Fractals Avoiding Fractal Sets

Jacob Denson

November 21, 2018

Our proof depends very little on the Euclidean structure of the plane, and as such, we rephrase the construction as a combinatorial problem on graphs. Recall that an n *uniform hypergraph* if a collection of *vertices* and *hyperedges*, where a hyperedge is a set of n distinct vertices. We say such a graph is *hypartite* if we can partition the vertex set into n sets $V_1, \dots, V_n$, such that each edge in the graph contains exactly one vertex from each set. An *independant set* is a subset of vertices containing no complete set of vertices in any edge. A *coloring* is a partition of the vertex set into finitely many classes, and we call each such class a *color*. The next lemma is a variant of Turán's theorem on the construction of independant sets, generalized for hypergraphs. For technical reasons, we need an extra restriction on the vertex set so it is 'uniformly' chosen over the graph, which is the reason for the coloring.

Lemma 1. *Let G be an n uniform hypartite graph together with a coloring partitioning each vertex set into size A sets. Then we can find an independent set U containing all but $(n/A^n)|E|$ of the colors.*

Proof. Form vertex sets $W_1 \subset V_1, \dots, W_n \subset V_n$ by randomly selecting a vertex from each color. Then each vertex in the graph is selected with probability $1/A$. For each edge $e = (v_1, \dots, v_n)$ in the graph, the chance that this edge connects vertices in the set we have constructed is

$$\begin{aligned} &\mathbf{P}(w_1 \in W_1, \dots, w_n \in W_n) \\ &\quad = \mathbf{P}(w_1 \in W_1) \dots \mathbf{P}(v_n \in W_n) = 1/A^n \end{aligned}$$

Thus if we let E' denote the set of all edges $e = (w_1, \dots, w_n)$ with $w_1 \in W_1, \dots, w_n$, then

$$\mathbf{E}|E'| = \sum_{e \in E} \mathbf{P}(e \in E') = \sum_{e \in E} 1/A^n = \frac{|E|}{A^n}$$

In particular, we may choose a *particular*, nonrandom choice $W_1, \dots, W_n$ for which $|E'| \leqslant |E|/A^n$. For each i, we form $U_i \subset W_i$ by removing all vertices $w \in W_i$ which are a vertex for some edge in E'. Then the vertices in $U_1, \dots, U_n$ form an independant set, and we have removed at most $n|E'| \leqslant (n/A^n)|E|$ representatives from the color classes. □

Corollary. *If $|V_1|, \dots, |V_n| \gtrsim N^a$, $|E| \lesssim N^b$, and $A \gtrsim N^c$, where $b < a + c(n-1)$, then as $N \to \infty$ we can find an independent set containing all but a fraction $o(1)$ of all colors.*

Proof. A simple calculation on the quantities of the previous lemma yields

$$\begin{aligned} \frac{\#(\text{colors removed})}{\#(\text{total colors})} &= \frac{(n/A^n)|E|}{|V|/A} \\ &= \frac{n|E|}{|V|A^{n-1}} \lesssim \frac{N^b}{N^{a+c(n-1)}} \end{aligned}$$

This is $o(1)$ if $b < a + c(n-1)$. □

We now apply these constructions on graphs to a problem which is quite clearly related to the fractal avoidance problem, and will form our key building block to constructing solutions to the problem. Given a fixed integer N, we subdivide $\mathbf{R}^d$ into a grid of sidelength $1/N$ cubes, the collection of such cubes we will denote by $\mathcal{I}$.

Theorem 1. *Suppose that $\dim_{\mathbf{M}}(Y) < \alpha$. If $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint collections of length $1/N$ cubes in $\mathcal{I}$, with $|\mathcal{I}_i| \gtrsim N^d$ for each i, then provided $\beta < (n-1)/(n-\alpha)$, we can find a collection of length $1/N^\beta$ cubes $\mathcal{J}_1, \dots, \mathcal{J}_n$ with $\mathcal{J}_1 \times \dots \times \mathcal{J}_n$ disjoint from Y and as $N \to \infty$, each $\mathcal{J}_i$ contains a cube in all but $o(1)$ of the cubes in $\mathcal{I}_i$.*

Proof. For sufficiently large N, if we partition $\mathbf{R}^{nd}$ into a grid of length $1/N^\beta$ cubes, and if $\mathcal{K}$ is the collection of all these cubes intersecting Y, then $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Similarily, we partition each $\mathcal{I}_i$ into a grid of length $1/N^\beta$ cubes $\mathcal{I}_i'$, using these intervals as vertices in a hypartite graph G with a hyperedge between $I_1 \in \mathcal{I}_1', \dots, I_n \in \mathcal{I}_n'$ if $I_1 \times \dots \times I_n \in \mathcal{K}$. We say two cubes in G are the same color if they are contained in a common cube in $\mathcal{I}_i$. Since each sidelength $1/N$ cube contains $N^{\beta d}/N^d = N^{(\beta-1)d}$ sidelength $1/N^\beta$ cubes, each color in G contains $N^{(\beta-1)d}$

vertices. Each vertex set V_i in G contains $|\mathcal{I}_i'| = N^{\beta-1}|\mathcal{I}_i| \gtrsim N^{\beta+(d-1)}$ vertices. Finally the number of edges is bounded above by $|\mathcal{K}| \lesssim N^{\alpha\beta}$. Thus in the parameters of the previous corollary, $a = \beta + (d-1)$, $b = \alpha\beta$, and $c = (\beta-1)d$, and in particular, we can find an independent set $\mathcal{J}_1 \subset \mathcal{I}_1, \dots, \mathcal{J}_n \subset \mathcal{I}_n$ containing all but $o(1)$ of the colors provided that $\alpha\beta < \beta + (d-1) + (\beta-1)d(n-1)$, which can be rearranged to give the inequality in the hypothesis. $\square$

s

References

[1] I. Z. Ruzsa *Difference Sets Without Squares*

[2] Tamás Keleti *A 1-Dimensional Subset of the Reals that Intersects Each of its Translates in at Most a Single Point*

[3] Robert Fraser, Malabika Pramanik *Large Sets Avoiding Patterns*

[4] Karl Dickman *On the Frequency of Numbers Containing Prime Factors of a Certain Relative Magnitude*

[5] B. Sudakov, E. Szemerédi, V.H. Vu *On a Question of Erdös and Moser*