Large Salem Sets Avoiding Nonlinear Configurations

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Abstract

We construct Salem sets with large dimension that avoid patterns described by the zero sets of a family of smooth functions, as well as rough families of patterns, complementing previous results constructing sets with large Hausdorff dimension. For a countable family of smooth functions $\{f_i : (\mathbf{T}^d)^{n-1} \to \mathbf{T}^d\}$ satisfying a modest geometric condition, we obtain a Salem subset of $\mathbf{T}^d$ with dimension $d/(n-3/4)$ whose cartesian product avoids points in the zero set of each function f_i with distinct coordinates. For a set $Z \subset \mathbf{T}^{dn}$ which is the countable union of a family of sets, each with lower Minkowski dimension s, we construct a Salem subset of $\mathbf{T}^d$ of dimension $(dn-s)/(n-1/2)$ whose Cartesian product does not intersect Z except at points with non-distinct coordinates.

1 Introduction

Geometric measure theory explores the relationship between the geometry of subsets of $\mathbf{R}^n$, and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what geometric information can be gathered from the Fourier analytic properties of these measures. A large body of research has focused on showing that the support of a measure with Fourier decay contain patterns, like a family of points forming an arithmetic progression. In this paper, we work in the opposite direction, showing that *most* sets supporting measures with a certain type of Fourier decay *do not* contain certain patterns. More precisely, given a set $Z \subset \mathbf{T}^{dn}$, we focus on showing that a 'generic' compact set $E \subset \mathbf{T}^d$ supporting a measure whose Fourier transform exhibits a quantitative decay bound also avoids the pattern defined by Z, in the sense that for any *distinct* points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.

A useful statistic associated with any Borel subset E of $\mathbf{T}^d$ is its *Fourier dimension*; given a finite Borel measure μ, its Fourier dimension $\dim_{\mathbf{F}}(\mu)$ is the supremum of all $s \in [0,d]$ such that $\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty$. The Fourier dimension of a Borel set E is then the supremum of $\dim_{\mathbf{F}}(\mu)$, where μ ranges over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most pattern avoiding sets constructed in

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the literature are not Salem, often having Fourier dimension zero. Nonetheless, the methods in this paper prove the exsitence of large Salem pattern avoiding sets.

The main inspiration for the results of this paper was the result of [1] on 'rough' patterns, which constructed, for any compact set $Z \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, a set E avoiding Z with

$$\dim_{\mathbf{H}}(E) = \max\left(\frac{dn-\alpha}{n-1}, d\right). \tag{1.1}$$

However, the sets E constructed using this method are not guaranteed to be Salem, and the construction is not even guaranteed to produce sets E with $\dim_{\mathbf{F}}(E) > 0$. Our goal was to modify the construction of [1] in order to ensure the resulting sets constructed were Salem. The baseline in the setting of Salem sets was Theorem 38 of [2], which constructed a Salem set E avoiding Z with

$$\dim_{\mathbf{F}}(E) = \max\left(\frac{dn-\alpha}{n}, d\right). \tag{1.2}$$

In this paper, we are only able to construct Salem sets with dimension matching that of (1.1) when Z exhibits translational symmetry (Theorem 1.3 of this paper), but for general sets we are still able to improve upon the dimension given in (1.2) (via Theorems 1.1 and 1.2 of this paper).

The methods in this paper are generic, in the sense of the Baire category theorem; we define a complete metric space $\mathcal{X}_\beta$ for each $\beta \in (0, d]$, which consists of all pairs (E, μ), where E is compact, μ is supported on E, and $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, and show that for an appropriate choice of β, the family of all pairs $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids a pattern is *comeager*, or *generic* in $\mathcal{X}_\beta$ (the complement of a set of first category). The approaches given in this paper can be modified to produce explicit pattern avoiding Salem sets via some limiting process, by applying the kinds of queuing methods found in [1], [2], [4], and [6]. But Baire category techniques allow us to focus on the more novel aspects of our argument.

Let us now introduce the theorems we will prove in this paper. Theorem 1.1 has the weakest conclusions, but works for the most general family of patterns.

Theorem 1.1. *Fix $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a compact set with lower Minkowski dimension at most α. Set*

$$\beta_0 = \min\left(\frac{dn-\alpha}{n-1/2}, d\right).$$

Then there exists a compact Salem set $E \subset \mathbf{T}^d$ with $\dim_{\mathbf{F}}(E) = \beta_0$, such that for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$. Moreover, if $\beta \leqslant \beta_0$, then the family of all pairs $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids the pattern generated by Z is comeager.

Under the stronger assumption that Z is a smooth hypersurface, satisfying an equation geometrically equivalent to Z being transverse to any axis-oriented hyperplane, we are able to improve the Fourier dimension bound obtained, though not quite enough to match the Hausdorff dimension bound obtained in [4], except in the fairly trivial case where $n = 2$.

Theorem 1.2. *Consider a smooth function $f : V \to \mathbf{T}^d$, where V is an open subset of $\mathbf{T}^{d(n-1)}$, such that for each $k \in \{1, \dots, n-1\}$, the matrix*

$$D_{x_k} f(x_1, \dots, x_{n-1}) = \left(\frac{\partial f_i}{\partial x_{kj}} \right)_{1 \leqslant i,j \leqslant d}$$

is invertible whenever $x_1, \dots, x_n$ are distinct and $(x_1, \dots, x_n) \in V$. Then there exists a compact Salem set $E \subset \mathbf{T}^d$ with dimension

$$\beta_0 = \begin{cases} d & : n = 2 \\ d/(n - 3/4) & : n \geqslant 3 \end{cases}$$

such that for any distinct points $x_1, \dots, x_n \in E$, with $x_1, \dots, x_{n-1} \in V$,

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

Moreover, if $\beta \leqslant \beta_0$, then the family of pairs $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids solutions to the equation $x_n = f(x_1, \dots, x_{n-1})$ for distinct points $x_1, \dots, x_n \in E$ is comeager.

Finally, we consider a situation in which our pattern exhibits some translational symmetry. Here we can construct Salem sets with dimension exactly matching the Hausdorff dimension results obtained in [1]. The simplest example of such of a pattern is that specified by an equation of the form $m_1 x_1 + \dots + m_n x_n = 0$, where at least two of the integers $m_1, \dots, m_n \in \mathbf{Z}$ is nonzero. But we can also consider more nonlinear patterns, such as those formed by solutions to an equation $m_1 x_1 + m_2 x_2 = f(x_3, \dots, x_n)$ for $m_1, m_2 \neq 0$, and a suitably regular function f. Even in the linear case, this theorem implies new results.

