Fractals Avoiding Fractal Sets

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Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Do large sets unavoidably contain certain patterns? The Lebesgue density theorem implies positive Lebesgue measure contains an affine copy of any finite configuration. But fractal dimensions prove more interesting. In [1], a subset of the line with full Hausdorff dimension is obtained avoiding translates. In [3], a subset of $\mathbf{R}^d$ exists with dimension $d/4$ such that no triple of points generate the angle $\pi/3$. On the other hand, [4] shows every subset of $\mathbf{R}^d$ with dimension exceeding 1 contains $d+1$ points in a hyperplane. For many cases, it remains unclear at what threshold high dimensional configurations are forced upon sets, or even if such a threshold exists at all.

This paper constructs large sets avoiding 'sparse' configurations. The configurations will still be dense in configuration space, which makes them nontrivial to avoid, but the sparsity is measured by a bound on their fractal dimension. To be precise, we construct sets with high Hausdorff dimension avoiding patterns with low modified box counting dimension.

Theorem 1. *Suppose Z is the countable union of compact sets with lower Minkowski dimension bounded by α. Then there exists a set $X \subset [0,1]^d$ with*

$$\dim_{\mathbf{H}}(X) = \min\left(\frac{nd-\alpha}{n-1}, d\right),$$

such that if $x_1, \dots, x_n \in X$ are distinct, $(x_1, \dots, x_n) \notin Z$.

There are already generic pattern avoidance methods in the literature. But they rely on the assumption that Z is a smooth manifold, or an algebraic hypersurface. The novel feature of our method is that we only exploit the fractal dimension of the configuration. We are therefore able to avoid arbitrarily irregular Z. Meanwhile, the Hausdorff dimension of X is competitive in relation to results obtained by more restricted techniques, remains robust under small changes in the dimension of Z, and even recovers a selection of the avoidance results as special cases. Our method is compared to the other pattern avoidance methods in Section 6.

The key idea to avoiding sparse configurations is a random mass equidistribution strategy, described via a discrete variant of the problem in Section 2. This discrete optimization problem is difficult to solve optimally, but a random choice is optimal enough for our purposes, and is likely tight for general versions of the problem. By overlaying the solution to the discretized problems at a sequence of scales, we obtain X as a Cantor-type set, at the end of Section 3. Exploiting the equidistribution of mass at discrete scales, in Section 4, we are able to show the set X has the required Hausdorff dimension regardless of how fast our sequence of scales decay. This technique occurs implicitly in at least one other Hausdorff dimension calculation in the literature, e.g. [2]. But we do not believe equidistribution has been explicitly identified as a way to maintain dimension combined with a rapid decay of scales in fractal constructions.

Remark. *The difficult setting of the theorem occurs when $\alpha \geqslant d$. If $\alpha < d$,*

$$X = \left\{ x \in [0,1]^d : \textit{For all } (z_1, \dots, z_n) \in Z, x \neq z_k \textit{ for any } k \right\}$$

gives a set with full Hausdorff dimension satisfying the properties of the theorem. In our proof, we will assume $d \leqslant \alpha < dn$, and thus we must find a set X with Hausdorff dimension $(dn - \alpha)/(n-1)$.

1 Notation and Terminology

- For a length l, $\mathcal{B}^d(l)$ denotes the family of all half open cubes in $\mathbf{R}^d$ with sidelength l and corners on the lattice $(l \cdot \mathbf{Z})^d$. That is,

$$\mathcal{B}^d(l) = \{[a_1, a_1 + l) \times \cdots \times [a_d, a_d + l) : a_i \in l \cdot \mathbf{Z}\}.$$

  If $E \subset \mathbf{R}^d$, $\mathcal{B}^d(l, E)$ is the family of cubes in $\mathcal{B}^d(l)$ intersecting E, i.e.

$$\mathcal{B}^d(l, E) = \{I \in \mathcal{B}^d(l) : I \cap E \neq \varnothing\}.$$

  For instance, $\mathcal{B}^d(l, \mathbf{Q}^d) = \mathcal{B}^d(l)$.

- The *lower Minkowski dimension* of a compact set $E \subset \mathbf{R}^d$ is

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{l \to 0} \frac{\log(\#(\mathcal{B}^d(l, E)))}{\log(1/l)}.$$

  Thus there is $l_k \to 0$ with $\#(\mathcal{B}^d(l_k, E)) = (1/l_k)^{\underline{\dim}_{\mathbf{M}}(E) + o(1)}$.

