Incidence Theorems in Arbitrary Characteristic

Jacob Denson

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In this lecture, we try and explore the Szemeredi-Trotter theorem from a non-topological perspective, i.e. we try and bound the quantities

$$I(P,L) = \sum_{p \in P} \text{ord}_p(L) \quad I(L) = \sum_p \text{ord}_p(L) - 1$$

Recall that in the plane, using topological techniques, such as polynomial partitioning, one is able to conclude that for any set of lines L and points P,

$$|I(P,L)| \lesssim |P|^{2/3}|L|^{2/3} + |P| + |L|$$

One way to test the dependence of this theorem on topological techniques is to try and prove the conjecture over an arbitrary field k, rather than over $\mathbf{R}$. Indeed, for an arbitrary field there are no precise topological properties one can exploit in a proof of an incidence bound. Using mathematical logic, we can even show this truly does test how powerful 'non topological methods' are on the real plane.

Example. *The completeness of affine and projective geometry shows that a statement over $\mathbf{R}$ is provable using only basic methods of affine or projective geometry if and only if that statement remains true over any field k. If we want to add 'synthetic' topological methods, such as exploiting the fact that a point is 'between' two other points, then the completeness in this setting shows that the statment is true when k is any ordered field if and only if there is a proof of this theorem.*

Of course, incidence theorems on finite fields have independant interest and have many applications outside of incidence geometry, so trying to prove incidence bounds over finite fields is a very important problem. Thus we work with incidence bounds in $\mathbf{P}^2$, over as a general a field K as possible.

To begin with, we recall the Szmeredi-Trotter theorem for bounding incidences in the plane. Using Cauchy-Schwartz together with an L^2 bound on incidences, we were able to prove the bound $I(P,L) \leq |L|^{3/4}|M|^{3/4}$. This proof was obtained *purely* using incidence methods, and holds over any affine / projective space. We then applied polynomial partitioning to sharpen this bound, but we cannot use this technique over arbitrary fields. And indeed, one cannot sharpen the result, at least in the setting of finite fields.

Example. *Let $k = \mathbf{F}_p$, with P and L consisting of all projective points and lines. Then $|P| = |L| = p^2 + p + 1$, and we count*

$$I(P,L) = (p+1)(p^2+p+1) = \frac{p+1}{(p^2+p+1)^{1/2}}|L|^{3/4}|M|^{3/4}$$

As $p \to \infty$, $I(P,L) \sim |L|^{3/4}|M|^{3/4}$, so we obtain sharpness.

This means there is no purely affine proof of the Szemeredi-Trotter bound in $\mathbf{R}^2$. But it remains an open question whether the only counterexamples in the two dimensional finite field setting are *only* obtained by taking very large sets, i.e. is there a threshold at which sparse sets of points and lines behave like Euclidean space.

Our goal in this talk is to discuss some results obtained by Koller for three dimensional incidences in arbitrary fields. A key idea here is to utilize the arithmetic genus to upper bound incidences, which is easily calculated using classical results of algebraic geometry.

We begin by discussing a variant of the Guth-Katz bound on $I(L)$, which works over a general class of fields. The proof given here is very close to what we discussed over distinct distances, but we must replace the parts of the proof which involve polynomial partitioning with alternative geometric results. We 'almost' succeed to do this.

Theorem 1 (Theorem 6)**.** *Let L be a set of projective lines in a field of characteristic 0, or of characteristic p where $|L| \leq p^2$, and let $c > 0$ be a constant. If no plane or smooth quadric contains more than $c|L|^{1/2}$ lines, then $I(L) \leq (29.1 + 0.5c)|L|^{3/2}$.*

This theorem differs from the Guth-Katz result in three important ways:

- Guth-Katz only holds when $K = \mathbf{R}$.
- Their bound counts the cardinality of incidences rather than multiplicity.
- Explicit constants are not obtained.

Nonetheless, we obtain Theorem 6 by working very similarly to Guth-Katz, except replacing applications of polynomial partitioning with a more geometric method.

