# Topology

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# Table Of Contents

# Chapter 1

# There and Back Again

Metaphor is at the heart of modern mathematics. Given a complex abstract problem, it is difficult to see beyong the logic. Via abstraction, we obtain analogies laden with intuition. Why try and see an infinite dimensional vector space, when we can see a three dimensional cartesian space which allows us to 'see the problem'. In the context of number theory, linear algebra, and algebraic geometry, abstract algebra is the analogy constructor. In analysis, topology is king, generalizing that intuitive notion of space which encloses calculus, differential geometry, functional, real, and complex analysis under a single umbrella.

Initially, the idea of limits, continuity, and neighbourhood arose in geometry and in the physical sciences, where it connects to the idea of experimental approximation. Since most of the measurements ins ciences are obtained numerically, it is not a surprise that these topological facts mostly remained in the realm of real and complex sciences. However, in the beginning of the 20th century, a general description of these ideas was found from a diverse range of fields.

**Example.** *It is well understood what we mean when we say a sequence of numbers $a_i$ converges to a point a. However, it becomes less clear what it means for a sequence of real valued functions $f_i$ to converge to a function $f$. One definition is pointwise convergence: $f_i$ converges to $f$ if $f_i(x)$ converges to $f(x)$ for each x. But then the equality*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x) = \int_{-\infty}^{\infty}\lim_{i\to\infty} f_n(x)$$

*need not hold, and does not even make sense in all cases. We could instead use uniform convergence: $f_i$ converges to $f$ if, for any $\varepsilon$, there is $m$ such that for any $n > m$, and for any $x$, $|f_n(x) - f(x)| < \varepsilon$. If the $f_i$ are integrable, and vanish outside a bounded region, then the equality above holds. Nonetheless, uniform convergence is much more restrictive than pointwise convergence, and neither is universally used over the other. It turns out that changing the definition of sequential convergence for functions corresponds to changing the topological structure of the space of all real-valued functions. Topology gives us the framework to naturally describe these kinds of structures on many classes of mathematical objects.*

**Example.** *Consider the space of all loops in the plane. It is natural to identify a loop with one of its parameterizations $\lambda$. We can then use the above definitions of convergence for functions to decide when loops converge to one another, and we ask what conditions on the convergence properties of $\lambda_n$ allow us to conclude*

$$\lim_{n \to \infty} \int_{\lambda_n} f(z)\, dz = \int_{\lambda} f(z)\, dz$$

*a relationship with useful applications in complex analysis*

Topology does not just have applications to the interchange of limit processes, but seems to be integral to any part of mathematics which involves 'the infinite'.

**Example.** *In recreational mathematics, one deals with 'rubber sheet' geometry problems. Given two shapes, if they were made of rubber would it be possible to stretch one shape into the other shape, without tearing or making holes in the objects. A rigorous mathematical definition of 'stretching' most naturally emerges from topology by the definition of a homeomorphism.*

**Example.** *The central theorem of Galois field theory is that the subgroups of automorphisms of a finite Galois field extension $F/E$ are in a natural bijection with subextensions $F/K/E$ of fields. However, this correspondence does not seem to hold if we allow $F/E$ to be infinite – there tends to be more subgroups then field extensions. Topology provides a 'finiteness' condition, known as compactness, which allows us to specify which groups correspond to subextensions, generalizing the fundamental theorem of Galois theory.*

Seeing as how general topology arose from a diverse range of fields, it is unsurprising to see that there are many definitions which specify the object studied in the field of topology. The following definition is due to the German mathematician Georg Cantor. Though it is not the only way to define what we mean by a topology, it is by far the most common.

## 1.1 The Topological Space, and other delights

Just as how numbers and rotations are the primary examples which abstract algebra generalizes, the primary example of a topological space, from which all the terminology is obtained, is the Euclidean plane. I'm sure that, at one point in your childhood, you owned a colouring book. In this tome of black and white, you delighted in colouring between in the lines which define a figure. Shapes emphasized by a black silouhette are easily distinguished and filled by a nice blue, green, or pink fluorescent marker. Mathematically, we distinguish between shapes which contain their silouhette and shapes that do not. Those shapes that do contain their 'boundary', as it is formally called, are called closed. The shapes that contain none of their boundary are open. Consider the interior of a circle (not including an outline). At any point on this shape, there is an infinitude of points from your pen tip to the edge of the circle – you'll never finish precisely coloring the shape in, if you don't include the boundary; you'll either overshoot by a tiny bit if you try and get close to the boundary, or undershoot. Sets which do not contain their boundary are known as open. The foundations of topology are obtained by abstracting the qualities that these 'edgeless' shapes possess.

A **topological space** is a set $X$, together with a **topological structure** $\Omega$, a designated collection of subsets, known as *open sets*, satisfying

1. Both $\varnothing$ and $X$ are open.
2. If $\{A_i\}_{i\in I}$ is an arbitrary family of open sets, then their union $\bigcup_{i\in I} A_i$ is also open.
3. If $\{A_i\}_{i=1}^n$ is a finite collection of open sets, then their intersection $\bigcap_{i=1}^n A_i$ is also open (it is sufficient to show that $A\cap B$ is open if $A$ and $B$ are open).

You should learn the definition of open-ness by rote. It is the most crucial idea in topology; everything else we shall deal with is defined in terms of open sets and the qualities they possess amongst themselves. Right now, it is an abstract axiomatic set of principles, but as we study topology, an intuitive spatial relationship will emerge.

Together with the definition comes an onslaught of terminology: to learn a field, you must first speak its language. All terminology will be expanded fully later on. This is just to provide an easy reference.

