

# Salem Sets Avoiding Rough Configurations

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## 1 Introduction

Recall that a set  $X \subset \mathbf{R}^d$  is a *Salem set* of dimension  $t$  if it has Hausdorff dimension  $t$ , and for every  $\varepsilon > 0$ , there exists a probability measure  $\mu_\varepsilon$  supported on  $X$  such that

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{t-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty. \quad (1.1)$$

It is a result of the Poisson summation formula that if  $\mu_\varepsilon$  is compactly supported, then (1.1) is equivalent to the equation

$$\sup_{k \in \mathbf{Z}^d} |k|^{t-\varepsilon} |\widehat{\mu_\varepsilon}(k)| < \infty. \quad (1.2)$$

Our goal in these notes is to obtain high dimensional Salem sets avoiding rough configurations. Here is an initial result.

**Theorem 1.** *Let  $Z \subset [0, 1]^{dn}$  be the countable union of sets, each with lower Minkowski dimension at most  $s$ . Then there exists a Salem set  $X \subset \mathbf{R}^d$  of dimension*

$$t = \frac{nd - s}{n},$$

*such that for any  $n$  distinct elements  $x_1, \dots, x_n \in X$ ,  $(x_1, \dots, x_n) \notin Z$ .*

We rely on a random selection approach, like in our paper on rough configurations, to obtain such a result, since such random selections give high probability bounds on the Fourier transform of the measures we study.

## 2 Concentration Inequalities

Define a convex function  $\psi_2 : [0, \infty) \rightarrow [0, \infty)$  by  $\psi_2(t) = e^{t^2} - 1$ , and a corresponding Orlicz norm on the family of scalar valued random variables  $X$  over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \leq 1\}.$$

The family of random variables with  $\|X\|_{\psi_2(L)} < \infty$  are known as *subgaussian random variables*. Here are some important properties:

- If  $\|X\|_{\psi_2(L)} \leq A$ , then for each  $t \geq 0$ ,

$$\mathbf{P}(|X| \geq t) \leq 10 \exp(-t^2/10A^2).$$

Thus Subgaussian random variables have Gaussian tails.

- If  $|X| \leq A$  almost surely, then  $\|X\|_{\psi_2(L)} \leq 10A$ . Thus bounded random variables are subgaussian.
- If  $X_1, \dots, X_N$  are *independent*, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leq 10 \left( \|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2 \right)^{1/2}.$$

This is an equivalent way to state *Hoeffding's Inequality*, and we refer to an application of this inequality as an application of Hoeffding's inequality.

**Remark 2.** *The constants involved in these statements are suboptimal, but will suffice for our purposes. Proofs can be found in Chapter 2 of [1].*

Roughly speaking, we can think of a random variable  $X$  with  $\|X\|_{\psi_2(L)} \leq A$  as deviating from  $[-A, A]$  with extremely low probability. The Orlicz norm thus provides a convenient way to quantify concentration phenomena.

## 3 A Family of Cubes

Fix two integer-valued sequences  $\{K_m : m \geq 1\}$  and  $\{M_m : m \geq 1\}$ . For convenience, we also define  $N_m = K_m M_m$  for  $m \geq 1$ . We then define two sequences of real numbers  $\{l_m : m \geq 0\}$  and  $\{r_m : m \geq 0\}$ , by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each  $m, d \geq 0$ , we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string  $\sigma = \sigma_0 \sigma_1 \dots \sigma_{2k} \in \Sigma_m^d$ , we define a vector  $a(\sigma) \in (l_m \mathbf{Z})^d$  by setting

$$a(\sigma) = \sigma_0 + \sum_{k=1}^m \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k$$

Then each string  $\sigma \in \Sigma_m^d$  can be identified with the sidelength  $l_m$  cube  $Q(\sigma)$  with left-hand corner lies at  $a(\sigma)$ , i.e. the cube

$$Q(\sigma) = \prod_{i=1}^d [a(\sigma)_i, a(\sigma)_i + l_m].$$

Similarly, for each string  $\sigma = \sigma_0 \dots \sigma_{2m-1} \in \Pi_m^d$ , we define a vector  $a(\sigma) \in (r_m \mathbf{Z})^d$  by setting, for each  $1 \leq j \leq d$ ,

$$a(\sigma) = \sigma_0 + \left( \sum_{k=1}^{m-1} \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k \right) + \sigma_{2m-1} \cdot r_m,$$

and then define a sidelength  $r_m$  cube

$$R(\sigma) = \prod_{i=1}^d [a(\sigma)_i, a(\sigma)_i + r_m].$$

We let  $\mathcal{Q}_m^d = \{Q(\sigma) : \sigma \in \Sigma_m^d\}$ , and  $\mathcal{R}_m^d = \{R(\sigma) : \sigma \in \Pi_m^d\}$ . We now list some important properties of this collection of cubes:

- For each  $m$ , the two collections  $\mathcal{Q}_m^d$  and  $\mathcal{R}_m^d$  form covers of  $\mathbf{R}^d$ .
- If  $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$ , then either  $Q_1$  and  $Q_2$  have disjoint interiors, or one cube is contained in the other. Similarly, if  $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$ , then either  $R_1$  and  $R_2$  have disjoint interiors, or one cube is contained in the other.

