Salem Sets Avoiding Rough Configurations

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May 15, 2020

Geometric measure theory explores the relationship between the geometry of a subset of Euclidean space, and regularity properties of the family of Borel measures supported on that set. From the perspective of harmonic analysis, it is often popular to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a set. In this paper, we study the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a 'rough pattern', in the sense of [3], we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any finite Borel measure μ on $\mathbf{R}^d$ is it's *Fourier dimension* $\dim_{\mathbf{F}}(\mu)$, defined as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup \left\{ |\hat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d \right\} < \infty. \tag{1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. This definition is analogous to the definition of the Hausdorff dimension; For $x \in \mathbf{R}^d$ and $\varepsilon > 0$, we let $B_\varepsilon(x)$ denote the open ball of radius ε centered at x. Then $\dim_{\mathbf{H}}(\mu)$ is the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup \left\{ \mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{R}^d, \varepsilon > 0 \right\} < \infty, \tag{2}$$

and then define $\dim_{\mathbf{H}}(E)$ to be the supremum of $\dim_{\mathbf{H}}(\mu)$ over all Borel measure μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most classical constructions of sets with fractional Hausdorff dimension are not Salem sets, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem.

Theorem 1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset [0,1]^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n - 1}, d\right)$$

such that for any distinct points $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$. Moreover, there is a measure μ supported on X such that for each $\xi \in \mathbf{Z}^d$,

$$|\hat{\mu}(\xi)| \leqslant \log(1 + |\xi|)^{1/2} |\xi|^{-\beta}.$$

Remark 2. *Theorem 1 strengthens the main result of [3]. Unlike in [3], the case of the problem $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $X - \pi(Z)$ is not necessarily a Salem set.*

A well-known result in this pattern avoidance setting is that sets with large Fourier dimension satisfy many algebraic relations. More precisely, if integer coefficients $m_1, \ldots, m_n \in \mathbf{Z}$ are fixed, and we consider a compact set $E \subset \mathbf{R}$ with $\dim_{\mathbf{F}}(E) > 2/n$, then the sum set $m_1 E + \cdots + m_n E$ contains an open interval. It follows by a slight modification of these coefficients that if $E \subset \mathbf{R}$ and $\dim_{\mathbf{F}}(E) > 2/n$, then there exists $m_1, \ldots, m_n \in \mathbf{Z}$, distinct points $x_1, \ldots, x_n \in E$, and an additional integer $a \in \mathbf{Z}$, such that

$$m_1 x_1 + \cdots + m_n x_n = a. \tag{3}$$

It is an interesting to determine how tight this result is. In [2], T.W. Körner constructs a set E with Fourier dimension $1/(n-1)$ such that for non-zero $m \in \mathbf{Z}^n$, and $a \in \mathbf{Z}$, E does not contain distinct points $x_1, \ldots, x_n$ solving (3). If, for each nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$, we consider the set

$$Z_{m,a} = \{(x_1, \ldots, x_n) \in [0,1]^n : m_1 x_1 + \cdots + m_n x_n = a\},$$

then $Z_{m,a}$ is a subset of an $n-1$ dimensional hyperplane, and thus can be easily seen to have Minkowski dimension $n-1$. It follows that we can apply 1 to $Z = \bigcup \{Z_{m,a} : m \neq 0, a \in \mathbf{Z}\}$ to obtain a Salem set $X \subset [0,1)$ of dimension

$$\frac{n - (n-1)}{n-1} = \frac{1}{n-1},$$

such that $(x_1, \ldots, x_n) \notin Z$ for each distinct $x_1, \ldots, x_n \in X$. Thus X avoids solutions to (3) for all nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$. Thus we see Theorem 1

generalizes Körner's result, and thus shows the result depends little on the arithmetic properties of the pattern Körner avoids, but rather, depends only on the 'thickness' of the family of tuples $(x_1, \ldots, x_n)$ satisfying the pattern.

Since we are working with compactly supported sets, working in $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, which has the advantage of being compact. For a finite measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{k \in \mathbf{R}^d} |\hat{\mu}(k)||k|^{s/2} < \infty. \tag{4}$$

We can then define the Fourier dimension of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Similarily, $\mathbf{T}^d$ has a natural quotient metric induced from $\mathbf{R}^d$, so we can consider open balls $B_\varepsilon(x + \mathbf{Z}^d)$, and thus define the Hausdorff dimension of finite Borel measures and sets on $\mathbf{T}^d$. It is a simple consequence of either the Poisson summation formula or the uncertainty principle that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1) is equivalent to the more discrete condition

$$\sup_{k \in \mathbf{Z}^d} |\hat{\mu}(k)||k|^{s/2} < \infty. \tag{5}$$

A proof is given in [4, Lemma 39]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x) \, d\mu^*(x) = \int_{\mathbf{R}^d} f(x + \mathbf{Z}^d) \, d\mu(x),$$

Another application of the uncertainty principle or Poisson summation thus shows that $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$. Thus Theorem 1 is clearly equivalent to it's periodic variant.

Theorem 3. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset \mathbf{T}^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n - 1}, d\right)$$

such that for any distinct points $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$. Moreover, there is a measure μ supported on X such that for each $\xi \in \mathbf{Z}^d$,

$$|\hat{\mu}(\xi)| \leqslant \log(1 + |\xi|)^{1/2} |\xi|^{-\beta}.$$

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To construct our set, we rely on a Baire-category argument for the purpose of our construction. Thus we consider a complete metric space X, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in X$, the set E satisfies the conclusions of Theorem 3, in the sense that the set of pairs $(E, \mu) \in X$ failing to satisfy the conclusions of Theorem 3 form a set of first category in X. It follows that Theorem 3 holds in a 'generic' sense for elements of X.

Once we have setup the appropriate metric space X, our approach is quite similar to our construction in [3], relying on a random selection procedure, which is now exploited in this paper to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach, is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [3]. But the Baire category argument makes our proof simpler to read, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern.

1 Notation

- Given a metric space X, $x \in X$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For a given set $E \subset X$ and $\varepsilon > 0$, we let
$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x).$$

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let
$$|x| = \min\{|x + n| : n \in \mathbf{Z}\},$$
and for $x \in \mathbf{T}^d$, we let $|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}$. The standard metric on $\mathbf{T}^d$ then takes the distance between $x, y \in \mathbf{T}^d$ is $|x - y|$.

- For a positive integer N, we let $[N] = \{1, \dots, N\}$

- We will need to consider a standard mollifier. Throughout the paper, we fix a smooth probability density $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for

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$|x| \geqslant 2/5$, and such that for each $x \in \mathbf{T}^d$

$$\sum_{k \in \{0,1\}^d} \phi(x + k/2) = 2^d.$$

For each $\varepsilon \in (0, 1)$, we can then define $\phi_\varepsilon \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_\varepsilon(x) = \begin{cases} \varepsilon^{-d}\phi(x/\varepsilon) & : |x| < \varepsilon, \\ 0 & : \text{otherwise}. \end{cases}$$

Then ϕ_ε is a smooth probability density, $\phi_\varepsilon(x) = 0$ for $|x| \geqslant \varepsilon$, and if $\varepsilon = 1/N$, and $x \in \mathbf{T}^d$,

$$\sum_{k \in [2N]^d} \phi_{1/N}(x + k/2N) = (2N)^d. \tag{6}$$

As $\varepsilon \to 0$, ϕ_ε converges weakly to the Dirac delta function at the origin, so that for each $k \in \mathbf{Z}^d$,

$$\lim_{\varepsilon \to 0} \widehat{\phi}_\varepsilon(k) = 1. \tag{7}$$

Moreover, for each $\alpha > 0$, there exists $C_\alpha > 0$ such that for all $\varepsilon > 0$ and non-zero $k \in \mathbf{Z}^d$,

$$|\widehat{\phi}_\varepsilon(k)| \leqslant \frac{C_\alpha}{\varepsilon^\alpha |k|^\alpha}, \tag{8}$$

which can be proven by using the rescaling symmetries of the Fourier transform on $\mathbf{R}^d$.

