

Salem Sets Avoiding Rough Configurations

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1 Introduction

Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension t if it has Hausdorff dimension t , and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{t-\varepsilon} |\hat{\mu}_\varepsilon(\xi)| < \infty. \quad (1.1)$$

It is a result of the Poisson summation formula that if μ_ε is compactly supported, then (1.1) is equivalent to the equation

$$\sup_{k \in \mathbf{Z}^d} |k|^{t-\varepsilon} |\hat{\mu}_\varepsilon(k)| < \infty. \quad (1.2)$$

Our goal in these notes is to obtain high dimensional Salem sets avoiding rough configurations.

Theorem 1. *Let $Z \subset [0, 1]^{dn}$ be the countable union of sets, each with lower Minkowski dimension at most s . Then there exists a Salem set $X \subset \mathbf{R}^d$ of dimension*

$$t = \frac{nd - s}{n},$$

such that for any n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

We rely on a random selection approach, like in our paper on rough configurations, to obtain such a result, since such random selections give high probability bounds on the Fourier transform of the measures we study.

2 Concentration Inequalities

Define a convex function $\psi_2 : [0, \infty) \rightarrow [0, \infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \leq 1\}.$$

The family of random variables with $\|X\|_{\psi_2(L)} < \infty$ are known as *subgaussian random variables*. Here are some important properties:

- If $\|X\|_{\psi_2(L)} \leq A$, then for each $t \geq 0$,

$$\mathbf{P}(|X| \geq t) \leq 10 \exp(-t^2/10A^2).$$

Thus Subgaussian random variables have Gaussian tails.

- If $|X| \leq A$ almost surely, then $\|X\|_{\psi_2(L)} \leq 10A$. Thus bounded random variables are subgaussian.
- If X_1, \dots, X_N are *independent*, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leq 10 \left(\|X_1\|_{\psi_2(L)}^2 + \dots + \|X_N\|_{\psi_2(L)}^2 \right)^{1/2}.$$

This is an equivalent way to state *Hoeffding's Inequality*, and we refer to an application of this inequality as an application of Hoeffding's inequality.

Remark 2. *The constants involved in these statements are suboptimal, but will suffice for our purposes. Proofs can be found in Chapter 2 of [1].*

Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leq A$ as a variable whose magnitude exceeds A with extremely low probability. The Orlicz norm thus provides a convenient way to quantify concentration phenomena.

3 A Family of Cubes

Fix two integer-valued sequences $\{K_m : m \geq 1\}$ and $\{M_m : m \geq 1\}$. For convenience, we also define $N_m = K_m M_m$ for $m \geq 1$. We then define two sequences of real numbers $\{l_m : m \geq 0\}$ and $\{r_m : m \geq 0\}$, by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each $m, d \geq 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $\sigma = \sigma_0 \sigma_1 \dots \sigma_{2k} \in \Sigma_m^d$, we define a vector $a(\sigma) \in (l_m \mathbf{Z})^d$ by setting

$$a(\sigma) = \sigma_0 + \sum_{k=1}^m \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k$$

Then each string $\sigma \in \Sigma_m^d$ can be identified with the sidelength l_m cube $Q(\sigma)$ with left-hand corner lies at $a(\sigma)$, i.e. the cube

$$Q(\sigma) = \prod_{i=1}^d [a(\sigma)_i, a(\sigma)_i + l_m].$$

Similarly, for each string $\sigma = \sigma_0 \dots \sigma_{2m-1} \in \Pi_m^d$, we define a vector $a(\sigma) \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leq j \leq d$,

$$a(\sigma) = \sigma_0 + \left(\sum_{k=1}^{m-1} \sigma_{2k-1} \cdot r_k + \sigma_{2k} \cdot l_k \right) + \sigma_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R(\sigma) = \prod_{i=1}^d [a(\sigma)_i, a(\sigma)_i + r_m].$$

We let $\mathcal{Q}_m^d = \{Q(\sigma) : \sigma \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R(\sigma) : \sigma \in \Pi_m^d\}$. We now list some important properties of this collection of cubes:

- For each m , the two collections \mathcal{Q}_m^d and \mathcal{R}_m^d form covers of \mathbf{R}^d .
- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.

- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q . Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R .

We say a set $E \subset \mathbf{R}^d$ is \mathcal{Q}_m discretized if it is a union of cubes in \mathcal{Q}_m^d , and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarly, we say a set $E \subset \mathbf{R}^d$ is \mathcal{R}_m discretized if it is a union of cubes in \mathcal{R}_m^d , and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{\sigma \in \Sigma_m^d : Q(\sigma) \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{\sigma \in \Pi_m^d : R(\sigma) \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j , and a third index $\sigma \in \Pi_m^d$, such that $R_\sigma \cap Q_i, R_\sigma \cap Q_j \neq \emptyset$.

4 A Family of Mollifiers

We now consider a family of C^∞ mollifiers, which we will use to ensure the Fourier transform of the measure we study have appropriate decay.

Lemma 3. *There exists a non-negative, C^∞ function ψ supported on $[-1, 1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi = 1, \quad (4.1)$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x + n) = 1. \quad (4.2)$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0, 1]$, such that $\alpha(1/2 + x) = \alpha(1/2 - x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \leq \alpha(x) \leq 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|) = 1 - \alpha(1 - |x|).$$

Symmetry considerations imply that $\int_{\mathbf{R}} \alpha + \beta = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x + m) + \beta(x + m) = 1. \quad (4.3)$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

The function $\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d)$ then satisfies the constraints of the lemma. \square

Fix some choice of ψ given by Lemma 3. Since ψ is C^∞ and compactly supported, then for each $t \in [0, \infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \quad (4.4)$$

Now we rescale the mollifier. For each integer $m \geq 1$, we let

$$\psi_m(x) = l_m^{-d} \cdot \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (4.1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \quad (4.5)$$

Equation (4.2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \quad (4.6)$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \cdot \xi), \quad (4.7)$$

In particular, (4.7) implies that for each $t \geq 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)| |\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)| |\xi|^t. \quad (4.8)$$

Intuitively, $\{\psi_m\}$ is a ‘uniform’ family of wave packets, with ψ_m supported in phase space on $[-l_m, l_m]^d$, and in frequency space, essentially supported on $[-l_m^{-1}, l_m^{-1}]^d$.

5 Comparison to Previous Paper

As in our previous paper, our proof of Theorem 1 will involve constructing a configuration avoiding set X by considering a nested decreasing family of sets $\{X_m : m \geq 0\}$, where $X_m \subset [0, 1]^d$ is a \mathcal{Q}_m discretized set, and then

setting $X = \bigcap_{m \geq 0} X_m$. We find a strong cover of Z by sets $\{B_m\}$, where B_m is \mathcal{Q}_m discretized. Provided X_m^d is disjoint from strongly non-diagonal cubes in B_m , we conclude that for any n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We now show that the technique of our last paper as stated fails to produce Salem sets.

Let us recap the approach of our last paper. To form X_{m+1} , we chose a cube $Q_R \in \mathcal{Q}_{m+1}(R)$ uniformly at random, for each $R \in \mathcal{R}_{m+1}(X_m)$. We then let $Y_{m+1} = \bigcup Q_R$. If $s \geq d$, and

$$K_{m+1} \approx M_{m+1}^{\frac{s-d}{dn-s}}, \quad (5.1)$$

then with non-zero probability, we proved there is $X_{m+1} \subset Y_{m+1}$ such that X_{m+1}^d avoids strongly non-diagonal cubes in B_{m+1} , and X_{m+1} contains at least half of the cubes in $\mathcal{Q}_{m+1}(Y_{m+1})$. Then X_{m+1} will be the union of at least M_{m+1}^{-d} cubes with sidelength l_{m+1} . Provided that $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$, we have

$$M_{m+1}^{-d} \approx r_{m+1}^{-d} \approx l_{m+1}^{-\frac{dn-s}{n-1}}.$$

Thus X has lower Minkowski dimension at most $(dn - s)/(n - 1)$, and a more involved analysis shows the set has Hausdorff dimension exactly equal to $(dn - s)/(n - 1)$.