- For each cube  $Q \in \mathcal{Q}_m$ , there is a unique cube  $Q^* \in \mathcal{R}_m$  with  $Q \subset Q^*$ . We refer to  $Q^*$  as the *parent cube* of  $Q$ . Similarly, if  $R \in \mathcal{R}_m$ , there is a unique cube in  $\mathcal{Q}_{m-1}$  with  $R \subset R^*$ , and we refer to  $R^*$  as the *parent cube* of  $R$ .

We say a set  $E \subset \mathbf{R}^d$  is  $\mathcal{Q}_m$  discretized if it is a union of cubes in  $\mathcal{Q}_m^d$ , and we then let  $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$ . Similarly, we say a set  $E \subset \mathbf{R}^d$  is  $\mathcal{R}_m$  discretized if it is a union of cubes in  $\mathcal{R}_m^d$ , and we then let  $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$ . We set  $\Sigma_m(E) = \{\sigma \in \Sigma_m^d : Q(\sigma) \in \mathcal{Q}_m(E)\}$ , and  $\Pi_m(E) = \{\sigma \in \Pi_m^d : R(\sigma) \in \mathcal{R}_m(E)\}$ . We say a cube  $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$  is *strongly non diagonal* if there does not exist two distinct indices  $i, j$ , and a third index  $\sigma \in \Pi_m^d$ , such that  $R_\sigma \cap Q_i, R_\sigma \cap Q_j \neq \emptyset$ .

## 4 A Family of Mollifiers

We now consider a family of  $C^\infty$  mollifiers, which we will use to ensure the Fourier transform of the measure we study have appropriate decay.

**Lemma 3.** *There exists a non-negative,  $C^\infty$  function  $\psi$  supported on  $[-1, 1]^d$  such that*

$$\int_{\mathbf{R}^d} \psi = 1, \quad (4.1)$$

and for each  $x \in \mathbf{R}^d$ ,

$$\sum_{n \in \mathbf{Z}^d} \psi(x + n) = 1. \quad (4.2)$$

*Proof.* Let  $\alpha$  be a non-negative,  $C^\infty$  function compactly supported on  $[0, 1]$ , such that  $\alpha(1/2 + x) = \alpha(1/2 - x)$  for all  $x \in \mathbf{R}$ ,  $\alpha(x) = 1$  for  $x \in [1/3, 2/3]$ , and  $0 \leq \alpha(x) \leq 1$  for all  $x \in \mathbf{R}$ . Then define  $\beta$  to be the non-negative,  $C^\infty$  function supported on  $[-1/3, 1/3]$  defined for  $x \in [-1/3, 1/3]$  by

$$\beta(x) = 1 - \alpha(|x|) = 1 - \alpha(1 - |x|).$$

Symmetry considerations imply that  $\int_{\mathbf{R}} \alpha + \beta = 1$ , and for each  $x \in \mathbf{R}$ ,

$$\sum_{m \in \mathbf{Z}} \alpha(x + m) + \beta(x + m) = 1. \quad (4.3)$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

The function  $\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d)$  then satisfies the constraints of the lemma.  $\square$

Fix some choice of  $\psi$  given by Lemma 3. Since  $\psi$  is  $C^\infty$  and compactly supported, then for each  $t \in [0, \infty)$ , we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \quad (4.4)$$

Now we rescale the mollifier. For each integer  $m \geq 1$ , we let

$$\psi_m(x) = l_m^{-d} \cdot \psi(l_m \cdot x).$$

Then  $\psi_m$  is supported on  $[-l_m, l_m]^d$ . Equation (4.1) implies that for each  $x \in \mathbf{R}^d$ ,

$$\int_{\mathbf{R}^d} \psi_m = 1. \quad (4.5)$$

Equation (4.2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \quad (4.6)$$

An important property of the rescaling in the frequency domain is that for each  $\xi \in \mathbf{R}^d$ ,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \cdot \xi), \quad (4.7)$$

In particular, (4.7) implies that for each  $t \geq 0$ ,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)| |\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)| |\xi|^t. \quad (4.8)$$

Intuitively,  $\{\psi_m\}$  is a ‘uniform’ family of wave packets, with  $\psi_m$  supported in phase space on  $[-l_m, l_m]^d$ , and in frequency space, essentially supported on  $[-l_m^{-1}, l_m^{-1}]^d$ .