- Elements of a topological space are called **points**.
- A subset is **closed** if its complement is open.
- A **neighbourhood** of a point $x \in X$ is a set enveloping an open set containing $x$.[1]
- A **limit point** of a subset $U$ of $X$ is a point $p$ such that every open set containing $p$ contains points in $U - \{p\}$. The set of limit points of a subset $U$ is denoted $U'$, and is called the derived set of $U$.
- If we have two topological structures $\Delta$ and $\Omega$, then we say $\Delta$ is **finer** than $\Omega$, or $\Omega$ is coarser, if $\Omega \subset \Delta$. Spaces are comparable in the same way that rocks, pebbles, and grains of sand are.
- A function $f : X \to Y$ between two topological spaces is **continuous** if $f^{-1}(U)$ is open for every open set $U$ in $Y$. A homeomorphism is a continuous bijective function whose inverse is continuous.

Topology also defines additional **separation axioms** which ensure your spaces are more reasonable. If your space does not have at least the very low level properties, then be wary for non-intuitive spatial structure.

- A topological space is **Kolmogorov**, or **T0**, if, for any points $x \neq y$, either $x$ has a neighbourhood not containing $y$, or $y$ has a neighbourhood not containing $x$. Almost every Topological space 'occuring in real life' has this property, and if it doesn't there is normally a trick, described later, which allows us to remove this property.

[1] Some mathematicians enforce a neighbourhood itself to be open. Normally, this does not cause issues, but when reading other works of topology, make sure you remember which definition the author is using.

- A topological space is **Frechet**, or **T1** if, for any points $x \neq y$, we may find a neighbourhood of $x$ not containing $y$. Equivalently, a **Frechet** space is one in which points are closed.

- A topological space is **Hausdorff**, or **T2** if, for any points $x \neq y$, we may find disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$. This is the property which represents 'unique convergence' of sequences, which we shall discuss later. Spaces which are Hausdorff are the 'normal' spaces in topology. Some topologists consider the Hausdorff property in the definition of a topological space, to avoid pathological examples.

- A topological space is **Regular Hausdorff**, or **T3**, if it is Hausdorff, and when $C$ is a closed set not containing a point $x$, we may find disjoint open sets $U$ and $V$, such that $C \subset U$, and $x \in V$.

- A topological space is **Normal Hausdorff**, or **T4**, if it is Hausdorff and if two disjoint closed sets $C$ and $D$ can be spearated by disjoint open neighbourhoods $U$ and $V$, such that $C \subset U$, $D \subset V$. Normal spaces are useful because they are useful for constructing families of continuous functions on spaces – if $C$ and $D$ are disjoint and closed in a normal space $X$, there exists a continuous function $f : X \to \mathbf{R}$ such that $f(c) = 0$ for $c \in C$, $f(d) = 1$ for $d \in D$.

We defined openness as above because these properties hold on the Euclidean plane (that is, for the 'coloring book topology'). We define a shape to be open if we can never colour up to an edge. Precisely, A shape (a subset of the plane) is open if we may draw a circle around every point in the shape, and all points in the interior of the circle are contained in the shape itself. Intuition should tell you why the properties of open sets hold for this space – we shall define this precisely later. In this topology, the finite assumption of open intersections is crucial. If we take the intersection of an infinite number of open sets, we may no longer have an open set – the boundary may have been stretched too thin. Consider the intersection of the 'infinite venn diagram below'.

INFINITE CIRCLES INTERSECTING ONLY AT A POINT.

Let us define the Euclidean topology precisely.

**Example.** *The Euclidean topology on $\mathbf{R}^2$ is defined such that a set $A$ is open if it contains open circles around every point. Precisely, if $x \in A$, there is a $\varepsilon > 0$ such that the circle*

$$B_\varepsilon(x) = \{y \in \mathbf{R}^2 : d(x,y) < \varepsilon\}$$

*is a subset of $A$, where $d$ is the function measuring the distance between two points. If we swap 2 with $n$, we obtain the Euclidean topology in any dimension. These are the standard topologies whenever the real line, plane, and higher dimensional analogues are mentioned.*

Given any set $X$ there are two spatial extremities we can use to form a topology. We either choose the minimal number of open sets, or the maximal number. The 'discrete topology' on $X$ lets the topological structure $\Omega$ be equal to the power set $\mathcal{P}(X)$. In this case, every subset of the plane is open, and the axioms of trivially satisfied. The 'lumpy topology' on $X$ has minimum structure, with topological structure $\Omega' = \{\emptyset, X\}$. The fact that these are topologies is verified by trivial set theory. The discrete topology is named based on how closeness factors into topological spaces. 'limit points', as we have defined them above, indicate points which reach infinitely close to a given set. This is as close to distance as we can get in topology – we cannot tell if two points are miles or metres away, but only if they are infinitisimally close. The lumpy topology is named because topologically, every point is a limit point of every set (except in the case where the set is empty).

**Exercise 1.1.** *Let $X$ be the ray $[0,\infty)$, and let $\Omega$ consist of $\emptyset$, $X$, and all rays $(a,\infty)$ with $a \geqslant 0$. Then $\Omega$ is a topological structure, which can only distinguish a point from another point if it lies further down the plane (but not the converge).*

**Example.** *Let $X$ be a plane. Let $\Sigma$ consist of $\emptyset$, $X$, and all open disks with center at the origin. This is a topological structure, which only measures convergence to the origin, but cannot distinguish any other point.*

The most important example of a topological space is $\mathbf{R}$, the space of real numbers. Many other examples will stem from this space. The topological structure consists of the empty set, and unions of intervals $(a,b)$, where $a < b$ can be infinite. We have $\mathbf{R} = (-\infty,\infty)$. Its a bit fiddly, but you should be able to show that the intersection of two intervals is the union of intervals, and thus that the intersection property holds in general. This topology is the standard topology on $\mathbf{R}$.

**Example.** *Show that every open set in* $\mathbf{R}$ *can be broken into the disjoint union of open intervals. Bonus points: Show that this union is countable!*

*Proof.* The fact that the union is countable stems from the fact that every interval contains a rational which we may uniquely identify with the interval in order to count the set. $\square$

Closed sets of $\mathbf{R}$ are not so simple to classify. Later on, we will see the Cantor set, a very strange closed set of $\mathbf{R}$.

