Geometry

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Part I

Euclid

I'm writing these notes so that I can understand Euclidean geometry better. We'll build up the axioms from the ground up, so I can understand Euclid's work from the ground up. Thus these notes probably won't be useful for someone trying to understand Euclid themselves, because it's just my ramblings about the subject.

Chapter 1

Book I

Basic Euclidean geometry consists of three objects: Points, Lines (both finite lines with endpoints, an infinite lines with no extremities), and Circles (defined by a point and a radius). Classically, these objects were seen as distinct, but with the power of set theory, it is easier to model lines and circles as sets of points. This has the advantage of making things notationally simple. These is no real logical difference between switching to this notation – any theorem provable in one system is provable in the other. However, we'll avoid from using set theory too much, to avoid making the exposition too austere.

Euclid was the first to pioneer the axiomatic method in mathematics. However, the philosophy behind his proofs was different to ours. At the end of the day, his arguments attack a particular model of the planar geometry find in our world, and he proves things like a physicist, adopted some methods of proof not explicitly stated in his assumptions. This causes problems for us when we try and look at his proofs from a modern day perspective. We will eventually look at other logical systems for geometry, but for now a naive approach will be most useful.

Most of Euclid's proofs concern constructions of certain figures in the planes. Rather than a proof of existence, Euclid literally builds these figures from the ground up. In the early parts of the text these figures will all be defined by a simple curve consisting of straight lines, so that we may describe such a figure by the sequence of points which define the figure. If $X_1, \dots, X_n$ are points, then $X_1 \dots X_n$ will denote the figure obtained by drawing the line X_1X_2, then X_2X_3, and so on, finishing off by drawing X_nX_1. Two such figures will be considered equal if we may obtain the points of one from the points of the other by performing a cycle permutation of the points. For instance, a **triangle** is just a sequence of distinct points ABC, and $ABC = BCA = CAB$, and we can abuse the notation, denoting a line between two points A and B as $AB = BA$. The question of whether this is a unique description of such a line is settled by the first axiom of geometry.

Axiom 1. *There is a unique straight line between any pair of points, having those points as endpoints.*

Euclid does not assume that the straight line which exists between the points is unique, but later he uses the fact that a finite line is defined by its endpoints, so we can only assume that he really wants this fact to hold. In order to discuss the lengths of lines, we shall be required to discuss circles at points, and so we introduce the second axiom.

Axiom 2. *A circle may be described with any centre and radius.*

A circle is *defined* by its centre and radius, so the circle which exists by this axiom is unique. Note that circles with a different radii and the same centre may still be equal. Indeed, this happens exactly when the two radii have the same length, a concept we will very shortly discuss.

Euclid defines an **equilateral triangle** as 'a triangle whose three sides are equal', which he really means as saying the *magnitude*, or length, of the sides are equal. In Euclid's synthetic geometry, there do not exist real numbers to assign length to, and as is well known most Greek's did not even believe in irrational numbers. But we shall find that we can get away with much of the theory of magnitude without ever mentioning the concept of a number, which gives a certain sense of satisfaction.

Right now, we only need equality in the length of lines, and we shall discuss a very aggreeable manner in checking equality. If we have two lines AB and AC with a common point, we can check if they have equal length by checking if the circles constructed with centre A and radii AB and AC are equal. This gives us an equivalence relation on the set of lines extending out from A. We shall require that this equivalence relation describes exactly the set of circles with centre A, so that a point C lies on the circle with centre A and radius AB if and only if the length of AC is equal to the length of AB.

Axiom 3. *If C lies on the circle with radius A and radii AB, then AC has the same length as AB.*

In order to generalize equality of length of arbitrary lines, we just make the relation transitive. The relation is already reflexive and symmetric, so this generates an equivalence relation on the set of all lines in the plane. Thus we see that the only basic way to check if two lines AB and CD are equal is to form a sequence of lines beginning at B, and ending at C, which are all equal to one another as lines extending from the same basepoint.

Theorem 1.1. *Any finite line lies on an equilateral triangle.*

Proof. To prove the existence of an equilateral triangle at a line AB, Euclid constructs the circle with radius AB and centre A, and the circle with centre B and radius AB, and considers their point of intersection C. Since C lies on the first circle, AB has the same length as AC, and since C lies on the second circle, CB has the same length as AB. But then the lines AB, BC, and CA describe an equilateral triangle, and so ABC is the triangle required. $\square$

There is only one problem remaining in this proof. There is nothing saying that the two circles given will have a common point of intersection. We could describe an axiom which supplies us with such a point, but this axiom would probably be more general than the theorem itself. Indeed, the existence of a point on the intersection of two circles with the same radius but different centres is equivalent to the theorem we set out to prove. Thus we shall have to settle on the fact that theorem one must be treated as an assumption from our current viewpoint.

Theorem 1.2. *Given a point A and line BC, to construct a line extending from A with the same length as BC.*

Proof. Construct an equilateral triangle ABD on the line AB. Then construct the circle with centre B and radius BC. Find an intersection point E on the circle which either lies on the line BD, or extends the line, and then construct the circle with centre B and radius BE. Extend the line DA from the extremity A to an intersection point F on the circle. We claim AF has the same length as BC. Indeed, the length of DF is the sum of the length of DA and AF, and the length of DE is the sum of DB and BE. Since the length of DF is equal to DE, since they both lie on the same circle extending from D, and the length of DA is equal to the length of DB, we may subtract to conclude that the length of AF is the same as the length of BE. But BE has the same length of BC, which is all that is required to show AF has the same length as BC. $\square$