## 5 Comparison to Previous Paper

As in our previous paper, our proof of Theorem 1 will involve constructing a configuration avoiding set  $X$  by considering a nested decreasing family of sets  $\{X_m : m \geq 0\}$ , where  $X_m \subset [0, 1]^d$  is a  $\mathcal{Q}_m$  discretized set, and then

setting  $X = \bigcap_{m \geq 0} X_m$ . We find a strong cover of  $Z$  by sets  $\{B_m\}$ , where  $B_m$  is  $\mathcal{Q}_m$  discretized. Provided  $X_m^d$  is disjoint from strongly non-diagonal cubes in  $B_m$ , we conclude that for any  $n$  distinct elements  $x_1, \dots, x_n \in X$ ,  $(x_1, \dots, x_n) \notin Z$ . We now show that the technique of our last paper as stated fails to produce Salem sets.

Let us recap the approach of our last paper. To form  $X_{m+1}$ , we chose a cube  $Q_R \in \mathcal{Q}_{m+1}(R)$  uniformly at random, for each  $R \in \mathcal{R}_{m+1}(X_m)$ . We then let  $Y_{m+1} = \bigcup Q_R$ . If  $s \geq d$ , and

$$K_{m+1} \approx M_{m+1}^{\frac{s-d}{dn-s}}, \quad (5.1)$$

then with non-zero probability, we proved there is  $X_{m+1} \subset Y_{m+1}$  such that  $X_{m+1}^d$  avoids strongly non-diagonal cubes in  $B_{m+1}$ , and  $X_{m+1}$  contains at least half of the cubes in  $\mathcal{Q}_{m+1}(Y_{m+1})$ . Then  $X_{m+1}$  will be the union of at least  $M_{m+1}^{-d}$  cubes with sidelength  $l_{m+1}$ . Provided that  $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$ , we have

$$M_{m+1}^{-d} \approx r_{m+1}^{-d} \approx l_{m+1}^{-\frac{dn-s}{n-1}}.$$

Thus  $X$  has lower Minkowski dimension at most  $(dn - s)/(n - 1)$ , and a more involved analysis shows the set has Hausdorff dimension exactly equal to  $(dn - s)/(n - 1)$ .

The approach detailed in the last paragraph is *not* guaranteed to produce a set with Fourier dimension  $t$ . Because  $X_{m+1}$  is random, it exhibits pseudorandomness properties with high probability. In particular, it supports probability measures whose Fourier transform has sharp decay. However, since the choice of the set  $Y_{m+1}$  is *not* chosen randomly from  $X_{m+1}$ , depending heavily on the set  $Z$  and the discretized set  $B_{m+1}$ , the set  $Y_{m+1}$  will in general not possess pseudorandomness properties. For instance, if  $\mu$  is the probability measure induced by normalizing Lebesgue measure restricted to  $X_{m+1}$ , then with high probability,

$$\|\widehat{\mu}\|_{L^\infty(\mathbf{R}^d)} \approx l_m^t.$$

If  $\nu$  is the probability measure induced by normalizing Lebesgue measure restricted to  $Y_{m+1}$ , then it is still possible for us to have

$$\|\widehat{\nu}\|_{L^\infty(\mathbf{R}^d)} \gtrsim 1.$$

For instance, this will be true if  $\mathcal{Q}_{m+1}(X_{m+1}) - \mathcal{Q}_{m+1}(Y_{m+1})$  is a thickening of a subset of an arithmetic progression. Thus the method of our previous paper is not able to reliably produce Salem sets without further analysis on the pseudorandom properties of the sets  $\{B_m\}$  we have to avoid.

In this paper, we take a different approach which avoids us having to analyze the pseudorandomness of the sets  $B_m$ . Instead of (5.1), we choose

$$K_{m+1} \approx M_{m+1}^{\frac{s}{dn-s}}.$$

Notice that  $M_{m+1}^{\frac{s}{dn-s}} \geq M_{m+1}^{\frac{s-d}{dn-s}}$ , so the set  $Y_{m+1}$  we will obtain will be a thinner set than  $X_m$ . In particular,  $Y_{m+1}$  will be covered by at most  $M_{m+1}^{-d}$  sidelength  $l_{m+1}$  cubes, and if  $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$

$$r_{m+1}^{-d} \approx l_{m+1}^{-t}$$

sidelength  $l_{m+1}$  cubes, which implies  $X$  will have upper Minkowski dimension at most  $t$ . However, as a result, because the set  $Y_{m+1}$  is thinner, we find that  $Y_{m+1}^d$  is disjoint from the cubes in  $B_{m+1}$  with high probability. In particular, we can set  $X_{m+1} = Y_{m+1}$ . This means that  $X_{m+1}$  will be pseudorandom, and we should therefore expect  $X$  to be a Salem set of dimension  $t$ . The remainder of this paper is devoted to showing that these heuristics are correct.

