Salem Sets Avoiding Rough Configurations

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1 Introduction

Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension t if it has Hausdorff dimension t, and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{t-\varepsilon} |\widehat{\mu}_\varepsilon(\xi)| < \infty. \tag{1.1}$$

It is a result of the Poisson summation formula that if μ_ε is compactly supported, then (1.1) is equivalent to the equation

$$\sup_{k \in \mathbf{Z}^d} |k|^{t-\varepsilon} |\widehat{\mu}_\varepsilon(k)| < \infty. \tag{1.2}$$

Our goal in these notes is to obtain high dimensional Salem sets avoiding rough configurations.

Theorem 1. *Let $Z \subset [0,1]^{dn}$ be the countable union of sets, each with lower Minkowski dimension at most s. Then there exists a Salem set $X \subset \mathbf{R}^d$ of dimension*

$$t = \frac{nd - s}{n},$$

such that for any n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

We rely on a random selection approach, like in our paper on rough configurations, to obtain such a result, since such random selections give high probability bounds on the Fourier transform of the measures we study.

2 Concentration Inequalities

Define a convex function $\psi_2 : [0,\infty) \to [0,\infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{A \in (0,\infty) : \mathbf{E}(\psi_2(|X|/A)) \leqslant 1\}.$$

The family of random variables with $\|X\|_{\psi_2(L)} < \infty$ are known as *subgaussian random variables*. Here are some important properties:

- If $\|X\|_{\psi_2(L)} \leqslant A$, then for each $t \geqslant 0$,

  $$\mathbf{P}(|X| \geqslant t) \leqslant 10 \exp\left(-t^2/10A^2\right).$$

  Thus Subgaussian random variables have Gaussian tails.

- If $|X| \leqslant A$ almost surely, then $\|X\|_{\psi_2(L)} \leqslant 10A$. Thus bounded random variables are subgaussian.

- If $X_1, \dots, X_N$ are *independent*, then

  $$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leqslant 10 \left(\|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2\right)^{1/2}.$$

  This is an equivalent way to state *Hoeffding's Inequality*, and we refer to an application of this inequality as an application of Hoeffding's inequality.

Remark 2. *The constants involved in these statements are suboptimal, but will suffice for our purposes. Proofs can be found in Chapter 2 of [1].*

Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leqslant A$ as a variable whose magnitude exceeds A with extremely low probability. The Orlicz norm thus provides a convenient way to quantify concentration phenomena.

3 A Family of Cubes

Fix two integer-valued sequences $\{K_m : m \geqslant 1\}$ and $\{M_m : m \geqslant 1\}$. For convenience, we also define $N_m = K_m M_m$ for $m \geqslant 1$. We then define two sequences of real numbers $\{l_m : m \geqslant 0\}$ and $\{r_m : m \geqslant 0\}$, by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each $m, d \geqslant 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $\sigma = \sigma_0 \sigma_1 \dots \sigma_{2k} \in \Sigma_m^d$, we define a vector $a(\sigma) \in (l_m \mathbf{Z})^d$ by setting

$$a(\sigma) = \sigma_0 + \sum_{k=1}^m \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k$$

Then each string $\sigma \in \Sigma_m^d$ can be identified with the sidelength l_m cube $Q(\sigma)$ with left-hand corner lies at $a(\sigma)$, i.e. the cube

$$Q(\sigma) = \prod_{i=1}^d \left[a(\sigma)_i, a(\sigma)_i + l_m \right].$$

Similarly, for each string $\sigma = \sigma_0 \dots \sigma_{2m-1} \in \Pi_m^d$, we define a vector $a(\sigma) \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leqslant j \leqslant d$,

$$a(\sigma) = \sigma_0 + \left(\sum_{k=1}^{m-1} \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k \right) + \sigma_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R(\sigma) = \prod_{i=1}^d \left[a(\sigma)_i, a(\sigma)_i + r_m \right].$$

We let $\mathcal{Q}_m^d = \{Q(\sigma) : \sigma \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R(\sigma) : \sigma \in \Pi_m^d\}$. We now list some important properties of this collection of cubes:

- For each m, the two collections $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ form covers of $\mathbf{R}^d$.
- If $Q_1, Q_2 \in \bigcup_{m=0}^\infty \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^\infty \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.

- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarly, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{\sigma \in \Sigma_m^d : Q(\sigma) \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{\sigma \in \Pi_m^d : R(\sigma) \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j, and a third index $\sigma \in \Pi_m^d$, such that $R_\sigma \cap Q_i, R_\sigma \cap Q_j \neq \varnothing$.

4 A Family of Mollifiers

We now consider a family of C^∞ mollifiers, which we will use to ensure the Fourier transform of the measure we study have appropriate decay.

Lemma 3. *There exists a non-negative, C^∞ function ψ supported on $[-1,1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi = 1, \tag{4.1}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{4.2}$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0,1]$, such that $\alpha(1/2 + x) = \alpha(1/2 - x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \leqslant \alpha(x) \leqslant 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|) = 1 - \alpha(1 - |x|).$$

Symmetry considerations imply that $\int_{\mathbf{R}} \alpha + \beta = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x+m) + \beta(x+m) = 1. \tag{4.3}$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

The function $\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d)$ then satisfies the constraints of the lemma. $\square$

Fix some choice of ψ given by Lemma 3. Since ψ is C^∞ and compactly supported, then for each $t \in [0, \infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \tag{4.4}$$

Now we rescale the mollifier. For each integer $m \geqslant 1$, we let

$$\psi_m(x) = l_m^{-d} \cdot \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (4.1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \tag{4.5}$$

Equation (4.2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \tag{4.6}$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \cdot \xi), \tag{4.7}$$

In particular, (4.7) implies that for each $t \geqslant 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)||\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)||\xi|^t. \tag{4.8}$$

Intuitiely, $\{\psi_m\}$ is a 'uniform' family of wave packets, with ψ_m supported in phase space on $[-l_m, l_m]^d$, and in frequency space, essentially supported on $[-l_m^{-1}, l_m^{-1}]^d$.

5 Comparison to Previous Paper

As in our previous paper, our proof of Theorem 1 will involve constructing a configuration avoiding set X by considering a nested decreasing family of sets $\{X_m : m \geqslant 0\}$, where $X_m \subset [0,1]^d$ is a $\mathcal{Q}_m$ discretized set, and then setting $X = \bigcap_{m \geqslant 0} X_m$. We find a strong cover of Z by sets $\{B_m\}$, where B_m is $\mathcal{Q}_m$ discretized. Provided X_m^d is disjoint from strongly non-diagonal cubes in B_m, we conclude that for any n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We now show that the technique of our last paper as stated fails to produce Salem sets.

Let us recap the approach of our last paper. To form X_{m+1}, we chose a cube $Q_R \in \mathcal{Q}_{m+1}(R)$ uniformly at random, for each $R \in \mathcal{R}_{m+1}(X_m)$. We then let $Y_{m+1} = \bigcup Q_R$. If $s \geqslant d$, and

$$K_{m+1} \approx M_{m+1}^{\frac{s-d}{dn-s}}, \tag{5.1}$$

then with non-zero probability, we proved there is $X_{m+1} \subset Y_{m+1}$ such that X_{m+1}^d avoids strongly non-diagonal cubes in B_{m+1}, and X_{m+1} contains at least half of the cubes in $\mathcal{Q}_{m+1}(Y_{m+1})$. Then X_{m+1} will be the union of at least M_{m+1}^{-d} cubes with sidelength l_{m+1}. Provided that $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$, we have

$$M_{m+1}^{-d} \approx r_{m+1}^{-d} \approx l_{m+1}^{-\frac{dn-s}{n-1}}.$$

Thus X has lower Minkowski dimension at most $(dn-s)/(n-1)$, and a more involved analysis shows the set has Hausdorff dimension exactly equal to $(dn-s)/(n-1)$.