1 Proof of Theorem 6

Given L, standard polynomial counting techniques enable us to find a surface S containing all lines in L with degree $d \leq (6|L|)^{1/2} - 2$. Decompose S into irreducible components $S_1, \dots, S_k$, and let L_i denote the set of lines that are contained in S_i, but not in $S_1, \dots, S_{i-1}$.

Lemma 2.
$$I(L) = \underbrace{\sum_{i=1}^{K} I(L_i)}_{} + \underbrace{\sum_{i=2}^{k} |L_i \cap (L_1 \cup \cdots \cup L_{i-1}))|}_{}$$

The first term is Internal Incidences, the second External Incidences.

Proof. Note that $\{L_i\}$ is a partition of L, and $\text{ord}_p(L) = \sum \text{ord}_p(L_i)$, so

$$\begin{aligned}\sum_{i=1}^{k} I(L_i) &= \sum_{i=1}^{k} \sum_{p \in L_i} \text{ord}_p(L_i) - 1 \\ &= \sum_{p \in L} \text{ord}_p(L) - |\{i : p \in L_i\}| \\ &= I(L) - \sum_{i=2}^{k} |L_i \cap (L_1 \cup \cdots \cup L_{i-1})|. \end{aligned}$$
$\square$

Bounding external incidences is quite easy. And bounding the internal incidences when the surface is ruled is done analogously to Guth-Katz. Carrying out these calculations gives

$$I(L) \le 0.5c \cdot |L|^{3/2} + 1.23 \cdot |L|^{3/2} + \sum_{\text{non-ruled } S_i} I(L_i)$$

Because this is so similar to Guth-Katz, we omit the details. All the calculations to obtain this bound can be done over arbitrary fields. What is important is bounding incidences over non-ruled surfaces, which is where Guth-Katz applied polynomial partitioning. The novel feature of Kollár's approach is that this can be completely avoided, provided one takes a recourse in looking at the arithmetic genus of a surface.

2 Arithmetic Genus

The multiplicative incidence number is a projective invariant of a family of lines. In particular, it seems plausible that the incidence number is related to other classical projective invariants in geometry. Kollar's observation was that the arithmetic genus of the union of liens is closely related to the incidence number, and furthermore, is not too difficult to calculate.

Recall the following notation:

- Let $R = k[x_0, \dots, x_d]$.
- If M is finitely generated over R, the *Hilbert function* is $H_M(t) = \dim_k(M_t)$.
- H_M agrees with a degree $\le d$ polynomial $P_M(t)$ for sufficiently large t.
- If X is a projective variety/subscheme of $\mathbf{P}^n$ corresponding to some homogenous ideal I, then $k[X] = R/I$ is the homogenous coordinate ring, and we let $H_X = H_{R/I}$, $P_X = P_{R/I}$.

- P_X describes *all* additive invariants of X, i.e. invariants satisfying the rank nullity theorem. For instance,
  - $\deg(P_X) = \dim(X)$.
  - The leading coefficient is $\deg X/(\dim X)!$
  - The constant coefficient is the Euler characteristic $\chi(X, \mathcal{O})$.
- If X is a projective curve, then one calls $g(X) = 1 - \chi(X, \mathcal{O})$ the *arithmetic genus* of X. This agrees with the topological genus if X is a smooth curve over the complex numbers.

Lemma 3. *If $H = V(f)$ is a projective hypersurface of degree a, then*

$$P_{X \cap H}(t) = P_X(t) - P_X(t-a)$$

Proof. If f is homogenous of degree a, let H be the projective variety corresponding to f. If f is not a zero-divisor of X, the map $g \mapsto fg$ induces an exact sequence of vector spaces

$$0 \to k[X]_{t-a} \to k[X]_t \to k[X \cap H]_t \to 0$$

Thus $P_X(t-a) + P_{X \cap H}(t) = P_X(t)$, for $t \geq a$. □

Consider projective hypersurfaces $H_1, \dots, H_{n-1}$, with degrees $a_1, \dots, a_{n-1}$. If $C = H_1 \cap \dots \cap H_{n-1}$ is one-dimensional, we say it is a *complete intersection curve*.

