Category Theory

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Chapter 1

Basic Definitions

Category Theory is the language of transformations. A great many objects share some common formal behaviour with respect to the functions defined on them. Rather than looking at particular elements of particular groups, rings, and sets, we just look at the objects themselves, and describe the functions connecting them. A **category** $\mathcal{C}$ consists of objects $\text{Obj}(\mathcal{C})$, such that for each pair of objects $(A, B) \in \text{Obj}(\mathcal{C})$ we have a collection of morphisms $\text{Mor}(A, B)$, which are pairwise disjoint, together with a composition operation. We write $f \in \text{Mor}(A, B)$ as $f : A \to B$. For each triplet A, B, C of objects, we have an associative composition map

$$\circ : \text{Mor}(B, C) \times \text{Mor}(A, B) \to \text{Mor}(A, C)$$

For any object A, we have a morphism $\text{id}_A \in \text{Mor}(A, A)$, such that $\text{id} \circ f = f$ for any $f : B \to A$, and $g \circ \text{id}_A = g$ for any $g : A \to B$. Morphisms in $\text{Mor}(A, A)$ will be known as **endomorphisms**, and the set of such objects will be abbreviated to $\text{End}(A)$. With the operation of composition, $\text{End}(A)$ becomes a monoid, and all monoids can be realized as endomorphisms over some object in a particular category. An **operation** of a monoid M on an object A is a monoid homomorphism $f : M \to \text{End}(A)$.

Example. *Perhaps the most basic category is the category* **Set** *of sets, whose objects are sets, and whose morphisms are set-theoretic functions. Category theory can be seen as a generalization of this category, and most often categories will be seen as a subcategory of this category.*

Example. *Algebra makes extensive use of category theory. The category* **Grp** *of groups has groups as objects, and whose morphisms are group homomorphisms.*

One similarily defines the categories **Rng** *and* **Vect** *of rings and vectors, with ring homomorphisms and linear transformations as morphisms.*

Example. *Often useful for constructing illuminating examples, every partially ordered set X can be given the structure of a category such that, if $x \geqslant y$ in X, there is precisely one morphism from x to y.*

Example. *Category theory is also useful in analysis. Let* **Top** *be the category of topological spaces, whose morphisms are continuous maps. One may specialize to the category* **Man** *of manifolds, or even further to* $\mathbf{Man}^\infty$*, which consists of differentiable manifolds with differentiable maps as morphisms.*

An **isomorphism** in a category is a morphism $f : A \to B$ for which there is $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. It follows trivially that g is unique, for if h is another inverse, then

$$g = g \circ \mathrm{id}_B = g \circ f \circ h = \mathrm{id}_A \circ h = h$$

we denote g by f^{-1}. Examples of isomorphism are algebraic isomorphisms, bijective maps, homeomorphisms, and diffeomorphisms, all in one concept. The set of isomorphisms from an object A to itself will be denoted $\mathrm{Aut}(A)$. It is a group, and all groups are isomorphic to automorphisms over some object in a category. By an **operation** or **representation** of a group G on an object A in a category we mean a group homomorphism $f : G \to \mathrm{Aut}(A)$. Linear reputations, which are the representations of a group over the category of vector spaces, but also permutation representations, which are representations over the category of sets. Isomorphisms really are 'the same object' in a categorical sense, because there are natural bijections between the morphisms of the object. Let $f : X \to Y$ be an isomorphism. The map $g \mapsto f \circ g \circ f^{-1}$ is then a bijection between $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$. Similarily, $g \mapsto g \circ f$ is a bijection between $\mathrm{Mor}(Y, A)$ and $\mathrm{Mor}(X, A)$.

Example. *If M is a differentiable manifold, and X a differentiable vector field, then X induces a representation of the additive group* $\mathbf{R}$ *on the diffeomorphisms of M, obtained by taking t to the map which perturbs a point to where it would be in t seconds after the vector field acts on the space.*

Other useful maps to describe are **sections**, maps $f : X \to Y$ which are *left invertible*, such that there are maps $g : Y \to X$ such that $g \circ f = \mathrm{id}_A$. The *right invertible* elements are called **retractions**.

