Logarithmic Improvements in L^p Bounds for Eigenfunctions at the Critical Exponent

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Given a compact Riemannian manifold M, we can consider the Laplace-Beltrami operator on M. This operator breaks $L^2(M) = \bigoplus_{\lambda \in \Lambda_M} E_\lambda$, where Λ_M is a discrete set of non-negative numbers, $\dim(E_\lambda) < \infty$, $E_\lambda \subset C^\infty(M)$, and for each $u \in E_\lambda$, $\Delta u = -\lambda^2 u$. A natural question is to study the asymptotic behaviour of elements of E_λ as $\lambda \to \infty$. In particular, in this talk, we focus on determining the asymptotic behaviour of the L^p norms of functions in E_λ, i.e. the study of the quantities

$$A(\lambda, p) = \sup_{f \in E_\lambda} \frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}}.$$

Understanding the quantities $A(\lambda, p)$ is a natural analogue of the Tomas-Stein theorem, since that theorem is equivalent to establishing that for any function $f \in L^2(\mathbf{R}^d)$ with $\Delta f = -\lambda^2 f$,

$$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p,d)},$$

where

$$\delta(p,d) = \begin{cases} d(\frac{1}{2} - \frac{1}{p}) - 1/2 & : p_c \leq p \leq \infty \\ \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & : 2 \leq p \leq p_c \end{cases},$$

where $p_c = 2(d+1)/(d-1)$ is the *critical exponent* for the Tomas-Stein theorem. In particular, at the critical exponent we have $\delta(p_c, d) = 1/p_c$:

In 1988, Sogge showed that $A(\lambda, p) \lesssim \lambda^{\delta(p,d)}$ on a general compact Riemannian manifold, actually showing the stronger conclusion that the operator norm of the *spectral band projection operators*

$$\chi_{[\lambda,\lambda+1]} f = \sum_{\lambda \leq \lambda' \leq \lambda+1} f_{\lambda'}$$

from $L^2(M)$ to $L^p(M)$ has magnitude $O(\lambda^{\delta(p,d)})$. This is more powerful than an L^p bound on an individual eigenfunction because the Weyl law says that the dimension of $\bigoplus_{\lambda \leq \lambda' \leq \lambda + O(1)} E_{\lambda'}$ is $\Theta(\lambda^{d-1})$, and thus the orthogonal spaces $\{E_\lambda\}$ 'remain orthogonal' in $L^p(M)$ to some capacity, i.e.

$$\left\| \sum_{\lambda \leq \lambda' \leq \lambda+1} a_{\lambda'} e_{\lambda'} \right\|_{L^p(M)} \lesssim \lambda^{\delta(p,d)} \left(\sum_{\lambda \leq \lambda' \leq \lambda+1} |a_{\lambda'}|^2 \right)^{1/2}.$$

The *exponent* here is tight for any manifold M, and any exponent p, i.e.

$$\|\chi_{[\lambda,\lambda+1]}\|_{L^2(M) \to L^p(M)} \gtrsim \lambda^{\delta(p,d)}.$$

However, that does *not* mean that the bound $A(\lambda, p) \lesssim \lambda^{\delta(p,d)}$ is tight in general, since an extremizer for the operator norm above could be composed of a sum of eigenvalues corresponding to several different clustered 'resonant' eigenvalues.

The bound *is* sharp for the sphere S^d. The eigenfunctions here are the spherical harmonics. The decomposition of $L^2(M)$ is into the eigenspaces E_{λ_n}, where $\lambda_n = \sqrt{n(d+n-1)}$. For our purposes, there are two important elements of E_{λ_n}, i.e. the *high weight harmonic*

$$H_n(x) = (x_1 + ix_2)^d$$

and the *zonal harmonic* $Z_n(x) = L_n(x_n)$, where L_n is the Legendre polynomial of degree n. The high weight harmonic is concentrated near the equator $x_3 = \cdots = x_d = 0$, which means the harmonic should have large L^p norm for low L^p because most points on a sphere are close to the equator. The zonal harmonic is concentrated near the poles of the sphere, which should give large L^p norm for large p. And indeed, we find that $\|Z_n\|_{L^p(S^n)} \gtrsim \lambda^{\delta(n,p)}$ for $p_c \leq p \leq \infty$, and $\|H_n\|_{L^p(S^n)} \gtrsim \lambda^{\delta(n,p)}$ for $2 \leq p_c \leq p_c$.

