Sets Avoiding Patterns with Fourier Decay

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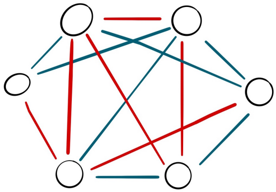
General Research Question



- Phenomenon: Structure appears in suitably large objects.
- Like Ramsey Theory, but with a more analytical foundation, e.g. Geometric Measure Theory / Harmonic Analysis.

Examples

- How large can a subset X of $\mathbf{R}^d$ be such that there does not exist four distinct points $x_1, x_2, x_3, x_4 \in X$ which form a parallelogram, i.e. satisfy $x_2 - x_1 = x_4 - x_3$.
- How large can a subset X of $\mathbf{R}^d$ be such that no three distinct points $x_1, x_2, x_3 \in X$ form a right angle, i.e satisfy $(x_2 - x_1) \cdot (x_3 - x_1) = 0$.
- How large can an additive group $G \subset \mathbf{R}^d$ be, such that $G \cap \mathbf{Q}^d = \{0\}$.
- Our problem isn't well specified: No subset of $\mathbf{R}^d$ with positive measure can satisfy the constraints of these problems, but we can find discrete sets of arbitrarily large cardinality which do satisfy these constraints.

What Other Definitions of ‘Largeness’ Can We Use?

Fractional Dimension

- *Fractional dimensions* measure largeness / thickness of sets. Standard fractional dimension are defined in terms of coverings.
  - Roughly speaking, a set $X \subset \mathbf{R}^d$ has *Minkowski dimension s* if it can be covered by at most r^{-s} balls of radius r, for arbitrarily small $r > 0$.
  - Again working roughly, a set $X \subset \mathbf{R}^d$ has *Hausdorff dimension s* if it can be covered by a family of arbitrarily small balls $\{B_1(r_1), B_2(r_2), \dots\}$, where $\sum_{i=1}^{\infty} r_i^s < \infty$.
- If $|X| > 0$, $\dim_{\mathbf{H}}(X) = \underline{\dim}_{\mathbf{M}}(X) = d$.
- If $\#(X) < \infty$, $\dim_{\mathbf{H}}(X) = \underline{\dim}_{\mathbf{M}}(X) = 0$.

Fourier Dimension

- A compact set X has *Fourier dimension* at least s if there exists a Borel probability measure μ supported on X such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$$

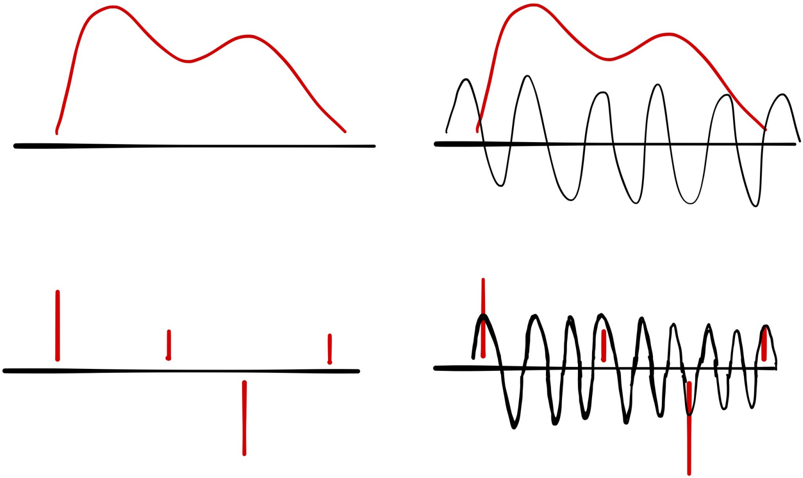
for $\xi \in \mathbf{R}^d$. Then $\dim_{\mathbf{F}}(X)$ is the supremum of such values s.