## 6 Discrete Lemma

We now proceed to solve a discretized version of Theorem 1.

**Proposition 4.** *Fix  $s \in [1, dn)$  and  $\varepsilon \in [0, (dn - s)/2)$ . Let  $T \subset [0, 1]^d$  be a non-empty,  $\mathcal{Q}_m$  discretized set, and let  $\mu_T$  be a smooth measure compactly supported on  $T$ . Let  $B \subset \mathbf{R}^{dn}$  be a non-empty,  $\mathcal{Q}_{m+1}$  discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}. \quad (6.1)$$

*Then there exists a large constant  $C(\mu_T, l_m, n, d, s, \varepsilon)$ , such that if*

$$K_{m+1}, M_{m+1} \geq C(\mu_T, l_m, n, d, s, \varepsilon, l_m), \quad (6.2)$$

*and*

$$M_{m+1}^{\frac{s}{dn-s} + c\varepsilon} \leq K_{m+1} \leq 2M_{m+1}^{\frac{s}{dn-s} + c\varepsilon}, \quad (6.3)$$

where

$$c = \frac{6dn}{(dn - s)^2},$$

then there exists a  $\mathcal{Q}_{m+1}$  discretized set  $S \subset T$  together with a smooth probability measure supported on  $S$  such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists  $i$  such that  $Q_i \notin \mathcal{Q}_{m+1}(S)$ .

(B)  $\mu_S(\mathbf{R}^d) \geq \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$ .

(C) If  $|k| \leq 10l_{m+1}^{-d}$ ,  $|\widehat{\mu}_T(k) - \widehat{\mu}_S(k)| \leq r_{m+1}^{d/2} \log(M_{m+1})$ .

(D) If  $|k| \geq 10l_{m+1}^{-d}$ ,  $|\widehat{\mu}_S(k)| \leq |k|^{-d/2}$ .

**Remark 5.** To make the statement of Proposition (4) more clean, we have hidden the explicit choice of constant  $C(\mu_T, l_m, n, d, s, \varepsilon)$ . But this constant can certainly be made explicit; such a choice can be made by ensuring that (6.2) implies (6.5), (6.11), (6.17), (6.27), (6.28), and (6.29) all hold.

*Proof of Proposition 4.* First, we describe the construction of the set  $S$ , and the measure  $\mu_S$ . For each string  $\sigma \in \Pi_{m+1}^d$ , let  $j_\sigma$  be a random integer vector chosen from  $\{0, \dots, K_{m+1} - 1\}^d$ , such that the family  $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$  is an independent family of random variables. Then it is certainly true that for any  $j \in [K_{m+1}]^d$ ,

$$\mathbf{P}(j_\sigma = j) = K_{m+1}^{-d}. \quad (6.4)$$

Then  $\sigma j_\sigma \in \Sigma_{m+1}^d$ . We can thus define a measure  $\mu_S$  such that, for each  $x \in \mathbf{R}^d$ ,

$$d\mu_S(x) = r_{m+1}^d \sum_{\sigma \in \Pi_{m+1}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then  $S$  is  $\mathcal{Q}_{m+1}$  discretized,  $\mu_S$  is supported on  $S$ , and  $S \subset T$ . Our goal is to show that, with non-zero probability, some choice of the family of indices  $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$  yields a set  $S$  and a measure  $\mu_S$  satisfying Properties (A) and (B) of Proposition 4. In our calculations, it will help us to decompose the



measure  $\mu_S$  into components roughly supported on sidelength  $r_{m+1}^d$  cubes. For each  $\sigma \in \Pi_{m+1}(T)$ , define a measure  $\mu_\sigma$  such that for each  $x \in \mathbf{R}^d$ ,

$$d\mu_\sigma(x) = r_{m+1}^d \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

Then  $\mu_S = \sum_{\sigma \in \Pi_{m+1}^d(T)} \mu_\sigma$ . We shall split the proof of Properties (A), (B), and (C) into several more managable lemmas.

**Lemma 6.** *If*

$$M_{m+1} \geq \left( 3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right)^2, \quad (6.5)$$

*then*  $\mu_S(\mathbf{R}^d) \geq \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$ .