On the other hand, the bound is *not* tight on the torus $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Indeed, the eigenspaces here are E_n, where E_n is the linear span of the exponentials $e_m(x) = e^{imx}$, where $|m| = n$. The eigenspace E_n therefore has dimension equal to the number of representations of n^2 as a sum formed from the squares of d integers, and the number of solutions is, heuristically (by Vinogradov type results), $O(n^{d-2})$. It therefore follows from the triangle inequality that

$$\begin{aligned} \|\sum a_m e_m\|_{L^\infty(\mathbf{T}^d)} &\lesssim \sum |a_m| \|e_m\|_{L^\infty(\mathbf{T}^d)} \\ &\lesssim n^{\frac{d-2}{2}} \left(|a_m|^2\right)^{1/2} \\ &\lesssim n^{\frac{d-2}{2}} \|\sum a_m e_m\|_{L^2(\mathbf{T}^d)}. \end{aligned}$$

Thus

$$A(\lambda, \infty) \lesssim \lambda^{\delta(\lambda,\infty) - \frac{1}{2}},$$

i.e. we get an improvement in the exponent, and interpolation gives an improvement in the exponent of the form

$$A(\lambda, p) \lesssim \lambda^{\delta(p,d) - \varepsilon(p,d)}$$

where $\varepsilon(p,d) > 0$ for all $2 < p < \infty$. So in this case, the result is certainly *not tight*.

In fact, we should expect that in general, this bound is *not* tight. In Zhongkai's talk on (Sogge, Zelditch, 2009), and in Ben and Jaume's talk on (Sogge, Zelditch), we saw that for a generic manifold M, we should expect that $A(\lambda,p) = o(\lambda^{\delta(p,d)})$, namely, we should only expect the bound $A(\lambda,p) \lesssim \lambda^{\delta(p,d)}$ to be tight if there exists many recurrent geodesics on M.

On negatively curved compact manifolds, recurrent geodesics are quite rare. For instance, on the spaces of the form $\mathbf{H}/\Gamma$ studied by James and Rajula (Marshall, 2016), there are only countably many periodic geodesics. And on a negatively curved, simply connected compact manifold, there do not exist *any* geodesics that return to their original position. Indeed, if we take three points on a geodesic that returns to the same point it started at, these three points form a triangle with total angle greater than 2π, which is impossible by the Gauss-Bonnet theorem. Thus it makes sense to make improvements to the bound above on a general compact, negatively curved manifold.

In Hongki's talk (Sogge, 2017), we saw that on a negatively curved manifold, we could obtain the improvement

$$A(\lambda,p) \lesssim \lambda^{\delta(p,d)}/(\log\log\lambda)^{\sigma(p,d)}$$

for any $2 < p < \infty$. In Chamsol's talk (Blair, Sogge, 2018), we saw that some Kakeya-Nikodym estimates yielded the logarithmic improvements $A(\lambda,p) \lesssim \lambda^{\delta(p,d)}/(\log\lambda)^{\sigma(p,d)}$ for $2 < p < p_c$. The goal of this talk is to discuss the methods of (Blair, Sogge, 2019), which allow us to obtain such a bound for $p = p_c$, and thus to obtain that $\sigma(p,d) > 0$ for all $2 < p < \infty$. We will do this by obtaining bounds of the form

$$\|\chi_{[\lambda,\lambda+1/T]}\|_{L^2(M)\to L^{p_c}(M)} \lesssim \lambda^{1/p_c}/(\log\lambda)^{\sigma(p_c,d)}$$

where $T \sim \log\lambda$.