**Exercise 1.2.** *Let $X$ be a set, and let $\Omega$ consist of all subsets of $X$ whose complement is finite, and $\varnothing$. Show that $\Omega$ is a topological structure, called the $T_1$ topology.*

*Proof.* If $X-A$ is finite, and $X-B$ is finite, then $X-(A\cap B) = (X-A)\cup(X-B)$ is a finite union of finite sets, hence finite. If $X - A$ is finite, and $B$ is any other subset, then $X - (A \cup B) \subset X - A$ is finite. $\square$

This example should show us that our definition of topology is not trivial in the slightest.

**Exercise 1.3.** *Consider the set $\mathbf{Z}$, and let $\Omega$ be the set of all unions of arithmetical sequences of the form $a\mathbf{Z}+b = \{ax+b : x \in \mathbf{Z}\}$, where $a \neq 0$. Show that $\Omega$ is a topological structure, and that there are infinitely many prime numbers.*

*Proof.* Since every integer is in some arithmetic sequence, we know that $\mathbf{Z}$ itself is open, as is the empty set. Here is an arbitrary intersection of two sequences

$$[a\mathbf{Z} + b] \cap [m\mathbf{Z} + n]$$

If $x$ can be written $ay+b$ and $mz+n$ via two integers $y$ and $z$, then $x+am\mathbf{Z}$ is also contained in the intersection. so that the intersection of two sequences is the union of other arithmetical sequences. Hence the set of all union of arithmetical sequences specifies a topology.

This topology is a very strange one. Consider the factorial sequence

$$1!,\ 2!,\ 3!, \dots = 1,\ 2,\ 6,\ 24,\ 120,\ 720, \dots$$

Any open set containing 0 contains some arithmetical sequence $a\mathbf{Z}$, and the factorial sequence will eventually end up in this sequence since the sequence accumulates all integers as factors. Therefore this sequence is 'infinitely close' to 0 in this toplogy, even though in the canonical topology on the integers the sequence is one of the one that grows faster than any other function. The main importance of this topological space is that it allows us to show there are infinitely many primes. Suppose there are only finitely many, which we may write as $\{p_1, p_2, \dots, p_n\}$. Then

$$A = \bigcap_{k=1}^{n} p_k \mathbf{Z}$$

is the finite intersection of open sets, and is thus open. Since every non-zero non-unit integer can be written as the product of primes, it follows that the complement $A^c = \{-1, 0, 1\}$. In our topology, the complement of open sets is open (check this via the basis elements), so that $A^c$ is the union of open sets, which is clearly not true. Hence there must be infinitely many primes. □

## 1.2 Closed topological sets

A set in a topological space $X$ is closed if it is the complement of an open set. A closed interval $[a,b]$ is closed in $\mathbf{R}$, since $[a,b]^c = (-\infty, a) \cup (b, \infty)$. To specify the axioms of a topology, we could have taken as a primitive notion closedness instead of openness. Since

$$\Big(\bigcup_{i \in I} A_i\Big)^c = \bigcap_{i \in I} A_i^c$$

$$\Big(\bigcap_{i \in I} A_i\Big)^c = \bigcup_{i \in I} A_i^c)$$

we could have taken a topology as a collection of closed sets such that arbitrary intersections and finite unions of closed sets are closed.

**Exercise 1.4.** *Prove that $[0,1)$ is not closed nor open in $\mathbf{R}$, yet it is the union of closed sets and the intersection of open ones.*

*Proof.* Suppose $[0,1)$ was open, so it is equal to the union of intervals $\bigcup (a_i, b_i)$. We must have $0 \in (a_i, b_i)$ for some $i$, but then $a_i < 0$, and thus $a_i/2 < 0 \in [0,1)$, a contradiction. Conversely, suppose $[0,1)$ was closed, so $(-\infty, 0) \cup [1, \infty)$ was open. A similar argument to the one above shows that this is impossible.

We have

$$[0,1) = \bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, 1)$$

So we cannot take arbitrary unions of closed sets nor intersections of open sets. $\square$

**Exercise 1.5.** *Prove that* $\{0\} \cup \{\frac{1}{n} : n \in \mathbf{Z}\}$ *is closed in* $\mathbf{R}$.

*Proof.*

$$\left[\{0\} \cup \left\{\frac{1}{n} : n \in \mathbf{Z}\right\}\right]^c = (-\infty, 0) \cup (1, \infty) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \dots$$

The complement is open, as it is just the union of open sets. $\square$

Closed sets are seen as 'containing the points' infinitely close to each other. Here we establish what exactly this means. Recall that $x$ is a limit point of a set $A$ if every open set containing $x$ contains some point in $A$.

**Exercise 1.6.** *If $C$ is a closed set if and only if set of limit points $C'$ is a subset of $C$.*

*Proof.* Let $C$ be a closed set. Then $C^c$ is open, so if $x$ is not in $C$, $x$ is not a limit point of $C$. We have shown what we needed to show. Now suppose $C'$ is contained in $C$, where $C$ is an arbitrary set. If $x$ is not an element of $C$, then it follows that there is an open set $U_x$ containing $x$ not containing any point in $C$. The union

$$\bigcup_{x \notin C} U_x$$

is an open set containing no elements of $C$, and any point not in $C$. Thus the union is just $C^c$, and we have shown $C^c$ is open, hence $C$ is closed. $\square$

### 1.2.1 An excursion – Cantor's set

This excursion shows us that weird closed sets exist even in the most simple topologies – here the real numbers $\mathbf{R}$.

