Salem Sets Avoiding Rough Configurations

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Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension s if it has Hausdorff dimension s, and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that for all $\xi \in \mathbf{R}^d$,

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty.$$

Our goal in these notes is to obtain, for each set $Z \subset \mathbf{R}^{dn}$ with Minkowski dimension s, a Salem set $X \subset \mathbf{R}^d$ with dimension

$$\frac{nd - s}{s},$$

such that for each set of n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We hope that we can rely on the random selection approach of our paper on rough configurations to obtain such a result.

1 Concentration Inequalities

Define a convex function $\psi_2 : [0, \infty) \to [0, \infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{ A \in (0, \infty) : \mathbf{E}(\psi_2(|X|/A)) \leq 1 \}.$$

The family of random variables $\psi_2(L)$ are known as *subgaussian random variables*. Here are some important properties:

- (Gaussian Tails): If $\|X\|_{\psi_2(L)} \leq A$, then for each $t \geq 0$,

$$\mathbf{P}(|X| \geq t) \leq 10 \exp\left(-t^2/10A^2\right).$$

- (Bounded Variables are Subgaussian): For any random X,

$$\|X\|_{\psi_2(L)} \leq 10\|X\|_{L^\infty}.$$

- (Union Bound) If $X_1, \dots, X_N$ are random variables, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leq \|X_1\|_{\psi_2(L)} + \dots + \|X_N\|_{\psi_2(L)}.$$

- (Hoeffding's Inequality): If $X_1, \dots, X_N$ are *independent* random variables, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leq 10\left(\|X_1\|^2_{\psi_2(L)} + \dots + \|X_N\|^2_{\psi_2(L)}\right)^{1/2}.$$

The Orlicz norm is a convenient notation to summarize calculations involving the principle of concentration of measure. Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leq A$ as essentially always lying in the interval $[-A, A]$, very rarely deviating outside this interval.

2 A Family of Cubes

Fix sequences of integers $\{K_m : m \geq 1\}$ and $\{M_m : m \geq 1\}$, and set $N_m = K_m M_m$. We then define two sequences of real numbers $\{l_m : m \geq 0\}$ and $\{r_m : m \geq 0\}$, by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each $m, d \geq 0$, we define two collections of strings

$$\Sigma^d_m = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \dots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi^d_m = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \dots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $i \in \Sigma^d_m$, we define a vector $a_i \in (l_m \mathbf{Z})^d$ by setting

$$a_i = i_0 + \sum_{k=1}^m i_{2k-1} \cdot r_k + i_{2k} \cdot l_k$$

Then each string $i \in \Sigma^d_m$ can be identified with the sidelength l_m cube

$$Q_i = \prod_{j=1}^d [a_{ij}, a_{ij} + l_m].$$

centered at a_i. Similarily, for each string $i \in \Pi_m^d$, we define a vector $a \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leq j \leq d$,

$$a_i = i_0 + \left(\sum_{k=1}^{m-1} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k \right) + i_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R_i = \prod_{j=1}^{d} [a_{ij}, a_{ij} + r_m].$$

We let $\mathcal{Q}_m^d = \{Q_i : i \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R_i : i \in \Pi_m^d\}$. Here are some important properties of this collection of cubes:

- For each m, $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ are covers of $\mathbf{R}^d$.
- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarily, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.
- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarily, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarily, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{i \in \Sigma_m^d : Q_i \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{i \in \Pi_m^d : R_i \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j, and a third index $k \in \Pi_m^d$, such that $R_k \cap Q_i, R_k \cap Q_j \neq \varnothing$.

3 A Family of Mollifiers

We now consider a family of mollifiers, which we will use to smooth out the Fourier transform of the measures we study.