To gain intuition about the intricacies of the critical exponent, let us compute some examples that show this result is tight, let us consider an analogous problem in Euclidean setting, namely, bounding the operators

$$\chi_{[\lambda,\lambda+1/T]}f = \int_{\lambda\le|\xi|\le\lambda+1} \widehat{f}(\xi)e^{2\pi i\xi\cdot x}.$$

from $L^p(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$. Here simply rescaling the variables shows that we should expect to have

$$\|\chi_{[\lambda,\lambda+1/T]}\|_{L^p(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \lesssim (1/T)\lambda^{\delta(p,d)}$$

In particular, if $T = \log\lambda$, the operator norm is $O(\lambda^{\delta(p,d)}/\log\lambda)$, which hints that this is the value of T we should aim for if we expect to get logarithmic improvement in L^p bounds for eigenfunctions.

1 Some Familiar Tools

Set $P = \sqrt{-\Delta}$. As with many talks in this summer school, To begin our analysis, we introduce a very useful tool to understand a multiplier operator of the form

$$mf = m\,(P)\,f = \sum_{\lambda \in \Lambda_M} m(\lambda) f_\lambda,$$

This tool is just the Fourier inversion formula in disguise, i.e. we can write

$$m\,(P)\,f = \int_{-\infty}^{\infty} \widehat{m}(t) e^{2\pi i t P} f \; dt.$$

The function $u = e^{2\pi i t P} f$ solves the *half wave equation* $\partial_t u = 2\pi i P u$ with initial conditions f, and so one can understand the behaviour of the multiplier $m(P)$ by studying the behaviour of the half wave equation at times where the mass of $\widehat{m}$ is concentrated.

The uncertainty principle hints that to understand $\chi_{[\lambda,\lambda+1/T]}(P)$, we must understand the behaviour of the wave equation on times $|t| \lesssim T$. We can make this intuition even more precise by replacing the multiplier $\chi_{[\lambda,\lambda+1/T]}(P)$ with the operator $\rho_\lambda(\lambda') = \rho(T(\lambda - \lambda'))$ whose Fourier transform is more well behaved, for some fixed even function $\rho \in \mathcal{S}(\mathbf{R})$, with $\rho(0) = 1$ and with $\text{supp}(\widehat{\rho})$ on $1/4 \leq |t| \leq 1/2$. Then $|m_\lambda| \lesssim |\rho_\lambda|$, which implies that

$$\|m_\lambda\|_{L^2(M) \to L^{p_c}(M)} \lesssim \|\rho_\lambda\|_{L^2(M) \to L^{p_c}(M)}.$$

Thus it suffices to bound the multiplier operators ρ_λ, whose Fourier transform is supported on $T/4 \leq |t| \leq T/2$, and it is for these times that we must understand the solution to the wave equation.

The best control of the wave equation occurs for small times, when we have a *parametrix* for the half-wave equation. More precisely, if (a, U) is a coordinate chart, and $K \subset U$ is compact, then there exists T_0 such that for $|t| \leq T_0$, we have a *parametrix* for the wave equation for initial data on K. Namely, we can write $e^{2\pi i t P}$ as an oscillatory integral, namely for $f \in L^2(M)$ with $\text{supp}(f) \subset K$,

$$e^{2\pi i t P} f \approx \int \int v(t, x, y, \xi) e^{2\pi i \Phi(t,x,y,\xi)} f(y) \; dy \; d\xi$$

where:

- v is a symbol of order zero such that $\text{supp}_x(v)$ is a compact subset of U.
- If $\nabla_\xi \Phi(t, x, y, \xi) = 0$, then

  $$\nabla_y \Phi(t, x, y, \xi) = \xi \quad \text{and} \quad \nabla_x \Phi(t, x, y, \xi) = \exp_y(t\xi).$$

  Thus stationary phase hints that the mass of the wave equation travels microlocally along the geodesics of the manifold M.