*Proof.* Fix  $\sigma \in \Pi_{m+1}^d$ . If  $j_0, j_1 \in \{0, \dots, K_{m+1} - 1\}^d$ , then

$$|a(\sigma j_0) - a(\sigma j_1)| = |j_0 - j_1| \cdot l_{m+1} \leq (\sqrt{d} \cdot K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1}. \quad (6.6)$$

Together with (4.5), (6.6) implies

$$\begin{aligned} & \left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_0)) \mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_1)) \mu_T(x) \right| \\ & \leq r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) |\mu_T(x + a(\sigma j_0)) - \mu_T(x + a(\sigma j_1))| \\ & \leq \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1} \\ & = \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad (6.7)$$

Thus (6.7) implies that for each  $\sigma$ ,

$$|\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \leq \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \quad (6.8)$$

Furthermore, (4.6) implies

$$\begin{aligned} & \sum_{\sigma \in \Pi_{m+1}^d} \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \\ & = r_{m+1}^d \sum_{(\sigma, j) \in \Sigma_{m+1}^d} \mathbf{P}(j_\sigma = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) d\mu_T(x) \\ & = \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left( \sum_{(\sigma, j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\ & = \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \mu_T(\mathbf{R}^d) = \mu_T(\mathbf{R}^d). \end{aligned} \quad (6.9)$$

For all but at most  $3^d r_{m+1}^{-d}$  indices  $\sigma \in \Pi_{m+1}^d$ ,  $\mu_\sigma = 0$  almost surely. Thus we can apply the triangle inequality together with (6.8) and (6.9) to conclude that

$$\begin{aligned}
|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| &= \left| \sum_{\sigma \in \Pi_{m+1}^d} [\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))] \right| \\
&\leq \sum_{\sigma \in \Pi_{m+1}^d} |\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \\
&\leq (3^d r_{m+1}^{-d}) \left( \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \\
&= \left( 3^d \sqrt{d} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \cdot r_{m+1} \\
&= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.
\end{aligned} \tag{6.10}$$

Thus (6.5) and (6.10) imply that,  $|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| \leq M_{m+1}^{-1/2}$ .  $\square$

**Lemma 7.** *If*

$$M_{m+1} \geq (10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)})^{1/\varepsilon}, \tag{6.11}$$

*then*

$$\mathbf{P}(S \text{ does not satisfies Property (A)}) \leq 1/10.$$

*Proof.* For any cube  $Q \in \Sigma_{m+1}^d$ , there are at most  $3^d$  indices  $\sigma j \in \Sigma_{m+1}^d$  such that  $Q_{\sigma j} \cap Q \neq \emptyset$ , and so a union bound together with (6.4) gives

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S)) \leq \sum_{Q_{\sigma j} \cap Q \neq \emptyset} \mathbf{P}(j_\sigma = j) \leq 3^d K_{m+1}^{-d}. \tag{6.12}$$

Without loss of generality, removing cubes from  $B$  if necessary, we may assume all cubes in  $B$  are strongly non-diagonal. Let  $Q = Q_1 \times \dots \times Q_n \in \mathcal{Q}_{m+1}(B)$  be such a cube. Since  $Q$  is strongly diagonal, the events  $\{Q_k \in S\}$  are independent from one another for  $k \in \{1, \dots, n\}$ , which together with (6.12) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_1 \in S) \dots \mathbf{P}(Q_n \in S) \leq 3^{dn} K_{m+1}^{-dn}. \tag{6.13}$$

Taking expectations over all cubes in  $B$ , and applying (6.1) and (6.13) gives

$$\begin{aligned}
\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leq \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\
&\leq l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\
&= \frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}.
\end{aligned} \tag{6.14}$$

Since  $\varepsilon \leq (dn - s)/2$ , we conclude

$$\begin{aligned} (dn - s - \varepsilon) \left( \frac{s}{dn - s} + c\varepsilon \right) &= s + \varepsilon \left( c(dn - s - \varepsilon) - \frac{s}{dn - s} \right) \\ &\geq s + \varepsilon \left( \frac{c(dn - s)}{2} - \frac{s}{dn - s} \right) \\ &= s + \varepsilon \frac{3dn - s}{dn - s} \geq s + 2\varepsilon. \end{aligned}$$

Applying (6.3), we therefore conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geq M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \geq M_{m+1}^{s+2\varepsilon}.$$

Combined with (6.11), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leq \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^\varepsilon} \leq 1/10. \quad (6.15)$$

We can then apply Markov's inequality with (6.14) and (6.15) to conclude

$$\begin{aligned} \mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \emptyset) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geq 1) \\ &\leq \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) \\ &\leq 1/10. \end{aligned} \quad \square$$

**Lemma 8.** Set  $D = \{k \in \mathbf{Z}^d : |k| \leq 10l_{m+1}^{-1}\}$ . Then if

$$K_{m+1} \leq M_{m+1}^{\frac{2dn}{dn-s}}, \quad (6.16)$$

and

$$M_{m+1} \geq \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \quad (6.17)$$

then

$$\mathbf{P}\left(\|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1})\right) \leq 1/10 \quad (6.18)$$