Theorem 1.3. *Fix $0 \leqslant \alpha < dn$, $a \in \mathbf{Q} - \{0\}$, and a locally Lipschitz function $S : V \to \mathcal{E}$, where V is an open subset of $\mathbf{T}^{d(n-2)}$ and $\mathcal{E}$ is the family of all compact subset of $\mathbf{T}^d$, equipped with the Hausdorff distance metric. Suppose that the sets $S(x_1, \dots, x_{n-2})$* locally uniformly *have lower Minkowski dimension at most $\alpha - d$, in the sense that for any $\lambda < \alpha$, and any closed set $W \subset V$, there exists a decreasing sequence $\{r_i\}$ with $\lim_{i \to \infty} r_i = 0$ such that for $x \in W$, $|S(x)_{r_i}| \leqslant r_i^{d(n-2)-\lambda}$ for all $x \in V$. Set*

$$\beta_0 = \min\left(\frac{dn - \alpha}{n - 1}, d \right).$$

Then there exists a compact Salem set $E \subset \mathbf{T}^d$ with $\dim_{\mathbf{F}}(E) = \beta_0$, such that for any distinct points $x_1, \dots, x_n \in E$, with $(x_1, \dots, x_{n-2}) \in V$,

$$x_n - a x_{n-1} \notin S(x_1, \dots, x_{n-2}).$$

Moreover, if $\beta \leqslant \beta_0$, then the family of all pairs $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids the pattern generated by Z is comeager.

An archetypical example of a function S to which Theorem 1.3 applies is obtained by setting $S(x_1, \dots, x_{n-2}) = \{f(x_1, \dots, x_{n-2})\}$ for some Lipschitz continuous function f, in which case Theorem 1.3 constructs Salem sets E of dimension $(dn - 1)/(n - 1)$ avoiding solutions to the equation $x_n - ax_{n-1} = f(x_2, \dots, x_{n-1})$. The advantage of considering a 'multi-valued function' S instead of a 'single-valued' function f in Theorem 1.3 is that it enables us to consider problems in which we avoiding a family of equations of the form $x_n - ax_{n-1} = f_i(x_1, \dots, x_{n-2})$, where $\{f_i : i \in I\}$ is an *uncountable* family of Lipschitz functions with uniformly bounded Lipschitz constant, such that the set $S(x_1, \dots, x_{n-2}) = \{f_i(x_1, \dots, x_{n-2}) : i \in I\}$ locally uniformly has lower Minkowski dimension at most $\alpha - d$, so that the assumptions of Theorem 1.3 apply to the function S. An application of this property to avoiding uncountable families of linear patterns is detailed in Section 3.

Remarks 1.4.

1. *Because we are using Baire category techniques, the results we obtain remain true when, instead of avoiding a single pattern, we avoid a countable family of patterns. This is because the countable intersection of comeager sets is comeager. For instance, the conclusion of Theorem 1.1 holds when Z is replaced by a* countable union *of compact sets, each with lower Minkowski dimension at most α. Similar generalizations apply to Theorem 1.2 and Theorem 1.3.*

2. *If $0 \leqslant \alpha < d$, then the pattern avoiding set $[0,1]^d - \pi_i(Z)$ has full Hausdorff dimension d, where $\pi_i(x_1, \dots, x_n) = x_i$ is projection onto a particular coordinate. Thus the pattern avoidance problem is trivial in this case for Hausdorff dimension. This is no longer true when studying Fourier dimension, since $[0,1]^d - \pi_i(Z)$ need not be a Salem set, nor even have particularly large Fourier dimension compared to the sets guaranteed by Theorem 1.1.*

   That this is true is hinted at in Example 8 of [3], where it is shown that there exists a set $X \subset [0,1]$ which is the countable union of a family of compact sets $\{X_k\}$ with $\sup_k \dim_{\mathbf{M}}(X_k) \leqslant 3/4$, such that $\dim_{\mathbf{F}}([0,1] - X) \leqslant 3/4$. Thus $[0,1] - X$ is not a Salem set. If we let F be any countable union of compact sets with Minkowski dimension zero, and we set
   $$Z = \bigcup_{i=0}^{n-1} F^i \times X \times F^{n-i-1},$$
   then Z is a countable union of compact sets with Minkowski dimension at most $3/4$, whereas
   $$\dim_{\mathbf{F}}([0,1] - \pi_i(X)) \leqslant \dim_{\mathbf{F}}([0,1] - X) \leqslant 3/4$$
   for each $i \in \{1, \dots, n\}$. Thus the trivial solution obtained by removing a projection of Z onto a particular coordinate axis does not necessarily give a pattern avoiding set with optimal Fourier dimension in this setting. Applying Theorem 1.1 directly to Z shows that a generic Salem set $E \subset \mathbf{T}$ of dimension $(n - 3/4)/(n - 1/2)$ avoids Z, which exceeds the dimension of the trivial construction for all $n > 1$. In fact, a generic Salem set $E \subset \mathbf{T}$ with dimension 1 will avoid Z, since any subset of $\mathbf{T} - F$ will avoid Z, and Theorem 1.1 applied with $Z = F$ proves that a generic Salem set E of dimension 1 will be contained in $\mathbf{T} - F$.

3. *If $n = 2$, the avoidance problem for a continuous function $f : V \to \mathbf{T}^d$ is essentially trivial. If there exists $x \in \mathbf{T}^d$ such that $f(x) \neq x$, there there exists an open set U*

around x such that $U \cap f(U) = \varnothing$. Then U has full Fourier dimension, and avoids solutions to the equation $y = f(x)$. On the other hand, if $f(x) = x$ for all x, then there are no distinct x and y in $[0,1]$ such that $y = f(x)$, and so the problem is also trivial. But it is a less trivial to argue that a generic *set with full Fourier dimension avoids this pattern, which is proved in Theorem 1.2, so we still obtain nontrivial information in this case.*

The study of the kinds of patterns in sets which occur in this paper is a fairly local problem, because for the most interesting examples (e.g. arithmetic progressions, equilateral triangles), these patterns exist at all scales. This implies that working in the domain $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d$. But working in $\mathbf{T}^d$ has several technical and notational advantages over $\mathbf{R}^d$, which is why in this paper we have chosen to work with the pattern avoidance pattern in this setting. Let us briefly describe how one can reduce the relation of Fourier dimension and pattern avoiding in $\mathbf{R}^d$ to $\mathbf{T}^d$. Given a Borel measure μ on $\mathbf{R}^d$, we define the Fourier dimension $\dim_{\mathbf{F}}(\mu)$ of μ to be the supremum of all $s \in [0,d]$ such that $\sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported finite measure on $\mathbf{R}^d$, and we consider the *periodization* μ^* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x)\, d\mu^*(x) = \int_{\mathbf{R}^d} f(x)\, d\mu(x), \tag{1.3}$$

then $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. A proof is given in Lemma 39 of [2]. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$ (this can be done, for instance, by noticing the similarity between the Frostman measure conditions). Using these results, one can reduce the study of patterns on $\mathbf{R}^{dn}$ to patterns on $\mathbf{T}^{dn}$, and thus obtain analogous results to Theorems 1.1, 1.2, and 1.3 for patterns in $\mathbf{R}^d$.

2 Notation

- Given a metric space X, a point $x \in X$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. For $x \in X$, we let δ_x denote the Dirac delta measure at x. For a given set $E \subset X$ and $\varepsilon > 0$, we let

  $$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

  denote the ε-*thickening* of the set E.

- A subset of a metric space X is of *first category*, or *meager* in X if it is the countable union of closed sets with empty interior, and is *comeager* if it is the complement of such a set. We say a property holds *quasi-always*, or a property is *generic* in X, if the set of points in X satisfying that property is comeager. The Baire category theorem then states that any comeager set in a complete metric space is dense.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let

  $$|x| = \min\{|x + n| : n \in \mathbf{Z}\},$$

and for $x \in \mathbf{T}^d$, we let

$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

The canonical metric on $\mathbf{T}^d$ is then given by $d(x,y) = |x-y|$, for $x, y \in \mathbf{T}^d$.

