

# Large Salem Sets Avoiding Nonlinear Configurations

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## Abstract

We construct Salem sets with large dimension that avoid patterns described by the zero sets of a family of smooth functions, as well as rough families of patterns, complementing previous results constructing sets with large Hausdorff dimension. For a (possibly uncountable) family of uniformly Lipschitz functions  $\{f_i : (\mathbb{T}^d)^{n-2} \rightarrow \mathbb{T}^d\}$ , we obtain a Salem subset of  $\mathbb{T}^d$  with dimension  $d/(n-1)$  avoiding nontrivial solutions to the equation  $x_n - x_{n-1} = f_i(x_1, \dots, x_{n-2})$ . For a countable family of smooth functions  $\{f_i : (\mathbb{T}^d)^{n-1} \rightarrow \mathbb{T}^d\}$  satisfying a modest geometric condition, we obtain a Salem subset of  $\mathbb{T}^d$  with dimension  $d/(n-3/4)$  avoiding nontrivial solutions to the equation  $x_n = f(x_1, \dots, x_{n-1})$ . For a set  $Z \subset \mathbb{T}^{dn}$  which is the countable union of a family of sets, each with lower Minkowski dimension  $s$ , we obtain a Salem subset of  $\mathbb{T}^d$  of dimension  $(dn-s)/(n-1/2)$  which avoids the ‘rough pattern’ specified by  $Z$ , i.e. whose Cartesian product does not intersect  $Z$  except at points with non-distinct coordinates. The key techniques to obtain the existence of these sets is the study of random exponential sums using probabilistic concentration bounds.

## 1 Introduction

Geometric measure theory explores the relationship between the geometry of subsets of  $\mathbb{R}^n$ , and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what geometric information can be gathered from the Fourier analytic properties of these measures. A large body of research focuses on showing that the support of a measure with Fourier decay contain patterns, such as a family of points forming an arithmetic progression. In this paper, we work in the opposite direction, showing that *most* sets supporting measures with a certain type of Fourier decay *do not* contain certain patterns. More precisely, given a set  $Z \subset \mathbb{T}^{dn}$ , we focus on showing that a ‘generic’ compact set  $E \subset \mathbb{T}^d$  supporting a measure whose Fourier transform exhibits a quantitative decay bound also avoids the pattern defined by  $Z$ , in the sense that for any *distinct* points  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \notin Z$ . Common examples include sets of the form

$$Z = \{(x_1, \dots, x_n) \in \mathbb{T}^{dn} : x_n = f(x_1, \dots, x_n)\},$$

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in which case the set  $E$  does not contain solutions to the equation  $x_n = f(x_1, \dots, x_n)$  for distinct points  $x_1, \dots, x_n \in E$ .

A useful statistic associated with any Borel subset  $E$  of  $\mathbb{T}^d$  is its *Fourier dimension*; given a finite Borel measure  $\mu$ , its Fourier dimension  $\dim_{\mathbb{F}}(\mu)$  is the supremum of all  $s \in [0, d]$  such that  $\sup_{\xi \in \mathbb{Z}^d} |\hat{\mu}(\xi)| |\xi|^{s/2} < \infty$ . The Fourier dimension of a Borel set  $E$  is then the supremum of  $\dim_{\mathbb{F}}(\mu)$ , where  $\mu$  ranges over all Borel probability measures  $\mu$  with  $\text{supp}(\mu) \subset E$ . A particularly tractable family of sets in this scheme are *Salem sets*, sets whose Fourier dimension agrees with their Hausdorff dimension. Most pattern avoiding sets constructed in the literature are not Salem, often having Fourier dimension zero. Nonetheless, the methods in this paper are able to prove the existence of large Salem pattern avoiding sets.

Our paper is part of a body of literature on *pattern avoidance problems*: given a set  $Z \subset \mathbb{T}^{dn}$ , the pattern avoidance problem for  $Z$  is to construct a *pattern avoiding* set  $E \subset \mathbb{T}^d$ , which means that for any *distinct*  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \in Z$ , which is as large as possible with respect to some particular criteria relevant to the problem, such as the Hausdorff or Fourier dimension. The main inspiration for the results of this paper was the result of [2] on ‘rough’ patterns, which constructed, for any set  $Z \subset \mathbb{T}^{dn}$  formed from the countable union of compact sets with lower Minkowski dimension at most  $\alpha$ , a set  $E$  avoiding  $Z$  with

$$\dim_{\mathbb{H}}(E) = \max \left( \frac{dn - \alpha}{n - 1}, d \right). \quad (1.1)$$

However, the sets  $E$  constructed using this method are not guaranteed to be Salem, and the construction is not even guaranteed to produce sets  $E$  with  $\dim_{\mathbb{F}}(E) > 0$ . Our goal was to modify the construction of [2] in order to ensure the resulting sets constructed were Salem. The baseline in the setting of Salem sets was Theorem 38 of [1], which constructed a Salem set  $E$  avoiding  $Z$  with

$$\dim_{\mathbb{F}}(E) = \max \left( \frac{dn - \alpha}{n}, d \right). \quad (1.2)$$

In this paper, we are only able to construct Salem sets with dimension matching (1.1) when  $Z$  exhibits translational symmetry (Theorem 1.3 of this paper), but for more general sets we are still able to improve upon the dimension given in (1.2) (via Theorems 1.1 and 1.2).

The methods in this paper are generic, in the sense of the Baire category theorem; we define a complete metric space  $\mathcal{X}_{\beta}$  for each  $\beta \in (0, d]$ , which consists of all pairs  $(E, \mu)$ , where  $E$  is a compact set,  $\mu$  is a Borel probability measure supported on  $E$ , and  $\dim_{\mathbb{F}}(\mu) \geq \beta$  (which implies  $\dim_{\mathbb{F}}(E) \geq \beta$ ), and show that for an appropriate choice of  $\beta$ , the family of all pairs  $(E, \mu) \in \mathcal{X}_{\beta}$  such that  $E$  is Salem and avoids a pattern is *comeager*, or *generic* in  $\mathcal{X}_{\beta}$  (the complement of a set of first category).

Many other approaches to constructing pattern avoiding sets, such as [2], [1], [4], and [5], construct *explicit* pattern avoiding Salem sets via various *queuing techniques*. The approaches in this paper can be modified to produce explicit pattern avoiding Salem sets via these kinds of methods. But we feel that sticking with Baire category techniques in this paper allows us to avoid the technical numerology that goes into these arguments so that we can focus on the more novel aspects of our analysis.

Let us now introduce the three main results of this paper. Theorem 1.1 has the weakest conclusions and has the simplest proof, but works for the most general family of patterns.

**Theorem 1.1.** Fix  $0 \leq \alpha < dn$ , and let  $Z \subset \mathbb{T}^{dn}$  be a compact set with lower Minkowski dimension at most  $\alpha$ . Set

$$\beta_0 = \min \left( \frac{dn - \alpha}{n - 1/2}, d \right).$$

Then there exists a compact Salem set  $E \subset \mathbb{T}^d$  with  $\dim_{\mathbb{F}}(E) = \beta_0$ , such that for any distinct points  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \notin Z$ . Moreover, if  $\beta \leq \beta_0$ , then the family of all pairs  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids the pattern generated by  $Z$  is comeager.

Now we consider the problem of avoiding a pattern defined by an equation of the form  $x_n = f(x_1, \dots, x_{n-1})$ . Under the assumption that  $f$  is smooth, and satisfies a regularity condition geometrically equivalent to the graph of  $f$  being transverse to any axis-oriented hyperplane, we are able to improve the Fourier dimension bound obtained, though not quite enough to match the Hausdorff dimension bound obtained in [4], except in the fairly trivial case where  $n = 2$ .

**Theorem 1.2.** Consider a smooth function  $f : V \rightarrow \mathbb{T}^d$ , where  $V$  is an open subset of  $\mathbb{T}^{d(n-1)}$ , such that for each  $k \in \{1, \dots, n-1\}$ , the matrix

$$D_{x_k} f(x_1, \dots, x_{n-1}) = \left( \frac{\partial f_i}{\partial (x_k)_j} \right)_{1 \leq i, j \leq d}$$

is invertible whenever  $x_1, \dots, x_{n-1}$  are distinct and  $(x_1, \dots, x_{n-1}) \in V$ . Then there exists a compact Salem set  $E \subset \mathbb{T}^d$  with dimension

$$\beta_0 = \begin{cases} d & : n = 2 \\ d/(n - 3/4) & : n \geq 3 \end{cases}$$

such that for any distinct points  $x_1, \dots, x_n \in E$ , with  $x_1, \dots, x_{n-1} \in V$ ,

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

Moreover, if  $\beta \leq \beta_0$ , then the family of pairs  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids solutions to the equation  $x_n = f(x_1, \dots, x_{n-1})$  for distinct points  $x_1, \dots, x_n \in E$  is comeager.

Finally, we consider a situation in which our pattern exhibits some translational symmetry. Here we can construct Salem sets with dimension exactly matching the Hausdorff dimension results obtained in [2]. The simplest example of such of a pattern is that specified by an equation of the form  $m_1 x_1 + \dots + m_n x_n = s$ , where at least two of the integers  $m_1, \dots, m_n \in \mathbb{Z}$  is nonzero, and  $s$  ranges over a low dimension set in  $\mathbb{T}^d$ . But we can also consider more nonlinear patterns, such as those formed by solutions to an equation  $m_1 x_1 + m_2 x_2 = f(x_3, \dots, x_n)$  for  $m_1, m_2 \neq 0$ , and a Lipschitz function  $f$ . Even in the case of linear patterns, this theorem implies new results.

**Theorem 1.3.** Fix  $d(n-1) \leq \alpha < dn$ ,  $a \in \mathbb{Q} - \{0\}$ , and a locally Lipschitz function  $T : V \rightarrow \mathcal{E}$ , where  $V$  is an open subset of  $\mathbb{T}^{d(n-2)}$ , and where  $\mathcal{E}$  is the family of all compact subsets of  $\mathbb{T}^d$ , equipped with the Hausdorff distance metric. Suppose that the sets  $T(x_1, \dots, x_{n-2})$  locally uniformly have lower Minkowski dimension at most  $\alpha - d(n-1)$ , in the sense that

for any  $\gamma > \alpha$ , and any closed set  $W \subset V$ , there exists a decreasing sequence  $\{r_i\}$  with  $\lim_{i \rightarrow \infty} r_i = 0$  such that for  $x \in W$ ,  $|T(x)_{r_i}| \leq r_i^{nd-\gamma}$  for all  $x \in V$ . Set

$$\beta_0 = \min \left( \frac{dn - \alpha}{n - 1}, d \right).$$

Then there exists a compact Salem set  $E \subset \mathbb{T}^d$  with  $\dim_{\mathbb{F}}(E) = \beta_0$ , such that for any distinct points  $x_1, \dots, x_n \in E$ , with  $(x_1, \dots, x_{n-2}) \in V$ ,

$$x_n - ax_{n-1} \notin T(x_1, \dots, x_{n-2}).$$

Moreover, if  $\beta \leq \beta_0$ , then the family of all pairs  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids the pattern generated by  $Z$  is comeager.

Intuitively, if for each  $(x_1, \dots, x_{n-2}) \in V$ ,  $T(x_1, \dots, x_{n-2})$  is a set of lower Minkowski dimension  $\alpha - (n - 1)d$ , then one might expect the set

$$Z = \{(x_1, \dots, x_n) : x_1 - ax_2 \in T(x_1, \dots, x_{n-2})\}$$

to have dimension  $\alpha$ . This is the case, for instance, if  $n = 2$ , and  $T$  is a single set with lower Minkowski dimension  $\alpha$ . This is why Theorem 1.3 is formulated in terms of sets of dimension  $\alpha - (n - 1)d$ , because it formulates the result in a way that is more comparable to the way Theorem 1.1 of this paper is stated, and to the way the main result of [2] is stated.

An archetypical example of a function  $T$  to which Theorem 1.3 applies is obtained by setting  $T(x_1, \dots, x_{n-2}) = \{f(x_1, \dots, x_{n-2})\}$  for some Lipschitz continuous function  $f$ , in which case Theorem 1.3 constructs Salem sets  $E$  of dimension  $(dn - 1)/(n - 1)$  avoiding solutions to the equation  $x_n - ax_{n-1} = f(x_2, \dots, x_{n-1})$ . The advantage of considering a ‘multi-valued function’  $T$  instead of a ‘single-valued’ function  $f$  in Theorem 1.3 is that it enables us to consider problems in which we are avoiding a family of equations of the form  $x_n - ax_{n-1} = f_i(x_1, \dots, x_{n-2})$ , where  $\{f_i : i \in I\}$  is an *uncountable* family of Lipschitz functions with uniformly bounded Lipschitz constant, such that the set  $T(x_1, \dots, x_{n-2}) = \{f_i(x_1, \dots, x_{n-2}) : i \in I\}$  locally uniformly has lower Minkowski dimension at most  $\alpha - d$ , so that the assumptions of Theorem 1.3 apply to the function  $T$ . An application of this property to avoiding uncountable families of linear patterns is detailed in Section 3.

#### Remarks 1.4.

1. It is quite surprising that we are able to generically improve the bound (1.2) for arbitrary families of nonlinear patterns. In [10], for each  $\beta$ , a probability measure is constructed on the space of all  $\beta$ -dimensional subsets of  $\mathbb{T}^d$ . Theorem 1.1 of that paper implies that when  $Z \subset \mathbb{T}^{dn}$  is the set consisting of all translations and dilations of a tuple of points  $(y_1, \dots, y_n) \in \mathbb{T}^{dn}$ , then  $Z$  has dimension  $\alpha = d + 1$ , and for any  $\beta > (dn - \alpha)/n$ , almost every compact  $\beta$ -dimensional set  $E \subset \mathbb{T}^d$  does not avoid the pattern  $Z$ . On the other hand, Theorem 1.1 implies that for any  $\beta \leq (dn - \alpha)/(n - 1/2)$ , a generic element of  $\mathcal{X}_\beta$  avoids the pattern specified by  $Z$ . Thus the theorems in this paper imply that generic pattern avoidance in a Baire category sense differs from generic pattern avoidance in a probabilistic sense for  $(dn - \alpha)/n \leq \beta \leq (dn - \alpha)/(n - 1/2)$ .

2. Because we are using Baire category techniques, the results we obtain remain true when, instead of avoiding a single pattern, we avoid a countable family of patterns. This is because the countable union of sets of first category is a set of first category. As an example of this property, we note that the conclusion of Theorem 1.1 holds when  $Z$  is replaced by a countable union of compact sets, each with lower Minkowski dimension at most  $\alpha$ . Similar generalizations apply to Theorem 1.2 and Theorem 1.3.
3. If  $0 \leq \alpha < d$ , then the pattern avoiding set  $[0, 1]^d - \pi_i(Z)$  has full Hausdorff dimension  $d$ , where  $\pi_i(x_1, \dots, x_n) = x_i$  is projection onto a particular coordinate. Thus the pattern avoidance problem is trivial in this case for Hausdorff dimension. This is no longer true when studying Fourier dimension, since  $[0, 1]^d - \pi_i(Z)$  need not be a Salem set, nor even have particularly large Fourier dimension compared to the sets guaranteed by Theorem 1.1.

That this is true is hinted at in Example 8 of [3], where it is shown that there exists a set  $X \subset [0, 1]$  which is the countable union of a family of compact sets  $\{X_k\}$  with  $\sup_k \dim_{\mathbb{M}}(X_k) \leq 3/4$ , such that  $\dim_{\mathbb{F}}([0, 1] - X) \leq 3/4$ . Thus  $[0, 1] - X$  is not a Salem set, since  $[0, 1]$  has Hausdorff dimension one. If we let  $F$  be any countable union of compact sets with Minkowski dimension zero, and we set

$$Z = \bigcup_{i=0}^{n-1} F^i \times X \times F^{n-i-1},$$

then  $Z$  is a countable union of compact sets with Minkowski dimension at most  $3/4$ , whereas

$$\dim_{\mathbb{F}}([0, 1] - \pi_i(X)) \leq \dim_{\mathbb{F}}([0, 1] - X) \leq 3/4$$

for each  $i \in \{1, \dots, n\}$ . Thus the trivial solution obtained by removing a projection of  $Z$  onto a particular coordinate axis does not necessarily give a pattern avoiding set with optimal Fourier dimension in this setting. Applying Theorem 1.1 directly to  $Z$  shows that a generic Salem set  $E \subset \mathbb{T}$  of dimension  $(n - 3/4)/(n - 1/2)$  avoids  $Z$ , which exceeds the dimension of the trivial construction for all  $n > 1$ . In fact, a generic Salem set  $E \subset \mathbb{T}$  with dimension 1 will avoid  $Z$ , since any subset of  $\mathbb{T} - F$  will avoid  $Z$ , and Theorem 1.1 applied with  $Z = F$  proves that a generic Salem set  $E$  of dimension 1 will be contained in  $\mathbb{T} - F$ .