**Exercise 1.7.** *Consider the Cantor set $K$ defined below:*

$$K = \left\{ x \in \mathbf{R} : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \text{ with } a_k = 0 \text{ or } 2 \right\}$$

*Show that $K$ can be inductively defined as $\bigcap_{i=1}^{\infty} K_i$, where $K_0 = [0,1]$, and*

$$K_{i+1} = \frac{K_i}{3} \cup \left( \frac{2}{3} + \frac{K_i}{3} \right)$$

*Proof.* Let us show first that $K \subset \bigcap_{i=1}^{\infty} K_i$. Here is an arbitrary decimal expansion in the Cantor set

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where each $a_k = 0$ or 2. Then $x \in K_0 = [0,1]$, since

$$0 = \sum_{k=1}^{\infty} \frac{0}{3^k} \leqslant \sum_{k=1}^{\infty} \frac{a_K}{3^k} \leqslant \sum_{k=1}^{\infty} \frac{2}{3^k} = 2(\frac{1}{1-1/3} - 1) = 1$$

For an induction, assume $K \subset K_n$. Suppose $a_1 = 0$. Then

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} \cdots = \frac{1}{3}(a_2 + \frac{a_2}{3} + \dots) = y/3$$

where $y \in K$, so $y \in K_n$ by hypothesis. Then $x \in K_n/3 \subset K_{n+1}$. If $a_1 = 2$,

$$x = 2/3 + \frac{1}{3}(\frac{a_2}{3} + \frac{a_3}{3^2} + \dots) = 2/3 + y/3$$

where $y \in K$. Therefore $y \in K_n$, and $x \in 2/3 + K_n/3 \subset K_{n+1}$. This exactly proves that $K \subset K_{n+1}$. By induction, we have shown $K \subset \bigcap_{i=1}^{\infty} K_i$.

Now we must show, conversely, that $\bigcap K_i \subset K$. Let $x \in \bigcap K_i$. We will build a correct sequence of digits that converge to $x$. Let $x_0 = 0$, and

$$x_n = \max \left\{ a_k \in \{0,2\} : \sum_{k=0}^{n-1} x_k/3^k + a_k/3^n \leqslant x \right\}$$

Our first claim:

$$\sum_{k=0}^{\infty} x_k/3^k = x$$

We shall prove by induction that if $x \in K_n$, then $x - \sum_{k=1}^{\infty} x_k/3^k < 1/3^n$, If $x$ is in all $n$, then $x - \sum_{k=1}^{\infty} x_k/3^k$ must be equal to zero, since it is smaller than $1/3^n$ for all $n$, and greater than or equal to zero. This is trivially true for $K_0$, since any sum is between 0 and 1, and $x$ is also between 0 and 1. Assume this is true for all $y \in K_n$. Consider $x \in K_{n+1}$. Assume that there is $y = 3x$ in $K_n$. Then $x_1 = 0$ (since otherwise $x > 1/3$, and $y > 1$), and if we consider the expansion of $y$, we will see that $y_n = x_{n+1}$. This follows because

$$\sum_{k=1}^{n-1} y_k/3^{k+1} + a/3^n = \sum_{k=1}^{n} x_k/3^k + a/3^{n+1} \leqslant x$$

holds if and only if the inequality

$$3\left(\sum_{k=1}^{n-1} x_k/3^k + a_k/3^n\right) = \sum_{k=1}^{n-1} y_k/3^k + a_k/3^n \leqslant 3x = y$$

holds, assuming for an induction that $y_k = x_{k+1}$ for $k \leqslant n-1$. If

$$y - \sum_{k=1}^{\infty} y_k/3^k < 1/3^n$$

then $y/3 - \sum_{k=1}^{\infty} y_k/3^{k+1} = x - \sum_{k=1}^{n} x_k/3^k < 1/3^{n+1}$. If $x = 2/3 + y$, where $y \in K_n$, the same technique establishes the inequality. Our claim is thus proved. $\square$

**Exercise 1.8.** *Show that the Cantor set $K$ is closed in* $\mathbf{R}$.

*Proof.* We will show each $K_i$ is closed. Obviously $K_0 = [0,1]$ is closed. If $K_n$ is closed, then $K_{n+1} = K_n/3 \cup 2/3 + K_n/3$ is the union of two closed sets, since the maps $x \to x/3$ and $x \to x + 2/3$ are homeomorphisms of the real line (which we will define later), and map closed sets to closed sets, the claim is proved. $\square$

## 1.3 The Basis of a space

Normally, a topology is not given via specifying every single open set in the topology. Since open sets are constructed from other open sets, we may specify some archetypal sets, and provided these sets are sufficient to describe a topology, define a topology in terms of them.

Let $X$ be a set, and $\{\Omega_i\}_{i\in I}$ a family of topological structures on $X$. Consider the intersection of all structures,

$$\Delta = \bigcap_{i\in I} \Omega_i$$

Surely $\varnothing$ and $X$ itself are elements of $\Delta$, since they are an element of each $\Omega_i$. If $\{A_i\}_{i\in I} \in \Omega_i$ for each $i$, then surely $\bigcup_{i\in I} A_i \in \Omega_i$, so that if $\{A_i\}_{i\in I} \in \Delta$, we also have $\bigcup_{i\in I} A_i \in \Delta$. Similarily, if $\{A_i\}_{i\in I} \in \Delta$, then $\bigcap_{i=1}^n A_i \in \Delta$, so that $\Delta$ is a topological space. This fact will allow us to generate structures from generating sets, as is done in many areas of mathematics.

If $X$ is a set, and $D$ is a family of subsets, then we may consider

$$\Delta = \{\Omega \in \mathcal{P}(X) : D \subset \Omega \text{ and D is a topological space}\}$$

Taking $\Phi = \bigcap \Delta$, we obtain the coarsest topological space containing $D$, called the topology generated by $D$. If $\Omega$ is a topological space containing $D$, then $\Phi \subset \Delta$ also, by the construction above.

When $D$ has certain nice properties, the topological space generated by $D$ is much simpler to work with:

**Definition.** Let $D$ be a subset of a topological space $X$. Suppose that

1. $\varnothing, X \in D$.
2. If $A$ and $B$ are in $D$, and $A \cap B$ is non-empty, containing a point $x$, then there is a set $C$ in $D$ containing $x$ with $C \subset A, B$.