- If $s < \dim_{\mathbf{H}}(X)$, then $|\widehat{\mu}(\xi)||\xi|^{s/2}$ is small for *most* ξ, i.e.

$$\frac{|\{\xi \in B_R : |\widehat{\mu}(\xi)| \geq |\xi|^{-s/2}\}|}{|B_R|} = o(1).$$

But a uniform bound is not always possible.

- In general $\dim_{\mathbf{F}}(X) \leq \dim_{\mathbf{H}}(X) \leq \dim_{\mathbf{M}}(X)$.

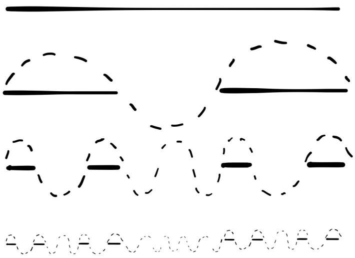


An Example

- Let C be the middle thirds Cantor set.
- For each n, C is covered by 2^n intervals of length $1/3^n$.
- Recall a set has Minkowski dimension s if it can be covered by r^{-s} intervals of length r. Here $r = 1/3^n$, and

$$2^n = 3^{n \log_3 2} = r^{-\log_3 2}.$$

  This suggests that $\dim_{\mathbf{H}}(C) = \dim_{\mathbf{M}}(C) = \log_3 2 \approx 0.63$.
- On the other hand, $\dim_{\mathbf{F}}(C) = 0$, since C is highly correlated with waves of frequency 3^n.



Salem Sets

- If, at each stage of the Cantor set construction, instead of taking the middle third J from each length l interval I, we remove $l \cdot t_I + J$, where $t_I \in [-1/6, 1/6]$ is selected uniformly at random, then we find that

$$\dim_{\mathbf{F}}(X) = \dim_{\mathbf{H}}(X) = \dim_{\mathbf{M}}(X) = \log_3 2.$$

- A set is *Salem* if $\dim_{\mathbf{F}}(X) = \dim_{\mathbf{H}}(X)$.
- Main Focus of Talk: To construct Salem sets, the more probabilistic tools we can develop (especially concentration of measure / square root cancellation results) the better.

Now Let's Return to Pattern Avoidance

The General Problem

- **Avoidance Problem**: Given a set $Z \subset \mathrm{R}^{nd}$, find $X \subset \mathrm{R}^d$ with large *Fourier* dimension such that for distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We say X *avoids* Z.
- Let $Z = \{(x, y, z) \in (\mathbf{R}^d)^3 : (x - z) \cdot (y - z) = 0\}$.
  - $X \subset \mathbf{R}^d$ avoids Z iff X does not contain any right angles.
- For each $m \in \mathbf{Z}^n - \{0\}$ and $a \in \mathbf{Z}$, define

  $$Z(m, a) = \{(x_1, \dots, x_n) \in (\mathbf{R}^d)^n : m_1 x_1 + \cdots + m_n x_n = a\}.$$

  If $Z_n = \bigcup_{m \in \mathbf{Z}^n - \{0\}} \bigcup_{a \in \mathbf{Z}} Z(m, a)$, then $X \subset \mathbf{R}^d$ avoids Z_n for all $n > 0$ if and only if X generates a subgroup of $\mathbf{R}^d$ disjoint from $\mathbf{Q}^d - \{0\}$.

- Fourier Dimension often gives much more structural information about a set than Minkowski dimension does.
- (Keleti, 1998) There exist an 'independant' set $X \subset \mathbf{R}$ with $\dim_{\mathbf{H}}(X) = 1$ such that there exists no nontrivial solutions to $m_1 x_1 + \cdots + m_n x_n = 0$ for any $m \in \mathbf{Z}^n$ and $x_1, \dots, x_n \in X$.
- (Rudin, 1960) If $\dim_{\mathbf{F}}(X) \geq 1/n$, then there exists some $m \in \mathbf{Z}^n$ and some $x_1, \dots, x_n \in X$ such that $m_1 x_1 + \cdots + m_n x_n = 0$.