1.1 Universal Objects

Let $\mathcal{C}$ be a category. An **initial object** (or a universal repeller) X in the category is an object such that for any other object A, there is a unique map $f : X \to A$. A **final object** (or universal attractor) has unique maps $f : A \to X$. It is easy to see that any two initial and final objects in the same category are isomorphic. Here are some examples,

Example. *The trivial group* (0) *is both initial and final in the category of groups. Similarily, the trivial module is initial and final in the category of modules over a fixed ring.*

Example. *The empty set is an initial object in the category of sets, and a singleton is a final object in this category. The same is true in the category of topological spaces and differentiable manifolds.*

We shall describe other **universal objects** in this section, which are objects satisfying some extremal functorial property. Most of the time, one can reduce the understanding of such objects as those which are initial or final in some category related to another category. In the next example, we show that final objects are really initial objects in another category.

Example. *Given a category* $\mathcal{C}$*, consider the category* $\mathcal{C}^{rev}$*, with the same objects, but if* $f : A \to B$ *is a morphism in* $\mathcal{C}$*, then* $f^{rev} : B \to A$ *is a morphism in* $\mathcal{C}^{rev}$*. An initial object in* $\mathcal{C}^{rev}$ *is simply a final object in* $\mathcal{C}$*. Suppose we only know that initial objects are isomorphic. Let* A *and* B *be final objects in* $\mathcal{C}$*. Then* A *and* B *are final in* $\mathcal{C}^{rev}$*, so there is an isomorphism* $f^{rev} : B \to A$*, which converts back to an isomorphism* $f : A \to B$*.*

We shall make common use of a certain type of construction. Given any category $\mathcal{C}$, we may form a new category $\mathcal{C}^{\to}$, whose objects consist of the morphisms in $\mathcal{C}$, and such that a morphism between two morphisms $f : A \to B$ and $g : A' \to B'$ is a pair of maps $\phi : A \to A'$, $\psi : B \to B'$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"] & B \arrow[d, "\psi"] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

commutes. There are many variations to this category. For instance, if we fix an object X, and consider the category of morphisms $f : A \to X$, whose morphisms are just as morphisms in $\mathcal{C}^{\to}$, but forcing ψ to be the identity, so that the diagram

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes.

Given two objects A and B, we will construct a **product** object $A \times B$. Consider the category whose objects consist of triplets (X, f, g), where $f : X \to A$, and $g : X \to B$ are morphisms. A morphism between (X, f, g) and (Y, ρ, π) is a morphism $h : X \to Y$, such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ A & Y \arrow[l, "\rho"] \arrow[r, "\pi"'] & B \end{tikzcd}$$

commutes. A product for A and B is then a final object $(A \times B, \pi_A, \pi_B)$ in this category. It is clear that any products are not only isomorphic in the original category, but also isomorphic in the stronger sense in the category above. What we have argued is that if $f : X \to A$ and $g : X \to B$ are any two morphisms, then there exists a morphism $f \times g : X \to A \times B$, such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "f \times g"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commutes. One define products of arbitrary families $\{A_\alpha\}$ of objects, and denotes the final objects as $(\prod A_\alpha, \{\pi_\alpha\})$.

Example. *Given two groups G and H, one canonically defines the product $G \times H$ to be the set of all tuples (g, h), with $g \in G$ and $h \in H$, and with multiplication structure $(g, h)(x, y) = (gx, hy)$. The same trick works for products of rings, modules, vector spaces, and sets, where the associated operations are adjusted accordingly.*

Coproducts are obtained from the above definition by reversing the arrows. We consider the category of objects (X, f, g), where $f : A \to X$ and $g : B \to X$ are morphisms. An initial object in this category is the coproduct of A and B, denoted $A \coprod B$. Given $f : A \to X$ and $g : B \to X$, we have a unique map $f \coprod g : A \coprod B \to X$, such that

$$\begin{tikzcd} & X & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A \coprod B \arrow[u] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

We may also take coproducts $\coprod A_\alpha$ of an arbitrary family of objects $\{A_\alpha\}$.