Lemma 4. $g(C) = 1 + \frac{\sum a_i - (n+1)}{2} \prod_{i=1}^{n-1} a_i$.

Proof. Since

$$P_{\mathbf{P}^n}(t) = \binom{t+n}{n}$$

Then using the last lemma recursively, we conclude

$$P_C(t) = \left(\prod_{i=1}^{n-1} a_i \right) \cdot t - \frac{\sum a_i - (n+1)}{2} \prod_{i=1}^{n-1} a_i.$$

Thus

$$g(C) = 1 - \chi(C) = 1 + \frac{\sum a_i - (n+1)}{2} \prod_{i=1}^{n-1} a_i.$$ □

There are a few technicalities here, related to the fact that the intersections in the last two lemmas are *scheme theoretical*, rather than *set theoretical*. And the genus can disagree for these two objects.

Example. *Let $I_1 = (xy - zt)$, and $I_2 = (x^2 + xy - zt)$ be homogenous ideals, generating hyperboloids $H_1 = V(I_1)$ and $H_2 = V(I_2)$ in $\mathbf{P}^3$. Let $C = H_1 \cap H_2$ be the scheme theoretic intersection. Then $I_1 \oplus I_2 = (x^2, xy - zt)$, and an easy algebraic calculation shows $x \notin I_1 \oplus I_2$, but $x^2 \in I_1 \oplus I_2$. Thus $R/(I_1 \oplus I_2)$ is not a reduced ring, and the reduction of this ring is $R/I(x, zt)$ with corresponding 'reduced curve' C'. The last lemma shows $P_C(t) = 4t$, yet $P_{C'}(t) = 2(t+1) = 2t + 2$. Thus $g(C) = 1$, but $g(C') = -1$. Note that both of these differ from the topological genus of the set, which is zero.*

Note, that in this example $g(C') \leq g(C)$. Using standard techniques in scheme cohomology, we can show that for *any* complete intersection curve C, if C' is a reduced version of C, then

$$g(C') \leq g(C) = 1 + \frac{\sum a_i - (n+1)}{2} \prod_{i=1}^{n-1} a_i$$

We leave the details to the paper, which involves some basic sheaf cohomology. According to the paper, this inequality is intuitive: For families of curves, the arithmetic genus tends to jump up near the singular curves in the family, i.e. for curves where the scheme theoretic intersection disagrees with the set theoretic intersection.

So why should arithmetic genus tell us about the incidences of a set of lines? Recall that for a degree d curve C, $H_C(t) = dt + 1 - g(C)$. Thus among all degree d curves, those with small genus have the most degree t homogenous functions defined on them. If L is a *disjoint* union of projective lines $\{L_i\}$, then a degree t regular function on L is precisely given by degree t functions on each individual line. However, if the lines in L are allowed to intersect, then the degree t regular functions on each line must agree at the intersection points. Thus the dimension of the space of degree t regular functions must decrease, hence the genus increases. Thus there should be a relation between the incidence number and the arithmetic genus.

Now suppose L is formed from k projective lines $L_1, \dots, L_k$. For each line L_i, the embedding $e_i : L_i \to L$ induces a graded restriction homomorphism $e_i^* : k[L] \to k[L_i]$ given by $e_i^*(f) = f|_{L_i}$. We can put these together to obtain a map $e^* : k[L] \to \bigoplus k[L_i]$. We let $e_t^* : k[L]_t \to \bigoplus k[L_i]_t$ denote the restriction of e^* to $k[L]_t$.