- Our random construction involves a probabilistic concentration of measure argument. Define a convex function $\psi_2 : [0, \infty) \to [0, \infty)$ by setting

$$\psi_2(t) = e^{t^2} - 1,$$

The function ψ_2 induces an Orlicz norm on the family of scalar valued random variables over a probability space by setting, for each random variable X,

$$\|X\|_{\psi_2(L)} = \inf\{A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \leqslant 1\}.$$

The family of random variables with $\|X\|_{\psi_2(L)} < \infty$ are known as *sub-gaussian random variables*. Here are the important properties of sub-gaussian random variables which we use in this paper:

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- If $\|X\|_{\psi_2(L)} \leqslant A$, then for each $t \geqslant 0$,
$$\mathbf{P}\left(|X| \geqslant t\right) \leqslant 10 \exp\left(-t^2/10A^2\right).$$

Thus Subgaussian random variables have Gaussian tails.

- If $|X| \leqslant A$ almost surely, then $\|X\|_{\psi_2(L)} \leqslant 10A$. Thus bounded random variables are subgaussian.

- If $X_1, \ldots, X_N$ are *independent*, then
$$\|X_1 + \cdots + X_N\|_{\psi_2(L)} \leqslant 10\left(\|X_1\|^2_{\psi_2(L)} + \cdots + \|X_N\|^2_{\psi_2(L)}\right)^{1/2}.$$

This is an equivalent way to state *Hoeffding's Inequality*, and we refer to an application of this inequality as an application of Hoeffding's inequality.

Roughly speaking, if X is a random variable with $\|X\|_{\psi_2(L)} \leqslant A$, we can think of X as being sharply concentrated in the region $[-A, A]$. The Orlicz norm thus provides a convenient way to quantify concentration phenomena.

Remark 4. *The constants involved in these statements are suboptimal, but will suffice for our purposes. Proofs can be found in Chapter 2 of [5].*

2 A Complete Metric Space

Let us construct a metric space appropriate for our task. We proceed analogously to (TODO: KORNER CITE) We begin with two simple metric spaces:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance
$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$
then $(\mathcal{E}, d_H)$ forms a complete metric space.

- Given any positive sequence of real numbers $\{A(k)\}$, we let $M(A)$ consist of the class of all finite Borel measures μ supported on $\mathbf{T}^d$ such that the quantity
$$\|\mu\|_A = \sup_{k \neq 0} \frac{|\widehat{\mu}(k)|}{|A(k)|}$$

is finite. Then $\|\cdot\|_A$ is a norm, and $(M(A), \|\cdot\|_A)$ is a Banach space. It is important to note that if there is a sequence of measures $\mu_k \in M(A)$ and $\mu \in M(A)$ such that

$$\lim_{k \to \infty} \|\mu_k - \mu\|_A = 0,$$

then the sequence $\{\mu_k\}$ converges weakly to μ. We will assume that the choice of $\{A(k)\}$ is such that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for each $k \neq 0$,

$$\frac{1}{C_\varepsilon} \frac{1}{|k|^{\beta-\varepsilon}} \leqslant A(k) \leqslant C_\varepsilon \frac{1}{|k|^{\beta+\varepsilon}}. \tag{9}$$

In particular, (9) implies that $C^\infty(\mathbf{T}^d) \subset M(A)$, where $C^\infty(\mathbf{T}^d)$ is the space of smooth functions with support contained in $\mathbf{T}^d$.

From these spaces, we let X be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(A)$, where μ is a probability measure with $\mathrm{supp}(\mu) \subset E$, and

$$\lim_{|k| \to \infty} \frac{|\widehat{\mu}(k)|}{A(k)} = 0.$$

It is easy to see X is a closed subset of $\mathcal{E} \times M(A)$ under the product topology, and thus if we consider the product metric

$$d_X((E, \mu), (F, \nu)) = \max\left(d_H(E, F), \|\mu - \nu\|_A\right),$$

then (X, d_X) is a complete metric space. The main reason we enforced that $\widehat{\mu}(k) = o(A(k))$ is so that smooth functions are dense in this space.

Lemma 5. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in X.*

Proof. We just apply a mollification strategy. Given $(E, \mu) \in X$ and consider $\varepsilon > 0$. For each $\delta > 0$, consider $\mu_\delta = \mu * \phi_\delta$. Then $\mu_\delta \in C^\infty(\mathbf{T}^d)$, and if $\delta \leqslant \varepsilon$, then μ_δ is supported on $\overline{E_\varepsilon}$, and

$$d_H(E, \overline{E_\varepsilon}) \leqslant \varepsilon. \tag{10}$$

Since $(E, \mu) \in X$, there is N depending on μ such that for $|k| \geqslant N$, $|\widehat{\mu}(k)| \leqslant \varepsilon A(k)$. For $|k| \geqslant N$, we thus find that

$$|\widehat{\mu_\delta}(k) - \widehat{\mu}(k)| = |\widehat{\mu}(k)||\widehat{\phi_\delta}(k) - 1| \leqslant 2|\widehat{\mu}(k)| \leqslant 2\varepsilon A(k). \tag{11}$$

On the other hand, for suitably small δ, (7) implies that for $|k| \leqslant N$,

$$|\hat{\widehat{\phi}}_\delta(k) - 1| \leqslant \varepsilon A(k).$$

But this implies that for $|k| \leqslant N$,

$$|\widehat{\mu}_\delta(k) - \widehat{\mu}(k)| = |\widehat{\mu}(k)||\hat{\widehat{\phi}}_\delta(k) - 1| \leqslant \varepsilon A(k). \tag{12}$$

Thus we conclude that for suitably small δ, $\|\widehat{\mu}_\delta - \widehat{\mu}\|_A \leqslant 2\varepsilon$. But combining (10), (11), and (12), we conclude

$$d_X((E, \mu), (\overline{E_\varepsilon}), \mu_\delta) \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. $\qquad\square$

Lemma 6. *For quasi-all $(E, \mu) \in X$, $supp(\mu) = E$.*

Proof. For each closed set $K \subset \mathbf{T}^d$ with nonempty interior, let $A(K)$ be the family of all $(E, \mu) \in X$ such that $E \cap K \neq \varnothing$ and $\mu(K) = 0$. The set $A(K)$ is clearly closed in X. Moreover, this set is nowhere dense; Fix $(E, \mu) \in A(K)$ and $\varepsilon > 0$. Then we can find a smooth probability density $\psi \in C^\infty(\mathbf{T}^d)$ supported on $K^\circ \cap E_\varepsilon$. We then define $\mu_\varepsilon = (1 - \varepsilon)\mu + \varepsilon\psi$. Now

$$\|\mu - \mu_\varepsilon\|_A \leqslant \varepsilon \left(\|\mu\|_A + \|\psi\|_A \right) \lesssim \varepsilon.$$