4. If  $n = 2$ , the problem of avoiding solutions to the equation  $y = f(x)$  for a continuous function  $f : V \rightarrow \mathbb{T}^d$  is essentially trivial. If there exists  $x \in \mathbb{T}^d$  such that  $f(x) \neq x$ , then there exists an open set  $U$  around  $x$  such that  $U \cap f(U) = \emptyset$ . Then  $U$  has full Fourier dimension, and avoids solutions to the equation  $y = f(x)$ . On the other hand, if  $f(x) = x$  for all  $x$ , then there are no distinct  $x$  and  $y$  in  $[0, 1]$  such that  $y = f(x)$ , and so the problem is also trivial. But it is a less trivial to argue that a generic set with full Fourier dimension avoids this pattern, which is proved in Theorem 1.2, so we still obtain nontrivial information in this case.

Working on patterns in the domain  $\mathbb{R}^d$  is not significantly different from working over  $\mathbb{T}^d$ . The latter domain has several technical and notational advantages, which is why in this

paper we have chosen to work with the pattern avoidance pattern in this setting. But there is no theoretical obstacle in applying the techniques described here to prove the existence of pattern avoiding sets in  $\mathbb{R}^d$ . Let us briefly describe how this can be done. Given a Borel measure  $\mu$  on  $\mathbb{R}^d$ , we define the Fourier dimension  $\dim_{\mathbb{F}}(\mu)$  of  $\mu$  to be the supremum of all  $s \in [0, d]$  such that  $\sup_{\xi \in \mathbb{R}^d} |\hat{\mu}(\xi)| |\xi|^{s/2} < \infty$ . It is a simple consequence of the Poisson summation formula that if  $\mu$  is a compactly supported finite measure on  $\mathbb{R}^d$ , and we consider the *periodization*  $\mu^*$  of  $\mu$ , i.e. the finite Borel measure on  $\mathbb{T}^d$  such that for any  $f \in C(\mathbb{T}^d)$ ,

$$\int_{\mathbb{T}^d} f(x) d\mu^*(x) = \int_{\mathbb{R}^d} f(x) d\mu(x), \quad (1.3)$$

then  $\dim_{\mathbb{F}}(\mu^*) = \dim_{\mathbb{F}}(\mu)$ . A proof is given in Lemma 39 of [1]. Since  $\mu$  is compactly supported, it is also simple to see that  $\dim_{\mathbb{H}}(\mu^*) = \dim_{\mathbb{H}}(\mu)$ . It follows that if  $E$  is a compact subset of  $[0, 1]^d$ , and  $\pi : [0, 1]^d \rightarrow \mathbb{T}^d$  is the natural projection map, then  $\dim_{\mathbb{F}}(E) = \dim_{\mathbb{F}}(\pi(E))$  and  $\dim_{\mathbb{H}}(E) = \dim_{\mathbb{H}}(\pi(E))$ . But these results imply we can reduce the study of patterns on  $\mathbb{R}^{dn}$  to patterns on  $\mathbb{T}^{dn}$ , and thus obtain analogous results to Theorems 1.1, 1.2, and 1.3 for patterns in  $\mathbb{R}^d$ .

## 2 Notation

- Given a metric space  $X$ , a point  $x \in X$ , and a positive number  $\varepsilon > 0$ , we let  $B_{\varepsilon}(x)$  denote the open ball of radius  $\varepsilon$  around  $x$ . For  $x \in X$ , we let  $\delta_x$  denote the Dirac delta measure at  $x$ . For a set  $E \subset X$  and  $\varepsilon > 0$ , we let  $E_{\varepsilon} = \bigcup_{x \in E} B_{\varepsilon}(x)$  denote the  $\varepsilon$ -*thickening* of the set  $E$ .

For two sets  $E_1, E_2 \subset X$ , we let  $d(E_1, E_2) = \inf\{d(x_1, x_2) : x_1 \in E_1, x_2 \in E_2\}$ , and then define the *Hausdorff distance*

$$d_{\mathbb{H}}(E_1, E_2) = \max \left( \sup_{x_1 \in E_1} d(x_1, E_2), \sup_{x_2 \in E_2} d(E_1, x_2) \right).$$

- A subset of a metric space  $X$  is of *first category*, or *meager* in  $X$  if it is the countable union of closed sets with empty interior, and is *comeager* if it is the complement of such a set (a countable intersection of open, dense sets). We say a property holds *quasi-always*, or a property is *generic* in  $X$ , if the set of points in  $X$  satisfying that property is comeager. The Baire category theorem then states that any comeager set in a complete metric space is dense.
- We let  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ . Given  $x \in \mathbb{T}$ , we let  $|x|$  denote the minimal absolute value of an element of  $\mathbb{R}$  lying in the coset of  $x$ . For  $x \in \mathbb{T}^d$ , we let  $|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}$ . The canonical metric on  $\mathbb{T}^d$  is then given by  $d(x, y) = |x - y|$ , for  $x, y \in \mathbb{T}^d$ .

For an axis-oriented cube  $Q$  in  $\mathbb{T}^d$ , we let  $2Q$  be the axis-oriented cube in  $\mathbb{T}^d$  with the same center and twice the sidelength.

We say a family of sets  $\mathcal{A} \subset \mathcal{P}(\mathbb{T}^d)$  is *monotone* if, whenever  $E \in \mathcal{A}$ , any subset of  $E$  is also an element of  $\mathcal{A}$ . The quintessential monotone statement for our purposes, given a set  $Z \subset \mathbb{T}^{dn}$ , is the collection of sets  $E \subset \mathbb{T}^d$  such that for any distinct points  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \notin Z$ .

- For  $\alpha \in [0, d]$  and  $\delta > 0$ , we define the  $(\alpha, \delta)$  *Hausdorff content* of a Borel set  $E \subset \mathbb{T}^d$  as

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{i=1}^{\infty} \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^{\infty} B_{\varepsilon_i}(x_i) \text{ and } 0 < \varepsilon_i \leq \delta \text{ for all } i \geq 1 \right\}.$$

The  $\alpha$  dimensional Hausdorff measure of  $E$  is equal to

$$H^\alpha(E) = \lim_{\delta \rightarrow 0} H_\delta^\alpha(E).$$

The Hausdorff dimension  $\dim_{\mathbb{H}}(E)$  of a Borel set  $E$  is then the supremum over all  $s \in [0, d]$  such that  $H^s(E) = 0$ .

For a measurable set  $E \subset \mathbb{T}^d$ , we let  $|E|$  denote its Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set  $E \subset \mathbb{T}^d$  as

$$\underline{\dim}_{\mathbb{M}}(E) = \liminf_{r \rightarrow 0} d - \log_r |E_r|.$$

Thus  $\underline{\dim}_{\mathbb{M}}(E)$  is the largest number such that for  $\alpha < \underline{\dim}_{\mathbb{M}}(E)$ , there exists a decreasing sequence  $\{r_i\}$  with  $\lim_{i \rightarrow \infty} r_i = 0$  and  $|E_{r_i}| \leq r_i^{d-\alpha}$  for each  $i$ .

- At several points in this paper we will need to employ probabilistic concentration bounds. In particular, we use *McDiarmid's inequality*. Let  $S$  be a set, let  $\{X_1, \dots, X_N\}$  be an independent family of  $S$ -valued random variables, and consider a function  $f : S^N \rightarrow \mathbb{C}$ . Suppose that for each  $i \in \{1, \dots, N\}$ , there exists a constant  $A_i > 0$  such that for any  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N \in S$ , and for each  $x_i, x'_i \in S$ ,

$$|f(x_1, \dots, x_i, \dots, x_N) - f(x_1, \dots, x'_i, \dots, x_N)| \leq A_i.$$

Then McDiarmid's inequality guarantees that for all  $t \geq 0$ ,

$$\mathbb{P}(|f(X_1, \dots, X_N) - \mathbb{E}(f(X_1, \dots, X_N))| \geq t) \leq 4 \exp \left( \frac{-2t^2}{A_1^2 + \dots + A_N^2} \right).$$

Proofs of McDiarmid's inequality for real-valued functions are given in many probability texts, for instance, in Theorem 3.11 of [12], but can be trivially extended to the real case by taking a union bound to the inequality for real and imaginary values of  $f$ .

A special case of McDiarmid's inequality is *Hoeffding's Inequality*. For the purposes of this paper, Hoeffding's inequality states that if  $\{X_1, \dots, X_N\}$  is a family of independent random variables, such that for each  $i$ , there exists a constant  $A_i \geq 0$  such that  $|X_i| \leq A_i$  almost surely, then for each  $t \geq 0$ ,

$$\mathbb{P}(|(X_1 + \dots + X_N) - \mathbb{E}(X_1 + \dots + X_N)| \geq t) \leq 4 \exp \left( \frac{-t^2}{2(A_1^2 + \dots + A_N^2)} \right).$$

## 3 Applications of our Results

### 3.1 Arithmetic Patterns

An important problem in current research on pattern avoidance is to construct sets  $E$  which avoid *linear patterns*, i.e. sets  $E$  which avoid solutions to equations of the form

$$m_1x_1 + \cdots + m_nx_n = 0 \quad (3.1)$$

for distinct points  $x_1, \dots, x_n \in E$ . This is a scenario in which we have robust *upper bounds* on the dimension of pattern avoiding sets. It is simple to prove that if  $E \subset \mathbb{T}^d$  and  $\dim_{\mathbb{F}}(E) > 2d/n$ , and  $m_1, \dots, m_n$  are non-zero integers, then  $m_1E + \cdots + m_nE$  is an open subset of  $\mathbb{T}^d$  (a simple modification of Proposition 3.14 of [8]), which implies that there exists *some* choice of integers  $m_1, \dots, m_n$  and distinct points  $x_1, \dots, x_n \in E$  such that  $m_1x_1 + \cdots + m_nx_n = 0$ . Recently, under the same assumptions, Liang and Pramanik have shown [7] that for  $d = 1$ , one can choose these integers  $m_1, \dots, m_n$  to satisfy  $m_1 + \cdots + m_n = 0$ .

If  $\dim_{\mathbb{F}}(E) > d/n$ , and  $m_1, \dots, m_n \neq 0$ , then  $m_1E + \cdots + m_nE$  has *positive Lebesgue measure* [8, Proposition 3.14]. This does not contradict that sets avoiding linear patterns exist beyond the dimension  $d/n$ , but indicates some of the difficulty in pushing past the barrier of  $d/n$  obtained in (1.2). The first success in pushing past this barrier was the main result of [6], which showed that for each  $n > 0$ , there exists a set  $E \subset \mathbb{T}$  with Fourier dimension  $1/(n-1)$  such that for any integers  $m_1, \dots, m_n \in \mathbb{Z}$ , not all zero, and any distinct  $x_1, \dots, x_n \in \mathbb{T}^d$ ,  $m_1x_1 + \cdots + m_nx_n \neq 0$ . The technique used to control Fourier decay in that paper (bounding first derivatives of distribution functions associated with the construction of the set) relies heavily on the one dimensional nature of the problem. The results of this paper give a  $d$ -dimensional generalization of Körner's result, as well as extending this result to consider avoiding certain uncountable families of linear equations.

**Theorem 3.1.** *Consider any set  $S \subset \mathbb{T}^d$  formed from the countable union of compact sets with lower Minkowski dimension zero, and let  $\beta_0 = d/(n-1)$ . Then there exists a Salem set  $E \subset \mathbb{T}^d$  of dimension  $d/(n-1)$  such that for any  $s \in S$ , any distinct  $x_1, \dots, x_n \in E$ , and any integers  $m_1, \dots, m_n \in \mathbb{Z}$ ,  $m_1x_1 + \cdots + m_nx_n \neq s$ . Moreover, for any  $\beta \leq \beta_0$ , and for a generic Salem set  $(E, \mu) \in \mathcal{X}_\beta$ , the set  $E$  has this property.*

*Proof.* Without loss of generality, we may assume  $S$  is compact, and it will suffice to show that a generic Salem set  $(E, \mu) \in \mathcal{X}_\beta$  avoids solutions to equations of the form

$$x_n - a_{n-1}x_{n-1} = s + a_3x_3 + \cdots + a_nx_n, \quad (3.2)$$

with  $a_2, \dots, a_n \in \mathbb{Q}$ ,  $s \in S$ , and where either  $a_{n-1} \neq 0$ , or  $a_2 = a_3 = \cdots = a_n = 0$ . If  $a_2 \neq 0$ , then Theorem 1.3 applies directly to the equation

$$x_1 + a_2x_2 \in T(x_3, \dots, x_n), \quad (3.3)$$

where  $T(x_3, \dots, x_n) = S - a_3x_3 - \cdots - a_nx_n$ . Thus we conclude that the set of  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids solutions to (3.3) is comeager. On the other hand, if  $a_2 = a_3 = \cdots = a_n = 0$ , then the equation is precisely  $x_1 \in S$ ,

$$x_1 \in S, \quad (3.4)$$



and it follows from Theorem 1.1 with  $Z = S$  and  $n = 1$  that the set of  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids solutions to (3.4) is comeager. Taking countable unions shows that the set of all  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  is Salem and avoids all  $n$ -variable linear equations is comeager.  $\square$

**Remark 3.2.** *For particular linear patterns, it is certainly possible to improve the result of Theorem 3.1. For instance, Schmerkin [9] constructed a set  $E \subset \mathbb{T}$  with  $\dim_{\mathbb{F}}(E) = 1$  which contains no three numbers forming an arithmetic progression. Liang and Pramanik [7] constructed, for any finite family of translation-invariant linear functions  $\{f_i\}$ , a set  $E \subset \mathbb{T}$  with  $\dim_{\mathbb{F}}(E) = 1$  such that for distinct  $x_1, \dots, x_n \in E$ ,  $f_i(x_1, \dots, x_n) \neq 0$ . This same paper even constructs a set with Fourier dimension close to one avoiding an uncountable family of translation-invariant linear functions, though only those that are of a very special form. The advantage of Theorem 3.1 is that it applies to a very general family of uncountably many linear equations, though one does not obtain a tight Fourier dimension bound compared to the results of [7] and [9].*

The arguments in this paper are heavily inspired by the techniques of [6], but augmented with some more robust probabilistic concentration inequalities and stationary phase techniques, which enables us to push the results of [6] to a much more general family of patterns. In particular, Theorem 1.3 shows that the results of that paper do not depend on the rich arithmetic structure of the equation  $m_1x_1 + \dots + m_nx_n = 0$ , but rather only on a simple translation invariance property of the pattern. We are unable to close the gap between the upper bound  $2d/n$  of sets avoiding  $n$ -variable linear equations for  $n \geq 3$ , which would seem to require utilizing the full linear nature of the equations involved much more heavily than the very weak linearity assumption that Theorem 3.1 requires.

### 3.2 Isosceles Triangles on Curves

Theorems 1.1, 1.2, and 1.3 can be applied to find sets avoiding linear patterns, but the main power of these results that they can be applied to ‘nonlinear’ patterns which are not necessarily related to the arithmetic structure of  $\mathbb{T}^d$ , differing from most other results in the field. In this section we consider a standard problem of this kind, avoiding isosceles triangles on curves; given a simple segment of a curve given by a smooth map  $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ , we say a set  $E \subset [0, 1]$  *avoids isosceles triangles* if for any distinct values  $t_1, t_2, t_3 \in [0, 1]$ ,  $|\gamma(t_1) - \gamma(t_2)| \neq |\gamma(t_2) - \gamma(t_3)|$ . Then  $E$  avoids isosceles triangles if and only if  $\gamma(E)$  does not contain any three points forming the vertices of an isosceles triangle. In [4], methods are provided to construct sets  $E \subset [0, 1]$  with  $\dim_{\mathbb{H}}(E) = \log 2 / \log 3 \approx 0.63$  such that  $\gamma(E)$  does not contain any isosceles triangles, but  $E$  is not guaranteed to be Salem. We can use Theorem 1.2 to construct *Salem sets*  $E \subset [0, 1]$  with  $\dim_{\mathbb{F}}(E) = 4/9 \approx 0.44$ .