Example. *Given two groups, G and H, the coproduct is the free product $G * H$, which is a quotient of the monoid of all finite words with elements in G and H (assumed disjoint) whose operation is concatenation. Consider the equivalence which identifies (g, g') with $g * g'$, and $h * h'$ with hh', and if e is the identity in G, and e' the identity in H', then identify $e * h$ and $h * e$ with h, and $e' * g$ and $g * e'$ with g. Extend this to semigroup congruence. The monoid formed is the free product, and is a group, for $G*H$ is generated by $G \cup H$, and each $g \in G$ and $h \in H$ has an inverse in $G * H$. We have canonical embeddings $i_G : G \to G * H$ mapping g to itself, and $i_H : H \to G * H$ mapping h to itself.*

Example. *Given two modules M and N over an abelian ring R, the coproduct is the direct sum $M \oplus N$, which is the set $M \times N$ (where (m, n) is denoted $m \oplus n$) with operations $(m \oplus n) + (x \oplus y) = (m + x) \oplus (n + y)$.*

Products and Coproducts are the most basic constructions in category theory, but we have some other occasionally useful objects. Fix an object Z in a category, and consider the category whose objects are morphisms $f : A \to Z$ and $g : B \to Z$, and a morphism between morphisms is a map $h : A \to B$ such that the standard diagram commutes. A product of morphisms in this category is called the **fibre product**, and satisfies the diagram

$$\begin{tikzcd} & A \times_Z B \arrow[dl, "\pi_f"'] \arrow[dr, "\pi_g"] \arrow[dd, "f \times_Z g"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

π_f is known as the pullback of f by g, and π_g the pullback of g by f. Similarily, we may consider **fiber coproducts**, or **pushouts**.

Example. *Fibre products exist in the category of groups. Let $f : G \to K$ and $g : H \to K$ be two maps. Let $G \times_K H = \{(x, y) \in G \times H : f(x) = g(y)\}$, and let π_f and π_g be the standard projections, then define*

$$f \times_K g = f \circ \pi_f = g \circ \pi_g$$

Let $\pi : L \to G$, $\rho : L \to H$, and $\psi : L \to K$ be three maps such that

$$f \circ \pi = g \circ \rho = \psi$$

Then we may consider $\pi \times \rho : L \to G \times H$, and the image of $\pi \times \rho$ is contained in $G \times_K H$, for $f(\pi(x)) = g(\rho(x))$, hence we may consider $\pi \times_K \rho : L \to G \times_K H$, obtained by restricting the domain. This map is unique, for the product map is unique. The normal product $G \times H$ does not satisfy the property of $G \times_K H$, because there may not be gloablly definable morphisms π_f and π_g making the diagram commute.

Example. *If Z is a final object in the category, then $X \times_Z Y$ is precisely $X \times Y$. This is because given $f_X : X \to Z$ and $f_Y : Y \to Z$, and $g_X : A \to X$, $g_Y : A \to Y$, the universal property of the product implies there is a map $g : A \to X \times Y$, and the projections commute as desired because the final object makes the final commuting parts of the diagram trivial.*

1.2 Functors and Natural Transformations

The main reason to rigorously define groups is to define what a homomorphism is, so we can consider groups with similar structure. Categories were invented to define Functors and natural transformations. A **(Covariant) Functor** F between two categories $\mathcal{C}$ and $\mathcal{D}$ is an association of an object X in $\mathcal{C}$ with an object $F(X)$ in $\mathcal{D}$, and associating a morphism $f : X \to Y$ with a morphism $F(f) : F(X) \to F(Y)$, such that

$$F(g \circ f) = F(g) \circ F(f)$$

whenever $g \circ f$ is defined. A **Contravariant Functor** associative a morphism $f : X \to Y$ with a morphism $F(f) : F(Y) \to F(X)$ such that

$$F(g \circ f) = F(f) \circ F(g)$$

Often $F(f)$ is denoted f_* in the case of a covariant functor, and f^* for a contravariant functor. Functors are the natural 'morphisms' of categories, and together form a category whose objects are themselves categories, denoted **Cat**. A functor is **faithful** if the map between morphisms is injective for each pair of objects, and **full** if the map is surjective for each pair of objects. A subcategory of a category is called full if the inclusion functor is full.

