Radial Multipliers

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March 9, 2021

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Chapter 1

Heo, Nazarov, and Seeger

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's paper 2011 *Radial Fourier Multipliers in High Dimensions* [1]. Recall that if $m \in L^\infty(\mathbf{R}^d)$ is the symbol of a Fourier multiplier operator T_m, then we let $\|m\|_{M^p(\mathbf{R}^d)}$ denote the operator norm of T_m from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. The goal of this paper is to show that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero, then

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2(d-1)}{d+1}\right),$$

where the implicit constant depends on p and η. Since

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\mathrm{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}$$

we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family $\{\mathrm{Dil}_t \eta\}$.

In Garrigós and Seeger's 2007 paper *Characterizations of Hankel Multipliers*, it was proved that if $\eta \in \mathcal{S}(\mathbf{R}^d)$ is a nonzero, radial Schwartz function, then

$$\|m\|_{M^p_{\mathrm{rad}}(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2d}{d+1}\right),$$

where $M^p_{\mathrm{rad}}(\mathbf{R}^d)$ is the operator norm of T_m from $L^p_{\mathrm{rad}}(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. Thus, at least in the range $p \in (1, 2(d-1)/(d+1))$, boundedness of T_m on radial functions is equivalent to boundedness on all functions.

1.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a convolution inequality, which can also be used to prove local smoothing results for the wave equation.

Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial Schwartz function $\psi \in \mathcal{S}(\mathbf{R}^d)$. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $f_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$, which we view as a smoothened indicator function on a thickness ≈ 1 annulus of radius r centered at x. Our goal is to prove the following inequality.

Lemma 1.1. *For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

Why is Lemma 1.1 useful? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier given by some function $\tilde{m} : [1,\infty) \to \mathbf{C}$, and we set $a_r(x) = \tilde{m}(r) f(x)$ for some $f : \mathbf{R}^d \to \mathbf{C}$. Then it is simple to check that

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr = K * \psi * f$$

where $K(x) = |x|^{1-d} m(x)$. In this setting, Lemma 1.1 says that

$$\|K * \psi * f\|_{L^p(\mathbf{R}^d)} \lesssim \|m\|_{L^p(\mathbf{R}^d)} \|f\|_{L^p(\mathbf{R}^d)},$$

which is clearly related to the convolution bound we want to show if $\psi = \widehat{\eta}$, provided that we are dealing with a multiplier supported away from the origin. To understand Lemma 1.1 it suffices to prove the following discretized estimate.

Theorem 1.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1,\infty)$, which is* discretized *in the sense that $|(x_1,r_1)-(x_2,r_2)| \geqslant 1$ for each distinct pair (x_1,r_1) and (x_2,r_2) in $\mathcal{E}$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r)\in\mathcal{E}} |a_r(x)|^p r^{p-1} \right)^{1/p},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independant of $\mathcal{E}$.

Proof of Lemma 1.1 from Lemma 1.2. For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$,

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} = \int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx})\, dr\, dx$$

Minkowski's inequality thus implies that

$$\begin{aligned}
\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} &\leqslant \int_{[0,1)^d} \int_0^1 \| \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} \mathrm{Trans}_{n,m}(af_{rx}) \|_{L^p(\mathbf{R}^d)}\, dr\, dx \\
&\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} |a_r(x)|^p r^{p-1} \right)^{1/p} dr\, dx \\
&\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} |a_r(x)|^p r^{p-1}\, dr\, dx \right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.
\end{aligned}$$

□

Lemma 1.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r)\in\mathcal{E}} a_r(x) f_{xr}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 1.2 is implied by the following Lemma.

Lemma 1.3. *For any $1 \leqslant p < 2(d-1)/(d+1)$ and $k \geqslant 1$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim 2^{k(d-1)} \#(\mathcal{E}_k)^{1/p} = 2^k \cdot (2^{k(d-p-1)} \#(\mathcal{E}_k))^{1/p}.$$

Proof of Lemma 1.2 from Lemma 1.3. Applying a dyadic interpolation result (Lemma 2.2 of the paper), Lemma 1.3 implies that

$$\| \sum_{(x,r)\in\mathcal{E}} f_{xr} \|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum 2^{kp} 2^{k(d-p-1)} \#(\mathcal{E}_k) \right)^{1/p} = \left(\sum 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}$$

This is a restricted strong type bound for Lemma 1.2, which we can then interpolate. □

If ψ is compactly supported, and r is sufficiently large depending on the size of this support, then f_{xr} is supported on an annulus with centre x, radius r, and thickness $O(1)$. Thus $\|f_{xr}\|_{L^p(\mathbf{R}^d)} \sim r^{(d-1)/p}$, which implies that

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \|_{L^p(\mathbf{R}^d)} \gtrsim 2^{k(d-1)/p} \#(\mathcal{E}_k)^{1/p}.$$

Thus this bound can only be true if $p \geqslant 1$, and becomes tight when $p = 1$, where we actually have

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr} \|_{L^1(\mathbf{R}^d)} \sim 2^{k(d-1)} \#(\mathcal{E}_k)$$

because there can be no constructive interference in the L^1 norm. Understanding the sum in Lemma 1.3 for $1 < p < 2(d-1)/(d+1)$ will require an understanding of the interference patterns of annuli with comparable radius. We will use almost orthogonality principles to understand these interference patterns.