The approach detailed in the last paragraph is *not* guaranteed to produce a set with Fourier dimension t . Because X_{m+1} is random, it exhibits pseudorandomness properties with high probability. In particular, it supports probability measures whose Fourier transform has sharp decay. However, since the choice of the set Y_{m+1} is *not* chosen randomly from X_{m+1} , depending heavily on the set Z and the discretized set B_{m+1} , the set Y_{m+1} will in general not possess pseudorandomness properties. For instance, if μ is the probability measure induced by normalizing Lebesgue measure restricted to X_{m+1} , then with high probability,

$$\|\hat{\mu}\|_{L^\infty(\mathbf{R}^d)} \approx l_m^t.$$

If ν is the probability measure induced by normalizing Lebesgue measure restricted to Y_{m+1} , then it is still possible for us to have

$$\|\hat{\nu}\|_{L^\infty(\mathbf{R}^d)} \gtrsim 1.$$

For instance, this will be true if $\mathcal{Q}_{m+1}(X_{m+1}) - \mathcal{Q}_{m+1}(Y_{m+1})$ is a thickening of a subset of an arithmetic progression. Thus the method of our previous paper is not able to reliably produce Salem sets without further analysis on the pseudorandom properties of the sets $\{B_m\}$ we have to avoid.

In this paper, we take a different approach which avoids us having to analyze the pseudorandomness of the sets B_m . Instead of (5.1), we choose

$$K_{m+1} \approx M_{m+1}^{\frac{s}{dn-s}}.$$

Notice that $M_{m+1}^{\frac{s}{dn-s}} \geq M_{m+1}^{\frac{s-d}{dn-s}}$, so the set Y_{m+1} we will obtain will be a thinner set than X_m . In particular, Y_{m+1} will be covered by at most M_{m+1}^{-d} sidelength l_{m+1} cubes, and if $K_{m+1}, M_{m+1} \gg K_1, M_1, \dots, K_m, M_m$

$$r_{m+1}^{-d} \approx l_{m+1}^{-t}$$

sidelength l_{m+1} cubes, which implies X will have upper Minkowski dimension at most t . However, as a result, because the set Y_{m+1} is thinner, we find that Y_{m+1}^d is disjoint from the cubes in B_{m+1} with high probability. In particular, we can set $X_{m+1} = Y_{m+1}$. This means that X_{m+1} will be pseudorandom, and we should therefore expect X to be a Salem set of dimension t . The remainder of this paper is devoted to showing that these heuristics are correct.

6 Discrete Lemma

We now proceed to solve a discretized version of Theorem 1.

Proposition 4. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn - s)/2)$. Let $T \subset [0, 1]^d$ be a non-empty, \mathcal{Q}_m discretized set, and let μ_T be a smooth measure compactly supported on T . Let $B \subset \mathbf{R}^{dn}$ be a non-empty, \mathcal{Q}_{m+1} discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}. \quad (6.1)$$

Then there exists a large constant $C(\mu_T, l_m, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geq C(\mu_T, l_m, n, d, s, \varepsilon, l_m), \quad (6.2)$$

and

$$M_{m+1}^{\frac{s}{dn-s} + c\varepsilon} \leq K_{m+1} \leq 2M_{m+1}^{\frac{s}{dn-s} + c\varepsilon}, \quad (6.3)$$

where

$$c = \frac{6dn}{(dn - s)^2},$$

then there exists a \mathcal{Q}_{m+1} discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) $\mu_S(\mathbf{R}^d) \geq \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

(C) If $|k| \leq 10l_{m+1}^{-d}$, $|\widehat{\mu}_T(k) - \widehat{\mu}_S(k)| \leq r_{m+1}^{d/2} \log(M_{m+1})$.

(D) If $|k| \geq 10l_{m+1}^{-d}$, $|\widehat{\mu}_S(k)| \leq |k|^{-d/2}$.