Moreover, μ_ε is supported on $\overline{E_\varepsilon}$, so

$$d_X((E, \mu), (\overline{E_\varepsilon}, \mu_\varepsilon)) \lesssim \varepsilon.$$

If we let $A = \bigcup_{n=1}^\infty A(K_n)$, where $\{K_n\}$ is the family of all closed cubes in $\mathbf{T}^d$ whose corners have rational coordinates, then quasi-all $(E, \mu) \in X$ belong to A^c. But A^c is precisely the family of pairs (E, μ) with $\operatorname{supp}(\mu) = E$. $\qquad\square$

It is a general heuristic that quasi-all elements of a metric space are as 'thin as possible'. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of X to be as low as possible. For each $(E, \mu) \in X$, the condition that $\mu \in M(A)$, combined with (9) implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so we might expect the quasi-all $\mu \in M(A)$ have Fourier dimension *equal to* β. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that quasi-all $\mu \in M(A)$ have Hausdorff dimension β.

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Lemma 7. *For quasi-all $(E, \mu) \in X$, E is a Salem set of dimension β.*

Proof. For each $\alpha > \beta$, let $A(\alpha, \varepsilon) = \{(E, \mu) \in \mathcal{G} : H^\alpha(E) < \varepsilon\}$, where H^α is the Hausdorff measure of dimension α on $\mathbf{T}^d$, i.e.

$$H^\alpha(E) = \liminf_{\delta \to 0} \left\{ \sum_{k=1}^\infty \varepsilon_k^\alpha : E \subset \bigcup_{k=1}^\infty B_{\varepsilon_k}(x_k) \text{ and } \varepsilon_k \leqslant \delta \text{ for all } k \right\}.$$

Then $A(\alpha, \varepsilon)$ is an open set, and

$$\bigcap_{n=1}^\infty \bigcap_{m=1}^\infty A(\beta + 1/n, 1/m)$$

is precisely the family of $(E, \mu) \in X$ such that E has Hausdorff dimension β. Certainly any E in this family must have $H^\alpha(E) = 0$ for all $\alpha > \beta$, so $\dim_{\mathbf{H}}(E) \leqslant \beta$. But the condition that $\mu \in M(A)$ implies $\dim_{\mathbf{F}}(\mu) \geqslant \beta$. Thus

$$\beta \geqslant \dim_{\mathbf{H}}(E) \geqslant \dim_{\mathbf{F}}(E) \geqslant \dim_{\mathbf{F}}(\mu) \geqslant \beta,$$

hence all these quantities are equal to β. Thus it suffices to show that $A(\alpha, \varepsilon)$ is dense in X for $\alpha > \beta$ and $\varepsilon > 0$.

Fix $(E, \mu) \in X$ and $\varepsilon > 0$. Without loss of generality, we may assume $\mu \in C^\infty(\mathbf{T}^d)$. Consider large integers N and M, and write $\varepsilon = 1/NM$. Then construct a family of independant, uniformly distributed random variables $X_a \in [M]^d$ for each $a \in [2N]^d$. Define a measure ν such that for each $x \in \mathbf{T}^d$,

$$d\nu(x) = \left(\sum_{a \in [2N]^d} \phi_\varepsilon \left(x - \frac{a}{2N} - \frac{X_a}{2NM} \right) \right) d\mu(x).$$

Adding a discrete number of points to the support of ν if necessary to cover the set E by a ε net, ν is supported on a set F with $d_H(E, F) \leqslant \varepsilon$ which also satisfies

$$H^\alpha(F) \leqslant (2N)^d/(NM)^\alpha = 2^d N^{d-\alpha} M^{-\alpha}.$$

$\square$

3 BLAH

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Fix $W \subset \mathbf{T}^d$ with $\underline{\dim}_{\mathbf{M}}(W) \leqslant \alpha$. Then

$$\liminf_{\varepsilon \to 0} \log_\varepsilon |W(\varepsilon)| \leqslant \alpha.$$

Thus there exists arbitrarily small $\varepsilon > 0$ and arbitrarily small $\delta > 0$ such that

$$|W_\varepsilon| \leqslant \varepsilon^{nd-\alpha-\delta}.$$

Proof. Let $X_1, \ldots, X_N$ be uniformly distributed on $\mathbf{T}^d$. For each $i \in \{1, \ldots, N\}^n$ such that $i_1, \ldots, i_n$ are distinct, consider the random vector $X_i = (X_{i_1}, \ldots, X_{i_n})$. Then X_i is uniformly distributed on $\mathbf{T}^{nd}$, and so

$$\mathbf{P}(d(X_i, W) \leqslant \varepsilon) \leqslant \varepsilon^{nd-\alpha-\delta}.$$

If M denotes the number of indices i such that $d(X_i, W) \leqslant \varepsilon$, then by linearity of expectation we conclude that $\mathbf{E}(M) \leqslant N^n \varepsilon^{nd-\alpha-\delta}$. Applying Markov's inequality, we conclude that

$$\mathbf{P}(M \geqslant 4N^n \varepsilon^{nd-\alpha-\delta}) \leqslant 1/4.$$

Now consider the measure $\nu = \delta_{X_1} + \cdots + \delta_{X_N}$. For each $k \in \mathbf{Z}^d$,

$$\widehat{\nu}(k) = e^{2\pi i k \cdot X_1} + \cdots + e^{2\pi i k \cdot X_N}.$$

Thus $\widehat{\nu}(k)$ is the sum of N bounded independent random variables, so we can apply Hoeffding's inequality to conclude that for each $t \geqslant 0$

$$\mathbf{P}(|\widehat{\nu}(k)| \geqslant t) \leqslant 2\exp(-2t^2/N).$$

A union bound allows us to conclude that

$$\mathbf{P}\left(\text{for all } |k| \leqslant R,\ |\widehat{\nu}(k)| \leqslant t\right) \geqslant 1 - 2^{d+1} R^d \exp(-2t^2/N).$$

Thus provided

$$2^{d+1} R^d \exp(-2t^2/N) \leqslant 1/4, \tag{13}$$

we conclude that there exists a choice of $X_1, \ldots, X_N \in \mathbf{T}^d$ such that if

$$I = \{i \in \{1, \ldots, N\}^n : i_1, \ldots, i_n \text{ are distinct and } d(X_i, W) \leqslant \varepsilon\},$$

then $\#(I) \leqslant 4N^n \varepsilon^{nd-\alpha-\delta}$, and for all $|k| \leqslant R$, $|\hat{\nu}(k)| \leqslant t$. In particular, if we choose

$$N = \lfloor \varepsilon^{-\frac{nd-\alpha-\delta}{n-1/2}} \rfloor,$$

choose $R = \varepsilon^{-10}$, and choose

$$t = 5d^{1/2} \cdot \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2},$$

then using the fact that $\varepsilon \leqslant 1/2$, we conclude that (13) is satisfied. The standard (L^1, L^∞) bound for the Fourier transform then implies that for each $k \in \mathbf{Z}^d$,

$$\left\| \sum_{i \in I} \widehat{\delta_{X_{i_1}}} \right\|_{L^\infty(\mathbf{Z}^d)} \leqslant \#(I) \leqslant 4N^n \varepsilon^{nd-\alpha-\delta} \leqslant 4N^{1/2} \leqslant 4\varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}}.$$