- Adopting the terminology of [6], we say a collection of sets $U_1, U_2, \dots$ is a *strong cover* of some set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k.
- Given a cube $I \in \mathcal{B}^{dn}(l)$, there are unique cubes $I_1, \dots, I_n \in \mathcal{B}^d(l)$ such that $I = I_1 \times \cdots \times I_n$. We say I is *non diagonal* if $I_1, \dots, I_n$ are distinct.

2 Avoidance at Discrete Scales

We avoid Z by considering an infinite sequence of scales. At each scale, we solve a discretized version of the problem. Combining these solutions then solves the original problem. This section describes the discretized avoidance technique. The technique is the *core* part of our construction, and the Hausdorff dimension we achieve is a direct result of our success in the discrete setting.

Fix two dyadic scales $l > s$. In the discrete setting, Z is replaced by a union of sidelength s cubes, denoted K. Our goal is to take a set E, which is a union of sidelength l cubes, and carve out a union of sidelength s cubes F such that F^n is disjoint from the non-diagonal cubes of K.

In order to ensure the Hausdorff dimension calculations of X go smoothly, it is crucial that the mass of F is spread uniformly over E in the discrete setting. We can achieve this by trying to include a equal portion of mass in each sidelength r subcube of E, for some intermediary scale r with $l > r > s$. The next lemma shows that we can select a equal portion of mass from *almost* all of the sidelength r cubes.

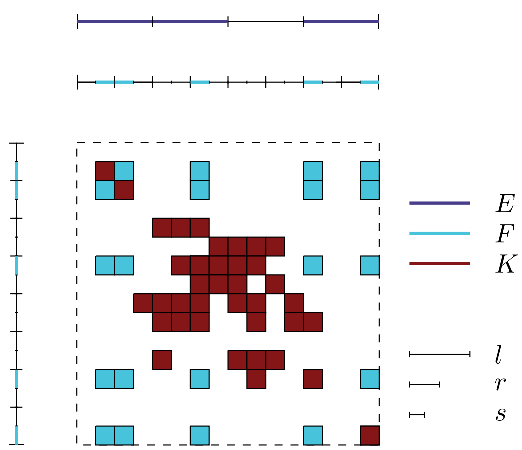

Figure 1: An example choice of F satisfying the Lemma below, where $d = 1$ and $n = 2$. F satisfies the non-concentration and avoidance property, as well as containing an interval from all but 3 of the intervals in $\mathcal{B}(E, r)$.

Lemma 1. *Fix three dyadic lengths $l > r > s$. Let E be a union of cubes in*

$\mathcal{B}^d(l)$*, and* K *a union of cubes in* $\mathcal{B}^{dn}(s)$*. Then there exists* $F \subset E$*, which is a union of cubes in* $\mathcal{B}^d(s)$*, such that*

(a) Avoidance: *For any distinct* $I_1, \dots, I_n \in \mathcal{B}^d(s, F)$*,* $I_1 \times \cdots \times I_n \notin \mathcal{B}^{dn}(s, K)$*.*

(b) Non Concentration: $\mathcal{B}^d(s, F)$ *contains at most one subcube of each cube in* $\mathcal{B}^d(s, E)$*.*

(c) Equidistribution: $\mathcal{B}^d(r, E)$ *contains a subcube from all but at most* $|K|r^{-dn}$ *of the cubes in* $\mathcal{B}^d(r, E)$*.*

Proof. Form a random set U by selecting a sidelength s cube from each sidelength r cube uniformly at random. More precisely, set

$$U = \bigcup \{J_I : I \in \mathcal{B}^d(r, E)\},$$

where J_I is an element selected uniformly randomly from $\mathcal{B}^d(s, I)$. U certainly satisfies the equidistribution and non-concentration properties, but not the avoidance property. We will show that with non-zero probability, we can obtain the avoidance property by removing at most $|K|r^{-dn}$ cubes from U.

For any $J \in \mathcal{B}^d(s, E)$, there is a unique $I \in \mathcal{B}^d(l, E)$ such that $J \subset I$. Then

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_I = J) = (s/r)^d.$$

Since any two elements of $\mathcal{B}^d(s, U)$ lie in distinct cubes of $\mathcal{B}^d(r)$, the only chance that a *non-diagonal* cube J in $\mathcal{B}^{dn}(s, K)$ is a subset of U^n is if $J_1, \dots J_n$ all lie in separate cubes of $\mathcal{B}^d(r)$. They each have an independent chance of being added to U, and so

$$\mathbf{P}(J \subset U^n) = \mathbf{P}(J_1 \subset U) \cdots \mathbf{P}(J_n \subset U) = (s/r)^{dn}.$$