Remark 5. To make the statement of Proposition (4) more clean, we have hidden the explicit choice of constant $C(\mu_T, l_m, n, d, s, \varepsilon)$. But this constant can certainly be made explicit; such a choice can be made by ensuring that (6.2) implies (6.5), (6.11), (6.17), (6.27), (6.28), and (6.29) all hold.

Proof of Proposition 4. First, we describe the construction of the set S , and the measure μ_S . For each string $\sigma \in \Pi_{m+1}^d$, let j_σ be a random integer vector chosen from $\{0, \dots, K_{m+1} - 1\}^d$, such that the family $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ is an independent family of random variables. Then it is certainly true that for any $j \in [K_{m+1}]^d$,

$$\mathbf{P}(j_\sigma = j) = K_{m+1}^{-d}. \quad (6.4)$$

Then $\sigma j_\sigma \in \Sigma_{m+1}^d$. We can thus define a measure μ_S such that, for each $x \in \mathbf{R}^d$,

$$d\mu_S(x) = r_{m+1}^d \sum_{\sigma \in \Pi_{m+1}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}_{m+1}^d : \mu_S(Q) > 0\},$$

then S is \mathcal{Q}_{m+1} discretized, μ_S is supported on S , and $S \subset T$. Our goal is to show that, with non-zero probability, some choice of the family of indices $\{j_\sigma : \sigma \in \Pi_{m+1}^d\}$ yields a set S and a measure μ_S satisfying Properties (A) and (B) of Proposition 4. In our calculations, it will help us to decompose the

measure μ_S into components roughly supported on sidelength r_{m+1}^d cubes. For each $\sigma \in \Pi_{m+1}(T)$, define a measure μ_σ such that for each $x \in \mathbf{R}^d$,

$$d\mu_\sigma(x) = r_{m+1}^d \psi_{m+1}(x - a(\sigma j_\sigma)) \cdot d\mu_T(x).$$

Then $\mu_S = \sum_{\sigma \in \Pi_{m+1}^d(T)} \mu_\sigma$. We shall split the proof of Properties (A), (B), and (C) into several more managable lemmas.

Lemma 6. *If*

$$M_{m+1} \geq \left(3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right)^2, \quad (6.5)$$

then $\mu_S(\mathbf{R}^d) \geq \mu_T(\mathbf{R}^d) - M_{m+1}^{-1/2}$.

Proof. Fix $\sigma \in \Pi_{m+1}^d$. If $j_0, j_1 \in \{0, \dots, K_{m+1} - 1\}^d$, then

$$|a(\sigma j_0) - a(\sigma j_1)| = |j_0 - j_1| \cdot l_{m+1} \leq (\sqrt{d} \cdot K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1}. \quad (6.6)$$

Together with (4.5), (6.6) implies

$$\begin{aligned} & \left| r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_0)) \mu_T(x) - r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_1)) \mu_T(x) \right| \\ & \leq r_{m+1}^d \int_{\mathbf{R}^d} \psi_{m+1}(x) |\mu_T(x + a(\sigma j_0)) - \mu_T(x + a(\sigma j_1))| \\ & \leq \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1} \\ & = \sqrt{d} \cdot r_{m+1}^{d+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad (6.7)$$

Thus (6.7) implies that for each σ ,

$$|\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \leq \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \quad (6.8)$$

Furthermore, (4.6) implies

$$\begin{aligned} & \sum_{\sigma \in \Pi_{m+1}^d} \mathbf{E}(\mu_\sigma(\mathbf{R}^d)) \\ & = r_{m+1}^d \sum_{(\sigma, j) \in \Sigma_{m+1}^d} \mathbf{P}(j_\sigma = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a(\sigma j_\sigma)) d\mu_T(x) \\ & = \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left(\sum_{(\sigma, j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\ & = \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \mu_T(\mathbf{R}^d) = \mu_T(\mathbf{R}^d). \end{aligned} \quad (6.9)$$