Define a probability measure

$$\mu = \frac{1}{N - \#(I)} \left(\nu - \sum_{i \in I} \delta_{X_{i_1}} \right),$$

If ε is sufficiently small, so that N is sufficiently large, then for each $k \in \mathbf{Z}^d$ with $|k| \leqslant R = \varepsilon^{-10}$,

$$|\hat{\mu}(k)| \leqslant \frac{1}{N - 4N^{1/2}} \left(5d^{1/2} \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2} + 4\varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \right)$$
$$\leqslant 9.5d^{1/2} N^{-1} \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2} \leqslant 10d^{1/2} \varepsilon^{\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2}. \quad \square$$

Following through with the proof leads to a set with Fourier dimension

$$\frac{nd - \alpha}{n - 1/2}$$

Which isn't quite enough for what we want.

5   A Family of Cubes

Fix two integer-valued sequences $\{K_m : m \geqslant 1\}$ and $\{M_m : m \geqslant 1\}$. For convenience, we also define $N_m = K_m M_m$ for $m \geqslant 1$. We then define two sequences of real numbers $\{l_m : m \geqslant 0\}$ and $\{r_m : m \geqslant 0\}$, by

$$l_m = \frac{1}{N_1 \ldots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \ldots N_{m-1} M_m}.$$

For each $m, d \geqslant 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $\sigma = \sigma_0 \sigma_1 \ldots \sigma_{2k} \in \Sigma_m^d$, we define a vector $a(\sigma) \in (l_m \mathbf{Z})^d$ by setting

$$a(\sigma) = \sigma_0 + \sum_{k=1}^{m} \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k$$

Then each string $\sigma \in \Sigma_m^d$ can be identified with the sidelength l_m cube $Q(\sigma)$ with left-hand corner lies at $a(\sigma)$, i.e. the cube

$$Q(\sigma) = \prod_{i=1}^{d} [a(\sigma)_i, a(\sigma)_i + l_m].$$

Similarily, for each string $\sigma = \sigma_0 \ldots \sigma_{2m-1} \in \Pi_m^d$, we define a vector $a(\sigma) \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leqslant j \leqslant d$,

$$a(\sigma) = \sigma_0 + \left(\sum_{k=1}^{m-1} \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k \right) + \sigma_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R(\sigma) = \prod_{i=1}^{d} [a(\sigma)_i, a(\sigma)_i + r_m].$$

We let $\mathcal{Q}_m^d = \{Q(\sigma) : \sigma \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R(\sigma) : \sigma \in \Pi_m^d\}$. We now list some important properties of this collection of cubes:

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- For each m, the two collections $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ form covers of $\mathbf{R}^d$.

- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.

- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarly, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{\sigma \in \Sigma_m^d : Q(\sigma) \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{\sigma \in \Pi_m^d : R(\sigma) \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j, and a third index $\sigma \in \Pi_m^d$, such that $R_\sigma \cap Q_i, R_\sigma \cap Q_j \neq \varnothing$.

6   A Family of Mollifiers

It will be useful to consider a family of C^∞ mollifiers, which we will use to ensure the Fourier transform of the measures we study have appropriate decay.

Lemma 8. *There exists a non-negative, C^∞ function ψ supported on $[-1,1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi = 1, \tag{14}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{15}$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0, 1]$, such that $\alpha(1/2 + x) = \alpha(1/2 - x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \leqslant \alpha(x) \leqslant 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|) = 1 - \alpha(1 - |x|).$$

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Symmetry considerations imply that $\int_{\mathbf{R}} \alpha + \beta = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x + m) + \beta(x + m) = 1. \tag{16}$$

If we set
$$\psi_0(x) = \alpha(x) + \beta(x),$$

The function $\psi(x_1, \ldots, x_d) = \psi_0(x_1) \ldots \psi_0(x_d)$ then satisfies the constraints of the lemma. $\qquad\square$

Fix some choice of ψ given by Lemma 8. Since ψ is C^∞ and compactly supported, then for each $t \in [0, \infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\hat{\psi}(\xi)| < \infty. \tag{17}$$

Now we rescale the mollifier. For each integer $m \geqslant 1$, we let

$$\psi_m(x) = l_m^{-d} \cdot \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (14) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \tag{18}$$

Equation (15) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \tag{19}$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,
$$\widehat{\psi_m}(\xi) = \hat{\psi}(l_m \cdot \xi), \tag{20}$$

In particular, (20) implies that for each $t \geqslant 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)| |\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\hat{\psi}(\xi)| |\xi|^t. \tag{21}$$

Intuitively, $\{\psi_m\}$ is a 'uniform' family of wave packets, with ψ_m supported in phase space on $[-l_m, l_m]^d$, and in frequency space, essentially supported on $[-l_m^{-1}, l_m^{-1}]^d$.

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7 Comparison to Previous Paper

As in our previous paper, our proof of Theorem 1 will involve constructing a configuration avoiding set X by considering a nested decreasing family of sets $\{X_m : m \geqslant 0\}$, where $X_m \subset [0,1]^d$ is a $\mathcal{Q}_m$ discretized set, and then setting $X = \bigcap_{m \geqslant 0} X_m$. We find a strong cover of Z by sets $\{B_m\}$, where B_m is $\mathcal{Q}_m$ discretized. Provided X_m^d is disjoint from strongly non-diagonal cubes in B_m, we conclude that for any n distinct elements $x_1, \ldots, x_n \in X$, $(x_1, \ldots, x_n) \notin Z$. We now show that the technique of our last paper as stated fails to produce Salem sets.

Let us recap the approach of our last paper. To form X_{m+1}, we chose a cube $Q_R \in \mathcal{Q}_{m+1}(R)$ uniformly at random, for each $R \in \mathcal{R}_{m+1}(X_m)$. We then let $Y_{m+1} = \bigcup Q_R$. If $s \geqslant d$, and

$$K_{m+1} \approx M_{m+1}^{\frac{s-d}{dn-s}}, \tag{22}$$

then with non-zero probability, we proved there is $X_{m+1} \subset Y_{m+1}$ such that X_{m+1}^d avoids strongly non-diagonal cubes in B_{m+1}, and X_{m+1} contains at least half of the cubes in $\mathcal{Q}_{m+1}(Y_{m+1})$. Then X_{m+1} will be the union of at least M_{m+1}^{-d} cubes with sidelength l_{m+1}. Provided that $K_{m+1}, M_{m+1} \gg K_1, M_1, \ldots, K_m, M_m$, we have

$$M_{m+1}^{-d} \approx r_{m+1}^{-d} \approx l_{m+1}^{-\frac{dn-s}{n-1}}.$$

Thus X has lower Minkowski dimension at most $(dn - s)/(n - 1)$, and a more involved analysis shows the set has Hausdorff dimension exactly equal to $(dn - s)/(n - 1)$.