In this case, the topological space $\Phi$ generated by $D$ has a following property. A set $U$ is open in $\Phi$ if and only if it is the union of sets in $D$. We call $D$ a **basis** for the topology $\Phi$ if it satisfies properties (1) and (2), and a **subbasis** if it is just a generating set.

**Exercise 1.9.** *Can two distinct topological structures have the same base? That is, does the base of a topology uniquely define a topology.*

*Proof.* No. Let $X$ be a set, and $\Delta$, $\Omega$ two topological structures with the same base. Then $\Delta$ must be finer than $\Omega$, since $\Delta$ is the minimal structure containing the base. Similarily, $\Omega$ must be finer than $\Delta$, so we conclude the two structures are equal. $\square$

**Exercise 1.10.** *Prove that there is no minimal topological basis for* **R**.

*Proof.* Let $D$ be a basis for **R**, and let $U \in D$ be a set in $D$ not equal to $\varnothing$ or $X$, which is therefore open in **R** and contains an interval $(a, b)$. Consider $D - U$. The first property of a basis is satisfied, and if $x$ is contained both in two subsets $A$ and $B$ in $D$, and if it were the case that $x \in U \subset A \cap B$ (without loss of generality, $x \in (a, b)$), then since $(a+\varepsilon, b-\varepsilon)$ is the union of sets in $D$, there must be some $U' \in D - U$ containing $x$, since $(a+\varepsilon, b-\varepsilon) \subsetneq U$. Thus $D - U$ is a basis, and generates a topological space. Obviously, $D$ is finer than $D - U$. Let $U = \bigcup (a_i, b_i)$. There exists $\delta$ such that, for each $0 < \varepsilon < \delta$, $(a_i + \varepsilon, b_i - \varepsilon)$ is open in **R** and is not equal to $U$, so $(a_i + \varepsilon, b_i - \varepsilon) = \bigcup U_i$, for some $U_i \in D - U$. But then

$$U = \bigcup (a_i, b_i) = \bigcup_{i \in I} \bigcup_{\delta > \varepsilon > 0} \bigcup_{j \in J_i} U_{i,\varepsilon,j}$$

And $U$ is still open in the topology generated by $D - U$. $\square$

A basis for a discrete space $X$ is just $X$ itself, since any open set in a discrete space is the union of some points in $X$. A basis for **R** is the set of all intervals $(a, b)$.

**Exercise 1.11.** *Show that two bases $D$ and $D'$ generate the same topological structure if every element of $D'$ is the union of $D$, and vice versa.*

## 1.4 Topology and Convergence

Analysis takes topology and uses it to study limit operations. For instance, when we initially began studying the topology of **R**, we defined convergence (with $\varepsilon$'s and $\delta$'s), and then proceeded to define open and closed sets as a corollary. Most of the applications of topology to analysis deal with convergence, so it is natural to wonder whether it is possible to define all topologies by the convergent sequences that result.

**Definition.** Let $\Omega$ and $\Psi$ be two topologies on a single set $X$. We will say $\Omega$ and $\Psi$ are **sequentially equivalent** when the sequences that converge in $(X, \Omega)$ are exactly the same as those in $(X, \Psi)$, and to exactly the same points.

Does it follow that, when two topologies $\Omega$ and $\Psi$ are sequentially equivalent, $\Omega = \Psi$? The next example shows this is, unfortunately, not the case.

**Example.** *Let $X$ consist of all countable ordinals, together with the first uncountable ordinal, denoted $\omega_1$. Let $\Omega$ be the order topology on $X$, and let $\Psi$ be the topology generated by $\Omega \cup \{\{\omega_1\}\}$. Surely $\Psi \neq \Omega$, since $(\omega, \omega_1] \neq \{\omega_1\}$ for any choice of $\omega$, yet $\Omega$ is sequentially equivalent to $\Psi$. Take any sequence of ordinals $\{x_i\}_{i \in \mathbf{N}}$ in $X$. Since the relative topologies on $[0, \omega_1)$ generated by $\Omega$ and $\Psi$ are the same, if $x < \omega_1$, then $x_i \to x$ in $(X, \Omega)$ if and only if the sequence converges to $x$ in $(X, \Psi)$. Suppose that $x_i \to \omega_1$ in $(X, \Psi)$. This means precisely that there is some $m \in \mathbf{N}$ such that, for $n > m$, $x_n = \omega_1$. Suppose that this is not true of some sequence $\{x_i\}_{i \in \mathbf{N}}$, that is, $x_k \neq \omega_1$ for arbitrarily large integers $k$. Then we may select some subsequence $\{y_i\}_{i \in \mathbf{N}}$, such that $y_k \neq \omega_1$ for any $k$. But then, as the countable union of countable ordinals, $\bigcup_{i \in \mathbf{N}} y_i$ is countable, so $\omega_1 > \bigcup_{i \in \mathbf{N}} y_i \geqslant y_k$, for any $k$, and thus $y_i \nrightarrow \omega_1$ in $\Omega$, hence $x_i \nrightarrow \omega_1$.*

Topologists were exiled from sequential paradise as soon as arbitrary unions of open sets were allowed to be open, since this allowed uncountability to enter into topology. It is still possible to find salvation, nonetheless. One solution is to remain in a class of topologies which are determined by their convergent sequences.

**Definition.** Let $(X, \Omega)$ be an arbitrary topological space. We say $U \subset X$ is **sequentially open** if every sequence which converges to $x \in U$ is eventually in $U$. We say $C \subset X$ is **sequentially closed** if it contains all sequential limits. A **sequential space** is a space $(X, \Omega)$ where sequentially open sets are open, or equivalently, if all sequentially closed sets are closed.

