Salem Sets
Avoiding Nonlinear Configurations

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Research Problem: Can Large Sets Avoid Patterns?

More specifically: If a set $S \subset \mathbb{R}^d$ has large *fractal dimension*, does it contain patterns? The main focus of this project is on the construction of counterexamples: for a given function f, can we construct large sets S such that S *does not contain* distinct points $x_1, \dots, x_n$ satisfying $f(x_1, \dots, x_n) = 0$, i.e. such that S avoids zeroes of f?

- If $f(x_1, x_2, x_3) = (x_1 - x_2) - (x_2 - x_3)$, then sets avoiding zeroes of f do not contain three term arithmetic progressions.
- If $f(x_1, x_2, x_3) = |x_1 - x_2|^2 - |x_2 - x_3|^2$, then sets in $\mathbb{R}^d$ avoiding zeroes of f do not contain the vertices of any isosceles triangle.

Mainly, this project constructs large *Salem sets* avoiding zeroes of *nonlinear* functions.

There are several fractal dimensions, and they differ subtly in the properties they measure. The *Hausdorff dimension* $\dim_{\mathbb{H}}(S)$ of a set $S \subset \mathbb{R}^d$ intuitively measures the possibility of distributing mass onto S in a way that does not concentrate too strongly around points. The *Fourier dimension* $\dim_{\mathbb{F}}(S)$ of a set $S \subset \mathbb{R}^d$ measures the possiblity, not only of avoiding mass concentration at points, but also of avoiding mass concentration near families of equally spaced points, i.e. concentration 'at a particular frequency', as measured quantitatively through the Fourier transform: a set S has $\dim_{\mathbb{F}}(S) > \alpha$ precisely when one can find a probability measure μ with $\text{supp}(\mu) \subset S$ such that $|\widehat{\mu}(\xi)| \leq |\xi|^{-\alpha/2}$.

If S has large Hausdorff dimension, then one can distribute mass on S not concentrated near points is also not concentrated near 'most frequencies'. But in order to have large Fourier dimension, a distribution of mass must avoid concentrating near *all frequencies*. It is always true that $\dim_{\mathbb{F}}(S) \leq \dim_{\mathbb{H}}(S)$ for any set $S \subset \mathbb{R}^d$, but the reverse is often *not true* if the set is clustered 'near particular frequencies'.
TODO: Picture of Cantor Set, Hyperplane, Curved Surface

Salem Sets: Structure vs. Randomness

A set is *Salem* if it's Fourier dimension agrees with it's Hausdorff dimension. This is a common feature of *random sets*, which tend to avoid clustering near equally spaced points with high probability. On the other hand, it is *suprisingly difficult* to find Salem sets without employing randomness in some way, since adding *structure* to a set can possibly introduce clustering near certain frequencies in very subtle ways, which makes it very difficult to compute Fourier dimensions. In particular, *nonlinear structure* is especially difficult to understand, as indicated by the following open problems:

- There are very few explicit (i.e. nonrandom) examples of Salem sets. Pretty much the only examples occur from the theory of Diophantine approximation (Cite Kauffman, Hambrook, Fraser, etc). For $d > 2$, it remains an open problem to construct Salem sets $S \subset \mathbb{R}^d$ of dimension s for general values $s \in [0, d]$.
- We do not know the Fourier dimension of $\{x + x^2 : x \in C\}$, where C is the Cantor set, whereas we know the set has Hausdorff dimension $\log_3(2)$.

While there are many constructions of sets with large *Hausdorff dimension* avoiding the zeroes of nonlinear functions f (Cite:), most constructions of large Salem sets avoiding functions f focus on the case when f is linear, e.g. on the study of arithmetic progressions or other linear relations between points. Nonetheless, here we focus mostly on nonlinear functions f.

Theorem. *Suppose $f : (\mathbf{R}^d)^n \to \mathbf{R}^d$ is given by*

$$f(x^1, \dots, x^n) = x^1 - g(x^2, \dots, x^n),$$

where $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$ is smooth, and $D_{x^k} g = (\partial g^i / \partial x^k_j)$ is an invertible matrix for all $2 \leq k \leq n$. Then we can construct a Salem set $S \subset \mathbb{R}^d$ with

$$\dim_{\mathbf{F}}(S) = \frac{d}{n - 3/4}$$

avoiding solutions to f.

Under these assumptions, (TODO) constructs sets S with

$$\dim_{\mathbb{H}}(S) \geq d/(n-1)$$

avoiding zeroes to f, and we conjecture the theorem above can be improved to this bound in the setting of Salem sets.

Constructing Salem Sets

One can view the construction method as a random interval dissection method iterating on different scales, ala the construction of a Cantor set. The main importance is working with intervals is that we can *discretize* the problem.

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References
Ekstrom Survey

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