**Theorem 3.3.** *For any smooth map  $\gamma : [0, 1] \rightarrow \mathbb{R}^d$  with  $\gamma'(x) \neq 0$  for all  $x \in [0, 1]$ , there exists a Salem set  $E \subset [0, 1]$  with  $\dim_{\mathbb{F}}(E) = 4/9$  which avoids isosceles triangles.*

*Proof.* Assume without loss of generality (working on a smaller portion of the curve if necessary) that there exists a constant  $C \geq 1$  such that for any  $t, s \in [0, 1]$ ,

$$|\gamma(t) - \gamma(s) - (t - s)\gamma'(0)| \leq C(t - s)^2, \quad (3.5)$$

$$1/C \leq |\gamma'(t)| \leq C, \quad (3.6)$$

and

$$|\gamma'(t) - \gamma'(s)| \leq C|t - s|. \quad (3.7)$$

Let  $\varepsilon = 1/2C^3$ , and let

$$F(t_1, t_2, t_3) = |\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2. \quad (3.8)$$

A simple calculation using (3.5) and (3.6) reveals that for  $0 \leq t_1, t_2 \leq \varepsilon$ ,

$$\left| \frac{\partial F}{\partial t_1} \right| = 2 |(\gamma(t_1) - \gamma(t_2)) \cdot \gamma'(t_1)| \geq (2/C)|t_2 - t_1| - 2C|t_2 - t_1|^2 \geq (1/C)|t_2 - t_1|. \quad (3.9)$$

This means that  $\partial F / \partial t_1 \neq 0$  unless  $t_1 = t_2$ . Thus the implicit function theorem implies that there exists a countable family of smooth functions  $\{f_i : U_i \rightarrow [0, 1]\}$ , where  $U_i \subset [0, \varepsilon]^2$  for each  $i$  and  $f_i(t_2, t_3) \neq t_3$  for any  $(t_2, t_3) \in U_i$ , such that if  $F(t_1, t_2, t_3) = 0$  for distinct points  $t_1, t_2, t_3 \in [0, \varepsilon]$ , then there exists an index  $i$  with  $(t_2, t_3) \in U_i$  and  $t_1 = f_i(t_2, t_3)$ . Differentiating both sides of the equation

$$|\gamma(f_i(t_2, t_3)) - \gamma(t_2)|^2 = |\gamma(t_2) - \gamma(t_3)|^2 \quad (3.10)$$

in  $t_2$  and  $t_3$  shows that

$$\frac{\partial f_i}{\partial t_2}(t_2, t_3) = \frac{(\gamma(f_i(t_2, t_3)) - \gamma(t_3)) \cdot \gamma'(t_2)}{(\gamma(f_i(t_2, t_3)) - \gamma(t_2)) \cdot \gamma'(f_i(t_2, t_3))} \quad (3.11)$$

and

$$\frac{\partial f_i}{\partial t_3}(t_2, t_3) = \frac{-(\gamma(t_2) - \gamma(t_3)) \cdot \gamma'(t_3)}{(\gamma(f_i(t_2, t_3)) - \gamma(t_2)) \cdot \gamma'(f_i(t_2, t_3))}. \quad (3.12)$$

In order to apply Theorem 1.2, we must show that the partial derivatives in 3.11 and 3.12 are both non-vanishing for  $t_2, t_3 \in [0, \varepsilon]$ . We calculate using (3.5), (3.6) and (3.7) that

$$\begin{aligned} |(\gamma(f_i(t_2, t_3)) - \gamma(t_3)) \cdot \gamma'(t_2)| &\geq |(\gamma(f_i(t_2, t_3)) - \gamma(t_3)) \cdot \gamma'(t_3)| \\ &\quad + |(\gamma(f_i(t_2, t_3)) - \gamma(t_3)) \cdot (\gamma'(t_2) - \gamma'(t_3))| \\ &\geq (1/C)|f_i(t_2, t_3) - t_3| - C^2|f_i(t_2, t_3) - t_3||t_2 - t_3| \\ &\geq (1/C - C^2\varepsilon)|f_i(t_2, t_3) - t_3| \\ &\geq (1/2)|f_i(t_2, t_3) - t_3|. \end{aligned} \quad (3.13)$$

Since  $f_i(t_2, t_3) \neq t_3$  for all  $(t_2, t_3) \in U_i$ , it follows from (3.11) and (3.13) that if  $(t_2, t_3) \in U_i$  with  $t_2 \neq t_3$ ,

$$\frac{\partial f_i}{\partial t_2}(t_2, t_3) \neq 0. \quad (3.14)$$

A similar calculation to (3.9) shows that for  $t_2, t_3 \in [0, \varepsilon]$ ,

$$|(\gamma(t_2) - \gamma(t_3)) \cdot \gamma'(t_3)| \geq (1/C)|t_2 - t_3|. \quad (3.15)$$

Combining (3.12) with (3.15) shows that for  $t_2 \neq t_3$  with  $(t_2, t_3) \in U_i$ ,

$$\frac{\partial f_i}{\partial t_3}(t_2, t_3) \neq 0. \quad (3.16)$$

Now (3.14) and (3.16) imply that each function in the family  $\{f_i\}$  satisfy the hypothesis of Theorem 1.2. Thus that theorem implies that for  $\beta = 4/9$ , each index  $i$ , and a generic element of  $(E, \mu) \in \mathcal{X}_\beta$ , the set  $E$  is Salem and for any distinct  $t_1, t_2, t_3 \in E \cap [0, \varepsilon]$ ,  $f_i(t_1, t_2, t_3) \neq 0$ . This means precisely that  $|\gamma(t_1) - \gamma(t_2)| \neq |\gamma(t_2) - \gamma(t_3)|$  for any distinct  $t_1, t_2, t_3 \in E$ . Thus we conclude we can find a Salem set  $E \subset [0, \varepsilon]$  with  $\dim_{\mathbb{R}}(E) = 4/9$  such that  $\gamma(E)$  does not contain the vertices of any isosceles triangles.  $\square$

Theorem 1.1 can also be used to construct sets with a slightly smaller dimension avoiding isosceles triangles on a rougher family of curves. If we consider a Lipschitz function  $\gamma : [0, 1] \rightarrow \mathbb{R}^{d-1}$ , where there exists  $M < 1$  with  $|\gamma(t) - \gamma(s)| \leq M|t - s|$  for each  $t, s \in [0, 1]$ , then Theorem 3 of [2] guarantees that the set

$$Z = \left\{ (x_1, x_2, x_3) \in [0, 1]^3 : \begin{array}{l} (x_1, \gamma(x_1)), (x_2, \gamma(x_2)), (x_3, \gamma(x_3)) \\ \text{form the vertices of an isosceles triangle.} \end{array} \right\}$$

has lower Minkowski dimension at most two. Thus Theorem 1.1 guarantees that there exists a Salem set  $E \subset [0, 1]$  with  $\dim_{\mathbb{R}}(E) = 2/5 = 0.4$  such that  $\gamma(E)$  avoids all isosceles triangles. The main result of [4] constructs a set  $E \subset [0, 1]$  with  $\dim_{\mathbb{H}}(E) = 0.5$  such that  $\gamma(E)$  avoids all isosceles triangles, but this set is not guaranteed to be Salem.

## 4 A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task and establish a set of tools for obtaining convergence in this metric space. In later sections we will fix a specific choice of  $\beta$  to avoid a particular pattern. But in this section we let  $\beta$  be an arbitrary fixed number in  $(0, d]$ . Our approach in this section is heavily influenced by [6]. However, we employ a Fréchet space construction instead of the Banach space construction used in [6], which enables us to use softer estimates in our arguments, with the disadvantage that we can obtain only Fourier dimension bounds in Theorems 1.1, 1.2, and 1.3 at the endpoint dimensions  $\beta_0$  considered in the theorems, rather than the explicit decay estimates determined in Theorem 2.4 of [6]:

- We let  $\mathcal{E}$  denote the family of all compact subsets of  $\mathbb{T}^d$ . If we consider the Hausdorff distance  $d_{\mathbb{H}}$  between sets, then  $(\mathcal{E}, d_{\mathbb{H}})$  forms a complete metric space.
- We let  $M(\beta)$  consist of all finite Borel measures  $\mu$  on  $\mathbb{T}^d$  such that for each  $\lambda \in [0, \beta)$ ,

$$\|\mu\|_{M(\lambda)} = \sup_{\xi \in \mathbb{Z}^d} |\widehat{\mu}(\xi)| |\xi|^{\lambda/2}$$

is finite. Then  $\|\cdot\|_{M(\lambda)}$  is a seminorm on  $M(\beta)$  for each  $\lambda \in [0, \beta)$ , and the collection of all such seminorms gives  $M(\beta)$  the structure of a Frechét space. Under this topology, a sequence of probability measures  $\{\mu_k\}$  converges to a probability measure  $\mu$  in  $M(\beta)$  if and only if for any  $\lambda \in [0, \beta)$ ,  $\lim_{k \rightarrow \infty} \|\mu_k - \mu\|_{M(\lambda)} = 0$ .

We now let  $\mathcal{X}_\beta$  be the collection of all pairs  $(E, \mu) \in \mathcal{E} \times M(\beta)$ , where  $\mu$  is a probability measure such that  $\text{supp}(\mu) \subset E$ . Then  $\mathcal{X}_\beta$  is a closed subset of  $\mathcal{E} \times M(\beta)$  under the product metric, and thus a complete metrizable space. We remark that for any  $\lambda \in [0, \beta)$  and  $(E, \mu) \in \mathcal{X}_\beta$ ,

$$\lim_{|\xi| \rightarrow \infty} |\widehat{\mu}(\xi)| |\xi|^{\lambda/2} = 0. \quad (4.1)$$

Thus  $\dim_{\mathbb{F}}(E) \geq \dim_{\mathbb{F}}(\mu) \geq \beta$  for each  $(E, \mu) \in \mathcal{X}_\beta$ . This means that  $\mathcal{X}_\beta$  can be thought of as describing a family of sets with Fourier dimension at least  $\beta$ , together with an associated measure which provides a certificate proving the lower bound on the Fourier dimension of the set.

Lemma 4.1, the main result of this section, reduces density arguments in  $\mathcal{X}_\beta$  to the construction of large discrete subsets in  $\mathbb{T}^d$  with well-behaved Fourier analytic properties. Let us intuitively describe the context of Lemma 4.1, and why its conclusion is intuitive. Consider a large integer  $N$ , and suppose there is a discrete family of  $N$  points  $S = \{x_1, \dots, x_N\}$  such that  $S_r$  does not contain any incidences of a particular pattern. Then  $S_r$  is a union of  $N$  balls of radius  $r$ , so if  $N \approx r^{-\beta}$ ,  $S_r$  behaves like an  $r$ -thickening of a set with Hausdorff dimension  $\beta$ . To understand the Fourier analytic properties of  $S$ , we take exponential sums. For any such set  $S$ , taking in absolute values gives a trivial bound on the exponential sum

$$\left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot x_k} \right| \leq 1. \quad (4.2)$$

If one can improve upon this bound, we think of  $S$  as having additional Fourier analytic structure. The best case to hope for a ‘generic’ set  $S$  is that we have a *square root cancellation bound*

$$\left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot x_k} \right| \lesssim N^{-1/2}, \quad (4.3)$$

which roughly holds if the points  $\{x_k\}$  do not exhibit any kind of periodicity, the main quantity measured in  $\mathcal{X}_\beta$ . If  $\kappa > 0$  is fixed, and a slightly weaker version of equation (4.3) holds for all  $|\xi| \lesssim (1/r)^{1+\kappa}$ , then  $S_r$  behaves like an  $r$ -thickening of a set with *Fourier dimension*  $\beta$ . The main consequence of this is that the behaviour of  $\{x_1, \dots, x_K\}$  should then be a generic property of  $\mathcal{X}_\beta$ .

**Lemma 4.1.** *Let  $\mathcal{A}$  be a monotone family of subsets of  $\mathbb{T}^d$ . Fix  $\beta > 0$ ,  $\kappa > 0$ , and a large constant  $C > 0$ . Suppose that for all small  $\delta > 0$  and  $\lambda \in [0, \beta)$ , there are arbitrarily large finite sets  $S = \{x_1, \dots, x_N\}$ , positive numbers  $\{a(x_1), \dots, a(x_N)\}$  with  $\sum_{k=1}^N a(x_k) \geq N/2$ , and quantities  $r > 0$  with  $N \geq (1/2)r^{-\lambda}$  such that the following properties hold:*

- (1)  $S_r \in \mathcal{A}$ .
- (2) For each  $\xi \in \mathbb{Z}^d - \{0\}$  with  $|\xi| \leq (1/r)^{1+\kappa}$ ,

$$\left| \frac{1}{N} \sum_{k=1}^N a(x_k) e^{2\pi i \xi \cdot x_k} \right| \leq CN^{-1/2} \log(N) + \delta |\xi|^{-\lambda/2}$$

Then  $\{(E, \mu) \in \mathcal{X}_\beta : E \in \mathcal{A}\}$  is dense in  $\mathcal{X}_\beta$ .

**Remark 4.2.** We will be able to take  $\delta = 0$  for the applications of Lemma 4.1 in Theorems 1.1 and 1.3. We only need to take  $\delta > 0$  when applying this result to Theorem 1.2, since we need to employ oscillatory integral bounds which give an additional decaying factor, rather than solely relying on probabilistic arguments to get the  $O(N^{-1/2} \log(N))$  upper bound.

Throughout this section, we will apply mollification. So we fix a smooth, non-negative function  $\phi \in C^\infty(\mathbb{T}^d)$  such that  $\phi(x) = 0$  for  $|x| \geq 2/5$  and  $\int_{\mathbb{T}^d} \phi(x) dx = 1$ . For each  $r \in (0, 1)$ , we can then define  $\phi_r \in C^\infty(\mathbb{T}^d)$  by writing

$$\phi_r(x) = \begin{cases} r^{-d} \phi(x/r) & : |x| < r, \\ 0 & : \text{otherwise.} \end{cases}$$

The following standard properties hold:

- (1) For each  $r \in (0, 1)$ ,  $\phi_r$  is a non-negative smooth function with

$$\int_{\mathbb{T}^d} \phi_r(x) dx = 1, \tag{4.4}$$

and  $\phi_r(x) = 0$  for  $|x| \geq r$ .

- (2) For any  $r \in (0, 1)$ ,

$$\|\hat{\phi}_r\|_{L^\infty(\mathbb{Z}^d)} = 1. \tag{4.5}$$

- (3) For each  $\xi \in \mathbb{Z}^d$ ,

$$\lim_{r \rightarrow 0} \hat{\phi}_r(\xi) = 1. \tag{4.6}$$

- (4) For each  $T > 0$ , for all  $r > 0$ , and for any non-zero  $\xi \in \mathbb{Z}^d$ ,

$$|\hat{\phi}_r(\xi)| \lesssim_T r^{-T} |\xi|^{-T}. \tag{4.7}$$

We will prove Lemma 4.1 after a series of more elementary lemmas which give results about the metric space  $\mathcal{X}_\beta$ .

**Lemma 4.3.** The set of all  $(E, \mu) \in \mathcal{X}_\beta$  with  $\mu \in C^\infty(\mathbb{T}^d)$  and  $\text{supp}(\mu) = E$  is dense in  $\mathcal{X}_\beta$ .

*Proof.* Let

$$\tilde{\mathcal{X}}_\beta = \{(E, \mu) \in \mathcal{X}_\beta : \text{supp}(\mu) = E\}. \tag{4.8}$$

We begin by proving that the set of all  $(E, \mu) \in \tilde{\mathcal{X}}_\beta$  such that  $\mu \in C^\infty(\mathbb{T}^d)$  is dense in  $\tilde{\mathcal{X}}_\beta$ . Fix  $(E_0, \mu_0) \in \tilde{\mathcal{X}}_\beta$ . For each  $r \in (0, 1)$ , consider the convolved measure  $\mu_r = \mu_0 * \phi_r$ . Then  $\mu_r \in C^\infty(\mathbb{T}^d)$  and  $\text{supp}(\mu_r) = E_r$ . We claim that  $\lim_{r \rightarrow 0} (E_r, \mu_r) = (E_0, \mu_0)$ , which would complete the proof. Since  $d_{\mathbb{H}}(E_0, E_r) \leq r$ , we find that  $\lim_{r \rightarrow 0} E_r = E_0$  holds with respect to the Hausdorff metric. Now fix  $\lambda \in (0, \beta]$  and  $\delta > 0$ . For each  $\xi \in \mathbb{Z}^d$ ,  $|\hat{\mu}_r(\xi)| = |\hat{\phi}_r(\xi)| |\hat{\mu}_0(\xi)|$ , so

$$|\xi|^{\lambda/2} |\hat{\mu}_r(\xi) - \hat{\mu}_0(\xi)| = |\xi|^{\lambda/2} |\hat{\phi}_r(\xi) - 1| |\hat{\mu}_0(\xi)|. \tag{4.9}$$

We control (4.9) using the fact that  $|\hat{\mu}_0(\xi)|$  is small when  $\xi$  is large, and  $|\hat{\phi}_r(\xi) - 1|$  is small when  $\xi$  is small. Since  $(E_0, \mu_0) \in \mathcal{X}_\beta$ , we can apply (4.1) to find  $R > 0$  such that for  $|\xi| \geq R$ ,

$$|\xi|^{\lambda/2} |\hat{\mu}_0(\xi)| \leq \delta/2. \quad (4.10)$$

Combining (4.9), (4.10), and (4.5), for  $|\xi| \geq R$  we find that

$$|\xi|^{\lambda/2} |\hat{\mu}_r(\xi) - \hat{\mu}_0(\xi)| \leq \delta. \quad (4.11)$$

On the other hand, (4.6) shows that there exists  $r_0 > 0$  such that for  $r \leq r_0$  and  $|\xi| \leq R$ ,

$$|\xi|^{\lambda/2} |\hat{\phi}_r(\xi) - 1| \leq \delta. \quad (4.12)$$

The  $(L^1, L^\infty)$  bound for the Fourier transform implies that  $|\hat{\mu}_0(\xi)| \leq \mu_0(\mathbb{T}^d) = 1$ , which combined with (4.12) gives that for  $r \leq r_0$  and  $|\xi| \leq R$ ,

$$|\xi|^{\lambda/2} |\hat{\mu}_r(\xi) - \hat{\mu}_0(\xi)| \leq \delta. \quad (4.13)$$

Putting together (4.11) and (4.13) shows that for  $r \leq r_0$ ,  $\|\mu_r - \mu_0\|_{M(\lambda)} \leq \delta$ . Since  $\delta$  and  $\lambda$  were arbitrary, we conclude that  $\lim_{r \rightarrow 0} \mu_r = \mu_0$ . Thus the set of all pairs  $(E, \mu) \in \tilde{\mathcal{X}}_\beta$  with  $\mu \in C^\infty(\mathbb{T}^d)$  is dense in  $\tilde{\mathcal{X}}_\beta$ .