Lemma 1. *There exists a non-negative, C^∞ function ψ supported on $[-1,1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi(x)\, dx = 1, \tag{1}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{2}$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0,1]$, such that $\alpha(1/2+x) = \alpha(1/2-x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \le \alpha(x) \le 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|).$$

Symmetry considerations imply that $\int \alpha(x) + \beta(x) = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x+m) + \beta(x+m) = 1. \tag{3}$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

then ψ_0 satisfies the required constraints, at least in the one dimensional case. In general, define

$$\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d). \qquad \square$$

Fix some choice of ψ given by Lemma 1. Since ψ is C^∞ and compactly supported, then for each $t \in [0, \infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \tag{4}$$

Now we rescale the mollifier. For each $m > 0$, we let

$$\psi_m(x) = l_m^{-d} \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \tag{5}$$

Equation (2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \tag{6}$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \xi), \tag{7}$$

In particular, (7) implies that for each $t \ge 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)||\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)||\xi|^t. \tag{8}$$

Thus, uniformly in m, $\widehat{\psi_m}$ decays sharply outside of the box $[-l_m^{-1}, l_m^{-1}]^d$, a manifestation of the Heisenberg uncertainty principle.

4 Discrete Lemma

We now consider a discrete form of the Fourier bound argument, which we can apply iteratively to obtain a Salem set avoiding configurations.

Lemma 2. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn-s)/2)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth probability measure compactly supported on T, together with a constant $A \geq 1$ such that for each $m \in \mathbf{Z}^d$,*

$$|\widehat{\mu_T}(m)| \leq A \cdot |m|^{a\varepsilon - \frac{dn-s}{2n}}.$$

where

$$a = \frac{3d + 2dn - 2s}{dn}.$$

Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that

$$\#(\mathcal{Q}_{m+1}(B)) \leq (1/l_{m+1})^{s+\varepsilon}. \tag{9}$$

Then there exists a large constant $C(\mu_T, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geq C(\mu_T, n, d, s, \varepsilon), \tag{10}$$

and

$$M_{m+1}^{\frac{s}{dn-s} + c\varepsilon} \leq K_{m+1} \leq 2M_{m+1}^{\frac{s}{dn-s} + c\varepsilon}, \tag{11}$$

where

$$c = \frac{6dn}{(dn-s)^2},$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) For any $m \in \mathbf{Z}^d$,

$$|\widehat{\mu}(m)| \leq (1 + M_{m+1}^{-1/2})[A + 10^d M_{m+1}^{-\varepsilon}]|m|^{c\varepsilon - \frac{dn-s}{2n}}.$$

Proof of Lemma 2. First, we describe the construction of the set S, and the measure μ_S. For each $i \in \Pi^d_{m+1}$, let j_i be a random integer vector chosen from $[K_{m+1}]^d$, such that the family $\{j_i : i \in \Pi^d_{m+1}\}$ is independent. Then it is certainly true for any $j \in [K_{m+1}]^d$ that

$$\mathbf{P}(j_i = j) = K_{m+1}^{-d}. \tag{12}$$

We define a measure ν_S such that, for each $x \in \mathbf{R}^d$,

$$d\nu_S(x) = r^d_{m+1} \sum\nolimits_{i \in \Pi^d_{m+1}} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

We then normalize, defining

$$\mu_S = \frac{\nu_S}{\nu_S(\mathbf{R}^d)}.$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}^d_{m+1} : \mu_S(Q) > 0\},$$

then S is $\mathcal{Q}_{m+1}$ discretized, μ_S is supported on S, and $S \subset T$. Our goal is to show, with non-zero probability, some choice of $\{j_i\}$ yields a set S satisfying Properties (A) and (B) of the Lemma.

In our calculations, it will help us to decompose the measure ν_S into components roughly supported on sidelength r^d_{m+1} cubes. For each $i \in \Pi_{m+1}(T)$, define a measure ν_i such that for each $x \in \mathbf{R}^d$,

$$d\nu_i(x) = r^d_{m+1} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

Then $\nu_S = \sum_{i \in \Pi^d_{m+1}(T)} \nu_i$. We shall split the proof of the statement into several, more managable lemmas.

Lemma 3. *If*

$$M_{m+1} \geq \left(3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}\right)^2, \tag{13}$$

then almost surely, $|\nu(\mathbf{R}^d) - 1| \leq M^{-1/2}_{m+1}$.