- (Körner, 2009) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1/(n-1)$ that contains no solutions to $m_1 x_1 + \cdots + m_n x_n = 0$ for any $m \in \mathbf{Z}^n$.
- (Schmerkin, 2015) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1$ that contains no three term arithmetic progressions, i.e. no nontrivial solutions to the equation $x_2 - x_1 = x_3 - x_2$.
- (Liang and Pramanik, 2019) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1$ that contains no solutions to a 'translation invariant' equation of the form $m_1 x_1 + \cdots + m_n x_n = m_0 x_0$, where $m_0, \dots, m_n \geq 0$ and $m_1 + \cdots + m_n = m_0$.

Results in Literature

- How does the geometry of Z help us?

Author	Geometry of Z	$\dim_{\mathbf{H}}(X)$
Mathé (2012)	A degree r algebraic hypersurface in $\mathbf{R}^{dn}$	d/r
Fraser and Pramanik (2016)	An $nd - m$ dimensional surface in $\mathbf{R}^{dn}$	$\frac{m}{n-1}$
Denson, Pramanik, and Zahl (2019)	A subset of $\mathbf{R}^{dn}$ with (lower) Minkowski dimension s	$\frac{dn-s}{n-1}$
Denson (2019)	A subset of $\mathbf{R}^n$ such that there exists a full rank linear map $\pi : \mathbf{R}^n \to \mathbf{R}^m$ where $\pi(Z)$ is s dimensional	$\frac{m-s}{m}$

- Can we modify these constructions to obtain Salem sets?

Salem Set Result

Theorem

If Z is a countable union of sets with (lower) Minkowski dimension bounded by s, we can find a Salem set X avoiding Z with

$$dim_{\mathsf{F}}(X) = \frac{nd - s}{n - 1/2}.$$

- The previous results find a set X with

$$\dim_{\mathbf{H}}(X) = \frac{nd - s}{n - 1}.$$

Salem Set Result

Theorem

If Z is a countable union of sets of the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_n = f(x_1, \dots, x_{n-1})\}$$

where $f : \mathbb{R}^{d(n-1)} \to \mathbb{R}^d$ is smooth, and the matrix $D_{x_k} f(x_1, \dots, x_{n-1}) = \left(\frac{\partial f^i}{\partial x_{kj}} \right)$ is invertible for each k and distinct $x_1, \dots, x_n \in \mathbb{R}^d$, then we can find a Salem set X avoiding Z with

$$dim_{\mathbf{F}}(X) = \frac{d}{n - 3/4}.$$

- The previous results find a set X with $\dim_{\mathbf{H}}(X) = \frac{d}{n-1}$.
- We will focus on the ideas behind this proof in this talk.

Applications

TODO

Isolating a Single Scale

- We apply Baire category techniques to isolate a 'single scale' of the problem at a time.
- We consider a complete metric space $\mathcal{X}_s$ which consists of measures μ such that for each $\varepsilon > 0$,

$$\sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}(\xi)||\xi|^{s/2-\varepsilon} < \infty.$$

  Thus $\text{supp}(\mu)$ is a set with Fourier dimension at least s.
- Our goal is to show that the set of measures μ such that $\text{supp}(\mu)$ avoids the pattern Z is a set of first category in $\mathcal{X}_s$, where $s = d/(n - 3/4)$.

- This means we must show that for any disjoint closed cubes $Q_1, \dots, Q_n$ in $[0,1]^d$ with common sidelength s, the family

$$\mathcal{Y}_{Q_1,\dots,Q_n} = \left\{ \mu \in \mathcal{X}_s : \begin{array}{c} \text{If } x_1 \in Q_1 \cap \operatorname{supp}(\mu), \dots, \\ x_n \in Q_n \cap \operatorname{supp}(\mu), x_n \neq f(x_1, \dots, x_{n-1}) \end{array} \right\}.$$

  is dense in $\mathcal{X}_s$.
- It suffices to show that for any disjoint family of closed cubes $Q_1, \dots, Q_n \subset [0,1]^d$, and $\varepsilon_1, \varepsilon_2 > 0$, there exists a compactly supported measure μ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2-\varepsilon_1} \leq \varepsilon_2.$$

  and if $x_1 \in Q_1 \cap \operatorname{supp}(\mu), \dots, x_n \in Q_n \cap \operatorname{supp}(\mu)$, then

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

  (The uncertainty principle implies we only need to look at integer frequencies).