Our proof will therefore be complete if we can show that  $\tilde{\mathcal{X}}_\beta$  is dense in  $\mathcal{X}_\beta$ . We prove this using a Baire category argument. For each closed cube  $Q \subset \mathbb{T}^d$ , let

$$A(Q) = \{(E, \mu) \in \mathbb{T}^d : (E \cap Q) = \emptyset \text{ or } \mu(Q) > 0\}.$$

Then  $A(Q)$  is an open set. If  $\{Q_k\}$  is a countable sequence enumerating all cubes with rational corners in  $\mathbb{T}^d$ , then

$$\bigcap_{k=1}^{\infty} A(Q_k) = \{(E, \mu) \in \mathcal{X}_\beta : \text{supp}(\mu) = E\}. \quad (4.14)$$

Thus it suffices to show that  $A(Q)$  is dense in  $\mathcal{X}_\beta$  for each closed cube  $Q$ . To do this, we fix  $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$ ,  $\lambda \in [0, \beta)$ , and  $\varepsilon > 0$ , and try and find  $(E, \mu) \in A(Q)$  with  $d_{\mathbb{H}}(E, E_0) \leq \varepsilon$  and  $\|\mu_0 - \mu\|_{M(\lambda)} \leq \varepsilon$ .

Because  $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$ , we know  $E_0 \cap Q \neq \emptyset$  and  $\mu_0(Q) = 0$ . Find a smooth probability measure  $\nu$  supported on  $(E_0)_\varepsilon \cap Q$  and, for  $t \in (0, 1)$ , define  $\mu_t = (1 - t)\mu_0 + t\nu$ . Then  $\text{supp}(\mu_t) \subset (E_0)_\varepsilon$ , so if we let  $E = \text{supp}(\nu) \cup \text{supp}(\mu)$ , then  $d_{\mathbb{H}}(E, E_0) \leq \varepsilon$ . Clearly  $(E, \mu_t) \in A(Q)$  for  $t > 0$ . And

$$\|\mu_t - \mu_0\|_{M(\lambda)} \leq t (\|\mu_0\|_{M(\lambda)} + \|\nu\|_{M(\lambda)}), \quad (4.15)$$

so if we choose  $t \leq \varepsilon \cdot (\|\mu_0\|_{M(\lambda)} + \|\nu\|_{M(\lambda)})^{-1}$  we find  $\|\mu_t - \mu_0\|_{M(\lambda)} \leq \varepsilon$ . Since  $\varepsilon$  was arbitrary, we conclude  $A(Q)$  is dense in  $\mathcal{X}_\beta$ .  $\square$

**Remark 4.4.** *The reason we must work with the metric space  $\mathcal{X}_\beta$  rather than the smaller space  $\tilde{\mathcal{X}}_\beta \subset \mathcal{X}_\beta$  is that  $\tilde{\mathcal{X}}_\beta$  is not a closed subset of  $\mathcal{X}_\beta$ , and so is not a complete metric space, preventing the use of the Baire category theorem. However, the last paragraph of the proof of Lemma 4.3 shows that quasi-all elements of  $\mathcal{X}_\beta$  belong to  $\tilde{\mathcal{X}}_\beta$ , so that one can think of  $\mathcal{X}_\beta$  and  $\tilde{\mathcal{X}}_\beta$  as being equal ‘generically’.*

The density argument of Lemma 4.1 requires constructing approximations to an arbitrary element of  $(E_0, \mu_0) \in \mathcal{X}_\beta$  by  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E \in A$ . We do this by multiplying  $\mu_0$  by a smooth function  $f \in C^\infty(\mathbb{T}^d)$  which cuts off parts of  $\mu_0$  which cause the support of  $\mu_0$  to fail to be in  $A$ . As long as  $\mu_0$  is appropriately smooth, and the Fourier transform of  $f$  decays appropriately quickly, the next lemma shows that  $f\mu_0 \approx \mu_0$ .

**Lemma 4.5.** *Consider a finite measure  $\mu_0$  on  $\mathbb{T}^d$ , as well as a smooth probability density function  $f \in C^\infty(\mathbb{T}^d)$ . If we define  $\mu = f\mu_0$ , then for any  $\lambda \in [0, d]$ ,*

$$\|\mu - \mu_0\|_{M(\lambda)} \lesssim_d \|\mu_0\|_{M(3d)} \|f\|_{M(\lambda)}.$$

*Proof.* Since  $\hat{\mu} = \hat{f} * \hat{\mu}_0$ , and  $\hat{f}(0) = 1$ , for each  $\xi \in \mathbb{Z}^d$  we have

$$|\xi|^{\lambda/2} |\hat{\mu}(\xi) - \hat{\mu}_0(\xi)| = |\xi|^{\lambda/2} \left| \sum_{\eta \neq \xi} \hat{f}(\xi - \eta) \hat{\mu}_0(\eta) \right|. \quad (4.16)$$

If  $|\eta| \leq |\xi|/2$ , then  $|\xi|/2 \leq |\xi - \eta| \leq 2|\xi|$ , so

$$|\xi|^{\lambda/2} |\hat{f}(\xi - \eta)| \leq \|f\|_{M(\lambda)} |\xi|^{\lambda/2} |\xi - \eta|^{-\lambda/2} \leq 2^{\lambda/2} \|f\|_{M(\lambda)} \lesssim_d \|f\|_{M(\lambda)}. \quad (4.17)$$

Thus the bound (4.17) implies

$$\begin{aligned} |\xi|^{\lambda/2} \left| \sum_{0 \leq |\eta| \leq |\xi|/2} \hat{f}(\xi - \eta) \hat{\mu}_0(\eta) \right| &\lesssim_{\mu_0, d} \|\mu_0\|_{M(d+1)} \|f\|_{M(\lambda)} \left( 1 + \sum_{0 < |\eta| \leq |\xi|/2} \frac{1}{|\eta|^{d+1}} \right) \\ &\lesssim_d \|\mu_0\|_{M(d+1)} \|f\|_{M(\lambda)} \leq \|\mu_0\|_{M(3d)} \|f\|_{M(\lambda)}. \end{aligned} \quad (4.18)$$

On the other hand, for all  $\eta \neq \xi$ ,

$$|\hat{f}(\xi - \eta)| \leq \|f\|_{M(\lambda)} |\xi - \eta|^{-\lambda} \leq \|f\|_{M(\lambda)}. \quad (4.19)$$

Thus we calculate that

$$\begin{aligned} |\xi|^{\lambda/2} \left| \sum_{\substack{|\eta| > |\xi|/2 \\ \eta \neq \xi}} \hat{f}(\xi - \eta) \hat{\mu}_0(\eta) \right| &\lesssim_{d, \mu_0} \|\mu_0\|_{M(3d)} \|f\|_{M(\lambda)} \cdot |\xi|^{\lambda/2} \sum_{|\eta| > |\xi|/2} \frac{1}{|\eta|^{3d}} \\ &\lesssim_d \|\mu_0\|_{M(3d)} \|f\|_{M(\lambda)}. \end{aligned} \quad (4.20)$$

Combining (4.16), (4.18) and (4.20) completes the proof.  $\square$

The bound in Lemma 4.5, if  $\|f\|_{M(\lambda)}$  is taken appropriately small, also implies that the Hausdorff distance between the supports of  $\mu$  and  $\mu_0$  are small.

**Lemma 4.6.** *Fix a probability measure  $\mu_0 \in C^\infty(\mathbb{T}^d)$  and  $\lambda \in [0, d]$ . For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $\mu \in C^\infty(\mathbb{T}^d)$ ,  $\text{supp}(\mu) \subset \text{supp}(\mu_0)$ , and  $\|\mu_0 - \mu\|_{M(\lambda)} \leq \delta$ , then  $d_{\mathbb{H}}(\text{supp}(\mu), \text{supp}(\mu_0)) \leq \varepsilon$ .*

*Proof.* Consider any cover of  $\text{supp}(\mu_0)$  by a family of radius  $\varepsilon/3$  balls  $\{B_1, \dots, B_N\}$ , and for each  $i \in \{1, \dots, N\}$ , consider a smooth function  $f_i \in C_c^\infty(B_i)$  such that there is  $s > 0$  with

$$\int f_i(x) d\mu_0(x) \geq s \quad (4.21)$$

for each  $i \in \{1, \dots, N\}$ . Fix  $A > 0$  with

$$\sum_{\xi \neq 0} |\hat{f}_i(\xi)| \leq A \quad (4.22)$$

for all  $i \in \{1, \dots, N\}$  as well. Set  $\delta = s/2A$ . If  $\|\mu_0 - \mu\|_{M(\lambda)} \leq \delta$ , we apply Plancherel's theorem together with (4.21) and (4.22) to conclude that

$$\begin{aligned} \left| \int f_i(x) d\mu(x) dx - \int f_i(x) d\mu_0(x) \right| &= \left| \sum_{\xi \in \mathbb{Z}^d} \hat{f}_i(\xi) (\hat{\mu}(\xi) - \hat{\mu}_0(\xi)) \right| \\ &\leq A \|\mu_0 - \mu\|_{M(\lambda)} \\ &\leq s/2. \end{aligned} \quad (4.23)$$

Thus we conclude from (4.21) and (4.23) that

$$\int f_i(x) d\mu(x) dx \geq \int f_i(x) d\mu_0(x) - s/2 \geq s/2 > 0. \quad (4.24)$$

Since equation (4.24) holds for each  $i \in \{1, \dots, N\}$ , the support of  $\mu$  intersects every ball in  $\{B_1, \dots, B_N\}$ . Combined with the assumption that  $\text{supp}(\mu) \subset \text{supp}(\mu_0)$ , this implies that  $d_{\mathbb{H}}(\mu_0, \mu) \leq \varepsilon$ .  $\square$

Now we have the technology to prove Lemma 4.1.

*Proof of Lemma 4.1.* Fix  $(E_0, \mu_0) \in \mathcal{X}_\beta$ . By Lemma 4.3, without loss of generality, we may assume that  $\mu_0 \in C^\infty(\mathbb{T}^d)$  and that  $\text{supp}(\mu_0) = E_0$ . Our goal, for any  $\lambda \in [0, \beta)$  and  $\delta_0 > 0$ , is to find  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E \in \mathcal{A}$ ,  $d_{\mathbb{H}}(E, E_0) \leq \delta_0$ , and  $\|\mu - \mu_0\|_{M(\gamma)} \leq \delta_0$ .

Fix  $\delta > 0$ ,  $\varepsilon > 0$ , and  $\lambda \in (\gamma, \beta)$ , and consider a set  $S = \{x_1, \dots, x_N\}$  and  $\{a_1, \dots, a_N\}$  satisfying the assumptions of the Lemma. Increasing the constant  $C$  if necessary, we may assume without loss of generality that  $\sum a_i = N$ . If we set

$$\eta = \frac{1}{N} \sum_{k=1}^N a(x_k) \delta_{x_k},$$

then  $\eta$  is a probability measure, and Property (2) implies that for  $|\xi| \leq (1/r)^{1+\kappa}$ ,

$$|\hat{\eta}(\xi)| \leq CN^{-1/2} \log(N) + \delta |\xi|^{-\lambda/2}. \quad (4.25)$$

Consider the function  $f = \eta * \phi_r$ , where  $\phi_r$  is the mollifier defined in the notation section. For each  $\xi \in \mathbb{Z}^d$ ,

$$\hat{f}(\xi) = \hat{\eta}(\xi) \hat{\phi}_r(\xi). \quad (4.26)$$



For  $|\xi| \leq 1/r$ , (4.25) and (4.5) together with (4.26) imply that there is  $\kappa_1 > 0$  depending on  $\beta, \lambda$ , and  $\gamma$  such that

$$|\widehat{f}(\xi)| \leq CN^{-1/2} \log(N) + \delta |\xi|^{-\lambda/2} \leq (CN^{-\kappa_1} + \delta) |\xi|^{-\gamma/2}. \quad (4.27)$$

Thus if  $N$  is suitably large, we conclude that for  $|\xi| \leq 1/r$ ,

$$|\widehat{f}(\xi)| \leq 2\delta |\xi|^{-\gamma/2}. \quad (4.28)$$

If  $(1/r) \leq |\xi| \leq (1/r)^{1+\kappa}$ , (4.7) implies  $|\widehat{\phi}_r(\xi)| \lesssim_\beta r^{-\beta/2} |\xi|^{-\beta/2}$ , which together with (4.5) and (4.25) applied to (4.26) allows us to conclude that there is  $\kappa_2 > 0$  depending on  $\beta, \lambda$ , and  $\gamma$ , such that

$$\begin{aligned} |\widehat{f}(\xi)| &= \delta |\xi|^{-\lambda/2} + O_\beta(N^{-1/2} \log(N) \cdot r^{-\beta/2} |\xi|^{-\beta/2}) \\ &\leq (\delta + O_{\beta, \kappa}(N^{-1/2} \log(N) \cdot r^{-\beta/2} |\xi|^{-(\beta-\lambda)/2})) |\xi|^{-\lambda/2} \\ &\leq (\delta + O_{\beta, \kappa}(N^{-1/2} \log(N) r^{-\lambda/2})) |\xi|^{-\lambda/2} \\ &\leq (\delta + O_{\beta, \kappa}(N^{-\kappa_2(\beta, \varepsilon)})) |\xi|^{-\gamma/2}. \end{aligned} \quad (4.29)$$

Thus if  $N$  is sufficiently large, then for  $(1/r) \leq |\xi| \leq (1/r)^{1+\kappa}$ ,

$$|\widehat{f}(\xi)| \leq 2\delta |\xi|^{-\gamma/2}. \quad (4.30)$$

Finally, if  $|\xi| \geq (1/r)^{1+\kappa}$ , we apply (4.7) for  $T \geq \beta/2$  together with the bound  $\|\widehat{\eta}\|_{L^\infty(\mathbb{Z}^d)} = 1$ , which follows because  $\eta$  is a probability measure, to conclude that

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_T r^{-T} |\xi|^{-T} \\ &= r^{-T} |\xi|^{\beta/2-T} \cdot |\xi|^{-\beta/2} \\ &\leq r^{-T} (1/r)^{(\beta/2-T)(1+\kappa)} \cdot |\xi|^{-\beta/2} \\ &= r^{\kappa T - (\beta/2)(1+\kappa)} \cdot |\xi|^{-\beta/2}. \end{aligned} \quad (4.31)$$

If we choose  $T > (\beta/2)(1 + 1/\kappa)$ , then as  $r \rightarrow 0$ ,  $r^{\kappa T - (\beta/2)(1+\kappa)} \rightarrow 0$ . Thus we conclude from (4.31) that if  $N$  is sufficiently large, then for  $|\xi| \geq (1/r)^{1+\kappa}$

$$|\widehat{f}(\xi)| \leq 2\delta |\xi|^{-\beta/2} \leq 2\delta |\xi|^{-\gamma/2}. \quad (4.32)$$

Combining (4.28), (4.30) and (4.32) shows that if  $N$  is sufficiently large,

$$\|f\|_{M(\gamma)} \leq 2\delta. \quad (4.33)$$

Intuitively, if  $\delta \ll 1$ , then the Fourier transform of  $f$  approximately looks like the Dirac delta function at the origin in  $\mathbb{T}^d$ , so we should expect  $f \approx 1$  on  $\mathbb{T}^d$ . In particular, we should expect that  $f\mu_0 \approx \mu_0$ . Since  $\text{supp}(f\mu_0) \subset \text{supp}(f) \subset S_r$ , we know that  $\text{supp}(f\mu_0) \in \mathcal{A}$ . Carrying out all these details numerically will complete the proof of density.