For all but at most $3^d r_{m+1}^{-d}$ indices $\sigma \in \Pi_{m+1}^d$, $\mu_\sigma = 0$ almost surely. Thus we can apply the triangle inequality together with (6.8) and (6.9) to conclude that

$$\begin{aligned}
|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| &= \left| \sum_{\sigma \in \Pi_{m+1}^d} [\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))] \right| \\
&\leq \sum_{\sigma \in \Pi_{m+1}^d} |\mu_\sigma(\mathbf{R}^d) - \mathbf{E}(\mu_\sigma(\mathbf{R}^d))| \\
&\leq (3^d r_{m+1}^{-d}) \left(\sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \\
&= \left(3^d \sqrt{d} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \right) \cdot r_{m+1} \\
&= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.
\end{aligned} \tag{6.10}$$

Thus (6.5) and (6.10) imply that, $|\mu_S(\mathbf{R}^d) - \mu_T(\mathbf{R}^d)| \leq M_{m+1}^{-1/2}$. \square

Lemma 7. *If*

$$M_{m+1} \geq (10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)})^{1/\varepsilon}, \tag{6.11}$$

then

$$\mathbf{P}(S \text{ does not satisfies Property (A)}) \leq 1/10.$$

Proof. For any cube $Q \in \Sigma_{m+1}^d$, there are at most 3^d indices $\sigma j \in \Sigma_{m+1}^d$ such that $Q_{\sigma j} \cap Q \neq \emptyset$, and so a union bound together with (6.4) gives

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S)) \leq \sum_{Q_{\sigma j} \cap Q \neq \emptyset} \mathbf{P}(j_\sigma = j) \leq 3^d K_{m+1}^{-d}. \tag{6.12}$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_k \in S\}$ are independent from one another for $k \in \{1, \dots, n\}$, which together with (6.12) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_1 \in S) \cdots \mathbf{P}(Q_n \in S) \leq 3^{dn} K_{m+1}^{-dn}. \tag{6.13}$$

Taking expectations over all cubes in B , and applying (6.1) and (6.13) gives

$$\begin{aligned}
\mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leq \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\
&\leq l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\
&= \frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}.
\end{aligned} \tag{6.14}$$

Since $\varepsilon \leq (dn - s)/2$, we conclude

$$\begin{aligned} (dn - s - \varepsilon) \left(\frac{s}{dn - s} + c\varepsilon \right) &= s + \varepsilon \left(c(dn - s - \varepsilon) - \frac{s}{dn - s} \right) \\ &\geq s + \varepsilon \left(\frac{c(dn - s)}{2} - \frac{s}{dn - s} \right) \\ &= s + \varepsilon \frac{3dn - s}{dn - s} \geq s + 2\varepsilon. \end{aligned}$$

Applying (6.3), we therefore conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geq M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \geq M_{m+1}^{s+2\varepsilon}.$$

Combined with (6.11), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leq \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^\varepsilon} \leq 1/10. \quad (6.15)$$

We can then apply Markov's inequality with (6.14) and (6.15) to conclude

$$\begin{aligned} \mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \emptyset) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geq 1) \\ &\leq \mathbf{E}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n))) \\ &\leq 1/10. \end{aligned} \quad \square$$

Lemma 8. Set $D = \{k \in \mathbf{Z}^d : |k| \leq 10l_{m+1}^{-1}\}$. Then if

$$K_{m+1} \leq M_{m+1}^{\frac{2dn}{dn-s}}, \quad (6.16)$$

and

$$M_{m+1} \geq \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \quad (6.17)$$

then

$$\mathbf{P}\left(\|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1})\right) \leq 1/10 \quad (6.18)$$

Proof. For each $\sigma \in \Pi_{m+1}^d$, and $k \in \mathbf{Z}$, define $X_{\sigma k} = \widehat{\mu}_\sigma(k) - \widehat{\mathbf{E}(\mu_\sigma)}(k)$. Applying (4.2) gives

$$\begin{aligned} \sum_{\sigma \in \Pi_{m+1}^d} \widehat{\mathbf{E}(\mu_\sigma)}(k) &= \sum_{\sigma \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \psi_{m+1}(x - a(\sigma j)) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) = \widehat{\mu}_T(k). \end{aligned} \quad (6.19)$$

