Large Sets Avoiding Rough Patterns

Jacob Denson* Malabika Pramanik* Joshua Zahl*

May 17, 2019

Abstract

The pattern avoidance problem seeks to construct a set $X \subset \mathbf{R}^d$ with large dimension that avoids a prescribed pattern. Examples of such patterns include three-term arithmetic progressions (solutions to $x_1 - 2x_2 + x_3 = 0$), or more general patterns of the form $f(x_1, \dots, x_n) = 0$. Previous work on the subject has considered patterns described by polynomials, or by functions f satisfying certain regularity conditions. We consider the case of 'rough' patterns, not prescribed by functional zeros.

There are several problems that fit into the framework of rough pattern avoidance. As a first application, if $Y \subset \mathbf{R}^d$ is a set with Minkowski dimension α, we construct a set X with Hausdorff dimension $1 - \alpha$ such that $X + X$ is disjoint from Y. As a second application, if C is a Lipschitz curve, we construct a set $X \subset C$ of dimension $1/2$ that does not contain the vertices of an isosceles triangle.

A major question in modern geometric measure theory is whether sufficiently large sets are forced to contain copies of certain patterns. Intuitively, one expects the answer to be yes, and many results in the literature support this intuition. For example, the Lebesgue density theorem implies that a set of positive Lebesgue measure contains an affine copy of every finite set. And any set $X \subset \mathbf{R}^2$ with Hausdorff dimension exceeding one must contain three collinear points. On the other hand, there is a distinct genre of results that challenges this intuition. Keleti [4] constructs a set $X \subset \mathbf{R}$ that avoids all solutions of the equation $x_2 - x_1 = x_4 - x_3$ with $x_1 < x_2 \leqslant x_3 < x_4$, and which consequently does not contain any nontrivial arithmetic progression. Maga [5] constructs a set $X \subset \mathbf{R}^2$ of full Hausdorff dimension such that no four points in X form the vertices of a parallelogram. The pattern avoidance problem (informally stated) asks: for a given pattern, how large can the dimension of a set $X \subset \mathbf{R}^d$ be before it is forced to contain a copy of this pattern?

One way to formalize the notion of a pattern is as follows. If $d \geqslant 1$ and $n \geqslant 2$ are integers, we define a pattern to be a set $Z \subset \mathbf{R}^{dn}$. We say that a set $X \subset \mathbf{R}^d$ avoids the pattern Z if for every n-tuple of distinct points $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$. For example, a set $X \subset \mathbf{R}^2$ does not contain three collinear points if and only if it avoids the pattern

$$Z = \{(x_0, x_1, x_2) \in \mathbf{R}^6 : \det(x_1 - x_0, x_2 - x_0) = 0\}.$$

*University of British Columbia, Vancouver BC, {denson, malabika, jzahl}@math.ubc.ca.

Similarly, a set $X \subset \mathbf{R}^2$ avoids the pattern

$$Z = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^8 : x_1 + x_4 = x_2 + x_3\}$$

if and only if no four points in X form the vertices of a (possibly degenerate) parallelogram.

A number of recent articles have established pattern avoidance results for increasingly general patterns. In [6], Máthé constructs a set $X \subset \mathbf{R}^d$ that avoids a pattern specified by a countable union of algebraic varieties of controlled degree. In [2], Fraser and the second author consider the pattern avoidance problem for countable unions of C^1 manifolds. In this paper, we consider the pattern avoidance problem for an even more general class of 'rough' patterns $Z \subset \mathbf{R}^{dn}$, that are the countable union of sets with controlled lower Minkowski dimension.

Theorem 1. *Suppose $\alpha \geqslant d$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a set $X \subset [0,1)^d$ with Hausdorff dimension at least $(nd - \alpha)/(n-1)$ such that whenever $x_1, \dots, x_n \in X$ are distinct, we have $(x_1, \dots, x_n) \notin Z$.*

Remarks.

1. When $\alpha < d$, the pattern avoidance problem is trivial, since $X = [0,1)^d - \pi(Z)$ is full dimensional and solves the pattern avoidance problem, where $\pi(x_1, \dots, x_n) = x_1$ is a projection map from $\mathbf{R}^{dn}$ to $\mathbf{R}^d$. Obtaining a full dimensional set in the case $\alpha = d$, however, is still interesting.

2. Theorem 1 is trivial when $\alpha = dn$, since we can set $X = \varnothing$. We will therefore assume that $\alpha < dn$ in our proof of the theorem, without loss of generality.

3. When Z is a countable union of smooth manifolds in $\mathbf{R}^{nd}$ of co-dimension m, we have $\alpha = nd - m$. In this case Theorem 1 yields a set in $\mathbf{R}^d$ with Hausdorff dimension at least $(nd - \alpha)/(n-1) = m/(n-1)$. This recovers Theorem 1.1 and 1.2 from [2], making Theorem 1 a generalization of these results.

4. Since Theorem 1 does not require any regularity assumptions on the set Z, it can be applied in contexts that cannot be addressed using previous methods. Two such applications, new to the best of our knowledge, have been recorded in Section 5; see Theorems 2 and 3 there.

The set X in Theorem 1 is obtained by constructing a sequence of approximations to X, each of which avoids the pattern Z at different scales. For a sequence of lengths $l_k \searrow 0$, we construct a nested family of sets $\{X_k\}$, where X_k is a union of cubes of sidelength l_k that avoids Z at scales close to l_n. The set $X = \bigcap X_k$ avoids Z at all scales. While this proof strategy is not new, our method for constructing the sets $\{X_k\}$ has several innovations that simplify the analysis of the resulting set $X = \bigcap X_k$. In particular, through a probabilistic selection process we are able to avoid the complicated queuing techniques used in [4] and [2], that required storage of data from each step of the iterated construction, to be retrieved at a much later stage of the construction process.