We start by applying Lemma 4.5 using (4.33), which implies that if  $\rho = f\mu_0$ , then

$$\|\rho - \mu_0\|_{M(\gamma)} \lesssim_{d, \mu_0} \|f\|_{M(\gamma)} \leq 2\delta. \quad (4.34)$$

Using (4.33) and the fact that  $\hat{\mu}_0 \in L^1(\mathbb{Z}^d)$  because  $\mu_0 \in C^\infty(\mathbb{T}^d)$ , we find that

$$\rho(\mathbb{T}^d) = (\hat{f} * \hat{\mu}_0)(0) \geq 1 - \sum_{|\xi| \neq 0} |\hat{f}(\xi)| |\hat{\mu}_0(-\xi)| \geq 1 - O_{\mu_0}(2\delta). \quad (4.35)$$

Thus if we define  $\mu = \rho/\rho(\mathbb{T}^d)$ , then for  $\delta \leq 1$ ,

$$\begin{aligned} \|\mu - \mu_0\|_{M(\gamma)} &\leq \|\mu - \rho\|_{M(\gamma)} + \|\rho - \mu_0\|_{M(\gamma)} \\ &= (1/\rho(\mathbb{T}^d) - 1) \cdot \|\rho\|_{M(\gamma)} + \|\rho - \mu_0\|_{M(\gamma)} \\ &\lesssim_{\mu_0} \delta \|\rho\|_{M(\gamma)} + \|\rho - \mu_0\|_{M(\gamma)} \\ &\lesssim \delta \|\mu_0\|_{M(\gamma)} + \|\rho - \mu_0\|_{M(\gamma)} \lesssim \delta. \end{aligned} \quad (4.36)$$

If we take  $\delta$  suitably small, (4.36) implies that  $\|\mu - \mu_0\|_{M(\gamma)} \leq \delta_0$ . Since  $\text{supp}(\mu) \subset \text{supp}(\mu_0)$ , Lemma 4.6 implies that if  $\delta$  is taken even smaller, then  $d_{\mathbb{H}}(E, E_0) \leq \delta_0$ . Thus if we set  $E = \text{supp}(\mu)$ , then  $E \in \mathcal{A}$  since  $E \subset S_r$  and  $\mathcal{A}$  is monotone, and since  $\delta_0$  and  $\gamma$  were arbitrary, this completes the proof of density.  $\square$

It is a useful heuristic that in a metric space whose elements are sets, quasi-all elements are as ‘thin as possible’. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of  $\mathcal{X}_\beta$  to be as low as possible. For each  $(E, \mu) \in \mathcal{X}_\beta$ , the condition that  $\mu \in M(\beta)$  implies that  $\dim_{\mathbb{F}}(\mu) \geq \beta$ , so  $\dim_{\mathbb{H}}(E) \geq \dim_{\mathbb{F}}(E) \geq \beta$ . Thus it is natural to expect that for quasi-all  $(E, \mu) \in M(\beta)$ , the set  $E$  has both Hausdorff dimension and Fourier dimension equal to  $\beta$ , i.e.  $E$  is a Salem set of dimension  $\beta$ . We will finish this Section by proving this fact. This will use the same kinds of probabilistic arguments we will later use to prove Theorems 1.1, 1.2, and 1.3, so we will begin by stating these probabilistic results in the generality that we will need later.

**Lemma 4.7.** *Fix a positive integer  $N$ , and  $\kappa > 0$ . Let  $X_1, \dots, X_N$  be independent random variables on  $\mathbb{T}^d$ , such that for each nonzero  $\xi \in \mathbb{Z}^d$ ,*

$$\sum_{k=1}^N \mathbb{E} \left( e^{2\pi i \xi \cdot X_k} \right) = 0. \quad (4.37)$$

*Then there exists a constant  $C$  depending on  $d$  and  $\kappa$  such that*

$$\mathbb{P} \left( \sup_{|\xi| \leq N^{1+\kappa}} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot X_k} \right| \geq C N^{-1/2} \log(N)^{1/2} \right) \leq 1/10.$$

**Remark 4.8.** *In particular, the assumptions of Lemma 4.7 hold if the random variables  $\{X_1, \dots, X_N\}$  are uniformly distributed on  $\mathbb{T}^d$ .*

*Proof.* For each  $\xi \in \mathbb{Z}^d$  and  $k \in \{1, \dots, N\}$ , consider the random variable

$$Y(\xi, k) = N^{-1} e^{2\pi i \xi \cdot X_k}.$$

Then for each  $\xi \in \mathbb{Z}^d$ ,

$$\sum_{k=1}^N Y(\xi, k) = \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot X_k}. \quad (4.38)$$

We also note that for each  $\xi \in \mathbb{Z}^d$  and  $k \in \{1, \dots, N\}$ ,

$$|Y(\xi, k)| = N^{-1}. \quad (4.39)$$

Moreover,

$$\sum_{k=1}^N \mathbb{E}(Y(\xi, k)) = 0. \quad (4.40)$$

Since the family of random variables  $\{Y(\xi, k)\}$  is independent for a fixed  $\xi$ , we can apply Hoeffding's inequality together with (4.38) and (4.39) to conclude that for all  $t \geq 0$ ,

$$\mathbb{P} \left( \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot X_k} \right| \geq t \right) \leq 2e^{-Nt^2/2}. \quad (4.41)$$

Taking a union bound obtained by applying (4.41) over all  $|\xi| \leq N^{1+\kappa}$  gives the existence of a constant  $C \geq 10$  depending on  $d$  and  $\kappa$  such that

$$\mathbb{P} \left( \sup_{|\xi| \leq N^{1+\kappa}} \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot X_k} \right| \geq t \right) \leq \exp \left( C \log(N) - \frac{5Nt^2}{C} \right). \quad (4.42)$$

But then setting  $t = CN^{-1/2} \log(N)^{1/2}$  in (4.42) completes the proof.  $\square$

**Lemma 4.9.** *For quasi-all  $(E, \mu) \in \mathcal{X}_\beta$ ,  $E$  is a Salem set of dimension  $\beta$ .*

*Proof.* We shall assume  $\beta < d$  in the proof, since when  $\beta = d$ ,  $E$  is a Salem set for any  $(E, \mu) \in \mathcal{X}_\beta$ , and thus the result is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that for quasi-all  $(E, \mu) \in \mathcal{X}_\beta$ ,  $E$  has Hausdorff dimension at most  $\beta$ . For each  $\alpha > \beta$  and  $\delta, s > 0$ , we let

$$\mathcal{A}(\alpha, \delta, s) = \{E \subset \mathbb{T}^d : H_\delta^\alpha(E) < s\}.$$

and set

$$A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{X}_\beta : E \in \mathcal{A}(\alpha, \delta, s)\}.$$

Then  $A(\alpha, \delta, s)$  is an open subset of  $\mathcal{X}_\beta$ , and

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(\beta + 1/n, 1/m, 1/k) \quad (4.43)$$

is precisely the family of  $(E, \mu) \in \mathcal{X}_\beta$  such that  $E$  has Hausdorff dimension at most  $\beta$ . Thus it suffices to show that  $A(\alpha, \delta, s)$  is dense in  $\mathcal{X}_\beta$  for all  $\alpha > \beta$ , all  $\delta > 0$ , and all  $s > 0$ . Since  $\mathcal{A}(\alpha, \delta, s)$  is a *monotone* family of subsets of  $\mathbb{T}^d$ , we may apply Lemma 4.1. Fix a large integer  $N$ , and set  $r = N^{-1/\beta}$ , so that  $N \geq (1/2)r^{-\lambda}$  satisfies the condition for Lemma 4.1 to apply to these quantities. Lemma 4.7 shows that there exists a constant  $C$  depending on  $\beta$  and  $d$ , as well as  $N$  points  $S = \{x_1, \dots, x_N\} \subset \mathbb{T}^d$  such that for each  $|\xi| \leq N^{1+\kappa}$ ,

$$\left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot x_k} \right| \leq CN^{-1/2} \log(N)^{1/2}. \quad (4.44)$$

Now  $S_r$  is a union of  $N$  balls of radius  $r$ , and thus if  $r \leq \delta$ ,

$$H_\delta^\alpha(S_r) \leq Nr^\alpha = N^{1-\alpha/\beta}. \quad (4.45)$$

Since  $\alpha > \beta$ , taking  $N$  appropriately large gives a set  $S_r$  with

$$H_\delta^\alpha(S_r) < s. \quad (4.46)$$

Thus  $S_r \in \mathcal{A}(\alpha, \delta, s)$  for sufficiently large integers  $N$ . But together with (4.44), this justifies that the hypothesis of Lemma 4.1 applies to this scenario. Thus that lemma implies that  $\mathcal{A}(\alpha, \delta, s)$  is dense in  $\mathcal{X}_\beta$ , completing the proof.  $\square$

This concludes the setup to the proof of Theorems 1.1 and 1.2. All that remains is to show that quasi-all elements of  $\mathcal{X}_\beta$  avoid the given set  $Z$  for a suitable parameter  $\beta$ ; it then follows from Lemma 4.9 that quasi-all elements of  $\mathcal{X}_\beta$  are *Salem* and avoid the given set  $Z$ . The advantage of Lemma 4.1, combined with a Baire category argument, is that we can reduce our calculations to a quantitative, discretized version of the problem.

## 5 Random Avoiding Sets for Rough Patterns

We begin by proving Theorem 1.1, which requires simpler calculations than Theorem 1.2. In the last section, our results held for an arbitrary  $\beta \in (0, d]$ . But in this section, we assume

$$\beta \leq \min \left( d, \frac{dn - \alpha}{n - 1/2} \right).$$

Then  $\beta$  is small enough to show that the pattern  $Z$  described in Theorem 1.1 is generically avoided in  $\mathcal{X}_\beta$ . The construction here is very similar to the construction in [2], albeit in a Baire category setting, and with modified parameters to ensure a Fourier dimension bound rather than just a Hausdorff dimension bound.

**Lemma 5.1.** *Let  $Z \subset \mathbb{T}^{dn}$  be a compact set with lower Minkowski dimension at most  $\alpha$ . Then for quasi-all  $(E, \mu) \in \mathcal{X}_\beta$ , for any distinct points  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \notin Z$ .*

*Proof.* For any  $s > 0$ , consider the set

$$\mathcal{B}(Z, s) = \left\{ E \subset \mathbb{T}^d : \begin{array}{l} \text{for all } x_1, \dots, x_n \in E \text{ such that} \\ |x_i - x_j| \geq s \text{ for } i \neq j, (x_1, \dots, x_n) \notin Z \end{array} \right\},$$

and

$$B(Z, s) = \{(E, \mu) \in \mathcal{X}_\beta : E \in \mathcal{B}(Z, s)\}.$$

Then  $B(Z, s)$  is open in  $\mathcal{X}_\beta$ , and

$$\bigcap_{k=1}^{\infty} B(Z, 1/k) \quad (5.1)$$

consists of the family of sets  $(E, \mu)$  such that for distinct  $x_1, \dots, x_n \in E$ ,  $(x_1, \dots, x_n) \notin Z$ . Now  $\mathcal{B}(Z, 1/k)$  is a monotone family of sets, which means that, after we verify the appropriate

hypotheses, we can apply Lemma 4.1 to prove  $B(Z, s)$  is dense in  $\mathcal{X}_\beta$  for each  $s > 0$ , which would complete the proof. Thus we must construct a set  $S = \{x_1, \dots, x_N\}$  for large  $N$  such that  $S_r \in \mathcal{B}(Z, s)$  and whose exponential sums satisfy a square root cancellation bound.

Since  $Z$  has lower Minkowski dimension at most  $\alpha$ , for any  $\gamma \in (\alpha, dn]$ , we can find arbitrarily small  $r \in (0, 1)$  such that

$$|Z_{2n^{1/2}r}| \leq r^{dn-\gamma}. \quad (5.2)$$

Pick  $\lambda \in [0, (dn - \gamma)/(n - 1/2))$ , and suppose that we can find an integer  $M \geq 10$  with

$$r^{-\lambda} \leq M \leq r^{-\lambda} + 1. \quad (5.3)$$

Let  $X_1, \dots, X_M$  be independent and uniformly distributed on  $\mathbb{T}^d$ . For each distinct set of indices  $k_1, \dots, k_n \in \{1, \dots, M\}$ , the random vector  $X_k = (X_{k_1}, \dots, X_{k_n})$  is uniformly distributed on  $\mathbb{T}^{nd}$ , and so (5.2) and (5.3) imply that

$$\mathbb{P}(d(X_k, Z) \leq 2n^{1/2}r) \leq |Z_{2n^{1/2}r}| \leq r^{dn-\gamma} \lesssim M^{-\frac{dn-\gamma}{\lambda}} \leq M^{-(n-1/2)}, \quad (5.4)$$

If  $M_0$  denotes the number of indices  $k$  such that  $d(X_k, Z) \leq 2n^{1/2}r$ , then by linearity of expectation, since there are at most  $M^n$  such indices, we conclude from (5.4) that there is a constant  $C$  such that

$$\mathbb{E}(M_0) \leq (C/10)M^{1/2}. \quad (5.5)$$

Applying Markov's inequality to (5.5), we conclude that

$$\mathbb{P}(M_0 \geq CM^{1/2}) \leq 1/10. \quad (5.6)$$

Fix some small  $\kappa > 0$ . Taking a union bound to (5.6) and the results of Lemma 4.7, we conclude that if  $M$  is sufficiently large, there exists  $M$  distinct points  $x_1, \dots, x_M \in \mathbb{T}^d$  and a constant  $C$  such that the following two statements hold:

(1) Let  $I$  be the set of indices  $k_n \in \{1, \dots, M\}$  with the property that we can find distinct indices  $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$  such that if  $X = (X_{k_1}, \dots, X_{k_n})$ , then  $d(X, Z) \leq 2n^{1/2}r$ . Then  $\#(I) \leq CM^{1/2}$ .

(2) For  $0 < |\xi| \leq M^{1+\kappa}$ ,

$$\left| \frac{1}{M} \sum_{k=1}^M e^{2\pi i \xi \cdot x_k} \right| \leq CM^{-1/2} \log(M)^{1/2}.$$

Now set  $S = \{x_k : k \notin I\}$  and let  $N = \#(S)$ . Then Property (1) implies that

$$N \geq M - \#(I) \geq M - CM^{1/2}. \quad (5.7)$$

Thus for  $M \geq 4C^2$ ,

$$N \geq (1/2)M \geq (1/2)r^{-\lambda}. \quad (5.8)$$

Property (1) and (2) imply that for  $0 < |\xi| \leq N^{1+\kappa}$ ,

$$\begin{aligned} \left| \frac{1}{N} \sum_{x \in S} e^{2\pi i \xi \cdot x} \right| &\leq \left| \frac{1}{N} \sum_{k=1}^N e^{2\pi i \xi \cdot x_k} \right| + \left| \frac{1}{N} \sum_{k \in I} e^{2\pi i \xi \cdot x_k} \right| \\ &\leq 2CM^{-1/2} \log(M)^{1/2} + \#(I)/N \\ &\lesssim N^{-1/2} \log(N)^{1/2} + N^{-1/2} \\ &\lesssim N^{-1/2} \log(N)^{1/2}. \end{aligned} \tag{5.9}$$

As long as we can show that  $S_r \in \mathcal{B}(Z, s)$ , then (5.8) and (5.9) allows us to apply Lemma 4.1, completing the proof that  $B(Z, s)$  is dense. To check this, consider  $n$  points  $y_1, \dots, y_n \in S_r$ , with  $|y_i - y_j| \geq r$  for any two indices  $i \neq j$ . We can therefore find distinct indices  $k_1, \dots, k_n \notin I$  such that for each  $i \in \{1, \dots, n\}$ ,  $|x_{k_i} - y_i| \leq r$ , which means if we set  $x = (x_{k_1}, \dots, x_{k_n})$  and  $y = (y_1, \dots, y_n)$ , then

$$|x - y| \leq n^{1/2}r. \tag{5.10}$$

Since  $k_n \notin I$ ,  $d(x, Z) \geq 2n^{1/2}r$ , which combined with (5.10) implies

$$d(y, Z) \geq d(x, Z) - |x - y| \geq n^{1/2}r. \tag{5.11}$$

Thus in particular we conclude  $y \notin Z$ . We have thus proved that  $S_r \in \mathcal{B}(Z, s)$ . Thus Lemma 4.1 implies that  $B(Z, s)$  is dense in  $\mathcal{X}_\beta$  for each  $s > 0$ , completing the proof.  $\square$

The Baire category theorem, applied to Lemma 5.1, completes the proof of Theorem 1.1. Before we move onto the proof of Theorem 1.2, let us discuss the main obstacle in the method above, which prevents us from obtaining a pattern avoiding Salem set of dimension

$$\frac{dn - \alpha}{n - 1},$$

like was obtained from Hausdorff dimension under the same conditions in [2]. In the proof of Lemma 5.1, in order to obtain the bound (5.9), we were forced to choose  $N \approx r^{-\frac{dn-\alpha}{n-1/2}}$  so that the set  $I$  in that proof satisfied  $\#(I) \leq N^{1/2}$ . This means that the trivial bound

$$\left| \sum_{k \in I} e^{2\pi i (\xi \cdot X_k)} \right| \leq \#(I) \tag{5.12}$$

obtained by the triangle inequality, was viable enough to control the Fourier dimension of the resulting pair  $(E, \mu)$ . On the other hand, if we were able to show that one could instead obtain the *square root cancellation bound*

$$\left| \sum_{k \in I} e^{2\pi i (\xi \cdot X_k)} \right| \lesssim \#(I)^{1/2}, \tag{5.13}$$

then we would only need the condition  $\#(I) \lesssim N$ , which would allow us to choose  $N \approx r^{-\frac{dn-\alpha}{n-1}}$ , which would lead to a proof of existence of a pattern avoiding set with larger Fourier dimension, matching that of the Hausdorff dimension bound obtained in [2]. Under stronger

assumptions on the pattern we are trying to avoid, which form the hypotheses of Theorem 1.2, we are able to justify that some kind of square root cancellation, like that of (5.13) takes place, though with an additional term that is only bounded appropriately for  $n > 2$  if we set  $N \approx r^{-\frac{dn-\alpha}{n-3/4}}$ . Under the hypothesis of Theorem 1.3, we are able to make this additional term vanish completely, which will enable us to set  $N \approx r^{-\frac{dn-\alpha}{n-1}}$ , and thus completely recover the dimension bound in [2] in the Salem setting.

