Algorithmic Aspects of the Brascamp Lieb Inequality

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- *Classical Complexity and Quantum Entanglement*
  Gurvits, 2004.
- *Algorithmic and Optimization Aspects of Brascamp-Lieb Inequalities, Via Operator Scaling*
  Garg, Gurvits, Oliveira, Wigderson, 2016.
- *A Deterministic Polynomial Time Algorithm For Non-Commutative Rational Identity Testing*
  Garg, Gurvits, Oliveira, Wigderson, 2016.

Lieb's Theorem

$$\int_{\mathbf{R}^n} \prod_{i=1}^{m} |f_i(B_i x)|^{p_i} \, dx \leq \mathrm{BL}(B, p) \cdot \prod_{i=1}^{m} \|f_i\|_{L^1(\mathbf{R}^{n_i})}^{p_i}.$$

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  - Can be exponentially many constraints, so inefficient.

Geometric Brascamp Lieb

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  - (Projection Property): $B_i B_i^* = I$.
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  - (Isotropy Property): $\sum_i p_i B_i^* B_i = I$.
- (Barthe, 1998), generalizing (Ball, 1989), showed that if (B, p) is geometric, $\mathrm{BL}(B, p) = 1$.

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$$\int_{\mathbf{R}^n} \prod_{i=1}^m |f_i(M_i B_i M x)|^{p_i} \, dx \leq \mathrm{BL}(B', p) \cdot \prod_{i=1}^m \|f_i\|_{L^1(\mathbf{R}^{n_i})}^{p_i}.$$

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$$\begin{aligned}\mathrm{BL}(B',p) &= \left(\sup_{A_1,\ldots,A_m \succ 0} \frac{\prod_i \det(A_i)^{p_i}}{\det\left(\sum_i p_i M^* B_i^* M_i^* A_i M_i B_i M\right)} \right)^{1/2} \\ &= \det(M)^{-1} \prod_i \det(M_i)^{-p_i} \cdot \mathrm{BL}(B,p).\end{aligned}$$

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- (BCCT) Geometric rescaling possible iff extremizers exist.

Main Result

- (Garg, Gurvits, Oliveira, Wigderson, 2016) If $\mathrm{BL}(B, p) < \infty$, we can rescale to (B', p) with $\mathrm{BL}(B', p) \leq 1 + \varepsilon$, for any $\varepsilon > 0$.

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  - Conversely, we can determine if $\mathrm{BL}(B, p) = \infty$, and find a violating subspace V to the condition $\dim(V) \leq \sum p_i \dim(B_i V)$ in this case, in $\mathrm{Poly}(\mathrm{Bits}(B, p), d)$ computations.

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  - Open Problem: Can we can improve this to $\mathrm{Poly}(\mathrm{Bits}(B,p), d, \log(1/\varepsilon))$ computations?

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$$(1-\varepsilon)\mathrm{BL}(B,p) \leq \mathrm{BL}(B_1,p) \leq (1+\varepsilon)\mathrm{BL}(B,p).$$

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  - If

$$R = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \gamma_n \end{pmatrix},$$

    then

$$\begin{aligned} \mathrm{Perm}(RAC) &= \mathrm{Perm}(\lambda_i A_{ij} \gamma_j) \\ &= (\lambda_1 \dots \lambda_n)(\gamma_1 \dots \gamma_n)\mathrm{Perm}(A) \\ &= \det(R)\det(C)\mathrm{Perm}(A). \end{aligned}$$

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- (Egorychev, 1981), (Falikman, 1981) If A is *doubly stochastic*, $e^{-n} \leq \text{Perm}(A) \leq 1$.
- If RAC is doubly stochastic, then

$$\text{Perm}(A) \leq \det(R)^{-1}\det(C)^{-1} \leq e^n \cdot \text{Perm}(A),$$

so $\text{Perm}(A) \approx \det(R)^{-1}\det(C)^{-1}$.

The Algorithm

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- Claim: If $\mathrm{Per}(A_0) > 0$, $d(A_i, \mathbf{S}) \to 0$, where $\mathbf{S}$ is the family of doubly stochastic matrices.
- For even (odd) i, let γ_{ij} be the jth row (column) sum of A_i, so that $\mathrm{Per}(A_{i+1}) = (\gamma_{i1} \ldots \gamma_{in})^{-1} \cdot \mathrm{Per}(A_i)$.

