Salem Sets Avoiding Rough Configurations

Jacob Denson

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1 Introduction

Geometric measure theory explores the relationship between the geometry of a subset of Euclidean space, and regularity properties of the family of Borel measures supported on that set. From the perspective of harmonic analysis, it is often popular to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a set. In this paper, we study the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a 'rough pattern', in the sense of [3], we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel set $E \subset \mathbf{R}^d$ is it's *Fourier dimension*. Given a finite Borel measure μ on $\mathbf{R}^d$, we define it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$, to be the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ |\widehat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d \right\} < \infty. \tag{1.1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most classical fractal sets are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem.

Theorem 1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset [0,1]^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n-1}, d \right)$$

such that for any distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. Moreover, there is a measure μ supported on X such that for each $\xi \in \mathbf{R}^d$,

$$|\widehat{\mu}(\xi)| \leqslant \log(1 + |\xi|)^{1/2} |\xi|^{-\beta}.$$

Remark. *Theorem 1 strengthens the main result of [3]. Unlike in [3], the case of the problem $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $X - \pi(Z)$ is not necessarily a Salem set, where $\pi(x_1, \dots, x_n) = x_1$ is projection onto the first coordinate. (TODO: CHECK THIS IS TRUE).*

A well-known result in this pattern avoidance setting is that sets with large Fourier dimension satisfy many algebraic relations. More precisely, if integer coefficients $m_1, \dots, m_n \in \mathbf{Z}$ are fixed, and we consider a compact set $X \subset \mathbf{R}$ with $\dim_{\mathbf{F}}(X) > 2/n$, then the sum set $m_1 X + \dots + m_n X$ contains an open interval. It follows by a slight modification of these coefficients that if $X \subset \mathbf{R}$ and $\dim_{\mathbf{F}}(X) > 2/n$, then there exists $m_1, \dots, m_n \in \mathbf{Z}$, distinct points $x_1, \dots, x_n \in X$, and an additional integer $a \in \mathbf{Z}$, such that

$$m_1 x_1 + \dots + m_n x_n = a. \tag{1.2}$$

It is an interesting to determine how tight this result is. In [2], T.W. Körner constructs a set X with Fourier dimension $1/(n-1)$ such that for non-zero $m \in \mathbf{Z}^n$, and $a \in \mathbf{Z}$, X does not contain distinct points $x_1, \dots, x_n$ solving (1.2). If, for each nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$, we consider the set

$$Z_{m,a} = \{ (x_1, \dots, x_n) \in [0,1]^n : m_1 x_1 + \dots + m_n x_n = a \},$$

then $Z_{m,a}$ is a subset of an $n-1$ dimensional hyperplane, and thus can be easily seen to have Minkowski dimension $n-1$. It follows that we can apply 1 to $Z = \bigcup \{ Z_{m,a} : m \neq 0, a \in \mathbf{Z} \}$ to obtain a Salem set $X \subset [0,1)$ of dimension

$$\frac{n-(n-1)}{n-1} = \frac{1}{n-1},$$

such that $(x_1, \dots, x_n) \notin Z$ for each distinct $x_1, \dots, x_n \in X$. Thus X avoids solutions to (1.2) for all nonzero $m \in \mathbf{Z}^n$ and $a \in \mathbf{Z}$. Thus we see Theorem 1 generalizes Körner's result, and thus shows the result depends little on the arithmetic properties of the pattern Körner avoids, but rather, depends only on the 'thickness' of the family of tuples $(x_1, \dots, x_n)$ satisfying the pattern.

Since we are working with *compact* sets avoiding patterns, working in $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, and working in this space has several advantages over the Euclidean case. For a finite measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)| |\xi|^{s/2} < \infty. \tag{1.3}$$

We can then define the Fourier dimension of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Similarily, $\mathbf{T}^d$ has a natural quotient metric induced from $\mathbf{R}^d$, so we can consider open balls $B_\varepsilon(x + \mathbf{Z}^d)$, and thus define the Hausdorff dimension of finite Borel measures and sets on $\mathbf{T}^d$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1.1) is equivalent to the more discrete condition

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)| |\xi|^{s/2} < \infty. \tag{1.4}$$

A proof is given in [4, Lemma 39]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x)\, d\mu^*(x) = \int_{\mathbf{R}^d} f(x + \mathbf{Z}^d)\, d\mu(x),$$

then (1.4) implies $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$. Thus Theorem 1 is clearly equivalent to it's periodic variant, introduced below.

Theorem 2. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $X \subset \mathbf{T}^d$ with dimension*

$$\beta = \min\left(\frac{nd - \alpha}{n-1}, d\right)$$

such that for any distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. Moreover, there is a measure μ supported on X such that for each $\xi \in \mathbf{Z}^d$,

$$|\widehat{\mu}(\xi)| \leqslant \log(1 + |\xi|)^{1/2} |\xi|^{-\beta}.$$

To construct our set, we rely on a Baire-category argument for the purpose of our construction. Thus we consider a complete metric space $\mathcal{X}$, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in \mathcal{X}$, the set E is pattern avoiding, and for each $\xi \in \mathbf{Z}^d$,

$$|\widehat{\mu}(\xi)| \lesssim \log(1 + |\xi|)^{1/2} |\xi|^{-\beta},$$

in the sense that the set of pairs (E, μ) which do not satisfy these properties is a set of first category in $\mathcal{X}$. It follows that Theorem 2 holds in a 'generic' sense for elements of $\mathcal{X}$.