TODO: DO WE STILL NEED THIS NOTATION? For an axis-oriented cube Q in $\mathbf{T}^d$, we let $2Q$ be the cube in $\mathbf{T}^d$ with the same center and twice the sidelength.

- For $\alpha \in [0,d]$ and $\delta > 0$, we define the (α, δ) *Hausdorff content* of a Borel set $E \subset \mathbf{T}^d$ as

$$H^\alpha_\delta(E) = \inf \left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } 0 < \varepsilon_i \leqslant \delta \text{ for all } i \geqslant 1 \right\}.$$

The α dimensional Hausdorff measure of E is equal to

$$H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infinum over all $s \in [0,d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0,d]$ such that $H^s(E) = 0$. Frostman's lemma (see [8], Chapter 8) says that if we define the Hausdorff dimension $\dim_{\mathbf{H}}(\mu)$ of a finite Borel measure μ as the supremum of all $s \in [0,d]$ such that

$$\sup \left\{ \mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{T}^d, \varepsilon > 0 \right\} < \infty,$$

then $\dim_{\mathbf{H}}(E)$ is the supremum of $\dim_{\mathbf{H}}(\mu)$, over all Borel probability measures μ supported on E. This gives a specification of the Hausdorff dimension analogous to the definition of the Fourier dimension of a set E given in the introduction.

For a measurable set $E \subset \mathbf{T}^d$, we let $|E|$ denote its Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set $E \subset \mathbf{T}^d$ as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{r \to 0} d - \log_r |E_r|.$$

Thus $\underline{\dim}_{\mathbf{M}}(E)$ is the largest number such that for $\alpha < \underline{\dim}_{\mathbf{M}}(E)$, there exists a decreasing sequence $\{r_i\}$ with $\lim_{i \to \infty} r_i = 0$ and $|E_{r_i}| \leqslant r_i^{d-\alpha}$ for each i.

- In this paper we will employ probabilistic concentration bounds several times. In particular, we use *McDiarmid's inequality*, trivially modified from the standard theorem to work with complex-valued functions. Let $\{X_1, \dots, X_N\}$ be an independent family of $\mathbf{T}^d$ valued random variables, and consider a function $f : (\mathbf{T}^d)^N \to \mathbf{C}$. Suppose that for each $i \in \{1, \dots, N\}$, there exists a constant $A_i > 0$ such that for any $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N \in \mathbf{T}^d$, and for each $x_i, x_i' \in \mathbf{T}^d$,

$$|f(x_1, \dots, x_i, \dots, x_N) - f(x_1, \dots, x_i', \dots, x_N)| \leqslant A_i.$$

Then McDiarmid's inequality guarantees that for all $t \geqslant 0$,

$$\mathbf{P}\left(|f(X_1, \dots, X_N) - \mathbf{E}(f(X_1, \dots, X_N))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-2t^2}{A_1^2 + \cdots + A_N^2} \right).$$

The complex-valued extension we have just stated is proved easily from the real-valued case by taking a union bound to the inequality for the real and imaginary values of f. Proofs of McDiarmid's inequality are given in many probability texts, for instance, in Theorem 3.11 of [5].

A special case of McDiarmid's inequality is *Hoeffding's Inequality.* For the purposes of this paper, Hoeffding's inequality states that if $\{X_1, \dots, X_N\}$ is a family of independent random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

$$\mathbf{P}\left(|(X_1 + \cdots + X_N) - \mathbf{E}(X_1 + \cdots + X_N)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-t^2}{2(A_1^2 + \cdots + A_N^2)}\right).$$

- Throughout this paper, we will need to consider a standard mollifier. So we fix a smooth, non-negative function $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$ and

$$\int_{\mathbf{T}^d} \phi(x)\, dx = 1.$$

For each $r \in (0,1)$, we can then define $\phi_r \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_r(x) = \begin{cases} r^{-d}\phi(x/r) & : |x| < r, \\ 0 & : \text{otherwise.} \end{cases}$$

The following standard properties hold:

(1) For each $r \in (0,1)$, ϕ_r is a non-negative smooth function with

$$\int_{\mathbf{T}^d} \phi_r(x)\, dx = 1, \tag{2.1}$$

and $\phi_r(x) = 0$ for $|x| \geqslant r$.

(2) For any $r \in (0,1)$,

$$\|\widehat{\phi_r}\|_{L^\infty(\mathbf{Z}^d)} = 1. \tag{2.2}$$

(3) For each $\xi \in \mathbf{Z}^d$,

$$\lim_{r \to 0} \widehat{\phi_r}(\xi) = 1. \tag{2.3}$$

(4) For each $T > 0$, for all $r > 0$, and for any non-zero $\xi \in \mathbf{Z}^d$,

$$|\widehat{\phi_r}(\xi)| \lesssim_T r^{-T} |\xi|^{-T}. \tag{2.4}$$

3 Applications of our Results

3.1 Arithmetic Patterns

An important problem in current research on pattern avoidance is to construct sets E which avoid *linear patterns*, i.e. avoiding nontrivial solutions to equations of the form

$$m_1 x_1 + \cdots + m_n x_n = 0.$$

In [7], Körner showed that for each $n > 0$, there exists a set $E \subset \mathbf{T}$ with Fourier dimension $1/(n-1)$ such that for any distinct $x_1, \dots, x_n \in E$, and any integers $m_1, \dots, m_n \in \mathbf{Z}$, not all zero, $m_1 x_1 + \dots + m_n x_n \neq 0$. The technique used to control Fourier decay in [7] (bounding the first derivative of an associated distribution function to a random quantity) relies heavily on the one dimensional nature of the problem. The results of this paper give a d-dimensional version of Körner's result, as well as extending this result to consider avoiding certain uncountable families of linear equations.

Theorem 3.1. *For any compact set $S \subset \mathbf{T}^d$ with lower Minkowski dimension zero, there exists a Salem set $E \subset \mathbf{T}^d$ of dimension $d/(n-1)$ such that for any $s \in S$, any distinct $x_1, \dots, x_n \in E$, and any integers $m_1, \dots, m_n \in \mathbf{Z}$, $m_1 x_1 + \dots + m_n x_n \neq s$.*

Proof. Without loss of generality we may assume that S is closed under dilations by rational numbers. Under these assumptions, it will suffice to construct a set E avoiding equations of the form

$$x_n - a_{n-1} x_{n-1} = s + a_3 x_3 + \dots + a_n x_n,$$

with $a_2, \dots, a_n \in \mathbf{Q}$, and where either $a_2 \neq 0$, or $a_2 = a_3 = \dots = a_n = 0$. To construct E, we apply a Baire category argument. If $a_2 \neq 0$, then Theorem 1.3 applies directly to the equation

$$x_1 + a_2 x_2 \in S(x_3, \dots, x_n), \tag{3.1}$$

where $S(x_3, \dots, x_n) = S - a_3 x_3 - \dots - a_n x_n$. Thus we conclude that if $\beta = d/(n-1)$, then the set of $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids solutions to (3.1) is comeager. On the other hand, if $a_2 = a_3 = \dots = a_n = 0$, then the equation is precisely

$$x_1 \in S, \tag{3.2}$$

and it follows from Theorem 1.1 with $Z = S$ and $n = 1$ that the set of $(E, \mu) \in \mathcal{X}_\beta$ such that E is Salem and avoids solutions to (3.2) is comeager. Since we only have to consider countably many equations of the form (3.1), and only a single equation of the form (3.2), it follows that for a generic element $(E, \mu) \in \mathcal{X}_\beta$, the set E satisfies the conclusions of the theorem. In particular, such a set E exists. $\square$

The arguments in this paper are heavily inspired by the techniques of [7], but augmented with some more robust probabilistic concentration inequalities and stationary phase techniques, which enables us to push the results of [7] to a much more general family of patterns. In particular, Theorem 1.3 shows that the results of that paper do not depend on the rich arithmetic structure of the equation $m_1 x_1 + \dots + m_n x_n = 0$, but rather only on a simple translation invariance property of the pattern.