Proof. If $j_0, j_1 \in [K_{m+1}]^d$, then

$$|a_{ij_0} - a_{ij_1}| = |j_0 - j_1| \cdot l_{m+1} \leq (\sqrt{d} K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1},$$

which, together with (5), implies

$$\begin{aligned} &\left| r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_0}) \mu_T(x) - r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_1}) \mu_T(x) \right| \\ &\quad \leq r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x) \left| \mu_T(x + a_{ij_0}) - \mu_T(x + a_{ij_1}) \right| \\ &\quad \leq \sqrt{d} \cdot r^{d+1}_{m+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1}(x) = \sqrt{d} \cdot r^{d+1}_{m+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \tag{14}$$

Thus (14) implies that almost surely, for each i,

$$|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))| \le \sqrt{d} \cdot r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \tag{15}$$

Furthermore, (6) implies

$$\begin{aligned}\sum_{i \in \Pi_{m+1}^d} \mathbf{E}(\nu_i(\mathbf{R}^d)) = r_{m+1}^d \textstyle\sum_{(i,j)\in\Sigma_{m+1}^d} &\mathbf{P}(j_i = j) \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij}) \mu_T(x)\, dx \\ &= \frac{r_{m+1}^d}{K_{m+1}^d} \int_{\mathbf{R}^d} \left(\textstyle\sum_{(i,j)\in\Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij}) \right) \mu_T(x)\, dx \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d} = 1.\end{aligned} \tag{16}$$

For all but at most $3^d \cdot r_{m+1}^{-d}$ indices i, $\nu_i = 0$ almost surely. Thus we can apply the triangle inequality together with (15) and (16) to conclude that almost surely,

$$\begin{aligned}|\nu_S(\mathbf{R}^d) - 1| &= \| \textstyle\sum_{i\in\Pi_{m+1}^d} \left[\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d)) \right] \|_{L^\infty} \\ &\le \textstyle\sum_{i\in\Pi_{m+1}^d} \|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))\|_{L^\infty} \\ &\le 3^d \sqrt{d} \cdot r_{m+1}^{-d} r_{m+1}^{d+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \\ &= 3^d \sqrt{d} \cdot r_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)} \\ &= \frac{3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.\end{aligned} \tag{17}$$

Thus (17) and (13) imply that almost surely, $|\nu_S(\mathbf{R}^d) - 1| \le M_{m+1}^{-1/2}$. □

Lemma 4. *If*

$$M_{m+1} \ge \left(10 \cdot 3^{dn} \cdot l_m^{-(s+\varepsilon)} \right)^{1/\varepsilon}, \tag{18}$$

then

$$\mathbf{P}\left(S \text{ does not satisfies Property (A)} \right) \le 1/10.$$

Proof. For any cube $Q_{ij} \in \Sigma_{m+1}^d$, there are at most 3^d pairs $(i_0, j_0) \in \Sigma_{m+1}^d$ such that $Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing$, and so a union bound together with (12) gives

$$\mathbf{P}(Q_{ij} \in \mathcal{Q}_{m+1}(S)) \le \textstyle\sum_{Q_{i_0 j_0} \cap Q_{ij} \neq \varnothing} \mathbf{P}(j_{i'} = j') \le 3^d K_{m+1}^{-d}. \tag{19}$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q = Q_{i_1 j_1} \times \cdots \times Q_{i_n j_n} \in \mathcal{Q}_{m+1}(B)$

be such a cube. Since Q is strongly diagonal, the events $\{Q_{i_k j_k} \in S\}$ are independent from one another, which together with (19) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_{i_1 j_1} \in S) \dots \mathbf{P}(Q_{i_n j_n} \in S) \le 3^{dn} K_{m+1}^{-dn}. \tag{20}$$

Taking expectations over all cubes in B, and applying (9) and (20) gives

$$\begin{aligned} \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\le \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\ &\le l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\ &= 3^{dn} l_m^{-(s+\varepsilon)} \frac{M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}. \end{aligned} \tag{21}$$