Example. *The association of a set S with the free abelian group $\mathbf{Z}\langle S\rangle$ is a covariant functor from* **Set** *to* **Ab**, *since a map $f : S \to T$ induces a map*

$$f_* : \mathbf{Z}\langle S\rangle \to \mathbf{Z}\langle T\rangle$$

defined by $f_*(\sum_{s\in S} n_i s) = \sum_{s\in S} n_i f(s)$.

Example. *In almost every category, the objects are sets equipped with some additional structure (a group is a set equipped with an operation, a topological space a set equipped with a family of open sets). A morphism is then a function between sets with some additional structure. Thus we may define a functor from the category to* **Set** *by mapping an object to the underlying set of that object, and a morphism to the underlying function that defines it. This is known as a forgetful functor, because it is obtained by forgetting information about the underlying category.*

Example. *Given a fixed object A in a category $\mathcal{C}$, we can define a* **contrafunctor of points** *from $\mathcal{C}$ to* **Set** *by associating with each object B the set $Mor(B,A)$, and with each morphism $f : B \to C$ the map from $Mor(C,A) \to Mor(B,A)$ mapping $g : C \to A$ to $g \circ f : B \to A$.*

Example. *Functors most naturally occur in algebraic topology, which studies associations of topological space in* **Top** *with some algebraic structure, for instance, in the categories* **Grp**, **Ab**, *and* **Rng**. *Functors were invented to discuss these associations. For instance, the homology is a covariant functor, associating each topological space X a graded $\mathbf{Z}$-module $H(X)$.*

Natural transformations are the natural maps relating functors. Given two functors F and G between two categories $\mathcal{C}$ and $\mathcal{D}$, a natural transformation is an association with each object $X \in \mathcal{C}$ a morphism $H(X) : F(X) \to G(X)$, such that for each morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "H(X)"'] & F(Y) \arrow[d, "H(Y)"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}$$

commutes. We may therefore consider isomorphisms of functors, known as **natural equivalences**. We will often say a functor itself is natural if it is naturally equivalent

Example. *The classic example of a natural transformation says that a finite dimensional vector space V it 'naturally isomorphic' to its double dual V^{**}. What we mean is that the endofunctor which associates V with V^{**}, and associates $f : V \to W$ with $f^{**} : V^{**} \to W^{**}$ defined by*

$$f^{**}(\phi) = \phi \circ f^*$$

is naturally equivalent to the identity functor. Given a vector space V, consider the 'double dual' map $(\cdot)^{**} : V \to V^{**}$, which maps $v \in V$ to $v^{**} : V^* \to \mathbf{F}$, defined by $v^{**}(f) = f(v)$. We claim this is a natural transformation. Fix $f : V \to W$. Then, for each $v \in V$, and $\phi \in W^*$,

$$f(v)^{**}(\phi) = \phi(f(v)) = (\phi \circ f)(v) = f^*(\phi)(v) = (v^{**} \circ f^*)(\phi) = f^{**}(v^{**})(\phi)$$

The natural transformation has an inverse if we restrict ourselves to finite dimensional vector spaces, since then the double dual is invertible. On the other hand, the association of V with V^* is 'unnatural'; there is no 'natural' choice of morphism between V and V^* which does not depend on a choice of basis. However, if we restrict ourselves to the category of Hilbert spaces, with isometries as morphisms, or the category of vector spaces equipped with a nondegenerate bilinear form, then V is naturally isomorphic to V^* (for infinite dimensional vector spaces, we need to interpret V^* as the dual space of continuous functionals).

An **equivalence of categories** is a functor F with an 'inverse functor' G such that $F \circ G$ and $G \circ F$ are both equivalence to the identity functor. Often, this is a better notion of saying two categories are 'equal' then the two categories being isomorphic, which is too strong a condition.