The Importance of Square Root Cancellation

- Fix $K > 0$ and $r > 0$. Let $x_1, \dots, x_K$ be points such that for $|\xi| \lesssim 1/r$, $|e^{2\pi i \xi \cdot x_1} + \dots + e^{2\pi i \xi \cdot x_K}| \lesssim K^{1/2}$. A trivial bound (triangle inequality) is $O(K)$, so we have 'square root cancellation'.
- Fix a mollifier $\phi \in C_c^\infty(\mathbf{R}^d)$, let $\phi_r(x) = r^{-d}\phi(x/r)$ and define
$$\mu(x) = \frac{\phi_r(x - x_1) + \dots + \phi_r(x - x_K)}{K}.$$
Then $\text{supp}(\mu)$ is covered by K radius r balls.
- Then
$$\widehat{\mu}(\xi) = K^{-1}\left(e^{2\pi i \xi \cdot x_1} + \dots + e^{2\pi i \xi \cdot x_K}\right)\widehat{\phi}(r\xi).$$
If $K = r^{-s}$ and r is sufficiently small, then
$$|\widehat{\mu}(\xi)| \leq K^{-1/2}|\widehat{\phi}(r\xi)| \leq r^{s/2}|\widehat{\phi}(r\xi)|$$
So if $|\xi| \leq 1/r$, $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$, and if $|\xi| \geq 1/r$, $\widehat{\phi}(r\xi)$ decays fast.
- $K^{-1/2}$ error (or even $K^{-1/2}\log(K)^{100}$) is perfectly fine.

An Interlude on Concentration Inequalities

Concentration Bounds

- Heuristic: A function of *many* independant random variables is tightly concentrated about it's mean (plus or minus it's variance).
- Where this is true: A sum $X_1 + \cdots + X_K$ of i.i.d. random variables, where K is large.
- Where this fails: $\sum_{k=1}^{\infty} X_k/2^k$, where $\{X_k\}$ are independent and uniformly distributed $\{0,1\}$ valued Bernoulli random variables.
- The distribution of this sum is uniform on $[0,1]$, so not tightly concentrated at all despite involving *infinitely many* random variables because X_k has much more influence on the overall result for small k vs for large k.

Concentration Bounds

Theorem (Hoeffding's Inequality)

Suppose $X_1, \ldots, X_K$ are independant random variables with $|X_i| \leq A$ for each i and $\sum \mathrm{E}(X_i) = 0$, then

$$\mathrm{P}\left(|X_1 + \cdots + X_K| \geq t\right) \leq 4 \exp\left(-2t^2/KA^2\right).$$

Thus $|X_1 + \cdots + X_K| \lesssim AK^{1/2}$ with high probability.

Concentration Bounds

Theorem (McDiarmid's Inequality)

Suppose $f : \mathrm{R}^K \to \mathrm{R}$ *is a function. Suppose that for each* $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_K \in \mathrm{R}$, *and any* $x_i, x_i' \in \mathrm{R}$,

$$|f(x_1, \ldots, x_i, \ldots, x_K) - f(x_1, \ldots, x_i, \ldots, x_K)| \leq A$$

Then if $X_1, \ldots, X_K$ *are a family of independant random variables,*

$$\mathrm{P}(|f(X_1, \ldots, X_K)) - \mathrm{E}(f(X_1, \ldots, X_N))|) \leq 4 \exp\left(-t^2/2A^2K\right).$$

Thus $|f(X_1, \ldots, X_K)) - \mathrm{E}(f(X_1, \ldots, X_N))| \lesssim AK^{1/2}$ *with high probability.*

The Avoidance Method

Thanks for listening!