The approach detailed in the last paragraph is *not* guaranteed to produce a set with Fourier dimension t. Because X_{m+1} is random, it exhibits psuedorandomness properties with high probability. In particular, it supports probability measures whose Fourier transform has sharp decay. However, since the choice of the set Y_{m+1} is *not* chosen randomly from X_{m+1}, depending heavily on the set Z and the discretized set B_{m+1}, the set Y_{m+1} will in general not possess psuedorandomness properties. For instance, if μ is the probability measure induced by normalizing Lebesgue measure restricted to X_{m+1}, then with high probability,

$$\|\widehat{\mu}\|_{L^\infty(\mathbf{R}^d)} \approx l_m^t.$$

If ν is the probability measure induced by normalizing Lebesgue measure restricted to Y_{m+1}, then it is still possible for us to have

$$\|\widehat{\nu}\|_{L^\infty(\mathbf{R}^d)} \gtrsim 1.$$

For instance, this will be true if $\mathcal{Q}_{m+1}(X_{m+1}) - \mathcal{Q}_{m+1}(Y_{m+1})$ is a thickening of a subset of an arithmetic progression. Thus the method of our previous paper is not able to reliably produce Salem sets without further analysis on the psuedorandom properties of the sets $\{B_m\}$ we have to avoid.

In this paper, we take a different approach which avoids us having to analyze the psuedorandomness of the sets B_m. Instead of (5.1), we choose

$$K_{m+1} \approx M_{m+1}^{\frac{s}{dn-s}}.$$

Notice that $M_{m+1}^{\frac{s}{dn-s}} \geqslant M_{m+1}^{\frac{s-d}{dn-s}}$, so the set Y_{m+1} we will obtain will be a thinner set than X_m. In particular, Y_{m+1} will be covered by at most M_{m+1}^{-d} sidelength l_{m+1} cubes, and if $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$¡

$$r_{m+1}^{-d} \approx l_{m+1}^{-t}$$

sidelength l_{m+1} cubes, which implies X will have upper Minkowski dimension at most t. However, as a result, because the set Y_{m+1} is thinner, we find that Y_{m+1}^d is disjoint from the cubes in B_{m+1} with high probability. In particular, we can set $X_{m+1} = Y_{m+1}$. This means that X_{m+1} will be pseudorandom, and we should therefore expect X to be a Salem set of dimension t. The remainder of this paper is devoted to showing that these heuristics are correct.

6 Discrete Lemma

We now proceed to solve a discretized version of Theorem 1.

Proposition 4. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn-s)/2)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth measure compactly supported on T. Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leqslant (1/l_{m+1})^{s+\varepsilon}. \tag{6.1}$$

Then there exists a large constant $C(\mu_T, l_m, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geqslant C(\mu_T, l_m, n, d, s, \varepsilon, l_m), \tag{6.2}$$

and

$$M_{m+1}^{\frac{s}{dn-s}+c\varepsilon} \leqslant K_{m+1} \leqslant 2M_{m+1}^{\frac{s}{dn-s}+c\varepsilon}, \tag{6.3}$$

where

$$c = \frac{6dn}{(dn-s)^2},$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) $\mu_S(\mathbf{R}^d) \geqslant \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

(C) If $|k| \leqslant 10 l_{m+1}^{-d}$, $|\widehat{\mu_T}(k) - \widehat{\mu_S}(k)| \leqslant r_{m+1}^{d/2} \log(M_{m+1})$.

(D) If $|k| \geqslant 10 l_{m+1}^{-d}$, $|\widehat{\mu_S}(k)| \leqslant |k|^{-d/2}$.