This approximation only holds for small times. The fact that $\{e^{2\pi i t P}\}$ is a semi-group means we can compose these operators to give integral representations for slightly larger times, but the 'fuzz' in the approximation grows worse and worse as we continue to compose these operators. There is a heuristic that suggests that his fuzz starts to dominate after the *Ehrenfast time* for a general geodesic flow; given f with $\text{supp}_\xi(f) \subset \{|\xi| \sim \lambda\}$, the fuzz starts to dominate past time $\log \lambda$. Thus we see that we are very close to fully exploiting the parametrix given here using the method above.

2 Microlocal Decomposition

On $\mathbf{R}^d$, for each $\theta \in (0,1)$, it is natural to decompose $\mathbf{R}^n$ into $N = O(1/\theta)$ disjoint conic sectors $\Gamma_1, \dots, \Gamma_N$ with aperture $O(\theta)$, and consider decompositions of the form $f = f_1 + \dots + f_N$, where $\widehat{f_i}$ is supported on Γ_i for each i. Our goal is to consider an analogous decomposition for functions on a Riemannian manifold, in a way that respects the geodesics, at least locally in space.

Let $\chi_t : T^*M \to T^*M$ denote the geodesic flow. Let us work in some local coordinates $(x_1, x') \in \mathbf{R} \times \mathbf{R}^{d-1}$. If $x_0 \in U \subset \mathbf{R}^{d-1}$, and $W \subset S^{d-1}_\xi$ is a small enough open neighborhood around $(1,0)$, then there is an open set $V \subset \mathbf{R}^d$ containing $\{0\} \times U$ such that there is a map $\Psi : V \times W \to U$, where $\Psi(x,\omega)$ is the point on U such that the geodesic passing through x in the direction ω passes through $\{0\} \times U$ in the direction $\Psi(x,\omega)$.

Let us now consider a small enough conic neighborhood Γ of some vector $\xi \in S^{d-1}$. Consider a maximal $O(\lambda^{-1/8})$ separated family of vectors ν on Γ, and consider a partition of unity $\{\psi_\nu\}$ on Γ such that the support of the functions ψ_ν is contained on a diameter $O(\lambda^{-1/8})$ cap centered at ν. Define a symbol $q_{\lambda,\nu}$ supported on a small neighbodhood of V such that on V,

$$q_{\lambda,\nu}(x,\xi) \approx \beta_\nu(\Psi(x, \xi/|\xi|_g))$$

Thus $q_{\lambda,\nu}$ is supported on a $O(\lambda^{-1/8})$ neighborhood of the geodesic passing through ν, at least on the set V, the collection $\{q_{\lambda,\nu}\}$ forms a partition of unity on V. Most importantly, the functions $\{q_{\lambda,\nu}\}$ are invariant under the geodesic flow. We also have bounds of the form

$$|\nabla_x^n \nabla_\xi^m q_{\lambda,\nu}(x,\xi)| \lesssim_{n,m} \lambda^{(1/8)(n+m)}.$$

Given these symbols, we define the semiclassical pseudodifferential operators Q_ν, which have kernel

$$\lambda^d \int q_{\lambda,\nu}(x,\xi) e^{2\pi i \lambda \xi \cdot (x-y)}\, d\xi.$$

In the remaining part of this lecture, we will see how to individually bound terms that occur in our understanding of the equation once decomposed using the operators $\{Q_\nu\}$. It is also very important that these decompositions can be recombined in an optimal way. On Thursday, Hong will show that these operators satisfy almost orthogonality conditions that will enable one to recombine these terms, and thus prove the required result.