3.2 Isosceles Triangles on Curves

Theorems 1.1 and 1.2 can be applied to find sets avoiding linear patterns, but the main power of these Theorems above other results in the field is that they can be applied to 'nonlinear' patterns which are not necessarily related to the arithmetic structure of $\mathbf{T}^d$, differing from most other results in the field. In this section we consider a standard problem of this kind, avoiding isosceles triangles on curves; given a simple segment of a curve given by a smooth map $\gamma : [0,1] \to \mathbf{R}^d$, we say a set $E \subset [0,1]$ *avoids isosceles triangles* if for any distinct values $t_1, t_2, t_3 \in [0,1]$, $|\gamma(t_1) - \gamma(t_2)| \neq |\gamma(t_2) - \gamma(t_3)|$. Then E avoids isoceles triangles if and only if $\gamma(E)$ does not contain any three points forming the vertices of an isosceles triangle. In [4], methods are provided to construct sets $E \subset [0,1]$ with $\dim_{\mathbf{H}}(E) = \log 2/\log 3 \approx 0.63$ such that $\gamma(E)$ does not contain any isosceles triangles, but E is not guaranteed to be Salem. We can use Theorem 1.2 to construct *Salem sets* $E \subset [0,1]$ with $\dim_{\mathbf{F}}(E) = 4/9 \approx 0.44$.

Theorem 3.2. *For any smooth map $\gamma : [0,1] \to \mathbf{R}^d$ with $\gamma'(x) \neq 0$ for all $x \in [0,1]$, there exists a Salem set $E \subset [0,1]$ with $\dim_{\mathbf{F}}(E) = 4/9$ which avoids isosceles triangles.*

Proof. Assume without loss of generality (working on a smaller portion of the curve if necessary) that there exists a constant $C >\geqslant 1$ such that for any $t, s \in [0,1]$,

$$|\gamma(t) - \gamma(s) - (t-s)\gamma'(0)| \leqslant C(t-s)^2, \tag{3.3}$$

$$1/C \leqslant |\gamma'(t)| \leqslant C, \tag{3.4}$$

and

$$|\gamma'(t) - \gamma'(s)| \leqslant C|t-s|. \tag{3.5}$$

Let $\varepsilon = 1/2C^3$, and let

$$F(t_1, t_2, t_3) = |\gamma(t_1) - \gamma(t_2)|^2 - |\gamma(t_2) - \gamma(t_3)|^2. \tag{3.6}$$

A simple calculation using (3.3) and (3.4) reveals that for $0 \leqslant t_1, t_2 \leqslant \varepsilon$,

$$\left| \frac{\partial F}{\partial t_1} \right| = 2\,|(\gamma(t_1) - \gamma(t_2)) \cdot \gamma'(t_1)| \geqslant (2/C)|t_2 - t_1| - 2C|t_2 - t_1|^2 \geqslant (1/C)|t_2 - t_1|. \tag{3.7}$$

This means that $\partial F/\partial t_1 \neq 0$ unless $t_1 = t_2$. Thus the implicit function theorem implies that there exists a countable family of smooth functions $\{f_i : U_i \to [0,1]\}$, where $U_i \subset [0,\varepsilon]^2$ for each i and $f_i(t_2, t_3) \neq t_3$ for any $(t_2, t_3) \in U_i$, such that if $F(t_1, t_2, t_3) = 0$ for distinct points $t_1, t_2, t_3 \in [0, \varepsilon]$, then there exists an index i with $(t_2, t_3) \in U_i$ and $t_1 = f_i(t_2, t_3)$. Differentiating both sides of the equation

$$|\gamma(f_i(t_2, t_3)) - \gamma(t_2)|^2 = |\gamma(t_2) - \gamma(t_3)|^2 \tag{3.8}$$

in t_2 and t_3 shows that

$$\frac{\partial f_i}{\partial t_2}(t_2, t_3) = \frac{(\gamma(f_i(t_2, t_3)) - \gamma(t_3)) \cdot \gamma'(t_2)}{(\gamma(f_i(t_2, t_3)) - \gamma(t_2)) \cdot \gamma'(f_i(t_2, t_3))} \tag{3.9}$$

and

$$\frac{\partial f_i}{\partial t_3}(t_2, t_3) = \frac{-(\gamma(t_2) - \gamma(t_3)) \cdot \gamma'(t_3)}{(\gamma(f_i(t_2, t_3)) - \gamma(t_2)) \cdot \gamma'(f_i(t_2, t_3))}. \tag{3.10}$$

In order to apply Theorem 1.2, we must show that the partial derivatives in 3.9 and 3.10 are both non-vanishing for $t_2, t_3 \in [0, \varepsilon]$. We calculate using (3.3), (3.4) and (3.5) that

$$\begin{aligned} |(\gamma(f_i(t_2,t_3)) - \gamma(t_3)) \cdot \gamma'(t_2)| &\geqslant |(\gamma(f_i(t_2,t_3)) - \gamma(t_3)) \cdot \gamma'(t_3)| \\ &\quad + |(\gamma(f_i(t_2,t_3)) - \gamma(t_3)) \cdot (\gamma'(t_2) - \gamma'(t_3))| \\ &\geqslant (1/C)|f_i(t_2,t_3) - t_3| - C^2|f_i(t_2,t_3) - t_3||t_2 - t_3| \\ &\geqslant (1/C - C^2\varepsilon)|f_i(t_2,t_3) - t_3| \\ &\geqslant (1/2)|f_i(t_2,t_3) - t_3|. \end{aligned} \tag{3.11}$$

Since $f_i(t_2, t_3) \neq t_3$ for all $(t_2, t_3) \in U_i$, it follows from (3.9) and (3.11) that if $(t_2, t_3) \in U_i$ with $t_2 \neq t_3$,

$$\frac{\partial f_i}{\partial t_2}(t_2, t_3) \neq 0. \tag{3.12}$$

A similar calculation to (3.7) shows that for $t_2, t_3 \in [0, \varepsilon]$,

$$|(\gamma(t_2) - \gamma(t_3)) \cdot \gamma'(t_3)| \geqslant (1/C)|t_2 - t_3|. \tag{3.13}$$