Remark 5. *To make the statement of Proposition* (4) *more clean, we have hidden the explicit choice of constant $C(\mu_T, l_m, n, d, s, \varepsilon)$. But this constant can certainly be made explicit; such a choice can be made by ensuring that* (6.2) *implies* (6.5), (6.11), (6.17), (6.27), (6.28), *and* (6.29) *all hold.*

Proof of Proposition 4. First, we describe the construction of the set S, and the measure μ_S. For each string $\sigma \in \Pi_{m+1}^d$, let j_σ be a random integer vector chosen from $\{0, \dots, K_{m+1} - 1\}^d$, such that the family $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ is an independent family of random variables. Then it is certainly true that for any $j \in [K_{m+1}]^d$,

$$\mathbf{P}(j_\sigma = j) = K_{m+1}^{-d}. \tag{6.4}$$

Then $\sigma j_\sigma \in \Sigma_{m+1}^d$. We can thus define a measure μ_S such that, for each $x \in \mathbf{R}^d$,

$$d\mu_S(x) = r_{m+1}^d \sum_{\sigma \in \Pi_{m+1}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then S is $\mathcal{Q}_{m+1}$ discretized, μ_S is supported on S, and $S \subset T$. Our goal is to show that, with non-zero probability, some choice of the family of indices $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ yields a set S and a measure μ_S satisfying Properties (A) and (B) of Proposition 4. In our calculations, it will help us to decompose the measure μ_S into components roughly supported on sidelength r_{m+1}^d cubes. For each $\sigma \in \Pi_{m+1}(T)$, define a measure μ_σ such that for each $x \in \mathbf{R}^d$,

$$d\mu_\sigma(x) = r_{m+1}^d \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

Then $\mu_S = \sum_{\sigma \in \Pi_{m+1}^d(T)} \mu_\sigma$. We shall split the proof of Properties (A), (B), and (C) into several more managable lemmas.

Lemma 6. *If*

$$M_{m+1} \geqslant \left(3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}\right)^2, \tag{6.5}$$

then $\mu_S(\mathbf{R}^d) \geqslant \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

Proof. Fix $\sigma \in \Pi_{m+1}^d$. If $j_0, j_1 \in \{0, \dots, K_{m+1} - 1\}^d$, then

$$|a(\sigma j_0) - a(\sigma j_1)| = |j_0 - j_1| \cdot l_{m+1} \leqslant (\sqrt{d} \cdot K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1}. \tag{6.6}$$

Together with (4.5), (6.6) implies

$$\begin{aligned}
&\left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_0)) \mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_1)) \mu_T(x) \right| \\
&\qquad \leqslant r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) \, |\mu_T(x + a(\sigma j_0)) - \mu_T(x + a(\sigma j_1))| \\
&\qquad \leqslant \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1} \\
&\qquad = \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}.
\end{aligned} \tag{6.7}$$

Thus (6.7) implies that for each σ,

$$|\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \leqslant \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \tag{6.8}$$

Furthermore, (4.6) implies

$$\begin{aligned}
&\textstyle\sum_{\sigma \in \Pi_{m+1}^d} \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \\
&\qquad = r_{m+1}^d \textstyle\sum_{(\sigma,j) \in \Sigma_{m+1}^d} \mathbf{P}(j_\sigma = j) \displaystyle\int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) d\mu_T(x) \\
&\qquad = \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left(\textstyle\sum_{(\sigma,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\
&\qquad = \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \mu_T(\mathbf{R}^d) = \mu_T(\mathbf{R}^d).
\end{aligned} \tag{6.9}$$

For all but at most $3^d r_{m+1}^{-d}$ indices $\sigma \in \Pi_{m+1}^d$, $\mu_\sigma = 0$ almost surely. Thus we can apply the triangle inequality together with (6.8) and (6.9) to conclude that

$$\begin{aligned}
|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| &= \left| \sum\nolimits_{\sigma \in \Pi_{m+1}^d} \left[\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \right] \right| \\
&\leqslant \sum\nolimits_{\sigma \in \Pi_{m+1}^d} \left| \mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \right| \\
&\leqslant (3^d r_{m+1}^{-d}) \left(\sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \qquad (6.10) \\
&= \left(3^d \sqrt{d} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \cdot r_{m+1} \\
&= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.
\end{aligned}$$

Thus (6.5) and (6.10) imply that, $|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| \leqslant M_{m+1}^{-1/2}$. □

Lemma 7. *If*

$$M_{m+1} \geqslant \left(10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)} \right)^{1/\varepsilon}, \qquad (6.11)$$

then

$$\mathbf{P}\left(S \text{ does not satisfies Property (A)} \right) \leqslant 1/10.$$