Once we have setup the appropriate metric space $\mathcal{X}$, our approach is quite similar to the construction in [3], relying on a random selection procedure, which is now exploited to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach, is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [3]. But the Baire category argument makes our proof simpler to read, giving us the 'infinitisimal argument' for free from the discrete case analysis, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern. Moreover, the proof of the Baire category theorem is in some senses, 'hidden' in the queuing method, so the two methods are roughly equivalent.

2 Notation

- For a positive integer N, we let $[N] = \{1, \dots, N\}$.
- Given a metric space Ω, $x \in \Omega$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let

$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

denote the ε-*thickening* of the set E. A subset of Ω is of *first category* in Ω if it is the countable union of closed sets with nonempty interior. We say a property holds *quasi-always*, or a property is *generic* in Ω if the set of points in Ω failing to satisfy that property form a set of first category.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let

$$|x| = \min\{|x+n| : n \in \mathbf{Z}\},$$

and for $x \in \mathbf{T}^d$, we let

$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

The canonical metric on $\mathbf{T}^d$ is then $d(x,y) = |x-y|$, for $x, y \in \mathbf{T}^d$.

- Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0,d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as

$$H^\alpha_\delta(E) = \inf \left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } \varepsilon_i \leqslant \delta \text{ for all } i \in \mathbf{N} \right\}.$$

The α dimensional Hausdorff measure of E is equal to

$$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0,d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0,d]$ such that $H^s(E) = 0$. Frostman's lemma says that if we define the Hausdorff dimension $\dim_{\mathbf{H}}(\mu)$ of a finite Borel measure μ as the supremum of all $s \in [0,d]$ such that

$$\sup \left\{ \mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{R}^d, \varepsilon > 0 \right\} < \infty, \tag{2.1}$$

then $\dim_{\mathbf{H}}(E)$ is the supremum of $\dim_{\mathbf{H}}(\mu)$, over all Borel probability measures μ supported on E, analogous to the definition of the Fourier dimension of a set E given in the introduction.

- For $\mathbf{E} = \mathbf{R}^d$ or $\mathbf{E} = \mathbf{T}^d$, and for a measurable set E, we let $|E|$ denote it's standard Lebesgue measure. We define the lower and upper Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon| \quad \text{and} \quad \overline{\dim}_{\mathbf{M}}(E) = \limsup_{\varepsilon \to 0} \log_\varepsilon |E_\varepsilon|$$

respectively.

- We will need to consider a standard mollifier. Throughout the paper, we fix a smooth probability density $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$, and such that for each $x \in \mathbf{T}^d$

$$\sum_{k \in \{0,1\}^d} \phi(x + k/2) = 2^d.$$

For each $\varepsilon \in (0,1)$, we can then define $\phi_\varepsilon \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_\varepsilon(x) = \begin{cases} \varepsilon^{-d}\phi(x/\varepsilon) & : |x| < \varepsilon, \\ 0 & : \text{otherwise.} \end{cases}$$

Then ϕ_ε is a smooth probability density, $\phi_\varepsilon(x) = 0$ for $|x| \geqslant \varepsilon$, and if $\varepsilon = 1/N$, and $x \in \mathbf{T}^d$,

$$\sum_{k \in [2N]^d} \phi_{1/N}(x + k/2N) = (2N)^d. \tag{2.2}$$

As $\varepsilon \to 0$, ϕ_ε converges weakly to the Dirac delta function at the origin, so that for each $k \in \mathbf{Z}^d$,

$$\lim_{\varepsilon \to 0} \widehat{\phi}_\varepsilon(k) = 1. \tag{2.3}$$

Moreover, for each $\alpha > 0$, there exists $C_\alpha > 0$ such that for all $\varepsilon > 0$ and non-zero $k \in \mathbf{Z}^d$,

$$|\widehat{\phi}_\varepsilon(k)| \leqslant \frac{C_\alpha}{\varepsilon^\alpha |k|^\alpha}. \tag{2.4}$$

- We will make several uses of *Hoeffding's Inequality* to control the deviations of independent families of random variables. The version of Hoeffding's inequality we use states that if $\{X_1, \dots, X_N\}$ is an independent family of random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

$$\mathbf{P}\left(|X_1 + \dots + X_N| \geqslant t\right) \leqslant 2\exp\left(\frac{-2t^2}{A_1^2 + \dots + A_N^2}\right).$$

Proofs are given in many probability textbooks, for instance, in Chapter 2 of [5].

3 A Complete Metric Space of Generically Salem Sets

Let us construct a metric space appropriate for our task. We proceed analogously to [2]. We shall form our metric space as a combination of two simpler metric spaces:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

$$d_H(E,F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

then $(\mathcal{E}, d_H)$ forms a complete metric space.

