SALEM SETS AVOIDING ROUGH CONFIGURATIONS

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Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension s if it has Hausdorff dimension s, and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that for all $\xi \in \mathbf{R}^d$,

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty.$$

Our goal in these notes is to obtain, for each set $Z \subset \mathbf{R}^{dn}$ with Minkowski dimension s, a Salem set $X \subset \mathbf{R}^d$ with dimension

$$\frac{nd - s}{s},$$

such that for each set of n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We hope that we can rely on the random selection approach of our paper on rough configurations to obtain such a result.

1. Orlicz Norm

We define a convex function $\psi_2 : [0, \infty) \to [0, \infty)$ by $\psi_2(t) = e^{t^2} - 1$, and define a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{ A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \le 1 \}.$$

The family of random variables $\psi_2(L)$ are known as *subgaussian random variables*. Here are some important properties:

- (Gaussian Tails): There exists a universal constant $c > 0$ such that for any random variable X, $\|X\|_{\psi_2(L)} \le A$ if and only if for each $t \ge 0$,
$$\mathbf{P}(|X| \ge t) \le 2 \exp\left(-ct^2/A^2\right).$$
- (Bounded Variables are Subgaussian): For any random X,
$$\|X\|_{\psi_2(L)} \lesssim \|X\|_{L^\infty}.$$
- (Centering) For any random variable X,
$$\|X - \mathbf{E}(X)\|_{\psi_2(L)} \lesssim \|X\|_{\psi_2(L)}.$$
- (Union Bound) If $X_1, \dots, X_N$ are random variables, then
$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \le \|X_1\|_{\psi_2(L)} + \dots + \|X_N\|_{\psi_2(L)}.$$

- (Hoeffding's Inequality): If $X_1, \dots, X_N$ are *independent* random variables, then
$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \lesssim \left(\|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2 \right)^{1/2}.$$

The Orlicz norm is a convenient notation to summarize calculations of subgaussian concentration inequalities. Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leq A$ as having the large majority of it's support in the interval $[-A, A]$, and having mass rapidly decaying outside of this interval.

2. A Family of Cubes

We fix sequences of integers $\{N_m : m \geq 1\}$ and $\{M_m : m \geq 1\}$. We then define two sequences of real numbers $\{l_m : m \geq 0\}$ and $\{r_m : m \geq 0\}$, by initially setting $l_0 = r_0 = 1$, and then, for each $m \geq 1$, setting $r_m = l_{m-1}/M_k$, and $l_m = r_m/N_m$. For each $m \geq 0$ and d, we define two collections of strings
$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times \dots \times [N_1]^d \times \dots \times [M_m]^d \times [N_m]^d$$
and
$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [N_1]^d \times \dots \times [N_{m-1}]^d \times [M_m]^d.$$
For each string $i \in \Sigma_m^d$, we define a vector $a_i \in (l_m \mathbf{Z})^d$ by setting
$$a_i = i_0 + \sum_{k=1}^m i_{2k-1} \cdot r_k + i_{2k} \cdot l_k$$
Then each string $i \in \Sigma_m^d$ can be identified with a sidelength l_m cube
$$Q_i = \prod_{j=1}^d [a_{ij}, a_{ij} + l_m].$$
centered at a_i. Similarily, for each string $i \in \Pi_m^d$, we define a vector $a \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leq j \leq d$,
$$a_i = i_0 + \left(\sum_{k=1}^{m-1} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k \right) + i_{2m-1} \cdot r_m,$$
and then define a sidelength r_m cube
$$R_i = \prod_{j=1}^d [a_{ij}, a_{ij} + r_m].$$
We let $\mathcal{Q}_m^d = \{Q_i : i \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R_i : i \in \Pi_m^d\}$. Then

- For each m, $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ are covers of $\mathbf{R}^d$.
- If $Q_1, Q_2 \in \bigcup_{m=0}^\infty \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^\infty \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.

- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarily, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarily, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{i \in \Sigma_m^d : Q_i \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{i \in \Pi_m^d : R_i \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if $Q_i^* \cap Q_j^* = \varnothing$ for each $i \neq j$.

