Math 340 Worksheet 1

The following are select solutions from the worksheet.

Exercise 1.
If $\det(A) = 3$ and $\det(B) = -1$, compute $\det(A^3B^2A(A^T)^{-2}B)$.

Solution 1.
Using the multiplicative property of the determinant, we find that

$$\begin{aligned}\det(A^3B^2A(A^T)^{-2}B) &= \det(A)^3\det(B)^2\det(A)\det(A^T)^{-2}\det(B)\\ &= \det(A)^4\det(A^T)^{-2}\det(B)^3.\end{aligned}$$

Using the fact that $\det(A^T) = \det(A)$, we find that

$$\det(A)^4\det(A^T)^{-2}\det(B)^3 = \det(A)^4\det(A)^{-2}\det(B)^3 = \det(A)^2\det(B)^3.$$

Now pluggin in the values above, we get that

$$\det(A)^2\det(B)^3 = 3^2(-1)^3 = -9.$$

Exercise 2.
Compute the determinant of the 5×5 matrix

$$A = \begin{pmatrix} 3 & -1 & 1 & -3 & 2 \\ 1 & -4 & -1 & 4 & 3 \\ -3 & 2 & 3 & 0 & -4 \\ -3 & 3 & -2 & -4 & 0 \\ -3 & 1 & -1 & 3 & 4 \end{pmatrix}$$

You may use the following facts:

$$\begin{vmatrix} -4 & -1 & 4 & 3 \\ 2 & 3 & 0 & -4 \\ 3 & -2 & -4 & 0 \\ 1 & -1 & 3 & 4 \end{vmatrix} = -67 \quad \begin{vmatrix} -1 & 1 & -3 & 2 \\ 2 & 3 & 0 & -4 \\ 3 & -2 & -4 & 0 \\ 1 & -1 & 3 & 4 \end{vmatrix} = 354 \quad \begin{vmatrix} -1 & 1 & -3 & 2 \\ -4 & -1 & 4 & 3 \\ 3 & -2 & -4 & 0 \\ 1 & -1 & 3 & 4 \end{vmatrix} = -294$$

$$\begin{vmatrix} -1 & 1 & -3 & 2 \\ -4 & -1 & 4 & 3 \\ 2 & 3 & 0 & -4 \\ 1 & -1 & 3 & 4 \end{vmatrix} = 300 \quad \text{and} \quad \begin{vmatrix} -1 & 1 & -3 & 2 \\ -4 & -1 & 4 & 3 \\ 2 & 3 & 0 & -4 \\ 3 & -2 & -4 & 0 \end{vmatrix} = 5$$

Solution 2.
Using the cofactor expansion, we have

$$\det(A) = (-67) - (354) + (-294) - (300) + (5) = -67 - 354 - 294 - 300 + 5 = -1010.$$

Exercise 3.
Use determinants to compute the area of the triangle with points $(1,2)$, $(-1,3)$, and $(0,-1)$. How about the area of the five sided polygon with vertices $(1,0)$, $(1,3)$, $(-1,4)$, $(-2,0)$, and $(0,-1)$ (Hint: Break things into smaller triangles and apply the techniques of the last part of the question).

Solution 3.
To find the first area, we use the fact that the area of the triangle is half the absolute value of the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The determinant of this matrix is 7, so the area of the triangle is $3/2$.

To find the second area, we note that $(0,0)$ is an interior point of the polygon. The interior of the polygon thus decomposes into the five triangles, the first with vertices $(0,0)$ $(1,0)$, and $(1,3)$, the second with vertices $(0,0)$ $(1,3)$, $(-1,4)$, the third with vertices $(0,0)$, $(-1,4)$, $(-2,0)$, the fourth with vertices $(0,0)$, $(-2,0)$, $(0,-1)$, and the fifth with vertices $(0,0)$, $(0,-1)$, $(1,0)$. The advantage of using $(0,0)$ as a vertex is that the determinants simplify to 2×2 determinants. In general, given a triangle consisting of three vertices $(0,0)$, (a,b), and (c,d), the area of the triangle is half the absolute value of the determinant of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ a & b & 1 \\ c & d & 1 \end{pmatrix}$$

Using the cofactor expansion *along columns* (which is equivalent to cofactor expansion along rows because – think over why this connects – the determinant of a matrix is the same as the determinant of it's transpose), we conclude that

$$\begin{vmatrix} 0 & 0 & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix} = 0\cdot\begin{vmatrix} b & 1 \\ d & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 0 & 1 \\ c & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Thus we find that the area is $(1/2)|ad-bc|$. The areas of the triangles above, in order, are $3/2$, $1/2$, 4, 1, and 1, and summing up these areas gives that the area of the overall polygon is 8.

Exercise 4.
Given $n\times n$ matrices A, B and C, decide if the following assertions are true or false, justify your answer.

1. $\det(A+B) = \det(A) + det(B)$.
2. If $\det(AB) = 0$, then $\det(A) = 0$ or $\det(B) = 0$.
3. If A is nonsingular and $A = A^{-1}$, then $\det(A) = 1$.

4. If $AB = AC$ and $\det(A) \neq 0$, then $B = C$.

Solution 4.

1. False, take $n = 2$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. we have $\det(A + B) = 1$ but $\det(A) + \det(B) = 0$.
2. True, just notice that $\det(AB) = \det(A)\det(B) = 0$.
3. False, Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We have $A = A^{-1}$ but $\det(A) = -1$.
4. True, if $\det(A) \neq 0$ then A is nonsingular. Multiplying at the left by A^{-1} we get $B = A^{-1}AB = A^{-1}AC = C$.