Combining (3.10) with (3.13) shows that for $t_2 \neq t_3$ with $(t_2, t_3) \in U_i$,

$$\frac{\partial f_i}{\partial t_3}(t_2, t_3) \neq 0. \tag{3.14}$$

Now (3.12) and (3.14) imply that each function in the family $\{f_i\}$ satisfy the hypothesis of Theorem 1.2. Thus that theorem implies that for $\beta = 4/9$, each index i, and a generic element of $(E, \mu) \in \mathcal{X}_\beta$, the set E is Salem and for any distinct $t_1, t_2, t_3 \in E$, $f_i(t_1, t_2, t_3) \neq 0$. This means precisely that $|\gamma(t_1) - \gamma(t_2)| \neq |\gamma(t_2) - \gamma(t_3)|$ for any distinct $t_1, t_2, t_3 \in E$. Thus we conclude we can find a Salem set $E \subset [0, \varepsilon]$ with $\dim_{\mathbf{F}}(E) = 4/9$ such that $\gamma(E)$ does not contain the vertices of any isosceles triangles. $\square$

Theorem 1.1 can also be used to construct sets with a slightly smaller dimension avoiding isosceles triangles on a rougher family of curves. If we consider a Lipschitz function $\gamma : [0, 1] \to \mathbf{R}^{d-1}$, where there exists $M < 1$ with $|\gamma(t) - \gamma(s)| \leqslant M|t - s|$ for each $t, s \in [0, 1]$, then Theorem 3 of [1] guarantees that the set

$$Z = \left\{ (x_1, x_2, x_3) \in [0,1]^3 : \begin{array}{c} (x_1, \gamma(x_1)), (x_2, \gamma(x_2)), (x_3, \gamma(x_3)) \\ \text{form the vertices of an isosceles triangle.} \end{array} \right\}$$

has lower Minkowski dimension at most two. Thus Theorem 1.1 guarantees that there exists a Salem set $E \subset [0, 1]$ with $\dim_{\mathbf{F}}(E) = 2/5 = 0.4$ such that $\gamma(E)$ avoids all isosceles triangles. The main result of [4] constructs a set $E \subset [0, 1]$ with $\dim_{\mathbf{H}}(E) = 0.5$ such that $\gamma(E)$ avoids all isosceles triangles, but this set is not guaranteed to be Salem.

4 A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task and establish a set of tools for obtaining convergence in this metric space. In later sections we will fix a specific choice of β to avoid a particular pattern. But in this section we let β be an arbitrary fixed element of $(0, d]$. Our approach in this section is heavily influenced by [7]. However, we employ a Fréchet space construction instead of the Banach space construction used in [7], which enables us to use softer estimates in our arguments, with the disadvantage that we can obtain only Fourier dimension bounds in Theorems 1.1 and 1.2, rather than the explicit decay estimates determined in [7]:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

  $$d_{\mathbf{H}}(E, F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

  then $(\mathcal{E}, d_{\mathbf{H}})$ forms a complete metric space.

- We let $M(\beta/2)$ consist of the class of all finite Borel measures μ on $\mathbf{T}^d$ such that for each $\varepsilon \in (0, \beta/2]$, the quantity

  $$\|\mu\|_{M(\beta/2-\varepsilon)} = \sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon}$$

  is finite. Then $\|\cdot\|_{M(\beta/2-\varepsilon)}$ is a seminorm on $M(\beta/2)$, and the collection of all such seminorms for $\varepsilon \in (0, \beta/2]$ gives $M(\beta/2)$ the structure of a Frechét space. Under this topology, a sequence of probability measures $\{\mu_k\}$ converges to a probability measure μ in $M(\beta/2)$ if and only if for any $\varepsilon > 0$, $\lim_{k\to\infty} \|\mu_k - \mu\|_{M(\beta/2-\varepsilon)} = 0$.

We now let $\mathcal{X}_\beta$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(\beta/2)$, where μ is a probability measure such that $\mathrm{supp}(\mu) \subset E$. Then $\mathcal{X}_\beta$ is a closed subset of $\mathcal{E} \times M(\beta/2)$ under the product metric, and thus a complete metrizable space. We remark that for any $\varepsilon > 0$ and $(E, \mu) \in \mathcal{X}_\beta$,

$$\lim_{|\xi|\to\infty} |\xi|^{\beta/2-\varepsilon}|\widehat{\mu}(\xi)| = 0, \tag{4.1}$$

which follows because $\|\mu\|_{M(\beta/2-\varepsilon/2)}$ is finite. Thus $\dim_{\mathbf{F}}(\mu) \geqslant \beta$ for each $(E, \mu) \in \mathcal{X}_\beta$.

We will pursue Baire category arguments in the metric space $\mathcal{X}_\beta$ to establish the construction of Salem sets. Since the main difficulty in establishing these arguments is to show certain subsets of $\mathcal{X}_\beta$ are dense, the next lemma enables us to reduce these arguments to the study of measures with a smooth density function.

Lemma 4.1. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Proof. Consider $(E, \mu) \in \mathcal{X}_\beta$. For each $r \in (0, 1)$, consider the convolved measure $\mu_r = \mu * \phi_r$. Then $\mu_r \in C^\infty(\mathbf{T}^d)$. We claim that $\lim_{r\to 0}(E_r, \mu_r) = (E, \mu)$. Since $d_{\mathbf{H}}(E, E_r) \leqslant r$, we find that $\lim_{r\to 0} E_r = E$ holds with respect to the Hausdorff metric. Now fix $\varepsilon_1 \in (0, \beta/2]$ and $\varepsilon > 0$. For each $\xi \in \mathbf{Z}^d$, $|\widehat{\mu_r}(\xi)| = |\widehat{\phi_r}(\xi)||\widehat{\mu}(\xi)|$, so

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu_r}(\xi) - \widehat{\mu}(\xi)| = |\xi|^{\beta/2-\varepsilon_1}|\widehat{\phi_r}(\xi) - 1||\widehat{\mu}(\xi)|. \tag{4.2}$$

We control (4.2) using the fact that $|\widehat{\mu}(\xi)|$ is small when ξ is large, and $|\widehat{\phi_r}(\xi) - 1|$ is small when ξ is small. Since $(E, \mu) \in \mathcal{X}_\beta$, we can apply (4.1) to find $R > 0$ such that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{4.3}$$

Combining (4.2), (4.3), and (2.2), for $|\xi| \geqslant R$ we find that

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu}_r(\xi) - \widehat{\mu}(\xi)| \leqslant 2\varepsilon. \tag{4.4}$$

On the other hand, (2.3) shows that there exists $r_0 > 0$ such that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\phi}_r(\xi) - 1| \leqslant \varepsilon. \tag{4.5}$$

The (L^1, L^∞) bound for the Fourier transform implies that $|\widehat{\mu}(\xi)| \leqslant \mu(\mathbf{T}^d) = 1$, which combined with (4.5) gives that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu}_r(\xi) - \widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{4.6}$$

Putting together (4.4) and (4.6) shows that for $r \leqslant r_0$, $\|\mu_r - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant 2\varepsilon$. Since ε and ε_1 were arbitrary, $\lim_{r\to 0} \mu_r = \mu$, completing the proof. □

Remark 4.2. *Let*

$$\tilde{\mathcal{X}}_\beta = \{(E,\mu) \in \mathcal{X}_\beta : supp(\mu) = E\}.$$

Suppose $(E_0, \mu_0) \in \tilde{\mathcal{X}}_\beta$. *Then, in the proof above, one may let replace* E_r *with the set* $\tilde{E}_r = supp(\mu_r)$, *since* $d_{\mathbf{H}}(E_0, \tilde{E}_r) \leqslant r$. *This means that the set of pairs* $(E,\mu) \in \tilde{\mathcal{X}}_\beta$ *with* $\mu \in C^\infty(\mathbf{T}^d)$ *is dense in* $\tilde{\mathcal{X}}_\beta$.