Proof. For any cube $Q \in \Sigma_{m+1}^d$, there are at most 3^d indices $\sigma j \in \Sigma_{m+1}^d$ such that $Q_{\sigma j} \cap Q \neq \varnothing$, and so a union bound together with (6.4) gives

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S)) \leqslant \sum\nolimits_{Q_{\sigma j} \cap Q \neq \varnothing} \mathbf{P}(j_\sigma = j) \leqslant 3^d K_{m+1}^{-d}. \qquad (6.12)$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_k \in S\}$ are independent from one another for $k \in \{1, \dots, n\}$, which together with (6.12) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_1 \in S) \dots \mathbf{P}(Q_n \in S) \leqslant 3^{dn} K_{m+1}^{-dn}. \qquad (6.13)$$

Taking expectations over all cubes in B, and applying (6.1) and (6.13) gives

$$\begin{aligned}
\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leqslant \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\
&\leqslant l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \qquad (6.14) \\
&= \frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}.
\end{aligned}$$

Since $\varepsilon \leqslant (dn-s)/2$, we conclude

$$\begin{aligned}(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right) &= s+\varepsilon\left(c(dn-s-\varepsilon)-\frac{s}{dn-s}\right)\\ &\geqslant s+\varepsilon\left(\frac{c(dn-s)}{2}-\frac{s}{dn-s}\right)\\ &= s+\varepsilon\frac{3dn-s}{dn-s} \geqslant s+2\varepsilon.\end{aligned}$$

Applying (6.3), we therefore conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geqslant M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \geqslant M_{m+1}^{s+2\varepsilon}.$$

Combined with (6.11), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leqslant \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^{\varepsilon}} \leqslant 1/10. \tag{6.15}$$

We can then apply Markov's inequality with (6.14) and (6.15) to conclude

$$\begin{aligned}\mathbf{P}(\mathcal{Q}_{k+1}(B)\cap\mathcal{Q}_{k+1}(S^n)\neq\varnothing) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B)\cap\mathcal{Q}_{k+1}(S^n))\geqslant 1)\\ &\leqslant \mathbf{E}(\#(\mathcal{Q}_{m+1}(B)\cap\mathcal{Q}_{m+1}(S^n)))\\ &\leqslant 1/10. \qquad\square\end{aligned}$$

Lemma 8. *Set $D = \{k \in \mathbf{Z}^d : |k| \leqslant 10 l_{m+1}^{-1}\}$. Then if*

$$K_{m+1} \leqslant M_{m+1}^{\frac{2dn}{dn-s}}, \tag{6.16}$$

and

$$M_{m+1} \geqslant \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \tag{6.17}$$

then

$$\mathbf{P}\left(\|\widehat{\mu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10 \tag{6.18}$$

Proof. For each $\sigma \in \Pi_{m+1}^d$, and $k \in \mathbf{Z}$, define $X_{\sigma k} = \widehat{\mu_\sigma}(k) - \widehat{\mathbf{E}(\mu_\sigma)}(k)$. Applying (4.2) gives

$$\begin{aligned}\sum_{\sigma\in\Pi_{m+1}^d}\widehat{\mathbf{E}(\mu_\sigma)}(k) &= \sum_{\sigma\in\Pi_{m+1}^d} l_{m+1}^d \sum_{j\in[K_{m+1}]^d}\int_{\mathbf{R}^d} e^{-2\pi i k\cdot x}\psi_{m+1}(x-a(\sigma j))\,d\mu_T(x)\\ &= \int_{\mathbf{R}^d} e^{-2\pi i k\cdot x}\,d\mu_T(x) = \widehat{\mu_T}(k).\end{aligned} \tag{6.19}$$

For each σ and k, the standard (L^1, L^∞) bound on the Fourier transform, combined with (6.8), shows

$$\begin{aligned}\|X_{\sigma k}\|_{\psi_2(L)} &\leqslant 10|X_{\sigma k}| \\ &\leqslant 10[|\mu_\sigma(\mathbf{R}^d)| + \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)] \\ &\leqslant 10^2\left(\mathbf{E}(\mu_\sigma)(\mathbf{R}^d) + \sqrt{d}\cdot r_{m+1}^{d+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{6.20}$$