For each σ and k , the standard (L^1, L^∞) bound on the Fourier transform, combined with (6.8), shows

$$\begin{aligned} \|X_{\sigma k}\|_{\psi_2(L)} &\leq 10|X_{\sigma k}| \\ &\leq 10[|\mu_\sigma(\mathbf{R}^d)| + \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)] \\ &\leq 10^2 \left(\mathbf{E}(\mu_\sigma)(\mathbf{R}^d) + \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (6.20)$$

For a fixed k , the family of random variables $\{X_{\sigma k} : \sigma \in \Pi_{m+1}^d\}$ are independent. Furthermore, $\sum X_{\sigma k} = \widehat{\mu_S}(k) - \widehat{\mathbf{E}(\mu_S)}(k)$. Equations (4.6) and (6.4) imply that

$$\begin{aligned} \mathbf{E}(\widehat{\mu_S}(k)) &= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \left(\sum_{(\sigma, j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a(\sigma j)) \right) d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} \widehat{\mu_T}(k) = \widehat{\mu_T}(k). \end{aligned} \quad (6.21)$$

Hoeffding's inequality, together with (6.20) and (6.21), imply that

$$\begin{aligned} \|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} &\leq 10^3 \sqrt{d} \left(\left(\sum \mathbf{E}(\mu_\sigma)(\mathbf{R}^d)^2 \right)^{1/2} + r_{m+1}^{d/2+1} \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned} \quad (6.22)$$

Equation (4.5) shows

$$\begin{aligned} \mathbf{E}(\mu_\sigma)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int \psi_{m+1}(x - a(ij)) d\mu_T(x) \\ &\leq r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad (6.23)$$

Combining (6.22) and (6.23) gives

$$\|\widehat{\mu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \leq 10^3 \sqrt{d} \left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \right] r_{m+1}^{d/2}. \quad (6.24)$$

We can then apply a union bound over the set D , which has cardinality at most $10^{d+1}l_{m+1}^{-d}$, together with (6.24) to conclude that

$$\begin{aligned} \mathbf{P} \left(\|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \\ \leq 10^{d+2} \cdot l_{m+1}^{-d} \exp \left(-\frac{\log(M_{m+1})^2}{10^7 d} \right) \\ = 10^{d+2} l_m^{-d} \exp \left(d \log(M_{m+1} K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d} \right). \end{aligned} \quad (6.25)$$

Combined with (6.16) and (6.17), (6.25) implies

$$\mathbf{P} \left(\|\widehat{\mu}_S - \widehat{\mu}_T\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \leq 1/10. \quad (6.26)$$

Thus $\widehat{\mu}_S$ and $\widehat{\mu}_T$ are highly likely to differ only by a negligible amount over small frequencies. \square

Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}_T(\xi)| |\xi|^t < \infty.$$

In light of (4.7), if we define, for each $t > 0$,

$$B(t) = \sup_{\xi \in \mathbf{R}^d} |\widehat{\psi}(\xi)| |\xi|^t,$$

then

$$\sup_{\xi \in \mathbf{R}^d} |\widehat{\psi_{m+1}}(\xi)| |\xi|^t = l_{m+1}^{-t} B(t).$$

Lemma 9. *Suppose that*

$$N_{m+1}^d \geq 10 \cdot 2^{3d/2+1} A(3d/2 + 1), \quad (6.27)$$

$$N_{m+1}^d \geq \frac{10 \cdot 2^{3d}}{1 + d/2} A(3d/2 + 1), \quad (6.28)$$

and

$$N_{m+1}^d \geq 10 \cdot 2^{7d/2+1} B(3d/2 + 1). \quad (6.29)$$

then if $|\eta| \geq 10l_{m+1}^{-1}$,

$$|\widehat{\mu}_S(\eta)| \leq \frac{1}{|\eta|^{d/2}}. \quad (6.30)$$

Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{\sigma \in \Pi_{m+1}^d \\ d(a(\sigma), T) \leq 2r_{m+1}^{-1}}} \delta_{a(ij_i)}.$$

Then $\mu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\mu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d , so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leq \alpha(\mathbf{R}^d) \leq 2^d$. Thus

$$|\widehat{\mu_S}(\eta)| \leq 2^d \int |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi. \quad (6.31)$$