The reason we must work with the metric space $\mathcal{X}_\beta$ rather than the smaller space $\tilde{\mathcal{X}}_\beta \subset \mathcal{X}_\beta$ is that $\tilde{\mathcal{X}}_\beta$ is not a closed subset of $\mathcal{X}_\beta$, and so is not a complete metric space, preventing the use of the Baire category theorem. However, as a consolation, quasi-all elements of $\mathcal{X}_\beta$ belong to $\tilde{\mathcal{X}}_\beta$, so that one can think of $\mathcal{X}_\beta$ and $\tilde{\mathcal{X}}_\beta$ as being equal 'generically'.

Lemma 4.3. *For quasi-all* $(E,\mu) \in \mathcal{X}_\beta$, $supp(\mu) = E$.

Proof. For each closed cube $Q \subset \mathbf{T}^d$, let

$$A(Q) = \{(E,\mu) \in \mathbf{T}^d : (E \cap Q) = \varnothing \text{ or } \mu(Q) > 0\}.$$

Then $A(Q)$ is an open set. If $\{Q_k\}$ is a sequence enumerating all cubes with rational corners in $\mathbf{T}^d$, then

$$\bigcap_{k=1}^{\infty} A(Q_k) = \{(E,\mu) \in \mathcal{X}_\beta : \operatorname{supp}(\mu) = E\}. \tag{4.7}$$

Thus it suffices to show that $A(Q)$ is dense in $\mathcal{X}_\beta$ for each closed cube Q. To do this, we fix $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$, $\varepsilon_1 \in (0, \beta/2]$, and $\varepsilon > 0$, and try and find $(E,\mu) \in A(Q)$ with $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu_0 - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Applying Lemma 4.1, we may assume without loss of generality that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Because $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$, we know $E \cap Q \neq \varnothing$ and $\mu(Q) = 0$. Find a smooth probability measure ν supported on $E_\varepsilon \cap Q$ and, for $t \in (0,1)$, define $\mu_t = (1-t)\mu_0 + t\nu$. Then $\operatorname{supp}(\mu_t) \subset E_\varepsilon$, so if we let $E = \operatorname{supp}(\nu) \cup \operatorname{supp}(\mu)$, then $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$. Clearly $(E, \mu_t) \in A(Q)$ for $t > 0$. And

$$\|\mu_t - \mu_0\|_{M(\beta/2-\varepsilon)} \leqslant t\left(\|\mu_0\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)}\right), \tag{4.8}$$

so if we choose $t \leqslant \varepsilon(\|\mu\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)})^{-1}$ we find $\|\mu_t - \mu\|_{M(\beta/2-\varepsilon)} \leqslant \varepsilon$. Since ε was arbitrary, we conclude $A(Q)$ is dense in $\mathcal{X}_\beta$. □

Combining Lemma 4.3 with Remark 4.2 gives the following simple corollary.

Corollary 4.4. *The family of (E, μ) with supp$(\mu) = E$ and $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Our main way of constructing approximations to $(E_0, \mu_0) \in \mathcal{X}_\beta$ is to multiply μ_0 by a smooth function $f \in C^\infty(\mathbf{T}^d)$. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which contribute to the formation of a pattern we are trying to avoid. As long as μ_0 is appropriately smooth, and the Fourier transform of f decays appropriately quickly, the next lemma shows that $f\mu_0 \approx \mu_0$.

Lemma 4.5. *Consider a finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function $f \in C^\infty(\mathbf{T}^d)$. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(\beta/2)} \lesssim_d \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}.$$

Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu_0}$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\xi|^{\beta/2} |\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)| = |\xi|^{\beta/2} \left| \sum_{\eta \neq \xi} \widehat{f}(\xi - \eta) \widehat{\mu_0}(\eta) \right|. \tag{4.9}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi - \eta| \leqslant 2|\xi|$, so

$$|\xi|^{\beta/2} |\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)} |\xi|^{\beta/2} |\xi - \eta|^{-\beta/2} \leqslant 2^{\beta/2} \|f\|_{M(\beta/2)} \lesssim_d \|f\|_{M(\beta/2)}. \tag{4.10}$$

Thus the bound (4.10) implies

$$\begin{aligned} |\xi|^{\beta/2} \left| \sum_{0 \leqslant |\eta| \leqslant |\xi|/2} \widehat{f}(\xi - \eta) \widehat{\mu_0}(\eta) \right| &\lesssim_{\mu_0, d} \|\mu_0\|_{M(d+1)} \|f\|_{M(\beta/2)} \left(1 + \sum_{0 < |\eta| \leqslant |\xi|/2} \frac{1}{|\eta|^{d+1}} \right) \\ &\lesssim_d \|\mu_0\|_{M(d+1)} \|f\|_{M(\beta/2)} \leqslant \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}. \end{aligned} \tag{4.11}$$

On the other hand, for all $\eta \neq \xi$,

$$|\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)} |\xi - \eta|^{-\beta} \leqslant \|f\|_{M(\beta/2)}. \tag{4.12}$$

Thus we calculate that

$$\begin{aligned} |\xi|^{\beta/2} \left| \sum_{\substack{|\eta| > |\xi|/2 \\ \eta \neq \xi}} \widehat{f}(\xi - \eta) \widehat{\mu_0}(\eta) \right| &\lesssim_{d, \mu_0} \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)} \cdot |\xi|^{\beta/2} \sum_{|\eta| > |\xi|/2} \frac{1}{|\eta|^{3d/2}} \\ &\lesssim_d \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}. \end{aligned} \tag{4.13}$$

Combining (4.9), (4.11) and (4.13) completes the proof. □

Remark 4.6. *Lemma 4.5 implies that if $\|\mu_0\|_{M(3d/2)} < \infty$, then*

$$|\mu_0(\mathbf{T}^d) - \mu(\mathbf{T}^d)| = |\widehat{\mu_0}(0) - \widehat{\mu}(0)| \lesssim_d \|\mu_0 - \mu\|_{M(0)} \|f\|_{M(0)} \lesssim_{d, \mu_0} \|f\|_{M(0)}.$$

The bound in Lemma 4.5, if $\|f\|_{M(\beta/2)}$ is taken appropriately small, also implies that the Hausdorff distance between the supports of μ and μ_0 is small.

