Algebraic Topology

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Chapter 1

Homotopy

1.1 Deformations

Verifying two topological spaces are homeomorphic is a fairly easy ordeal. One needs only find a single homeomorphism between them. The converse, verifying two topological spaces are not homeomorphic, is much more tricky; we need to show that *every* function from one space to the other is not a homeomorphism. One trick is to find fundamental topological properties which distinguish two topological spaces. Connectedness, Compactness, and Hausdorffiness are all preserved by homeomorphism, so two spaces in which these properties differ cannot be homeomorphic. Algebraic topology consists of deep techniques to distinguish topological spaces.

It shall turn out that most interesting spatial invariants are also invariant under a type of topological equivalence more general that homeomorphism. Consider two functions f and g between topological spaces X and Y[1]. Though f might not be equal to g, they may be in some sense topologically equal – we can continuouslly deform one to the other. Two continuous functions $f : X \to Y$ and $g : X \to Y$ are **homotopic**, written $f \simeq g$, if there exists a map $H : [0,1] \times X \to Y$, such that for all $x \in X$,

$$H_0(x) = f(x) \qquad H_1(x) = g(x)$$

We write the image of (t,x) under H as $H_t(x)$. H is a homotopy between f and g. Thus we may see a homotopy as a family of maps $H_t : X \to Y$

[1]From now on, we shall assume all functions continuous.

which vary continuously with respect to t. H is a homotopy **relative to** A if $H_t(a) = a$ for all $a \in A$. If f is homotopic to g relative to A, we write $f \simeq_A g$.

Category provides a unifying tool in algebraic topology. We aim to associate 'invariant algebraic objects' to each element of a topological space, and this is most naturally explained as a functor between categories. Category theory was invented by Samuel Eilenberg and Saunders Maclane to form the foundations of abstract homology theory. Our first category is formed from the category **Top** of topological spaces together with continuous maps. If we identify homotopic continuous maps, we obtain the homotopy quotient category **Toph**. The category of pairs of topological spaces (X, A), with $A \subset X$, and whose morphisms from (X, A) to (Y, B) are equivalence classes of maps from X to Y which are identified by homotopies relative to A. We call this category **RelToph**.

Example. *Any two $\mathbf{R}^n$-valued functions f and g defined on the same domain are homotopic. The map*

$$H_t(x) = tf(x) + (1-t)g(x)$$

is a homotopy between f and g. Thus the spaces $\mathbf{R}^n$ are the terminal objects in the homotopy category, since there is a unique homotopy class of functions from $\mathbf{R}^n$ to any other topological space.

Two spaces are **homotopy equivalent** if they are isomorphic in **Toph**; that is, X is homotopic to Y, written $X \simeq_h Y$ if there is a map $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identities on X and Y. g is known as the **homotopy inverse** of f. Since each $\mathbf{R}^n$ is a terminal object, each is homotopy equivalent to the other, by abstract nonsense. A map $f : X \to Y$ is a homotopy inverse relative to A if $A \subset X, Y$, and f is an isomorphism between (X, A) and (Y, B) in the category **RelToph**. In particular, we require

Example. *The n-sphere S^n is homotopy equivalent to $\mathbf{R}^{n+1} - \{0\}$. The embedding $i : S^n \to \mathbf{R}^{n+1} - \{0\}$ has a homotopy inverse $j : \mathbf{R}^{n+1} - \{0\} \to S^n$, defined by*

$$j(v) = v/\|v\|$$

One deforms the norm to show that $i \circ j$ is homotopic to $id_{\mathbf{R}}$. We take

$$H_t(x) = \left(\frac{t}{\|x\|} + (1-t) \right) x$$

as the required homotopy. The other end is trivial, since $j \circ i = id_{S^n}$.

A retraction is a map $r : X \to X$ for which $r^2 = r$. If $Y = \text{im}(X)$, and r is a homotopy isomorphism, then we say X **deformation retracts** to Y. The deformation retraction is **strong** if $X \simeq_Y Y$. We showed that $\mathbf{R}^{n+1} - \{0\}$ has a strong deformation retract onto S^n. A space is **contractible** or **null-homotopic** if it is deformation retracts to a point.

Example. *A subset X of a vector space is* **star-shaped** *if there is $x \in X$ such that if $y \in X$, the line segment between x and y is contained in X. Then X is (strongly) contractible to $\{x\}$, by the map*

$$H_t(y) = ty + (1-t)x$$

which shows every star shaped space is final.

Example. *A* **cycle** *in a graph Γ is a sequence of distinct vertices $v_1, \dots, v_n$, together with distinct edges $e_1, e_2, \dots, e_n$ such that e_i connects v_i and v_{i+1}, and e_n connects v_n and v_1. A* **tree** *is a connected graph with no cycles. Consider any particular tree Γ, and in that tree fix a vertex v. For any vertex w, there is then a unique path $(w, k_1, \dots, k_{n_w}, v)$ to v with edges $e_0, e_1, \dots, e_{n_w}$. Identify the edge e_i with its parameterization by the interval $[0,1]$, in the direction which leads to v. Take $m = max_w(n_w)$ to be the longest path length. We identity a topological path from w to v by a map which travels at a unit velocity. This can be described cryptically by*

$$c_w : [0, \infty) \to \Gamma \qquad c_w(t) = \begin{cases} e_{\lfloor t \rfloor}(t - \lfloor t \rfloor) & t < n_w + 1 \\ v & t \geqslant n_w + 1 \end{cases}$$

So that c_w moves from w to v at a unit velocity. Define

$$H^w : [0, m] \times (w, k_1, \dots, k_n, v) \to \Gamma \qquad H^w_t(c_w(u)) = c_w(t+u)$$

If w and u are vertices, then H^w agrees with H^u on the intersection of their domain, so we may put all the maps together to obtain a strong deformation retraction from Γ onto $\{v\}$, hence Γ is contractible.

1.2 Homotopy Extensions

We would like to make it easy to verify homotopy equivalence. Most theorems of this variety rely on a useful property. A tuple (X, A), with A a subspace of X, satisfies the **homotopy extension property** if, given any homotopy $H : [0,1] \times A \to Y$ between f and g, and given an extension $\tilde{f}$ of f to X, there is an extended homotopy $\widetilde{H} : [0,1] \times X \to Y$ between $\tilde{f}$ and some extension of g. More succinctly, (X,A) has the homotopy extension property if every map defined on $X \times \{0\} \cup A \times [0,1]$ extends to a map defined on $X \times [0,1]$.

Lemma 1.1. *(X,A) has the homotopy extension property if and only if there is a retraction from $X \times [0,1]$ onto $X \times \{0\} \cup A \times [0,1]$.*

Proof. If (X,A) has the homotopy extension property, one obtains a retract of $X \times [0,1]$ onto $X \times \{0\} \cup A \times [0,1]$ by extending the identity map on $X \times \{0\} \cup A \times [0,1]$. Conversely, if we have a homotopy H between the identity of id_X and a retract r onto $X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$, then every map $f : X \times \{0\} \cup A \times [0,1] \to Y$ extends to a map $f \circ r : X \times [0,1] \to Y$. $\square$

Corollary 1.2. *If (X,A) has the extension property, then $(X \times Z, A \times Z)$ has the homotopy extension property.*

Proof. If $r : X \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ is a retract, then we obtain a retract

$$\begin{aligned} X \times Z \times [0,1] &\cong (X \times [0,1]) \times Z \\ &\xrightarrow{r \times \mathrm{id}_Z} (X \times \{0\} \cup A \times [0,1]) \times Z \\ &\cong X \times Z \times \{0\} \cup A \times Z \times [0,1] \end{aligned}$$

Thus $(X \times Z, A \times Z)$ has the extension property. $\square$

Corollary 1.3. *If (X,A) has the extension property, then $(X \coprod B, A \coprod B)$ has the homotopy extension property.*

Proof. A homotopy between two functions from $A \coprod B$ to Y can be seen as two separate homotopies, one between functions from A to Y, and one from functions from B to Y. The first homotopy extensions to a homotopy from X to Y, which may be combined with the second function to form a homotopy between $X \coprod B$ in Y. Thus $(X \coprod B, A \coprod B)$ has the extension property. $\square$

Theorem 1.4. *If (X,A) has the homotopy extension property, and X is hausdorff, then A is closed in X.*

Proof. Given a map $f : A \to A$ between topological spaces, the set of x such that $f(x) = x$ is closed, for it is the inverse image of

$$\Delta = \{(x,x) : x \in A\}$$

under the map $f \times \mathrm{id}_A : A \to A \times A$, and Δ is closed in $A \times A$ if A is Hausdorff. A retraction r from $X \times [0,1]$ to $X \times \{0\} \cup A \times [0,1]$, then $X \times \{0\} \cup A \times [0,1]$ is closed in $X \times [0,1]$. But $A \times \{1\}$ is closed in $X \times \{0\} \cup A \times [0,1]$, so it is closed in $X \times [0,1]$, and this implies it is closed in $X \times \{1\}$, so A is closed in X. $\square$

Example. *Let $X = [0,1]$, and $A = \{1, 1/2, 1/3, \dots\}$. Suppose we had a retract*

$$r : [0,1] \times [0,1] \to [0,1] \times \{0\} \cup A \times [0,1]$$

Then $r(0,x) = \lim_{t\to\infty} r(1/t, x) = \lim_{t\to\infty}(1/t, x) = (0,x)$, for all $x \in X$. This is clearly impossible. Thus (X,A) does not have the homotopy extension property.

The next lemma sounds complicated. It is best to see it in diagram form, for then it becomes quite simple.