*Proof.* For each  $\sigma \in \Pi_{m+1}^d$ , and  $k \in \mathbf{Z}$ , define  $X_{\sigma k} = \widehat{\mu}_\sigma(k) - \widehat{\mathbf{E}}(\widehat{\mu}_\sigma)(k)$ . Applying (4.2) gives

$$\begin{aligned} \sum_{\sigma \in \Pi_{m+1}^d} \widehat{\mathbf{E}}(\widehat{\mu}_\sigma)(k) &= \sum_{\sigma \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \psi_{m+1}(x - a(\sigma j)) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) = \widehat{\mu}_T(k). \end{aligned} \quad (6.19)$$

For each  $\sigma$  and  $k$ , the standard  $(L^1, L^\infty)$  bound on the Fourier transform, combined with (6.8), shows

$$\begin{aligned} \|X_{\sigma k}\|_{\psi_2(L)} &\leq 10|X_{\sigma k}| \\ &\leq 10[|\mu_\sigma(\mathbf{R}^d)| + \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)] \\ &\leq 10^2 \left( \mathbf{E}(\mu_\sigma)(\mathbf{R}^d) + \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (6.20)$$

For a fixed  $k$ , the family of random variables  $\{X_{\sigma k} : \sigma \in \Pi_{m+1}^d\}$  are independent. Furthermore,  $\sum X_{\sigma k} = \widehat{\mu_S}(k) - \widehat{\mathbf{E}(\mu_S)}(k)$ . Equations (4.6) and (6.4) imply that

$$\begin{aligned} \mathbf{E}(\widehat{\mu_S}(k)) &= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \left( \sum_{(\sigma, j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \widehat{\mu_T}(k) = \widehat{\mu_T}(k). \end{aligned} \quad (6.21)$$

Hoeffding's inequality, together with (6.20) and (6.21), imply that

$$\begin{aligned} \|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} &\leq 10^3 \sqrt{d} \left( \left( \sum \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)^2 \right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (6.22)$$

Equation (4.5) shows

$$\begin{aligned} \mathbf{E}(\mu_\sigma)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int \psi_{m+1}(x - a(ij)) d\mu_T(x) \\ &\leq r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad (6.23)$$

Combining (6.22) and (6.23) gives

$$\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \leq 10^3 \sqrt{d} \left[ \|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right] r_{m+1}^{d/2}. \quad (6.24)$$

We can then apply a union bound over the set  $D$ , which has cardinality at most  $10^{d+1}l_{m+1}^{-d}$ , together with (6.24) to conclude that

$$\begin{aligned} \mathbf{P} \left( \|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \\ \leq 10^{d+2} \cdot l_{m+1}^{-d} \exp \left( -\frac{\log(M_{m+1})^2}{10^7 d} \right) \\ = 10^{d+2} l_m^{-d} \exp \left( d \log(M_{m+1} K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d} \right). \end{aligned} \quad (6.25)$$

Combined with (6.16) and (6.17), (6.25) implies

$$\mathbf{P} \left( \|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \leq 1/10. \quad (6.26)$$

Thus  $\widehat{\mu}_S$  and  $\widehat{\mu}_T$  are highly likely to differ only by a negligible amount over small frequencies.  $\square$

Since  $\mu_T$  is compactly supported, we can define, for each  $t > 0$ ,

$$A(t) = \sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}_T(\xi)| |\xi|^t < \infty.$$

In light of (4.7), if we define, for each  $t > 0$ ,

$$B(t) = \sup_{\xi \in \mathbf{R}^d} |\widehat{\psi}(\xi)| |\xi|^t,$$

then

$$\sup_{\xi \in \mathbf{R}^d} |\widehat{\psi_{m+1}}(\xi)| |\xi|^t = l_{m+1}^{-t} B(t).$$

**Lemma 9.** *Suppose that*

$$N_{m+1}^d \geq 10 \cdot 2^{3d/2+1} A(3d/2 + 1), \quad (6.27)$$

$$N_{m+1}^d \geq \frac{10 \cdot 2^{3d}}{1 + d/2} A(3d/2 + 1), \quad (6.28)$$

and

$$N_{m+1}^d \geq 10 \cdot 2^{7d/2+1} B(3d/2 + 1). \quad (6.29)$$

then if  $|\eta| \geq 10l_{m+1}^{-1}$ ,

$$|\widehat{\mu}_S(\eta)| \leq \frac{1}{|\eta|^{d/2}}. \quad (6.30)$$

*Proof.* Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{\sigma \in \Pi_{m+1}^d \\ d(a(\sigma), T) \leq 2r_{m+1}^{-1}}} \delta_{a(ij_i)}.$$