- Given any positive sequence of real numbers $\{A(k) : k \in \mathbf{Z}^d\}$, we let $M(A)$ consist of the class of all finite Borel measures μ supported on $\mathbf{T}^d$ such that the quantity

$$\|\mu\|_{M(A)} = \sup_{\xi \in \mathbf{Z}^d} \frac{|\widehat{\mu}(\xi)|}{|A(\xi)|}$$

is finite. Then $\|\cdot\|_A$ is a norm, and $(M(A), \|\cdot\|_A)$ is a Banach space.

In our argument, we will assume that the choice of $\{A(\xi)\}$ is such that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for each $\xi \neq 0$,

$$\frac{1}{C_\varepsilon} \frac{1}{|\xi|^{\beta/2+\varepsilon}} \leqslant A(\xi) \leqslant C_\varepsilon \frac{1}{|\xi|^{\beta/2-\varepsilon}}. \tag{3.1}$$

In particular, (3.1) implies that $C^\infty(\mathbf{T}^d) \subset M(A)$, and we will later show it implies that generic elements of $M(A)$ are 'β dimensional'.

We then define $\mathcal{X}$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(A)$, where μ is a probability measure with $\operatorname{supp}(\mu) \subset E$, and

$$\lim_{|\xi| \to \infty} \frac{|\widehat{\mu}(\xi)|}{A(\xi)} = 0.$$

It is easy to see $\mathcal{X}$ is a closed subset of $\mathcal{E} \times M(A)$ under the product topology, and thus if we consider the product metric

$$d_{\mathcal{X}}((E,\mu),(F,\nu)) = \max\left(d_H(E,F), \|\mu - \nu\|_A\right),$$

then $(\mathcal{X}, d_{\mathcal{X}})$ is a complete metric space.

Lemma 3. *The set of all (E,μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}$.*

Proof. We just apply a mollification strategy. Consider $(E,\mu) \in \mathcal{X}$ and $\varepsilon > 0$. For each $\delta > 0$, consider $\mu_\delta = \mu * \phi_\delta$. Then $\mu_\delta \in C^\infty(\mathbf{T}^d)$, and if $\delta \leqslant \varepsilon$, then μ_δ is supported on $\overline{E_\varepsilon}$, so

$$d_H(E, \overline{E_\varepsilon}) \leqslant \varepsilon. \tag{3.2}$$

Since $(E,\mu) \in \mathcal{X}$, there is N depending on μ such that for $|\xi| \geqslant N$,

$$|\widehat{\mu}(\xi)| \leqslant \varepsilon A(k).$$

For $|\xi| \geqslant N$, we thus find that

$$|\widehat{\mu_\delta}(\xi) - \widehat{\mu}(\xi)| = |\widehat{\mu}(k)||\widehat{\phi_\delta}(\xi) - 1| \leqslant 2|\widehat{\mu}(k)| \leqslant 2\varepsilon A(\xi). \tag{3.3}$$

On the other hand, for suitably small $\delta > 0$, (2.3) implies that for $|\xi| \leqslant N$,

$$|\widehat{\phi_\delta}(\xi) - 1| \leqslant \varepsilon A(\xi).$$

But this implies that for $|\xi| \leqslant N$,

$$|\widehat{\mu_\delta}(\xi) - \widehat{\mu}(\xi)| = |\widehat{\mu}(\xi)||\widehat{\phi_\delta}(\xi) - 1| \leqslant \varepsilon A(\xi). \tag{3.4}$$

Thus we conclude that for suitably small δ, $\|\widehat{\mu_\delta} - \widehat{\mu}\|_A \leqslant 2\varepsilon$. But combining (3.2), (3.3), and (3.4), we conclude

$$d_{\mathcal{X}}((E,\mu),(\overline{E_\varepsilon},\mu_\delta)) \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. $\square$

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}$ to be as low as possible. For each $(E, \mu) \in \mathcal{X}$, the condition that $\mu \in M(A)$, combined with (3.1), implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus we might expect that for quasi-all $(E, \mu) \in M(A)$, the set E has Fourier dimension *equal to* β. This turns out to be the correct conjecture.

Lemma 4. *For quasi-all* $(E, \mu) \in \mathcal{X}$, *E is a Salem set of dimension* β.

Proof. We shall assume $\beta < d$ in the proof, since the case $\beta = d$ is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that quasi-all $\mu \in M(A)$ have Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{G} : H^\alpha_\delta(E) < s\}$. Then $A(\alpha, \delta, s)$ is an open set, and

$$\bigcap_{n=1}^\infty \bigcap_{m=1}^\infty \bigcap_{k=1}^\infty A(\beta + 1/n, 1/m, 1/k)$$

is precisely the family of $(E, \mu) \in \mathcal{X}$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha, \delta, s)$ is dense in $\mathcal{X}$ for $\alpha \in (\beta, d)$ and $\delta, s > 0$.