## 6 Concentration Bounds for Smooth Surfaces

In this section we prove Theorem 1.2 using some more robust probability concentration calculations to justify the kinds of square root cancellation alluded to at the end of the last section. We set

$$\beta \leq \begin{cases} d & : n = 2 \\ d/(n - 3/4) & : n \geq 3 \end{cases}.$$

For such  $\beta$ , elements of the space  $\mathcal{X}_\beta$  will generically avoid patterns described by an equation  $x_n = f(x_1, \dots, x_{n-1})$ , where  $f$  satisfies the hypotheses of Theorem 1.2.

**Lemma 6.1.** *Let  $f : V \rightarrow \mathbb{T}^d$  satisfy the hypothesis of Theorem 1.2. Then for quasi-all  $(E, \mu) \in \mathcal{X}_\beta$ , and for any distinct points  $x_1, \dots, x_n \in E$ ,  $x_n \neq f(x_1, \dots, x_{n-1})$ .*

*Proof.* Given any family of disjoint, closed cubes  $R_1, \dots, R_n \subset \mathbb{T}^d$  such that  $(R_1 \times \dots \times R_n) \cap V$  is a closed set, we let

$$\mathcal{H}(R_1, \dots, R_n) = \{E \subset \mathbb{T}^d : \text{for all } x_i \in R_i \cap E, x_n \neq f(x_1, \dots, x_{n-1})\},$$

and let

$$H(R_1, \dots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : E \in \mathcal{H}(R_1, \dots, R_n)\}.$$

Then  $H(R_1, \dots, R_n)$  is an open subset of  $\mathcal{X}_\beta$ . For the purpose of a Baire category argument, this proof will follow by showing  $H(R_1, \dots, R_n)$  is dense in  $\mathcal{X}_\beta$  for any family of disjoint cubes  $\{R_1, \dots, R_n\}$ , each having common sidelength  $s$  for some  $s > 0$ , such that if  $Q_i = 2R_i$  for each  $i$ , then  $Q_2 \times \dots \times Q_n \subset V$ , and  $d(R_i, R_j) \geq 10s$  for each  $i \neq j$ . Since  $\mathcal{H}(R_1, \dots, R_n)$  is a monotone family of sets, we will prove this by applying Lemma 4.1. Thus for a suitable choice of  $r > 0$ , we must construct a large set  $S$  such that  $S_r \in \mathcal{H}(R_1, \dots, R_n)$  and whose exponential sums exhibit square root cancellation.

Since  $f$  is smooth, we can fix a constant  $L \geq 0$  such that for any  $x, y \in Q_1 \times \dots \times Q_{n-1}$ ,

$$|f(x) - f(y)| \leq L|x - y|. \quad (6.1)$$

Fix a family of non-negative bump functions  $\psi_0, \psi_1, \dots, \psi_n \in C^\infty(\mathbb{T}^d)$ , such that for  $i \in \{1, \dots, n\}$ ,  $\psi_i(x) = 1$  for  $x \in 1.5 \cdot R_i$ ,  $\psi_i(x) = 0$  for  $x \notin Q_i$ , and  $\psi_0(x) + \dots + \psi_n(x) = 1$  for  $x \in \mathbb{T}^d$ . For  $i \in \{0, \dots, n\}$ , let  $A_i = \int \psi_i(x) dx$  denote the total mass of  $\psi_i$ . Now fix a large integer  $M > 0$ , and consider a family of independent random variables

$$\{X_i(k) : 0 \leq i \leq n, 1 \leq k \leq M\},$$

where the random variable  $X_i(k)$  is chosen with respect to the probability density function  $A_i^{-1}\psi_i$ . Fix  $\lambda \in [0, \beta)$ , and choose  $r > 0$  such that  $r^{-\lambda} \leq M \leq r^{-\lambda} + 1$ , and then let  $I$  be the set of indices  $k_n \in \{1, \dots, M\}$  such that there are indices  $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$  with the property that

$$|X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leq 2n^{1/2} \cdot (L + 1) \cdot r. \quad (6.2)$$

Now (6.2) implies that if  $k_n \notin I$ , then for any  $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$ ,

$$|X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| > 2n^{1/2} \cdot (L + 1) \cdot r. \quad (6.3)$$

Thus if we set

$$S = \{X_i(k) : 0 \leq i \leq n-1, 1 \leq k \leq M\} \cup \{X_n(k) : k \notin I\}$$

Then we claim that  $S_r \in \mathcal{H}(R_1, \dots, R_n)$  for suitably small  $r$ ; to see this, suppose there were distinct  $y_1, \dots, y_n \in S_r$  such that  $y_1 \in R_1, \dots, y_n \in R_n$ , and  $y_n = f(y_1, \dots, y_{n-1})$ . We may pick  $x_1, \dots, x_n \in S$  such that  $|x_i - y_i| \leq r$  for each  $i$ . Since  $d(R_i, R_j) = 10s$  for  $i \neq j$ , if  $r \leq s$ , then it cannot be true that  $x_i = X_j(k)$  for some  $j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, M\}$ . Since  $\psi_i(x) = 1$  on  $1.5R_i$ , if  $r < 0.5s$ , we have  $d(\text{supp}(\psi_0), R_i) \geq 0.5s$  and so it also cannot be true that  $x_i = X_0(k)$  for some  $k \in \{1, \dots, M\}$ . Thus there must be  $k_i \in \{1, \dots, M\}$  such that  $x_i = X_i(k_i)$ . But by assumption  $k_n \notin I$ , so we have

$$|X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| > 2n^{1/2} \cdot (L + 1) \cdot r. \quad (6.4)$$

Thus (6.1) and (6.4) imply that

$$0 = |y_n - f(y_1, \dots, y_n)| \geq n^{1/2}(L + 1)r > 0, \quad (6.5)$$

which gives a contradiction, proving that  $S_r \in \mathcal{H}(R_1, \dots, R_n)$ . The remainder of the proof focuses on bounding exponential sums associated with  $S$ , so that we may apply Lemma 4.1.

Consider the random exponential sums

$$F(\xi) = \sum_{i \in \{0, \dots, n-1\}} \sum_{k=1}^M A_i e^{2\pi i \xi \cdot X_i(k)} + \sum_{k \notin I} A_n e^{2\pi i \xi \cdot X_n(k)}.$$

Controlling  $|F(\xi)|$  with high probability will justify an application of Lemma 4.1. To analyze  $F$ , introduce the auxillary exponential sums

$$G(\xi) = \sum_{i=0}^n \sum_{k=1}^M A_i e^{2\pi i \xi \cdot X_i(k)}$$

and

$$H(\xi) = \sum_{k \in I} A_n e^{2\pi i \xi \cdot X_n(k)}.$$



Then  $F(\xi) = G(\xi) - H(\xi)$ . Obtaining a bound on  $G(\xi)$  is simple. For non-zero  $\xi \in \mathbb{Z}^d$ ,

$$\begin{aligned}\mathbb{E}(G(\xi)) &= \sum_{i=0}^n M A_i \int (\psi_i(x)/A_i) e^{2\pi i \xi \cdot x} dx \\ &= M \sum_{i=0}^n \int \psi_i(x) e^{2\pi i \xi \cdot x} dx \\ &= M \int_{\mathbb{T}^d} e^{2\pi i \xi \cdot x} dx = 0.\end{aligned}\tag{6.6}$$

Applying Lemma 4.7, we conclude that for any fixed  $\kappa > 0$ , there is  $C > 0$  such that

$$\mathbb{P} \left( \sup_{|\xi| \leq N^{1+\kappa}} |G(\xi)| \geq C M^{1/2} \log(M)^{1/2} \right) \leq 1/10.\tag{6.7}$$

Analyzing  $H(\xi)$  requires a more subtle concentration bound, which we delegate to a series of lemmas following this proof. In Lemma 6.2, we will employ some concentration bounds to show that

$$\mathbb{P} \left( \sup_{|\xi| \leq N^{1+\kappa}} |H(\xi) - \mathbb{E}(H(\xi))| \geq C M^{1/2} \log(M)^{1/2} \right) \leq 1/10.\tag{6.8}$$

In Lemma 6.4 we show that for any  $\delta > 0$ , there exists  $r_1 > 0$  such that for  $r \leq r_1$  and any nonzero  $\xi \in \mathbb{Z}^d$ ,

$$|\mathbb{E}(H(\xi))| \leq \delta M |\xi|^{-\beta/2} + O(M^{1/2}).\tag{6.9}$$

Analogous to equation (5.4) in Lemma 5.1, for any indices  $k_1, \dots, k_n \in \{1, \dots, M\}$ , we have

$$\mathbb{P} \left( |X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leq 2n^{1/2} \cdot (L+1) \cdot r \right) \lesssim_{n,L} r^d \lesssim M^{-d/\lambda}.\tag{6.10}$$

Thus if  $M_0$  denotes the number of tuples of indices  $(k_1, \dots, k_n)$  such that (6.2) holds, then (6.10) implies that

$$\mathbb{E}(M_0) \lesssim M^{n-d/\lambda}\tag{6.11}$$

Applying Markov's inequality to (6.11), we conclude that there exists a constant  $C > 0$  such that

$$\mathbb{P}(M_0 \geq C M^{d/\lambda-n}) \leq 1/10.\tag{6.12}$$

Taking a union bound to (6.7), (6.8), and (6.12), and then applying (6.9), we conclude that there exists  $C > 0$  and a particular instantiation of the random variables  $\{X_i(k)\}$  such that for any  $0 < |\xi| \leq M^{1+\kappa}$ ,

$$|G(\xi)| \leq C M^{1/2} \log(M)^{1/2},\tag{6.13}$$

and

$$|H(\xi)| \leq C M^{1/2} \log(M)^{1/2} + \delta M |\xi|^{-\beta/2}.\tag{6.14}$$

And

$$\#(I) \leq C M^{d/\lambda-n}.\tag{6.15}$$

Since  $\lambda < \beta_0 < d/(n-1)$ , the inequality  $d/\lambda - n < 1$  holds. Thus (6.15) implies that for sufficiently large  $M > 0$ , if  $N = \#(S)$ , then

$$N \geq M - CM^{d/\lambda - n} \geq (1/2)M \geq (1/2)r^{-\lambda}. \quad (6.16)$$

Putting (6.9), (6.13), (6.14), and the fact that  $F(\xi) = G(\xi) + H(\xi)$  together, if we set  $a(X_i(k)) = A_i$  for each  $i$  and  $k$ , then

$$\left| \frac{1}{N} \sum_{x \in S} a(x) e^{2\pi i \xi \cdot x} \right| \lesssim CN^{1/2} \log(N)^{1/2} + \delta |\xi|^{-\beta/2}. \quad (6.17)$$

Since  $\sum_{x \in S} a(x) \geq N$ , the exponential sum bound (6.17) allows us to apply Lemma 4.1, together with (6.16) and the fact that  $S_r \in \mathcal{H}(R_1, \dots, R_n)$ , to conclude that  $H(R_1, \dots, R_n)$  is dense in  $\mathcal{X}_\beta$ .  $\square$

All that remains is to prove (6.8) and (6.9).

**Lemma 6.2.** *Let  $H(\xi)$  be the random exponential sum described in Lemma 6.1. Then*

$$\mathbb{P} \left( \sup_{|\xi| \leq M^{1+\kappa}} |H(\xi) - \mathbb{E}(H(\xi))| \geq CM^{1/2} \log(M)^{1/2} \right) \leq 1/10.$$

for some universal constant  $C > 0$ .

**Remark 6.3.** *Before we begin the proof of this lemma, let us describe the idea. As a random quantity,  $H(\xi)$  is a function of the independent random quantities  $\{X_i(k)\}$ , and so McDiarmid's inequality presents itself as a useful concentration bound. However, a naive application of McDiarmid's inequality fails here, because changing a single random variable  $X_i(k)$  for  $1 \leq i \leq n-1$  while fixing all other random variables can change  $H(\xi)$  by as much as  $O(M)$  (see Figure 1), which is far too much to obtain the square root cancellation bounds like we obtained in (6.7). On the other hand, it seems that a single variable  $X_i(k)$  only changes  $H(\xi)$  by  $O(M)$  when the other random variables  $\{X_1(k)\}$  are configured in a very particular way, which is unlikely to happen. Thus we should expect that adjusting a single random variable  $X_i(k)$  does not influence the value of  $H(\xi)$  much when averaged over the possible choices of  $\{X_1(k)\}$ , and then we can apply McDiarmid's inequality.*

*Proof.* Consider the random set  $\Omega$  of values  $x_n \in Q_n$  such that there are  $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$  with

$$|x_n - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leq 2n^{1/2} \cdot (L+1) \cdot r. \quad (6.18)$$

Then

$$H(\xi) = A_n \sum_{k=1}^M Z(k). \quad (6.19)$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_n(k)} & : X_n(k) \in \Omega, \\ 0 & : X_n(k) \notin \Omega \end{cases}.$$

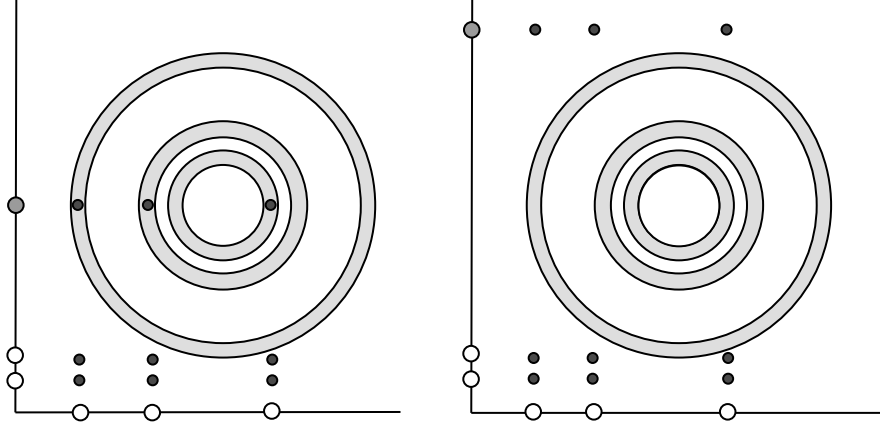


Figure 1: The two diagrams displayed indicate two instances of the process of defining the set  $S$  for the function  $f(x_1, x_2) = x_1^2 + x_2^2$ . Here  $M = 3$ ,  $n = 3$ , the values on the  $x$ -axis represent the values  $X_1(1), X_1(2)$ , and  $X_1(3)$ , the values on the  $y$ -axis represent the values  $X_2(1), X_2(2)$ , and  $X_2(3)$ , the dark points represent the family of all pairs  $(X_1(k_1), X_2(k_2))$ , and the annuli represent the  $r$ -neighborhoods of  $f^{-1}(X_3(1)), f^{-1}(X_3(2))$ , and  $f^{-1}(X_3(3))$ . In the left diagram,  $S = \{1, 2, 3\}$ , whereas in the right diagram  $S = \emptyset$ . Thus  $S$  can be completely changed simply by adjusting the position of the shaded point on the  $y$ -axis (adjusting one of the variables  $X_2(k)$ ), which completely changes the value of the exponential sums associated with  $S$ . Nonetheless, this only occurs if the other values are arranged in a highly particular manner, which is why we are able to show that the *average* value of the exponential sums associated with  $S$  over all possible choices of the variables  $\{X_1(1), X_1(2), X_1(3)\}$  is not changed drastically when we vary the other variables, which enables us to apply McDiarmid's inequality after this averaging process.

If  $\Sigma$  is the  $\sigma$ -algebra generated by the random variables

$$\{X_i(k) : i \in \{1, \dots, n-1\}, k \in \{1, \dots, M\}\},$$

then  $\Omega$  is measurable with respect to  $\Sigma$ . Thus the random variables  $\{Z(k)\}$  are *conditionally independent* given  $\Sigma$ . Since we have  $|Z(k)| \leq 1$  almost surely, Hoeffding's inequality thus implies that for all  $t \geq 0$ ,

$$\mathbb{P}(|H(\xi) - \mathbb{E}(H(\xi)|\Sigma)| \geq t) \leq 4 \exp\left(\frac{-t^2}{2M}\right). \quad (6.20)$$

It is simple to see that

$$\mathbb{E}(H(\xi)|\Sigma) = A_n M \int_{\Omega} \psi_n(x) e^{2\pi i \xi \cdot x} dx. \quad (6.21)$$

Since

$$\Omega = \bigcup \{T(X_1(k_1), \dots, X_{n-1}(k_{n-1}))_{2n^{1/2}(L+1)r} : 1 \leq k_1, \dots, k_{n-1} \leq N\}. \quad (6.22)$$

we see that varying each random variable  $X_i(k)$ , for  $1 \leq i \leq n-1$  while fixing the other random variables adjusts at most  $M^{n-2}$  of the balls forming  $\Omega$ , and thus varying  $X_i(k)$  while fixing the other random variables changes  $\mathbb{E}(H(\xi)|\Sigma)$  by at most

$$M \cdot 2 \cdot (4n^{1/2}(L+1)r)^d \cdot M^{n-2} \lesssim_{n,d,L} r^d M^{n-1} \lesssim 1. \quad (6.23)$$

Thus McDiarmid's inequality shows that there exists a constant  $C$  depending on  $d, n$ , and  $L$ , such that for any  $t \geq 0$ ,

$$\mathbb{P}(|\mathbb{E}(H(\xi)|\Sigma) - \mathbb{E}(H(\xi))| \geq t) \leq 4 \exp\left(\frac{-t^2}{CM}\right). \quad (6.24)$$

Combining (6.20) and (6.24), we conclude that there exists a constant  $C > 0$  such that for each  $\xi \in \mathbb{Z}^d$ ,

$$\mathbb{P}(|H(\xi) - \mathbb{E}(H(\xi))| \geq t) \leq 8 \exp\left(\frac{-Mt^2}{C}\right). \quad (6.25)$$

Applying a union bound to (6.25) over all  $0 < |\xi| \leq M^{1+\kappa}$  shows that there exists a constant  $C > 0$  such that

$$\mathbb{P}\left(\sup_{|\xi| \leq M^{1+\kappa}} |H(\xi) - \mathbb{E}(H(\xi))| \geq CM^{1/2} \log(M)^{1/2}\right) \leq 1/10. \quad \square$$

The analysis of (6.9) requires a more technical calculation.