Theorem 1.1. *A function $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of classes if it is fully faithful and every object $B \in \mathcal{D}$ is isomorphic to $F(A)$ for some $A \in \mathcal{C}$.*

Proof. s □

1.3 Adjoint Functors

Universal properties characterize objects in a category. Adjoint functors characterize functors. Two covariant functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ if, for each $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection $\tau_{AB} : \mathrm{Mor}(F(A), B) \to \mathrm{Mor}(A, G(B))$, such that for all $f : A_0 \to A_1$ in $\mathcal{C}$,

$$\begin{array}{ccc} \mathrm{Mor}(F(A_1), B) & \xrightarrow{F(f)^*} & \mathrm{Mor}(F(A_0), B) \\ \big\downarrow{\scriptstyle \tau_{A_1 B}} & & \big\downarrow{\scriptstyle \tau_{A_0 B}} \\ \mathrm{Mor}(A_1, G(B)) & \xrightarrow{f^*} & \mathrm{Mor}(A_0, G(B)) \end{array}$$

where $f^*(g) = g \circ f$ and $F(f)^*(g) = g \circ F(f)$. The diagram says exactly that $\tau_{A_0 B}(g \circ F(f)) = \tau_{A_1 B}(g) \circ f$, for $g : F(A_1) \to B$. We say F is a *left adjoint* for G, and G a *right adjoint* for F.

Example. *Let M, N, and P be A modules. Then we have a bijection from $Hom(M \otimes N, P)$ and $Hom(M, Hom(N, P))$, because if $f : M \otimes N \to P$, then f induces a map $f^\sharp : M \to Hom(N, P)$ by setting $f^\sharp(m)(n) = f(m \otimes n)$. Conversely, if $\varphi : M \to Hom(N, P)$ is a homomorphism, then we obtain a bilinear map $\varphi' : M \times N \to P$ by setting $\varphi'(m, n) = \varphi(m)(n)$, and therefore there is a homomorphism $f : M \otimes N \to P$ with $f(m \otimes n) = \varphi(m)(n)$, hence $f^\sharp = \varphi$. We have two functors $F : M \mapsto M \otimes N$ and $G : M \mapsto Hom(N, M)$. Given $g : M_1 \otimes N \to B$, and $f : M_0 \to M_1$, it suffices to show $(g \circ F(f))^\sharp = g^\sharp \circ f$. We calculate*

$$(g \circ F(f))^\sharp(m)(n) = (g \circ F(f))(m \otimes n) = g(f(m) \otimes n) = g^\sharp(f(m))(n)$$

So the two functors are adjoint.

Example. *If we have a morphism of rings $B \to A$, then every A module can be considered as a B module, so we get a functor F from the category of A modules to the category of B modules. This functor is right adjoint to the function $G(M) = M \otimes_B A$.*

Example. *If S is an abelian semigroup, one can consider the groupification G_S of S, obtained by the equivalence relation on $S \times S$ by setting $(a, b) \sim (c, d)$ if $a + d = c + b$. Then we embed S in $Grp(S)$ by the map $s \mapsto [s, 0]$. If $f : S \to H$ is a homomorphism, we can define a homomorphism $f : Grp(S) \to H$ by defining*

$f[a,b] = f(a) - f(b)$, and this is the unique homomorphism extending the map on S to $Grp(S)$. The map Grp is a functor from the category of abelian semigroups to the category of abelian groups. The functor F associating each abelian group H with itself as an abelian semigroup (a forgetful functor) is then a right-adjoint to the groupification functor. We have a bijection between $Hom(Grp(S), H)$ and $Hom(S, H)$, because every homomorphism $Grp(S) \to H$ restricts to a homomorphism $S \to H$, and every homomorphism $S \to H$ extends to $Grp(S) \to H$. The adjoint property follows automatically.

Chapter 2

Abelian Categories

In many categories, we can consider homological arguments. One naturally encounters the homology of vector spaces in linear algebra. In elementary algebraic topology, one obtained homological arguments using abelian groups. We now discuss the general categorical setting where we can consider these types of arguments. An **additive category** is a category such that for any two objects A and B, $\mathrm{Mor}(A,B)$ is an abelian group, such that addition distributes over composition, the category has finite products, and the category has a **zero object** (an object that is both initial and final). In such a scenario, $\mathrm{Mor}(A,B)$ is often denoted $\mathrm{Hom}(A,B)$. An **additive functor** between additive categories is a functor preserving addition.