Fix $(E, \mu_0) \in \mathcal{X}$, $\alpha \in (\beta, d)$, $\delta > 0$, and $s > 0$, and consider $\varepsilon > 0$. Without loss of generality, we may assume $\mu_0 \in C^\infty(\mathbf{T}^d)$. Fix some choice of $\alpha_0 \in (\beta, \alpha)$, chosen to give us an epsilon of room in our argument. Consider a positive integer N, and set M equal to the smallest integer larger than N^λ, where $\lambda = d/\alpha_0 - 1$. Let $\varepsilon_0 = 1/NM$. Then construct an independent family of random variables $\{j(i) : i \in [2N]^d\}$, where $j(i)$ is uniformly distributed in $[M]^d$ for each $i \in [2N]^d$. Define a measure ν such that for each $x \in \mathbf{T}^d$,

$$\nu(x) = \left(\sum_{i \in [2N]^d} \phi_{\varepsilon_0} \left(x - \frac{i}{2N} - \frac{j(i)}{2NM} \right) \right) \mu_0(x).$$

Then define a normalized probability measure $\mu = \nu/\nu(\mathbf{T}^d)$. We claim that there is a non-zero probability that μ is supported on a set F such that, if N is sufficiently large, then $d_{\mathcal{X}}((E, \mu_0), (F, \mu)) \leqslant \varepsilon$.

The measure μ is supported on E, so, letting $F = \operatorname{supp}(\nu) \cup \{x_1, \dots, x_K\}$, where $\{x_1, \dots, x_K\}$ is a ε_0 cover of E, we find $d_H(E, F) \leqslant \varepsilon_0$. In particular, if $N \geqslant 1/\varepsilon$, then

$$d_H(E, F) \leqslant \varepsilon_0 \leqslant \varepsilon. \tag{3.5}$$

The set F is covered by $(2N)^d$ radius ε_0 balls, in addition to K balls whose radius can be made as small as desired. Thus if $\varepsilon_0 \leqslant \delta$, which holds for $N \geqslant 1/\delta$,

$$H^\alpha_\delta(F) \leqslant (2N)^d/(NM)^\alpha \lesssim_d N^{d-\alpha} M^{-\alpha} \lesssim N^{d(1-\alpha/\alpha_0)}.$$

In particular, for any fixed $\delta > 0$, as $N \to \infty$, $H^\alpha_\delta(F) \to 0$, so for sufficiently large N, $H^\alpha_\delta(F) \leqslant s$. For such N, it thus follows that $(\mu, F) \in A(\alpha, \delta, s)$. Provided we can show for suitably large N, there is a non-zero probability that $\|\mu_0 - \mu\|_{M(A)} \leqslant \varepsilon$, then we will have shown that $d_{\mathcal{X}}((E, \mu_0), (F, \mu)) \leqslant \varepsilon$, which will complete the proof since ε was arbitrary. To accomplish this, we employ some standard calculations controlling the Fourier series of μ, proved in Theorem 6 of the Appendix to this paper.

Applying Theorem 6 with our choice of N and M, we know that for any $\varepsilon_1 > 0$, there exists a constant C and an integer N_0, both depending only on μ_0, d, and ε_1, such that for $N \geqslant N_0$, and for $|\xi| \leqslant (NM)^2 = N^{2d/\alpha_0}$,

$$|\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| \leqslant CN^{-d/2}\log(N)^{1/2}, \tag{3.6}$$

and such that for $|\xi| \geqslant \max\left(C, N^{(d/\alpha_0)(1+\varepsilon_1)}\right)$,

$$|\widehat{\mu}(\xi)| \leqslant |\xi|^{-d/2}. \tag{3.7}$$

To obtain an epsilon of room, fix $\alpha_1 \in (\beta, \alpha_0)$. If ε_1 is chosen such that $(d/\alpha_0)(1+\varepsilon_1) = d/\alpha_1$, and if for the resultant constant C, we have $N \geqslant C^{\alpha_1/d}$, then (3.7), together with (3.1), implies that for $|\xi| \geqslant N^{d/\alpha_1}$,

$$|\widehat{\mu}(\xi)| \leqslant |\xi|^{-d/2} \lesssim_{\alpha_1} A(\xi)|\xi|^{\alpha_1/2-d/2} \leqslant A(\xi)N^{d/2(1-d/\alpha_1)}. \tag{3.8}$$

In particular, since the implicit constants in (3.8) do not depend on N, we can take N sufficiently large to annihilate the implicit constants. Thus we conclude from (3.8) that for sufficiently large N depending only on α_1, the sequence $\{A(\xi)\}$, μ_0, d, and ε, if $|\xi| \geqslant N^{d/\alpha_1}$,

$$|\widehat{\mu(\xi)}| \leqslant (\varepsilon/2)A(\xi). \tag{3.9}$$

On the other hand, since $(E, \mu_0) \in \mathcal{X}$, there exists K such that for $|\xi| \geqslant K$,

$$|\widehat{\mu_0}(\xi)| \leqslant (\varepsilon/2)A(\xi). \tag{3.10}$$

Combining (3.9) and (3.10), we conclude that if $N \geqslant K^{\alpha_1/d}$, then for $|\xi| \geqslant N^{d/\alpha_1}$,

$$|\widehat{\mu_0}(\xi) - \widehat{\mu}(\xi)| \leqslant \varepsilon A(\xi). \tag{3.11}$$