3 Bounding Individual Terms

Our goal is now to understand the operators $\{\rho_\lambda\}$ as $\lambda \to \infty$. Using the decompositions we just introduced, we write

$$\rho_\lambda = \sum_\nu (q_{\lambda,\nu} \circ \rho_\lambda).$$

For now, we understand each of these terms individually, without putting them back together. Thus we wish to understand an element of the form $q_\lambda \circ \rho_\lambda$, and in particular, we focus on obtaining estimates of the form

$$\|q_\lambda \circ \rho_\lambda\|_{L^{p_c,\infty}(M)} \lesssim \frac{\lambda^{1/p_c}}{(\log \lambda)^\varepsilon}.$$

Thus fix $\alpha > 0$, and consider a set $A \subset M$ such that for $x \in A$,

$$|(q_\lambda \circ \rho_\lambda) f(x)| \geq \alpha.$$

Our goal is to show that

$$\alpha |A|^{1/p_c} \lesssim \frac{\lambda^{1/p_c}}{(\log \lambda)^{\varepsilon/p_c}}.$$

Set

$$r = \lambda \alpha^{\frac{-4}{d-1}} (\log \lambda)^{-\frac{2}{d-1}}$$

Hongki's talk actually already addresses the case whe $\alpha \geq \lambda^{\frac{d-1}{4}+\frac{1}{8}}$, so it suffices to consider $\alpha < \lambda^{\frac{d-1}{4}+\frac{1}{8}}$. In this case, $r > \lambda^{-\frac{1}{2(d-1)}} (\log \lambda)^{-\frac{2}{d-1}} \gg \lambda^{-3/4}$.

Without loss of generality, throwing away all but a fixed percentage, say 10% of A, we may assume without loss of generality that we can write $A = A_1 \cup \cdots \cup A_N$, where each of the sets $\{A_i\}$ has diameter at most r, and $d(A_i, A_j) \gtrsim r$ for $i \neq j$.

Write $a_i = \chi_{A_i} \cdot \mathrm{Sgn}(Q_\lambda)$. Then

$$\alpha |A| \leq \left| \int_M (Q_\lambda \circ \rho_\lambda) f \cdot \mathrm{Sgn}(Q_\lambda) \chi_A \right| \lesssim \left(\int_M \left(\sum_{i=1}^N (\rho_\lambda \circ Q_\lambda^*) a_i \right)^2 \right)^{1/2}.$$

Then

$$\begin{aligned} \alpha^2 |A|^2 &\leq \sum_{i=1}^N \int |(\rho_\lambda \circ Q_\lambda^*) a_i|^2 + \sum_{i \neq j} \int \left(Q_\lambda \circ \rho_\lambda^2 \circ Q_\lambda^* \right) a_i \cdot a_j \\ &= I + II. \end{aligned}$$

Let us address each of these terms individually.

The decay estimates on the symbols defining the pseudodifferential operators $\{Q_\lambda^*\}$ imply that the kernels start to rapidly decay only outside of a $O(\lambda^{-7/8})$ neighbourhood of the diagonal. In Hongki's talk (Sogge, 2017), it was / will be shown that for any ball B of radius r,

$$\|\rho_\lambda\|_{L^2(B)\to L^2(M)} \lesssim r^{1/2}$$

This is used to bound equivalent expressions like that in I, except that Q_λ does not appear in the equation. But the same method there yields that

$$I \lesssim r \sum_{i=1}^N \int |Q_\lambda^* a_i|^2 \lesssim r|A| = \lambda \alpha^{-\frac{4}{d-1}} (\log \lambda)^{-\frac{2}{d-1}} |A|.$$

because the kernel of Q_λ^* begins to rapidly decay outside a $O(\lambda^{-7/8})$ neighborhood of the diagonal, and by assumption, $r \gg \lambda^{-3/4}$.

