$ 'should' prove the claim.

$\square$

6 Relation to Literature, and Future Work

Our result is part of a growing body of work finding general methods to find sets avoiding patterns. The main focus of this section is comparing our method to the two other major results in the literature. [3] constructs sets with dimension $k/(n-1)$ avoiding the zero sets of rank k C^1 functions. In [8], sets of dimension d/l are constructed avoiding a degree l algebraic hypersurface specified by a polynomial with rational coefficients.

We can view our result as a robust version of Pramanik and Fraser's result. Indeed, if we try and avoid the zero set of a C^1 rank k function, then we are really avoiding a dimension $dn - k$ dimensional manifold. Our method gives a dimension

$$\frac{dn - (dn - k)}{n - 1} = \frac{k}{n - 1}$$

set, which is exactly the result obtained in [3].

That our result generalizes [3] should be expected because the technical skeleton of our construction is heavily modeled after their construction technique. Their result also reduces the problem to a discrete avoidance problem. But they *deterministically* select a particular side length S cube in every side length R cube. For arbitrary Z, this selection procedure can easily be exploited for a particularly nasty Z, so their method must rely on smoothness in order to ensure some cubes are selected at each stage. Our discrete avoidance technique was motivated by other combinatorial optimization problems, where adding a random quantity prevents inefficient selections from being made in expectation. This allows us to rely purely on combinatorial calculations, rather than employing smoothness, and greatly increases the applicability of the sets Z we can apply our method to. Furthermore, it shows that the underlying problem is robust to changes in dimension; slightly 'thickening' Z only slightly perturbs the dimension of X.

One useful technique in [3], and its predecessor [5], is the use of a Cantor set construction 'with memory'; a queue in the construction algorithm for their sets allows storage of particular discrete versions of the problem to be stored, and then retrieved at a much later stage of the construction process. This enables them to 'separate' variables in the discrete version of the problem, i.e. instead of forming a single set F from a set E, they from n sets $F_1, \dots, F_n$ from disjoint sets $E_1, \dots, E_n$. The fact that our result is more general, yet does not rely on this technique is an interesting anomaly. An obvious advantage is that the description of the technique is much more simple. But an additional advantage is that we can attack 'one scale' of the problem at a time, rather than having to rely on stored memory from a vast number of steps before the current one. We believe that we can exploit the single scale approach to the problem to generalize our theorem to a much wider family of 'dimension α' sets Z, which we plan to discuss in a later paper.

As a generalization of the result in [3], our result has the same issues when compared to the result of [8]. When the parameter n is large, the dimension of our result suffers greatly, as with the n fold sum application in the last section. Furthermore, our result can't even beat trivial results if Z is almost full dimensional, as the next example shows.

Example. *Consider an α dimensional set of angles Y, and try and find $X \subset \mathbf{R}^2$ such that the angle formed from any collection of three points in X avoids Y. If we form the set*

$$Z = \left\{ (x,y,z) : \text{There is } \theta \in Y \text{ such that } \frac{(x-y)\cdot(x-z)}{|x-y||x-z|} = \cos\theta \right\}$$

Then we can find X avoiding Z. But one calculates that Z has dimension $3d + \alpha - 1$, which means X has dimension $(1-\alpha)/2$. Provided the set of angles does not contain π, the trivial example of a straight line beats our result.

Nonetheless, we still believe our method is a useful inspiration for new techniques in the 'high dimensional' setting. Most prior literature studies sets Z only implicitly as zero sets of some regular function f. The features of the function f imply geometric features of Z, which are exploited by these results. But some geometric features are not obvious from the functional perspective; in particular, the fractional dimension of the zero set of f is not an obvious property to study. We believe obtaining methods by looking at the explicit geometric structure of Z should lead to new techniques in the field, and we already have several ideas in mind when Z has geometric structure in addition to a dimension bound, which we plan to publish in a later paper.

We can compare our randomized selection technique to a discrete phenomenon that has been recently noticed, for instance in [2]. There, certain combinatorial problems can be rephrased as abstract problems on hypergraphs, and one can then generalize the solutions of these problems using some strategy to improve the result where the hypergraph is sparse. Our result is a continuous analogue of this phenomenon, where sparsity is represented by the dimension of the set Z we are trying to avoid. One can even view Lemma 1 as a solution to a problem about independent sets in hypergraphs. In particular, we can form a hypergraph by taking the cubes $\mathcal{B}^d_s(E)$ as vertices, and adding an edge $(I_1, \dots, I_n)$ between n distinct cubes $I_k \in \mathcal{B}^d_s(E)$ if $I_1 \times \cdots \times I_n$ intersects Z_s. An independent set of cubes in this hypergraph corresponds precisely to a set F with F^n disjoint except on a discretization of the diagonal. And so Lemma 1 really just finds a 'uniformly chosen' independent set in a sparse graph. Thus we really just applied the discrete phenomenon at many scales to obtain a continuous version of the phenomenon.

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