Number Theory

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Chapter 1

Generating Functions

Example. *Suppose we are working in a country with only a one, a two, and a three penny coin. Given an integer n, let $r(n)$ denote the number of ways that a person can be paid n pennies using these three coins. Since this is a question about the additivity of numbers, we can likely understand it using generating functions. Formally,*

$$r(n) = \#\{(a,b,c) : a + 2b + 3c = n\}$$

We note

$$\left(\sum_{a=0}^{\infty} z^a\right)\left(\sum_{b=0}^{\infty} z^{2b}\right)\left(\sum_{c=0}^{\infty} z^{3c}\right) = \sum_{a,b,c} z^{a+2b+3c} = \sum_{n=0}^{\infty} r(n) z^n$$

Thus, for $|z| < 1$,

$$\sum_{n=0}^{\infty} r(n) z^n = \frac{1}{(1-z)(1-z^2)(1-z^3)}$$

We can now perform a partial fraction decomposition, writing

$$\frac{1}{(1-z)(1-z^2)(1-z^3)} = \frac{1}{(1-z)^3(1+z)(\omega - z)(\omega + z)}$$

where $\omega = e(1/3)$ is a primitive third root of unity. Some intense linear algebra shows this is equal to

$$\frac{z+2}{9(z^2+z+1)} + \frac{17z^2 - 52z + 47}{72(1-z)^3} + \frac{1}{8(1+z)}$$

which can be further decomposed into

$$
\begin{aligned}
&-\frac{\omega^2+3\omega+2}{9(1-z/\omega)}+\frac{\omega^2-\omega+2}{9(1-z/\omega^2)} \\
&\quad+\frac{1}{6(1-z)^3}+\frac{1}{4(1-z)^2}+\frac{17}{72(1-z)}+\frac{1}{8(1+z)}
\end{aligned}
$$

where $\omega = e(1/3)$. *Taking power series and summing up, we find*

$$
\begin{aligned}
r(n) &= -\frac{\omega^2+3\omega+2}{9\omega^n}+\frac{\omega^2-\omega+2}{9\omega^{2n}}+\frac{(n+1)(n+2)}{12}+\frac{n+1}{4}+\frac{17}{72}+\frac{(-1)^n}{8} \\
&= \frac{6n^2+36n+47+9(-1)^n}{72}+\begin{cases} 0 & n \equiv 0 (mod\ 3) \\ -2/9 & n \equiv 1 \text{ or } 2 (mod\ 3) \end{cases} \\
&= \frac{(n+3)^2}{12}+\frac{9(-1)^n-7}{72}+\begin{cases} 0 & n \equiv 0 (mod\ 3) \\ -16/72 & n \equiv 1 \text{ or } 2 (mod\ 3) \end{cases}
\end{aligned}
$$

We know $r(n)$ *is an integer, and since*

$$
\frac{9+7+16}{72}=\frac{32}{72}<\frac{1}{2}
$$

So $r(n)$ *is the closest integer to* $(n+3)^2/12$.