For a fixed k, the family of random variables $\{X_{\sigma k} : \sigma \in \Pi_{m+1}^d\}$ are independent. Furthermore, $\sum X_{\sigma k} = \widehat{\mu_S}(k) - \widehat{\mathbf{E}(\mu_S)}(k)$. Equations (4.6) and (6.4) imply that

$$\begin{aligned}\mathbf{E}(\widehat{\mu_S}(k)) &= \frac{r_{m+1}^d}{K_{m+1}^d}\int_{\mathbf{R}^d} e^{-2\pi i k\cdot x}\left(\sum_{(\sigma,j)\in\Sigma_{m+1}^d}\psi_{m+1}(x - a(\sigma j))\right)d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d}\int_{\mathbf{R}^d} e^{-2\pi i k\cdot x}d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d}\widehat{\mu_T}(k) = \widehat{\mu_T}(k).\end{aligned} \tag{6.21}$$

Hoeffding's inequality, together with (6.20) and (6.21), imply that

$$\begin{aligned}&\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \\ &\quad\leqslant 10^3\sqrt{d}\left(\left(\sum \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)^2\right)^{1/2} + r_{m+1}^{d/2+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{6.22}$$

Equation (4.5) shows

$$\begin{aligned}\mathbf{E}(\mu_\sigma)(\mathbf{R}^d) &= l_{m+1}^d\sum_{j\in[K_{m+1}]^d}\int\psi_{m+1}(x - a(ij))d\mu_T(x) \\ &\leqslant r_{m+1}^d\|\mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{6.23}$$

Combining (6.22) and (6.23) gives

$$\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \leqslant 10^3\sqrt{d}\left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right] r_{m+1}^{d/2}. \tag{6.24}$$

We can then apply a union bound over the set D, which has cardinality at most $10^{d+1}l_{m+1}^{-d}$, together with (6.24) to conclude that

$$\begin{aligned}
&\mathbf{P}\left(\|\widehat{\mu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right)\\
&\qquad \leqslant 10^{d+2}\cdot l_{m+1}^{-d}\exp\left(-\frac{\log(M_{m+1})^2}{10^7 d}\right) \\
&\qquad = 10^{d+2}l_m^{-d}\exp\left(d\log(M_{m+1}K_{m+1})-\frac{\log(M_{m+1})^2}{10^7 d}\right).
\end{aligned} \tag{6.25}$$

Combined with (6.16) and (6.17), (6.25) implies

$$\mathbf{P}\left(\|\widehat{\mu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10. \tag{6.26}$$

Thus $\widehat{\mu_S}$ and $\widehat{\mu_T}$ are highly likely to differ only by a negligible amount over small frequencies. □

Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup_{\xi\in\mathbf{R}^d} |\widehat{\mu_T}(\xi)||\xi|^t < \infty.$$

In light of (4.7), if we define, for each $t > 0$,

$$B(t) = \sup_{\xi\in\mathbf{R}^d} |\widehat{\psi}(\xi)||\xi|^t,$$

then

$$\sup_{\xi\in\mathbf{R}^d} |\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t}B(t).$$

Lemma 9. *Suppose that*

$$N_{m+1}^d \geqslant 10\cdot 2^{3d/2+1}A(3d/2+1), \tag{6.27}$$

$$N_{m+1}^d \geqslant \frac{10\cdot 2^{3d}}{1+d/2}A(3d/2+1), \tag{6.28}$$

and

$$N_{m+1}^d \geqslant 10\cdot 2^{7d/2+1}B(3d/2+1). \tag{6.29}$$

then if $|\eta| \geqslant 10 l_{m+1}^{-1}$,

$$|\widehat{\mu_S}(\eta)| \leqslant \frac{1}{|\eta|^{d/2}}. \tag{6.30}$$

Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{\sigma \in \Pi_{m+1}^d \\ d(a(\sigma),T) \leqslant 2r_{m+1}^{-1}}} \delta_{a(ij_i)}.$$