**Lemma 1.1.** *Every first-countable space is sequential.*

*Proof.* Let $(X, \Omega)$ be a first countable space. Suppose $C$ is a sequentially closed set. Fix some limit point $x$ of $C$. Let $\{V_i\}_{i \in \mathbf{N}}$ be a countable neighbourhood base of $x$. Let $W_1 = V_1$, and $W_{n+1} = W_n \cap V_{n+1}$. Then $\{W_i\}_{i \in \mathbf{N}}$ is a countable neighbourhood base such that $i < j$ implies $W_i \supset W_j$. Since $x$ is a limit point of $C$, $W_i \cap C$ is non-empty, for any $i$. We may therefore define a choice function $s : \mathbf{N} \to C$ such that $s_i \in W_i \cap C$ for any integer $i$. By construction, $s_i \to x$, so $x \in C$. It follows that $C$ is closed. □

**Corollary 1.2.** *Every metric space is sequential.*

**Exercise 1.12.** *Let $X$ be an arbitrary set, and $(X, \Omega)$, $(X, \Psi)$ two sequential spaces. $\Omega$ and $\Psi$ are sequentially equivalent if and only if $\Omega = \Psi$.*

## 1.5 s

Surely this specification must involve the circles we were discussing as our fundamental example. Of course, these are not the only open sets, since the union of two balls is not necessarily an open ball, but we may use these as the fundamental sets from which all open sets are constructed.

The intuitive topology on $\mathbf{R}$ can be generalized to any linearly ordered set. In most cases, we may take the same open intervals. One case is more complicated; if the set contains a minimum or maximum element, then these elements are contained in no open intervals. We fix this by allowing in combination with open sets half closed rays $[-\infty, x)$ and $(x, \infty]$ into the definition of openess. We call the topology defined the order topology on the ordered set.

If $X$ is a topological space, and $Z$ is a subset, we can see intuitively how $Z$ may inherit the notion of space from $X$. We take open sets in $Z$ to be the intersection of open sets in $Z$ with $Z$, and we call this the subspace topology.

As examples, the topology $\mathbf{Z}$ inherits from the order topology on $\mathbf{R}$ the discrete topology (which is why we think of integers as being separated on the real line). Conversely, the set $\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$ inherits a completely different topology (zero is no longer open on its own).

Most of the time, when specifying a set, it is difficult to specify precisely the set of all open sets that define a topology. Since, when $\{C_i\}$ is a indexed set of topology, $\bigcap_{i \in I} C_i$ is also topology, we may, when given a subcollection of the power set of a set, generate a topology on that subset by taking the smallest topology which contains the subcollection. Special collections of subs ets $\mathcal{C}$ with the following properties are of increased importance:

1. Every element in the space is contained in one of the subsets

2. If $x$ is contained in two sets $C_1$ and $C_2$ in $\mathcal{C}$, there is a third set $C_3$ in $\mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Then the topology generated has the property that a set $A$ is open if and only if, for any element $x$ in $A$, there is a set $C$ in $\mathcal{C}$ containing $x$ such that $C \subseteq A$. We call such a collection a basis for the topology. Specific examples include open intervals in $\mathbf{R}$. We say $\mathcal{C}$ covers the topology it generates.

Unlike in linear algebra, a basis for a topology is not unique up to bijection. We cannot even always find a minimal basis for all topologies in terms of containment (consider open balls in $\mathbf{R}^n$). We can only promise that there is a basis with minimum cardinality, which exists due to the well ordering property of cardinal numbers.

**Theorem 1.3.** *Let $X$ be a topological space, and $Y$ a subset with the subspace topology. Then a subset $A$ is closed in $Y$ if and only if $A = B \cap Y$ for a closed set $B$ in $X$.*

*Proof.* Suppose $A = B \cap Y$ as in the theorem's statement. Then $X - B$ is open in $X$, so $(X - B) \cap Y$ is open in $Y$, and this set is just $Y - B$. But this $B \cap Y$, which is the complement of $Y - B$, is closed in $Y$.

Conversely, if $A$ is closed in $Y$, $Y - A$ is open in $Y$, hence $Y - A = V \cap Y$ for some open set $V$ in $X$. Since $V^c \cap Y = A$, we see that $A$ is the intersection of $Y$ with a closed set in $X$. $\square$

# Chapter 2

# Distances

In topology, we escrew the concept of an exact distance – we care only about the distance that lies infinitely small between different objects. Nonetheless, distances are more consistant with out intuitions about space, especially in the context of the euclidean plane and analysis, so it is useful to be able to construct topological space using an abstract definition of distance.

**Definition.** A **Metric** on a set $X$ is a function $d : X \times X \to \mathbf{R}$ such that for any $x, y, z \in X$

1. (Nondegeneracy) $d(x,y) \geqslant 0$, and $d(x,y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x,y) = d(y,x)$.
3. (The Triangle Inequality) $d(x,z) \leqslant d(x,y) + d(y,z)$.

Given a subset $A$ of $X$, we define $d(x,A) = \inf\{d(x,a) : a \in A\}$, and the **diameter** of $A$ to be $\operatorname{diam}(A) = \sup\{d(x,y) : x,y \in A\}$. A set is **bounded** if its diameter is finite. The **open ball** of radius $r$ and center $x$ is $B_r(x) = \{y \in X : d(x,y) < r\}$, and the **closed ball** $D_r(x) = \{y \in X : d(x,y) \leqslant r\}$. If we take the set of all open balls to be the basis of a topological space, we obtain a topological structure – a **metric space**.