Then  $\mu_S = (\alpha * \psi_{m+1})\mu_T$ . Thus we have  $\widehat{\mu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$ . The measure  $\alpha$  is the sum of at most  $2^d r_{m+1}^{-d}$  delta functions, scaled by  $r_{m+1}^d$ , so  $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leq \alpha(\mathbf{R}^d) \leq 2^d$ . Thus

$$|\widehat{\mu_S}(\eta)| \leq 2^d \int |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi. \quad (6.31)$$

If  $|\xi| \leq |\eta|/2$ ,  $|\eta - \xi| \geq |\eta|/2$ , and since (4.5) implies  $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leq 1$ , we find that for all  $t > 0$ ,

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \leq \frac{A(t)2^{t-d}}{|\eta|^{t-d}}. \quad (6.32)$$

Set  $t = 3d/2 + 1$ . Equation (6.32), together with (6.27), implies

$$\begin{aligned} & \int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \\ & \leq \frac{A(3d/2 + 1)2^{1+d/2}|\eta|^{-1}}{|\eta|^{d/2}} \\ & \leq \frac{A(3d/2 + 1)2^{1+d/2}l_{m+1}}{|\eta|^{d/2}} \\ & \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \end{aligned} \quad (6.33)$$

Conversely, if  $|\xi| \geq 2|\eta|$ , then  $|\eta - \xi| \geq |\xi|/2$ , so for each  $t > d$ ,

$$\begin{aligned} \int_{|\xi| \geq 2|\eta|} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi & \leq \int_{|\xi| \geq 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\ & \leq 2^d \int_{2|\eta|}^\infty r^{d-1-t} A(t) 2^t \\ & \leq \frac{4^d A(t)}{t-d} |\eta|^{d-t}. \end{aligned} \quad (6.34)$$

Set  $t = 3d/2 + 1$ . Equation (6.28), applied to (6.34), allows us to conclude

$$\int_{|\xi| \geq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \quad (6.35)$$

Finally, if  $t > 0$ , we use the fact that  $\|\widehat{\mu}_T\|_{L^\infty(\mathbf{R}^d)} \leq 1$  to conclude that

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \quad (6.36)$$

Set  $t = 3d/2 + 1$ . Then (6.36) and (6.29) imply

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \quad (6.37)$$

It then suffices to sum up (6.33), (6.35), and (6.37), and apply (6.31).  $\square$

*Proof of Proposition 4, Continued.* Let us now put all our calculations together. In light of Lemma 7 and Lemma 8, there exists some choice of  $j_\sigma$  for each  $\sigma$ , and a resultant non-random pair  $(\mu_S, S)$  such that  $S$  satisfies Property (A) of the Lemma, and  $\mu_S$  satisfies (6.18), implying that  $\mu_S$  satisfies Property (C) of the Lemma. But Lemma 6 shows that  $\mu_S$  always satisfies Property (B), and Lemma (9) shows Property (D) is also always satisfied. This completes the proof.  $\square$

## 7 Construction of the Salem Set

Let us now choose the parameters to construct our configuration avoiding set. First, we fix some preliminary parameters. Write  $Z \subset \bigcup_{i=1}^\infty Z_i$ , where  $Z_i$  has lower Minkowski dimension at most  $s$  for each  $i$ . Then choose an infinite sequence  $\{i_m : m \geq 1\}$  which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers  $\{\varepsilon_m : m \geq 1\}$ , with  $\varepsilon_m < (dn - s)/2$  for each  $m$ . We choose our parameters  $\{M_m\}$  and  $\{K_m\}$  inductively. First, set  $X_0 = [0, 1]^d$ , and  $\mu_0$  an arbitrary smooth probability measure supported on  $X_0$ . At the  $m$ th step of our construction, we have already found a set  $X_{m-1}$  and a measure  $\mu_{m-1}$ . We then choose  $K_m$  and  $M_m$  such that

$$K_m, M_m \geq C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m^{\frac{s}{dn-s} + c\varepsilon_m} \leq K_m \leq 2M_m^{\frac{s}{dn-s} + c\varepsilon},$$

and such that the set  $Z_{i_m}$  is covered by at most  $l_m^{-(s+\varepsilon_m)}$  cubes in  $\mathcal{Q}_m$ , the union of which, we define to be equal to  $B_m$ . We can then apply Proposition 4 with  $\varepsilon = \varepsilon_m$ ,  $T = X_{m-1}$ ,  $\mu_T = \mu_{m-1}$ , and  $B = B_m$ . This produces a  $\mathcal{Q}_{m+1}$  discretized set  $S \subset T$ , and a measure  $\mu_S$  supported on  $S$ . We define  $X_m = S$ , and  $\mu_m = \mu_S$ .