Lemma 4.7. *Fix a probability measure $\mu_0 \in C^\infty(\mathbf{T}^d)$. For any $\varepsilon > 0$, there exists $\delta > 0$ depending on μ_0, ε, and d, such that if $\mu \in C^\infty(\mathbf{T}^d)$, $supp(\mu) \subset supp(\mu_0)$, and $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, then $d_{\mathbf{H}}(supp(\mu), supp(\mu_0)) \leqslant \varepsilon$.*

Proof. Consider any cover of $\text{supp}(\mu_0)$ by a family of radius $\varepsilon/3$ balls $\{B_1, \dots, B_N\}$, and for each $i \in \{1, \dots, N\}$, consider a smooth function $f_i \in C_c^\infty(B_i)$ such that there is $s > 0$ with

$$\int f_i(x) d\mu_0(x) \geqslant s. \tag{4.14}$$

for each $i \in \{1, \dots, N\}$. Fix $A > 0$ with

$$\sum_{\xi \neq 0} |\widehat{f}_i(\xi)| \leqslant A \tag{4.15}$$

for all $i \in \{1, \dots, N\}$ as well. Set $\delta = s/2A$. If $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, we apply Plancherel's theorem together with (4.14) and (4.15) to conclude that

$$\begin{aligned} \left| \int f_i(x) d\mu(x)\; dx - \int f_i(x) d\mu_0(x) \right| &= \left| \sum_{\xi \in \mathbf{Z}^d} \widehat{f}_i(\xi) \left(\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi) \right) \right| \\ &\leqslant A \|\mu_0 - \mu\|_{M(\beta/2)} \leqslant s/2. \end{aligned} \tag{4.16}$$

Thus we conclude from (4.14) and (4.16) that

$$\int f_i(x) d\mu(x)\; dx \geqslant \int f_i(x) d\mu_0(x) - s/2 \geqslant s/2 > 0. \tag{4.17}$$

Since equation (4.17) holds for each $i \in \{1, \dots, N\}$, the support of μ intersects every ball in $\{B_1, \dots, B_N\}$. Combined with the assumption that $\text{supp}(\mu) \subset \text{supp}(\mu_0)$, this implies that $d_{\mathbf{H}}(\mu_0, \mu) \leqslant \varepsilon$. $\square$

To obtain a smooth function f to which we can apply Lemmas 4.5 and 4.7, we take a measure η, which is a linear combination of Dirac deltas, and set $f = \eta * \phi_r$. To obtain the appropriate control on $\widehat{f}$, it suffices to have a decay bound for $\widehat{\eta}$ for $|\xi| \leqslant 1/r$, and a weaker bound for $|\xi|$ slightly bigger than $1/r$, which is required before we can take complete advantage of the rapid decay of the Fourier transform of ϕ_r for $|\xi| \geqslant 1/r$.

Lemma 4.8. *Fix $C > 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, with $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ depending on all these quantities, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad \textit{for } 0 < |\xi| \leqslant (1/r) \tag{4.18}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2} \log(1/r)^{1/2} \qquad \textit{for } (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{4.19}$$

if we define $f(x) = (\eta * \phi_r)(x)$, then $\|f\|_{M(\beta/2 - \varepsilon_2)} \leqslant 2\varepsilon$.

Proof. For each $\xi \in \mathbf{Z}^d$,

$$\widehat{f}(\xi) = \widehat{\eta}(\xi)\widehat{\phi}_r(\xi). \tag{4.20}$$

For $|\xi| \leqslant 1/r$ we combine (4.18), (4.20) and (2.2) to conclude that

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{4.21}$$

If $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$, (2.4) implies $|\widehat{\phi}_r(\xi)| \lesssim_\beta r^{-\beta/2}|\xi|^{-\beta/2}$, which together with (4.19), (4.20), and (2.2), show that for $r \leqslant r_1$,

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_\beta \left(Cr^{\beta/2}\log(1/r)^{1/2}\right)\left(r^{-\beta/2}|\xi|^{-\beta/2}\right) \\ &\leqslant C\log(1/r)^{1/2} \cdot |\xi|^{-\beta/2} \\ &\leqslant Cr^{\varepsilon_2}\log(1/r)^{1/2} \cdot |\xi|^{\varepsilon_2 - \beta/2}. \end{aligned} \tag{4.22}$$

Since $Cr^{\varepsilon_2}\log(1/r)^{1/2} \to 0$ as $r \to 0$, we conclude from (4.22) that there exists $r_1 > 0$ such that for $r \leqslant r_1$ and $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{4.23}$$

If $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$, we apply (2.4) for $T \geqslant \beta/2$ together with the bound $\|\widehat{\eta}\|_{L^\infty(\mathbf{Z}^d)} = 1$, which follows because η is a probability measure, to conclude that

$$\begin{aligned} |\widehat{f}(\xi)| &\lesssim_T r^{-T}|\xi|^{-T} \\ &= r^{-T}|\xi|^{\beta/2 - T} \cdot |\xi|^{-\beta/2} \\ &\leqslant r^{-T}(1/r)^{(\beta/2 - T)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2} \\ &= r^{\varepsilon_1 T - (\beta/2)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2}. \end{aligned} \tag{4.24}$$

If we choose $T > (\beta/2)(1 + 1/\varepsilon_1)$, then as $r \to 0$, $r^{\varepsilon_1 T - (\beta/2)(1+\varepsilon_1)} \to 0$. Thus we conclude from (4.24) that there exists $r_2 > 0$ satisfying such that for $0 < r \leqslant r_2$ and $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{-\beta/2} \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{4.25}$$

All that remains is to combine (4.21), (4.23), and (4.25), defining $r_0 = \min(r_1, r_2)$. $\square$

Corollary 4.9. *Fix $C > 0$, a smooth probability measure $\mu_0 \in C^\infty(\mathbf{T}^d)$, and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, such that $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ and $\delta > 0$, depending on all these parameters, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \delta \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad \textit{for } 0 < |\xi| \leqslant (1/r) \tag{4.26}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2}\log(1/r)^{1/2} \qquad \textit{for } (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{4.27}$$

if we define $f(x) = (\eta * \phi_r)(x)$, and a probability measure

$$\mu = \frac{f\mu_0}{(f\mu_0)(\mathbf{T}^d)}$$

then $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_2)} \leqslant \varepsilon$.

Proof. Applying Lemma 4.5, there exists a constant $C' > 0$ depending on d and μ_0 such that for any $f \in C^\infty(\mathbf{T}^d)$,

$$\|f\mu_0 - \mu_0\|_{M(0)} \leqslant C'\|f\|_{M(0)} \quad \text{and} \quad \|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant C'\|f\|_{M(\beta/2-\varepsilon)}. \tag{4.28}$$

Let

$$\delta = \frac{1}{C'} \cdot \min\left(\frac{\varepsilon}{2}, \frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right).$$

If we apply Lemma 4.8, then there exists r_0 such that for $r \leqslant r_0$,

$$\|f\|_{M(0)}, \|f\|_{M(\beta/2-\varepsilon_2)} \leqslant \delta \tag{4.29}$$

In light of (4.28), (4.29) implies that

$$\|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon/2, \tag{4.30}$$

and

$$\|f\mu_0 - \mu_0\|_{M(0)} \leqslant \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right). \tag{4.31}$$

As discussed in Remark 4.6, (4.31) implies that

$$\left|1 - (f\mu_0)(\mathbf{T}^d)\right| \leqslant \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right). \tag{4.32}$$

But now (4.30) and (4.32) show that

$$\begin{aligned}\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} &\leqslant \|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon)} + \|\mu - f\mu_0\|_{M(\beta/2-\varepsilon)} \\ &\leqslant (\varepsilon/2) + \left(1 - \frac{1}{(f\mu_0)(\mathbf{T}^d)}\right)\|\mu_0\|_{M(\beta/2-\varepsilon)} \\ &\leqslant (\varepsilon/2) + (\varepsilon/2) \leqslant \varepsilon. \end{aligned}$$

□

A useful technique to find functions with small Fourier coefficients is to consider a family of random functions composed from many independent random variables.