Since $\varepsilon \le (dn-s)/2$, we conclude

$$\begin{aligned} (dn-s-\varepsilon)\left(\frac{s}{dn-s} + c\varepsilon\right) &= s + \varepsilon\left(c(dn-s-\varepsilon) - \frac{s}{dn-s}\right) \\ &\ge s + \varepsilon\left(\frac{c(dn-s)}{2} - \frac{s}{dn-s}\right) \\ &= s + \varepsilon \frac{3dn-s}{dn-s} \ge s + 2\varepsilon. \end{aligned}$$

Applying (11) together with this bound, we conclude that

$$K_{m+1}^{dn-s-\varepsilon} \ge M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \ge M_{m+1}^{s+2\varepsilon}.$$

Combined with (18), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \le \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^{\varepsilon}} \le 1/10. \tag{22}$$

We can then apply Markov's inequality with (21) and (22) to conclude

$$\begin{aligned} \mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \ne \varnothing) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \ge 1) \\ &\le \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) \\ &\le 1/10. \end{aligned}$$ □

Lemma 5. *Set* $D = \{m \in \mathbf{Z}^d : |m| \le 10 l_{m+1}^{-1}\}$. *Then if*

$$K_{m+1} \le M_{m+1}^{\frac{2dn}{dn-s}}, \tag{23}$$

and

$$M_{m+1} \ge \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \tag{24}$$

then

$$\mathbf{P}\left(\|\widehat{\nu_T} - \widehat{\mu_T}\|_{L^\infty(D)} \ge r_{m+1}^{d/2} \log(M_{m+1})\right) \le 1/10$$

Proof. For each $i \in \Pi^d_{m+1}$, and $m \in \mathbf{Z}$, define $X_{im} = \widehat{\nu_i}(m) - \widehat{\mathbf{E}(\nu_i)}(m)$. Applying (2) gives

$$\begin{aligned}\sum_{i\in\Pi^d_{m+1}} \widehat{\mathbf{E}(\nu_i)}(m) &= \sum_{i\in\Pi^d_{m+1}} l^d_{m+1} \sum_{j\in[K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x} \psi_{m+1}(x-a_{ij}) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x} d\mu_T(x) = \widehat{\mu_T}(m).\end{aligned} \tag{25}$$

For each i and m, the standard (L^1, L^∞) bound on the Fourier transform, combined with (15), shows

$$\begin{aligned}\|X_{im}\|_{\psi_2(L)} &\leq 10\|X_{im}\|_{L^\infty} \\ &\leq 10[\|\nu_i(\mathbf{R}^d)\|_{L^\infty} + \mathbf{E}(\nu_i)(\mathbf{R}^d)] \\ &\leq 10^2\left(\mathbf{E}(\nu_i)(\mathbf{R}^d) + r^{d+1}_{m+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{26}$$

For a fixed m, the family of random variables $\{X_{im}\}$ are independent. Furthermore, $\sum X_{im} = \widehat{\nu}(m) - \widehat{\mathbf{E}(\nu)}(m)$. Equations (6) and (12) imply that

$$\begin{aligned}\mathbf{E}(\widehat{\nu_S}(m)) &= \frac{r^d_{m+1}}{K^d_{m+1}} \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x}\left(\sum_{(i,j)\in\Sigma^d_{m+1}} \psi_{m+1}(x-a_{ij})\right) d\mu_T(x) \\ &= \frac{r^d_{m+1} l^{-d}_{m+1}}{K^d_{m+1}} \int_{\mathbf{R}^d} e^{-2\pi i m\cdot x} d\mu_T(x) \\ &= \frac{r^d_{m+1} l^{-d}_{m+1}}{K^d_{m+1}} \widehat{\mu_T}(m) = \widehat{\mu_T}(m).\end{aligned} \tag{27}$$

Hoeffding's inequality, together with (26) and (27), imply that

$$\begin{aligned}&\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \\ &\quad\leq 10^3\sqrt{d}\left(\left(\sum \mathbf{E}(\nu_i)(\mathbf{R}^d)^2\right)^{1/2} + r^{d/2+1}_{m+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{28}$$