**Lemma 6.4.** *Let  $H(\xi)$  be the random exponential sum described in Lemma 6.1. Then there exists  $C > 0$  such that for any  $\delta > 0$ , there exists  $M_0 > 0$  such that for  $M \geq M_0$ ,*

$$|\mathbb{E}(H(\xi))| \leq \delta M |\xi|^{-\beta/2} + CM^{1/2}.$$

*Proof.* We break the analysis of  $\mathbb{E}(H(\xi))$  into two cases, depending on whether  $n = 2$  or  $n > 2$ . The major difference here is that when  $n > 2$ ,  $\beta < d$ , whereas when  $n = 2$ ,  $\beta = d$ , i.e. we are constructing a full dimensional set, so that some argument that work for the case  $n > 2$  fail when  $n = 2$ . On the other hand, the analysis of patterns when  $n = 2$  is more trivial than the analysis for  $n > 2$ , which makes this argument more simple in other respects.

Let's start with the case  $n = 2$ . Using the fact that the family  $\{X_n(k) : 1 \leq k \leq K\}$  are identically distributed, we calculate that

$$\begin{aligned}\mathbb{E}(H(\xi)) &= \mathbb{E}(M \cdot A_2 \cdot e^{2\pi i \xi \cdot X_2(1)} \cdot \mathbb{I}(1 \in I)) \\ &= M \cdot A_2 \int_{\mathbb{T}^d} \psi_2(x) \mathbb{P}(1 \in I | X_2(1) = x) e^{2\pi i \xi \cdot x} dx.\end{aligned}\tag{6.26}$$

For each  $x \in \mathbb{T}^d$ , a change of variables formula implies that

$$\begin{aligned}\mathbb{P}(1 \in I | X_1(1) = x) &= 1 - \left(1 - \int_{f^{-1}(B_{(L+1)r}(x))} \psi_1(x_1) dx_1\right)^M \\ &= 1 - \left(1 - \int_{B_{(L+1)r}(x)} \frac{(\psi_1 \circ f^{-1})(x_2)}{|\det(Df)(f^{-1}(x_2))|} dx_2\right)^M \\ &= 1 - \left(1 - \int_{B_{(L+1)r}(x)} \tilde{\psi}_1(x_2) dx_2\right)^M,\end{aligned}\tag{6.27}$$

where we have introduced  $\tilde{\psi}_1$  for notational convenience. If we define

$$g(x) = \mathbb{P}(1 \in I | X_1(1) = x),$$

then  $\mathbb{E}(H(\xi)) = MA_2 \cdot \widehat{\psi_2 g}(\xi)$ . We can obtain a bound on this quantity by bounding the partial derivatives of  $\psi_2 g$ . Bernoulli's inequality implies that

$$g(x) = 1 - \left(1 - \int_{B_{(L+1)r}(x)} \tilde{\psi}_1(x_2) dx_2\right)^M \lesssim_L M r^d \leq M^{1-d/\lambda}.\tag{6.28}$$

On the other hand, for any multi-index  $\alpha > 0$ ,  $\partial^\alpha g(x)$  is a sum of terms of the form

$$(-1)^m \frac{M!}{(M-m)!} \left(1 - \int_{B_{(L+1)r}(x)} \tilde{\psi}_1(x_2) dx_2\right)^{M-m} \left(\prod_{i=1}^m \int_{B_{(L+1)r}(x)} \partial_{\alpha_i} \tilde{\psi}_1(x_2) dx_2\right),\tag{6.29}$$

where  $\alpha_i \neq 0$  for any  $i$  and  $\alpha = \alpha_1 + \dots + \alpha_m$ . This implies  $0 < m \leq |\alpha|$  for any terms in the sum. Now the bound  $|\partial_{\alpha_i} \tilde{\psi}_1(x_2)| \lesssim_{\alpha_i} 1$  implies that

$$\left|\int_{B_{(L+1)r}(x)} \partial_{\alpha_i} \tilde{\psi}_1(x_2) dx_2\right| \lesssim_{\alpha_i} r^d.\tag{6.30}$$

Applying (6.30) to (6.29) enables us to conclude that

$$|\partial_\alpha g(x)| \lesssim_\alpha \max_{0 < m \leq |\alpha|} M^m r^{md} \leq M^{1-d/\lambda},\tag{6.31}$$

Since the fact that  $\psi_2 \in C^\infty(\mathbb{T}^d)$  implies that  $\|\partial_\alpha \psi_2\|_{L^\infty(\mathbb{T}^d)} \lesssim_\alpha 1$  for any multi-index  $\alpha$ , the product rule applied to (6.31) implies that  $\|\partial_\alpha(\psi_2 g)\|_{L^\infty(\mathbb{T}^d)} \lesssim_\alpha M^{1-d/\lambda}$  for all  $\alpha > 0$ , which means that for any  $T > 0$  and  $\xi \neq 0$ ,

$$|\mathbb{E}(H(\xi))| \lesssim_T M^{2-d/\lambda} |\xi|^{-T}. \quad (6.32)$$

In particular, setting  $T = \beta/2$ , fixing  $\delta > 0$ , and then choosing  $M_0$  appropriately, if  $M \geq M_0$ , (6.32) shows that for  $r \leq r_0$ ,

$$|\mathbb{E}(H(\xi))| \leq \delta M |\xi|^{-\beta/2}. \quad (6.33)$$

This completes the proof in the case  $n = 2$ .

Now we move on to the case  $n \geq 3$ . A version of equation (6.26) continues to hold in this setting, namely that

$$\mathbb{E}(H(\xi)) = A_n M \int_{\mathbb{T}^d} \psi_n(x) \mathbb{P}(1 \in I | X_n(1) = x_n) e^{2\pi i \xi \cdot x_n} dx_n. \quad (6.34)$$

However, the analysis of this equation is made more complicated by the lack of an explicit formula for  $\mathbb{P}(1 \in I | X_n(1) = x_n)$ . For a set  $E \subset \mathbb{T}^{d(n-1)}$ , let  $A(E)$  denote the event that there exists  $k_1, \dots, k_{n-1}$  such that  $(X_1(k_1), \dots, X_{n-1}(k_{n-1})) \in E$ . Then

$$\mathbb{P}(1 \in I | X_1(1) = x) = \mathbb{P}(A(f^{-1}(B_{(L+1)r}(x_n)))). \quad (6.35)$$

For any cube  $Q \in \mathbb{T}^{d(n-1)}$  and any indices  $1 \leq k_1, \dots, k_{n-1} \leq K$ , set  $k = (k_1, \dots, k_{n-1})$  and let  $A(Q; k)$  denote the event that  $(X_1(k_1), \dots, X_{n-1}(k_{n-1})) \in Q$ . Then

$$A(Q) = \bigcup_k A(Q; k). \quad (6.36)$$

For any cube  $Q$  and index  $k$ ,

$$\mathbb{P}(A(Q; k)) = \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1}, \quad (6.37)$$

and so

$$\sum_k \mathbb{P}(A(Q; k)) = M^{n-1} \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1}. \quad (6.38)$$

An application of inclusion exclusion to (6.38) thus shows that

$$\begin{aligned} & \left| \mathbb{P}(A(Q)) - M^{n-1} \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1} \right| \\ & \leq \sum_{k \neq k'} \mathbb{P}(A(Q; k) \cap A(Q; k')). \end{aligned} \quad (6.39)$$

For each  $k, k'$ , the quantity  $\mathbb{P}(A(Q; k) \cap A(Q; k'))$  depends on the number of indices  $i$  such that  $k_i = k'_i$ . In particular, if  $I \subset \{1, \dots, n-1\}$  is the set of indices where the quantity agrees, then

$$\mathbb{P}(A(Q; k) \cap A(Q; k')) = \left( \prod_{i \in I} \int_{Q_i} \psi_i(x) dx \right) \cdot \left( \prod_{i \notin I} \left( \int_{Q_i} \psi_i(x) dx \right)^2 \right). \quad (6.40)$$

In particular, if  $Q$  has sidelength  $l$  and  $\#(I) = m$ , then  $\mathbb{P}(A(Q; k) \cap A(Q; k')) \lesssim l^{d(2n-m-2)}$ . For each  $m$ , there are at most  $M^{2n-m-2}$  pairs  $k$  and  $k'$  with  $\#(I) = m$ . And so provided  $l^d \leq 1/M$ ,

$$\sum_{k \neq k'} \mathbb{P}(A(Q; k) \cap A(Q; k')) \lesssim \sum_{m=0}^{n-2} (M \cdot l^d)^{2n-m-2} \lesssim M^n l^{dn}. \quad (6.41)$$

Thus we conclude from (6.39) and (6.41) that

$$\mathbb{P}(A(Q)) = M^{n-1} \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1} + O(M^n l^{dn}). \quad (6.42)$$

For a particular  $x_n \in \mathbb{T}^d$ , let  $E = f^{-1}(B_{(L+1)r}(x_n))$ . Since  $f$  is a submersion,  $E$  is contained in a  $O(r)$ -thickening of a  $d(n-2)$  dimensional surface in  $\mathbb{T}^{d(n-1)}$ . Applying the Whitney covering lemma, we can find a family of almost disjoint dyadic cubes  $\{Q_{ij} : j \geq 0\}$  such that

$$E = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{n_i} Q_{ij}, \quad (6.43)$$

where for each  $i \geq 0$ ,  $Q_{ij}$  is a sidelength  $r/2^i$  cube, and  $n_i \lesssim (r/2^i)^{-d(n-2)}$ . It follows from (6.43) that

$$A(E) = \bigcup_{i,j} A(Q_{ij}). \quad (6.44)$$

Since  $n \geq 3$ , we can use (6.42) to calculate that

$$\begin{aligned} & \left| \sum_{i,j} \mathbb{P}(A(Q_{ij})) - M^{n-1} \int_E \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx \right| \\ & \lesssim \sum_{i=0}^{\infty} (r/2^i)^{-d(n-2)} \cdot (M^n (r/2^i)^{dn}) \\ & \lesssim r^{2d} M^n \leq M^{-1/2}. \end{aligned} \quad (6.45)$$

Thus an inclusion exclusion bound together with (6.44) and (6.45) implies that

$$\begin{aligned} & \left| \mathbb{P}(A(E)) - M^{n-1} \int_E \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx \right| \\ & \lesssim M^{-1/2} + \sum_{(i_1, j_1) \neq (i_2, j_2)} \mathbb{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})). \end{aligned} \quad (6.46)$$

The quantity  $\mathbb{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2}))$  depends on the relation between the various sides of  $Q_{i_1 j_1}$  and  $Q_{i_2 j_2}$ . Without loss of generality, we may assume that  $i_1 \geq i_2$ . If  $I(Q_{i_1 j_1}, Q_{i_2 j_2})$  is the set of indices  $1 \leq k \leq n-1$  where  $Q_{i_1 j_1 k} \subset Q_{i_2 j_2 k}$ , and  $\#(I(Q_{i_1 j_1}, Q_{i_2 j_2})) = m$ , then

$$\begin{aligned} \mathbb{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})) & \lesssim (M(r/2^{i_1})^d)^m \cdot (M(r/2^{i_1})^d \cdot M(r/2^{i_2})^d)^{n-m-1} \\ & = 2^{-d[(n-1)i_1 + (n-m-1)i_2]} (M r^d)^{2n-m-2}. \end{aligned} \quad (6.47)$$

The condition that  $D_{x_k}f$  is invertible for all  $k$  on the domain of  $f$  implies that any axis-oriented plane in  $\mathbb{T}^{dn}$  intersects transversally with the level sets of  $f$ . In particular, this means that the intersection of a  $O(r/2^{i_1})$  thickening of a codimension  $dm$  axis-oriented hyperplane intersects a  $O(r/2^{i_1})$  thickening of  $\partial E$  (which has codimension  $d$ ) in a set with volume  $O((r/2^{i_1})^d(r/2^{i_1})^{dm})$ , and intersects a  $O(r/2^{i_2})$  thickening of  $\partial E$  in a set with volume  $O((r/2^{i_2})^d(r/2^{i_1})^{dm})$ . As a particular example of this, for any distinct indices  $j_1, \dots, j_m \in \{1, \dots, n-1\}$ , and any family of integers  $0 \leq n_{11}, \dots, n_{md} \leq 2^{i_1}/r$ , the set

$$\left\{ x \in E : \frac{n_{11}}{2^{i_1}} \leq x_{j_1 1} \leq \frac{(n_{11} + 1)}{2^{i_1}}, \dots, \frac{n_{md}}{2^{i_1}} \leq x_{j_m d} \leq \frac{(n_{md} + 1)}{2^{i_1}} \right\} \quad (6.48)$$

contains at most

$$O((r/2^{i_1})^d(r/2^{i_1})^{dm}(r/2^{i_1})^{-d(n-1)}) = O(2^{d(n-m-2)i_1}r^{-d(n-m-2)}) \quad (6.49)$$

sidelength  $r/2^{i_1}$  dyadic cubes in the decomposition of  $E$ , and at most

$$O((r/2^{i_2})^d(r/2^{i_1})^{dm}(r/2^{i_2})^{-d(n-1)}) = O(2^{d(n-2)i_2-(dm)i_1}r^{-d(n-m-2)}) \quad (6.50)$$

sidelength  $r/2^{i_2}$  dyadic cubes in the decomposition of  $E$ . Letting the integers  $\{n_{kl}\}$  vary over all possible choices we conclude from (6.49) and (6.50) that for each  $i_1$  and  $i_2$  there are at most

$$\begin{aligned} & O((2^{i_1}/r)^{dm} (2^{d(n-m-2)i_1}r^{-d(n-m-2)}) (2^{d(n-2)i_2-(dm)i_1}r^{-d(n-m-2)})) \\ & = O(2^{d(n-m-2)i_1+d(n-2)i_2}r^{-d(2n-m-4)}) \end{aligned} \quad (6.51)$$

pairs  $Q_{i_1 j_1}$  and  $Q_{i_2 j_2}$  with  $I(Q_{i_1 j_1}, Q_{i_2 j_2}) = m$ . Thus we conclude from (6.47) and (6.51) that

$$\begin{aligned} & \sum_{(i,j) \neq (i',j')} \mathbb{P}(A(Q_{ij}) \cap A(Q_{i'j'})) \\ & \lesssim \sum_{m=0}^{n-2} \sum_{i_1 \geq i_2} (2^{d(n-m-2)i_1+d(n-2)i_2}r^{-d(2n-m-4)}) \\ & \quad (2^{-d((n-1)i_1+(n-m-1)i_2)}(Mr^d)^{2n-m-2}) \\ & \lesssim r^{2d} \sum_{m=0}^{n-2} M^{2n-m-2} \sum_{i_1 \geq i_2} 2^{-d(m+1)i_1+d(m-1)i_2} \\ & \lesssim \sum_{m=0}^{n-2} M^{2n-m-2} r^{2d} \\ & \lesssim M^{2(n-1)} r^{2d} \lesssim M^{-1/2}. \end{aligned} \quad (6.52)$$

Returning to the bound in (6.46), (6.52) implies that

$$\left| \mathbb{P}(A(E)) - M^{n-1} \int_E \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1} \right| \lesssim M^{-1/2}. \quad (6.53)$$



Returning even further back to (6.34), recalling that  $E = f^{-1}(B_r(x_n))$ , (6.53) implies

$$\left| \mathbb{E}(H(\xi)) - A_n \cdot M^n \int_{\mathbb{T}^d} \psi_n(x_n) \int_{f^{-1}(B_r(x_n))} \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) e^{2\pi i \xi \cdot x_n} dx_1 \dots dx_n \right| \lesssim M^{1/2}. \quad (6.54)$$

Applying the co-area formula, writing  $\psi(x) = \psi_1(x_1) \dots \psi_n(x_n)$ , we find