If $\mathcal{J}$ denotes the family of all non-diagonal cubes of $\mathcal{B}^{dn}(s, K)$ contained in U^n, then, letting J range over the non-diagonal cubes of $\mathcal{B}^{dn}(s, K)$, we find

$$\mathbf{E}(\#(\mathcal{J})) = \sum\nolimits_J \mathbf{P}(J \subset U^n) \leqslant |\mathcal{B}^{dn}(s, K)|(s/r)^{dn} = |K|r^{-dn}.$$

In particular, this means that out of all possible outcomes for the set U, there is at least one *particular* U_0 we can choose for which the corresponding set of cubes $\mathcal{I}_0$ satisfies $\#(\mathcal{I}_0) \leqslant \mathbf{E}(\mathcal{I}) = |\mathcal{B}^{dn}(s, K)|(s/r)^{dn} = |K|r^{-dn}$.

We now define $F = U_0 - \{I_1 : I \in \mathcal{I}_0\}$. As a subset of U_0, F inherits the non-concentration property. We have removed at most $|K|r^{-dn}$ cubes from U_0, and since U_0 contains an cube from *every* cube in $\mathcal{B}^d(r, E)$, F satisfies the equidistribution property. Finally, since we have removed a single side from every non-diagonal cube in U_0^n intersecting K, F satisfies the avoidance property. So our construction is complete. □

Remark. *The existence of* U_0 *was justified using a randomized selection process. Nonetheless, it's existence can be made constructive: We simply interate through all possible outcomes of* U *and pick one minimizing the cardinality of* $\mathcal{J}$*. As a result, the set* X *in our theorem is obtained by explicit, constructive means.*

If Z has dimension α, we will obtain bounds of the form $|K| \leqslant 2^{dn} s^{dn-\gamma}$, where γ convering to α as $s \to 0$. For convenience, we will also set r to be the closest power of two to s^{λ}, for some $\lambda \in (0,1)$. The size of λ is directly related to the Hausdorff dimension of the set X we will obtain. The next corollary calculates how large λ can be if F must be equidistributed over a constant fraction of cubes in $\mathcal{B}^d(r,E)$. The error term $5A \log_s |E|$ will be made insignificant by the rapid decay of the values s used in our construction.

Corollary 1. *Maintain the last lemma's setup, in addition to three parameters* $\lambda \in (0,1]$, $\gamma \in [d, dn)$, *and* $m > 0$. *Suppose* $r = 2^{-\lfloor \lambda \log_2(1/s) \rfloor}$, $|E| \leqslant 1/2$, *and* $|K| \leqslant 2^{dn} s^{dn-\gamma}$. *If*

$$0 < \lambda \leqslant \frac{dn-\gamma}{d(n-1)} - 5m \log_s |E| \, ,$$

then F *is equidistributed over all but a fraction* $1/2^m$ *of the cubes in* $\mathcal{B}^d(r,E)$.

Proof. The inequality for λ implies

$$dn - \gamma - \lambda d(n-1) \geqslant 5d(n-1)m \log_s |E|.$$

Since r is within a factor of two from s^{λ}, we compute

$$\begin{aligned}
\frac{|\{I \in \mathcal{B}^d(r,E) : \mathcal{B}^d(s,I) \cap \mathcal{B}^d(s,F) = \varnothing\}|}{|\mathcal{B}^d(s,E)|} &\leqslant \frac{|K| r^{-dn}}{|E| r^{-d}} \\
\leqslant (2^{dn} s^{dn-\gamma}) |E|^{-1} r^{-d(n-1)} &\leqslant (2^{dn} s^{dn-\gamma}) |E|^{-1} (s/2)^{-\lambda d(n-1)} \\
\leqslant 2^{dn+\lambda d(n-1)} |E|^{5d(n-1)m-1} &\leqslant 2^{dn+d(n-1)-(5d(n-1)m-1)} \leqslant 1/2^m.
\end{aligned}$$

The last inequality was obtained because $n \geqslant 2$, $d \geqslant 1$, and $m \geqslant 1$, so

$$\begin{aligned}
[dn+&d(n-1) - (5d(n-1)m-1)] + m \\
&\leqslant 2dn + 1 - d + (1 - 5d(n-1))m \\
&\leqslant 2dn + (1 - 5d(n-1)) \\
&\leqslant 5d - 3dn + 1 \leqslant 0.
\end{aligned}$$

Thus $dn + d(n-1) - (5d(n-1)m-1) \leqslant -m$. □

Remark. *We emphasize that the discrete method is the core of our avoidance technique. The remaining argument is modular. Indeed, this part of our paper was based on the construction method of [2]. If for a special case of* Z, *one can improve the lemma so fewer cubes are discarded, then the remaining parts of our paper can likely be applied near verbatim to yield a set* X *with a larger Hausdorff dimension. For instance, a variation on the argument in [3] shows that if* Z *is a degree* m *algebraic hypersurface, and* $K = \mathcal{B}^{dn}(l,Z)$, *then a different selection strategy at the discrete scale allows us to set* $\lambda \approx 1/m$. *Following through the remainder of our proof replicates the main result of the paper.*

3 Fractal Discretization

Now we apply the discrete result at many scales. The fact that Z is the countable union of compact sets with Minkowski dimension α implies that we can find an efficient *strong cover* of Z by cubes restricted to lie at a sequence of dyadic scales l_k converging to zero arbitrarily fast.