At the same time, our construction continues to share certain features with [2]. For example, between each pair of scales l_{k-1} and l_k, we carefully select an intermediate scale r_k. The set $X_k \subset X_{k-1}$ avoids Z at scale l_k, and it is 'evenly distributed' at scale r_k: the set X_k is a union of intervals of length l_k whose midpoints resemble (a large subset of) an arithmetic progression of step size r_k. The details of a single step of this construction are described in Section 2. In Section 3, we explain how the length scales l_k and r_k for X are chosen, and prove its avoidance property. In Section 4 we analyze the size of X and show that it satisfies the conclusions of Theorem 1.

1 Frequently Used Notation and Terminology

1. A *dyadic length* is a number l of the form 2^{-k} for some non-negative integer k.

2. Given a length $l > 0$, we let $\mathcal{B}_l^d$ denote the set of all half open cubes in $\mathbf{R}^d$ with sidelength l and corners on the lattice $(l \cdot \mathbf{Z})^d$, i.e.

$$\mathcal{B}_l^d = \{[a_1, a_1 + l) \times \cdots \times [a_d, a_d + l) : a_k \in l \cdot \mathbf{Z}\}.$$

   If $E \subset \mathbf{R}^d$, we let $\mathcal{B}_l^d(E)$ denote the set of cubes in $\mathcal{B}_l^d$ intersecting E, i.e.

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : I \cap E \neq \varnothing\}.$$

3. The *lower Minkowski dimension* of a bounded set $Z \subset \mathbf{R}^d$ are defined as

$$\underline{\dim}_{\mathbf{M}}(Z) = \liminf_{l \to 0} \frac{\log(\# \mathcal{B}_l^d(Z))}{\log(1/l)}.$$

4. If $0 \leqslant \alpha$ and $\delta > 0$, we define the dyadic Hausdorff content of a set $E \subset \mathbf{R}^d$ as

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{k=1}^\infty l_k^\alpha : E \subset \bigcup_{k=1}^\infty I_k \right\},$$

   where the infinum is taken over all families of intervals $\{I_k\}$ such that for all k, $I_k \in \mathcal{B}_{l_k}^d$, and l_k is a dyadic length with $l_k \leqslant \delta$. The α-dimensional dyadic Hausdorff measure H^α on $\mathbf{R}^d$ is $H^\alpha(E) = \lim_{\delta \to 0} H_\delta^\alpha(E)$, and the *Hausdorff dimension* of a set E is $\dim_{\mathbf{H}}(E) = \inf\{\alpha \geqslant 0 : H^\alpha(E) = 0\}$.

5. Given $I \in \mathcal{B}_l^{dn}$, we can decompose I as $I_1 \times \cdots \times I_n$ for unique cubes $I_1, \dots, I_n \in \mathcal{B}_l^d$. We say I is *strongly non-diagonal* if the cubes $I_1, \dots, I_n$ are distinct. Strongly non-diagonal cubes will play an important role in Section 2, when we solve a discrete version of Theorem 1.

6. Adopting the terminology of [3], we say a collection of sets $\{U_k\}$ is a *strong cover* of a set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k. This idea will be useful in Section 3.

7. A *Frostman measure* of dimension α is a non-zero compactly supported probability measure μ on $\mathbf{R}^d$ such that for every cube I of sidelength l, $\mu(I) \lesssim l^\alpha$. Note that a measure μ satisfies this inequality for every cube I if and only if it satisfies the inequality for cubes whose sidelengths are dyadic lengths. *Frostman's lemma* says that

$$\dim_{\mathbf{H}}(E) = \sup \left\{ \alpha : \begin{array}{c} \text{there is a Frostman measure of} \\ \text{dimension } \alpha \text{ supported on } E \end{array} \right\}.$$

2 Avoidance at Discrete Scales

In this section we describe a method for avoiding Z at a single scale. We apply this technique in Section 3 at many scales to construct a set X avoiding Z at all scales. This single scale avoidance technique is the core building block of our construction, and the efficiency with which we can avoid Z at a single scale has direct consequences on the Hausdorff dimension of the set X obtained in Theorem 1.

At a single scale, we solve a discretized version of the problem, where all sets are unions of cubes at two dyadic lengths $l > s$. In this discrete setting, Z is replaced by a discretized version of itself, a union of cubes in $\mathcal{B}_s^{dn}$ denoted by G. Given a set E, which is a union of cubes in $\mathcal{B}_l^d$, our goal is to construct a set $F \subset E$ that is a union of cubes in $\mathcal{B}_s^d$, such that F^n is disjoint from strongly non-diagonal cubes (see Definition 5) in $\mathcal{B}_s^{dn}(G)$. Using the setup introduced at the end of the introduction, we will later choose $l = l_k$, $s = l_{k+1}$, and $E = X_k$. The set X_{k+1} will be defined as the set F constructed.

In order to ensure the final set X obtained in Theorem 1 has large Hausdorff dimension regardless of the rapid decay of scales used in the construction of X, it is crucial that F is uniformly distributed at intermediate scales between l and s. We achieve this by decomposing E into sub-cubes in $\mathcal{B}_r^d$ for some intermediate scale $r \in [s, l]$, and distributing F as evenly among these intermediate sub-cubes as possible. This is possible assuming a mild regularity condition on the number of cubes in G, i.e. Equation (2.1).