Lemma 1.5. *Let $A \subset X$, and suppose there is a map $f : Z \to A$, and a homeomorphism h from M_f onto a closed neighbourhood N of A in X with $h([a]) = a$ for all $a \in A$, and $h(M_f - [Z \times \{0\}])$ an open neighbourhood of A. Then (X,A) has the homotopy extension property.*

Proof. First note $([0,1], \{0,1\})$ has the homotopy extension property, since we have a retraction from $[0,1]^2$ to $[0,1] \times \{0\} \cup \{0,1\} \times [0,1]$, obtained by projection from $(0,2)$. Thus $((Z \times [0,1]) \coprod A, (Z \times \{0,1\}) \coprod A)$ has the homotopy extension property, and consider the particular retraction

$$((Z \times [0,1]) \coprod A) \times [0,1] \xrightarrow{r} ((Z \times [0,1]) \coprod A) \times \{0\} \cup ((Z \times \{0,1\}) \coprod A) \times [0,1]$$

Let $\pi : ((Z \times [0,1]) \coprod A) \times [0,1] \to M_f \times [0,1]$ be the projection map onto the quotient. Then

$$(\pi \circ r)(z, 1, t') = [z, 1, t'] = [f(z), t'] = (\pi \circ u)(f(z), t')$$

inducing a map

$$u : M_f \times [0,1] \to M_f \times [0,1]$$

which is a retraction, since u is the unique map making the diagram below commute

$$\begin{array}{ccc} ((Z\times[0,1])\coprod A)\times[0,1] & \xrightarrow{r} & ((Z\times[0,1])\coprod A)\times\{0\}\cup((Z\times\{0,1\})\coprod A)\times[0,1] \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M_f\times[0,1] & \xrightarrow{\quad u \quad} & M_f\times[0,1] \end{array}$$

and since $r^2 = r$, u^2 also makes the diagram commute. We retract onto

$$\begin{aligned} &\pi(((Z\times[0,1])\coprod A)\times\{0\}\cup((Z\times\{0,1\})\coprod A)\times[0,1]) \\ &\quad = M_f\times\{0\}\cup\pi(((Z\times\{0\})\coprod A)\times[0,1])) \end{aligned}$$

this implies $(M_f, \pi(Z\times\{0\}\cup A))$ has the homotopy extension property. By homeomorphism, $(N,(h\circ\pi)(Z\times\{0\}\cup A)) = (N,(h\circ\pi)(Z\times\{0\})\cup A)$ also has the homotopy extension property. Let $\widetilde{Z} = (h\circ\pi)(Z\times\{0\})$.

Consider any map $g : X \to Y$ and a homotopy $H : [0,1]\times A \to Y$ with $g|_A = H_0$. Let $K : [0,1]\times(X-N)\cup\widetilde{Z}$ be the constant homotopy on $(X-N)\cup\widetilde{H}$. Then $(H\cup K)|_N$ is a continuous map (since the domain of both sets is closed and disjoint) defined on $[0,1]\times\widetilde{Z}\cup A$, and therefore extends to a homotopy $G : [0,1]\times N \to Y$. G agress with K on the domain of definition, so G extends to $\widetilde{G} : [0,1]\times X \to Y$. Clearly $\widetilde{G}$ is the homotopy we wanted. □

Theorem 1.6. *If (X,A) is a CW pair, then (X,A) has the homotopy extension property.*

Proof. We prove that $X\times\{0\}\cup A\times[0,1]$ is actually a deformation retraction of $X\times[0,1]$. There is an easy deformation retraction

$$r : \mathbf{D}^n\times[0,1] \to \mathbf{D}^n\times\{0\}\cup\partial\mathbf{D}^n\times[0,1]$$

obtained by sliding down the stereographic projection over a time interval. Thus $(\mathbf{D}^n, \partial\mathbf{D}^n)$ has the homotopy extension property. We may combine these deformations to obtain a map from $X_n\times[0,1]$ to $X_n\times\{0\}\cup(X_{n-1}\cup A_n)\times[0,1]$. Let H_n be the deformation retraction on a shortened interval $[1/2^{n+1}, 1/2^n]$. We may put all the H_n's together to form a homotopy $H : X\times[0,1] \to X\times\{0\}\cup A\times[0,1]$, by inductively contracting each H_n, and then leaving higher dimensions alone. H is continuous on each skeleton $X_n\times[0,1]$, so it is continuous on the whole skeleton by the weak topology on X. Thus (X,A) has the homotopy extension property. □

Theorem 1.7. *If (X, A) has the homotopy extension property and A is contractible, then the quotient map $\pi : X \to X/A$ is a homotopy equivalence.*

Proof. Let $H : [0,1] \times X \to X$ be an extension of the homotopy between the contraction of id_A to a point, with $H_0 = \mathrm{id}_X$. Then, since $H_t(A) \subset A$ for all t, $(\pi \circ H)(a,t) = (\pi \circ H)(a',t)$ for all $a, a' \in A$, $t \in [0,1]$. Thus H is perturbed to a homotopy on the quotient

$$G : [0,1] \times X/A \to X/A$$

where G_t satisfies the diagram.

$$\begin{tikzcd} X \arrow[r, "H_t"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ X/A \arrow[r, "G_t"] & X/A \end{tikzcd}$$

Since $H_1 : X \to X$ maps A to a point, we obtain a function $g : X/A \to X$ satisfying the commutative diagram

$$\begin{tikzcd} X \arrow[r, "H_1"] \arrow[d, "\pi"] & X \\ X/A \arrow[ur, "g"] & \end{tikzcd}$$

and by combining the two diagrams, $\pi \circ g \circ \pi = \pi \circ H_1 = G_1 \circ \pi$, so by the uniqueness of G_1, $g \circ \pi = G_1$. Thus G gives us a homotopy between $g \circ \pi$ and the identity. On the other end, $\pi \circ g$ is homotopic to the identity by the map H, by using the triangle above. $\square$

Theorem 1.8. *If (X, A) and (Y, A) satisfy the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence with $f|_A = id_A$, then f is a homotopy equivalence relative to A.*

Theorem 1.9. *If (X_1, A) is a CW pair, and $f, g : A \to X_0$ are homotopic attaching maps, then*

$$X_0 \coprod_f X_1 \simeq X_0 \coprod_g X_1$$

by homotopy equivalences which restrict to the identity on X_0, by a homotopy relative to X_0.

Proof. UNFINISHED $\square$

One may visualize homotopy for lower dimensional spaces in the following way. Let $f : X \to Y$ be a continuous map. Define the **mapping cylinder**

$$M_f = \Big[(X \times [0,1]) \coprod Y\Big] / ((x,1) \sim f(x))$$

where $(x,1) \sim f(x)$.

Lemma 1.10. *If $f : X \to Y$ is a homotopy equivalence, and if π is the projection onto the quotient, then M_f retracts to both $\pi(X \times \{0\}) = \tilde{X}$ and $\pi(Y) = \tilde{Y}$.*

Proof. Surely M_f retracts onto Y by sliding down X, regardless of whether f is a homotopy equivalence. Let $r([x,t]) = [f(x)]$, $r([y]) = [y]$ be the retraction. Let $g : Y \to X$ be a homotopy inverse for f. Let $H : X \times [0,1] \to X$ be a homotopy between id_X and $g \circ f$. Then

$$H(x,1) = (g \circ f)(x) = g(f(x))$$

Since H agrees with g on the quotient of M_f, we induce a function

$$\tilde{H} : M_f \to X \to X \times \{0\}$$

This is a retraction onto $\tilde{X}$, since $H(x,0) = x$. Let $K : Y \times [0,1] \to Y$ be a homotopy between id_Y and $f \circ g$. Consider the map

$$G([x,s],t) = [K(f(x),t)] \qquad G([y],t) = [K(y,t)]$$

G is a homotopy between r and

$$k([x,s]) = G([x,s],1) = [K(f(x),1)] = [(f \circ g \circ f)(x)]$$

$$k([y]) = G([y],1) = [(f \circ g)(y)]$$

□

Thus two spaces are homotopic if and only if they are both deformation retracts of a bigger space.

Chapter 2

Fundamental Groups

2.1 The Fundamental Groupoid

In this chapter, we use homotopy to find a useful algebraic structure describing spaces. Two paths $\lambda : [0,1] \to X$ and $\gamma : [0,1] \to X$ are **path homotopic** if $\lambda(0) = \gamma(0)$, $\lambda(1) = \gamma(1)$, if there is a homotopy $\{\mu_t : [0,1] \to X\}$ between λ and γ such that $\mu_t(0) = \lambda(0)$, $\mu_t(1) = \lambda(1)$ for all $t \in [0,1]$. Thus a homotopy is a continuous deformation of the paths which fixes endpoints. $\{\mu_t\}$ is known as a **path homotopy**, and we write $\lambda \simeq_p \gamma$. It is simple to verify that $\simeq_p$ is an equivalence relation.

Example. *Any two paths λ and γ in $\mathbf{R}^n$ with the same start and endpoint are path homotopic, by the path homotopy*

$$\mu_t(u) = t\lambda(u) + (1-t)\gamma(u)$$

which is continuous, since multiplication and addition are continuous in $\mathbf{R}^n$.

Given a space X, we shall define a category $\Pi(X)$, whose objects consists of points in X, and whose morphisms consist of homotopy classes of paths beginning at one point and ending at another. Let λ be a path beginning at x and ending at y, and a path γ beginning at y and ending at z. We define the composed path

$$(\gamma * \lambda)(t) = \begin{cases} \lambda(2t) & t \leqslant 1/2 \\ \gamma(2t-1) & t \geqslant 1/2 \end{cases}$$

If $\lambda \simeq_p \gamma$, and $\alpha \simeq_p \beta$, then $\lambda * \alpha \simeq_p \gamma * \beta$, so $*$ really acts on the morphisms in $\Pi(X)$, which are equivalence classes of paths. Composition of paths is not an associative operation, but when we project down to the quotient structure, we do have associativity

$$[\lambda] * ([\gamma] * [\mu]) = [\lambda * (\gamma * \mu)] = [(\lambda * \gamma) * \mu] = ([\lambda] * [\gamma]) * [\mu]$$

it is easily verified that, while $\lambda * (\gamma * \mu) \neq (\lambda * \gamma) * \mu$, the two paths are path homotopic. Finally, we need to identify the 'identity paths' in $\Pi(X)$. Given an point $x \in X$, consider the constant path e_x, which remains at x at all time points. Then the path $e_x * c$ begins at one point, moves along c at twice the rate than normal, and settles down at the end, waiting for half the time. We shall vary this speed continuously to construct a path homotopy between c and $e_x * c$. Consider the path homotopy

$$\mu_t(u) = \begin{cases} c(\frac{2t}{1+u}) & t \leqslant \frac{1+u}{2} \\ c(1) & t \geqslant \frac{u}{2} \end{cases}$$

Similarily, $c * e_x \simeq_p c$, by the homotopy

$$\mu_t(u) = \begin{cases} c(0) & t \leqslant \frac{1-u}{2} \\ c(\frac{2t+u-1}{1+u}) & t \geqslant \frac{1-u}{2} \end{cases}$$

so that e_x is the identity morphism in $\Pi(X)$, and $\Pi(X)$ is a category.

We call a category where every morphism is invertible a **groupoid**. Given a path $\gamma : [0,1] \to X$, consider the map $\overline{\gamma} : [0,1] \to X$, defined by $\overline{\gamma}(t) = \gamma(1-t)$. We claim that $[\overline{\gamma}] = [\gamma]^{-1}$. This is because once γ is composed with $\overline{\gamma}$, only the beginning point is fixed, so we can 'pull' the path down to a point, by the homotopy

$$\mu_t(u) = \begin{cases} c(2ut) & t \leqslant 1/2 \\ c(2u(1-t)) & t \geqslant 1/2 \end{cases}$$

This verifies that $[\overline{\gamma} \circ \gamma] = [e_x]$. That $[\gamma \circ \overline{\gamma}] = [e_x]$ follows because $\overline{\overline{\gamma}} = \gamma$. For this reason, $\Pi(X)$ is known as the **fundamental groupoid** of X.