The analysis of II is where the negative curvature comes into play, and we use the fact that Q_λ is invariant under the geodesic flow. If $K(w,z)$ is the integral kernel of $Q_\lambda \circ \rho_\lambda^2 \circ Q_\lambda^*$, then our goal is to obtain a bound of the form

$$|K(w,z)| \lesssim \frac{1}{T} \left(\frac{\lambda}{\lambda^{-1} + d_g(w,z)} \right)^{\frac{d-1}{2}} + c(\lambda) \lambda^{\frac{d-1}{2}},$$

where $c(\lambda) \to 0$ at least as fast as some small power of $(\log \lambda)^{-1}$, but not faster than $(\log \lambda)^{-1}$ itself. A trivial bound thus implies that

$$\begin{aligned} II &\lesssim \left(\sup_{i \neq j} \sup_{w \in A_i, z \in A_j} |K(w,z)| \right) \sum_{i \neq j} \|a_i\|_{L^1} \|a_j\|_{L^1} \\ &\lesssim \left(\sup_{i \neq j} \sup_{w \in A_i, z \in A_j} |K(w,z)| \right) |A|^2 \\ &\lesssim (C_0 \alpha^2 + \lambda^{\frac{d-1}{2}} c(\lambda)) |A|^2, \end{aligned}$$

where we can make C_0 arbitrarily small by changing the separation parameter of the sets $\{A_i\}$. Combining these results gives that

$$\alpha^2 |A|^2 \lesssim \lambda \alpha^{-\frac{4}{d-1}} (\log \lambda)^{-\frac{2}{d-1}} |A| + (C_0 \alpha^2 + \lambda^{\frac{d-1}{2}} c(\lambda)) |A|^2,$$

and thus that for $\lambda^{\frac{d-1}{4}} c(\lambda)^{1/2} \lesssim \alpha \leq \lambda^{\frac{d-1}{4} + \frac{1}{8}}$,

$$\alpha |A|^{1/p_c} \lesssim \lambda^{1/p_c} (\log \lambda)^{-\frac{1}{d+1}}.$$

This completes the proof for the interesting values of α.

Now let's argue why we have estimates of the form

$$|K(w,z)| \lesssim \frac{1}{T} \left(\frac{\lambda}{\lambda^{-1} + d_g(w,z)} \right)^{\frac{d-1}{2}} + c(\lambda) \lambda^{\frac{d-1}{2}}.$$

Taking Fourier transforms,

$$\begin{aligned} Q_\lambda \circ \rho_\lambda^2 \circ Q_\lambda^* &= \frac{1}{T} \int \widehat{\rho^2}(t/T) e^{-2\pi i \lambda t} (Q_\lambda \circ e^{2\pi i t P} \circ Q_\lambda^*) \, dt \\ &\approx \frac{1}{T} \int_{-T}^{T} \widehat{\rho^2}(t/T) e^{-2\pi i \lambda t} (Q_\lambda \circ \cos(2\pi t P) \circ Q_\lambda^*) \, dt \end{aligned}$$

Thus we can write $K(w,z) = A(w,z) + B(w,z)$, where

$$A(w,z) = \frac{1}{T} \int_{-T}^{T} (1-\beta)(t) \left\{ \widehat{\rho} \right\}^2 (t/T) e^{2\pi i \lambda t} (Q_\lambda \circ \cos(tP) \circ Q_\lambda^*)(w,z) \, dt,$$

$$B(w,z) = \frac{1}{T} \int_{-T}^{T} \beta)(t) \left\{ \widehat{\rho} \right\}^2 (t/T) e^{2\pi i \lambda t} (Q_\lambda \circ \cos(tP) \circ Q_\lambda^*)(w,z) \, dt,$$

where $\beta(t) = 1$ for small t, and is compactly supported.

Now we have to employ the *Hadamard parametrix* for the wave equation, since $u = \cos(tP)f$ gives a solution to the wave operator. This gives that the kernel of $\cos(tP)$ is approximately equal to a constant multiple of (for times less than the injectivity radius of the manifold M)

$$|t| \cdot \zeta(x,y)(d_g(x,y)^2 - t^2)^{-\frac{n-3}{2}-1},$$

where

$$\frac{dV_g}{dx} = \zeta(x,y)^{-2}.$$

In order to use this kernel for *all times*, we replace M with a universal covering space $\tilde{M}$, and fix a universal domain $D \subset \tilde{M}$, so that we have a map $z \mapsto \tilde{z}$ from M to D. The operators Q_λ and Q_λ^* all extend to $\tilde{M}$ be employing periodicity, $\cos(tP)$ lifts to $\cos(t\tilde{P})$, and we have

