Fractals Avoiding Fractal Sets

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Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Consider a geometric point configuration, such as three points forming an isoceles triangle in the plane, or four points lying in a plane in $\mathbf{R}^3$. A natural problem is to determine the minimum size a collection of points must be before we can guarantee a subselection lies in a specified configuration. It is often the case that we can obtain arbitrarily large finite sets not containing these configurations. On the other hand, if the configuration is affine invariant, every positive measure set contains these configurations. Thus we need finer analytical measures of size for infinite sets, which is provided by the Hausdorff dimension. The general class of problems for finding high Hausdorff dimension sets avoiding configuration as a *configuration avoidance problems*. They can be seen as a continuous analogue to Ramsey theory, and seeing how the continuous setting differs from the discrete counterpart is what makes these problems interesting.

In this paper, we give techniques to *construct* sets with high Hausdorff dimension avoiding a very general class of geometric configurations. There are already generic pattern avoidance methods in the literature. We compare our method to them in section 6. But these rely on the non-singular nature of the configurations. The novel feature of our method is we can avoid configurations which have an *arbitrary* fractal quality to them. Meanwhile, the Hausdorff dimension of the set we construct still holds up in comparison to previous methods.

The key idea to our method is the introduction of a new geometric framework for pattern avoidance problems, described in section 1. A simple combinatorial argument, described in section 2, exploited repeatedly in section 3 via a queueing process leads directly to a pattern avoiding set. We believe this new geometric framework should help find further methods in the field, which we currently developing for publication in a later paper.

1 A Fractal Avoidance Framework

One way to think about generic pattern avoidance methods is to specify the pattern as the zero set of a function. For example,

- A set $X \subset \mathbf{R}$ contains no three term arithmetic progressions if and only if for any distinct $x, y, z \in X$,

  $$f(x,y,z) = x + z - 2y \neq 0$$

  This function vanishes if and only if there exists b and t such that $x = b$, $y = b + t$, and $z = b + 2t$.

- A set $X \subset \mathbf{R}^d$ contains the vertices of no isosceles triangles if and only if for any three distinct $x, y, z \in X$,

  $$f(x,y,z) = d(x,y) - d(y,z) \neq 0$$

  The problem of avoiding this function is similar to the last example, since isosceles triangles are planar variants of three term arithmetic progressions.

- A set $X \subset \mathbf{R}^d$ does not contain a family of angles $\{\alpha\}$ if and only if for any distinct $x, y, z \in X$, and α,

  $$f(x,y,z) = \frac{(x-z)\cdot(y-z)}{|x-z||y-z|} \neq \cos(\alpha)$$

  where the cosine formula for the dot product is used.

- A set $X \subset \mathbf{R}^d$ does not contain $d+1$ points in a lower dimensional hyperplane if and only if for any distinct $x_0, \dots, x_d \in X$,

  $$f(x_0,\dots,x_d) = \det(x_1 - x_0, \dots, x_d - x_0) \neq 0$$

  since $\det(x_1 - x_0, \dots, x_d - x_0) = 0$ only when the vectors $x_1 - x_0, \dots, x_d - x_0$ do not form a basis, and thus span a plane of dimension smaller than d.

These problems are summarized by a general framework.

The Configuration Avoidance Problem: Given a function $f : (\mathbf{R}^d)^n \to \mathbf{R}$ as input, find $X \subset \mathbf{R}^d$ such that for any *distinct* $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$, with as high a Hausdorff dimension as possible.

The functional framework is common in the literature. For instance, it is the viewpoint behind the methods of [2] and [3], who give results assuming various regularity conditions on the function f. It is the viewpoint of this paper that the function f contains extraneous information which is irrelevant to the problem. The only important information we need to extract from the function f is the geometric structure of it's zero set. If we denote the zero set of f by Z, the configuration avoidance problem becomes equivalent to another framework. It is the viewpoint of this paper that this framework is easier to work with, and leads to new general avoidance methods.