Lemma 1.4. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ and $r_1, r_2 \geqslant 1$,*

$$|\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

In particular,

$$|\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle| \lesssim \left(\frac{r_1 r_2}{|(x_1, r_1) - (x_2, r_2)|} \right)^{(d-1)/2}$$

Remark. Suppose $r_1 \leqslant r_2$. Then Lemma 1.4 implies that $f_{x_1 r_1}$ and $f_{x_2 r_2}$ are roughly uncorrelated, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. In particular, Laura Cladek's paper exploits this tangency information.

Proof. We write

$$\begin{aligned}
\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle &= \left\langle \widehat{f}_{x_1 r_1}, \widehat{f}_{x_2 r_2} \right\rangle \\
&= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbf{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now substituting in, assuming $A(s)$ vanishes to order $100N$ at the origin, we conclude that

$$\begin{aligned}\langle f_{x_1r_1}, f_{x_2r_2}\rangle &= C_d \left(\frac{r_1r_2}{|x_1-x_2|}\right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2-x_1|^{-n_3} \\ &\qquad \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1-n_2-n_3} e^{2\pi i(\tau_1 r_1+\tau_2 r_2+\tau_3|x_2-x_1|)s}\, ds \right\} \\ &\lesssim_N \left(\frac{r_1r_2}{|x_1-x_2|}\right)^{\frac{d-1}{2}} \left(1+\frac{1}{|x_1-x_2|^N}\right) \sum_\tau (1+|\tau_1r_1+\tau_2r_2+\tau_3|x_2-x_1||)^{-5N} \\ &\lesssim_N \left(\frac{r_1r_2}{|x_1-x_2|}\right)^{\frac{d-1}{2}} \left(1+\frac{1}{|x_1-x_2|^N}\right) \sum_\tau (1+|\tau_1\tau_3r_1+\tau_2\tau_3r_2+|x_2-x_1||)^{-5N}.\end{aligned}$$

This gives the result provided that $1+|x_1-x_2| \geqslant |r_1-r_2|/10$ and $|x_1-x_2| \geqslant 1$. If $1+|x_1-x_2| \leqslant |r_1-r_2|/10$, then the supports of $f_{x_1r_1}$ and $f_{x_2r_2}$ are disjoint, so the inequality is trivial. On the other hand, if $|x_1-x_2| \leqslant 1$, then the bound is trivial by the last sentence unless $|r_1-r_2| \leqslant 10$, and in this case the inequality reduces to the simple inequality

$$\langle f_{x_1r_1}, f_{x_2r_2}\rangle \lesssim_N (r_1r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $(d-1)/2$ in Lemma 1.4 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r)\in\mathcal{E}_k} f_{xr}$. To fix this, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in $\mathbf{R}^d \times [R, 2R)$ is of *density type* (u, R) if

$$\#(B \cap \mathcal{E}) \leqslant u \cdot \mathrm{diam}(B)$$

for each ball B in $\mathbf{R}^{d+1}$ with diameter $\leqslant R$. A covering argument then shows that for any ball B,

$$\#(B \cap \mathcal{E}) \lesssim_d u \cdot \left(1+\frac{\mathrm{diam}(B)}{R}\right)^d \cdot \mathrm{diam}(B).$$

(NOTE: WE MIGHT BE ABLE TO DO BETTER USING THE FACT THAT $\mathcal{E} \subset \mathbf{R}^d \times [R, 2R)$, USING THE VALUE R).

Theorem 1.5. *For any 1-separated set $\mathcal{E} \subset \mathbf{R}^d \times [R, 2R)$, we can consider a disjoint union $\mathcal{E} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each k and m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $\leqslant 2^m$ containing at least $2^m rad(B)$ points of $\mathcal{E}_k$, then*
$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'}).$$
- *For each k and m, there are disjoint balls $\{B_i\}$ of radius at most 2^k, such that*
$$\sum_i rad(B) \leqslant \frac{\#(\mathcal{E}_k)}{u}$$
such that $\bigcup B_i^*$ covers $\bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Vitali Covering. □

Given a sum $F = \sum_{(x,r) \in \mathcal{E}} f_{xr}$, decompose $\mathcal{E}$ as $\mathcal{E}_k(2^m)$, and define F_{km} to be the sum over $\mathcal{E}_k(2^m)$. It follows from the convering argument above that measure of the support of F_{km} is $O(2^{k(d-1)-m}\#(\mathcal{E}_k))$. We define $F_m = \sum_k F_{km}$. To Prove Lemma 1.3, it will suffice to prove the following L^2 estimate on F_m.