The approach detailed in the last paragraph is *not* guaranteed to produce a set with Fourier dimension t. Because X_{m+1} is random, it exhibits psuedorandomness properties with high probability. In particular, it supports probability measures whose Fourier transform has sharp decay. However, since the choice of the set Y_{m+1} is *not* chosen randomly from X_{m+1}, depending heavily on the set Z and the discretized set B_{m+1}, the set Y_{m+1} will in general not possess psuedorandomness properties. For instance, if μ is the probability measure induced by normalizing Lebesgue measure restricted to X_{m+1}, then with high probability,

$$\|\widehat{\mu}\|_{L^\infty(\mathbf{R}^d)} \approx l_m^t.$$

If ν is the probability measure induced by normalizing Lebesgue measure restricted to Y_{m+1}, then it is still possible for us to have

$$\|\widehat{\nu}\|_{L^\infty(\mathbf{R}^d)} \gtrsim 1.$$

For instance, this will be true if $\mathcal{Q}_{m+1}(X_{m+1}) - \mathcal{Q}_{m+1}(Y_{m+1})$ is a thickening of a subset of an arithmetic progression. Thus the method of our previous paper is not able to reliably produce Salem sets without further analysis on the psuedorandom properties of the sets $\{B_m\}$ we have to avoid.

In this paper, we take a different approach which avoids us having to analyze the psuedorandomness of the sets B_m. Instead of (22), we choose

$$K_{m+1} \approx M_{m+1}^{\frac{s}{dn-s}}.$$

Notice that $M_{m+1}^{\frac{s}{dn-s}} \geqslant M_{m+1}^{\frac{s-d}{dn-s}}$, so the set Y_{m+1} we will obtain will be a thinner set than X_m. In particular, Y_{m+1} will be covered by at most M_{m+1}^{-d} sidelength l_{m+1} cubes, and if $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$¦

$$r_{m+1}^{-d} \approx l_{m+1}^{-t}$$

sidelength l_{m+1} cubes, which implies X will have upper Minkowski dimension at most t. However, as a result, because the set Y_{m+1} is thinner, we find that Y_{m+1}^d is disjoint from the cubes in B_{m+1} with high probability. In particular, we can set $X_{m+1} = Y_{m+1}$. This means that X_{m+1} will be pseudorandom, and we should therefore expect X to be a Salem set of dimension t. The remainder of this paper is devoted to showing that these heuristics are correct.

8 Discrete Lemma

We now proceed to solve a discretized version of Theorem 1.

Proposition 9. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn - s)/2)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth measure compactly supported on T. Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leqslant (1/l_{m+1})^{s+\varepsilon}. \tag{23}$$

Then there exists a large constant $C(\mu_T, l_m, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geqslant C(\mu_T, l_m, n, d, s, \varepsilon), \tag{24}$$

and

$$M_{m+1}^{\frac{s}{dn-s}+c\varepsilon} \leqslant K_{m+1} \leqslant 2M_{m+1}^{\frac{s}{dn-s}+c\varepsilon}, \tag{25}$$

where

$$c = \frac{6dn}{(dn-s)^2},$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) *For any strongly non-diagonal cube*

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) $\mu_S(\mathbf{R}^d) \geqslant \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

(C) *If $|k| \leqslant 10l_{m+1}^{-d}$, $|\widehat{\mu_T}(k) - \widehat{\mu_S}(k)| \leqslant r_{m+1}^{d/2}\log(M_{m+1})$.*

(D) *If $|k| \geqslant 10l_{m+1}^{-d}$, $|\widehat{\mu_S}(k)| \leqslant |k|^{-d/2}$.*

Remark 10. *To make the statement of Proposition (9) more clean, we have hidden the explicit choice of constant $C(\mu_T, l_m, n, d, s, \varepsilon)$. But this constant can certainly be made explicit; such a choice can be made by ensuring that (24) implies (27), (33), (39), (49), (50), and (51) all hold.*

Proof of Proposition 9. First, we describe the construction of the set S, and the measure μ_S. For each string $\sigma \in \Pi_{m+1}^d$, let j_σ be a random integer vector chosen from $\{0, \ldots, K_{m+1} - 1\}^d$, such that the family $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ is an independent family of random variables. Then it is certainly true that for any $j \in [K_{m+1}]^d$,

$$\mathbf{P}(j_\sigma = j) = K_{m+1}^{-d}. \tag{26}$$

Then $\sigma j_\sigma \in \Sigma_{m+1}^d$. We can thus define a measure μ_S such that, for each $x \in \mathbf{R}^d$,

$$d\mu_S(x) = r_{m+1}^d \sum_{\sigma \in \Pi_{m+1}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then S is $\mathcal{Q}_{m+1}$ discretized, μ_S is supported on S, and $S \subset T$. Our goal is to show that, with non-zero probability, some choice of the family of indices

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$\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ yields a set S and a measure μ_S satisfying Properties (A) and (B) of Proposition 9. In our calculations, it will help us to decompose the measure μ_S into components roughly supported on sidelength r_{m+1}^d cubes. For each $\sigma \in \Pi_{m+1}(T)$, define a measure μ_σ such that for each $x \in \mathbf{R}^d$,

$$d\mu_\sigma(x) = r_{m+1}^d \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

Then $\mu_S = \sum_{\sigma \in \Pi_{m+1}^d(T)} \mu_\sigma$. We shall split the proof of Properties (A), (B), and (C) into several more managable lemmas.

Lemma 11. *If*

$$M_{m+1} \geqslant \left(3^d \sqrt{d} \cdot l_m \|\nabla\mu\|_{L^\infty(\mathbf{R}^d)} \right)^2, \tag{27}$$

then $\mu_S(\mathbf{R}^d) \geqslant \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

Proof. Fix $\sigma \in \Pi_{m+1}^d$. If $j_0, j_1 \in \{0, \ldots, K_{m+1} - 1\}^d$, then

$$|a(\sigma j_0) - a(\sigma j_1)| = |j_0 - j_1| \cdot l_{m+1} \leqslant (\sqrt{d} \cdot K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1}. \tag{28}$$

Together with (18), (28) implies

$$\begin{aligned}
&\left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_0))\mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_1))\mu_T(x) \right| \\
&\qquad \leqslant r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) \left| \mu_T(x + a(\sigma j_0)) - \mu_T(x + a(\sigma j_1)) \right| \\
&\qquad \leqslant \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1} \\
&\qquad = \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}.
\end{aligned} \tag{29}$$

Thus (29) implies that for each σ,

$$|\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \leqslant \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla\mu\|_{L^\infty(\mathbf{R}^d)}. \tag{30}$$

Furthermore, (19) implies

$$\begin{aligned}
\sum_{\sigma \in \Pi_{m+1}^d} &\mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \\
&= r_{m+1}^d \sum_{(\sigma,j) \in \Sigma_{m+1}^d} \mathbf{P}(j_\sigma = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_\sigma))d\mu_T(x) \\
&= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left(\sum_{(\sigma,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\
&= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \mu_T(\mathbf{R}^d) = \mu_T(\mathbf{R}^d).
\end{aligned} \tag{31}$$