Lemma 2. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α, and consider any decreasing sequence ε_k converging to zero with $\alpha + \varepsilon_k \leqslant dn$. Then there is a decreasing sequence of lengths $l_1, l_2, \dots$, and $\mathcal{B}^{dn}(l_k)$ sets Z_k such that Z is strongly covered by the sets Z_k and $|\mathcal{B}(Z_k, l_k)| \leqslant 2^{dn}/l_k^{\alpha+\varepsilon_k}$.*

Proof. Let Z be the union of sets Y_i with $\underline{\dim}_{\mathbf{M}}(Y_i) \leqslant \alpha$ for each i. Consider any sequence of integers $m_1, m_2, \dots$ which repeats each integer infinitely often. Given k, there are infinitely many lengths l with $\#(\mathcal{B}^{dn}(l, Y_{m_k})) \leqslant 1/l^{\alpha+\varepsilon_k}$. Replacing l with a dyadic number at most twice the size of l, there are infinitely many *dyadic* lengths l with $\#(\mathcal{B}^{dn}(l, Y_{m_k})) \leqslant 1/(l/2)^{\alpha+\varepsilon_k} \leqslant 2^{dn}/l^{\alpha+\varepsilon_k}$. In particular, we may fix a length l_k smaller than $l_1, \dots, l_{k-1}$. Then the union of the cubes in $\mathcal{B}^{dn}(l_k, Y_{m_k})$ forms the set Z_k. □

Remark. *In the proof, we are free to make l_k arbitrarily small in relation to the previous parameters $l_1, \dots, l_{k-1}$ we have chosen. For instance, later on when calculating the Hausdorff dimension, we will assume that $l_{k+1} \leqslant l_k^{k^2}$, and the argument above can be easily modified to incorporate this inequality. We will also find that setting $\varepsilon_k = c \cdot k^{-1}$ suffices to give the results we need, where c is a sufficiently small constant such that $\alpha + c \leqslant dn$.*

We can now construct X by avoiding the various discretizations of Z at each scale. The aim is to find a nested decreasing family of discretized sets X_k with $X = \lim X_k$. One condition guaranteeing that X avoids Z is that X_k^n is disjoint from *non diagonal* cubes in Z_k.

Lemma 3. *If for each k, X_k^n avoids non-diagonal cubes in Z_k, then $(x_1, \dots, x_n) \notin Z$ for any distinct $x_1, \dots, x_n \in X$.*

Proof. Let $z \in Z$ be given with $z_1, \dots, z_n$ are distinct. Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i, j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$. The point z is covered by cubes in infinitely many of collections Z_{k_m}. For suitably large N, the cube I in $\mathcal{B}^{dn}(l_{k_N})$ containing z is disjoint from Δ. But this means that I is non diagonal, and so $z \notin X_N^d$. In particular, z is not an element of X^n. □

It is now simple to see how we iteratively apply our discrete scale argument to construct X. First, we set $X_0 = [0, 1/2]^d$, so that $|X_0| \leqslant 1/2$. To obtain X_{k+1} from X_k, we set

$$E = X_k, \quad K = Z_{k+1}, \quad l = l_k, \quad s = l_{k+1}, \quad \text{and } \gamma = \alpha + \varepsilon_k = \alpha + c \cdot \varepsilon_k,$$

We set $r = r_{k+1}$, where r_{k+1} is the closest power of two to l_{k+1}^{λ}, and

$$\lambda = \beta_{k+1} := \frac{dn - \alpha}{d(n-1)} - \frac{\varepsilon_{k+1}}{d(n-1)} - 10(k+1)\log_{L_{k+1}} |X_k|.$$

We can now apply Corollary 1 to constructs a set F with F^n avoiding non diagonal cubes in Z_{k+1}, and containing a $\mathcal{B}^d(l_{k+1})$ subcube from all but a fraction $1/2^{2k+2}$ of the $\mathcal{B}^d(r_{k+1})$ cubes in I. We set $X_{k+1} = F$. Repeatedly doing this builds an infinite sequence of the X_k. Since X_k^n avoids Z_k, for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin X$.