If $|\xi| \leq |\eta|/2$, $|\eta - \xi| \geq |\eta|/2$, and since (4.5) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leq 1$, we find that for all $t > 0$,

$$\int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \leq \frac{A(t)2^{t-d}}{|\eta|^{t-d}}. \quad (6.32)$$

Set $t = 3d/2 + 1$. Equation (6.32), together with (6.27), implies

$$\begin{aligned} & \int_{0 \leq |\xi| \leq |\eta|/2} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi \\ & \leq \frac{A(3d/2 + 1)2^{1+d/2}|\eta|^{-1}}{|\eta|^{d/2}} \\ & \leq \frac{A(3d/2 + 1)2^{1+d/2}l_{m+1}}{|\eta|^{d/2}} \\ & \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \end{aligned} \quad (6.33)$$

Conversely, if $|\xi| \geq 2|\eta|$, then $|\eta - \xi| \geq |\xi|/2$, so for each $t > d$,

$$\begin{aligned} \int_{|\xi| \geq 2|\eta|} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| d\xi & \leq \int_{|\xi| \geq 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\ & \leq 2^d \int_{2|\eta|}^\infty r^{d-1-t} A(t) 2^t \\ & \leq \frac{4^d A(t)}{t-d} |\eta|^{d-t}. \end{aligned} \quad (6.34)$$

Set $t = 3d/2 + 1$. Equation (6.28), applied to (6.34), allows us to conclude

$$\int_{|\xi| \geq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \quad (6.35)$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu}_T\|_{L^\infty(\mathbf{R}^d)} \leq 1$ to conclude that

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \quad (6.36)$$

Set $t = 3d/2 + 1$. Then (6.36) and (6.29) imply

$$\int_{|\eta|/2 \leq |\xi| \leq 2|\eta|} |\widehat{\mu}_T(\eta - \xi)| |\widehat{\psi}_{m+1}(\xi)| d\xi \leq \frac{1}{10 \cdot 2^d \cdot |\eta|^{d/2}}. \quad (6.37)$$

It then suffices to sum up (6.33), (6.35), and (6.37), and apply (6.31). \square

Proof of Proposition 4, Continued. Let us now put all our calculations together. In light of Lemma 7 and Lemma 8, there exists some choice of j_σ for each σ , and a resultant non-random pair (μ_S, S) such that S satisfies Property (A) of the Lemma, and μ_S satisfies (6.18), implying that μ_S satisfies Property (C) of the Lemma. But Lemma 6 shows that μ_S always satisfies Property (B), and Lemma (9) shows Property (D) is also always satisfied. This completes the proof. \square

7 Construction of the Salem Set

Let us now choose the parameters to construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^\infty Z_i$, where Z_i has lower Minkowski dimension at most s for each i . Then choose an infinite sequence $\{i_m : m \geq 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers $\{\varepsilon_m : m \geq 1\}$, with $\varepsilon_m < (dn - s)/2$ for each m . We choose our parameters $\{M_m\}$ and $\{K_k\}$ inductively. First, set $X_0 = [0, 1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0 . At the m th step of our construction, we have already found a set X_{m-1} and a measure μ_{m-1} . We then choose K_m and M_m such that

$$K_m, M_m \geq C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m^{\frac{s}{dn-s} + c\varepsilon_m} \leq K_m \leq 2M_m^{\frac{s}{dn-s} + c\varepsilon},$$

and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in \mathcal{Q}_m , the union of which, we define to be equal to B_m . We can then apply Proposition 4 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a \mathcal{Q}_{m+1} discretized set $S \subset T$, and a measure μ_S supported on S . We define $X_m = S$, and $\mu_m = \mu_S$.