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For all but at most $3^d r_{m+1}^{-d}$ indices $\sigma \in \Pi_{m+1}^d$, $\mu_\sigma = 0$ almost surely. Thus we can apply the triangle inequality together with (30) and (31) to conclude that

$$
\begin{aligned}
|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| &= \left| \sum\nolimits_{\sigma \in \Pi_{m+1}^d} \left[\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \right] \right| \\
&\leqslant \sum\nolimits_{\sigma \in \Pi_{m+1}^d} \left| \mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \right| \\
&\leqslant (3^d r_{m+1}^{-d}) \left(\sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \\
&= \left(3^d \sqrt{d} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \cdot r_{m+1} \\
&= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.
\end{aligned}
\tag{32}
$$

Thus (27) and (32) imply that, $|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| \leqslant M_{m+1}^{-1/2}$. $\qquad\square$

Lemma 12. *If*

$$
M_{m+1} \geqslant \left(10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)} \right)^{1/\varepsilon},
\tag{33}
$$

then

$$
\mathbf{P}\left(S \text{ does not satisfy Property (A)} \right) \leqslant 1/10.
$$

Proof. For any cube $Q \in \Sigma_{m+1}^d$, there are at most 3^d indices $\sigma j \in \Sigma_{m+1}^d$ such that $Q_{\sigma j} \cap Q \neq \varnothing$, and so a union bound together with (26) gives

$$
\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S)) \leqslant \sum\nolimits_{Q_{\sigma j} \cap Q \neq \varnothing} \mathbf{P}(j_\sigma = j) \leqslant 3^d K_{m+1}^{-d}.
\tag{34}
$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_k \in S\}$ are independent from one another for $k \in \{1, \ldots, n\}$, which together with (34) implies that

$$
\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_1 \in S) \ldots \mathbf{P}(Q_n \in S) \leqslant 3^{dn} K_{m+1}^{-dn}.
\tag{35}
$$

Taking expectations over all cubes in B, and applying (23) and (35) gives

$$
\begin{aligned}
\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leqslant \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\
&\leqslant l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\
&= \frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}.
\end{aligned}
\tag{36}
$$

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Since $\varepsilon \leqslant (dn-s)/2$, we conclude

$$(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right) = s + \varepsilon\left(c(dn-s-\varepsilon)-\frac{s}{dn-s}\right)$$
$$\geqslant s + \varepsilon\left(\frac{c(dn-s)}{2}-\frac{s}{dn-s}\right)$$
$$= s + \varepsilon\frac{3dn-s}{dn-s} \geqslant s + 2\varepsilon.$$

Applying (25), we therefore conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geqslant M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \geqslant M_{m+1}^{s+2\varepsilon}.$$

Combined with (33), we conclude that

$$\frac{3^{dn}l_m^{-(s+\varepsilon)}M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leqslant \frac{3^{dn}l_m^{-(s+\varepsilon)}}{M_{m+1}^\varepsilon} \leqslant 1/10. \tag{37}$$

We can then apply Markov's inequality with (36) and (37) to conclude

$$\mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \varnothing) = \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geqslant 1)$$
$$\leqslant \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n)))$$
$$\leqslant 1/10. \qquad \square$$

Lemma 13. *Set* $D = \{k \in \mathbf{Z}^d : |k| \leqslant 10l_{m+1}^{-1}\}$. *Then if*

$$K_{m+1} \leqslant M_{m+1}^{\frac{2dn}{dn-s}}, \tag{38}$$

and

$$M_{m+1} \geqslant \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \tag{39}$$

then

$$\mathbf{P}\left(\|\widehat{\mu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10 \tag{40}$$

Proof. For each $\sigma \in \Pi_{m+1}^d$, and $k \in \mathbf{Z}$, define $X_{\sigma k} = \widehat{\mu_\sigma}(k) - \widehat{\mathbf{E}(\mu_\sigma)}(k)$. Applying (15) gives

$$\sum_{\sigma \in \Pi_{m+1}^d} \widehat{\mathbf{E}(\mu_\sigma)}(k) = \sum_{\sigma \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi ik\cdot x}\psi_{m+1}(x-a(\sigma j))d\mu_T(x)$$
$$= \int_{\mathbf{R}^d} e^{-2\pi ik\cdot x}d\mu_T(x) = \widehat{\mu_T}(k).$$

$$\tag{41}$$

For each σ and k, the standard (L^1, L^∞) bound on the Fourier transform, combined with (30), shows

$$
\begin{aligned}
\|X_{\sigma k}\|_{\psi_2(L)} &\leqslant 10|X_{\sigma k}| \\
&\leqslant 10[|\mu_\sigma(\mathbf{R}^d)| + \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)] \\
&\leqslant 10^2 \left(\mathbf{E}(\mu_\sigma)(\mathbf{R}^d) + \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right).
\end{aligned}
\tag{42}
$$

For a fixed k, the family of random variables $\{X_{\sigma k} : \sigma \in \Pi_{m+1}^d\}$ are independent. Furthermore, $\sum X_{\sigma k} = \widehat{\mu_S}(k) - \widehat{\mathbf{E}(\mu_S)}(k)$. Equations (19) and (26) imply that

$$
\begin{aligned}
\mathbf{E}(\widehat{\mu_S}(k)) &= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \left(\sum_{(\sigma,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\
&= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) \\
&= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \widehat{\mu_T}(k) = \widehat{\mu_T}(k).
\end{aligned}
\tag{43}
$$

Hoeffding's inequality, together with (42) and (43), imply that

$$
\begin{aligned}
&\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \\
&\qquad \leqslant 10^3 \sqrt{d} \left(\left(\sum \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)^2 \right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right).
\end{aligned}
\tag{44}
$$

Equation (18) shows

$$
\begin{aligned}
\mathbf{E}(\mu_\sigma)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int \psi_{m+1}(x - a(ij)) d\mu_T(x) \\
&\leqslant r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}.
\end{aligned}
\tag{45}
$$

Combining (44) and (45) gives

$$
\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \leqslant 10^3 \sqrt{d} \left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right] r_{m+1}^{d/2}.
\tag{46}
$$

We can then apply a union bound over the set D, which has cardinality at most $10^{d+1}l_{m+1}^{-d}$, together with (46) to conclude that

$$\mathbf{P}\left(\|\widehat{\mu_S} - \widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right)$$

$$\leqslant 10^{d+2}\cdot l_{m+1}^{-d}\exp\left(-\frac{\log(M_{m+1})^2}{10^7 d}\right) \tag{47}$$

$$= 10^{d+2}l_m^{-d}\exp\left(d\log(M_{m+1}K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d}\right).$$

Combined with (38) and (39), (47) implies

$$\mathbf{P}\left(\|\widehat{\mu_S} - \widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10. \tag{48}$$

Thus $\widehat{\mu_S}$ and $\widehat{\mu_T}$ are highly likely to differ only by a negligible amount over small frequencies. $\qquad\square$

Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup_{\xi\in\mathbf{R}^d}|\widehat{\mu_T}(\xi)||\xi|^t < \infty.$$

In light of (20), if we define, for each $t > 0$,

$$B(t) = \sup_{\xi\in\mathbf{R}^d}|\widehat{\psi}(\xi)||\xi|^t,$$

then

$$\sup_{\xi\in\mathbf{R}^d}|\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t}B(t).$$