**Example.** *The canonical metric on $\mathbf{R}^n$ is the euclidean metric*

$$d(x,y) = \sum_{i=1}^{n} (x_i - y_i)^2$$

*But we may consider many different metrics on $\mathbf{R}^n$.*

$$d(x,y) = \sum_{k=1}^{n} |y_i - x_i|$$

*The open balls of this metric are shaped like diamonds; the metric*

$$d(x,y) = \max_{i=1,\dots,n} |y_i - x_i|$$

*induces balls shaped like squares whose sides are oriented to the axes. The topologies induced by all three metrics are the same.*

**Example.** *On any set $X$, take $d(x,y) = \delta_{x,y}$. The topology induced is discrete.*

**Theorem 2.1.** *Every metric space is normal*

*Proof.* Let $A$ and $B$ be two closed, disjoint sets in a metric space. Consider $U = \{x : d(x,A) < d(x,B)\}$, and $V = \{x : d(x,B) < d(x,A)\}$. Then $U$ and $V$ are disjoint open sets (since $x \mapsto d(x,A)$ is continuous), with $U$ containing $A$ and $V$ containing $B$. □

**Theorem 2.2.** *A subset of a complete metric space is precompact if and only if it is totally bounded.*

*Proof.* Let $X$ be a totally bounded metric space. Let $\{x_i\}$ be a sequence in $X$. Choose a finite covering of $X$ by balls of radius 1. Select a subsequence $x_{1,i}$ of $x_i$ which lies in some specific ball $B_1$. Cover $B_1$ by a finite covering of radius $1/2$, and take a subsequence $x_{2,i}$ of $x_{1,i}$. Proceed inductively, considering subsequences $x_{n,i}$. Then $x_{i,i}$ is a cauchy sequence in $X$, and we have shown $X$ is precompact. □

**Theorem 2.3.** *Let $X$ be a compact metric space, and let $\{U_i\}$ be an open cover. Then the cover admits a* **Lebesgue number** *$\delta$ such that if $diam(Y) < \delta$, then $Y \subset U_i$.*

*Proof.* Let $U_1, \dots, U_n$ be a finite subcover of $X$, and let $C_i = X - U_i$. Define

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

Then $f$ is continuous, and is always positive. Since $f$ is defined on a compact set, it attains a minimum value $\delta > 0$. Suppose $Y$ is a set with $\operatorname{diam}(Y) < \delta$, containing points $y_1, \dots y_n$ in each $C_i$. But then

$$f(y_1) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) = \frac{(n-1)}{n} \delta < \delta$$

A contradiction which proves the claim. $\square$

# Chapter 3

# Constructions

Here we get to the visually interesting part of topology, providing methods to mold and curve the topological structures of your choosing, gluing, stretching, and all other kinds of fun stuff. It will explain how we get from a plane to a torus, or from $\mathbf{R}^2$ to $S^2$. The construction we will make can be shown in a very general manner.

**Definition.** Let $X$ be a topological space, and $\sim$ an equivalence relation on $X$. From this equivalence relation, we form the set $X/\sim$, and consider the projection mapping $\pi$ from $X$ to $X/\sim$. The quotient topology on $X/\sim$ is the coarsest topology that makes $\pi$ continuous, and makes $X/\sim$ the quotient space of $X$ and $\sim$. A set $A$ is open in $X/\sim$ if and only if $\pi^{-1}(A)$ is open in $X$.

If we have a surjective map $f : X \to Y$, where $X$ is some topological space, and $Y$ is any set, then we may construct a topology on $Y$ analogous to the quotient topology. First we consider the fibers of $X$ relative to the mapping $f$, and identify the quotient topology on this set. We obtain a bijective mapping $\overline{f}$. The quotient topology on $Y$ is the topology which makes $\overline{f}$ a homeomorphism. Since, in the context of topology, homeomorphisms preserve all important properties, we may as well consider this definition no different from the definition in terms of equivalence relations.

# Chapter 4

# Algebraic Topology

To verify that two topological spaces are homeomorphic, we need only find a single homeomorphism that connects the two spaces. On the contrary, to verify that two topological spaces are not homeomorphic, we need to somehow show that every function from one space to the other is not a homeomorphism, a computationally intractable problem. One trick we can use to separate topological spaces is to find fundamental topological properties which distinguish two topological spaces. Connectedness, Compactness, and Hausdorffiness are all preserved by homeomorphism, as does the topological properties of subspaces. Nonetheless, sometimes these properties are not enough to distinguish two spaces. This chapter shows a deep technique which is often useful for characterizing spaces.

Consider two functions $f$ and $g$ between topological spaces $X$ and $Y$. Though $f$ might not be equal to $g$, they may be in some sense topologically equal – we may be able to deform one to the other in a continuous fashion. This is a homotopy.

**Definition.** Let $f, g : X \to Y$ be two continuous functions. Define a topology on $Hom(X, Y)$ as a subspace of $Y^X$, which can be viewed as the product topology of $Y$ with itself $Y$ times. Then $f$ and $g$ are homotopic if there exists a path in $Hom(X, Y)$ which connects $f$ to $g$. Alternatively, these two functions are homotopic if there exists a continuous function $F : [0, 1] \times X \to Y$ such that for all $x$, $F(0, x) = f(x)$, and $F(1, x) = g(x)$.

The fact that homotopy is an equivalence relation will allow us to distill functions between spaces to their fundamental properties. We need to specialize our definition for it to be more of more use to us.

**Definition.** Two paths in $X$ are path homotopic if they have the same start and end point, and are homotopic to each other.

Let $f$ and $g$ be two paths in $X$, where the end point of $f$ is the start point of $g$. Then we may compose the two paths to form a new path $f * g$, defined by

$$(f * g)(x) = f(2x) : x \in [0,1/2] g(2x-1) : x \in [1/2,1]$$

By the pasting lemma, this function is a path which connects the start point of $f$ to the end point of $g$. Unfortunately, concatenation is not associative, we do not have that $f*(g*h) = (f*g)*h$. These paths are homotopic to each other, however, and moving to path homotopy classes makes the definition much simpler.

**Theorem 4.1.** *Let $f$ be path homotopic to $f'$, and $g$ path homotopic to $g'$. Then $f * g$ is homotopic to $f' * g'$.*

*Proof.* Let $F$ be the path homotopy from $f$ to $f'$, and $G$ the path homotopy from $g$ to $g'$. Define a homotopy $H$ between $f * g$ and $f' * g'$ by

$$H(\cdot, y) = F(\cdot, y) * G(\cdot, y)$$

More specifically

$$H(x,y) = \begin{cases} F(2x,y) & \text{if } x \in [0,1/2] \\ G(2x-1,y) & \text{if } x \in [1/2,1] \end{cases}$$

The pasting lemma guarentees this function is a new homotopy. $\square$

We now consider homotopy classes of paths, so when we talk about a path $f$, we are really talking about all paths homotopic to $f$.