3. A Family of Mollifiers

We now consider a family of mollifiers, which we will use to smooth out the Fourier transform of the measures we study. Begin by choosing a non-negative C^∞ function ψ supported on $[-1,1]^d$ such that

$$\int_{\mathbf{R}^d} \psi(x)\, dx = 1, \tag{1}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{2}$$

Since ψ is C^∞ and compactly supported, then for each $t \in [0,\infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \tag{3}$$

Now we rescale to obtain a mollifier at each scale of the argument. For each $k > 0$, we let

$$\psi_k(x) = l_k^{-d} \psi(l_k \cdot x).$$

Then ψ_k is supported on $[-l_k, l_k]^d$. Equation (1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_k = 1. \tag{4}$$

Equation (2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_k \cdot n) = l_k^{-d}. \tag{5}$$

For each $\xi \in \mathbf{R}^d$, $\widehat{\psi_k}(\xi) = \widehat{\psi}(l_k \xi)$, and so in particular, (3) implies that for each $t \in [0,\infty)$,

$$|\widehat{\psi_k}(\xi)| \lesssim_t l_k^{-t} |\xi|^{-t}. \tag{6}$$

These properties are sufficient to mollify the functions we consider.

4. Discrete Lemma

Lemma 1. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (n-s)/4)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth probability measure compactly supported on T, together with a constant C such that for each $m \in \mathbf{Z}^d$,*

$$|\widehat{\mu_T}(m)| \leq C|m|^{-s/2}.$$

Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}.$$

Then there exists a constant $C(\mu_T, n, s)$ such that if

$$M_{m+1} \geq C(T, \mu_T, n, d, s), \tag{7}$$

$$N_{m+1} \geq C(T, \mu_T, n, d, s) \cdot M_{m+1}^{\frac{s+\varepsilon}{dn-s-2\varepsilon}}, \tag{8}$$

and

$$N_{m+1} \geq 10^{1/\varepsilon}, \tag{9}$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

- *For any strongly non-diagonal cube*

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

  There exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.
- *For any $m \in \mathbf{Z}^d$,*

$$|\widehat{\mu}(m)| \leq (C + l_{m+1}^{-1})|m|^{-s/2}$$

Proof. For each $i \in \Pi_{m+1}^d$, let j_i be a random integer vector chosen from $[N_{m+1}]^d$, such that the family $\{j_i : i \in \Pi_{m+1}^d\}$ is independent. We define a measure ν_S by setting, for each $x \in \mathbf{R}^d$,

$$d\nu_S(x) = r_{m+1}^d \sum_{i \in \Pi_{m+1}^d} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

We then normalize, defining

$$\mu_S = \frac{\nu_S}{\nu_S(\mathbf{R}^d)}.$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then by definition, S is a $\mathcal{Q}_{m+1}$ discretized set, μ_S is supported on S, and $S \subset T$. It now suffices to show that with nonzero probability, S and μ_S satisfy the properties guaranteed by the lemma.

For each $i \in \Pi_{m+1}(T)$, define a random measure ν_i by setting

$$d\nu_i(x) = r_{m+1}^d \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

Then $\nu_S = \sum_{i\in\Pi_{m+1}(T)} \nu_i$. Note that if $j, j' \in [N_{m+1}]^d$, then

$$|a_{ij} - a_{ij'}| = |j - j'| \cdot l_{m+1} \lesssim_d N_{m+1} l_{m+1} = r_{m+1},$$

which implies

$$\begin{aligned} &\left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij}) \mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij'}) \mu_T(x) \right| \\ &\quad \le r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) \, |\mu_T(x + a_{ij}) - \mu_T(x + a_{ij'})| \\ &\quad \lesssim_d r_{m+1}^{d+1} \int_{\mathbf{R}^d} \psi_{m+1}(x) \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} = r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \tag{10}$$

Thus (10) implies that for each i,

$$\|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \lesssim_d r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \tag{11}$$

Furthermore, (5) implies

$$\begin{aligned} \sum_{i\in\Pi_{m+1}^d} \mathbf{E}(\nu_i(\mathbf{R}^d)) &= r_{m+1}^d \sum_{(i,j)\in\Sigma_{m+1}^d} \mathbf{P}(j_i = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij}) \mu_T(x)\, dx \\ &= \frac{r_{m+1}^d}{N_{m+1}^d} \int_{\mathbf{R}^d} \left(\sum_{(i,j)\in\Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij}) \right) \mu_T(x)\, dx \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{N_{m+1}^d} = 1. \end{aligned} \tag{12}$$

For all but $O_d(r_{m+1}^{-d})$ indices i, $\nu_i = 0$ almost surely. Thus we can apply the triangle inequality together with (11), and (12), we conclude

$$\begin{aligned} \|\nu_S(\mathbf{R}^d) - 1\|_{L^\infty} &= \| \sum_{i\in\Pi_{m+1}^d} \left[\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d)) \right] \|_{L^\infty} \\ &\le \sum_{i\in\Pi_{m+1}^d} \|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \\ &\lesssim_d r_{m+1}^{-d} r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \\ &= r_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \tag{13}$$

Thus if (7) holds for an appropriately chosen constant depending on d and $\|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}$, we can apply (13) to conclude

$$\|\nu_S(\mathbf{R}^d) - 1\|_{L^\infty} \le 1/2. \tag{14}$$

Thus normalizing by $\nu_S(\mathbf{R}^d)$ only introduces a neglible constant.