The Fractal Avoidance Problem: Given $Z \subset (\mathbf{R}^d)^n$, find a high dimensional set $X \subset \mathbf{R}^d$ such that if $x_1, \dots, x_n \in X$ are *distinct*, $(x_1, \dots, x_n) \notin Z$.

Because we are the first to introduce the fractal avoidance problem, a natural goal is to solve the generic problem with minimal assumptions on Z. Thus we let Z take the form of an arbitrary fractal, and the only assumptions we place on Z are it's fractal dimension.

Theorem 1. *If Z has Minkowski dimension $\alpha \geqslant d$, then there is X solving the fractal avoidance problem for Z with*

$$\dim_{\mathbf{H}}(X) = \frac{dn - \alpha}{n-1} = \frac{\operatorname{codim}_{\mathbf{H}}(Z)}{n-1}$$

Remark. *If Z has dimension $\alpha < d$, the set obtained from $\mathbf{R}^d$ by removing the projections of Z onto each coordinate has full Hausdorff dimension and trivially solves the fractal avoidance problem. Thus we need not consider these parameters in our theorem.*

Due to the lack of any *rigid* geometric information about the set Z, we are led to avoid Z by discretization. At the discrete scale, we can efficiently avoid Z by randomly choosing X, which we detail in the next section. Exploiting this technique repeatedly at all scales gives a complete avoiding set.

2 Avoidance at a Single Scale

We now develop a discrete technique used to construct solutions to the fractal avoidance problem. We assume Z has been discretized into cubes, all of the same fixed length. We then aim to construct a discretized version of X avoiding these cubes.

For a length L, we let $\mathcal{B}(L,d)$ denote the partition of $\mathbf{R}^d$ into the family of all half open cubes with corners on the lattice $(\mathbf{Z}/L)^d$. If the dimension d is clear, or unnecessary to the argument, we abbreviate $\mathcal{B}(L,d)$ as $\mathcal{B}(L)$. Families of cubes are denoted using calligraphic font, and the non-calligraphic version of the same character denotes the union of the family. For instance, we might have $\mathcal{I} \subset \mathcal{B}(L)$, and then I is the union of all cubes in $\mathcal{I}$.

Lemma 1. *Consider three dyadic scales $L \gg R \gg S$, as well as two collections $\mathcal{I} \subset \mathcal{B}(L,d)$ and $\mathcal{K} \subset \mathcal{B}(S,dn)$. Then there exists $\mathcal{J} \subset \mathcal{B}(S,d)$ with $J \subset I$, such that for any distinct $J_1, \dots, J_n \in \mathcal{J}$, $J_1 \times \dots \times J_n \notin \mathcal{K}$. Furthermore, for all but at most $|K|R^{-dn}$ of the elements of $\mathcal{B}(R,d)$ intersecting I, $J \cap \mathcal{B}(R,d)$ is nonempty.*

Proof. Form a random set U by selecting uniformly randomly, from each $\mathcal{B}(R)$ subcube of I, a single subcube in $\mathcal{B}(S)$. The probability that any $\mathcal{B}(R)$ subcube is selected is $(S/R)^d$. Since any two $\mathcal{B}(S)$ subcubes of U lie in distinct elements of $\mathcal{B}(R)$, the only chance that a $\mathcal{B}(S)$ subcube I of K with distinct sides intersects U^n is if $I_1, \dots, I_n$ all lie in separate cubes in $\mathcal{B}(R)$. Then the chance that each occurs is independant of one another, and so

$$\mathbf{P}(I \subset U^n) = \mathbf{P}(I_1 \subset U) \dots \mathbf{P}(I_n \subset U) = (S/R)^{dn}$$

If E denotes the number of $\mathcal{B}(S)$ subcubes I of K contained in U^n,

$$\mathbf{E}(E) = \sum_{I \subset K} \mathbf{P}(I \subset U^n) = [|K|S^{-dn}][(S/R)^{dn}] = |K|R^{-dn}$$

If, for each $\mathcal{B}(S)$ subcube I of U^n, we remove I_1 from U, we obtain a set J with $J_1 \times \dots \times J_n$ disjoint from K for any distinct $\mathcal{B}(R)$ subcubes J_i of J. The cube J contains an cube from all but E sidelength R cubes. In particular, we can select some nonrandom choice of U such that $E \leqslant |K|R^{-dn}$, which gives the required J. ☐

If we obtain $\mathcal{K}$ by discretizing the fractal Z, then the size is obviously related to the dimension of Z. Quantifying this precisely leads directly to the quantity which will become the Hausdorff dimension of X.