The set of loops at a point (automorphisms in the category at a fixed object) form a group. For $x \in X$, the automorphism group at $x \in \Pi(X)$ will be denoted $\pi_1(X,x)$. If X is path connected, then all objects in $\Pi(X)$ are isomorphic, and thus the group of loops at a point is invariant of which point we choose. We call this group[1] the **fundamental group** of X, denoted $\pi_1(X)$.

Example. *For a convex set X in $\mathbf{R}^n$, any loop λ can be contracted to a constant map, so $\pi_1(X)$ is the trivial group.*

A space is **simply connected** if it is path-connected and has trivial fundamental group. Every convex subset of $\mathbf{R}^n$ (and in general, any topological vector space) is simply connected. In terms of the fundamental groupoid, a space is simply connected if every object in the fundamental groupoid is initial.

Example. *If $n \geqslant 2$, then S^n is simply connected. Consider any particular curve γ, with start-point x and end-point y. Fix $z \neq x, y$, and pick a convex chart (u, U) around z not containing x nor y. Then $\gamma^{-1}(U)$ is an open subset of $[0,1]$, and thus a union of certain intervals (a_i, b_i), which we may have countably many of. Nonetheless, $\gamma^{-1}(z)$ is a compact subset, so is contained in only finitely many $(a_{i_1}, b_{i_1}), \dots, (a_{i_n}, b_{i_m})$. Suppose $m = 1$. Construct a continuous path from $\gamma(a_{i_1})$ to $\gamma(b_{i_m})$ which remains in U, and does not touch z. This is possible because it is possible in any connected, open subset of $\mathbf{R}^n$, for $n \geqslant 2$, and U is homeomorphic to such a subset. Since U is a convex subset, it is simply connected, and γ is path homotopic to the modified path γ', which does not touch z. In general, for $m > 1$, we remove the intersection intervals by induction. But if γ does not touch z, then γ remains in $S^n - \{z\}$, which is homeomorphic to $\mathbf{R}^n$, and thus γ is path homotopic to any other path which connects x and y and does not touch z. But then $\gamma \simeq_p \lambda$ for any other path λ, for λ is path homotopic to a path which does not touch z.*

Theorem 2.1. *Given two path connected spaces X and Y, we have*

$$\Pi(X \times Y) \cong \Pi(X) \times \Pi(Y)$$

Proof. Given a paths $\gamma : [0,1] \to X \times Y$, we have two paths $\pi_X \circ \gamma : [0,1] \to X$, $\pi_Y \circ \gamma : [0,1] \to Y$, for which $\gamma = (\pi_X \circ \gamma) \times (\pi_Y \circ \gamma)$ We have

$$\pi_X \circ (\gamma * \lambda) = (\pi_X \circ \gamma) * (\pi_X \circ \lambda)$$

[1] I suppose a pedantist would argue this is really a class of isomorphic groups, but ... meh

and if $\gamma \simeq_p \lambda$, then $\pi_X \circ \gamma \simeq_p \pi_X \simeq_p \lambda$ and $\pi_Y \circ \gamma \simeq_p \pi_Y \circ \lambda$, so that map

$$[\gamma] \mapsto ([\pi_X \circ \gamma], [\pi_Y \circ \gamma])$$

is a well defined functor from $\Pi(X \times Y)$ to $\Pi(X) \times \Pi(Y)$, since the image of $e_{(x,y)}$ is (e_x, e_y). It is easily verified to be an isofunctor. $\square$

Corollary 2.2. $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

For some forthcoming examples, we shall assume $\pi_1(S^1) = \mathbf{Z}$. This will never factor into formal proofs until we perform the calculation, so it does not cause a logical issue. The reason for this is that it is in general very difficult to calculate the fundamental group of spaces, and we require examples for some of the theory.

Example. *The torus $\mathbf{T}^2$ can be described as the product $S^1 \times S^1$. Hence*

$$\pi_1(\mathbf{T}^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbf{Z}^2$$

In general, $\pi_1(\mathbf{T}^n) = \pi_1(S^1 \times \cdots \times S^1) \cong \mathbf{Z}^n$.

2.2 Induced Homomorphisms

The map Π converts objects in **Top** to object in the category **Grpd** of groupoids. If $f : X \to Y$ is a map, and γ is a path in X, then we define $f_*(\gamma) = f \circ \gamma$. If $\gamma \simeq_p \lambda$, then $f \circ \gamma \simeq_p f \circ \lambda$. Since

$$f_*(\gamma * \lambda) = f \circ (\gamma * \lambda) = (f \circ \gamma) * (f \circ \lambda)$$

the map f_* is a functor between $\Pi(X)$ and $\Pi(Y)$. It follows that Π is actually a functor, since $(g \circ f)_* = g_* \circ f_*$, and $(\mathrm{id}_X)_*$ is the identity map on $\Pi(X)$. It follows that if two spaces are homeomorphic, then they have isomorphic fundamental groupoids.

Theorem 2.3. *A retraction $r : X \to A$ and an embedding $i : A \to X$ induces a faithful functor $i_* : \Pi(A) \to \Pi(X)$. If r is a strong deformation retraction, then i_* is also full.*

Proof. If r is a retraction, then $r \circ i = \mathrm{id}_A$, so

$$r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{A*} = \mathrm{id}_{\Pi(A)}$$

Thus i_* has a left inverse, and is therefore injective, hence faithful. Conversely, if $H : X \times [0,1] \to X$ is a deformation retraction between id_X and r, then any path γ between a and b in A is path homotopic to $r_*(\gamma)$ via the map

$$G(x,t) = H(\gamma(x),t)$$

Thus i_* is surjective. □

Example. *If r retracts a simply connected space X to a connected subset A, then A is simply connected. This implies that there is no retraction from* $\mathbf{D}$ *onto S^1.*

Retractions give strong relations between the fundamental group between spaces. For $a \in A$, the map $(i \circ r)_* : \pi_1(X,a) \to \pi_1(X,a)$ is a retraction onto $\pi_1(A,a)$, viewed as a subset of $\pi_1(X,a)$. If $\pi_1(A,a)$ is normal in $\pi_1(X,a)$, then

$$\pi_1(X,a) \cong \pi_1(A,a) \times \ker((i \circ r)_*)$$

More generally, if $\pi_1(A,a)$ is not normal, then we must instead take the semidirect product

$$\pi_1(X,a) \cong \pi_1(A,a) \rtimes \ker((i \circ r)_*)$$

Thus we can find whether there is a retraction to a subspace by comparing fundamental groups.

While the fundamental groupoid is a more sophisticated and general construction, the fundamental group is normally easier to compute with. Since the fundamental group is basepoint dependant, it is best to consider the construction as a functor on the category $\mathbf{Top}_*$ of **pointed topological spaces**, whose objects are pairs (X,x_0), with $x_0 \in X$ is a fixed point, and whose morphisms are **basepoint proserving maps**, $f : (X,x_0) \to (Y,y_0)$ which are continuous maps from X to Y which map x_0 to y_0. Given f, f_* can be seen as a map from $\pi(X,x_0) \to (Y,y_0)$, so π is a functor. A **basepoint preserving homotopy** between two maps $f,g : (X,x_0) \to (Y,y_0)$ is a homotopy $H : X \times [0,1] \to X$ between f and g such that $H(x_0,t) = x_0$ for all $t \in [0,1]$. If f is basepoint homotopic to g, then $f_* = g_*$ on the fundamental groups, since

$$f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma])$$

A basepoint preserving homotopy equivalence therefore induces an isomorphism between the fundamental groups at each point.

We shall show that general homotopy equivalences preserve the fundamental group. The trick to this is showing that, even though we do not preserve a point x_0 in the homotopy equivalence, the path obtained by following the image of x_0 allows us to construct a homotopy between the two sets.

Lemma 2.4. *If H is a homotopy between f and g, and $h : [0,1] \to Y$ is the path $h(t) = h(x_0, t)$, then the diagram below commutes.*

$$\begin{tikzcd} & \pi_1(Y, g(x_0)) \arrow[d, "\beta_h : [\gamma] \mapsto [h * \gamma * \overline{h}]"] \\ \pi_1(X, x_0) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & \pi_1(Y, f(x_0)) \end{tikzcd}$$

Proof. There is a path homotopy G between $h * (g \circ \gamma) * \overline{h}$ and $f \circ \gamma$. TO see this, define a path h_t to be a segment of the path h, defined by $h_t(u) = h(tu)$. Then take G to be

$$G(u,t) = (h_t * H(\cdot, t)_*(\gamma) * \overline{h_t})(u)$$

We compute $G(0,t) = (h_0 \circ f_*(\gamma) \circ \overline{h_0})(t)$, a path which is path homotopic to $f \circ \gamma$ by reparameterization, and $G(1,t) = (h \circ g_*(\lambda) \circ \overline{h})(t)$, which is the path $h * (g \circ \gamma) * \overline{h}$. Thus $\beta_h \circ g_*([\gamma]) = [h * (g \circ \gamma) * \overline{h}] = [f \circ \gamma]$. □

We note that $\beta_h : [\gamma] \mapsto [h * \gamma * \overline{h}]$ is an isomorphism from $\pi_1(Y, g(x_0)$ to $\pi_1(Y, f(x_0))$, since if $[h * \gamma * \overline{h}] = [e_x]$, then $[\gamma] = [e_x]$ by composing inverses, so the map is injective and $[h * (\overline{h} * \gamma * h) * \overline{h}] = [\gamma]$, so the map is surjective.

Theorem 2.5. *if $f : X \to Y$ is a homotopy equivalence, then*

$$f_* : \pi(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism for each $x_0 \in X$.

Proof. Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x_0)) \xrightarrow{f_*} \pi_1(Y, (f \circ g \circ f)(x_0))$$

Since $g \circ f$ is homotopic to id_X, it follows that $(g \circ f)_*$ is conjugation by h for some path h, and is therefore an isomorphism, so f_* is injective. The same argument shows $(f \circ g)_*$ is an isomorphism, so f_* is also surjective. □

Corollary 2.6. *A homotopy equivalence $f : X \to Y$ induces a full functor f_* from $\Pi(X)$ to $\Pi(Y)$.*

2.3 Van Kampen's Theorem

2.4 Covering Spaces

We have uncovered some basic mechanisms which govern the fundamental groups of a space, but we still haven't computed any interesting fundamental groups. The general problem of finding fundamental groups is provably computationally intractable, so it makes sense that these groups are hard to calculate. Algebraic topology must strike a balance with finding algebraic structures which are both easy to calculate, and powerful enough to distinguish spaces.