For each $i \in \Pi_{m+1}^d$, let

$$S_i = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \nu_i(Q) > 0\}.$$

Then $S = \bigcup_{i\in\Pi_{m+1}^d} S_i$. Because j_i is selected uniformly from $[N_{m+1}]^d$ for each i, and ψ_{m+1} is supported on $[-l_{m+1}, l_{m+1}]^d$,

$$S_i \subset \bigcup \{R_{i_0} : R_{i_0} \cap R_i \neq \varnothing\}.$$

For any cube $Q_{ij} \in \Sigma^d_{m+1}$, there are $O_d(1)$ pairs (i_0, j_0) such that $Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing$, and so a union bound gives

$$\mathbf{P}(Q_{ij} \in \mathcal{Q}_{m+1}(S)) \leq \textstyle\sum_{Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing} \mathbf{P}(j_{i'} = j') \lesssim_d N_{m+1}^{-d}.$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_{i_1 j_1} \times \cdots \times Q_{i_n j_n}$ be a strongly non-diagonal cube in $\mathcal{Q}_{m+1}(B)$. Since Q is strongly diagonal, the events $\{Q_{i_k j_k} \in S\}$ are independent from one another, which implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_{i_1 j_1} \in S) \dots \mathbf{P}(Q_{i_n j_n} \in S) \lesssim_{d,n} N_{m+1}^{-dn}. \tag{15}$$

Taking expectations over all cubes in B, and applying (15) gives

$$\begin{aligned}\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\lesssim_{d,n} \#(\mathcal{Q}_{m+1}(B)) \cdot N_{m+1}^{-dn} \\ &\leq l_{m+1}^{-(s+\varepsilon)} N_{m+1}^{-dn} \\ &= \frac{M_{m+1}^{s+\varepsilon} l_m^{-(s+\varepsilon)}}{N_{m+1}^{dn-s-\varepsilon}}.\end{aligned} \tag{16}$$

If (8) holds, for an appropriately chosen constant depending only on l_m, d, n, and s, we can apply Markov's inequality together with (9) and (16) to conclude

$$\begin{aligned}\mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \varnothing) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geq 1) \\ &\leq \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) \\ &\leq 1/N_{m+1}^{\varepsilon} \leq 1/10.\end{aligned} \tag{17}$$

Thus $\mathcal{Q}_{k+1}(S^n)$ is disjoint from $\mathcal{Q}_{k+1}(B)$ with high probability.

Now we analyze the Fourier transform of the measure ν. For each $i \in \Pi^d_{m+1}$, and $m \in \mathbf{Z}$, define $X_{im} = \widehat{\nu_i}(m) - \widehat{\mathbf{E}(\nu_i)}(m)$. Note that

$$\begin{aligned}\sum_{i \in \Pi^d_{m+1}} \widehat{\mathbf{E}(\nu_i)}(m) &= \sum_{i \in \Pi^d_{m+1}} l^d_{m+1} \sum_{j \in [N_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} \psi_{m+1}(x - a_{ij}) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i m \cdot x} d\mu_T(x) = \widehat{\mu_T}(m).\end{aligned} \tag{18}$$

For each i and m, the standard (L^1, L^∞) bound on the Fourier transform, combined with (11), shows

$$\begin{aligned}\|X_{im}\|_{\psi_2(L)} \leq \|X_{im}\|_{L^\infty} &\leq \|\nu_i(\mathbf{R}^d)\|_{L^\infty} + \mathbf{E}(\nu_i)(\mathbf{R}^d) \\ &\lesssim_d \mathbf{E}(\nu_i)(\mathbf{R}^d) + r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{19}$$

For a fixed m, the family of random variables $\{X_{im}\}$ are independent. Furthermore, $\sum X_{im} = \widehat{\nu}(m) - \widehat{\mathbf{E}(\nu)}(m)$, and

$$\begin{aligned}\mathbf{E}(\widehat{\nu_S}(m)) &= \frac{r_{m+1}^d}{N_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x} \left(\sum_{(i,j)\in\Sigma_{m+1}^d} \psi_{m+1}(x-a_{ij}) \right) d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{N_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x} d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{N_{m+1}^d} \widehat{\mu_T}(m) = \widehat{\mu_T}(m).\end{aligned}$$

Thus we may apply Hoeffding's inequality to (19) to conclude that

$$\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \lesssim_d \left(\sum \mathbf{E}(\nu_i)(\mathbf{R}^d)^2\right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}.$$

