Sets, Patterns, and Fourier Decay

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Fourier Analysis and Patterns in Sets

- What can one learn about the geometry of a compact set $E \subset \mathbb{T}^d$ via analytical properties of probability measures μ supported on E?
- A set E has *Minkowski dimension* s if $|N_\delta(E)| \lesssim \delta^{d-s}$.
- A set E has *Hausdorff dimension* s if for any $t < s$, E supports a probability measure μ_t with

  $$\sum_{k \neq 0} |\widehat{\mu_t}(k)|^2 |k|^{t-d} < \infty.$$

  Very similar to Minkowski dimension, but 'multiscale'.
- A set has *Fourier Dimension* s if it supports μ_t with $|\widehat{\mu_t}(k)| \lesssim |k|^{-t/2}$ for *all* n.
- $\dim_{\mathbb{F}}(E) \leq \dim_{\mathbb{H}}(E) \leq \dim_{\mathbb{M}}(E)$.

Pattern Avoidance

- If $\dim(E)$ is large, does E 'contain patterns'.
- Basic Example: If $\dim(E)$ is large, are there $m_1, \dots, m_n \in \mathbb{Z}$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$? (Can large sets be linearly independent over $\mathbb{Q}$)
- (Keleti, 1999) There is $E \subset \mathbb{T}$ with $\dim_{\mathbb{H}}(E) = 1$ such that for any $m_1, \dots, m_n$ and distinct $x_1, \dots, x_n \in E$, $m_1 x_1 + \dots + m_n x_n \neq 0$.
- If $\dim_{\mathbb{F}}(E) > 0$, there is n, $m_1, \dots, m_n \in \mathbb{Z}$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$.
  - ($E + \dots + E$ actually contains an interval for some large sum)
  - Consider μ with $\text{supp}(\mu) \subset E$ and $|\widehat{\mu}(k)| \lesssim |k|^{-\varepsilon}$.
- If $\dim_{\mathbb{F}}(E) > 2/n$, then there are $m_1, \dots, m_n$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$.

Independent Sets

- (Rudin, 1960): There exists $E \subset \mathbb{T}$ and a finite Borel measure μ with $\mathrm{supp}(\mu) \subset E$ such that E is independent but $|\widehat{\mu}(k)| \to 0$ as $|k| \to \infty$.
- (Körner, 2007): There exists independent E supporting measures converging to zero as 'fast as possible'.
- (Körner, 2009): There exists $E \subset \mathbb{T}$ with $\dim_{\mathbb{F}}(E) = 1/(n-1)$ such that E avoids solutions to all n-term linear equations.

Arithmetic Progressions ($x_1 - 2x_2 + x_3 = 0$)

- (Łaba and Pramanik, 2007): For some small $\varepsilon > 0$, if $|\widehat{\mu}(k)| \leq C_1|k|^{-(1-\varepsilon)/2}$ and $\mu((x, x+r)) \leq C_2 r^\alpha$ for appropriate C_1, C_2, and α, supp(μ) contains arithmetic progressions.
- (Schmerkin, 2015): There is $E \subset \mathbb{T}$ avoiding arithmetic progressions with $\dim_{\mathbb{F}}(E) = 1$.
- (Liang and Pramanik, 2020): Generalized Schmerkin's construction to all translation-invariant patterns.

Fourier Dimension and Nonlinear Patterns

- (Henriot and Łaba and Pramanik, 2015): For certain linear maps $A_1, \dots, A_n$ and polynomials Q, there is $\varepsilon > 0$ such that if $E \subset \mathbb{T}$ and $\dim_{\mathbb{F}}(E) \geq 1 - \varepsilon$, E contains a family of points of the form

$$\{x, x + A_1 y, \dots, x + A_{n-1} y, x + A_n y + Q(y)\}.$$

  The pattern $\{x, x+t, x+t^2\}$ is *not* covered.
- (Fraser and Guo and Pramanik, 2019): If $\deg(f) > 1$ and $f(0) = 0$, then patterns of the form $\{x, x+t, x+f(t)\}$ exist in $\mathrm{supp}(\mu)$ if μ satisfies explicit estimates ala Łaba and Pramanik.
- (Kuca, Orponen, Sahlsten, Preprint 2021): If $E \subset \mathbb{T}^2$ and $\dim_{\mathbb{H}}(E) \geq 2 - \varepsilon$, then E contains solutions to $y_2 - x_2 = (y_1 - x_1)^2$ for distinct $x, y \in E$.

Sets Avoiding Nonlinear Patterns for Hausdorff Dimension

- Find large $E \subset \mathbb{T}^d$ such that for distinct $x_1, \dots, x_n \in E$,

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

Author	**Property of f**	$\dim_{\mathbb{H}}(X)$
Mathé (2017)	A degree r polynomial	d/r
Fraser Pramanik (2018)	f is C^1	$m/(n-1)$
D. Pramanik Zahl (2020)	f Lipschitz	$m/(n-1)$
D. (2020)	$f = g \circ \pi$ where the linear map $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$ is is surjective	$1/(m-1)$

- Can we modify these constructions to obtain Salem sets?

Main Result

Theorem

Suppose $f(x_1, \ldots, x_{n-1})$ *is* C^{d+1}, *and for each* $1 \leq i \leq n-1$,

$$D_{x_k} f = \left(\frac{\partial f_i}{\partial x_{kj}} \right)$$

is invertible. Then there exists $E \subset \mathbb{T}^d$ *with*

$$dim_{\mathbb{F}}(E) = \frac{d}{n - 3/4}$$

avoiding solutions to the equation $x_n = f(x_1, \ldots, x_{n-1})$.

- (Fraser and Pramanik, 2016) obtains a set $E \subset \mathbb{R}$ with

$$\dim_{\mathbb{H}}(E) = \frac{d}{n-1}.$$

Linear Result

Theorem

Suppose f is Lipschitz. Then there exists $E \subset \mathbb{T}^d$ with

$$dim_{\mathbb{F}}(E) = \frac{d}{n-1}$$

avoiding solutions to the equation

$$x_n - x_{n-1} = f(x_1, \ldots, x_{n-2}).$$