Then $\mu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\mu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d, so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leqslant \alpha(\mathbf{R}^d) \leqslant 2^d$. Thus

$$|\widehat{\mu_S}(\eta)| \leqslant 2^d \int |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi. \tag{6.31}$$

If $|\xi| \leqslant |\eta|/2$, $|\eta - \xi| \geqslant |\eta|/2$, and since (4.5) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$, we find that for all $t > 0$,

$$\int_{0 \leqslant |\xi| \leqslant |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{A(t)2^{t-d}}{|\eta|^{t-d}}. \tag{6.32}$$

Set $t = 3d/2 + 1$. Equation (6.32), together with (6.27), implies

$$\begin{aligned} \int_{0 \leqslant |\xi| \leqslant |\eta|/2} & |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \\ &\leqslant \frac{A(3d/2+1)2^{1+d/2}|\eta|^{-1}}{|\eta|^{d/2}} \\ &\leqslant \frac{A(3d/2+1)2^{1+d/2} l_{m+1}}{|\eta|^{d/2}} \\ &\leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \end{aligned} \tag{6.33}$$

Conversely, if $|\xi| \geqslant 2|\eta|$, then $|\eta - \xi| \geqslant |\xi|/2$, so for each $t > d$,

$$\begin{aligned} \int_{|\xi| \geqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)| \, d\xi &\leqslant \int_{|\xi| \geqslant 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\ &\leqslant 2^d \int_{2|\eta|}^\infty r^{d-1-t} A(t) 2^t \\ &\leqslant \frac{4^d A(t)}{t-d} |\eta|^{d-t}. \end{aligned} \tag{6.34}$$

Set $t = 3d/2 + 1$. Equation (6.28), applied to (6.34), allows us to conclude

$$\int_{|\xi| \geqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)| \, d\xi \leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \tag{6.35}$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu_T}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$ to conclude that

$$\int_{|\eta|/2 \leqslant |\xi| \leqslant 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \tag{6.36}$$

Set $t = 3d/2 + 1$. Then (6.36) and (6.29) imply

$$\int_{|\eta|/2 \leqslant |\xi| \leqslant 2|\eta} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \tag{6.37}$$

It then suffices to sum up (6.33), (6.35), and (6.37), and apply (6.31). □

Proof of Proposition 4, Continued. Let us now put all our calculations together. In light of Lemma 7 and Lemma 8, there exists some choice of j_σ for each σ, and a resultant non-random pair (μ_S, S) such that S satisfies Property (A) of the Lemma, and μ_S satisfies (6.18), implying that μ_S satisfies Property (C) of the Lemma. But Lemma 6 shows that μ_S always satisfies Property (B), and Lemma (9) shows Property (D) is also always satisfied. This completes the proof. □

7 Construction of the Salem Set

Let us now choose the parameters to construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^\infty Z_i$, where Z_i has lower Minkowski dimension at most s for each i. Then choose an infinite sequence $\{i_m : m \geqslant 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers $\{\varepsilon_m : m \geqslant 1\}$, with $\varepsilon_m < (dn - s)/2$ for each m. We choose our parameters $\{M_m\}$ and $\{K_k\}$ inductively. First, set $X_0 = [0, 1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0. At the mth step of our construction, we have already found a set X_{m-1} and a measure μ_{m-1}. We then choose K_m and M_m such that

$$K_m, M_m \geqslant C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m^{\frac{s}{dn-s}+c\varepsilon_m} \leqslant K_m \leqslant 2M_m^{\frac{s}{dn-s}+c\varepsilon},$$

and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in $\mathcal{Q}_m$, the union of which, we define to be equal to B_m. We can then apply Proposition 4 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$, and a measure μ_S supported on S. We define $X_m = S$, and $\mu_m = \mu_S$.