Lemma 1. *Fix two distinct dyadic lengths l and s, with $l > s$. Let $E \subseteq [0,1)^d$ be a nonempty union of cubes in $\mathcal{B}_l^d$, and let $G \subset \mathbf{R}^{dn}$ be a nonempty union of cubes in $\mathcal{B}_s^{dn}$ such that*

$$(l/s)^d \leqslant \# \mathcal{B}_s^{dn}(G) \leqslant \frac{1}{2}(l/s)^{dn}. \tag{2.1}$$

Then there exists a dyadic length $r \in [s, l]$ of size

$$r \sim \left(l^{-d} s^{dn} \# \mathcal{B}_s^{dn}(G) \right)^{\frac{1}{d(n-1)}}, \tag{2.2}$$

and a set $F \subset E$ that is a nonempty union of cubes in $\mathcal{B}_s^d(E)$ satisfying the following three properties:

1. Avoidance*: For any choice of distinct cubes $J_1, \dots, J_n \in \mathcal{B}_s^d(F)$, $J_1 \times \cdots \times J_n \notin \mathcal{B}_s^{dn}(G)$.*

2. Non-Concentration*: For every $I' \in \mathcal{B}_r^d(E)$, there is at most one $J \in \mathcal{B}_s^d(F)$ with $J \subset I'$.*

3. Large Size*: For every $I \in \mathcal{B}_l^d(E)$, $\# \mathcal{B}_s^d(F \cap I) \geqslant \# \mathcal{B}_r^d(I)/2 = (l/r)^d/2$.*

Remark. Property 1 says that F avoids strongly non-diagonal cubes in $\mathcal{B}_s^{dn}(G)$. Properties 2 and 3 together imply that for every $I \in \mathcal{B}_l^d(E)$, at least half the cubes $I' \in \mathcal{B}_r^d(I)$ contribute a single sub-cube of sidelength s to F; the rest contribute none.

Proof. Let r be the smallest dyadic length at least as large as R, where

$$R = \left(2l^{-d}s^{dn}\#\,\mathcal{B}_s^{dn}(G)\right)^{\frac{1}{d(n-1)}}. \tag{2.3}$$

This choice of r satisfies (2.2). The inequalities in (2.1) ensure that $r \in [s, l]$; more precisely, the left inequality in (2.1) implies R is bounded from below by s, and the right inequality implies R is bounded from above by l. The minimality of r ensures $s \leqslant r \leqslant l$.

For each $I' \in \mathcal{B}_r^d(E)$, let $J_{I'}$ be an element of $\mathcal{B}_s^d(I)$ chosen uniformly at random; these choices are independent as I' ranges over the elements of $\mathcal{B}_r^d(E)$. Define

$$U = \bigcup \left\{ J_{I'} : I' \in \mathcal{B}_r^d(E) \right\},$$

and

$$\mathcal{K}(U) = \{K \in \mathcal{B}_s^{dn}(G) : K \in U^n, K \text{ strongly non-diagonal}\}.$$

Note that the sets U and $\mathcal{K}(U)$ are random sets, in the sense that they depend on the random variables $\{J_{I'}\}$. Define

$$F(U) = U - \{\pi(K) : K \in \mathcal{K}(U)\}, \tag{2.4}$$

where $\pi : \mathbf{R}^{dn} \to \mathbf{R}^d$ is the projection map $(x_1, \dots, x_n) \mapsto x_1$, for $x_i \in \mathbf{R}^d$. Thus π sends the cube $J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}$ to the cube $J_1 \in \mathcal{B}_s^d$. Given any strongly non-diagonal cube $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$, either $K \notin \mathcal{B}_s^{dn}(U^n)$, or $K \in \mathcal{B}_s^{dn}(U^n)$. If the former occurs then $K \notin \mathcal{B}_s^{dn}(F(U)^n)$ since $F(U) \subset U$, so $\mathcal{B}_s^{dn}(F(U)^n) \subset \mathcal{B}_s^{dn}(U^n)$. If the latter occurs then $K \in \mathcal{K}(U)$, and since $\pi(K) = J_1$, $J_1 \notin \mathcal{B}_s^d(F(U))$. In either case, $K \notin \mathcal{B}_s^{dn}(F(U)^n)$, so $F(U)$ satisfies Property 1. By construction, U contains at most one subcube $J \in \mathcal{B}_s^{dn}$ for each $I \in \mathcal{B}_l^{dn}(E)$. Since $F(U) \subset U$, $F(U)$ satisfies Property 2. Thus the set $F(U)$ satisfies Properties 1 and 2 regardless of which values are assumed by the random variables $\{J_{I'}\}$. Next we will show that with non-zero probability, the set $F(U)$ satisfies Property 3.

For each cube $J \in \mathcal{B}_s^d(E)$, there is a unique 'parent' cube $I' \in \mathcal{B}_r^d(E)$ such that $J \subset I'$. Since I' contains $(r/s)^d$ elements of $\mathcal{B}_s^d(E)$, and $J_{I'}$ is chosen uniformly at random from $\mathcal{B}_s^d(I)$,

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_{I'} = J) = (s/r)^d.$$

The cubes $J_{I'}$ are chosen independently, so if $J_1, \dots, J_k$ are distinct cubes in $\mathcal{B}_s^d(E)$, then

$$\mathbf{P}(J_1, \dots, J_k \in U) = \begin{cases} (s/r)^{dk} & \text{if } J_1, \dots, J_k \text{ have distinct parents,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Let $K = J_1 \times \cdots \times J_n \in \mathcal{B}_s^{dn}(G)$. If the cubes $J_1, \dots, J_n$ are distinct, we deduce from (2.5) that

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1, \dots, J_n \in U) \leqslant (s/r)^{dn}. \tag{2.6}$$

By (2.6) linearity of expectation, and (2.3),

$$\mathbf{E}(\#\mathcal{K}(U)) = \sum_{K \in \mathcal{B}_s^{dn}(G)} \mathbf{P}(K \subset U^n) \leqslant \#\,\mathcal{B}_s^{dn}(G) \cdot (s/r)^{dn} \leqslant 0.5 \cdot (l/r)^d.$$

In particular, there exists at least one (non-random) set U_0 such that

$$\#\mathcal{K}(U_0) \leqslant \mathbf{E}(\#\mathcal{K}(U)) \leqslant 0.5 \cdot (l/r)^d. \tag{2.7}$$