The theory of covering spaces is deeply connected to field theory. Galois' correspondence shows that subextensions of a field extension corresponds to subgroups of the Galois group. If one knows the subextensions, one may calculate the Galois group. In the theory of fundamental groups, one corresponds covering spaces of a space, which correspond to subgroups of the fundamental group of the space.

A **covering space** is a space E together with a surjective map $p : E \to B$, such that there exists an open cover $\{U_\alpha\}$ with $p^{-1}(U_\alpha)$ is the disjoint union of open sets in E known as **folds** or **sheets**, with each fold mapping homeomorphically onto U_α by the map p. The cardinality of the number of folds at a point is locally finite, so is constant on a connected covering space. An **n-fold cover** has n folds at each point.

Example. *The primordial example of a covering space is* $\mathbf{R}$ *over* S^1*, bound by the projection* $p(t) = e^{it}$*.* p *is an open map, since it is a differentiable and has full rank at every point.For each* x*, the inverse image of any open arc in* S^1 *splits into countably many disjoint intervals mapping homeomorphically onto the arc. Thus p really is a cover. One can view this covering space as an infinite helix which wraps around the circle.*

Example. S^1 *is also a cover for* S^1*, together with the map* $p(z) = z^n$*. One visualizes this as a finite helix which wraps around* S^1 *n times, then connects back with itself. Alternatively, take the curve around the torus which wraps around n times, and project it down to the circle used to form the surface of revolution of the torus.*

It turns out that the covers above are the only covers on S^1, and each corresponds to a unique subgroup of $\pi_1(S^1)$, which we will (eventually) show to be $\mathbf{Z}$. The cover by $\mathbf{R}$ corresponds to $\mathbf{Z}$ itself, and the loop which revolves n times around the torus corresponds to $n\mathbf{Z}$.

Example. *Covers of 2-oriented graphs?*

The primary technique in covering space theory is that covers enables us to transfer functions from the base space into the extension space. A **lift** of a map $f : X \to B$ is a map $\tilde{f} : X \to E$ for which $p \circ \tilde{f} = f$. The primary techniques of covering spaces result from the existence of lifts.

2.4.1 Lifting

Theorem 2.7 (Homotopy Lifting Lemma)**.** *Given a covering space $p : E \to B$, and a homotopy $H : X \times [0,1] \to B$ between f and g, then a lift $\tilde{f}$ of f induces a unique lifted homotopy $\tilde{H} : X \times [0,1] \to E$ between $\tilde{f}$ and some lift of g.*

Proof. We shall construct a lift locally around $\{x\} \times [0,1]$ for each $x \in X$. Provided these lifts are unique, we can put them all together to form a homotopy on the whole space. Fix $x \in X$. For each t, pick a neighbourhood U_t of x, and $t \in [a_t, b_t]$ for which $H(U_t \times [a_t, b_t])$ is contained in some U_α. The compactness of $\{x\} \times [0,1]$ allows us to cover this by finitely many $U_t \times [a_t, b_t]$. Taking the intersection of the U_t, we find a neighbourhood N and $0 = t_0 < \dots < t_n = 1$ such that $H(N \times [t_i, t_{i+1}]) \subset U_\alpha$ for some α. Assume we have constructed $\tilde{H}$ on $[0, t_n]$ (which we already have, for $n = 0$, since we have the lift $\tilde{f}$). We know $H(N \times [t_n, t_{n+1}]) \subset U_\alpha$ for some α, so pick a homeomorphic V containing (x, t_n) in $p^{-1}(H(N \times [t_n, t_{n+1}]))$. By choosing N to be smaller, we may assume that $\tilde{H}(N \times \{t_n\})$ is contained in V. Now extend $\tilde{H}$ by composing H with the homeomorphism $p^{-1} : U_\alpha \to V$. After finitely many steps, we obtain a lift $\tilde{H}$ in a neighbourhood of x.

To verify uniqueness, we assume, without loss of generality, that x consists of a single point. If X consists of more than one point, we find by the single point theorem that the homotopy H lifts uniquely on the fibre of each $x \in X$, and by combining all x, we find the homotopy generally lifts uniquely. In the singular case, a homotopy can be viewed as a path in B. Suppose $\tilde{H}$ and $\tilde{H}'$ are two lifts of H. Pick $0 = t_0 < t_1, \dots < t_n = 1$ such that $H([t_i, t_{i+1}])$ is in some U_α. Assume by induction that $\tilde{H}$ and $\tilde{H}'$ agree on $[0, t_i]$. Since $[t_n, t_{n+1}]$ is connected, $\tilde{H}([t_n, t_{n+1}]$ must be contained in one fold of U_α. The same is true of $\tilde{H}'([t_n, t_{n+1}])$, and this must be the same fold, since $\tilde{H}(t_n) = \tilde{H}'(t_n)$. This implies that $\tilde{H} = \tilde{H}'$ on $[t_n, t_{n+1}]$, for there is only one way to define the maps on the fold such that they lift H. By induction, we verify the claim. □

The case where X consists of a point is useful in of itself.

Corollary 2.8 (Path Lifting Lemma)**.** *Given a path γ beginning at $b \in B$, a point $e \in p^{-1}(b)$ induces a unique path $\tilde{\gamma}$ lifting γ, beginning at e.*

Similarily, a path homotopy H between γ and λ lifts to a unique path homotopy $\tilde{H}$ given a particular point $e \in p^{-1}(\gamma(0)) = p^{-1}(\lambda(0))$. This has a useful application.

Theorem 2.9. *A cover $p : E \to B$ induces a faithful functor $p_* : \Pi(E) \to \Pi(B)$.*

Proof. If $\gamma \simeq_p \lambda$ in B start at $b \in B$, then the path homotopy H between γ and λ lifts to a homotopy between the lifts of γ and λ at each $e \in p^{-1}(b)$. Thus p_* is injective on each set of morphisms with a given source and target. □

Note that we may consider covers $p : (E, e_0) \to (B, b_0)$ in **Top.**, and the last theorem gives us a corollary for the fundamental group.

Corollary 2.10. *A cover $p : (E, e_0) \to (B, b_0)$ induces an injective homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$.*

Theorem 2.11. *The number of sheets of a connected cover $p : (E, e_0) \to (B, b_0)$ is the index of $p_*(\pi(E, e_0))$ in $\pi_1(B, b_0)$.*

Proof. Let γ and λ be loops at b_0. Then γ and λ lift uniquely to paths $\tilde{\gamma}$ and $\tilde{\lambda}$ beginning at e_0. If $\tilde{\lambda}$ and $\tilde{\gamma}$ have the same endpoint, then $[\lambda]$ and $[\gamma]$ are conjugate relative to $p_*(\pi(E, e_0))$, for

$$[\gamma] = [\lambda] * [\overline{\lambda} * \gamma]$$

and $\overline{\lambda} * \gamma$ lifts to a loop at e_0. □

A **universal cover** is a cover $p : E \to B$ for which E is simply connected.

Corollary 2.12. *The number of sheets in a universal cover $p : (E, e_0) \to (B, b_0)$ is the cardinality of $\pi_1(B, b_0)$.*

Example. *The universal cover $p : \mathbf{R} \to S^1$ tells us $\pi_1(S^1)$ is countable. The projections $\pi : S^n \to \mathbf{RP}^n$ is a 2-fold cover which is universal for $n \geqslant 2$, so $\pi_1(\mathbf{RP}^n) \cong \mathbf{Z}_2$.*

Theorem 2.13. *Given a cover $p : (E, e_0) \to (B, b_0)$, and a path-connected, locally path-connected X, a map $f : (X, x_0) \to (B, b_0)$ lifts to $\tilde{f} : (X, x_0) \to (E, e_0)$ if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$*

Proof. If $\tilde{f}$ exists, then $f_* = p_* \circ \tilde{f}_*$, so the subset relation holds. Conversely, for $x \in X$, consider a path γ from x_0 to x. Then $f_*(\gamma)$ is a path from b_0 to b, which lifts uniquely to a path from e_0 to another point, which we shall define to be $\tilde{f}(x)$. If λ is another path from x_0 to x, then $\overline{\gamma} * \lambda$ is a loop at e_0, so $f_*(\overline{\gamma} * \lambda) = \overline{f_*(\gamma)} * f_*(\lambda)$ is a loop at b_0 which lifts to a loop at e_0, which we may then compose with the lift of γ, which is path homotopic to the lift of γ and thus moves to the same endpoint. $\tilde{f}$ satisfies our needs provided it is continuous. If U_α is an open prefold in B, fix $x \in f^{-1}(U_\alpha)$ and pick a path connected $x \in V \subset f^{-1}(U_\alpha)$. If y is also in V, then there is a path γ between x and y, which induces a path between $\tilde{f}(x)$ and $\tilde{f}(y)$ in $p^{-1}(U_\alpha)$. Thus $\tilde{f}(x)$ and $\tilde{f}(y)$ are in the same fold, so that locally $\tilde{f}$ is just $p^{-1} \circ f$, and therefore continuous. □

Lemma 2.14 (Lifting Lemma). *If $p : E \to B$ is a cover, and $f : X \to B$ have two lifts $\tilde{f}$ and $\tilde{f}'$ that agree at a point, then provided X is connected $\tilde{f} = \tilde{f}'$.*

Proof. Let $C = \{x \in X : \tilde{f}(x) = \tilde{f}'(x)\}$. Then C is open, for if $\tilde{f}(x) = \tilde{f}'(x)$, pick a sheet V around $f(x)$ which projects by p onto some presheet U_α. Then if W is a neighbourhood of x such that $\tilde{f}(W), \tilde{f}'(W) \subset V$, then on W we must have $\tilde{f} = \tilde{f}'$, since there is only one way to define the lift here. A similar construction shows C^c is open, so $C = X$, since it is open, closed, and nonempty. □

2.4.2 Existence of a Universal Cover

Universal covers are the nicest covers to possess, and we would like to find them when they exist. Like with algebraic closures and splitting extensions, we will find the universal cover is unique up to cover isomorphism. First, lets construct the universal cover.

