# Fractals avoiding Fractal Sets

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# Pattern Avoidance Problems

- How large can the Hausdorff dimension of a Q vector subspace of $\mathbb{R}^n$ be not containing any points in $\mathbb{Q}^n$.
- What is the largest Hausdorff dimension of a subset of $\mathbb{R}^d$ such that the distances between any two points is irrational?
- Given any equation $f$, what is the largest Hausdorff dimension of $X \subset \mathbb{R}^d$ such that for any distinct $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$.

# Two Observations

- Problems can be summarized as finding $X$ such that $X^n$ avoids a given set $Z$, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathrm{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \ldots, n\}\}$$

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  - For any $n$, if $Z_n = \{x \in \mathrm{R}^{nd} : \text{there is } a_1, \dots, a_n \text{ s.t. } a_1 x_1 + \cdots + a_n x_n \in \mathrm{Q}\}$, then $X$ generates a vector space avoiding the rationals if and only if $X^n \cap Z_n \subset \Delta$ for all $n$.

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  - If $Z = \{(x, y) : d(x, y) \in \mathbb{Q}\} \subset \mathbb{R}^{2d}$, then points in $X$ have irrational distances if and only if $X^2 \cap Z \subset \Delta$.
  - If $Z = f^{-1}(0)$, $X$ avoids zeroes of $f$ for distinct values if and only if $X^n \cap Z \subset \Delta$.

# The Generic Problem

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- Pramanik and Fraser (2016): If $Z$ is a smooth hypersurface of dimension $nd - d$, we can find $X$ with dimension $d/(n-1)$.

# Increasing the Difficulty. . .

*What if the Patterns are Fractally Specified...*

# Main Result

**Theorem**

*If $Z$ is the countable union of sets with lower Minkowski dimension bounded by $\alpha \geq d$, we can find $X$ with $X^n \cap Z \subset \Delta$ and*

$$\dim_{\mathrm{H}}(X) = \frac{nd - \alpha}{n - 1} = \frac{codim(Z)}{n - 1}$$

# Low Rank Avoidance

## Theorem

*If we have countably many sets $Z_i \subset \mathbb{R}^{n_i d}$ with linear transformations $T_i : \mathbb{R}^{n_i d} \to \mathbb{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension $\beta_i$, we can find $X$ with $X^{n_i} \cap Z_i \subset \Delta$ for each $i$ and*

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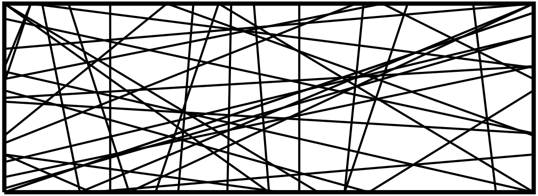
- Want to push the $2k - 1$ to $k - 1$, at least for $k \geq 2$. Know this is true for certain families of examples.

# Advantages of Our Method



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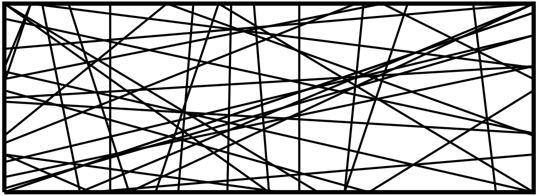

- More robust generalization of Pramanik and Fraser's result, showing that we can 'thicken' or 'thin' $Z$, we get stable effects on the Hausdorff dimension of $X$.
- Uncountable unions of regular sets are allowed!

# Applications

- Given a subset $Y$ of $\mathrm{R}^d$ which is the countable union of sets with Minkowski dimension $\alpha$, we can find a Q vector subspace $X$ of $\mathrm{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from $Y$.

# Applications

- Given a subset $Y$ of $\mathrm{R}^d$ which is the countable union of sets with Minkowski dimension $\alpha$, we can find a $\mathrm{Q}$ vector subspace $X$ of $\mathrm{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from $Y$.
- We can find a full dimensional subset of $\mathrm{R}^d$ avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $x \in X^n, y \in \mathrm{Q}^n$. No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.

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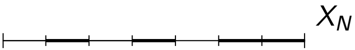
- Random Disection.

# Discretization of Scales




- Construct $X$ as a Cantor set by limits of interval sets $X_N$.