In other words, $F(U_0) \subset U_0$ is obtained by removing at most $0.5 \cdot (l/r)^d$ cubes in $\mathcal{B}_s^d$ from U_0. For each $I \in \mathcal{B}_l^d(E)$, we know that $\# \mathcal{B}_s^d(I \cap U_0) = (l/r)^d$. Combining this with (2.7), we arrive at the estimate

$$\begin{aligned}
\# \mathcal{B}_s^d(I \cap F(U_0)) &= \mathcal{B}_s^d(I \cap U_0) - \#\{\pi(K) : K \in \mathcal{K}(U_0), \pi(K) \in F(U_0)\} \\
&\geqslant \mathcal{B}_s^d(I \cap U_0) - \#(\mathcal{K}(U_0)) \\
&\geqslant (l/r)^d - 0.5 \cdot (l/r)^d \geqslant 0.5 \cdot (l/r)^d
\end{aligned}$$

In other words, $F(U_0)$ satisfies Property 3. Setting $F = F(U_0)$ completes the proof. $\square$

Remarks.

1. While Lemma 1 uses probabilistic arguments, the conclusion of the lemma is not a probabilistic statement. In particular, one can find a suitable F constructively by checking every possible choice of U (there are finitely many) to find one particular choice U_0 which satisfies (2.7), and then defining F by (2.4). Thus the set we obtain in Theorem 1 exists by purely constructive means.

2. At this point, it is possible to motivate the numerology behind the dimension bound $\dim(X) \geqslant (dn-\alpha)/(n-1)$ from Theorem 1, albeit in the context of Minkowski dimension. We will pause to do so here before returning to the proof of Theorem 1. For simplicity, let $\alpha > d$, and suppose that $Z \subset \mathbf{R}^{dn}$ satisfies

$$\# \mathcal{B}_s^{dn}(Z) \sim s^{-\alpha} \quad \text{for every } s \in (0,1]. \tag{2.8}$$

   Let $l = 1$, $E = [0,1)^d$, and let $s > 0$ be a small parameter. If s is chosen sufficiently small compared to d, n, and α, then (2.1) is satisfied with $G = \bigcup \mathcal{B}_s^{dn}(Z)$. We can then apply Lemma 1 to find a dyadic scale $r \sim s^{(dn-\alpha)/d(n-1)}$ and a set F that avoids the strongly non-diagonal cubes of $\mathcal{B}_s^{dn}(Z)$. The set F is a union of approximately $r^{-d} \sim s^{-(dn-\alpha)/(n-1)}$ cubes of sidelength s. Thus informally, the set F resembles a set with Minkowski dimension α when viewed at scale s.

   The set X constructed in Theorem 1 will be obtained by applying Lemma 1 iteratively at many scales. At each of these scales, X will resemble a set of Minkowski dimension $(dn-\alpha)/(n-1)$. A careful analysis of the construction (performed in Section 4) shows that X actually has Hausdorff dimension at least $(dn-\alpha)/(n-1)$.

3. Lemma 1 is the core method in our avoidance technique. The remaining argument is fairly modular. If, for a special case of Z, one can improve the result of Lemma 1 so that r is chosen on the order of $s^{\beta/d}$, then the remaining parts of our paper can be applied near verbatim to yield a set X with Hausdorff dimension β, as in Theorem 1.

3 Fractal Discretization

In this section we construct the set X from Theorem 1 by applying Lemma 1 at many scales. Since Z is a countable union of bounded sets with Minkowski dimension at most α, there exists a strong cover (see Definition 6) of Z by cubes restricted to a sequence of dyadic lengths $\{l_k\}$, with a quantitative bound on the number of cubes at each scale. We fix a cover so that the scales l_k converge to 0 very quickly.

Lemma 2. *Let $Z \subset \mathbf{R}^{dn}$ be a countable union of bounded sets with Minkowski dimension at most α, and let $\epsilon_k \searrow 0$ with $2\epsilon_k < dn - \alpha$ for all k. Then there exists a sequence of lengths $\{l_k\}$ and a strong cover of Z by a sequence of sets $\{Z_k\}$, such that*

1. Discreteness*: For all $k \geqslant 0$, Z_k is a union of cubes in $\mathcal{B}^{dn}_{l_k}$.*
2. Sparsity*: For all $k \geqslant 0$, $l_k^{-d} \leqslant \mathcal{B}^{dn}_{l_k}(Z_k) \leqslant l_k^{-\alpha-\epsilon_k}$.*
3. Rapid Decay*: For all $k > 1$,*

$$l_k^{dn-\alpha-\varepsilon_k} \leqslant 0.5 \cdot l_{k-1}^{dn} \quad \textit{and} \tag{3.1}$$
$$l_k^{\epsilon_k} \leqslant l_{k-1}^{2d}. \tag{3.2}$$

Proof. We can write

$$Z = \bigcup_{i=1}^{\infty} Y_i \quad \text{with } \underline{\dim}_{\mathbf{M}}(Y_i) \leqslant \alpha \text{ for each } i.$$

Consider the d dimensional hyperplane

$$H = \{(x_1, \dots, x_1) : x_1 \in [0,1)^d\}$$

We then set $Y_i' = Y_i \cup H$. In particular, this means for any l,

$$\# \mathcal{B}^{nd}_l(Y_i') \geqslant \# \mathcal{B}^{nd}_l(H) = l^{-d}, \tag{3.3}$$

yet we still know $\underline{\dim}_{\mathbf{M}}(Y_i') \leqslant \alpha$ for each index i. Let $\{i_k\}$ be a sequence of integers that repeats each integer infinitely often.