4 Dimension Bounds

All that remains in our argument is showing X has the right Hausdorff dimension. At the discrete scale l_k, X looks like a $d\beta_k$ dimensional set. If the lengths l_k rapidly converge to zero, then we can ensure $\beta_k \to \beta$, where

$$\beta = \frac{dn - \alpha}{d(n-1)}.$$

Thus X looks $d\beta = (dn - \alpha)/(n-1)$ dimensional at the discrete scales l_k, which is the Hausdorff dimension we want. To obtain the complete dimension bound, it then suffices to interpolate to get a $d\beta$ dimensional behaviour at all intermediary scales. We won't be penalized here by making the gaps between discrete scales too large, because the uniform way that we have selected cubes in consecutive scales implies that between the scales l_k and l_{k+1}^{β}, X behaves like a full dimensional set. This section fills in the details to this argument.

Lemma 4. *$\beta_k = \beta - O(1/k)$.*

Proof. We must show

$$\beta - \beta_k = \frac{\varepsilon_{k+1}}{d(n-1)} + 10(k+1)\log_{l_{k+1}} |X_k| = O(1/k).$$

Since $\varepsilon_k = c \cdot k^{-1}$, this term has the right decay for free. On the other hand, we need the lengths to tend to zero rapidly to make the other error term decay to zero. Since $l_{k+1} \leqslant l_k^{k^2}$, we find

$$(k+1)\log_{l_{k+1}} |X_k| \leqslant \frac{(k+1)\log l_k}{\log l_{k+1}} \leqslant \frac{(k+1)\log l_k}{k^2 \log l_k} = \frac{k+1}{k^2} = O(1/k).$$

Thus both components of the error term are $O(1/k)$. $\square$

The most convenient way to look at X's dimension at various scales is to use Frostman's lemma. We construct a non-zero measure μ supported on X such that for all $\varepsilon > 0$, for all lengths l, and for all $I \in \mathcal{B}^d(l)$, $\mu(I) \lesssim_\varepsilon l^{d\beta - \varepsilon}$. We can then understand the behaviour of X at a scale l by looking at μ's behaviour restricted to cubes at the scale l.

To construct μ, we take a sequence of measures μ_k, supported on X_k, and then take a weak limit. We initialize this construction by setting μ_0 to be the uniform probability measure on $X_0 = [0, 1/2]^d$. We then define μ_{k+1}, supported on X_{k+1}, by modifying the distribution of μ_k. First, we throw away the mass of the $\mathcal{B}^d(l_k)$ cubes I for which half of the elements of $\mathcal{B}^d(I, r_{k+1})$ fail to contain a part of X_{k+1}. For the cubes I with more than half of the cubes $\mathcal{B}^d(I, r_{k+1})$ containing a part of X_{k+1}, we distribute the mass $\mu_k(I)$ uniformly over the subcubes of I in X_{k+1}, giving the distribution of μ_{k+1}.

A glance at the cumulative distribution functions of the μ_k shows these measures converge weakly to a function μ. For any $I \in \mathcal{B}^d(l_k)$, we find $\mu(I) \leqslant \mu_k(I)$, which will be useful for passing from bounds on the discrete measures to bounds on the final measure. This occupies our attention for the remainder of this section.

Lemma 5. *If $I \in \mathcal{B}^d(l_k)$, then*

$$\mu(I) \leqslant \mu_k(I) \leqslant 2^k \left[\frac{r_k r_{k-1} \dots r_1}{l_{k-1} \dots l_1} \right]^d.$$

Proof. Consider $I \in \mathcal{B}^d(l_{k+1})$, $J \in \mathcal{B}^d(l_k)$. If $\mu_k(I) > 0$, J contains an element of $\mathcal{B}^d(l_k)$ in at least half of the cubes in $\mathcal{B}^d(r_k, J)$. Thus the mass of J distributes itself evenly over at least $2^{-1}(l_{k-1}/r_k)^d$ cubes, which gives that $\mu_k(I) \leqslant 2(r_k/l_{k-1})^d \mu_{k-1}(J)$. Expanding this recursive inequality, using that μ_0 has total mass one as a base case, we obtain exactly the result we need. □

Corollary 2. *The measure μ is positive.*

Proof. To prove this result, it suffices to show that the total mass of μ_k is bounded below, independently of k. At each stage k,

$$\#(\mathcal{B}^d(X_k, l_k)) \leqslant \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} \right]^d.$$