Lemma 5. *For $t \gg 0$, $g(L) = \dim(\operatorname{coker}(e_t^*)) + k - 1$.*

Proof. Both L and $L^* = \coprod L_i$ have the same degree, so P_L differs from P_{L^*} by a constant coefficient. If we consider

$$0 \to k[L]_t \to \bigoplus k[L_i]_t \to \operatorname{coker}(e_t^*) \to 0$$

The rank nullity theorem then implies that

$$\dim k[L]_t + \dim(\operatorname{coker}(e_t^*)) = \sum_{i=1}^{k} \dim k[L_i]_t$$

Thus $\dim(\operatorname{coker}(e_t^*)) = H_L(t) - H_{L^*}(t)$. Note that both L and L^* have the same degree, so H_L differs from H_{L^*} by a constant for sufficiently large t. We calculate that $P_{L^*}(t) = k(t+1)$, so $g(L^*) = 1-k$. Thus $g(L) = \dim(\operatorname{coker}(e^*)) + k - 1$. □

Example. *Suppose L is a family of k lines intersecting at the origin in $\mathbf{P}^3$. Since e preserves elements at ∞, e^* preserves the affine degree of any element of $k[L]$, i.e. the degree of $f(1, x_1, x_2, x_3)$ is the same as $(e_i^* f)(1, t_0, t_1)$ for each i. If we decompose*

$$k[L]_t = k[L]_{t1} \oplus \cdots \oplus k[L]_{tt} \quad k[L_i]_t = k[L_i]_t = k[L_i]_{t1} \oplus \cdots \oplus k[L_i]_{tt}$$

where $k[L]_{ts}$ is the space of homogenous polynomials with projective degree t and affine degree s, and $e^*_{ts} : k[L]_{ts} \to \bigoplus k[L_i]_{ts}$ then

$$\dim coker(e^*_t) = \bigoplus_{s=1}^{t} \dim(coker(e^*_{ts}))$$

We use the elementary estimate

$$\dim(coker(e^*_{ts})) \geq \sum_{i=0}^{k} \dim k[L_i]_{ts} - \dim k[L]_{ts} \geq k - \binom{s+2}{2} \geq k - (s+2)^2/2$$

This estimate is only useful if $i \leq (2k)^{1/2} - 2$. For $s = 0$, we obtain the $\dim(coker(e^*_t)) \geq k-1$. This will suffice for our purposes, but we can also calculate that

$$\dim(coker(e^*_t)) \geq \sum_{i=1}^{\lfloor (2k)^{1/2} - 2 \rfloor} k - (i+2)^2/2 \geq 0.7 \cdot (k-1)^{3/2}$$

In particular, $g(L) \geq 0.7 \cdot (k-1)^{3/2} + (k-1)$, which is more useful in point-incidence type bounds.

We can put this together to get bounds on general k element line incidences L. Given any point of incidence $p \in L$, let $L(p)$ denote all lines through p. We consider the restriction map $f^* : k[L] \to \bigoplus k[L(p)]$, and e^* factors through f^*, and this factor map on each $L(p)$ is precisely the one considered in the last example. Thus

$$\dim(\mathrm{coker}(e^*)) \geq \sum \dim(\mathrm{coker}(e^*_p)) \geq \sum_p (\mathrm{ord}_p(L) - 1)$$

Thus $g(L) \geq \sum_p (\mathrm{ord}_p(L) - 1)$.

Now we put together this calculation with the theory of complete intersection curves. Let L' be lines lying on the non-ruled components S' of S. If T denotes the surface of degree $\leq 11d - 24$ corresponding to the flecnode polynomial of S, then $S \cap T$ contains all the lines L. Monge's theorem implies that T contains no component of S, so $S \cap T$ is one dimensional, and thus a complete intersection curve. Thus

$$\begin{aligned} I(L') &= \sum \mathrm{ord}_p(L) - 1 \leq g(L) \leq g(S \cap T) \\ &\leq 0.5 \cdot (d(11d-24)(d + (11d-24) - 2)) \leq 66d^3 \leq 66(6|L'|)^{3/2} \leq 970|L'|^{3/2} \leq 970|L|^{3/2} \end{aligned}$$

We therefore find that $I(L) \leq c|L|^{3/2} + 1000|L|^{3/2}$