**Theorem 4.2.** $[f] * ([g] * [h]) = ([f] * [g]) * [h]$.

# Chapter 5

# Compactification

In the following, we consider only locally connected, locally compact, connected Hausdorff spaces.

**Definition.** An **end** of a $X$ is a map $\varepsilon$ defined on compact subsets of $X$, such that $\varepsilon(C)$ is a connected component of $X - C$ for each compact $C$, and $C \subset D$ implies $\varepsilon(D) \subset \varepsilon(C)$. Denote the set of all ends on $X$ by $\mathcal{E}(X)$.

**Definition.** The end compactification of a space $X$ is the space $\mathbf{X} = X \cup \mathcal{E}(X)$, where a set is open if it is open in $X$, or if it is of the form $U_{\varepsilon(C)} := \varepsilon(C) \cup \{\varepsilon' \in \mathcal{E}(X) : \varepsilon'(C) = \varepsilon(C)\}$, where $C$ is compact.

**Lemma 5.1.** *The end compactification is Hausdorff.*

*Proof.* If $\varepsilon, \varepsilon \in \mathcal{E}(X)$ are two unequal ends, then there is some compact set $C$ for which $\varepsilon(C) \neq \varepsilon'(C)$. But then $U_{\varepsilon(C)}$ and $U_{\varepsilon'(C)}$ are disjoint. If $x, y \in X$, then they can surely be separated in the end compactification because $X$ is Hausdorff. If $x$ is a point in $X$, and $\varepsilon$ is an end, then because $X$ is locally compact, $x$ possesses a precompact neighbourhood $V$ of $x$, and then $U_{\varepsilon(\overline{V})}$ is disjoint from $V$. $\square$

**Lemma 5.2.** **R** *has two ends, the 'left' and 'right' ends.*

*Proof.* $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n, n]$, and each $[-n, n]$ is compact. We contend every end $\varepsilon$ on **R** is defined by its action on $[-n, n]$. If $C$ is any compact set, then $C$ is contained in an interval of the form $[-n, n]$. Clearly, $\varepsilon(C)$ must be the unique connected extension of $\varepsilon([-n, n])$, since $\varepsilon(C) \supset \varepsilon([-n, n])$. In fact, $\varepsilon$ is defined solely by its action on $[-1, 1]$, since $[-1, 1] \subset [-2, 2] \subset \ldots$. Since the two choices $\varepsilon([-x, x]) = (-\infty, x)$ and $\varepsilon([-x, x]) = (x, \infty)$ constitute ends, the space has two ends. □

In general, if a space $X$ can be written as $C_1 \subset C_2 \subset \cdots \to X$, where each $C_i$ is compact, then all ends are defined by their action on $C_1$. We shall call such a space **hemicompact**. Not all choices of components of $X - C_1$ will work, however.

**Lemma 5.3.** *The end compactification of a hemicompact space is compact.*

*Proof.* s □

# Chapter 6

# Compactification

In the following, we consider only locally connected, locally compact, connected Hausdorff spaces.

**Definition.** An **end** of a $X$ is a map $\varepsilon$ defined on compact subsets of $X$, such that $\varepsilon(C)$ is a connected component of $X - C$ for each compact $C$, and $C \subset D$ implies $\varepsilon(D) \subset \varepsilon(C)$. Denote the set of all ends on $X$ by $\mathcal{E}(X)$.

**Definition.** The end compactification of a space $X$ is the space $\mathbf{X} = X \cup \mathcal{E}(X)$, where a set is open if it is open in $X$, or if it is of the form $U_{\varepsilon(C)} := \varepsilon(C) \cup \{\varepsilon' \in \mathcal{E}(X) : \varepsilon'(C) = \varepsilon(C)\}$, where $C$ is compact.

**Lemma 6.1.** *The end compactification is Hausdorff.*

*Proof.* If $\varepsilon, \varepsilon \in \mathcal{E}(X)$ are two unequal ends, then there is some compact set $C$ for which $\varepsilon(C) \neq \varepsilon'(C)$. But then $U_{\varepsilon(C)}$ and $U_{\varepsilon'(C)}$ are disjoint. If $x, y \in X$, then they can surely be separated in the end compactification because $X$ is Hausdorff. If $x$ is a point in $X$, and $\varepsilon$ is an end, then because $X$ is locally compact, $x$ possesses a precompact neighbourhood $V$ of $x$, and then $U_{\varepsilon(\overline{V})}$ is disjoint from $V$. $\square$

**Lemma 6.2.** $\mathbf{R}$ *has two ends, the 'left' and 'right' ends.*

*Proof.* $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n,n]$, and each $[-n,n]$ is compact. We contend every end $\varepsilon$ on $\mathbf{R}$ is defined by its action on $[-n,n]$. If $C$ is any compact set, then $C$ is contained in an interval of the form $[-n,n]$. Clearly, $\varepsilon(C)$ must be the unique connected extension of $\varepsilon([-n,n])$, since $\varepsilon(C) \supset \varepsilon([-n,n])$. In fact, $\varepsilon$ is defined solely by its action on $[-1,1]$, since $[-1,1] \subset [-2,2] \subset \dots$. Since the two choices $\varepsilon([-x,x]) = (-\infty,x)$ and $\varepsilon([-x,x]) = (x,\infty)$ constitute ends, the space has two ends. $\square$

In general, if a space $X$ can be written as $C_1 \subset C_2 \subset \cdots \to X$, where each $C_i$ is compact, then all ends are defined by their action on $C_1$. We shall call such a space **hemicompact**. Not all choices of components of $X - C_1$ will work, however.

**Lemma 6.3.** *The end compactification of a hemicompact space is compact.*

*Proof.* s $\square$