Lemma 1.6. *Suppose $\mathcal{E}$ is a set with density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r) \in \mathcal{E}} f_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim 2^{m/(d-1)} \sqrt{\log(2 + 2^m)} 2^{k(d-1)/2} \cdot \#(\mathcal{E}_k)^{1/2}.$$

Proof of Lemma 1.3 from Lemma 1.6. We have

$$\|F_{km}\|_{L^2(\mathbf{R}^d)} \lesssim 2^{m/(d-1)} \sqrt{\log(2 + 2^m)} 2^{k(d-1)/2} \#(\mathcal{E}_k)^{1/2}.$$

If we interpolate this bound with the support bound for F_{km}, we conclude that for $0 < p \leqslant 2$,

$$\begin{aligned}
\|F_{km}\|_{L^p(\mathbf{R}^d)} &\leqslant |\mathrm{Supp}(F_{km})|^{1/p-1/2} \|F_{km}\|_{L^2(\mathbf{R}^d)} \\
&\lesssim (2^{(k(d-1)-m)})^{1/p-1/2} 2^{m/(d-1)} \sqrt{\log(2 + 2^m)} 2^{k(d-1)/2} \#(\mathcal{E}_k)^{1/2} \\
&\lesssim 2^{m(1/p_d - 1/p)} \sqrt{\log(2 + 2^m)} \cdot 2^{k(d-1)/p} \#(\mathcal{E}_k)^{1/2}.
\end{aligned}$$

where $p_d = 2(d-1)/(d+1)$. This bound is summable in m for $p < p_d$, which enables us to conclude that

$$\|F_k\|_{L^p(\mathbf{R}^d)} \lesssim_p 2^{k(d-1)/p} \#(\mathcal{E}_k)^{1/2}.$$

NOTE: THIS SEEMS LIKE A TYPO. Thus for $1 \leqslant p < p_d$, we obtain the bound stated in Lemma 1.3. $\square$

Proving 1.6 is where the weak-orthogonality bounds from Lemma 1.4 come into play.

Proof of Lemma 1.6. Split the interval $[2^k, 2^{k+1}]$ into $\lesssim 2^{(1-\alpha)k}$ intervals of length $2^{\alpha k}$, for some α to be optimized later. For appropriate integers a, let $I_a = [2^k + (a-1)2^{\alpha k}, 2^k + a 2^{\alpha k}]$. Let $\mathcal{E}_a = \{(x,r) \in \mathcal{E} : r \in I_a\}$, and write $F = \sum f_{xr}$, and $F_a = \sum_{(x,r)\in\mathcal{E}_a} f_{xr}$. Without loss of generality, splitting up the sum appropriately, we may assume that the set of a such that $\mathcal{E}_a$ is nonempty is 10-separated. We calculate that

$$\|F\|_{L^2(\mathbf{R}^d)}^2 = \sum_a \|F_a\|_{L^2(\mathbf{R}^d)}^2 + 2 \sum_{a_1 < a_2} |\langle F_{a_1}, F_{a_2} \rangle|$$

Given $a_1 < a_2$, $(x_1, r_1) \in \mathcal{E}_{a_1}$, and (x_2, r_2) such that $\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle \neq 0$, then $|x_1 - x_2| \leqslant 2^{k+2}$. Since $|r_1 - r_2| \leqslant 2^{k+1}$ follows because $r_1, r_2 \in [2^k, 2^{k+1}]$, it follows that $|(x_1, r_1) - (x_2, r_2)| \leqslant 3 \cdot 2^{k+1}$. For each such pair, since we may assume that $a_2 - a_1 \geqslant 10$ without loss of generality, it follows that $|r_1 - r_2| \geqslant 2^{\alpha k}$, and so applying Lemma 1.4 together with the density property, we conclude that for $d \geqslant 4$,

$$\begin{aligned}
|\langle f_{x_1 r_1}, F_{a_2} \rangle| &\leqslant \sum_{l=1}^{(1-\alpha)k+1} \sum_{2^l 2^{\alpha k} \leqslant |(x_1, r_1) - (x_2, r_2)| \leqslant 2^{l+1} 2^{\alpha k}} \langle f_{x_1 r_1}, f_{x_2 r_2}| \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} (2^m 2^l 2^{\alpha k}) \left(\frac{2^{2k}}{2^l 2^{\alpha k}} \right)^{(d-1)/2} \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} 2^m (2^k)^{(d-1)-(d-3)/2\alpha} 2^{-(d-3)/2 \cdot l} \\
&\lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha}.
\end{aligned}$$

Summing over all choices of x_1 and r_1, we conclude that

$$2 \sum_{a_1 < a_2} |\langle F_{a_1}, F_{a_2} \rangle| \lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha} \#(\mathcal{E}).$$

On the other hand, TODO $\square$

1.2 Cladek's Improvement

Bibliography

[1] Andreas Seeger Yaryong Heo, Fëdor Nazrov. Radial fourier multipliers in high dimensions. 2011.