The lengths $\{l_k\}$ and sets $\{Z_k\}$ are defined inductively. As a base case, set $l_0 = 1$ and $Z_0 = [0,1)^d$. Suppose that the lengths $l_0, \dots, l_{k-1}$ have been chosen. Since $\underline{\dim}_{\mathbf{M}}(Y_{i_k}) \leqslant \alpha$, Definition 3 implies that there exists arbitrarily small lengths l which saitsfy

$$\# \mathcal{B}^{dn}_l(Y_{i_k}') \leqslant l^{-\alpha - \frac{\varepsilon_k}{2}}. \tag{3.4}$$

In particular, we can choose $l = l_k$ small enough to satisfy (3.1), (3.2), and (3.4), so Property 3 is satisfied. We then set $Z_k = \bigcup \mathcal{B}^{dn}_{l_k}(Y_{i_k}')$. For any $z \in Z$, there is some i such that $z \in Y_i$, and there is a subsequence of integers $k_1, k_2, \dots$ such that $i_{k_j} = i$ for all j. But then $z \in Y_i \subset Y_i' \subset Z_{i_{k_j}}$, so z is contained in the infinite sequence of sets $Z_{i_{k_j}}$. In particular, $z \in \limsup Z_i$, and since z was arbitrary, $Z \subset \limsup Z_i$, so $\{Z_i\}$ is a strong cover of Z. Now Property 1 is satisfied by construction of the $\{Z_i\}$. And Property 2 is implied by (3.3) and (3.4). ∎

To construct X, we consider a nested, decreasing family of discretized sets $\{X_k\}$, where X_k is a union of cubes in $\mathcal{B}^d_{l_k}(X_k)$. We then set $X = \bigcap X_k$. The goal is to choose X_k such that X_k^n is disjoint from *strongly non diagonal* cubes in Z_k.

Lemma 3. *Let $Z \subset \mathbf{R}^{dn}$, let $\{Z_k\}$ be a sequence of sets that strongly cover Z, and let $\{l_k\}$ be a sequence of lengths converging to zero. For each index k, let X_k be a union of cubes in $\mathcal{B}^d_{l_k}$. Suppose that for each k, X_k^n avoids strongly non-diagonal cubes in $\mathcal{B}^{dn}_{l_k}(Z_k)$. If $X = \bigcap X_k$, then for any distinct $x_1, \dots, x_n \in X$, we have $(x_1, \dots, x_n) \notin Z$.*

Proof. Let $z \in Z$ be a point with distinct coordinates $z_1, \dots, z_n$. Define

$$\Delta = \{(w_1, \dots, w_n) \in \mathbf{R}^{dn} : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$, where d is the Hausdorff distance between Δ and z. Since $\{Z_k\}$ strongly covers Z, there is a subsequence $\{k_m\}$ such that $z \in Z_{k_m}$ for every index m. Since $l_k \searrow 0$ and thus $l_{k_m} \searrow 0$, if m is sufficiently large then $\sqrt{dn} \cdot l_{k_m} < d(\Delta, z)$. Note that $\sqrt{dn} \cdot l_{km}$ is the diameter of a cube in $\mathcal{B}^{dn}_{l_{k_m}}$. For such a choice of m, if $I \in \mathcal{B}^{dn}_{l_{k_m}}(Z_{k_m})$ is the (unique) cube in $\mathcal{B}^{dn}_{l_{k_m}}$ containing z, then $I \cap \Delta = \varnothing$. But this means I is strongly non-diagonal. Since X_{k_m} avoids the strongly non-diagonal cubes of Z_{k_m}, we conclude that $z \notin X^n_{k_m}$. In particular, this means $z \notin X^n$. □

All that remains is to apply the discrete lemma to choose the sets X_k.

Lemma 4. *Given a sequence of dyadic length scales l_k obeying,* (3.1), (3.2)*, and* (3.4) *as above, there exists a sequence of sets $\{X_k\}$ and a sequence of dyadic intermediate scales $\{r_k\}$ with $l_k \leqslant r_k \leqslant l_{k-1}$ for each $k \geqslant 1$, such that each set X_k is a union of cubes in $\mathcal{B}^d_{l_k}(X_{k-1})$ that avoids the strongly non-diagonal cubes of $\mathcal{B}^{dn}_{l_k}(Z_k)$. Furthermore, for each index $k \geqslant 1$ we have*

$$r_k \lesssim l_k^{(dn-\alpha-\epsilon_k)/d(n-1)}, \tag{3.5}$$
$$\# \mathcal{B}^d_{l_k}(X_k \cap I) \geqslant 0.5 \cdot (l_{k-1}/r_k)^d \quad \textit{for each } I \in \mathcal{B}^d_{l_{k-1}}(X_{k-1}), \tag{3.6}$$
$$\# \mathcal{B}^d_{l_k}(X_k \cap I') \leqslant 1 \quad \textit{for each } I' \in \mathcal{B}^d_{r_k}(X_{k-1}). \tag{3.7}$$

Proof. We construct X_k by induction, using Lemma 1 as building block. Set $X_0 = [0,1)^d$. Next, suppose that the sets $X_0, \dots, X_{k-1}$ have been defined. Our goal is to apply Lemma 1 to $E = X_{k-1}$ and $G = Z_k$ with $l = l_{k-1}$ and $s = l_k$. This will be possible once we verify the hypothesis (2.1), which in this case takes the form

$$(l_{k-1}/l_k)^d \leqslant \# \mathcal{B}^{dn}_{l_k}(Z_k) \leqslant 0.5 \cdot (l_{k-1}/l_k)^{dn}. \tag{3.8}$$