Equation (5) shows

$$\begin{aligned}\mathbf{E}(\nu_i)(\mathbf{R}^d) &= l^d_{m+1} \sum_{j\in[K_{m+1}]^d} \int \psi_{m+1}(x-a_{ij}) d\mu_T(x) \\ &\leq r^d_{m+1}\|\mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{29}$$

Combining (28) and (29) gives

$$\|\widehat{\nu}(m) - \widehat{\mu_T}(m)\|_{\psi_2(L)} \leq 10^3\sqrt{d}\left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right] r^{d/2}_{m+1}. \tag{30}$$

We can then apply a union bound over $D = \{m \in \mathbf{Z}^d : |m| \leq 10 l_{m+1}^{-1}\}$, which has cardinality at most $10^{d+1} l_{m+1}^{-d}$, together with (30) to conclude that

$$\begin{aligned} \mathbf{P}&\left(\|\widehat{\nu} - \widehat{\mu_T}\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \\ &\leq 10^{d+2} \cdot l_{m+1}^{-d} \exp\left(- \frac{\log(M_{m+1})^2}{10^7 d} \right) \\ &= 10^{d+2} l_m^{-d} \exp\left(d \log(M_{m+1} K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d} \right). \end{aligned} \tag{31}$$

Combined with (23) and (24), (31) implies

$$\mathbf{P}\left(\|\widehat{\nu} - \widehat{\mu_T}\|_{L^\infty(D)} \geq r_{m+1}^{d/2} \log(M_{m+1}) \right) \leq 1/10. \tag{32}$$

Thus $\widehat{\nu}$ and $\widehat{\mu_T}$ are highly likely to differ only by a neglible amount over small frequencies. □

Lemma 6. *If*

$$M_{m+1}^d K_{m+1}^d \geq A(d+1+s/2) 2^{1+d+s/2} l_m^d, \tag{33}$$

$$K_{m+1}^d M_{m+1}^d \geq \frac{l_m^d 8^d A(d+1+s/2)}{1+s/2}, \tag{34}$$

and

$$K_{m+1}^d M_{m+1}^d \geq l_m^d 2^{3d+s/2+1} B(d+s/2+1), \tag{35}$$

then almost surely, if $|m| \geq 10 l_{m+1}^{-1}$,

$$|\widehat{\nu_T}(m)| \leq \frac{1}{2|\eta|^{s/2}}.$$

Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{i \in \Pi_{m+1}^d \\ d(a_i, T) \leq 2 r_{m+1}^{-1}}} \delta_{a_{i j_i}}.$$

Then $\nu_S = (\alpha * \psi_{m+1}) \mu_T$. Thus we have $\widehat{\nu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup |\widehat{\mu_T}(\xi)| |\xi|^t < \infty.$$

In light of (7), if we define, for each $t > 0$,

$$B(t) = \sup |\widehat{\psi}(\xi)| |\xi|^t,$$

then

$$\sup |\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t} B(t).$$

The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d, so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \le \alpha(\mathbf{R}^d) \le 2^d$. Thus

$$|\widehat{\nu_S}(\eta)| \le 2^d \int |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi. \tag{36}$$

If $|\xi| \le |\eta|/2$, $|\eta - \xi| \ge |\eta|/2$, so for all $t > 0$, and since (5) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \le 1$, we find

$$\int_{0 \le |\xi| \le |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \le \frac{A(t) 2^{t-d}}{|\eta|^{t-d}}. \tag{37}$$

Set $t = d + 1 + s/2$. Equation (37), together with (33), implies

$$\begin{aligned}\int_{0 \le |\xi| \le |\eta|/2} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi &\le \frac{A(d+1+s/2) 2^{1+s/2} |\eta|^{-1}}{|\eta|^{s/2}} \\ &\le \frac{A(d+1+s/2) 2^{1+s/2} l_{m+1}}{|\eta|^{s/2}} \\ &\le \frac{1}{10 \cdot 2^d} \frac{1}{|\eta|^{s/2}}.\end{aligned} \tag{38}$$