Lemma 14. *Suppose that*

$$N_{m+1}^d \geqslant 10\cdot 2^{3d/2+1}A(3d/2 + 1), \tag{49}$$

$$N_{m+1}^d \geqslant \frac{10\cdot 2^{3d}}{1 + d/2}A(3d/2 + 1), \tag{50}$$

and

$$N_{m+1}^d \geqslant 10\cdot 2^{7d/2+1}B(3d/2 + 1). \tag{51}$$

then if $|\eta| \geqslant 10l_{m+1}^{-1}$,

$$|\widehat{\mu_S}(\eta)| \leqslant \frac{1}{|\eta|^{d/2}}. \tag{52}$$

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Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{\sigma \in \Pi_{m+1}^d \\ d(a(\sigma),T) \leqslant 2r_{m+1}^{-1}}} \delta_{a(ij_i)}.$$

Then $\mu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\mu_S} = (\hat\alpha \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d, so $\|\hat\alpha\|_{L^\infty(\mathbf{R}^d)} \leqslant \alpha(\mathbf{R}^d) \leqslant 2^d$. Thus

$$|\widehat{\mu_S}(\eta)| \leqslant 2^d \int |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi. \tag{53}$$

If $|\xi| \leqslant |\eta|/2$, $|\eta - \xi| \geqslant |\eta|/2$, and since (18) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$, we find that for all $t > 0$,

$$\int_{0 \leqslant |\xi| \leqslant |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{A(t)2^{t-d}}{|\eta|^{t-d}}. \tag{54}$$

Set $t = 3d/2 + 1$. Equation (54), together with (49), implies

$$\begin{aligned}
\int_{0 \leqslant |\xi| \leqslant |\eta|/2} & |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \\
&\leqslant \frac{A(3d/2 + 1)2^{1+d/2}|\eta|^{-1}}{|\eta|^{d/2}} \\
&\leqslant \frac{A(3d/2 + 1)2^{1+d/2}l_{m+1}}{|\eta|^{d/2}} \\
&\leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}.
\end{aligned} \tag{55}$$

Conversely, if $|\xi| \geqslant 2|\eta|$, then $|\eta - \xi| \geqslant |\xi|/2$, so for each $t > d$,

$$\begin{aligned}
\int_{|\xi| \geqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\hat\psi_{m+1}(\xi)| \, d\xi &\leqslant \int_{|\xi| \geqslant 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\
&\leqslant 2^d \int_{2|\eta|}^\infty r^{d-1-t} A(t)2^t \\
&\leqslant \frac{4^d A(t)}{t - d}|\eta|^{d-t}.
\end{aligned} \tag{56}$$

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Set $t = 3d/2 + 1$. Equation (50), applied to (56), allows us to conclude

$$\int_{|\xi| \geqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)| \, d\xi \leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \tag{57}$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu_T}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$ to conclude that

$$\int_{|\eta|/2 \leqslant |\xi| \leqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{2^{d+t}B(t)}{|\eta|^{t-d}}. \tag{58}$$

Set $t = 3d/2 + 1$. Then (58) and (51) imply

$$\int_{|\eta|/2 \leqslant |\xi| \leqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \tag{59}$$

It then suffices to sum up (55), (57), and (59), and apply (53). $\qquad\square$

Proof of Proposition 9, Continued. Let us now put all our calculations together. In light of Lemma 12 and Lemma 13, there exists some choice of j_σ for each σ, and a resultant non-random pair (μ_S, S) such that S satisfies Property (A) of the Lemma, and μ_S satisfies (40), implying that μ_S satisfies Property (C) of the Lemma. But Lemma 11 shows that μ_S always satisfies Property (B), and Lemma (14) shows Property (D) is also always satisfied. This completes the proof. $\qquad\square$

9 Construction of the Salem Set

Let us now choose the parameters to construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^\infty Z_i$, where Z_i has lower Minkowski dimension at most s for each i. Then choose an infinite sequence $\{i_m : m \geqslant 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers $\{\varepsilon_m : m \geqslant 1\}$, with $\varepsilon_m < (dn - s)/2$ for each m. We choose our parameters $\{M_m\}$ and $\{K_k\}$ inductively. First, set $X_0 = [0,1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0. At the mth step of our construction, we have already found a set X_{m-1} and a measure μ_{m-1}. We then choose K_m and M_m such that

$$K_m, M_m \geqslant C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that
$$M_m^{\frac{s}{dn-s}+c\varepsilon_m} \leqslant K_m \leqslant 2M_m^{\frac{s}{dn-s}+c\varepsilon},$$
and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in $\mathcal{Q}_m$, the union of which, we define to be equal to B_m. We can then apply Proposition 9 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$, and a measure μ_S supported on S. We define $X_m = S$, and $\mu_m = \mu_S$.

The last paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$. Just as in our previous paper, it is easy to see X must be a configuration avoiding set. Given any $(x_1, \ldots, x_n) \in Z$, there are infinitely many integers m_k such that $(x_1, \ldots, x_n) \in B_{m_k}$. If $|x_i - x_j| \geqslant \varepsilon$ for each $i \neq j$, and $r_{m_k} \leqslant \varepsilon/2$, then $(x_1, \ldots, x_n)$ is contained in a strongly non-diagonal cube in $\mathcal{Q}_{m_k}(B_k)$, and as such $X^n \subset X_k^n$ does not contain $(x_1, \ldots, x_n)$.

10 Proof that X is Salem

We now show X is Salem, completing the proof of Theorem 1. Since the masses of the sequence of measures $\{\mu_m\}$ is uniformly bounded, there is some subsequence μ_{m_i} which converges weakly to some measure μ. Repeated applications of Property (B) of Proposition 9 imply

$$\mu(\mathbf{R}^d) = \lim_{i\to\infty} \mu_{m_i}(\mathbf{R}^d) \geqslant 1 - \sum_{m=1}^{\infty} M_m^{-1/2}.$$

In particular, μ is a non-zero measure if the sequence $\{M_m\}$ is rapidly increasing. Moreover, for each $k \in \mathbf{Z}^d$,

$$\widehat{\mu}(k) = \lim_{i\to\infty} \widehat{\mu_{m_i}}(k).$$

Thus

$$|\widehat{\mu}(k)| \leqslant |\widehat{\mu_0}(k)| + \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)|.$$

Fix $\varepsilon > 0$. Since $l_m \leqslant 2^{-m}/10$, we find that for $m \geqslant \log(k)$, $|k| \leqslant 10 l_{m+1}^{-1}$. Thus we can apply Property (C) and (D) of Proposition 9 to conclude

$$\sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)|$$

$$\leqslant 2\log(k)|k|^{-d/2} + \sum_{m=\log(k)}^{\infty} r_{m+1}^{d/2} \log(M_{m+1})$$

$$\lesssim_\varepsilon |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} |k|^{t/2-\varepsilon} r_{m+1}^{d/2} \log(M_{m+1}) \right)$$

$$\leqslant |k|^{\varepsilon - t/2} \left(1 + 10^{t/2-\varepsilon} \sum_{m=\log(k)}^{\infty} l_{m+1}^{\varepsilon - t/2} r_{m+1}^{d/2} \log(M_{m+1}) \right)$$

$$\lesssim_\varepsilon |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{1}{K_{m+1}^\varepsilon} \frac{K_{m+1}^{t/2}}{M_{m+1}^{d/2-t/2}} \right)$$

$$\lesssim |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^\varepsilon} \frac{M_{m+1}^{(t/2)\left(\frac{s}{dn-s}\right)}}{M_{m+1}^{d/2-t/2}} \right)$$

$$= |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^\varepsilon} \right) \lesssim_\varepsilon |k|^{\varepsilon - t/2}.$$