Conversely, if $|\xi| \leqslant N^{d/\alpha_1}$, then we pick $\alpha_2 \in (\beta, \alpha_1)$ to obtain another epsilon of room, and apply (3.6) with (3.1) to conclude

$$\begin{aligned} |\widehat{\mu_0}(\xi) - \widehat{\mu}(\xi)| &\leqslant CN^{-d/2}\log(N)^{1/2} \\ &\leqslant \left[CN^{d/2(\alpha_2/\alpha_1-1)}\log(N)^{1/2}\right] \cdot |\xi|^{-\alpha_2/2} \\ &\lesssim_{\alpha_2} \left[CN^{d/2(\alpha_2/\alpha_1-1)}\log(N)^{1/2}\right] \cdot A(\xi). \end{aligned} \tag{3.12}$$

Since implicit constants here also do not depend on N, if N is taken sufficiently large, we conclude from (3.12) that for $|\xi| \leqslant N^{d/\alpha_1}$,

$$|\widehat{\mu_0}(\xi) - \widehat{\mu}(\xi)| \leqslant \varepsilon A(\xi). \tag{3.13}$$

Combining (3.11) and (3.13) shows that for sufficiently large N,

$$\|\mu_0 - \mu\|_{M(A)} \leqslant \varepsilon,$$

which completes the argument. □

All that now remains is to show that quasi-all elements of $\mathcal{X}$ avoid the given set Z; just as with the proof above, we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

4 Random Avoiding Sets

Thus there exists arbitrarily small $\varepsilon > 0$ and arbitrarily small $\delta > 0$ such that

$$|W_\varepsilon| \leqslant \varepsilon^{nd-\alpha-\delta}.$$

Lemma 5. *Consider $W \subset \mathbf{T}^{nd}$ with $\underline{\dim}_{\mathbf{M}}(W) \leqslant \alpha$, and consider the set*

$$A()$$

$$\liminf_{\varepsilon \to 0} \log_\varepsilon |W(\varepsilon)| \leqslant \alpha.$$

Proof. Let $X_1, \dots, X_N$ be uniformly distributed on $\mathbf{T}^d$. For each $i \in \{1, \dots, N\}^n$ such that $i_1, \dots, i_n$ are distinct, consider the random vector $X_i = (X_{i_1}, \dots, X_{i_n})$. Then X_i is uniformly distributed on $\mathbf{T}^{nd}$, and so

$$\mathbf{P}(d(X_i, W) \leqslant \varepsilon) \leqslant \varepsilon^{nd-\alpha-\delta}.$$

If M denotes the number of indices i such that $d(X_i, W) \leqslant \varepsilon$, then by linearity of expectation we conclude that $\mathbf{E}(M) \leqslant N^n \varepsilon^{nd-\alpha-\delta}$. Applying Markov's inequality, we conclude that

$$\mathbf{P}(M \geqslant 4N^n \varepsilon^{nd-\alpha-\delta}) \leqslant 1/4.$$

Now consider the measure $\nu = \delta_{X_1} + \dots + \delta_{X_N}$. For each $k \in \mathbf{Z}^d$,

$$\widehat{\nu}(k) = e^{2\pi i k \cdot X_1} + \dots + e^{2\pi i k \cdot X_N}.$$

Thus $\widehat{\nu}(k)$ is the sum of N bounded independent random variables, so we can apply Hoeffding's inequality to conclude that for each $t \geqslant 0$

$$\mathbf{P}(|\widehat{\nu}(k)| \geqslant t) \leqslant 2\exp(-2t^2/N).$$

A union bound allows us to conclude that

$$\mathbf{P}\left(\text{for all } |k| \leqslant R,\ |\widehat{\nu}(k)| \leqslant t\right) \geqslant 1 - 2^{d+1} R^d \exp(-2t^2/N).$$

Thus provided

$$2^{d+1} R^d \exp(-2t^2/N) \leqslant 1/4, \tag{4.1}$$

we conclude that there exists a choice of $X_1, \dots, X_N \in \mathbf{T}^d$ such that if

$$I = \{i \in \{1, \dots, N\}^n : i_1, \dots, i_n \text{ are distinct and } d(X_i, W) \leqslant \varepsilon\},$$

then $\#(I) \leqslant 4N^n \varepsilon^{nd-\alpha-\delta}$, and for all $|k| \leqslant R$, $|\widehat{\nu}(k)| \leqslant t$. In particular, if we choose

$$N = \lfloor \varepsilon^{-\frac{nd-\alpha-\delta}{n-1/2}} \rfloor,$$

choose $R = \varepsilon^{-10}$, and choose

$$t = 5d^{1/2} \cdot \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2},$$

then using the fact that $\varepsilon \leqslant 1/2$, we conclude that (4.1) is satisfied. The standard (L^1, L^∞) bound for the Fourier transform then implies that for each $k \in \mathbf{Z}^d$,

$$\left\| \sum_{i \in I} \widehat{\delta_{X_{i_1}}} \right\|_{L^\infty(\mathbf{Z}^d)} \leqslant \#(I) \leqslant 4N^n \varepsilon^{nd-\alpha-\delta} \leqslant 4N^{1/2} \leqslant 4\varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}}.$$