The right hand side follows from Property 2 of Lemma 2 and (3.2). On the other hand, Property 2 and the fact that $l_{k-1} \leqslant 1$ implies that

$$(l_{k-1}/l_k)^d \leqslant l_k^{-d} \leqslant \# \mathcal{B}^{dn}_{l_k}(Z_k),$$

establishing the left inequality in (3.8). Applying Lemma 1 as described above now produces a dyadic length

$$r \sim \left(l_{k-1}^{-d} l_k^{dn} \# \mathcal{B}^{dn}_{l_k}(Z_k)\right)^{\frac{1}{d(n-1)}} \tag{3.9}$$

and a set $F \subset X_{k-1}$ that is a union of cubes in $\mathcal{B}^d_{l_k}$. The set F satisfies Properties 1, 2, and 3 from the statement of Lemma 1. Define $r_k = r$ and $X_k = F$. The estimate (3.5) on r_k follows from (3.9) using the known bounds (3.2) and (3.4):

$$r_k \lesssim \left(l_{k-1}^{-d} l_k^{dn-\alpha-0.5\epsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-d} l_k^{0.5\epsilon_k} l_k^{dn-\alpha-\epsilon_k}\right)^{\frac{1}{d(n-1)}} = \left(l_{k-1}^{-2d} l_k^{\epsilon_k}\right)^{\frac{1}{2d(n-1)}} l_k^{\frac{dn-\alpha-\epsilon_k}{d(n-1)}} \lesssim l_k^{\frac{dn-\alpha-\epsilon_k}{d(n-1)}}.$$

The requirements (3.6) and (3.7) follow from Properties 2 and 3 of Lemma 1 respectively. $\square$

Now we have defined the sets $\{X_k\}$, we set $X = \bigcap X_k$. Since X_k avoids strongly non-diagonal cubes in Z_k, Lemma 3 implies that if $x_1, \dots, x_n \in X$ are distinct, then $(x_1, \dots, x_n) \notin Z$. To finish the proof of Theorem 1, we must show that $\dim_{\mathbf{H}}(X) \geqslant (dn - \alpha)/(n-1)$. This will be done in the next section.

4 Dimension Bounds

To complete the proof of Theorem 1, we must show that $\dim_{\mathbf{H}}(X) \geqslant (dn - \alpha)/(n-1)$. In view of Definition 7, we will do this by constructing a Frostman measure of appropriate dimension supported on X.

We start by defining a premeasure on $\bigcup_{i=1}^\infty \mathcal{B}^d_{l_i}[0,1)^d$. Set $\mu([0,1)^d) = 1$. Suppose now that $\mu(I)$ has been defined for all cubes in $\mathcal{B}^d_{l_{k-1}}[0,1)^d$, and let $J \in \mathcal{B}^d_{l_k}$. Consider the unique 'parent cube' $I \in \mathcal{B}^d_{l_{k-1}}$ for which $J \subset I$. Define

$$\mu(J) = \begin{cases} \mu(I)/\#\,\mathcal{B}^d_{l_k}(X_k \cap I) & \text{if } J \subset X_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Observe that for each index $k \geqslant 1$ and each $I \in \mathcal{B}^d_{l_{k-1}}$,

$$\sum_{J \in \mathcal{B}^d_{l_k}(I)} \mu(J) = \sum_{J \in \mathcal{B}^d_{l_k}(X_k \cap I)} \mu(J) = \mu(I). \tag{4.2}$$

In particular, for each index k we have

$$\sum_{I \in \mathcal{B}_{l_k}} \mu(I) = 1.$$

By a standard argument involving the Caratheodory extension theorem [1, Proposition 1.7], the premeasure μ extends to a measure on the Borel subsets of $[0,1)^d$. Note that for each $k \geqslant 1$, μ is supported on X_k. Thus μ is supported on $\bigcap X_k = X$. To complete the proof of Theorem 1 we will show that μ is a Frostman measure of dimension $(dn - \alpha)/(n-1) - \epsilon$ for every $\epsilon > 0$.

Lemma 5. *For each $k \geqslant 1$ and each $J \in \mathcal{B}^d_{l_k}(X)$,*

$$\mu(J) \lesssim l_k^{\frac{dn-\alpha}{n-1} - \eta_k}, \quad \textit{where} \quad \eta_k = \frac{n+1}{2(n-1)} \cdot \epsilon_k \searrow 0 \textit{ as } k \to \infty.$$

Proof. Let $J \in \mathcal{B}^d_{l_k}$ and let $I \in \mathcal{B}^d_{l_{k-1}}$ be the parent cube of J. Since μ is a probability measure, we have $\mu(I) \leqslant 1$. Combining (4.1), (3.6), (3.5), and (3.2) we obtain

$$\mu(J) \leqslant \frac{2r_k^d}{l_{k-1}^d}\mu(I) \leqslant \frac{2r_k^d}{l_{k-1}^d} \lesssim \frac{l_k^{\frac{dn-\alpha-\epsilon_k}{n-1}}}{l_{k-1}^d} = l_k^{\frac{dn-\alpha}{n-1}-\eta_k}\left(l_k^{0.5\epsilon_k}/l_{k-1}^d\right) \leqslant l_k^{\frac{dn-\alpha}{n-1}-\eta_k}. \qquad \square$$

Corollary 1. *For each $k \geqslant 1$ and each $I' \in \mathcal{B}^d_{r_k}(X_{k-1})$,*

$$\mu(I') \lesssim (r_k/l_{k-1})^d \cdot l_{k-1}^{\frac{dn-\alpha}{n-1}-\eta_{k-1}}. \tag{4.3}$$

Proof. Let us fix a cube $I' \in \mathcal{B}^d_{r_k}(X_{k-1})$, and let I denote its unique parent cube in $\mathcal{B}^d_{l_{k-1}}(X_{k-1})$. According to (3.7), I' contains at most one cube in $\mathcal{B}^d_{l_k}(I)$; let us denote this cube by J if it exists. Then the mass distribution rule given by (4.1) dictates that

$$\mu(I') = \mu(X_k \cap I') = \begin{cases} \mu(J) = \mu(I)/\#\,\mathcal{B}^d_{l_k}(X_k \cap I) & \text{if } \#\,\mathcal{B}^d_{l_k}(X_k \cap I') = 1, \\ 0 & \text{if } \#\,\mathcal{B}^d_{l_k}(X_k \cap I') = 0. \end{cases}$$

Using the estimate (3.6) and applying Lemma 5 to $I \in \mathcal{B}^d_{l_{k-1}}(X)$, we arrive at the claimed bound (4.3). $\square$

Lemma 5 and Corollary 1 allow us to control the behavior of μ at all scales.

