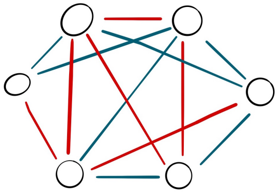
Salem Sets Avoiding Patterns

Jacob Denson
The University of Wisconsin-Madison

March 20, 2021

General Research Question



- Phenomenon: Structure appears in suitably large objects.
- Like Ramsey Theory, but with a more analytical foundation, e.g. Geometric Measure Theory / Harmonic Analysis.

Examples

- How large can a subset X of $\mathbf{R}^d$ be such that no right angle is formed by any three points in X.
- How large can a subset of $\mathbf{R}^d$ be, such that the distances between any two points is irrational?
- How large can $X \subset \mathbf{R}^d$ be such that X does not contain 2^d points forming a paralleliped?
- How large can a subgroup G of $\mathbf{R}^d$ be such that G does not contain any points in $\mathbf{Q}^d$?

Problem Isn't Well Posed

- If $S \subset \mathbf{R}^d$ has positive measure, it cannot avoid these patterns.
- We can find discrete sets $S \subset \mathbf{R}^d$ with $\#(S)$ arbitrarily large avoiding these patterns.
- To make the problem well posed, we need a measure of size 'between' cardinality and Lebesgue measure.

Dimension Theory

- We use *fractal dimension* to measure largeness / thickness.
  - It takes N^1 sidelength $1/N$ intervals to cover $[0,1]$.
  - It takes N^2 sidelength $1/N$ squares to cover $[0,1]^2$.
  - It takes N^3 sidelength $1/N$ cubes to cover $[0,1]^3$.
- A set $X \subset \mathbf{R}^d$ has *Minkowski dimension* at least s if it takes at least $\Omega(N^s)$ radius $1/N$ balls to cover X.
- *Hausdorff dimension* $\approx$ Minkowski dimension for compact X.

The Problem

- **Avoidance Problem**: Given $Z \subset (\mathbf{R}^d)^n$, find $X \subset \mathrm{R}^d$ with large dimension such that for distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We say X *avoids* Z.
- Let $Z = \{(x, y, z) \in (\mathbf{R}^d)^3 : (x - z) \cdot (y - z) = 0\}$.
  - $X \subset \mathbf{R}^d$ avoids Z iff X does not contain any right angles.
- How does the geometry of Z help us?
- e.g. Z is a degree 2 algebraic hypersurface in last example.

Results in Literature

- Mathé (2012): If $Z \subset (\mathbf{R}^d)^n$ is an algebraic hypersurfaces specified by a rational coefficient polynomial with degree at most r, then we can $X \subset \mathbf{R}^d$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = d/r.$$
- Fraser and Pramanik (2016): If $Z \subset (\mathbf{R}^d)^n$ is a smooth hypersurface with dimension at most m, we can find $X \subset \mathbf{R}^d$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = \frac{nd - m}{n - 1}.$$
- What if we use less rigid geometric information, i.e. the fractal dimension of the set Z?

Our Results

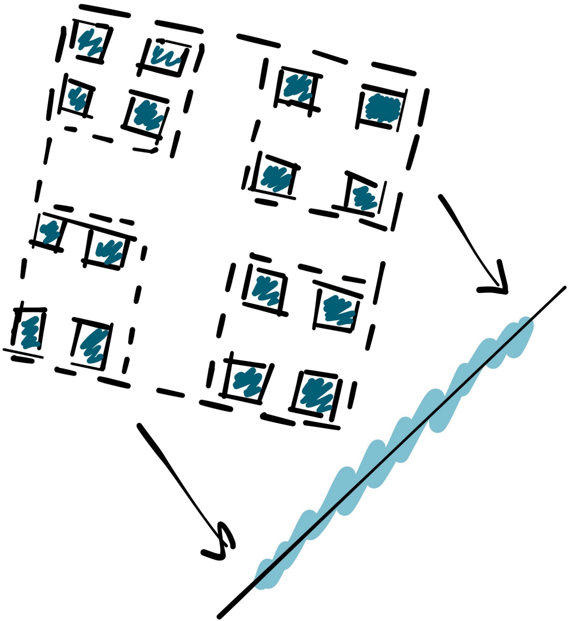
- D, Pramanik, and Zahl (2019): If $Z \subset (\mathbf{R}^d)^n$ is a set with Minkowski dimension bounded by s, we can find $X \subset \mathbf{R}^d$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = \frac{nd - s}{n - 1}.$$
- D (2019): If $Z \subset \mathbf{R}^n$, and $\pi : \mathbf{R}^n \to \mathbf{R}^m$ is a rational coefficient projection map such that $\pi(Z)$ has Minkowski dimension bounded by s, then we can find $X \subset \mathbf{R}$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = \frac{m - s}{m}.$$

Applications

- Given a subgroup $H \subset \mathbf{R}$, is it possible to find $G \subset \mathbf{R}$ such that $G + H = \mathbf{R}$?
- **Theorem:** Let $H \subset \mathbf{R}$ be a set with Minkowski dimension s. Then we can find an additive subgroup $G \subset \mathbf{R}$ such that $G \cap H \subset \{0\}$ and such that $\dim_{\mathbf{H}}(G) = 1 - s$.
  - It seems likely that whatever higher dimensional generalization of our results should construct $G \subset \mathbf{R}^d$ with Hausdorff dimension $d - s$ for any group H of Minkowski dimension s.