Define a probability measure

$$\mu = \frac{1}{N - \#(I)} \left(\nu - \sum_{i \in I} \delta_{X_{i_1}} \right),$$

If ε is sufficiently small, so that N is sufficiently large, then for each $k \in \mathbf{Z}^d$ with $|k| \leqslant R = \varepsilon^{-10}$,

$$\begin{aligned} |\widehat{\mu}(k)| &\leqslant \frac{1}{N - 4N^{1/2}} \left(5d^{1/2} \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2} + 4\varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \right) \\ &\leqslant 9.5 d^{1/2} N^{-1} \varepsilon^{-\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2} \leqslant 10 d^{1/2} \varepsilon^{\frac{nd-\alpha-\delta}{2n-1}} \log(1/\varepsilon)^{1/2}. \end{aligned}$$

□

Following through with the proof leads to a set with Fourier dimension

$$\frac{nd - \alpha}{n - 1/2}$$

Which isn't quite enough for what we want.

5 Appendix: Random Segments of Measures

Several times, in this paper, we rely on a simple random segmentation applied to a given smooth measure. Here we collect some simple calculations we use quite often to bound the Fourier transforms of these random segmentations.

Theorem 6. *Let μ_0 be a smooth measure on $\mathbf{T}^d$, and let ϕ be a mollifier with the properties describes in Section 2. Consider a positive integer K, and some $\varepsilon_0 > 0$, and take any independant family of K random vectors $\{X_i : i \in I\}$ in $\mathbf{T}^d$, and define a smooth measure*

$$\nu(x) = \left[\sum_{i \in I} \phi_{\varepsilon_0}(x - X_i) \right] \mu_0(x).$$

Then consider the probability measure $\mu(x) = \nu(x)/\nu(\mathbf{T}^d)$. If, for each $x \in \mathbf{T}^d$,

$$\frac{1}{K} \sum_{i \in I} \mathbf{E} \left[\phi_{\varepsilon_0}(x - X_i) \right] = 1, \tag{5.1}$$

then there exists an integer K_0 depending solely on μ_0 and d, such that for $K \geqslant K_0$, the following properties hold:

(A) *Fix $\beta > 0$, and define*

$$D = \{\xi \in \mathbf{Z}^d : |\xi| \leqslant K^{1/\beta}\}.$$

Then there exists a constant C, depending only on μ_0, d, and β, such that with probability greater than or equal to $1/2$,

$$\|\widehat{\mu_0} - \widehat{\mu}\|_{L^\infty(D)} \leqslant CK^{-1/2}\log(K)^{1/2}.$$

(B) *For any $\varepsilon_1, \varepsilon_2 > 0$, there exists a constant C, depending solely on μ_0, d, ε_1, and ε_2, such that if $|\xi| \geqslant \max(C, (1/\varepsilon_0)^{1+\varepsilon_1})$, then $|\widehat{\mu}(\xi)| \leqslant \varepsilon_2 \cdot |\xi|^{-d/2}$.*

Remark. *Property (A) gives bounds for the behaviour of $\widehat{\mu}$ on small frequencies, whereas Property (B) bounds the behaviour of $\widehat{\mu}$ on large frequencies.*

Remark. *The simplest example of inputs to which this lemma applies is obtained by fixing two integers N and M, letting $I = [2N]^d$ (so $K = N^d$, and for each $i \in [2N]^d$, defining $X_i = i/N + j(i)/NM$, where $j(i)$ is chosen from $[M]^d$ uniformly at random, such that the family $\{j(i) : i \in [2N]^d\}$ is independant. Equation* (5.1) *is then justified by* (2.2)*.*

Remark. *Another example is obtaing by considering K independant and uniformly distributed random variables $\{X_1, \dots, X_K\}$ on $\mathbf{T}^d$.*

We split our proof into several, more managable lemmas.