Lemma 6. *For every $\alpha \in [d, dn)$, and for each $\epsilon > 0$, there is a constant C_ϵ so that for all dyadic lengths $l \in (0,1]$ and all $I \in \mathcal{B}^d_l$, we have*

$$\mu(I) \leqslant C_\epsilon l^{\frac{dn-\alpha}{n-1}-\epsilon}. \tag{4.4}$$

Proof. Fix $\epsilon > 0$. Since $\eta_k \searrow 0$ as $k \to \infty$, there is a constant C_ϵ so that $l_k^{-\eta_k} \leqslant C_\epsilon l_k^{-\epsilon}$ for each $k \geqslant 1$. For instance, if ε_k is decreasing, we could choose $C_\epsilon = l_{k_0}^{-\eta_{k_0}}$, where k_0 is the largest integer for which $\eta_{k_0} \geqslant \epsilon$. Next, let k be the (unique) index so that $l_{k+1} \leqslant l < l_k$. We will split the proof of (4.4) into two cases, depending on the position of l within $[l_{k+1}, l_k]$.

Case 1: If $r_{k+1} \leqslant l < l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}^d_{r_{k+1}}$. By Corollary 1,

$$\begin{aligned} \mu(I) &\lesssim (l/r_{k+1})^d (r_{k+1}/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &= (l/l_k)^d l_k^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &= l^{\frac{dn-\alpha}{n-1}} (l/l_k)^{\frac{\alpha-d}{n-1}} l_k^{-\eta_k} \\ &\leqslant l^{\frac{dn-\alpha}{n-1}-\eta_k} \\ &\leqslant C_\epsilon l^{\frac{dn-\alpha}{n-1}-\epsilon}. \end{aligned} \tag{4.5}$$

The penultimate inequality is a consequence of our assumption $\alpha \geqslant d$.

Case 2: If $l_{k+1} \leqslant l \leqslant r_{k+1}$, we can cover I by a single cube in $\mathcal{B}^d_{r_{k+1}}$. By (3.7), each cube in $\mathcal{B}^d_{r_{k+1}}$ contains at most one cube $I_0 \in \mathcal{B}^d_{l_{k+1}}(X_{k+1})$, so by Lemma 5,

$$\mu(I) \leqslant \mu(I_0) \lesssim l_{k+1}^{\frac{dn-\alpha}{n-1}-\eta_{k+1}} \leqslant C_\epsilon l_{k+1}^{\frac{dn-\alpha}{n-1}-\epsilon} \leqslant C_\epsilon l^{\frac{dn-\alpha}{n-1}-\epsilon} \qquad \square$$

Applying Frostman's lemma to Lemma 6 gives $\dim_{\mathbf{H}}(X) \geqslant \frac{dn-\alpha}{n-1} - \epsilon$ for every $\epsilon > 0$, which concludes the proof of Theorem 1.

5 Applications

As discussed in the introduction, Theorem 1 generalizes Theorems 1.1 and 1.2 from [2]. In this section, we present two applications of Theorem 1 in settings where previous methods do not yield any results.

5.1 Sum-sets avoiding specified sets

Theorem 2. *Let $Y \subset \mathbf{R}^d$ be a countable union of sets of Minkowski dimension at most $\beta < d$. Then there exists a set $X \subset \mathbf{R}^d$ with Hausdorff dimension at least $d - \beta$ such that $X + X$ is disjoint from Y.*

Proof. Define $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x, y) : x + y \in Y\} \quad \text{and} \quad Z_2 = \{(x, y) : y \in Y/2\}.$$

Since Y is a countable union of sets of Minkowski dimension at most β, Z is a countable union of sets with lower Minkowski dimension at most $d + \beta$. Applying Theorem 1 with $n = 2$ and $\alpha = d + \beta$ produces a set $X \subset \mathbf{R}^d$ with Hausdorff dimension $2d - (d + \beta) = d - \beta$ such that $(x, y) \notin Z$ for any $x, y \in X$ with $x \neq y$. We claim that $X + X$ is disjoint from Y. To see this, first suppose $x, y \in X$, $x \neq y$. Since X avoids Z_1, we conclude that $x + y \notin Y$. Suppose now that $x = y \in X$. Since $(x, y) \notin Z_2$ for any $x, y \in X$, $x \notin Y/2$ for any $x \in X$. Thus for any $x \in X$, $x + x = 2x \notin Y$. This completes the proof. ☐

5.2 Subsets of Lipschitz curves avoiding isosceles triangles

In [2], Fraser and the second author prove that if $\gamma \subset \mathbf{R}^n$ is a simple C^2 curve with non-vanishing curvature, then there exists a set $S \subset \gamma$ of Hausdorff dimension 1/2 that does not contain the vertices of an isosceles triangle. Using Theorem 1, we generalize this result to Lipschitz curves.

Theorem 3. *Let $f \colon [0, 1) \to \mathbf{R}^{n-1}$ be Lipschitz. Then there is a set $X \subset [0, 1)$ of Hausdorff dimension 1/2 so that the set $\{(t, f(t)) : t \in X\}$ does not contain the vertices of an isosceles triangle.*

Proof. Set

$$Z = \left\{ (x_1, x_2, x_3) \in [0, 1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \text{form the vertices of an isosceles triangle} \end{array} \right\}.$$

In the next lemma, we show Z has lower Minkowski dimension at most two. By Theorem 1, there is a set $X_1 \subset [0, 1]$ of Hausdorff dimension 1/2 so that for each distinct $x_1, x_2, x_3 \in X_1$, we have $(x_1, x_2, x_3) \notin Z$. This is precisely the statement that for each $x_1, x_2, x_3 \in X$, the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ do not form the vertices of an isosceles triangle. To complete the proof, let $X = \{(x, f(x)) : x \in X_1\}$. ☐

Lemma 7. *Let $f\colon [0,1) \to \mathbf{R}^{n-1}$ be Lipschitz. Then the set*

$$Z = \left\{ (x_1, x_2, x_3) \in [0,1]^3 : \begin{array}{c} (x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)) \\ \textit{form the vertices of an isosceles triangle} \end{array} \right\}.$$

has Minkowski dimension at most two.