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{f^{-1}(B_r(x_n))} \psi(x) e^{2\pi i \xi \cdot x_n} dx_1 \dots dx_n \\ &= \int_{B_r(0)} \int_{\mathbb{T}^d} \int_{f^{-1}(x+v)} \psi(x) e^{2\pi i \xi \cdot x_n} dH^{n-2}(x_1, \dots, x_{n-1}) dx_n dv \\ &= \int_{B_r(0)} \int_{\mathbb{T}^{d(n-1)}} \psi(x, f(x) - v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} |Jf(x)| dx dv \\ &= \int_{B_r(0)} \int_{\mathbb{T}^{d(n-1)}} \tilde{\psi}(x, v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} dx dv. \end{aligned} \quad (6.55)$$

where  $\tilde{\psi}(x, v) = \psi(x, f(x) - v) \cdot |Jf(x)|$ , and  $Jf$  is the rank- $d$  Jacobian of  $f$ . A consequence of (6.55) in light of (6.54) is that it reduces the study of  $\mathbb{E}(H(\xi))$  to a standard oscillatory integral. In particular, we note that  $Df$  is surjective on the domain of  $f$ , which implies the oscillatory integral in  $x$  has no stationary points for  $\xi \neq 0$ . Applying Proposition 4 of Chapter 8 of [11], we conclude that for all  $|v| \leq 1$  and  $T > 0$ ,

$$\left| \int_{\mathbb{T}^{d(n-1)}} \tilde{\psi}(x, v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} dx \right| \lesssim_T |\xi|^{-T}. \quad (6.56)$$

Now the bound in (6.56) can be applied with (6.55) to conclude that

$$\left| \int_{\mathbb{T}^d} \int_{f^{-1}(B_r(x))} \psi(x) e^{2\pi i \xi \cdot x_n} dx_2 \dots dx_n dx_1 \right| \lesssim_T r^d |\xi|^{-T}. \quad (6.57)$$

In particular, taking  $T = \beta/2$  here, combined with (6.54), (6.57), we find that

$$|\mathbb{E}(H(\xi))| \lesssim M^n r^d |\xi|^{-\beta/2} + M^{1/2} \lesssim M^{3/4} |\xi|^{-\beta/2} + M^{1/2}. \quad (6.58)$$

Thus there exists  $C > 0$  such that for any  $\delta > 0$ , there exists  $r_0 > 0$  such that for  $r \leq r_0$ , and any nonzero  $\xi \in \mathbb{Z}^d$ ,

$$|\mathbb{E}(H(\xi))| \leq \delta M |\xi|^{-\beta/2} + CM^{1/2}. \quad \square$$

The proof of Lemma 6.4 is the only obstacle preventing us from constructing a Salem set  $X$  avoiding the pattern defined by  $Z$  with

$$\dim_{\mathbb{F}}(X) = \frac{d}{n-1}.$$

All other aspects of the proof carry through for  $\beta = d/(n-1)$ . The problem with Lemma 6.4 in this scenario is that if  $n \geq 3$  and  $\beta > d/(n-3/4)$ , there is too much ‘overlap’ between the various cubes we use in our covering argument; thus the inclusion-exclusion argument found in this proof cannot be used to control  $\mathbb{E}(H(\xi))$  in a significant way. We believe our method can construct Salem sets with Fourier dimension  $d/(n-1)$ , but new tools are required to improve the estimates on  $\mathbb{E}(H(\xi))$ . In the next section, we are able to bypass this by using a trick which will imply that, for the analogous exponential sum  $H(\xi)$ , we have  $\mathbb{E}(H(\xi)) = 0$  for all  $\xi \neq 0$ , so we need no careful analysis of this quantity.

## 7 Expectation Bounds for Translational Patterns

The proof of Theorem 1.3 uses very similar arguments to Theorem 1.2. The concentration bound arguments will be very similar to those applied in the last section. The difference here is that the translation-invariance of the pattern can be used to bypass estimating the expected values like those which caused us the most difficulty in Theorem 1.3. We can therefore construct Salem sets avoiding the pattern with dimension exactly matching the Hausdorff dimension of the sets which would be constructed using the method of [2]. In this section, let

$$\beta \leq \min \left( \frac{dn - \alpha}{n - 1}, d \right).$$

We then show that generic elements of  $\mathcal{X}_\beta$  avoid patterns satisfying the assumptions of Theorem 1.3.

**Lemma 7.1.** *Fix  $a \in \mathbb{Q} - \{0\}$ , and let  $T : V \rightarrow \mathcal{E}$  satisfy the assumptions of Theorem 1.3. Then for quasi-all  $(E, \mu) \in \mathcal{X}_\beta$ , and any distinct points  $(x_1, \dots, x_n) \in E$ ,*

$$x_n - ax_{n-1} \notin T(x_1, \dots, x_{n-2}).$$

*Proof.* Set

$$W = \{(x_1, \dots, x_n) \in \mathbb{T}^{2d} \times V : x_n - ax_{n-1} \in T(x_1, \dots, x_{n-2})\}.$$

The assumption that  $T$  is a locally Lipschitz map, and thus continuous, implies that for any disjoint, closed cubes  $R_1, \dots, R_n \subset \mathbb{T}^d$  such that  $R_1 \times \dots \times R_{n-2} \subset V$ ,  $(R_1 \times \dots \times R_n) \cap W$  will be a closed set. It follows that if we set

$$\mathcal{H}(R_1, \dots, R_n) = \{E \subset \mathbb{T}^d : (R_1 \times \dots \times R_n) \cap W \cap E^n = \emptyset\}$$

and

$$H(R_1, \dots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : E \in \mathcal{H}(R_1, \dots, R_n)\},$$

then  $H(R_1, \dots, R_n)$  is an open subset of  $\mathcal{X}_\beta$ , and  $\mathcal{H}(R_1, \dots, R_n)$  is a monotone family of sets. The proof will be complete will be proved that for each positive integer  $m$ , and any choice of cubes  $R_1, \dots, R_n$  with common sidelength  $1/2am$ , with  $d(R_i, R_j) \geq 10/am$  for  $i \neq j$ , and with  $Q_1 \times \dots \times Q_{n-2}$  a closed subset of  $V$ , where  $Q_i = 2R_i$ , then the set  $H(R_1, \dots, R_n)$  is dense in  $\mathcal{X}_\beta$ .

To prove  $H(R_1, \dots, R_n)$  is dense, we may assume without loss of generality that each set in the image of  $T$  is  $1/m$  periodic, i.e. for any  $x \in V$  and  $b \in \mathbb{Z}^d$ ,  $S(x) + b/m = S(x)$ . To see why this is true, we note that the set-valued function

$$\tilde{T}(x_1, \dots, x_{n-2}) = \bigcup_{0 \leq b_1, \dots, b_d < ma_2} T(x_1, \dots, x_{n-2}) + b/m$$

satisfies the same continuity and Minkowski dimension bounds that were required for  $T$  in the assumptions of Theorem 1.3, and proving density arguments for  $\tilde{T}$  would imply the same density results for  $T$ .

Fix a large integer  $M > 0$ ,  $\lambda \in [0, \beta)$ ,  $\gamma \in [0, \alpha)$  and pick  $r > 0$  such that  $r^{-\lambda} \leq M \leq r^{-\lambda} + 1$ , and for  $x \in W$ ,

$$|T(x)| \leq r^{dn-\gamma}.$$

Since  $T$  is a Lipschitz map, we may fix  $L > 0$  such that for  $x_1, x_2 \in R_1 \times \cdots \times R_{n-2}$ ,

$$d_{\mathbb{H}}(T(x_1), T(x_2)) \leq L|x_1 - x_2|.$$

For  $1 \leq i \leq n$ , consider a family of independent random variables  $\{X_i(k) : 1 \leq k \leq M\}$ , such that  $X_i(k)$  is uniformly distributed on  $Q_i$  for each  $i$ , as well as another independent family of random variables  $\{X_0(k) : 1 \leq k \leq M\}$  uniformly distributed in  $\mathbb{T}^d - (2R_1 - \cdots - 2R_n)$ . Let  $I$  be the set of indices  $k_n \in \{1, \dots, N\}$  such that there are indices  $k_1, \dots, k_{n-1} \in \{1, \dots, N\}$  with the property that

$$|d_{\mathbb{H}}(X_n(k_n) - aX_{n-1}(k_{n-1}), T(X_1(k_1), \dots, X_{n-1}(k_{n-2})))| \leq 2n^{1/2} \cdot (L + 1) \cdot r. \quad (7.1)$$

If

$$S = \{X_i(k) : 0 \leq i \leq n-1, 1 \leq k \leq N\} \cup \{X_n(k) : k \notin I\}.$$

then the Lipschitz condition on  $T$  implies that  $S_r \in \mathcal{H}(W; R_1, \dots, R_n)$  for suitably small  $r$ . The proof of this is analogous to the proof of the same property for the set  $S$  in Lemma 6.1, which also selected points to remove on the ‘double’ of the cubes  $R_i$ , though with respect to a different family of probability distributions.

We claim that for each  $x \in 2R_n$ , the quantity

$$P(x) = \mathbb{P}(1 \in I | X_n(1) = x)$$

is *independent* of  $x$ . To see this, we note that because all sets in the image of  $T$  are  $1/m$  periodic, the quantity

$$d_{\mathbb{H}}(x - aX_{n-1}(k_{n-1}), T(X_1(k_1), \dots, X_{n-2}(k_{n-2})))$$

depends only on the value of  $x - aX_{n-1}(k_{n-1})$  in  $\mathbb{T}^d / (1/am) \cdot \mathbb{Z}^d$ . Because  $aX_{n-1}(k_{n-1})$  is uniformly distributed in  $R_n$ , it follows that the distribution of the random variable  $x - aX_{n-1}(k_{n-1})$  is independent of  $x$  modulo  $\mathbb{T}^d / (1/am) \cdot \mathbb{Z}^d$ , which shows  $P(x)$  is independent of  $x$ . Let  $P$  denote the common quantity of the values  $P(x)$ . A union bound shows that

$$\begin{aligned} P &= \mathbb{P} \left( \bigcup_{k_1, \dots, k_{n-1}} \left\{ d_{\mathbb{H}}(aX_{n-1}(k_{n-1}), T(X_1(k_1), \dots, X_{n-2}(k_{n-2}))) \leq 2n^{1/2} \cdot (L + 1) \cdot r \right\} \right) \\ &\leq \sum_{k_1, \dots, k_{n-1}} \mathbb{P} \left( \left\{ d_{\mathbb{H}}(aX_{n-1}(k_{n-1}), T(X_1(k_1), \dots, X_{n-2}(k_{n-2}))) \leq 2n^{1/2} \cdot (L + 1) \cdot r \right\} \right) \\ &\lesssim_{a,m} M^{n-1} \left( \max_{x \in R_1 \times \cdots \times R_{n-2}} |T(x)| \right) \leq M^{n-1} r^{dn-\gamma} \lesssim M^{n-1-(dn-\gamma)/\lambda} \end{aligned} \quad (7.2)$$

Because  $(n-1) - (dn-\gamma)/\lambda < 0$ , (7.2) implies that for suitably large integers  $M$  depending on  $n$ ,  $d$ ,  $\lambda$ , and  $\gamma$ ,

$$P \leq 1/2. \quad (7.3)$$

Set  $A_0 = |\mathbb{T}^d - Q_1 - \cdots - Q_n|P$ , set  $A_i = |Q_i|P$  if  $i \in \{1, \dots, n-1\}$  and  $i < n$ , and let  $A_n = |Q_n|$ . Define

$$F(\xi) = \sum_{i=0}^{n-1} \sum_{k=1}^M A_i e^{2\pi i \xi \cdot X_i(k)} + \sum_{k \notin I} A_n e^{2\pi i \xi \cdot X_n(k)}.$$

The choice of coefficients is made so that for any  $\xi \neq 0$ ,  $\mathbb{E}(F(\xi)) = 0$ . Indeed, we have

$$\begin{aligned}\mathbb{E}(F(\xi)) &= \frac{MA_0}{|\mathbb{T}^d - Q_1 - \dots - Q_n|} \int_{\mathbb{T}^d - Q_1 - \dots - Q_n} e^{2\pi i \xi \cdot x} dx \\ &\quad + \sum_{i=1}^{n-1} \frac{MA_i}{|Q_i|} \int_{Q_i} e^{2\pi i \xi \cdot x} dx \\ &\quad + \frac{MA_n}{|Q_n|} \int_{R_n} P(x) e^{2\pi i \xi \cdot x} dx \\ &= MP \int_{\mathbb{T}^d} e^{2\pi i \xi \cdot x} = 0.\end{aligned}$$

Lemma 7.2, which follows from a very similar argument to Lemma 6.2 in the last section, implies that

$$\begin{aligned}\mathbb{P} \left( \sup_{|\xi| \leq M^{1+\kappa}} |F(\xi) - \mathbb{E}(F(\xi))| \geq CM^{1/2} \log(M)^{1/2} \right) \\ = \mathbb{P} \left( \sup_{|\xi| \leq M^{1+\kappa}} |F(\xi)| \geq CM^{1/2} \log(M)^{1/2} \right) \leq 1/10.\end{aligned}$$

Set  $N = \#(S)$ . Then

$$N \geq M \gtrsim (1/2)r^{-\lambda}. \quad (7.4)$$

In particular, there exists some instantiation of the random variables  $\{X_i(k)\}$  such that if  $a(X_i(k)) = A_i$ , then

$$\left| \frac{1}{N} \sum_{x \in S} a(x) e^{2\pi i \xi \cdot x} \right| \leq CN^{1/2} \log(N)^{1/2} \leq CN^{1/2} \log(N)^{1/2}. \quad (7.5)$$

Since  $\sum_{x \in S} a(x) \geq N/2$ , (7.5) is an exponential sum bound we can use to apply Lemma 4.1, together with (7.4) and the fact that  $S_r \in \mathcal{H}(R_1, \dots, R_n)$ , implies that  $H(R_1, \dots, R_n)$  is dense in  $\mathcal{X}_\beta$ , completing the proof.  $\square$

All that remains to prove Theorem 1.3 is to prove Lemma 7.2.

**Lemma 7.2.** *For any  $\kappa > 0$ , there exists  $C > 0$  such that*

$$\mathbb{P} \left( \sup_{|\xi| \leq M^{1+\kappa}} |F(\xi) - \mathbb{E}(F(\xi))| \geq CM^{1/2} \log(M)^{1/2} \right) \leq 1/10$$

*Proof.* Consider the random set  $\Omega$  of values  $x_n \in Q_n$  such that there are  $k_1, \dots, k_{n-1} \in \{1, \dots, M\}$  with

$$x_n \in T(X_1(k_1), \dots, X_{n-1}(k_{n-1}))_{2n^{1/2}(L+1)r}. \quad (7.6)$$

Then

$$F(\xi) = \frac{1}{M} \sum_{k=1}^M Z(k). \quad (7.7)$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_n(k)} & : X_n(k) \notin \Omega, \\ 0 & : X_n(k) \in \Omega \end{cases}.$$

If  $\Sigma$  is the  $\sigma$  algebra generated by the random variables

$$\{X_i(k) : i \in \{1, \dots, n-1\}, k \in \{1, \dots, M\}\},$$

then  $\Omega$  is measurable with respect to  $\Sigma$ . Thus the random variables  $\{Z(k)\}$  are *conditionally independent* given  $\Sigma$ . Since we have  $|Z(k)| \leq 1$  almost surely, Hoeffding's inequality thus implies that for all  $t \geq 0$ ,

$$\mathbb{P}(|H(\xi) - \mathbb{E}(H(\xi)|\Sigma)| \geq t) \leq 4 \exp\left(\frac{-t^2}{2M}\right). \quad (7.8)$$

It is simple to see that

$$\mathbb{E}(H(\xi)|\Sigma) = M \int_{\Omega} \psi_n(x) e^{2\pi i \xi \cdot x} dx. \quad (7.9)$$

Since

$$\Omega = \bigcup \{T(X_1(k_1), \dots, X_{n-1}(k_{n-1}))_{2n^{1/2}(L+1)r} : 1 \leq k_1, \dots, k_{n-1} \leq K\}. \quad (7.10)$$

we see that varying each random variable  $X_i(k)$ , for  $1 \leq i \leq n-1$  while fixing the other random variables adjusts at most  $M^{n-2}$  of the sets forming  $\Omega$ , each of volume  $O_{d,n,L}(r^{dn-\alpha})$ , and thus varying  $X_i(k)$  while fixing the other random variables changes  $\mathbb{E}(H(\xi)|\Sigma)$  by at most

$$M \cdot O_{d,n,L}(r^{dn-\alpha}) \cdot M^{n-1} \lesssim 1. \quad (7.11)$$

Thus McDiarmid's inequality shows that there exists  $C > 0$  such that for any  $t \geq 0$ ,

$$\mathbb{P}(|\mathbb{E}(H(\xi)|\Sigma) - \mathbb{E}(H(\xi))| \geq t) \leq 4 \exp\left(\frac{-t^2}{CM}\right). \quad (7.12)$$

Combining (7.8) and (7.12), we conclude that there exists  $C > 0$  such that for each  $\xi \in \mathbb{Z}^d$ ,

$$\mathbb{P}(|H(\xi) - \mathbb{E}(H(\xi))| \geq t) \leq 8 \exp\left(\frac{-t^2}{CM}\right). \quad (7.13)$$

Applying a union bound to (7.13) over all  $0 < |\xi| \leq M^{1+\kappa}$  shows that there exists a constant  $C > 0$  such that

$$\mathbb{P}\left(\sup_{|\xi| \leq M^{1+\kappa}} |H(\xi) - \mathbb{E}(H(\xi))| \geq CM^{1/2} \log(M)^{1/2}\right) \leq 1/10. \quad \square$$

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