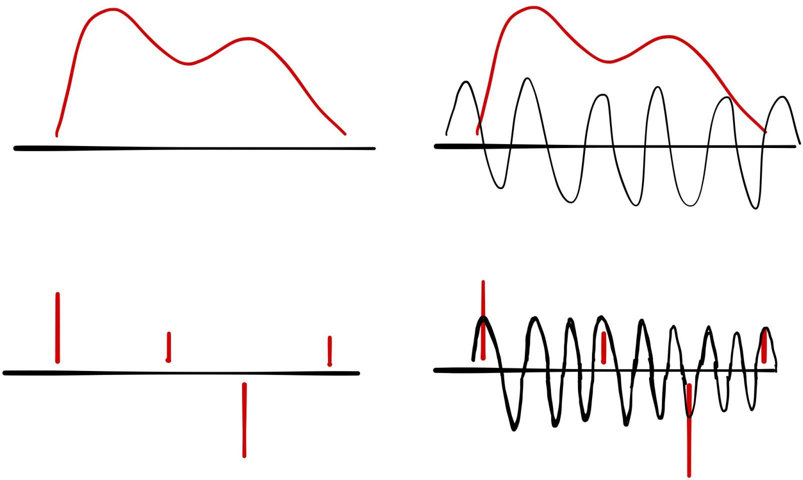
Applications

- Since we use 'rough' geometric information about Z, our method can even consider 'avoidance problems on fractals'.
  - Consider the Cantor dust E, with $\dim_{\mathbf{M}}(E) = 1$.
  - If $\pi : E \to \mathbf{R}$ is a projection onto the line at a 45° angle, $\pi(E)$ is an interval.
  - Let
$$Z = \left\{ (x, y, z) \in \pi(E)^3 : \begin{array}{c} \text{there is } x_0, y_0, z_0 \in E \\ \text{s.t. } (x_0 - z_0) \cdot (y_0 - z_0) = 0 \end{array} \right\}.$$
  - Basic considerations suggest that $\dim_{\mathbf{M}}(Z) = 2$, so that we can find $X \subset \pi(E)$ avoiding Z with Haudsorff dimension $1/2$.
  - Then $\pi^{-1}(X)$ avoids right angles and $\dim_{\mathbf{H}}(\pi^{-1}(X)) \geq 1/2$.
- (D, Pramanik, Zahl, 2019) used this technique to bound the existence of isosceles triangles on Lipschitz curves.



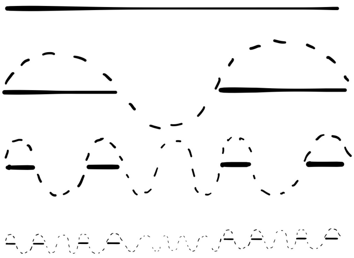
Fourier Dimension

- A set $X \subset \mathbf{R}^d$ has *Fourier dimension* at least s if there exists a finite measure μ supported on X such that $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$.
- Often gives much more structural information about a set than Minkowski dimension does.
- (Keleti, 1998) There exist an 'independent' set X with full Hausdorff dimension such that there exists no nontrivial solutions to $m_1x_1 + \dots + m_nx_n = 0$ for any $m \in \mathbf{Z}^n$ and $x_1, \dots, x_n \in X$.
- (Rudin, 1960) If X has Fourier dimension greater than $1/n$, then there exists some $m \in \mathbf{Z}^n$ and some $x_1, \dots, x_n \in X$ such that $m_1x_1 + \dots + m_nx_n = 0$.



Getting Fourier Dimension Bounds Is Hard

- For any set X, $\dim_{\mathbf{F}}(X) \leq \dim_{\mathbf{H}}(X) \leq \dim_{\mathbf{M}}(X)$.
- X is a *Salem Set* if $\dim_{\mathbf{F}}(X) = \dim_{\mathbf{H}}(X)$.
- All sets are Salem 'generically', but for most explicit constructions, the Fourier dimension is equal to zero.
- If X is the Cantor set, then $\dim_{\mathbf{H}}(X) = \dim_{\mathbf{M}}(X) = \log_3(2)$, but $\dim_{\mathbf{F}}(X) = 0$.



- Heuristic: Typically need 'square root cancellation' to obtain optimal Fourier decay, e.g. by using randomness.

Salem Set Result

Theorem (2021, D)

If Z has lower Minkowski dimension bounded by s, we can find X avoiding Z with

$$dim_{\mathrm{H}}(X) = dim_{\mathrm{F}}(X) = \frac{nd - s}{n - 1/2}.$$

Theorem (2021, D)

If Z is the countable union of sets Z_n of the form

$$Z_n = \{(y, x_1, \dots, x_{n-1}) \in (\mathrm{R}^d)^n : y = f_n(x_1, \dots, x_{n-1})\},$$

where $f_n \in C^\infty((\mathrm{R}^d)^n, \mathrm{R}^d)$, and for $1 \leq i \leq n-1$, $D_{x_i} f_n$ is an invertible matrix, then we can find X avoiding Z with

$$dim_{\mathrm{H}}(X) = dim_{\mathrm{F}}(X) = \frac{d}{n - 3/4}.$$

- D, Pramanik, and Zahl (2019): If $Z \subset (\mathbf{R}^d)^n$ is a set with Minkowski dimension bounded by s, we can find $X \subset \mathbf{R}^d$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = \frac{nd - s}{n - 1}.$$
- Fraser and Pramanik (2016): If $Z \subset (\mathbf{R}^d)^n$ is a smooth hypersurface with dimension at most m, we can find $X \subset \mathbf{R}^d$ avoiding Z with
$$\dim_{\mathbf{H}}(X) = \frac{nd - m}{n - 1}.$$

Thanks for listening!

If you're interesting in hearing more, contact me at jcdenson@wisc.edu.