Proof. By translating and rescaling the range of f, which does not change the Minkowski dimension of the graph, we may assume withous loss of generality that f is $1/10$ Lipschitz, and that it's graph is contained in $[0,1]^n$. Fix $0 < \delta < 1$. It suffices to show that

$$\#\mathcal{B}^3_\delta(Z) \lesssim \delta^{-2} \log(1/\delta). \tag{5.1}$$

We have

$$|\mathcal{B}^3_\delta(Z)| = \sum_{I_1 \in \mathcal{B}^1_\delta([0,1])} \sum_{k=0}^{\log_2(1/\delta)} \sum_{\substack{I_2 \in \mathcal{B}^1_\delta([0,1]) \\ \mathrm{dist}(I_1, I_2) \sim \delta 2^k}} \#\{I_3 \in \mathcal{B}^1_\delta([0,1]) : I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)\}.$$

In the above expression we abuse notation slightly and say that $\mathrm{dist}(I_1, I_2) \sim \delta$ if $I_1 = I_2$; this will not affect our estimates.

Note that for each $I_1 \in \mathcal{B}^1_\delta([0,1])$, there are roughly $(\delta 2^k)/\delta = 2^k$ intervals $I_2 \in \mathcal{B}^1_\delta([0,1])$ with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$. Thus to establish (5.1), it suffices to prove that for each $I_1 \in \mathcal{B}^1_\delta([0,1])$ and each $I_2 \in \mathcal{B}^1_\delta([0,1])$ with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$, we have

$$\#\{I_3 \in \mathcal{B}^1_\delta([0,1]) : I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)\} \lesssim 2^{-k}/\delta. \tag{5.2}$$

For each distinct $p, q \in [0,1]^n$, define

$$H_{p,q} = \left\{ z \in \mathbf{R}^n : \left(z - \frac{p+q}{2}\right) \cdot (p - q) = 0 \right\}.$$

This is the hyperplane passing through the midpoint of p and q that is perpendicular to the line passing through p and q. We will call $H_{p,q}$ the perpendicular bisector of p and q.

Fix a choice of intervals I_1 and I_2 with $\mathrm{dist}(I_1, I_2) \sim \delta 2^k$. Let $\tilde{I}_1$ and $\tilde{I}_2$ denote the twofold dilates of I_1 and I_2, respectively. Note that if $I_3 \in \mathcal{B}^1_\delta([0,1])$ with $I_1 \times I_2 \times I_3 \in \mathcal{B}^3_\delta(Z)$, then there are points $x_j \in \tilde{I}_j,\ i = 1,2,3$ so that

$$(x_3, f(x_3)) \in H_{(x_1, f(x_1)),(x_2, f(x_2))}.$$

Consider the set

$$S_{I_1, I_2} = [0,1]^n \cap \bigcup_{\substack{x_1 \in \tilde{I}_1 \\ x_2 \in \tilde{I}_2}} H_{(x_1, f(x_1)),(x_2, f(x_2))}.$$

For each $x_1 \in \tilde{I}_1$ and $x_2 \in \tilde{I}_2$, the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ makes an angle $\leqslant 1/10$ with the e_1 direction. Thus the hyperplane $H_{(x_1, f(x_1)),(x_2, f(x_2))}$ makes an angle $\leqslant 1/10$ with the hyperplane spanned by the $e_2, \ldots, e_n$ directions. Since $\tilde{I}_1$ and $\tilde{I}_2$ are intervals of length $\leqslant 3\delta$ that are $\sim \delta 2^k$ separated, S_{I_1,I_2} is contained in the $\sim 2^{-k}$ neighborhood of a hyperplane that makes an angle $\leqslant 1/10$ with the $e_2, \ldots, e_n$ directions.

Suppose that $x_3, x_3' \in [0,1]$ satisfy

$$(x_3, f(x_3)) \in S_{I_1,I_2} \quad \text{and} \quad (x_3', f(x_3')) \in S_{I_1,I_2}. \tag{5.3}$$

Since f is 1/10-Lipschitz, we must have

$$|f(x_3) - f(x_3')| \leqslant \frac{1}{10}|x_3 - x_3'|.$$

On the other hand, by (5.3) and the fact that S_{I_1,I_2} is contained in the $\sim 2^{-k}$ neighborhood of a hyperplane that makes an angle $\leqslant 1/10$ with the $e_2, \ldots, e_n$ directions, we have

$$|f(x_3) - f(x_3')| \geqslant 10|x_3 - x_3'| - O(2^{-k}).$$

We conclude that $|x_3 - x_3'| \lesssim 2^{-k}$. This establishes (5.2). We conclude that (5.1) holds, so Z has lower Minkowski dimension at most 2. $\square$

References

[1] Kenneth Falconer, *Fractal geometry: Mathematical foundations and applications*, Wiley, 2003.

[2] Robert Fraser and Malabika Pramanik, *Large sets avoiding patterns*, Anal. PDE **11** (2018), 1083–1111.

[3] Nets Hawk Katz and Terrence Tao, *Some connections between Falconer's distance set conjecture, and sets of Furstenburg type*, New York J. Math. **7** (2001), 149–187.

[4] Tamás Keleti, *A 1-dimensional subset of the reals that intersects each of its translates in at most a single point*, Real Anal. Exchange **24** (1999), 843–845.

[5] Péter Maga, *Full dimensional sets without given patterns*, Real Anal. Exchange **36** (2010), 79–90.

[6] András Máthé, *Sets of large dimension not containing polynomial configurations*, Adv. Math. **316** (2017), 691–709.