Conversely, if $|\xi| \ge 2|\eta|$, then $|\eta - \xi| \ge |\xi|/2$, so for each $t > d$,

$$\begin{aligned}\int_{|\xi| \ge 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)| \, d\xi &\le \int_{|\xi| \ge 2|\eta|} \frac{A(t) 2^t}{|\xi|^t} \\ &\le 2^d \int_{2|\eta|}^\infty r^{d-1-t} A(t) 2^t \\ &\le \frac{4^d A(t)}{t-d} |\eta|^{d-t}.\end{aligned} \tag{39}$$

Set $t = d + 1 + s/2$. Equation (34), applied to (39), allows us to conclude

$$\int_{|\xi| \ge 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi}_{m+1}(\xi)| \, d\xi \le \frac{1}{10 \cdot 2^d} \frac{1}{|\eta|^{s/2}}. \tag{40}$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu_T}\|_{L^\infty(\mathbf{R}^d)} \le 1$ to conclude that

$$\int_{|\eta|/2 \le |\xi| \le 2|\eta|} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \le \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \tag{41}$$

Set $t = d + s/2 + 1$. Then (41) and (35) imply

$$\int_{|\eta|/2 \le |\xi| \le 2|\eta} |\widehat{\mu_T}(\eta - \xi)||\widehat{\psi_{m+1}}(\xi)| \, d\xi \le \frac{1}{10 \cdot 2^d} \frac{1}{|\xi|^{s/2}}. \tag{42}$$

Summing up (38), (40), and (42), we conclude from (36) that if $|\eta| \geq 10 l_{m+1}^{-1}$, then

$$|\widehat{\nu_S}(\eta)| \leq \frac{1}{2|\eta|^{s/2}}. \tag{43}$$

□

Proof of Lemma 2, Continued. Let us now put all our calculations together. In light of Lemma 4 and Lemma 5, there exists some choice of j_i for each i, and a resultant non-random pair (ν_S, S) such that S satisfies Property (A) of the Lemma, and for any $m \in \mathbf{Z}^d$ with $|m| \leq 10 l_{m+1}^{-1}$,

$$|\widehat{\nu_S}(m) - \widehat{\mu_T}(m)| \leq r_{m+1}^{d/2} \log(M_{m+1}). \tag{44}$$

Now

$$r_{m+1}^{d/2} \log(M_{m+1}) = \left(l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \right) l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}.$$

Equation (11) implies

$$\begin{aligned}
& l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \\
&\quad = \frac{l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} \log(M_{m+1}) K_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}}{M_{m+1}^{a\varepsilon + \frac{s}{2n}}} \\
&\quad \leq \left[l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} \right] \log(M_{m+1}) M_{m+1}^{(\frac{s}{dn-s} + c\varepsilon)(\frac{dn-s}{2n} - a\varepsilon) - a\varepsilon - \frac{s}{2n}}.
\end{aligned}$$

Now

$$\begin{aligned}
\left(\frac{s}{dn-s} + c\varepsilon \right) \left(\frac{dn-s}{2n} - a\varepsilon \right) - a\varepsilon &\leq \left[\frac{(dn-s)c}{2n} - \left(\frac{s}{dn-s} + 1 \right) a \right] \varepsilon \\
&= \left[\frac{d(3-na)}{(dn-s)} \right] \varepsilon \\
&\leq -2\varepsilon.
\end{aligned}$$

Thus, if we assume that

$$A(l_m) \leq X^{\varepsilon} / \log(X) \tag{45}$$

then we conclude

$$\begin{aligned}
r_{m+1}^{d/2} \log(M_{m+1}) &\leq \left[l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} \log(M_{m+1}) M_{m+1}^{-\varepsilon} \right] M_{m+1}^{-\varepsilon} l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon} \\
&\leq M_{m+1}^{-\varepsilon} l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}
\end{aligned}$$

Thus we conclude that if $|m| \leq 10 l_{m+1}^{-1}$,

$$\begin{aligned}
|\widehat{\nu_S}(m)| &\leq |\widehat{\nu_S}(m) - \widehat{\mu_T}(m)| + |\widehat{\mu_T}(m)| \\
&\leq r_{m+1}^{d/2} \log(M_{m+1}) + A|m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq l_{m+1}^{-\frac{dn-s}{2n} - c\varepsilon + \varepsilon} + A|m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq [A + 10^d M_{m+1}^{-\varepsilon}]|m|^{c\varepsilon - \frac{dn-s}{2n}}.
\end{aligned}$$