The last paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$. Just as in our previous paper, it is easy to see X must be a configuration avoiding set. Given any $(x_1, \dots, x_n) \in Z$, there are infinitely many integers m_k such that $(x_1, \dots, x_n) \in B_{m_k}$. If $|x_i - x_j| \geqslant \varepsilon$ for each $i \neq j$, and $r_{m_k} \leqslant \varepsilon/2$, then $(x_1, \dots, x_n)$ is contained in a strongly non-diagonal cube in $\mathcal{Q}_{m_k}(B_k)$, and as such $X^n \subset X_k^n$ does not contain $(x_1, \dots, x_n)$.

8 Proof that X is Salem

We now show X is Salem, completing the proof of Theorem 1. Since the masses of the sequence of measures $\{\mu_m\}$ is uniformly bounded, there is some subsequence μ_{m_i} which converges weakly to some measure μ. Repeated applications of Property (B) of Proposition 4 imply

$$\mu(\mathbf{R}^d) = \lim_{i\to\infty} \mu_{m_i}(\mathbf{R}^d) \geqslant 1 - \sum_{m=1}^{\infty} M_m^{-1/2}.$$

In particular, μ is a non-zero measure if the sequence $\{M_m\}$ is rapidly increasing. Moreover, for each $k \in \mathbf{Z}^d$,

$$\widehat{\mu}(k) = \lim_{i\to\infty} \widehat{\mu_{m_i}}(k).$$

Thus

$$|\widehat{\mu}(k)| \leqslant |\widehat{\mu_0}(k)| + \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)|.$$

Fix $\varepsilon > 0$. Since $l_m \leqslant 2^{-m}/10$, we find that for $m \geqslant \log(k)$, $|k| \leqslant 10 l_{m+1}^{-1}$. Thus we can apply Property (C) and (D) of Proposition 4 to conclude

$$\begin{aligned}
&\sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)| \\
&\quad \leqslant 2\log(k)|k|^{-d/2} + \sum_{m=\log(k)}^{\infty} r_{m+1}^{d/2} \log(M_{m+1}) \\
&\quad \lesssim_\varepsilon |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} |k|^{t/2-\varepsilon} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
&\quad \leqslant |k|^{\varepsilon - t/2} \left(1 + 10^{t/2-\varepsilon} \sum_{m=\log(k)}^{\infty} l_{m+1}^{\varepsilon - t/2} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
&\quad \lesssim_\varepsilon |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{1}{K_{m+1}^{\varepsilon}} \frac{K_{m+1}^{t/2}}{M_{m+1}^{d/2-t/2}} \right) \\
&\quad \lesssim |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m (t/2)}}{K_{m+1}^{\varepsilon}} \frac{M_{m+1}^{(t/2)\left(\frac{s}{dn-s}\right)}}{M_{m+1}^{d/2-t/2}} \right) \\
&\quad = |k|^{\varepsilon - t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m (t/2)}}{K_{m+1}^{\varepsilon}} \right) \lesssim_\varepsilon |k|^{\varepsilon - t/2}.
\end{aligned}$$

The last inequality follows because $\varepsilon_m \to 0$, and so the series is summable if the sequence $\{K_m\}$ increases rapidly enough. Since μ_0 is smooth and compactly supported, we find

$$\sup_{k \in \mathbf{Z}^d} |k|^{t/2-\varepsilon} |\widehat{\mu}(k)| \lesssim_\varepsilon 1 + \sup_{k \in \mathbf{Z}} |k|^{t/2-\varepsilon} |\widehat{\mu_0}(k)| < \infty.$$

Since $\varepsilon > 0$ was arbitrary, this shows that the Fourier dimension of X is at least t. Because X_m is the union of $(M_1 \dots M_m)^d$ sidelength l_m cubes, one can easily show using (6.3) that the lower Minkowski dimension of X is upper bounded by t. But these two bounds imply that the Hausdorff dimension, Fourier dimension, and Minkowski dimension are all equal to t. Thus X is Salem of dimension t.

References

[1] Roman Vershynin, *High Dimensional Probability*, Cambridge Series in Statistical and Probabilistic Mathematics, 2018.