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- Thus $\mathrm{Per}(A_i)$ is bounded, monotonic, converges to $P \leq 1$.
- If $\mathrm{Per}(A_i) \geq P - \varepsilon$ for $\varepsilon \ll 1$, then

$$P \geq \mathrm{Per}(A_{i+1}) \geq (1 + C \cdot \Delta_i) \cdot \mathrm{Per}(A_i) \geq (1 + C \cdot \Delta_i)(P - \varepsilon).$$

Thus $\Delta_i \lesssim \varepsilon$. Taking $\varepsilon \to 0$ shows $\Delta_i \to 0$.

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  - Really just more robust form of AGM inequality.
  - If γ_{ij} are the row sums, then because the column sums are all one,

$$\frac{1}{n} \sum_j \gamma_{ij} = 1.$$

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- **Proof that Per**$(A_i) \leq 1$**:**
  - Simple inductive argument: the hypothesis is that if each row of a matrix B all sum up to less than one, then $\mathrm{Per}(B) \leq 1$.
- **Proof that Per**$(A_{i+1}) \geq (1 + C\Delta_i) \cdot$ **Per**(A_i)**:**
  - Really just more robust form of AGM inequality.
  - If γ_{ij} are the row sums, then because the column sums are all one,
$$\frac{1}{n} \sum_j \gamma_{ij} = 1.$$
  - AGM implies $\gamma_{i1} \ldots \gamma_{in} \geq 1$, and monotonicity follows from
$$\mathrm{Per}(A_{i+1}) = (\gamma_{i1} \ldots \gamma_{in})^{-1} \mathrm{Per}(A_i).$$

And now, back to our regularly scheduled programming

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  - We obtain a sequence $B \to B_1 \to B_2 \to \ldots$.

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- (1) and (2) follow from techniques in the study of *positive operators*.

Positive Operators

- A linear map $T : M_n \to M_n$ is *completely positive* if there are $n \times n$ matrices $B_1, \ldots, B_K$ and $p_i > 0$ such that

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- Given T, we have $T^*(A) = \sum p_i B_i^* A B_i$.

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- $\mathrm{BL}(B, p) < \infty$ can only hold if $\sum p_i = 1$.
- Consider optimizing the quantity

$$\inf_{A \succ 0} \frac{\det(\sum p_i B_i^* A B_i)}{\det(A)}$$

analogous to

$$\mathrm{BL}(B, p) = \sup_{A_1, \ldots, A_m \succ 0} \sqrt{\frac{\prod_i \det(A_i)^{p_i}}{\det(\sum p_i \cdot B_i^* A_i B_i)}},$$

if all A_i are equal.

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- Positive operators are well studied in the quantum information theory literature, so reduction of BL to this theory is useful.

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  - $B_i B_i^* = I$ if and only if $T^*(I) = I$.
- If (B, p) is a Brascamp-Lieb datum with associated operator $T : M_n \to M_n$, then (B, p) is geometric if and only if T is *doubly stochastic*, i.e. $T(I) = I$ and $T^*(I) = I$.

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- If $\mathrm{Cap}(T) > 0$, iteration yields a rescaling arbirarily close to a doubly stochastic operator, in $\mathrm{Poly}(\mathrm{Bits}(B), 1/\varepsilon)$ time.

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- If $\mathrm{Cap}(T) > 0$, there is $d > 0$ and $d \times d$ matrices C_i s.t.

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- Invariant theory shows we can choose $d \leq n^4[(n+1)!]^2$.

The Invariant Theory

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- We have a group action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on tuples $B = (B_1, \ldots, B_m)$, such that

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- Invariant Theory: Find the ring R of all 'invariant polynomials' $f(B)$ such that

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  for all $(M, N) \in \mathrm{SL}_n \times \mathrm{SL}_n$ and all tuples B.
- Note: $f_C(B) = \det(\sum C_i \otimes B_i)$ is an invariant homogeneous polynomial under this action for any C_i.

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  - Thus $d \leq d_0$.

Rank Decreasing Operators

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  (1) T is rank non-decreasing if and only if T_U is rank non-decreasing for all U.
  (2) $\text{Cap}(T) = \inf_U \text{Cap}(T_U)$.
- To prove (1) and (2), use a simple trick: Given $A \succeq 0$, find U diagonalizing $T(A)$. Then $T(A) = T_U(A)$.

Thanks For Listening!