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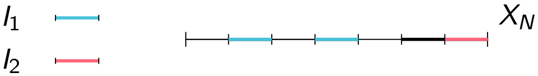


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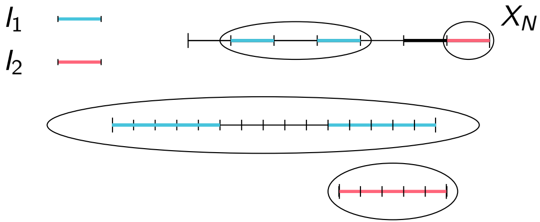


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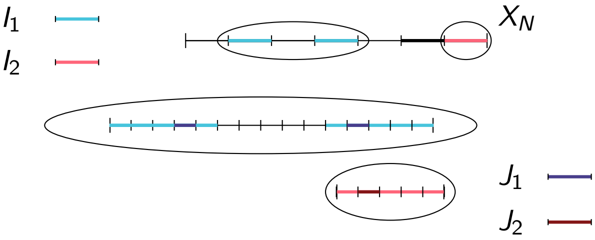


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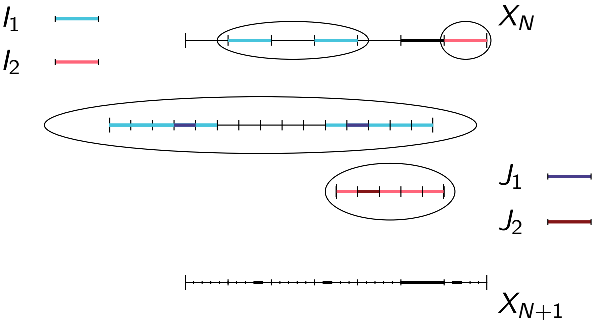


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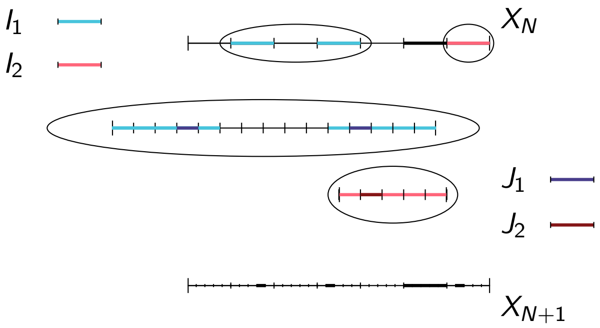


- Construct $X$ as a Cantor set by limits of interval sets $X_N$.
- **Discrete Problem**: Given disjoint unions of length $L$ intervals $I_1, \ldots, I_n$, find $J_i$ for each $i$ containing a part of each interval in $I_i$ such that $J_1 \times \cdots \times J_n$ is disjoint from $Z$.

# Queueing

- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set $X$ gives a fractal avoiding set $X$.

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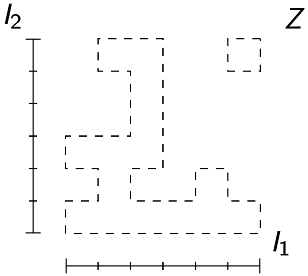
- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set $X$ gives a fractal avoiding set $X$.
- **Reason**: If we consider distinct $x_1, \dots, x_n \in X$, there are intervals $I_1, \dots, I_n$ with $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \cdots \times J_n$ cannot be contained in $Z$.

# Exploiting Randomness

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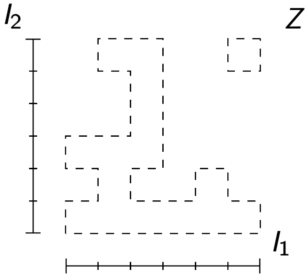


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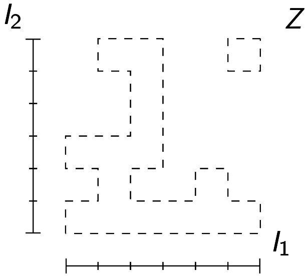


- How do we prove the discrete scale argument?
- Aside from $Z$'s dimension, we have little structural knowledge.
- Random choices of the $J_k$ avoid $Z$ effectively.
- We essentially obtain for all but a fraction $o(1)$ of the length $L$ intervals in $I_k$, $J_k$ contains a length $L^\beta$ section of each, where $\beta = d(n-1)/(nd-\alpha)$. This ratio gives the Hausdorff dimension bound $(nd-\alpha)/(n-1)$ for $X$.

# Conclusion

So What's Next?