Corollary. *Consider the notation introduced in the last theorem. Fix $0 < \beta < 1$, and suppose R is the closest dyadic number to S^β. Furthermore, suppose that $|\mathcal{K}| \leqslant S^{-\gamma}$, and*

$$0 < \beta \leqslant \frac{dn - \gamma - \log_S |I| - \log_S(O(A))}{d(n-1)}$$

Then the set J obtained in the last lemma contains a portion of all but a fraction A of all $\mathcal{B}(R)$ subcubes of I.

Proof. The inequality here is equivalent to

$$dn - \gamma - \beta(n-1)d \geqslant \log_S |I| + \log_S(O(A))$$

Since R is within a factor of two from S^β, we compute

$$\begin{aligned}\frac{\#(\mathcal{B}(R) \text{ subcubes not selected from})}{\#(\text{all } \mathcal{B}(R) \text{ subcubes})} &= \frac{|K|R^{-dn}}{|I|R^{-d}} \\ &\leqslant |I|^{-1} S^{dn-\gamma} R^{-(n-1)d} \leqslant |I|^{-1} S^{dn-\gamma} (S/2)^{-d(n-1)} \\ &\leqslant 2^{\beta d} |I|^{-1} S^{\log_S |I| + \log_S(O(A))} = 2^{\beta d} O(A) \leqslant A.\end{aligned}$$

At the end, we chose the constant in the $O(A)$ term to be on the order of $2^{-\beta d}$. ☐

3 Construction of Avoiding Set

To construct a set X solving the fractal avoidance problem, we fix a decreasing series of dyadic scales $L_1, L_2, \dots$, which enable us to discretize the problem. We will specify the exact values of these scales later. Since Z has Minkowski dimension α, for each N we can find $\mathcal{Z}_N \subset \mathcal{B}(L_N)$ such that $Z \subset Z_N$ for each N, and $|\mathcal{Z}_N| \leqslant 1/L_N^{\alpha+\varepsilon_N}$, where $\varepsilon_N \to 0$ as $N \to \infty$. Given this setup, we construct a nested family of discretized sets $X_1 \supseteq X_2 \supseteq \dots$, with $X = \lim X_N$. One condition that guarantees that X solves the fractal avoidance problem is that X_N^n is disjoint from *non diagonal* cubes in Z_N. By a non-diagonal cube, we mean a cube $I \subset \mathcal{B}(L_N, dn)$ such that the projections $I_i \in \mathcal{B}(L_N, d)$ onto the d dimensional coordinate axis of $(\mathbf{R}^d)^n$ are distinct.

Lemma 2. *If for each N, X_N^n avoids non-diagonal cubes in Z_N, X solves the fractal avoidance problem for Z.*

Proof. Let $z \in Z$ be given with $z_1, \dots, z_n$ are distinct. Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i, j \text{ such that } w_i = w_j\}$$

Then $d(\Delta, z) > 0$. In particular, this means that for suitably large N, the cube I in $\mathcal{B}(L_N)$ containing z is disjoint from Δ. But this means that I is non diagonal. Thus $z \notin X_N^d$, and so in particular, is not an element of X^n. □

As an initial set, for lack of an interesting choice, we let $X_0 = [0,1]^d$. Given X_{N-1}, we define X_N recursively. To do this, we apply the discrete corollary of the last section. The parameters we use are $I = X_{N-1}$, $K = Z_N$, $L = L_{N-1}$, $S = L_N$, and R the closest dyadic number to $L_N^{\beta_N}$, which from now on we denote by R_N. The exponent β_N is defined as

$$\frac{dn - \alpha - \varepsilon_N - \log_{L_N}|X_{N-1}| - \log_{L_N}(O(1/2^{2N+2}))}{(n-1)d}$$