We shall call a space X is a **semilocally simply connected** if every point x has a neighbourhood U for which the inclusion map induces a trivial morphism $i_* : \pi_1(U,x) \to \pi_1(X,x)$. This is a necessary condition for the existence of a universal cover, for if $p : E \to B$ is universal, then b has a neighbourhood U homeomorphic to some neighbourhood V of e in E. Each loop in U then lifts to a loop in V, and this lifted loop is nullhomotopic in E. Projecting this nullhomotopy down by p gives us a nullhomotopic loop in B. We shall construct universal covers for semilocally simply connected, path connected, locally path connected spaces.

Theorem 2.15. *Every semilocally simply connected, locally path connected, path connected space has a universal cover.*

Proof. Suppose we have a universal cover $p : (E, e_0) \to (B, b_0)$. Then we know that each homotopy class of paths in B lifts uniquely to a homotopy class of paths in E starting at e_0, and conversely, each point e in E corresponds to the lift of the projection of the unique homotopy class of paths from e_0 to e. Thus a good place to start construction a universal cover for B seems to be $\Pi(B)$. In particular, consider

$$\tilde{B} = \coprod_{e \in E} \mathrm{Mor}_{\Pi(B)}(e_0, e)$$

We have a surjective projection $p([\gamma]) = \gamma(1)$. We shall assign a topology to $\tilde{B}$ making the space a simply connected cover of B, for which p is a continuous projection.

Let $\mathcal{U}$ be a collection of all path connected subsets U of X such that the embedding $i_* : \Pi(U) \to \Pi(X)$ is trivial. Then $\mathcal{U}$ is a basis for X if X is locally path connected and semilocally simply-connected, for if $V \subset U$ is path connected, then the injection $\Pi(V) \to \Pi(U) \to \Pi(X)$ must be trivial. We shall use $\mathcal{U}$ to construct a topology on $\tilde{B}$. Given a path class $[\gamma]$ and $U \in \mathcal{U}$, define

$$U_{[\gamma]} = \{[\lambda * \gamma] : \lambda \text{ is a path in } U \text{ with } \lambda(0) = \gamma(1)\}$$

Then $p|_{U_{[\gamma]}}$ is onto U, and injective for if $(\lambda * \gamma)(1) = (\mu * \gamma)(1)$, then λ and μ both have the same end point, so $\lambda \simeq_p \mu$ in U, and $[\lambda * \gamma] = [\mu * \gamma]$. Furthermore, if $[\lambda] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\lambda]}$. This shows that $U_{[\gamma]}$ forms a basis for a topology on $\tilde{B}$, and p is locally a homeomorphism, since for $V \subset U$, $p(V_{[\gamma]}) = V$. $p^{-1}(V) = \bigcup_{[\gamma]} V_{[\gamma]}$, which are disjoint or equal. This completes the construction of the universal cover. $\square$

The existence of the universal cover give rise to a large variety of different covers, corresponding to subgroups of $\pi_1(B)$.

Theorem 2.16. *If B is path connected, locally path connected, and semilocally simply connected, then for every subgroup $H < \pi_1(B, b_0)$ there is a cover $p_H : (E, e_0) \to (B, b_0)$ such that $(p_H)_*(\pi_1(E, e_0)) = H$.*

Proof. Take a quotient on $\tilde{B}$, as in the last proof, that identifies $[\gamma]$ and $[\lambda]$ if $\gamma(1) = \lambda(1)$, and $[\overline{\lambda} * \gamma] \in H$, to form the space $\tilde{B}_H$. If $[\alpha * \lambda] \in U_{[\lambda]}$ is identified with $[\beta * \gamma] \in U_{[\gamma]}$, then $U_{[\gamma]}$ is identified with $U_{[\lambda]}$, for if $[\mu]$ is identified with $[\nu]$, then $[\eta * \mu]$ is identified with $[\eta * \nu]$ and $[\mu * \eta]$ is identified with $[\nu * \eta]$ if ν is a loop at e_0. Thus $[\delta * \lambda] = [\delta * \overline{\alpha} * \alpha * \lambda]$ is identified with $[\delta * \overline{\alpha} * \beta * \gamma]$. The natural basepoint is the image of the constant map $[e_{e_0}]$, which is projected onto e_0. Points identified project to the same point by p, so we obtain $p_H : (\tilde{B}_H, [e_{e_0}]) \to (B, b_0)$, which is still a cover, for we have identified entire sheets at once. We have

$$(p_H)_*(\pi_1(\tilde{B}_H, [e_{e_0}])) = H$$

for a loop γ at b_0 lifts to a loop at $[e_{e_0}]$ if and only if $[\gamma]$ is contained in H. $\square$

2.4.3 Isomorphisms of Covering Spaces

Now we turn to the question of uniqueness of covering spaces. A morphism between covering spaces $p : E \to B$ and $p' : E' \to B$ is a map $f : E \to E'$ such that the diagram below commutes

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

For a fixed base space B, we form the category $\mathbf{Cover}_B$ of covering spaces over B together with covering morphisms.

Theorem 2.17. *If B is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : E \to B$ and $p_2 : E' \to B$ are isomorphic via an isomorphism $f : E \to E'$ taking $e_0 \in p^{-1}(b_0)$ to $e_1 \in p^{-1}(b_0)$ if and only if $(p_1)_*(\pi_1(E, e_0)) = (p_2)_*(\pi_1(E', e_1))$.*

Proof. If f exists, the relation must hold, since $p_2 \circ f = p_1$. Conversely, if

$$(p_1)_*(\pi_1(E, e_0)) = (p_2)_*(\pi_1(E', e_1))$$

By the lifting criterion, we may lift p_1 to a map $\tilde{p_1} : (E, e_0) \to (E', e_1)$. Conversely, we may lift p_2 to $\tilde{p_2} : (E', e_1) \to (E, e_0)$, since lifts must be unique, $\tilde{p_2} \circ \tilde{p_1}$ and $\tilde{p_1} \circ \tilde{p_2}$ must be the identity, so $\tilde{p_1}$ is an isomorphism between the two covering spaces. □

We have justified the first half of the classification theorem for covering spaces. Here is the rest.

Theorem 2.18. *If (B, b_0) is a path-connected, locally path-connected, semilocally simply connected pointed space, then there is a bijection between isomorphism classes of path-connected covering spaces $p : (E, e_0) \to (B, b_0)$ and subgroups of $\pi_1(B, b_0)$, obtained by associative p with $p_*(\pi(E, e_0))$. When basepoints are ignored, one obtains a bijection between isomorphism classes of spaces $p : E \to B$ and conjugacy classes of subgroups of $\pi_1(B, b_0)$.*

Proof. We need only prove the last statement, since we have justified the isomorphism property. Given the projection $p : (E, e_0) \to (B, b_0)$, consider a path $\tilde{\gamma}$ from e_0 to e_1, with $e_1 \in p^{-1}(b_0)$. Then $\tilde{\gamma}$ projects to some $\gamma \in \pi_1(B, b_0)$. Let $H_0 = p_*(\pi_1(E, e_0))$, and $H_1 = p_*(\pi_1(E, e_1))$. Then $\gamma * H_0 * \overline{\gamma} \subset H_1$, since $\gamma * h * \overline{\gamma}$ lifts to a path that begins at e_1, goes to e_0, and then returns to e_1. But by symmetry, $\overline{\gamma} * H_1 * \gamma \subset H_0$, so that H_0 and H_1 are conjugate. Conversely, given $H_1 = \gamma * H_0 * \overline{\gamma}$, consider the isomorphism from $p : (E, e_0) \to (B, b_0)$ to $p : (E, e_1) \to (B, b_0)$, where e_1 is the endpoint of the lift of $\overline{\gamma}$ beginning at e_0. □

2.4.4 Deck Transformations

Let B be a base space with $b_0 \in B$. The set of automorphisms of a cover $p : E \to B$ in $\mathbf{Cover}_B$ is the group of **deck transformations** of the space, denoted $G(E)$. $G(E)$ acts faithfully on $p^{-1}(b_0)$, for any automorphism f maps some $e_0 \in p^{-1}(b_0)$ to $e_1 \in p^{-1}(b_0)$, since f must satisfy the commutative diagram, and if E is connected, f is uniquely determined by its action on e_0, for automorphisms are lifts of the identity map. The map p_* maps automorphisms $f \in G(E)$ to automorphisms $p_*(f)$ of $\Pi(B)$, fixing objects, but permuting maps.

A **normal** covering space is a space $p : (E, e_0) \to (B, b_0)$ such that for each $e_1 \in p^{-1}(b_0)$, there is an automorphism $f \in G(E)$ such that $f(e_0) = e_1$. As in Galois theory, normal extensions correspond to normal subgroups of the particular group studied.

Theorem 2.19. *$p : (E, e_0) \to (B, b_0)$ is a normal cover if and only if $p_* \pi_1((E, e_0))$ is a normal subgroup of $\pi_1(B, b_0)$.*

Proof. Let $H = \pi_1(B, b_0)$. If $\gamma \in \pi_1(B, b_0)$ lifts to a path $\tilde{\gamma}$ from e_0 to $e_1 \in p^{-1}(b_0)$, then we have seen that

$$p_*(\pi_1(E, e_1)) = [\gamma] * p_*(\pi_1(E, e_0)) * [\overline{\gamma}]$$

Thus if there is an automorphism $f \in G(E)$ which maps e_0 to e_1, then $p_*(\pi_1(E, e_1)) = p_*(\pi_1(E, e_0))$, so $[\gamma] \in N(H)$. Conversely, if $[\gamma] \in N(H)$, then γ lifts to a path from e_0 to some e_1 for which $p_*(\pi_1(E, e_1)) = p_*(\pi_1(E, e_0))$, so there is $f \in G(E)$ which maps e_0 to e_1 by the characterization of covering spaces.

Consider the map $\varphi : N(H) \to G(E)$ which associatives with each $[\gamma] \in N(H)$ the unique automorphism f which maps e_0 to whatever the lift of γ ends at. Then φ is a surjective homomorphism, whose kernel is the set of maps which lift to loops at e_0. But this is precisely H. Thus $G(E) \cong N(H)/H$. If H is normal, then every loop at $\pi_1(E, e_0)$ lifts to a map which induces an automorphism of E, so the cover must be normal. Conversely, if the cover is normal, then the projections of each fundamental group at $e_1 \in p^{-1}(b_0)$ must be the same, so that $p_*(\pi_1(E, e_0))$ must be a normal subgroup of $\pi_1(B, b_0)$. □

Corollary 2.20. *If $p : (E, e_0) \to (B, b_0)$ is universal, then $G(E) \cong \pi_1(B, b_0)$.*

The map which associates with a cover $p : (E, e_0) \to (B, b_0)$ the $\pi(B, b_0)$-set $p^{-1}(b_0)$, and with each equivalence the associated permutation of $p^{-1}(b_0)$ is a functor. It is

The functor which assigns to each covering space $p : E \to B$ the set

Consider a particular covering space $p : E \to (B, b)$, and let $e \in p^{-1}(b)$.