The last paragraph recursively generates an infinite sequence  $\{X_m\}$ . We set  $X = \bigcap X_m$ . Just as in our previous paper, it is easy to see  $X$  must be a configuration avoiding set. Given any  $(x_1, \dots, x_n) \in Z$ , there are infinitely many integers  $m_k$  such that  $(x_1, \dots, x_n) \in B_{m_k}$ . If  $|x_i - x_j| \geq \varepsilon$  for each  $i \neq j$ , and  $r_{m_k} \leq \varepsilon/2$ , then  $(x_1, \dots, x_n)$  is contained in a strongly non-diagonal cube in  $\mathcal{Q}_{m_k}(B_k)$ , and as such  $X^n \subset X_{m_k}^n$  does not contain  $(x_1, \dots, x_n)$ .

## 8 Proof that $X$ is Salem

We now show  $X$  is Salem, completing the proof of Theorem 1. Since the masses of the sequence of measures  $\{\mu_m\}$  is uniformly bounded, there is some subsequence  $\mu_{m_i}$  which converges weakly to some measure  $\mu$ . Repeated applications of Property (B) of Proposition 4 imply

$$\mu(\mathbf{R}^d) = \lim_{i \rightarrow \infty} \mu_{m_i}(\mathbf{R}^d) \geq 1 - \sum_{m=1}^{\infty} M_m^{-1/2}.$$

In particular,  $\mu$  is a non-zero measure if the sequence  $\{M_m\}$  is rapidly increasing. Moreover, for each  $k \in \mathbf{Z}^d$ ,

$$\widehat{\mu}(k) = \lim_{i \rightarrow \infty} \widehat{\mu_{m_i}}(k).$$

Thus

$$|\widehat{\mu}(k)| \leq |\widehat{\mu_0}(k)| + \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)|.$$



Fix  $\varepsilon > 0$ . Since  $l_m \leq 2^{-m}/10$ , we find that for  $m \geq \log(k)$ ,  $|k| \leq 10l_{m+1}^{-1}$ . Thus we can apply Property (C) and (D) of Proposition 4 implies

$$\begin{aligned}
& \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)| \\
& \leq 2 \log(k) |k|^{-d/2} + \sum_{m=\log(k)}^{\infty} r_{m+1}^{d/2} \log(M_{m+1}) \\
& \lesssim_{\varepsilon} |k|^{\varepsilon-t/2} \left( 1 + \sum_{m=\log(k)}^{\infty} |k|^{t/2-\varepsilon} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
& \leq |k|^{\varepsilon-t/2} \left( 1 + 10^{t/2-\varepsilon} \sum_{m=\log(k)}^{\infty} l_{m+1}^{\varepsilon-t/2} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
& \lesssim_{\varepsilon} |k|^{\varepsilon-t/2} \left( 1 + \sum_{m=\log(k)}^{\infty} \frac{1}{K_{m+1}^{\varepsilon}} \frac{K_{m+1}^{t/2}}{M_{m+1}^{d/2-t/2}} \right) \\
& \lesssim |k|^{\varepsilon-t/2} \left( 1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^{\varepsilon}} \frac{M_{m+1}^{(t/2)(\frac{s}{dn-s})}}{M_{m+1}^{d/2-t/2}} \right) \\
& = |k|^{\varepsilon-t/2} \left( 1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^{\varepsilon}} \right) \lesssim_{\varepsilon} |k|^{\varepsilon-t/2}.
\end{aligned}$$

The last inequality follows because  $\varepsilon_m \rightarrow 0$ , and so the series is summable if the sequence  $\{K_m\}$  increases rapidly enough. Since  $\mu_0$  is smooth and compactly supported, we find

$$\sup_{k \in \mathbf{Z}^d} |k|^{t/2-\varepsilon} |\widehat{\mu}(k)| \lesssim_{\varepsilon} 1 + \sup_{k \in \mathbf{Z}} |k|^{t/2-\varepsilon} |\widehat{\mu_0}(k)| < \infty.$$

Since  $\varepsilon > 0$  was arbitrary, this shows that the Fourier dimension of  $X$  is at least  $t$ . Because  $X_m$  is the union of  $(M_1 \dots M_m)^d$  sidelength  $l_m$  cubes, one can easily show using (6.3) that the lower Minkowski dimension of  $X$  is upper bounded by  $t$ . But these two bounds imply that the Hausdorff dimension, Fourier dimension, and Minkowski dimension are all equal to  $t$ . Thus  $X$  is Salem of dimension  $t$ .

## References

- [1] Roman Vershynin, *High Dimensional Probability*, Cambridge Series in Statistical and Probabilistic Mathematics, 2018.