The last inequality follows because $\varepsilon_m \to 0$, and so the series is summable if the sequence $\{K_m\}$ increases rapidly enough. Since μ_0 is smooth and compactly supported, we find

$$\sup_{k \in \mathbf{Z}^d} |k|^{t/2-\varepsilon} |\hat{\mu}(k)| \lesssim_\varepsilon 1 + \sup_{k \in \mathbf{Z}} |k|^{t/2-\varepsilon} |\hat{\mu_0}(k)| < \infty.$$

Since $\varepsilon > 0$ was arbitrary, this shows that the Fourier dimension of X is at least t. Because X_m is the union of $(M_1 \dots M_m)^d$ sidelength l_m cubes, one can easily show using (25) that the lower Minkowski dimension of X is upper bounded by t. But these two bounds imply that the Hausdorff dimension, Fourier dimension, and Minkowski dimension are all equal to t. Thus X is Salem of dimension t.

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11 Appendix: Random Segments of Smooth Functions

Several times, in this paper, we rely on a simple random segmentation applied to a given smooth measure. In this appendix, we collect the properties of this segmentation that are used repeatedly.

Theorem 15. *Let μ_0 be a smooth measure on $\mathbf{T}^d$, and let ϕ be a mollifier with the properties described in Section 1. Consider integers N, M, and set $\varepsilon = 1/NM$. Consider an independant family of random variables $\{j(i) : i \in [2N]^d\}$, where $j(i)$ is uniformly distributed in $[M]^d$ for each i, and then define a smooth measure ν such that for each $x \in \mathbf{T}^d$,*

$$d\nu(x) = \left[\sum_{i \in [2N]^d} \phi_\varepsilon \left(x - \frac{i}{2N} - \frac{j(i)}{2NM} \right) \right] d\mu_0(x).$$

Then $\nu(\mathbf{T}^d) = (2N)^d(1 + O(1/N))$, where the implicit constant depends on μ_0 and d. Moreover, if we define a probability measure $\mu = \nu/\nu(\mathbf{T}^d)$, and if we set $D = \{k \in \mathbf{Z}^d : |k| \leqslant 10/N\}$, then

$$\mathbf{P} \left(\|\mu - \mu_0\|_{L^\infty(D)} \geqslant N^{d/2} \log(M) \right) \leqslant 1/10,$$

and if $|k| \geqslant 10/N$, $|\widehat{\mu}(k)| \leqslant |k|^{-d/2}$.

We split our proof of the theorem into several, more managable lemmas.

Lemma 16. *For any choice of the random variables $\{j(a)\}$,*

$$\nu(\mathbf{T}^d) = (2N)^d(1 + O(1/N)),$$

where the implicit constants depend on μ_0 and d.

Proof. For each $i \in [2N]^d$, and $j \in [M]^d$, write

$$Y(i, j) = \int_{\mathbf{T}^d} \phi_\varepsilon \left(x - \frac{i}{2N} - \frac{j}{2NM} \right) d\mu_0(x)$$

$$= \int_{\mathbf{T}^d} \phi_\varepsilon(x) \, d\mu_0 \left(x + \frac{i}{2N} + \frac{j}{2NM} \right).$$

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Note that
$$\nu(\mathbf{T}^d) = \sum_{a \in [2N]^d} Y(a, X_a). \qquad (60)$$

For each choice $j_1, j_2 \in [M]^d$, $|j_1/2NM - j_2/2NM| \lesssim_d 1/N$. Thus Hölder's inequality, and the mean value theorem applied to μ imply that

$$|Y(i, j_1) - Y(i, j_2)| = \int_{\mathbf{T}^d} \phi_\varepsilon(x) \left(d\mu_0 \left(x + \frac{i}{2N} + \frac{j_1}{2NM} \right) - d\mu_0 \left(x + \frac{i}{2N} + \frac{j_2}{2NM} \right) \right)$$
$$\lesssim_d N^{-1} \|\phi_\varepsilon\|_{L^1(\mathbf{T}^d)} \|\nabla \mu_0\|_{L^\infty(\mathbf{T}^d)} \lesssim_{\mu_0} N^{-1}.$$

In particular, this implies $|Y(i, X_i) - \mathbf{E}(Y(i, X_i))| \lesssim_{d,\mu_0} N^{-1}$. Summing up this inequality over all $i \in [2N]^d$ and applying (60) gives

$$|\nu(\mathbf{T}^d) - \mathbf{E}(\nu(\mathbf{T}^d))| \lesssim_{d,\mu_0} N^{d-1}.$$

However, (6) implies

$$\mathbf{E}(\nu(\mathbf{T}^d)) = \frac{(2NM)^d}{M^d} = (2N)^d,$$

Thus we conclude that

$$\nu(\mathbf{T}^d) = (2N)^d \left(1 + O_{d,\mu_0}(1/N) \right). \qquad \square$$

Proof. For each $i \in [2N]^d$, define ν_i to be the smooth measure such that for each $x \in \mathbf{T}^d$,

$$d\nu_i(x) = \phi_\varepsilon \left(x - \frac{i}{2N} - \frac{j(i)}{2NM} \right) d\mu_0(x).$$

Then for each $k \in \mathbf{Z}^d$, define $X_{ik} = \hat{\nu}_i(k)$. For each i and k, the standard (L^1, L^∞) bound on the Fourier transform, combined with a mean value-theorem type bound, imply that

$$|X_{ik}| \leqslant \nu_i(\mathbf{T}^d) \lesssim_{\mu_0,d} \mathbf{E}(\nu_i(\mathbf{T}^d)) + N^{-1}. \qquad (61)$$

We also calculate using Hölder's inequality that

$$|\mathbf{E}\,\nu_i(\mathbf{T}^d)| \lesssim \|\mu\|_{L^\infty(\mathbf{T}^d)}. \qquad (62)$$

Combining (61) and (62), we conclude that

$$\sum_i \left(\mathbf{E}(\nu_i(\mathbf{T}^d)) + N^{-1} \right)^2 \lesssim \sum_i \mathbf{E}(\nu_i(\mathbf{T}^d))^2 + N^{-2} \lesssim_{\mu_0} N^d. \qquad (63)$$

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For a fixed k, the family of random variables $\{X_{ik} : i \in [2N]^d\}$ are independant and moreover, $\sum X_{ik} = \widehat{\nu}(k)$. Thus Hoeffding's inequality together with (63) implies that there exists a constant C, depending on μ and d, such that for each $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\nu}(k) - \mathbf{E}(\widehat{\nu}(k))| \geqslant t\right) \leqslant 2\exp\left(-Ct^2/N^d\right).$$

In particular, applying a union bound over all $|k| \leqslant 10/N$ $\qquad\square$

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