For suitably small choices of L_N relative to $|X_{N-1}|$ and $1/2^{2N+2}$, $0 < \beta_N < 1$. Thus the discrete corollary we described in the last section constructs a set J avoiding nondiagonal cubes in $\mathcal{Z}_N$, and containing a sidelength L_N portion from all but a fraction $1/2^{2N+2}$ of the $\mathcal{B}(R_N)$ cubes in X_N. Naturally, we set $X_N = J$. The resulting $X = \lim X_N$ is therefore a solution to the fractal avoidance problem. In the next section, we show that for a sufficiently fast decaying set of lengths L_N, X has the Hausdorff dimension we need. This completes the description of our method.

4 Dimension Bounds

Now we show that the set X obtained is βd dimensional, where $\beta = (dn - \alpha)/d(n-1)$. To offset implicit obstructions to this dimension, we must choose the lengths L_N to decay suitably rapidly. The constraints on L_N will emerge naturally from our arguments, but for the impatient, one such choice is to set L_N to be the closest dyadic number to $2^{-N^2}(L_1 \dots L_{N-1})^{Nd(1-\beta)}$. The construction we considered looks like a $d\beta_N$ dimensional set at the discrete scales L_N. Another very useful fact is that the construction looks *full* dimensional between the scales L_{N-1} and R_N because of the relative uniformity by which we have taken intervals. This enables us to let the scales L_N to decrease arbitrarily rapidly, without penalizing us for initially looking at the Hausdorff dimension at discrete scales.

An initial requirement to get the βd dimensional result is to show that $\beta_N \to \beta$ as $N \to \infty$. Otherwise, we do not even get a βd dimensional result at the discrete scales. To obtain this, we just require the L_N decrease rapidly in proportion to the exponential 2^{-2N} and $|X_{N-1}|$. We know $\varepsilon_N = o(1)$ provided $L_N \to 0$, which is easy to obtain. If $L_N \lesssim_C 2^{-CN}$ for all large $C > 0$, then $\log_{L_N}(O(1/2^{2N+2})) = o(1)$. Given the bound $L_N \lesssim_C |X_{N-1}|^C$, we conclude $\log_{L_N}|X_{N-1}| = o(1)$. These two constraints thus imply that $\beta_N \to \beta$ as $N \to \infty$. Thus X behaves like a βd dimensional set at discrete scales. We first quantify this behaviour by working with a measure supported on X, then interpolate to obtain the behaviour at all scales.

Using Frostman's lemma, to prove X has dimension βd, it suffices to find a non-zero measure μ supported on X such that for all $\varepsilon > 0$, for all lengths L, and for all sidelength L cube I, $\mu(I) \lesssim_\varepsilon L^{\beta d - \varepsilon}$. To construct μ, we rely on a variant of the mass distribution principle, i.e. as the weak limit of measures μ_N supported on the discrete sets X_N. Initially, we put the uniform probability measure μ_0 on $X_0 = [0,1]^d$. We then define μ_N, supported on X_N, by modifying the distribution of μ_{N-1}. First, we throw away the mass of the $\mathcal{B}(L_{N-1})$ cubes I in X_{N-1} for which X_N fails to contain a $\mathcal{B}(L_N)$ cube in more than half of the $\mathcal{B}(R_N)$ subcubes of I. For the remaining cubes I, we uniformly distribute the mass $\mu_{N-1}(I)$ over the cubes in X_N contained in I. Throwing away mass is necessary to avoid the mass of μ_N clumping in undesirable places. It is easy to see from the cumulative distribution functions of the μ_N that μ_N converges weakly to a limit μ. The measure μ has the property that for any $I \in \mathcal{B}(L_N)$, $\mu(I) \leqslant \mu_N(I)$, which is useful from passing from discrete results about our construction to properties of the final measure. The measure μ is the measure for which we will ultimately obtain a Frostman type inequality.