Lemma 7. *There exists a constant C depending on μ_0 and d such that*

$$\mathbf{P}\left(|\nu(\mathbf{T}^d) - K| \geqslant CK^{1/2}\right) \leqslant 1/10.$$

Proof. For each $i \in \{1, \dots, K\}$, set

$$Y_i = \int_{\mathbf{T}^d} \phi_{\varepsilon_0}(x - X_i)\, d\mu_0(x).$$

Then

$$|Y_i| \leqslant \|\mu_0\|_{L^\infty(\mathbf{T}^d)} \lesssim_{\mu_0} 1. \tag{5.2}$$

Moreover,

$$\sum_{i \in I} Y_i = \nu(\mathbf{T}^d). \tag{5.3}$$

The family of random variables $\{Y_i : 1 \leqslant i \leqslant K\}$ are independant, so we can apply Hoeffding's inequality together with (5.2) and (5.3) to conclude that there exists a constant C depending only on μ_0 and d, such that for any $t \geqslant 0$,

$$\mathbf{P}\left(|\nu(\mathbf{T}^d) - \mathbf{E}(\nu(\mathbf{T}^d))| \geqslant t\right) \leqslant 2\exp\left(-\frac{5t^2}{C^2 K}\right). \tag{5.4}$$

Noting that (5.1) implies $\mathbf{E}(\nu(\mathbf{T}^d)) = K$, it suffices to set $t = CK^{1/2}$ in (5.4). □

Lemma 8. *There exists a constant C, depending on μ_0 and d, such that*

$$\mathbf{P}\left(\|\widehat{\nu} - K\widehat{\mu_0}\|_{L^\infty(D)} \geqslant CK^{1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Proof. For each $i \in \{1, \ldots, K\}$, let $\nu_i(x) = \phi_{\varepsilon_0}(x - X_i)\mu_0(x)$. Then for each $\xi \in \mathbf{Z}^d$, define $X_{i\xi} = \widehat{\nu}_i(\xi)$. For each i and ξ, the standard (L^1, L^∞) bound on the Fourier transform implies

$$|X_{i\xi}| \leqslant \nu_i(\mathbf{T}^d) \leqslant \|\mu_0\|_{L^\infty(\mathbf{T}^d)}. \tag{5.5}$$

Thus the variables $|X_{i\xi}|$ are uniformly bounded by a constant depending only on μ_0. For a fixed ξ, the family of random variables $\{X_{i\xi} : i \in \{1, \ldots, K\}\}$ are independant and moreover,

$$\sum_{i \in I} X_{i\xi} = \widehat{\nu}(\xi). \tag{5.6}$$

Thus Hoeffding's inequality together with (5.5) and (5.6) imply that

$$\mathbf{P}\left(|\widehat{\nu}(\xi) - \mathbf{E}(\widehat{\nu}(\xi))| \geqslant t\right) \leqslant 2\exp\left(\frac{-t^2}{K\|\mu_0\|_{L^\infty(\mathbf{T}^d)}^2}\right) \tag{5.7}$$

for all $t \geqslant 0$. Applying a union bound to (5.7) over all $|\xi| \leqslant K^{1/\beta}$, we conclude that there exists a constant C, depending only on d and μ_0, such that for each $t \geqslant 0$,

$$\mathbf{P}\left(\|\widehat{\nu} - \mathbf{E}(\widehat{\nu})\|_{L^\infty(D)} \geqslant t\right) \leqslant \exp\left(C\log(K)^2 - \frac{5t^2}{CK}\right). \tag{5.8}$$

We can make C as large as we want, so in particular, we assume $C \geqslant 1$. Noting that (5.1) implies $\mathbf{E}(\widehat{\nu}) = K \cdot \widehat{\mu_0}$ and setting $t = CK^{1/2}\log(K)^{1/2}$ in (5.8) completes the proof. □

We now control the behaviour of ν at high frequencies.

Lemma 9. *For any $\varepsilon > 0$, there exists a constant C depending on ε, d, and μ_0, such that if $|\xi| \geqslant \max(C, (1/\varepsilon_0)^{1+\varepsilon})$, then $|\widehat{\nu}(\xi)| \leqslant |\xi|^{-d/2}$.*

Proof. Define a finite measure

$$\alpha(x) = \sum_{i \in I} \delta\left(x - X_i\right).$$

Then $\nu = (\alpha * \phi_{\varepsilon_0})\mu$, so

$$\widehat{\nu} = \left(\widehat{\alpha} \cdot \widehat{\phi_{\varepsilon_0}}\right) * \widehat{\mu}. \tag{5.9}$$

The standard (L^1, L^∞) bound for the Fourier transform shows that

$$\|\widehat{\alpha}\|_{L^\infty(\mathbf{Z}^d)} \leqslant \alpha(\mathbf{T}^d) \leqslant K. \tag{5.10}$$

Combining (5.9) with (5.10) shows that for each $\xi \in \mathbf{Z}^d$,

$$|\widehat{\nu}(\xi)| \leqslant K \sum_{\eta \in \mathbf{Z}^d} |\widehat{\phi_{\varepsilon_0}}(\eta)||\widehat{\mu_0}(\xi - \eta)|. \tag{5.11}$$

Since μ_0 is smooth, for any $T > 0$ and $\xi \in \mathbf{Z}^d$,

$$|\widehat{\mu_0}(\xi)| \lesssim_T |\xi|^{-T}. \tag{5.12}$$

If $|\eta| \leqslant |\xi|/2$, $|\xi - \eta| \geqslant |\xi|/2$, so (5.12) implies that $|\widehat{\mu_0}(\xi - \eta)| \lesssim_T |\eta|^{-T}$ for all $T > 0$. Combined with the trivial bound $\|\widehat{\phi_{\varepsilon_0}}\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1$ we find that

$$\sum_{0 \leqslant |\eta| \leqslant |\xi|/2} |\widehat{\phi_{\varepsilon_0}}(\eta)||\widehat{\mu_0}(\xi - \eta)| \lesssim_{d,T} \frac{1}{|\xi|^{T-d}}. \tag{5.13}$$