Since $|\widehat{\nu_S}(m)| \leq |m|^{-\frac{dn-s}{2n}} \leq |m|^{c\varepsilon - \frac{dn-s}{2n}}$ holds automatically for $|m| \geq 10 l_{m+1}^{-1}$, we conclude that for all $m \in \mathbf{Z}^d$,

$$|\widehat{\nu_S(m)}| \leq (A + 10^d M_{m+1}^{-\varepsilon})|m|^{c\varepsilon - \frac{dn-s}{2n}}.$$

Applying Lemma 3, we conclude that for all $m \in \mathbf{Z}^d$,

$$\begin{aligned}
|\widehat{\mu_S(m)}| &\leq \frac{A + 10^d M_{m+1}^{-\varepsilon}}{1 - M_{m+1}^{-1/2}} |m|^{c\varepsilon - \frac{dn-s}{2n}} \\
&\leq (1 + M_{m+1}^{-1/2})[A + 10^d M_{m+1}^{-\varepsilon}]|m|^{c\varepsilon - \frac{dn-s}{2n}}.
\end{aligned}$$ □

5 Construction of the Salem Set

Let us now construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^\infty Z_i$, where Z_i has lower Minkowski dimension at most s for each i. Then choose an infinite sequence $\{i_m : m \geq 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence $\{\varepsilon_m : m \geq 1\}$, with $\varepsilon_m < (dn-s)/2$ for each m.

We choose our parameters $\{M_m\}$ and $\{K_k\}$ inductively. First, set $X_0 = [0,1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0. At the m'th step of our construction, we have found a set X_{m-1} and a measure μ_{m-1}. We then choose K_m and M_m such that

$$K_m, M_m \geq C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m^{\frac{s}{dn-s} + c\varepsilon_m} \leq K_m \leq 2M_m^{\frac{s}{dn-s} + c\varepsilon},$$

and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in $\mathcal{Q}_m$, the union of which, we define to be equal to B_m. We can then apply Lemma 2 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$, and a measure μ_S supported on S. We define $X_m = S$, and $\mu_m = \mu_S$.

The preceding paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$, and pick an arbitrary measure μ, and some subsequence μ_{i_k}, such that $\mu_{i_k} \to \mu$ weakly. It then follows from pointwise convergence of the Fourier transform that for each $m \in \mathbf{Z}^d$, and each $\varepsilon > 0$,

$$\sup_{m \in \mathbf{Z}^d} |\widehat{\mu}(m)||m|^{\frac{dn-s}{2n}-\varepsilon} \leq \limsup_{i \to \infty} \sup_{m \in \mathbf{Z}^d} |\widehat{\mu_i}(m)||m|^{\frac{dn-s}{2n}-\varepsilon}.$$

Fix $\varepsilon > 0$. For each m, define

$$A_{m,\varepsilon} = \sup |\widehat{\mu_M}(m)||m|^{\frac{dn-s}{2n}-\varepsilon}.$$

Since each measure μ_M is smooth, all these quantities are finite. Since $\varepsilon_m \to 0$, there is M such that if $m \geq M$, then $a\varepsilon_m \leq \varepsilon$. Property (B) of Lemma (2) implies that for each $m \geq M$,

$$A_{m+1,\varepsilon} \leq (1 + M_{m+1}^{-1/2})(A_{m,\varepsilon} + 10^d M_{m+1}^{-\varepsilon_{m+1}}).$$

If the sequence $\{M_m\}$ increases rapidly enough, this recursive relationship guarantees that $\sup_{m \to \infty} A_{m,\varepsilon} < \infty$. Thus, for each $\varepsilon > 0$,

$$|\widehat{\mu}(m)| \lesssim_\varepsilon |m|^{\varepsilon - \frac{dn-s}{2n}}.$$