The last paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$. Just as in our previous paper, it is easy to see X must be a configuration avoiding set. Given any $(x_1, \dots, x_n) \in Z$, there are infinitely many integers m_k such that $(x_1, \dots, x_n) \in B_{m_k}$. If $|x_i - x_j| \geq \varepsilon$ for each $i \neq j$, and $r_{m_k} \leq \varepsilon/2$, then (x_1, \dots, x_n) is contained in a strongly non-diagonal cube in $\mathcal{Q}_{m_k}(B_k)$, and as such $X^n \subset X_{m_k}^n$ does not contain (x_1, \dots, x_n) .

8 Proof that X is Salem

We now show X is Salem, completing the proof of Theorem 1. Since the masses of the sequence of measures $\{\mu_m\}$ is uniformly bounded, there is some subsequence μ_{m_i} which converges weakly to some measure μ . Repeated applications of Property (B) of Proposition 4 imply

$$\mu(\mathbf{R}^d) = \lim_{i \rightarrow \infty} \mu_{m_i}(\mathbf{R}^d) \geq 1 - \sum_{m=1}^{\infty} M_m^{-1/2}.$$

In particular, μ is a non-zero measure if the sequence $\{M_m\}$ is rapidly increasing. Moreover, for each $k \in \mathbf{Z}^d$,

$$\widehat{\mu}(k) = \lim_{i \rightarrow \infty} \widehat{\mu_{m_i}}(k).$$

Thus

$$|\widehat{\mu}(k)| \leq |\widehat{\mu_0}(k)| + \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)|.$$

Fix $\varepsilon > 0$. Since $l_m \leq 2^{-m}/10$, we find that for $m \geq \log(k)$, $|k| \leq 10l_{m+1}^{-1}$. Thus we can apply Property (C) and (D) of Proposition 4 to conclude

$$\begin{aligned}
& \sum_{m=0}^{\infty} |\widehat{\mu_{m+1}}(k) - \widehat{\mu_m}(k)| \\
& \leq 2 \log(k) |k|^{-d/2} + \sum_{m=\log(k)}^{\infty} r_{m+1}^{d/2} \log(M_{m+1}) \\
& \lesssim_{\varepsilon} |k|^{\varepsilon-t/2} \left(1 + \sum_{m=\log(k)}^{\infty} |k|^{t/2-\varepsilon} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
& \leq |k|^{\varepsilon-t/2} \left(1 + 10^{t/2-\varepsilon} \sum_{m=\log(k)}^{\infty} l_{m+1}^{\varepsilon-t/2} r_{m+1}^{d/2} \log(M_{m+1}) \right) \\
& \lesssim_{\varepsilon} |k|^{\varepsilon-t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{1}{K_{m+1}^{\varepsilon}} \frac{K_{m+1}^{t/2}}{M_{m+1}^{d/2-t/2}} \right) \\
& \lesssim |k|^{\varepsilon-t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^{\varepsilon}} \frac{M_{m+1}^{(t/2)(\frac{s}{dn-s})}}{M_{m+1}^{d/2-t/2}} \right) \\
& = |k|^{\varepsilon-t/2} \left(1 + \sum_{m=\log(k)}^{\infty} \frac{M_{m+1}^{c\varepsilon_m(t/2)}}{K_{m+1}^{\varepsilon}} \right) \lesssim_{\varepsilon} |k|^{\varepsilon-t/2}.
\end{aligned}$$

The last inequality follows because $\varepsilon_m \rightarrow 0$, and so the series is summable if the sequence $\{K_m\}$ increases rapidly enough. Since μ_0 is smooth and compactly supported, we find

$$\sup_{k \in \mathbf{Z}^d} |k|^{t/2-\varepsilon} |\widehat{\mu}(k)| \lesssim_{\varepsilon} 1 + \sup_{k \in \mathbf{Z}} |k|^{t/2-\varepsilon} |\widehat{\mu_0}(k)| < \infty.$$

Since $\varepsilon > 0$ was arbitrary, this shows that the Fourier dimension of X is at least t . Because X_m is the union of $(M_1 \dots M_m)^d$ sidelength l_m cubes, one can easily show using (6.3) that the lower Minkowski dimension of X is upper bounded by t . But these two bounds imply that the Hausdorff dimension, Fourier dimension, and Minkowski dimension are all equal to t . Thus X is Salem of dimension t .

References

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