Lemma 3. *If $N \geqslant 1$, and $I \in \mathcal{B}(L_N)$,*

$$\mu(I) \leqslant \mu_N(I) \leqslant 2^N \left[\frac{R_N R_{N-1} \dots R_1}{L_{N-1} \dots L_1}\right]^d$$

Proof. Consider $I \in \mathcal{B}(L_N)$, $J \in \mathcal{B}(L_{N-1})$. If $\mu_N(I) > 0$, this means that J contains a $\mathcal{B}(L_N)$ cube in at least half of the $\mathcal{B}(R_N)$ cubes it contains. Thus the mass of J distributes itself evenly over at least $2^{-1}(L_{N-1}/R_N)^d$ cubes, which gives that $\mu_N(I) \leqslant 2(R_N/L_{N-1})^d \mu_{N-1}(J)$. But then expanding this recursive inequality, we obtain exactly the result we need. □

Corollary. *The measure μ is positive.*

Proof. To prove this result, it suffices to show that the total mass of μ_N is bounded below, independantly of N. At each stage N, X_N consists of at most

$$\left[\frac{L_{N-1}\dots L_1}{R_N\dots R_1}\right]^d$$

$\mathcal{B}(L_N)$ cubes. Since only a fraction $1/2^{2N+2}$ of the $\mathcal{B}(R_N)$ cubes do not contain an interval in X_{N+1}, it is only for at most a fraction $1/2^{2N+1}$ of the $\mathcal{B}(L_N)$ cubes that X_{N+1} fails to contain a $\mathcal{B}(L_{N+1})$ cube from more than half of the $\mathcal{B}(R_{N+1})$ cubes. Using the last lemma, this means that in the passage from μ_N to μ_{N+1}, we discard a mass of at most $1/2^{N+1}$. Thus

$$\mu_N(\mathbf{R}^d) \geqslant 1 - \sum_{k=0}^{N} \frac{1}{2^{k+1}} \geqslant 1/2$$

Thus $\mu(\mathbf{R}^d) \geqslant 1/2$, and in particular, $\mu \neq 0$. □

We fix an increasing sequence λ_N with $\lambda_N < \beta_N$, and $\lambda_N \to \beta$ as $N \to \infty$. This gives us slightly more room to bound mass when obtaining the Frostman's lemma result. We set $\lambda_N - \beta_N = 1/N$ to obtain the choice of L_N given as an example.

Corollary. *If $L_N \ll 1$, $\mu(I) \leqslant L_N^{d\lambda_N}$ for $I \in \mathcal{B}(L_N)$.*

Proof. We can rewrite the inequality in the last problem as

$$\mu(I) \leqslant \left[2^N \left(\frac{R_{N-1}\dots R_1}{L_{N-1}\dots L_1}\right)^d R_N^d L_N^{-d\lambda_N}\right] L_N^{d\lambda_N}$$

Now $R_N^d L_N^{-d\lambda_N} \leqslant (2L_N^{\beta_N})^d L_N^{-d\lambda_N} \leqslant 2^d L_N^{d(\beta_N-\lambda_N)}$, which tends to zero as $L_N \to \infty$, while the remaining parameters are fixed. Thus if L_N is sufficiently small, we can bound the constant in the square brackets by 1, which is sufficient to obtain the inequality. □

This is the cleanest expression of the $d\beta$ dimensional behaviour at discrete scales. To get a general inequality of this form, we use the fact that our construction distributes uniformly across all intervals.