Conversely, if $|\eta| \geqslant 2|\xi|$, then $|\xi - \eta| \geqslant |\eta|/2$, so a simple dyadic partition of the sum onto annular regions where $|\eta| \sim 2^k|\xi|$, each bounded using (5.12), combined with the trivial bound $\|\widehat{\phi_{\varepsilon_0}}\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1$, shows that for each $T > d$,

$$\sum_{|\eta| \geqslant 2|\xi|} |\widehat{\phi_{\varepsilon_0}}(\eta)||\widehat{\mu_0}(\xi - \eta)| \lesssim_{d,T} \frac{1}{|\xi|^{T-d}}. \tag{5.14}$$

Finally, if $|\xi|/2 \leqslant |\eta| \leqslant 2|\xi|$, then we employ (2.4) together with the bound $\|\widehat{\mu_0}\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1$, to conclude that

$$\sum_{|\xi|/2 \leqslant |\eta| \leqslant 2|\xi|} |\widehat{\phi_{\varepsilon_0}}(\eta)||\widehat{\mu_0}(\xi - \eta)| \lesssim_{d,T} \frac{(1/\varepsilon_0)^T}{|\xi|^{T-d}} = \frac{1}{|\xi|} \frac{(1/\varepsilon_0)^T}{|\xi|^{T-3d/2}} \frac{1}{|\xi|^{d/2}}. \tag{5.15}$$

If $|\xi| \geqslant (1/\varepsilon_0)^{T/(T-3d/2)} = (1/\varepsilon_0)^{1+O(1/T)}$, we conclude from (5.15) that

$$\sum_{|\xi|/2 \leqslant |\eta| \leqslant 2|\xi|} |\widehat{\phi_{\varepsilon_0}}(\eta)||\widehat{\mu_0}(\xi - \eta)| \lesssim_{d,T} \frac{1}{|\xi|} \frac{1}{|\xi|^{d/2}}. \tag{5.16}$$

Combining (5.13), (5.14), and (5.16) with (5.11), we conclude that for each $\varepsilon > 0$, if T is taken suitably large relative to ε, there exists a constant C depending only on μ_0, d, and ε such that if $|\xi| \geqslant (1/\varepsilon_0)^{1+\varepsilon}$, then

$$|\widehat{\nu}(\xi)| \leqslant \frac{C}{|\xi|} \frac{1}{|\xi|^{d/2}}.$$

If, in addition, we take $|\xi| \geqslant C$, then we conclude $|\widehat{\nu}(\xi)| \leqslant |\xi|^{-d/2}$. $\square$

We are now ready to complete our proof of Theorem 6.

Proof of Theorem 6. A union bound applied to Lemma 7 and Lemma 8 imply that there exists a constant C, depending solely on μ_0 and d, such that with probability greater than 1/2, we have

$$|\nu(\mathbf{T}^d) - K| \leqslant CK^{1/2}, \tag{5.17}$$

and

$$\|\widehat{\nu} - K\widehat{\mu_0}\|_{L^\infty(D)} \lesssim_{\mu_0,d} K^{1/2} \log(K)^{1/2}. \tag{5.18}$$

But combining (5.17) and (5.18), we conclude that

$$\begin{aligned} \|\widehat{\mu} - \widehat{\mu_0}\|_{L^\infty(D)} &= K^{-1}\|K\widehat{\mu} - K\widehat{\mu_0}\|_{L^\infty(D)} \\ &\lesssim_{\mu_0,d} K^{-1}\|\nu(\mathbf{T}^d)\widehat{\mu} - K\widehat{\mu_0}\|_{L^\infty(\mathbf{T}^d)} + K^{-1/2} \\ &= K^{-1/2}\|\widehat{\nu} - K\widehat{\mu_0}\|_{L^\infty(\mathbf{T}^d)} + K^{-1/2} \\ &\lesssim_{\mu_0,d} K^{-1/2} \log(K)^{1/2}. \end{aligned} \tag{5.19}$$

But this gives Property (A).

Equation (5.17) implies that for suitably large K,

$$\nu(\mathbf{T}^d) \geqslant 1. \tag{5.20}$$

Lemma 9 implies that for each $\varepsilon > 0$, there exists C depending only on μ_0, d, and ε such that if $|\xi| \geqslant \max(C, (1/\varepsilon_0)^{1+\varepsilon})$, then

$$|\widehat{\nu}(\xi)| \leqslant |\xi|^{-d/2}. \tag{5.21}$$

But then (5.20) and (5.21) imply that

$$|\widehat{\mu}(\xi)| \leqslant |\widehat{\nu}(\xi)| \leqslant |\xi|^{-d/2},$$

which proves Property (B). □

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