$$\cos(tP)(w,z) = \sum_\alpha \cos(t\tilde{P})(\tilde{w}, \alpha(\tilde{z})).$$

By finite speed of propogation, it suffices to understand the deck transformations α with $d_g(\tilde{w}, \alpha) \lesssim T$. Now consider the operator

$$\tilde{A}_0(\tilde{x}, \tilde{y}) = \frac{1}{T} \int_{-T}^{T} (1-\beta)(t) \left(\widehat{\rho} \right)^2 (t/T) e^{2\pi i \lambda t} \cos(t\tilde{P})(\tilde{x}, \tilde{y}) \, dt,$$

Then

$$A(w,z) = \sum_\alpha U_\alpha(\tilde{w}, \tilde{z}),$$

where

$$U_a(\tilde{w}, \tilde{z}) = \int_{\alpha(D)} \int_D Q_\lambda(\tilde{w}, \tilde{z}) \tilde{A}_0(\tilde{x}, \tilde{y}) Q_\lambda^*(\tilde{y}, \alpha^{-1}(\tilde{z})) \, d\tilde{x} d\tilde{y}$$

Applying the Hadamard parametrix in $\mathbf{R}^d$, and then stationary phase gives that

$$\tilde{A}_0(\tilde{x},\tilde{y}) \approx \frac{\lambda^{\frac{d-1}{2}}}{T d_g(\tilde{x},\tilde{y})^{\frac{d-1}{2}}} \sum_{\pm} e^{\pm 2\pi i \lambda d_g(x,y)} \zeta(\tilde{x},\tilde{y}) a_{\lambda,\pm}(d_g(\tilde{x},\tilde{y})).$$

for some symbol $a_{\lambda,\pm}$ of order zero which vanishes for $d_g(\tilde{x},\tilde{y}) \gtrsim T$ and for $d_g(\tilde{x},\tilde{y}) \lesssim 1$, and *Since M has nonpositive curvatures*, the Günther comparison theorem says that if the sectional curvatures of M are strictly negative, and bounded from above by $-\kappa^2$, then

$$\zeta(\tilde{x},\tilde{y}) \lesssim \exp\left(-\frac{\kappa(d-1)}{2} d_g(\tilde{x},\tilde{y}) \right)$$

This is because the volumes of balls *increases exponentially* in a negatively curved space.

We find that

$$U_a(\tilde{w},\tilde{z}) = \sum_{\pm} \frac{\lambda^{\frac{5d-1}{2}}}{T} \int e^{2\pi i \lambda \phi_{\pm}(\tilde{w},\tilde{x},\tilde{y},\tilde{z},\eta,\zeta)} q_\lambda(\tilde{w},\tilde{x},\eta) \zeta(\tilde{x},\tilde{y}) a_{\pm,\lambda}(d_g(\tilde{x},\tilde{y})) q_\lambda^*(\tilde{y},\alpha^{-1}(\tilde{z}),\eta)$$

where

$$\phi_{\pm} = (\tilde{w}-\tilde{x})\cdot\eta \pm d_g(\tilde{x},\tilde{y}) + (\tilde{y}-\alpha^{-1}(\tilde{z}))\cdot\xi.$$

Applying stationary phase again shows that for $d_g(\alpha,w) \lesssim T$,

$$|U_\alpha(\tilde{w},\tilde{z})| \lesssim \frac{\lambda^{\frac{d-1}{2}}}{T} (\zeta(\tilde{w},\alpha(\tilde{z})) + \lambda^{-2})(1+d_g(\tilde{w},\alpha(\tilde{z})))^{-\frac{d-1}{2}}.$$

But we haven't used the fact that the $\{q_\lambda\}$ are concentrated on tubes yet. To do this, we use the fact that the terms are negligible unless things are recurrent, and most of the trms aren't recurrent because of negative curature again.