

Number Theory

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Chapter 1

Generating Functions

Example. Suppose we are working in a country with only a one, a two, and a three penny coin. Given an integer n , let $r(n)$ denote the number of ways that a person can be paid n pennies using these three coins. Since this is a question about the additivity of numbers, we can likely understand it using generating functions. Formally,

$$r(n) = \#\{(a, b, c) : a + 2b + 3c = n\}$$

We note

$$\left(\sum_{a=0}^{\infty} z^a\right) \left(\sum_{b=0}^{\infty} z^{2b}\right) \left(\sum_{c=0}^{\infty} z^{3c}\right) = \sum_{a,b,c} z^{a+2b+3c} = \sum_{n=0}^{\infty} r(n) z^n$$

Thus, for $|z| < 1$,

$$\sum_{n=0}^{\infty} r(n) z^n = \frac{1}{(1-z)(1-z^2)(1-z^3)}$$

We can now perform a partial fraction decomposition, writing

$$\frac{1}{(1-z)(1-z^2)(1-z^3)} = \frac{1}{(1-z)^3(1+z)(\omega-z)(\omega+z)}$$

where $\omega = e(1/3)$ is a primitive third root of unity. Some intense linear algebra shows this is equal to

$$\frac{z+2}{9(z^2+z+1)} + \frac{17z^2-52z+47}{72(1-z)^3} + \frac{1}{8(1+z)}$$

which can be further decomposed into

$$-\frac{\omega^2 + 3\omega + 2}{9(1 - z/\omega)} + \frac{\omega^2 - \omega + 2}{9(1 - z/\omega^2)} \\ + \frac{1}{6(1 - z)^3} + \frac{1}{4(1 - z)^2} + \frac{17}{72(1 - z)} + \frac{1}{8(1 + z)}$$

where $\omega = e(1/3)$. Taking power series and summing up, we find

$$r(n) = -\frac{\omega^2 + 3\omega + 2}{9\omega^n} + \frac{\omega^2 - \omega + 2}{9\omega^{2n}} + \frac{(n+1)(n+2)}{12} + \frac{n+1}{4} + \frac{17}{72} + \frac{(-1)^n}{8} \\ = \frac{6n^2 + 36n + 47 + 9(-1)^n}{72} + \begin{cases} 0 & n \equiv 0 \pmod{3} \\ -2/9 & n \equiv 1 \text{ or } 2 \pmod{3} \end{cases} \\ = \frac{(n+3)^2}{12} + \frac{9(-1)^n - 7}{72} + \begin{cases} 0 & n \equiv 0 \pmod{3} \\ -16/72 & n \equiv 1 \text{ or } 2 \pmod{3} \end{cases}$$

We know $r(n)$ is an integer, and since

$$\frac{9 + 7 + 16}{72} = \frac{32}{72} < \frac{1}{2}$$

So $r(n)$ is the closest integer to $(n+3)^2/12$.