Theorem 2. *Assume the last corollary holds. If $L \leqslant L_N$ is dyadic and $I \in \mathcal{B}(L)$, then $\mu(I) \leqslant 2L^{\lambda_N}$.*

Proof. We break our analysis into three cases, depending on the size of L in proportion to L_N and R_N:

- If $R_{N+1} \leqslant L \leqslant L_N$, we can cover I by $(L/R_{N+1})^d$ cubes in $\mathcal{B}(R_{N+1})$. For each of these cubes, we know the mass is bounded by at most $2(R_{N+1}/L_{N+1})^d$ times the mass of a $\mathcal{B}(L_N)$ cube. Thus

$$\begin{aligned}\mu(I) &\leqslant [(L/R_{N+1})^d][2(R_{N+1}/L_N)^d][L_N^{\lambda_N}]\\ &\leqslant 2L^d L_N^{d-\lambda_N} \leqslant 2L^{\lambda_N}\end{aligned}$$

- If $L_{N+1} \leqslant L \leqslant R_{N+1}$, we can cover I by a single cube in $\mathcal{B}(R_{N+1})$. Each cube in $\mathcal{B}(R_{N+1}, d)$ contains at most one cube in $\mathcal{B}(L_{N+1}, d)$ which is also contained in X_{N+1}, so $\mu(I) \leqslant L_{N+1}^{d\lambda_{N+1}} \leqslant L^{d\lambda_N}$.

- If $L \leqslant L_{N+1}$, there certainly exists M such that $L_{M+1} \leqslant L \leqslant L_M$, and one of the previous cases yields that $\mu(I) \leqslant 2L^{\lambda_M} \leqslant 2L^{\lambda_N}$.

This addresses all cases considered in the theorem. □

To use Frostman's lemma, we need the result $\mu(I) \lesssim L^{\lambda_N}$ for an *arbitrary* interval, not just one with $L \leqslant L_N$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. This is because if $L \geqslant L_N$, then $\mu(I)/L^{\lambda_N} \leqslant 1/L_N^{\lambda_N} \lesssim_N 1$, so $\mu(I) \lesssim_N L^{\lambda_N}$ holds automatically for all sufficiently large intervals. Thus the general bound is complete, and we have proven that there is a choice of parameters which constructs a set X with Hausdorff dimension no less than $(dn-\alpha)/(n-1)$. It is also easy to see X has *precisely* this dimension.

Theorem 3. $\dim_{\mathbf{H}}(X) \leqslant (dn-\alpha)/(n-1)$.

Proof. X_N is covered by at most

$$\left[\frac{L_{N-1}\dots L_1}{R_N\dots R_1}\right]^d$$

sidelength L_N cubes. It follows that if $\gamma > \beta$, then

$$H^{d\gamma}_{L_N}(X) \leqslant \left[\frac{L_{N-1}\dots L_1}{R_N\dots R_1} L_N^{\gamma}\right]^d \lesssim \left[\frac{L_{N-1}\dots L_1}{R_{N-1}\dots R_1} L_N^{\gamma-\beta}\right]^d$$

Thus if L_N is suitably small depending on previous constants, which we know to be true from the last corollary, we conclude that as $N \to \infty$, $H^{\gamma}(X)$ is finite. Since γ was arbitrary, taking it to β allows us to conclude that $\dim_{\mathbf{H}}(X) \leqslant d\beta$. □

5 Applications

6 Comparison with Other Generic Avoidance Schemes

In the past few years in the discrete setting it has been noticed that rephrasing particular questions in terms of abstract problems on hypergraphs allows one to extend various results into sparse analogues [4]. In this paper we consider a continuous analogue, where sparsity is represented in terms of the dimension of the set Z we are trying to avoid.

7 Concluding Remarks

Another goal of our current research programme is to show an example of a fractal avoidance problem where extra geometric conditions on Z leads to constructions with a higher Hausdorff dimension. This means that our framework isn't designed for a single method, but naturally incorporates further methods in the field. We consider a condition where Z is efficiently coverable by parallel hyperplanes of a fixed dimension.

Theorem 4. *If there is $k \geqslant 2$ and a linear $T : \mathbf{R}^{dn} \to \mathbf{R}^{kd}$ such that $T(Z)$ is α dimensional, with $\alpha \leqslant (d-1)k$, then there exists X with*

$$\dim_{\mathbf{H}}(X) = \frac{dk - \alpha}{2k - 1}$$

solving the fractal avoidance problem for Z.

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