Chapter 3

Homology

The fundamental group is powerful enough to classify surfaces, but is not refined enough to treat spaces which operate in higher dimensions. Low-dimensional paths carry little information in high-dimensional spaces. For instance, $\pi_1(S^n)$ is trivial for $n \geqslant 2$, yet S^n is obviously not homeomorphic to S^m for $n \neq m$. Another example occurs in the analysis of CW-complexes, where we see the fundamental group only depends on the 2-skeleton of the complex. To classify these spaces, we can generalize the fundamental group to the homotopy groups $\pi_n(X)$, and theoretically, these groups succeed in classifying spaces up to homotopy, but the higher dimensional groups are nigh impossible to work with; Much remains unknown about $\pi_n(S^m)$ for $n > m$, even though these spaces are some of the simplest manifolds. Thus a different algebraic invariant must be employed, one which is practical to compute and refined enough to classify commonly encountered spaces. Fortunately, there is such an invariant, known as homology. It is much more technical than the fundamental group, but the end-result is worth every penny. It is one of the jewels of 20th century mathematics.

The idea of homology is not that different than the fundamental group. A path homotopy between loops λ and γ can be seen as the image of a circle, whose boundary is $\lambda * \gamma^{-1}$. The fundamental group therefore determines the 'loops' in the space, up to those which bound a circle. Homology theory generalizes this idea by determining loops up to those which bound an arbitrary surface. This can be easily generalized, by taking higher dimensional submanifolds up to those which bound a manifold one dimension higher. These are the homology groups.

In the fundamental group, the order of composition matters, since we literally compose two loops by putting one after the other. Thus we obtain a non-abelian invariant, which both enriches and complicates the theory. Homology treats submanifolds combinatorially – we consider the sum of two manifolds abstractly as some form of union of the two manifolds. Thus the theory is abelian, which is one of the reasons the theory is easy to calculate.

There are many homology theories, only a certain few of which we will focus on. One can abstract these theories to a common core, studied under the name of homological algebra. To avoid too much abstract nonsense, we begin by concretely introducing homology theories, but abstraction is often useful, so we will introduce it when necessary.

3.1 Δ-Complexes

The most computationally practical homology theory is simplicial homology, which applies only to spaces known as Δ complexes, which are obtained from discrete, combinatorial operations. Geometrically, these spaces are seen as geometric polyhedrons attached together along faces. In this section, we describe an abstract formation of Δ-complexes.

Recall that an **n-simplex** is the convex hull of an ordered sequence of $n+1$ points $v_0, \dots, v_n$ in $\mathbf{R}^m$ which do not lie in any hyperplane of dimension smaller than n (equivalently, $v_1 - v_0, \dots, v_n - v_0$ are linearly independent). We denote this simplex by $[v_0, \dots, v_n]$, and the set of all n simplexes by $\mathcal{D}^n$. The standard simplices are

$$\Delta^n = [e_1, \dots, e_{n+1}] = \{(t_1, \dots, t_{n+1}) \in \mathbf{R}^{n+1} : \sum t_i = 1, t_1, \dots, t_n \geqslant 0\}$$

we see an n-simplex as the simplest example of an n-dimensional polyhedron. To work with such simplices, it is useful to introduce a standard coordinate system. Each point p in an n-simplex $[v_0, \dots, v_n]$ can be uniquely specified by a tuple of positive real numbers $(t_0, \dots, t_n) \in \mathbf{R}^{n+1}$ such that $\sum t_i = 1$, under the map

$$(t_0, \dots, t_n) \mapsto \sum t_i v_i$$

These are known as the **barycentric coordinates** of p with respect to $[v_0, \dots, v_n]$. The coordinates define an affine isomorphism of $[v_0, \dots, v_n]$ with Δ^n, so we think of all simplexes as isomorphic.

A **face** of a simplex $[v_0,\dots,v_n]$ is a simplex of the form $[v_0,\dots,\widehat{v_i},\dots,v_n]$, obtained by deleting a vertex in the generator. If S is a simplex, we let ∂S denote the boundary of S, the union of all faces of S, and $S^\circ = S - \partial S$ the interior. This notation preserves the standard use from seeing S as an n-manifold with boundary. An important connection between a simplex and its faces are the **boundary operators** $\partial_0,\dots,\partial_n$, defined by the map

$$\partial_i[v_0,\dots,v_n] = [v_0,\dots,\widehat{v_i},\dots,v_n]$$

A purely combinatorial argument shows $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. In the other direction, we have maps $\sigma_i : \partial_i(S) \to S$, defined in the barycentric coordinates by

$$(t_0,\dots,\widehat{t_i},\dots,t_n) \mapsto (t_0,\dots,0,\dots,t_n)$$

To be brief, a Δ-complex is obtained by taking a certain number of simplices, and gluing them along boundaries via the maps σ_i. We shall give an abstract notation which simplifies this procedure.

A Δ-**set** $\mathfrak{S}$ is a sequence of arbitrary sets

$$\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \dots$$

together with a collection of abstract maps $\partial_i : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$ for $i = 0,\dots,n$ which satisfy

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$$

for $i < j$. We shall interpret a Δ set to be a union of simplices, one n-simplex for each element of $\mathfrak{S}_n$, attached at boundaries upon which the simplices agree. If we define a Δ-**morphism** between two Δ-sets $\mathfrak{S}$ and $\mathfrak{M}$ to be a sequence $f = \{f_n : \mathfrak{S}_n \to \mathfrak{M}_n\}$ of functions which satisfy the family of commutative diagrams

$$\begin{tikzcd}
\mathfrak{S}_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_i"] & \mathfrak{M}_{n+1} \arrow[d, "\partial_i"] \\
\mathfrak{S}_n \arrow[r, "f_n"] & \mathfrak{M}_n
\end{tikzcd}$$

then the set of Δ sets form a category **Delta**. The geometric realization of a Δ-set can be realized as a functor from **Delta** to **Top** which maps $\mathfrak{S}$ to the space $|\mathfrak{S}|$, an adjunction space of

$$\coprod_{k=0}^{\infty} \mathfrak{S}_k \times \Delta^k$$

under the attaching maps $\sigma_i^S : S \times \partial_i \Delta^n \to \partial_i S \times \Delta^{n-1}$. Given a Δ-morphism $f : \mathfrak{S} \to \mathfrak{M}$, define $f_* : |\mathfrak{S}| \to |\mathfrak{M}|$ by

$$f_*([S,x]) = [f_n(S), x]$$

The map is well defined, for

$$f_*([S,\sigma_i(x)]) = [f_n(S), \sigma_i(x)] = [\partial_i f_n(S), x] = [f_{n-1}(\partial_i S), x] = f_*([\partial_i S, x])$$

It is also continuous, for it induces a map which is effectively the identity map on each component of the pre-quotient space. Given another Δ-morphism $g : \mathfrak{M} \to \mathfrak{O}$, if we let $h = g_* \circ f_*$, and we find

$$h([S,x]) = [(g_n \circ f_n)(S), x] = g([f_n(S), x]) = (g \circ f)_*([S,x])$$

So $|\cdot|$ truly is a functor. The advantage of Δ complexes is that they are a purely combinatorial object, so they are easy to calculate with.

Example. *A Δ-decomposition of a topological space X is a Δ-set $\mathfrak{S}$ such that $|\mathfrak{S}|$ is homeomorphic to X. As an example, let us compute a Δ-decomposition of* $\mathbf{T}$. *The decomposition is realized by the diagram*

$$\begin{tikzcd}
\alpha \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Lambda", near start] & \alpha \arrow[d, "a"] \\
\alpha \arrow[r, "b"'] \arrow[ur, "c"] \arrow[ur, phantom, "\Gamma", near end, shift right=3ex] & \alpha
\end{tikzcd}$$

we let

$$\mathfrak{S}_0 = \{\alpha\} \quad \mathfrak{S}_1 = \{a, b, c\} \quad \mathfrak{S}_2 = \{\Lambda, \Gamma\}$$

and define

$$\partial_i a = \partial_i b = \partial_i c = \alpha$$

$$\partial_0 \Lambda = c \;\; \partial_1 \Lambda = b \;\; \partial_2 \Lambda = a \quad\quad \partial_0 \Gamma = a \;\; \partial_1 \Gamma = b \;\; \partial_2 \Gamma = c$$

One can check on a case by case basis that the boundary equations hold, or just note that the equations hold by a choice of consistant orientation, easily visualized in three dimensions.

Example. *The standard CW structure on S^n cannot be used to construct a Δ-decomposition of S^n for $n > 1$, for in such a decomposition the boundaries of the n-simplexes must project to $n-1$ simplexes. To obtain a decomposition, we glue two n-simplices together along the boundary. We take $\mathfrak{S}_n = \{\Delta_1^n, \Delta_2^n\}$, two copies of the standard n-simplex. We then define $\mathfrak{S}_k$, for $i < n$, to consist of all the subsimplices $[e_{i_1}, \dots, e_{i_k}]$ of Δ^n, where $i_1 < i_2 < \dots < i_n$. Define*

$$\partial_j [e_{i_1}, \dots, e_{i_k}] = [e_{i_1}, \dots, \widehat{e_{i_j}}, \dots, e_{i_k}]$$

$$\partial_j \Delta_1^n = \partial_j \Delta_2^n = [e_0, \dots, \widehat{e_j}, \dots, e_n]$$

This delta-set is a decomposition of S^n.