Since only a fraction $1/2^{2k+2}$ of the cubes in $\mathcal{B}^d(r_k, X_k)$ do not contain an cube in X_{k+1}, it is only for at most a fraction $1/2^{2k+1}$ of the cubes in $\mathcal{B}^d(r_k, X_k)$ cubes that X_{k+1} fails to equidistribute over more than half of the cubes. But this means that we discard a total mass of at most

$$\left(\frac{1}{2^{2k+1}} \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} \right]^d \right) \left(2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1} \right]^d \right) \leqslant 1/2^{k+1}.$$

Thus

$$\mu_k(\mathbf{R}^d) \geqslant 1 - \sum_{i=0}^{k} \frac{1}{2^{i+1}} \geqslant 1/2.$$

This implies $\mu(\mathbf{R}^d) \geqslant 1/2$, and in particular, $\mu \neq 0$. □

Ignoring all parameters in the inequality for I which depend on indices smaller than k, we 'conclude' that $\mu_k(I) \lesssim r_k^d \lesssim l_k^{\beta d - O(1/k)}$. The equation $l_{k+1} \leqslant l_k^{k^2}$ implies l_k decays very rapidly, which enables us to ignore quantities depending on previous indices, and obtain a true inequality.

Corollary 3. *For all $I \in \mathcal{B}^d(l_k)$, $\mu(I) \leqslant \mu_k(I) \lesssim l_k^{d\beta - O(1/k)}$.*

Proof. Given ε, we find

$$\begin{aligned}\mu_k(I) &\leqslant 2^k \left[\frac{r_k \dots r_1}{l_{k-1} \dots l_1}\right]^d \leqslant \left(\frac{2^{d+k}}{l_{k-1}^{d(1-\beta_{k-1})} \dots l_1^{d(1-\beta_1)}}\right) l_k^{d\beta_k} \\ &\leqslant \left(2^{d+k} l_k^{\varepsilon} / l_{k-1}^{d(k-1)}\right) l_k^{d\beta_k - \varepsilon} \leqslant \left(2^{d+k} l_{k-1}^{\varepsilon k^2 - d(k-1)}\right) l_k^{d\beta_k - \varepsilon}.\end{aligned}$$

The open bracket term decays as $k \to \infty$ so fast that it still tends to zero if ε is not fixed, but is instead equal to $1/k$. Thus we actually find

$$\mu_k(I) = o(l_k^{d\beta_k - 1/k}) = o(l_k^{d\beta - O(1/k)}). \qquad \square$$

This is the cleanest expression of the $d\beta$ dimensional behaviour at discrete scales we will need. To get a general inequality of this form, we use the fact that our construction distributes uniformly across all cubes.

Theorem 2. *If $l \leqslant l_k$ is dyadic and $I \in \mathcal{B}^d(l)$, then $\mu(I) \lesssim l^{d\beta - O(1/k)}$.*

Proof. We use a covering argument, which breaks into cases depending on the size of l in proportion to l_k and r_k:

- If $r_{k+1} \leqslant l \leqslant l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}^d(r_{k+1})$. For each of these cubes, because the mass is equidistributed over r_{k+1} cubes, we know the mass is bounded by at most $2(r_{k+1}/l_{k+1})^d$ times the mass of a $\mathcal{B}^d(l_k)$ cube. Thus

  $$\mu(I) \lesssim (l/r_{k+1})^d (2(r_{k+1}/l_k)^d) l_k^{d\beta - O(1/k)} \leqslant 2l^d / l_k^{d + O(1/k) - d\beta} \leqslant 2l^{d\beta - O(1/k)}.$$

  where we used the fact that $d + O(1/k) - d\beta \geqslant 0$.

- If $l_{k+1} \leqslant l \leqslant r_{k+1}$, we can cover I by a single cube in $\mathcal{B}^d(r_{k+1})$. Each cube in $\mathcal{B}^d(r_{k+1})$ contains at most one cube in $\mathcal{B}^d(l_{k+1}, d)$ which is also contained in X_{k+1}, so

  $$\mu(I) \lesssim l_{k+1}^{d\beta - O(1/k)} \leqslant l^{d\beta - O(1/k)}.$$

- If $l \leqslant l_{k+1}$, there certainly exists M such that $l_{M+1} \leqslant l \leqslant l_M$, and one of the previous cases yields that

  $$\mu(I) \lesssim 2l^{d\beta - O(1/M)} \leqslant 2l^{d\beta - O(1/k)}. \qquad \square$$

To use Frostman's lemma, we need the result $\mu(I) \lesssim l^{d\beta - O(1/k)}$ for an *arbitrary* dyadic cube, not just one with $l \leqslant l_k$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. If $l \geqslant l_k$, then $\mu(I)/l^{d\beta-O(1/k)} \leqslant 1/l_k^{d\beta-O(1/k)} \lesssim_k 1$, so $\mu(I) \lesssim_k l^{d\beta-O(1/k)}$ holds automatically for all sufficiently large cubes. Thus $\dim_{\mathbf{H}}(X) \geqslant d\beta - O(1/k)$, and letting $k \to \infty$ gives $\dim_{\mathbf{H}}(X) \geqslant d\beta = (dn-\alpha)/(n-1)$.