# Extension to Hausdorff Dimension

- There is no obvious reason why our techniques should fail when $Z$ has *Hausdorff dimension* $\alpha$ rather than a Minkowski dimension bound.

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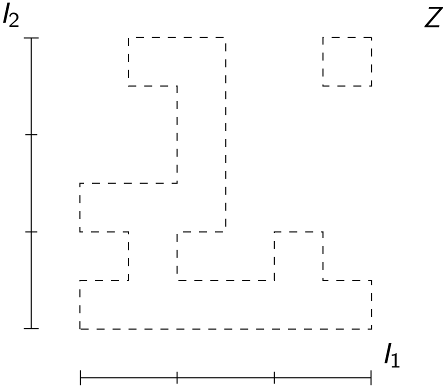
- We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

# Analogies with Hypergraphs



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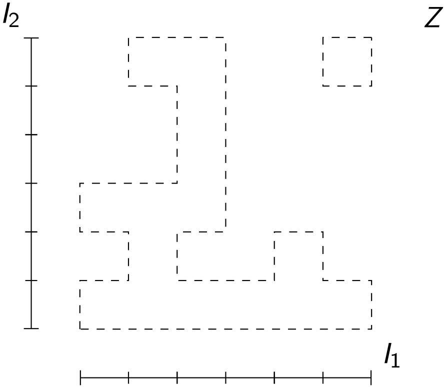

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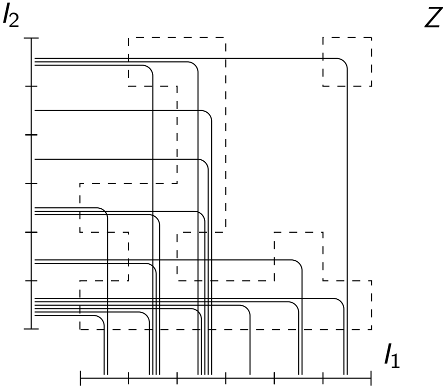

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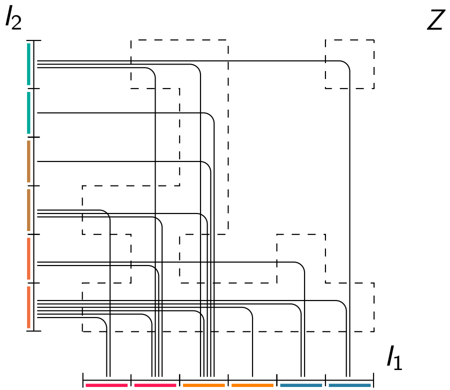

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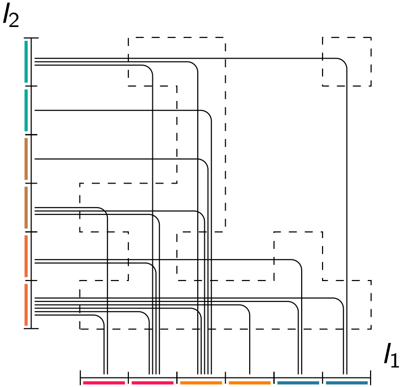


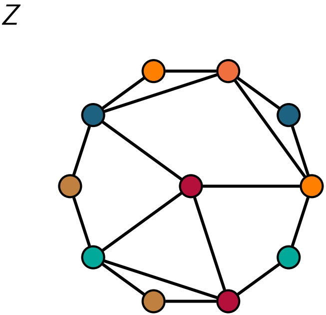


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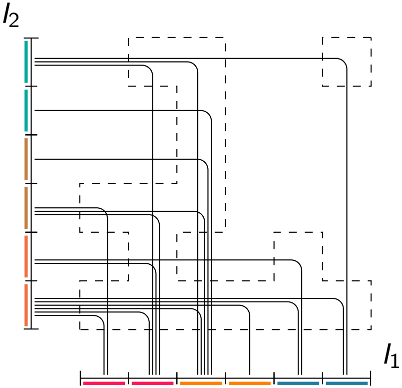


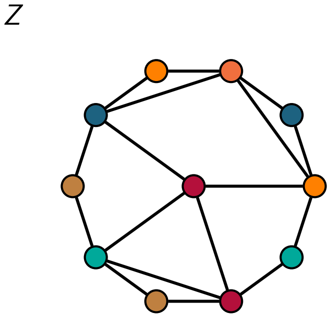


- The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- We are looking to using other methods on hypergraphs to improve the bound when $Z$ has certain structural properties (like for the low rank result).

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