Example. *We shall give a final example of a Δ-decomposition of $\mathbf{RP}^2$, described by the diagram*

$$\begin{tikzcd}
\alpha \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Lambda", near start] & \beta \\
\beta \arrow[ur, "c"] & \alpha \arrow[l, "b"] \arrow[u, "a"'] \arrow[ul, phantom, "\Gamma", near start]
\end{tikzcd}$$

which forces us to take

$$\mathfrak{S}_0 = \{\alpha, \beta\} \quad \mathfrak{S}_1 = \{a, b, c\} \quad \mathfrak{S}_2 = \{\Lambda, \Gamma\}$$

and define

$$\partial_0 a = \partial_0 b = \partial_0 c = \beta \quad \partial_1 a = \partial_1 b = \alpha \quad \partial_1 c = \beta$$

$$\partial_0 \Lambda = c \ \ \partial_1 \Lambda = b \ \ \partial_2 \Lambda = a \qquad \partial_0 \Gamma = c \ \ \partial_1 \Gamma = a \ \ \partial_2 \Gamma = b$$

3.2 Simplicial Chain Complexes

The main study of simplicial homology revolves around the functors C_n, from **Delta** to **Ab**, which associates with each Δ-set $\mathfrak{S}$ the free abelian group $C_n(\mathfrak{S}) = \mathbf{Z}\langle \mathfrak{S}_n \rangle$. Element of $C_n(\mathfrak{S})$ are known as *n*-chains. We define the **differentials** $d_n : C_n(\mathfrak{S}) \to C_{n-1}(\mathfrak{S})$ by

$$d_n(S) = \sum_{k=0}^{n} (-1)^k \partial_k S$$

for $S \in \mathfrak{S}$. Let $d_0 : C_0(\mathfrak{S}) \to (0)$ be the trivial map. Extending this to arbitrary linear combinations of the S, we obtain an infinite sequence

$$\dots C_n(\mathfrak{S}) \xrightarrow{d_n} \dots \xrightarrow{d_3} C_2(\mathfrak{S}) \xrightarrow{d_2} C_1(\mathfrak{S}) \xrightarrow{d_1} C_0(\mathfrak{S}) \xrightarrow{d_0} (0)$$

We may put all the i chains to obtain the group

$$C(\mathfrak{S}) = \bigoplus_{k=0}^{\infty} C_k(\mathfrak{S})$$

and thus put all d_i together to obtain $d : C(\mathfrak{S}) \to C(\mathfrak{S})$, which is notationally much neater.

Lemma 3.1. *For any simplicial complex $\mathfrak{S}$, $d^2 = 0$.*

Proof. We need only prove the identity for each element of the basis. Let $S \in \mathfrak{S}_n$ be a simplex. Then, using the identity $\partial_i \partial_j = \partial_{j-1}\partial_i$ for $i < j$,

$$\begin{aligned}
d^2(S) &= d\left(\sum_{i=0}^{n}(-1)^i \partial_i S\right) \\
&= \sum_{i=0}^{n}(-1)^i d(\partial_i S) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\partial_j \partial_i S) \\
&= \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}\partial_{i-1}\partial_j S + \sum_{i=0}^{n}\sum_{j=i}^{n-1}(-1)^{i+j}\partial_j\partial_i S \\
&= -\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j}\partial_i\partial_j S + \sum_{i=0}^{n}\sum_{j=i}^{n-1}(-1)^{i+j}\partial_j\partial_i S \\
&= -\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j}\partial_i\partial_j S + \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j}\partial_j\partial_i S = 0
\end{aligned}$$

□

We say a chain c is **closed** if $dc = 0$. Aside from the algebra, closed chains have a geometric visualization. Write

$$c = \sum \pm c_\alpha$$

where each c_α is a simplex. Take pairs $(c_\alpha \circ \sigma_i, c_\beta \circ \sigma_j)$ if such faces cancel each other out when taking the differential. We may pair off all c_α, since $dc = 0$. Fix some particular pairing, and consider the quotient of the disjoint union

$$\coprod_\alpha \Delta^n_\alpha$$

which identifies $\partial_i \Delta^n_\alpha$ and $\partial_j \Delta^n$ by the canonical mapping if $(c_\alpha \circ \sigma_i, c_\beta \circ \sigma_j)$ is in the pairing. Such a representation is a certainly a manifold on the interiors of the simplexes. It is also a manifold on the interiors of $\partial_i \Delta^n$, for it is here that two simplices are paired up at the boundary. Being careful, we even see that it is a manifold on the interiors of $\partial_j \partial_i \Delta^n$, but this fails if we take further boundaries.

If $db = c$, then we view the chain c as the boundary of the chain b. If this is true, the last theorem says we must necessarily have $dc = 0$. Thus if we find that $dc = 0$, then c is a potential boundary for some chain of a higher dimension.

3.3 Homology

A **chain complex** is a graded abelian group

$$A = \bigoplus_{i \in \mathbf{Z}} A_i$$

with a map $d : A \to A$ which decomposes into maps $d_i : A_i \to A_{i-1}$, such that $d^2 = 0$. d is known as the boundary operator. A **chain map** between chain complexes A and B is a morphism $f : A \to B$ which breaks into morphisms $f = \{f_i : A_i \to B_i\}$ such that $d \circ f = f \circ d$. This makes the set of chain complexes a category **Chain**, and we have basically argued the existence of a functor from **Delta** to **Chain**. In general, homology theories discuss functors into the category **Chain** (which could be sequences of modules over any ring, rather than abelian groups, or if we want to get real abstract, on any abelian category), and homological algebra gives us the backbone to discuss this theory.

If a chain complex induces an exact sequence

$$\dots \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \xrightarrow{d_{n-2}} \dots$$

Then we obtain a further diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & A_n \arrow[rr, "d_n"] \arrow[dr] & & A_{n-1} \arrow[rr, "d_{n-1}"] \arrow[dr] & & A_{n-2} \arrow[r, "d_{n-2}"] & \cdots \\
& & M_{n-1} \arrow[ur] \arrow[dr] & & M_{n-2} \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

where the diagonals are exact, defining $M_n = \text{im}(d_{n+1}) = \ker(d_n)$. Thus we obtain a family of short exact sequences

$$0 \to M_n \to A_{n-1} \to M_{n-1} \to 0$$

in general, we define the **n'th homology group**

$$H_n(A) = \ker(d_n) \,/\, \text{im}(d_{n+1})$$

Algebraically, the homology measures the degree of exactness of each joint in the sequence of derivation mappings. Geometrically, we see $H_n(A)$ as the space of all chains which could be seen as the boundary of a higher dimension chain, modulo those which actually are the boundary of the chain. Thus $H_n(A)$ 'counts holes', but in a very different manner than the fundamental group. We denote the composition of the functors from **Delta** to **Chain** to **Ab** by H_n^Δ.

Example. *When we 'compute the simplicial homology of a topological space' X, what we really mean is to compute the simplicial homology of some Δ-decomposition of X. We shall later prove that the homology is invariant of which decomposition we choose. As an example, we compute the simplicial homology of* $\mathbf{T}$. *We already have a decomposition, and we shall use the notation from this decomposition. One calculates the derivations as*

$$d_0(n\alpha) = 0 \quad d_1(na + mb + lc) = 0 \quad d_2(n\Gamma + m\Lambda) = n(a - b + c) + m(c - b + a)$$

Thus the kernel of d_0 is $\langle \alpha \rangle$, and the image of d_1 is (0), so

$$H_0^\Delta(\mathbf{T}) = \langle \alpha \rangle / (0) \cong \mathbf{Z}$$

The kernel of d_1 is $\langle a, b, c \rangle$, and the image of d_2 is $\langle a - b + c \rangle$, hence

$$H_1^\Delta(\mathbf{T})) = \langle a, b, c \rangle / \langle a - b + c \rangle$$

In the homology group,

$$[c] = [c - (a - b + c)] = [b - a]$$

so the homology group is spanned by $[a]$ and $[b]$, and this is a torsion free span, for if $na + mb \in \langle a - b + c \rangle$, then $n = m = 0$. It follows that $H_1^\Delta(\mathbf{T}) \cong \mathbf{Z}^2$. Finally, we find that the kernel of d_2 is $\langle \Lambda - \Gamma \rangle$, and the image of d_3 (which we see as a map from $C_3(\mathbf{T}) = (0)$) is trivial, so $H_2^\Delta(\mathbf{T}) \cong (0)$.

Example. *Let us compute the homology of $\mathbf{RP}^2$. The kernel of d_0 is $\langle \alpha, \beta \rangle$, whereas the image of d_1 is $\langle \beta - \alpha \rangle$, so that $H_0^\Delta(\mathbf{RP}^2)$ is spanned by $[\alpha]$, and is easily verified to have no torsion, and is thus isomorphic to $\mathbf{Z}$. We calculate that*

$$d_1(na + mb + lc) = (n + m)(\beta - \alpha)$$

So that the kernel of d_1 is $\langle c, b - a \rangle$. The image of d_2 is found by the calculation

$$d_2(m\Lambda + n\Gamma) = m(c - b + a) + n(c - a + b)$$

to be $\langle c - b + a, c - a + b \rangle = \langle 2c, c - a + b \rangle$. Since in the homology group

$$[b - a] = [c]$$

the group is spanned by $[c]$, but this group has torsion, since $[2c] = 0$, so $H_1^\Delta(\mathbf{RP}^2) \cong \mathbf{Z}_2$. Finally, we find the kernel of d_2 to be trivial, so that $H_2^\Delta(\mathbf{RP}^2) = (0)$.

Example. *Let us compute the homology group $H_n^\Delta(S^n)$. The kernel of the map is obviously generated by $\Delta_1^n - \Delta_2^n$, since*

$$d(\Delta_1^n) = d(\Delta_2^n) \neq 0$$

It follows that $H_n^\Delta(S^n) \cong \mathbf{Z}$. We shall eventually describe $H_i^\Delta(S^n)$ for $i < n$, but some theory is required to prevent undue calculation.

3.4 Singular Homology

Simplicial homology is easy to calculate, but there is a different homology theory which is useful for theorems. We shall eventually show the theories coincide, so that we may learn about simplicial homology through this separate theory, which we call singular homology.

Given a topological space X, let $C_n(X) = \mathbf{Z}\langle C(\Delta^n, X)\rangle$, the free abelian group generated by continuous maps from the standard n-simplex to X. If we define, for $c \in C(\Delta^n, X)$,

$$dc = \sum (-1)^k c \circ \sigma_k$$

then we obtain a differential from $C_n(X)$ to $C_{n-1}(X)$. That this differential satisfies $d^2 = 0$ follows from the same basic argument as in the simplicial complex case. Here we instead have the relation $\sigma_i \circ \sigma_j = \sigma_{j-1} \circ \sigma_i$ for $i < j$. Thus we obtain a chain complex $C(X)$, which induces the homology groups $H_n(X)$, known as the **singular homology group**.

Chapter 4

Appendix: CW-Complexes

Algebraic topology is difficult to approach from the perspective of all the topological spaces, since general topological spaces are very pathological. We restrict ourselves to nice spaces. Manifolds are pleasant, but we can get away with a more general construction. These are the CW-complexes.