Theorem 3. $\dim_{\mathbf{H}}(X) = (dn-\alpha)/(n-1)$.

Proof. X_k is covered by at most

$$\left[\frac{l_{k-1}\dots l_1}{r_k \dots r_1}\right]^d$$

sidelength l_k cubes. It follows that if $\gamma > \beta_k$, then

$$H^{d\gamma}_{l_k}(X) \leqslant \left[\frac{l_{k-1}\dots l_1}{r_k\dots r_1} l_k^{\gamma}\right]^d \lesssim \left[\frac{l_{k-1}\dots l_1}{r_{k-1}\dots r_1} l_k^{\gamma-\beta_k}\right]^d \leqslant l_k^{d(\gamma-\beta_k)}.$$

Since $l_k \to 0$ as $k \to \infty$, $H^{d\gamma}(X) = 0$. Since γ was arbitrary, $\dim_{\mathbf{H}}(X) \leqslant d\beta_k$, and since k was arbitrary, $\dim_{\mathbf{H}}(X) \leqslant d\beta$. ☐

5 Applications

Of course, our result already generalizes methods with interesting applications. But the most novel applications of our method occur when the configurations truly are a fractal set.

Example. *Let $Y \subset \mathbf{R}^d$ be the countable union of sets with lower Minkowski dimension upper bounded by α. Then the set $Y_0 = \{(x,y) : x+y \in Y\}$ is a countable union of sets with lower Minkowski dimension upper bounded by $d+\alpha$. Applying our lemma then gives a set X with Hausdorff dimension $d-\alpha$ such that for any distinct $x_1, x_2 \in X$, $x_1 + x_2 \notin Y$. Modifying our construction slightly makes it possible to construct X with $X+X$ avoiding Y completely. Less elegantly, we can also consider*

$$Y_1 = \{(x,y) : x+y \in Y\} \cup \{(x,y) : x \in Y/2\}$$

Then Y_1 is also the countable union of sets with lower Minkowski dimension bounded by $1+\alpha$, and X avoiding Y_1 has $X+X$ disjoint from Y.

We have ideas on fusing our result with inspiration from the result of [3] to obtain the more impressive result which will show, given a set Y with fractal dimension α, how to construct a set X, which is a $\mathbf{Q}$ vector space, disjoint from Y, with Hausdorff dimension $d-\alpha$. Thus given a $\mathbf{Q}$ subspace V of $\mathbf{R}^d$, we can always find a complementary $\mathbf{Q}$ vector space W with a complementary fractal dimension.

Example. *In [2], one shows that we can find a dimension* $1/2$ *subset of any smooth curve avoiding isoceles triangles. Applying much the same techniques as in [2], but applying our result (though we do not need to be as careful, since we do not care about smoothness), we can extend this result to find a dimension* $1/2$ *subset of any bi-Lipschitz curve avoiding isoceles triangles.*

Example. *However, our method is able to do something much more interesting than a simple generalization of the method of [2]. Suppose we have a set* $Y \subset \mathbf{R}^2$*, together with an orthogonal projection* $\pi : \mathbf{R}^2 \to \mathbf{R}$ *such that* $\pi(Y) = \mathbf{R}$*. Form the set*

$$Z = \left\{(x_1, x_2, x_3) \in Y^3 : d(y_1, y_2) = d(y_1, y_3)\right\}$$

We then set

$$Z' = \pi(Z) = \{(\pi(x_1), \pi(x_2), \pi(x_3)) : (x_1, x_2, x_3) \in Z\}$$

If Z *is the countable union of sets with Minkowski dimension at most* α*, then* Z' *also is. This enables us to find* X' *with* $(X')^3$ *disjoint from* Z*, with Hausdorff dimension* $\beta = (3d - \alpha)/2$*. If we define* X *by including a single element of* $\pi^{-1}(x)$ *for each* $x \in X'$*, then* $\dim_{\mathbf{H}}(X) \geqslant \dim_{\mathbf{H}}(X') = (3d - \alpha)/2$*, and* X *avoids isoceles triangles. Thus we have obtained the much more ambitious goal of finding a fractal subset of a 'fractal line' avoiding isoceles triangles, with the same dimension obtained in [2] assuming that the 'fractal line' is actually a smooth curve with non-vanishing curvature.*