Now taking in absolute values gives the inequality

$$\begin{aligned}\mathbf{E}(\nu_i(\mathbf{R}^d)) &= l_{m+1}^d \sum_{j\in[N_{m+1}]^d} \int \psi_{m+1}(x-a_{ij}) d\mu_T(x) \\ &\leq r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)},\end{aligned}$$

and so

$$\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \lesssim_d [\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}] r_{m+1}^{d/2}. \tag{20}$$

If (8) holds, for an appropriately chosen constant depending only on l_m, d, n, and s, then we can apply a union bound over $D = \{m \in \mathbf{Z}^d : |m| \leq 10 l_{m+1}^{-1}\}$ to conclude that there exists a constant $c(\mu_T, d)$ such that

$$\begin{aligned}\mathbf{P}\left(\|\widehat{\nu} - \widehat{\mu_T}\|_{L^\infty(D)} \geq r_{m+1}^{d/2}\log(M_{m+1})\right) &\lesssim_d l_{m+1}^{-d} \exp\left(-c(\mu_T,d)\log(M_{m+1})^2\right) \\ &\leq 1/10.\end{aligned}$$

Thus $\widehat{\nu}$ and $\widehat{\mu_T}$ are highly like to differ only by a miniscule amount over small frequencies.

Finally, it suffices to analyze the values of $\widehat{\nu_S}(m)$ when $|m| \geq 10 l_{m+1}^{-1}$. We note that if we define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{i\in\Pi_{m+1}^d \\ d(a_i,T)\leq 2r_{m+1}^{-1}}} \delta_{a_{ij_i}},$$

then $\nu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\nu_S} = \widehat{\mu_T} * (\widehat{\alpha} \cdot \widehat{\psi_{m+1}})$. Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup |\widehat{\mu_T}(\xi)||\xi|^t < \infty.$$

Similarily, since

$$\widehat{\psi_{m+1}}(\xi) = \widehat{\psi}\left(l_{m+1}^{-1}\xi\right),$$

if we set, for each $t > 0$,

$$B(t) = \sup |\widehat{\psi}(\xi)||\xi|^t,$$

then

$$\sup |\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t} B(t).$$

It now suffices to bound

$$\sup_{|\eta| \geq 10 l_{m+1}^{-1}} |\eta|^{s/2} \int \widehat{\mu_T}(\eta - \xi) \widehat{\alpha}(\xi) \widehat{\psi_{m+1}}(\xi)\, d\xi.$$

Since $\alpha(\mathbf{R}^d) \leq 2^d$, $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leq 2^d$, so it suffices to understand

$$\int |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi.$$

If $|\xi| \leq |\eta|/2$, $|\eta - \xi| \geq |\xi|/2$, so for all $t > 0$,

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leq \frac{A(t) 2^{t-d}}{|\eta|^{t-d}}.$$

If we set $t = d + 1 + s/2$ and apply (8) for an appropriate chosen constant depending only on d and μ_T, we conclude

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leq \frac{A(d + 1 + s/2) 2^{1 + s/2} |\eta|^{-1}}{|\eta|^{s/2}} \leq \frac{|\eta|^{-s/2}}{10 \cdot 2^d}.$$

Conversely, if $|\xi| \geq 2|\eta|$, then $|\eta - \xi| \geq |\xi|/2$, so for each $t > 0$,

$$\begin{aligned} \int_{|\xi| \geq 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)|\, d\xi &\leq \int_{|\xi| \geq 2|\eta|} \frac{A(t) 2^t}{|\xi|^t} \\ &\lesssim_d \int_{2|\eta|}^\infty r^{d-t} A(t) l_{m+1}^{-t'} 2^t. \end{aligned}$$

Provided $t > d + 1$, this integral is finite, and is

$$\lesssim_{d,t} A(t) 2^{d+1} |\eta|^{d+1-t}.$$

Setting $t = d + 2 + s/2$, and applying (8), we conclue

$$\int_{|\xi| \geq 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)|\, d\xi \leq \frac{|\eta|^{-s/2}}{10 \cdot 2^d}.$$

Finally, we conclude that for each $t > 0$,

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leq \frac{2^d |\eta|^d B(t) 2^t}{|\xi|^t}.$$

If we set $t = d + s/2 + 1$ and apply (8), we conclude

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leq \frac{|\xi|^{-s/2}}{10 \cdot 2^d}.$$

Summing up the three bounds, we conclude that if $|\eta| \geq 10 l_{m+1}^{-1}$, then

$$|\widehat{\nu_S}(\eta)| \leq |\eta|^{-s/2}.$$

$\square$