A **cell decomposition** of a space X together is a partition $\mathcal{C}$ of X into subsets C of X relatively homeomorphic to $B_{\mathbf{R}^n}$ for some $n \geqslant 0$. Elements of $\mathcal{C}$ are known as **cells**. A cell decomposition is **finite** if the partition is finite. The set of cells homeomorphic to $B_{\mathbf{R}^n}$, for a fixed $n \geqslant 0$, are the n-cells, denoted $\mathcal{C}_n$. The **dimension** of a decomposition, if it exists, is the largest n for which $\mathcal{C}_n$ is non-empty. The n-skeleton of a cell decomposition is the subspace $\bigcup \left(\bigcup_{i \leqslant n} \mathcal{C}_i\right)$ of X. A **cell complex** is a space X together with a fixed cell decomposition $\mathcal{C}$. In this case, we denote the n skeleton by X_n. A decomposition $\mathcal{C}$ of a Hausdorff space X is a **CW-decomposition** if

- (Extension Maps) For each cell $C \in \mathcal{C}_n$, there is a map $f : \mathbf{D}^n \to X$ such that $f|_{B_{\mathbf{R}^n}}$ is an embedding onto C. All zero cells are closed in X.

- (**C**losure finiteness) The closure of each n-cell intersects only finitely many other cells, and these other cells are contained in the $n-1$ skeleton.

- (**W**eak Topology) A subset D of X is closed if and only if $D \cap \overline{C}$ is closed in $\overline{C}$ for each cell $C \in \mathcal{C}$. Thus X possesses the weakest topology such that each map $f : \mathbf{D}^n \to X$ chosen above is continuous. This is superfluous if $\mathcal{C}$ is finite.

A **CW-complex** is a Hausdorff space X together with a fixed CW decomposition. It is the primary object of study in algebraic topology, since it is essentially a combinatorial object. The advantage of the weak topology is that $f : X \to Y$ is continuous if and only if $f|_{\overline{C}}$ is continuous for each $\overline{C}$, and thus on a certain disjoint union of $\mathbf{D}^n$, provided they are compatible with one another.

For each cell C in a CW complex $(X,\mathcal{C})$, fix a map $f_C : \mathbf{D}^n \to X$ which extends a homeomorphism, as in the definition of the complex. Then X is homeomorphic to a quotient of

$$\coprod_{k=0}^{\infty} \coprod_{C \in \mathcal{C}_n} \mathbf{D}^n$$

where x_C and y_D are identified if $f_C(x) = f_D(y)$. Since we have a surjective map g from the coproduct to X, by combining all f_C, we also have a surjective map $\tilde{g}$ from the quotient, which is injective by construction. It is also closed, since each f_C is closed (a map from a compact set is automatically closed), and the space has the weak topology. Thus $\tilde{g}$ is a homeomorphism.

Now let $(X,\mathcal{C})$ be a CW complex. Pick f_C for each $C \in \mathcal{C}_n$. Then

$$X_n \cong X_{n-1} \coprod \left(\coprod_{C \in \mathcal{C}_n} \mathbf{D}^n \right)$$

which identifies $x \in (C, \partial \mathbf{D}^n)$ with $f_C(x) \in X_{n-1}$. For each $i \leqslant j$, we have the embeddings $f_{ij} : X_i \to X_j$. Then we may consider the direct limit with respect to these mappings, and

$$X \cong \varinjlim X_i$$

where the direct limit $\varinjlim X_i$ is the quotient of $\coprod X_i$ obtained by identifying $x \in X_i$ with $f_{ij}(x)$, which has the topology such that A is closed if and only if the intersection of A and the image of X_i is closed in the image of X_i, viewed as homeomorphic to X_i. We have projection maps $f_i : X_i \to \varinjlim X_i$, which for $i < j$ satisfies the commutative diagram

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"'] & & X_j \arrow[dl, "f_j"] \\
& \varinjlim X_i &
\end{tikzcd}$$

because of how the quotient structure of $\varinjlim X_i$ is constructed, so by the weak topology on X, we obtain a continuous map $f : X \to \varinjlim$. The map is surjective, and injective, for each f_i is injective. The inverse map is also continuous, for if A is closed in X, $A \cap X_i$ is closed for each X_i, so $f(A \cap X_i) = f_i(A \cap X_i)$ is closed in the image of X_i, hence $f(A)$ is closed in X. Thus every CW complex is homeomorphic to a CW complex constructed inductively from a direct limit by attaching the boundary of n disks to the $n-1$ skeleton.

Example. *A* **graph** Γ *is a 1-dimensional CW complex, the simplest non-trivial example of a CW complex. zero cells are known as* **vertices**, *and one cells are known as* **edges**. *For each one cell C, there is a map $f_C : [0,1] \to \Gamma$ such that $f_C(0)$ and $f_C(1)$ are vertices, known as the* **ends** *of C. These ends are unique, for $f_C([0,1]) = \overline{C}$, so the ends can be identified as the elements in $\overline{C} - C$. An edge can connect a vertex to itself.*

Example. *The n-sphere S^n has a CW decomposition. Take a partition*

$$\mathcal{C} = \{\{(1,0,\dots,0)\}, S^n - \{(1,0,\dots,0)\}\}$$

By stereographic projection, we obtain a homeomorphism

$$\pi : S^n - \{(1,0,\dots,0)\} \to \mathbf{R}^n$$

$\mathbf{R}^n$ can be shrunk down to $B_{\mathbf{R}^n}$ by a map f. If $x \to \infty$ in $\mathbf{R}^n$, then $\pi^{-1}(x) \to (1,0,\dots,0)$, so the map

$$\pi^{-1} \circ f^{-1} : B_{\mathbf{R}^n} \to S^n$$

can be uniquely extended to $\mathbf{D}^n$ by mapping the boundary of the disk to $(1,0,\dots,0)$. Thus a CW complex for S^n consists of a one cell and a zero cell. The corresponding inductive construction takes $X_0 = \{x_0\}$, and attaches $\mathbf{D}^n$ to the point via the trivial map $f : \mathbf{D}^n \to \{x_0\}$. This follows because $S^n \cong \mathbf{D}^n/\partial\mathbf{D}^n$.

Example. *Real projective space $\mathbf{RP}^n$ has a CW decomposition. The space is the quotient of all lines through the origin in $\mathbf{R}^{n+1}$.*

$$\mathbf{RP}^n = (\mathbf{R}^{n+1} - \{0\})/(x \sim \lambda x : \lambda \in \mathbf{R} - \{0\}, x \in \mathbf{R}^{n+1} - \{0\})$$

The space may also be described, by throwing away redundant points, as

$$\mathbf{RP}^n \cong S^n/(x \sim -x : x \in S^n)$$

First, we notice that we may throw away half the points on the sphere, keeping only the top hemisphere of the sphere. Flattening this, we obtain that

$$\mathbf{RP}^n \cong \mathbf{D}^n/(x \sim -x : x \in \partial \mathbf{D}^n)$$

But $\partial \mathbf{D}^n \cong S^{n-1}$, and $\mathbf{RP}^{n-1}$ is obtained from S^{n-1} by attaching opposite points, so essentially

$$\mathbf{RP}^n = \mathbf{D}^n \coprod_f \mathbf{RP}^{n-1}$$

where $f : \partial \mathbf{D}^n \to \mathbf{RP}^{n-1}$ is just the projection map onto the quotient. Since $\mathbf{RP}^1 \cong S^1$ is a 1-dimensional CW complex, by a recursive construction, $\mathbf{RP}^n$ is obtained from an $n-1$ dimensional CW complex by attaching a single n-dimensional unit disk. It is interesting to take this to the extreme, and consider

$$\varinjlim \mathbf{RP}^n = \mathbf{RP}^\infty$$

This CW complex can be seen as the set of lines in $\mathbf{R}^\infty$ through the origin.

Example. *One can also consider complex projective space*

$$\mathbf{CP}^n = (\mathbf{C}^{n+1} - \{0\})/(x \sim \lambda x : \lambda \in \mathbf{C} - \{0\}, x \in \mathbf{C}^{n+1} - \{0\})$$

As with real projective space, we can flatten out the quotient to the sphere

$$\mathbf{CP}^n = S^{2n+1}/(x \sim \lambda x : |\lambda| = 1, x \in S^{2n+1})$$

One throws away duplicated points to obtain that the space is really

$$\mathbf{CP}^n = \mathbf{D}^{2n}/(x \sim \lambda x : x \in \partial \mathbf{D}^{2n}, |\lambda| = 1)$$

But, as with the real case, we can write $\mathbf{CP}^n = \mathbf{D}^{2n} \cup_f \mathbf{CP}^{2n-1}$, where the map $f : \partial \mathbf{D}^{2n} \to \mathbf{CP}^{2n-1}$ is just the projection, since $\partial \mathbf{D}^{2n} = S^{2n-1}$. We can then constructively build up a CW complex for all $\mathbf{CP}^n$, since $\mathbf{CP}^1 \cong S^2$ is a CW complex. It is interesting to note that the CW complex of $\mathbf{CP}^{2n}$ can be constructed only using even dimensional disks.

A **subcomplex** of a CW complex $(X, \mathcal{C})$ is a closed subspace A of X which is the union of some number of cells in X. A tuple (X, A), where A is a subcomplex of X, is known as a **CW pair**. Particular examples include $(\mathbf{CP}^i, \mathbf{CP}^j)$ and $(\mathbf{RP}^i, \mathbf{RP}^j)$, for $i > j$. S^i is a subcomplex of S^j if we give S^i a different CW structure, since S^{n+1} can be obtained from a CW complex for S^n by attaching two copies of $\mathbf{D}^n$ at the boundary. We may then consider

$$S^\infty = \varinjlim S^n$$

and $\mathbf{RP}^\infty$ can be constructed from S^∞ in the obvious way.

Theorem 4.1. *A compact subset of a CW complex X is contained in a finite subcomplex.*

Proof. s $\square$

Example. *Let $c : [0,1] \to X$ be a path between two vertices v and w. Then c corresponds to a unique discrete sequence of edges in X connecting v and w – a graph theoretic path. $c([0,1])$ is compact, and is therefore contained in a finite subcomplex.*

4.1 Operations on Complexes

There are some useful operations one can perform on CW complexes. Consider two complexes $(X, \mathcal{C})$ and $(Y, \mathcal{C}')$. We shall form the product complex $(X \times Y, \mathcal{C} \times \mathcal{C}')$, where if C is a n cell in X, and D is an m cell in Y, then $C \times D$ is an $n+m$ cell in $X \times Y$. These cells cover $X \times Y$. One verifies the properties of a CW complex quite simply. The only problem is that this construction has a slightly weaker topology than the product topology, in the case when we take infinite dimensional CW complexes. This rarely causes problems.

Let (X, A) be a CW pair, where $X = \varinjlim X_i$. We shall ascribe a CW complex $Y = \varinjlim Y_i$ homeomorphic to X/A. Let $\pi_i : X_i \to X$ embed X_i in X, and let f_n attach X_n to X_{n-1}. Define $Y_0 = \{x \in X_0 : \pi_1(x) \in A\} \cup \{a\}$, where a is a new point corresponding to the collapse of A.