We conjecture that if Y *is any set with fractal dimension one, then* Z' *will have dimension at most 2, leading to a dimension* $1/2$ *set* Z*. This is suggested by certain results on slices of measures, though we were not able to find an exact way to piece together these results to obtain this result.*

6 Relation to Literature, and Future Work

TODO: MOVE THIS This result is part of a growing body of work finding general methods to avoid patterns with particular geometric features. In [2] and [3], sets with large Hausdorff dimension are constructed avoiding patterns specified by smooth low-variable functions, and low degree polynomials.

The technical skeleton of our construction are heavily modelled after [2]. Reading this paper in tandem with ours provides an interesting contrast between the techniques of the function oriented configuration avoidance result, and the fractal avoidance result we use. Because of its heavy influence on our result, we begin our discussion of the literature with an in depth comparison of our method to theirs.

Our result is a direct generalization of the main result of [2], which says that if $Z \subset (\mathbf{R}^d)^n$ is a smooth surface of dimension $dn - d$, then we can find X with dimension $(n-1)^{-1}$ solving the fractal avoidance problem. Of course, such a Z has Minkowski dimension $dn - d$, and our result achieves the same dimension for X. In response to [2], our result says that the only really necessary feature of a smooth hypersurface to the avoidance problem, aside from other geometric features, is its dimension. Not only is our result more flexible, enabling the surface Z to have non smooth points, but we can also take advantage of the fact that the surface might have dimension different from $dn - d$. Better yet, we can 'thicken' or 'thin' Z by slightly increasing or decrease the Minkowski dimension, while stably affecting the Hausdorff dimension of the solution X we construct.

The technique leading to this generalization can be compared to a phenomenon that has recently been noticed in the discrete setting, i.e. [5]. There, certain combinatorial problems can be rephrased as abstract problems on hypergraphs, and by doing this one can often generalize the solutions of these problems into analogues on 'sparse versions' of these hypergraphs. One can see our result as a continuous analogue to this phenomenon, where sparsity is represented by the dimension of the set Z we are trying to avoid. One can even view Lemma 1 as a solution to a problem about independent sets in hypergraphs. In particular, we can form a hypergraph by taking the cubes $\mathcal{B}^d(F, S)$ as vertices, and adding an edge $(I_1, \dots, I_n)$ between n distinct cubes if $I_1 \times \dots \times I_n$ intersects W. Then the union of an independent set of cubes in this graph is precisely a set F with F^n disjoint except on the discretization of the diagonal. And so the goal of Lemma 1 is essentially to find a 'uniformly chosen' independent set in this graph. Thus we even applied the discrete phenomenon at many scales to obtain the continuous version of the phenomenon.

A useful technique used in [2], and its predecessor [1], is a Cantor set construction 'with memory'; a queue in their construction algorithm allows storage of particular configurations, to be retrieved and avoided at a much, much later step of the building process. The fact that our result is more general, yet we can discard the queueing method from our proof, is an interesting anomoly. Adding memory to the queueing set is certainly an important trick to remember when thinking of new constructions for fractal avoiding sets. It enables one to restrict the requirements of an analogy to Lemma 1 from carving out an avoiding set F from a single set E, to carving $F_1, \dots, F_n$ out of disjoint sets $E_1, \dots, E_n$, such that $F_1 \times \dots \times F_n$ avoids W. Nonetheless, it makes the construction much more complicated to describe, which makes understanding dimension bounds slightly more complicated, because its hard to 'grasp' precisely what configuration we are avoiding at each step of the construction. The fact that our algorithm is more general than [2], yet we can discard the queueing method, is an interesting anomoly. We have ideas on how to exploit the fact that we do not use queueing to generalize our theorem to much more wide family of 'dimension α' sets Z, which we plan to publish in a later result.

Aside from [2], another paper that takes the perspective of solving a generic fractal avoidance problem is [3], who finds a solution X to an avoidance problem with Z a degree k hypersurface with Hausdorff dimension d/k. If $k \geqslant n - 1$, then our result does better than Mathe's result, so where Mathe's result excels is when Z is a low dimensional hypersurface. Just like how the result of this paper is a sparse analogue of [2], we would like to publish a follow up result giving a sparse analogue to [3]. Just as our result is obtained by assuming Z is covered by a sparse family of cubes, a sprase analogue of [3] would give a result if Z is covered by a sparse family of thickened varieties from a pencil of low degree surfaces. We already have ideas we are refining on how to achieve this.

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