Lemma 4.10. *Fix a positive integer K. Let $X_1, \dots, X_K$ be independent random variables on $\mathbf{T}^d$, such that for each nonzero $\xi \in \mathbf{Z}^d$,*

$$\sum_{k=1}^{K} \mathbf{E}\left(e^{2\pi i \xi \cdot X_k}\right) = 0. \tag{4.33}$$

Set

$$\eta(x) = \frac{1}{K}\sum_{k=1}^{K} \delta_{X_k}(x)$$

and

$$B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| \leqslant K^{1+100/\beta}\}.$$

Then there exists a constant C depending on β and d, such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Remark 4.11. *In particular,* (4.33) *holds if the* $\{X_i\}$ *are uniformly distributed on* $\mathbf{T}^d$*.*

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \dots, K\}$, consider the random variable

$$Y(\xi, k) = K^{-1} e^{2\pi i (\xi \cdot X_k)}.$$

Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} Y(\xi, k) = \widehat{\eta}(\xi). \tag{4.34}$$

We also note that for each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \dots, K\}$,

$$|Y(\xi, k)| = K^{-1}. \tag{4.35}$$

Moreover,

$$\sum_{k=1}^{K} \mathbf{E}(Y(\xi, k)) = 0. \tag{4.36}$$

Since the family of random variables $\{Y(\xi, k)\}$ is independent for a fixed ξ, we can apply Hoeffding's inequality together with (4.34) and (4.35) to conclude that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\eta}(\xi)| \geqslant t\right) \leqslant 2e^{-Kt^2/2}. \tag{4.37}$$

A union bound obtained by applying (4.37) over all $|\xi| \leqslant K^{1+100/\beta}$ shows that there exists a constant $C \geqslant 10$ depending on d and β such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant t\right) \leqslant \exp\left(C\log(K) - \frac{5Kt^2}{C}\right). \tag{4.38}$$

But then, setting $t = CK^{-1/2}\log(K)^{1/2}$ in (4.38) completes the proof. □

Let us now consider the consequences of the square-root cancellation bound in Lemma 4.10. Given η as in that lemma, consider the smooth function $f = \eta * \phi_r$. The support of f consists of K radius r balls. Provided $K \approx r^{-\beta}$, the support therefore behaves like an r-thickening of a set with Minkowski dimension β. If

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2}\log(K)^{1/2}, \tag{4.39}$$

as is guaranteed with high probability in Lemma 4.10, we actually find that the support of f also behaves like an r-thickening of a set with *Fourier dimension* β as well, i.e. the hypotheses of Lemma 4.8 apply to f. This is one reason why constructions involving some kind of square-root cancellation are often a viable tactic to construct Salem sets, with random constructions involving many independent random variables being an important example.

Lemma 4.12. *Fix* $\varepsilon, \varepsilon_1 > 0$ *and* $C > 0$ *with* $\varepsilon_1 \leqslant \beta/2$*. Then there exists* $r_0 > 0$ *and* $C' > 0$ *depending on these quantities such that for* $r \leqslant r_0$*, if* $K \geqslant (1/C)r^{-\beta}$ *and* η *is a Borel probability measure with*

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2}\log(K)^{1/2}, \tag{4.40}$$

then

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon|\xi|^{\varepsilon_1-\beta/2} \qquad \textit{for } |\xi| \leqslant (1/r)$$

and

$$|\widehat{\eta}(\xi)| \leqslant C' r^{\beta/2}\log(1/r)^{1/2} \qquad \textit{for } (1/r) \leqslant |\xi| \leqslant (1/r)^2.$$

Proof. The function $x \mapsto x^{-1/2}\log(x)^{1/2}$ is decreasing for sufficiently large x. Thus if r_0 is chosen appropriately small, then for $|\xi| \leqslant (1/r)$ we find

$$\begin{aligned} |\widehat{\eta}(\xi)| &\leqslant CK^{-1/2}\log(K)^{1/2} \\ &\leqslant C^{3/2}r^{\beta/2}\log(r^{-\beta}/C) \\ &\leqslant \big(C^{3/2}r^{\varepsilon_1}\log(r^{-\beta}/C)\big)\cdot r^{\beta/2-\varepsilon_1} \\ &\leqslant \big(C^{3/2}r^{\varepsilon_1}\log(r^{-\beta}/C)\big)\cdot |\xi|^{\varepsilon_1-\beta/2}. \end{aligned} \tag{4.41}$$

If r_0 is appropriately small, then

$$(1/r)^2 \leqslant C^{2/\beta}K^{2/\beta} \leqslant K^{100/\beta} \tag{4.42}$$

Thus for $(1/r) \leqslant |\xi| \leqslant (1/r)^2$, $\xi \in B$, and so we find there is $C' > 0$ such that

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2} \leqslant Cr^{\beta/2}\log((1/C)r^{-\beta})^{1/2} \leqslant C'r^{\beta/2}\log(1/r)^{1/2}. \tag{4.43}$$

Together, (4.41) and (4.43) give the conclusions of the Lemma. □

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}_\beta$ to be as low as possible. For each $(E,\mu) \in \mathcal{X}_\beta$, the condition that $\mu \in M(\beta/2)$ implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus it is natural to expect that for quasi-all $(E,\mu) \in M(\beta/2)$, the set E has both Hausdorff dimension and Fourier dimension equal to β, i.e. E is a Salem set of dimension β. Let us use the calculations we have developed in this section to prove this statement.

Lemma 4.13. *For quasi-all $(E,\mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since when $\beta = d$, E is a Salem set for any $(E,\mu) \in \mathcal{X}_\beta$, and thus the result is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that for quasi-all $(E,\mu) \in \mathcal{X}_\beta$, E has Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, we let

$$A(\alpha,\delta,s) = \{(E,\mu) \in \mathcal{X} : H^\alpha_\delta(E) < s\}.$$

Then $A(\alpha,\delta,s)$ is an open subset of $\mathcal{X}_\beta$, and

$$\bigcap_{n=1}^\infty \bigcap_{m=1}^\infty \bigcap_{k=1}^\infty A(\beta+1/n, 1/m, 1/k) \tag{4.44}$$

is precisely the family of $(E,\mu) \in \mathcal{X}_\beta$ such that E has Hausdorff dimension at most β. Thus it suffices to show that $A(\alpha,\delta,s)$ is dense in $\mathcal{X}_\beta$ for $\alpha \in (\beta,d)$ and $\delta, s > 0$. Fix $(E_0,\mu_0) \in \mathcal{X}_\beta$,

$\alpha \in (\beta, d)$, $\delta > 0$, $s > 0$, and $\varepsilon_1 > 0$. We aim to show that for each $\varepsilon > 0$, there exists $(E, \mu) \in A(\alpha, \delta, s)$ such that $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Without loss of generality, in light of Lemma 4.1, we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a small value r, and then find an integer K such that $r^{-\beta} \leqslant K \leqslant r^{-\beta}+1$. Lemma 4.10 shows that there exists a constant C depending on β and d, as well as K points $x_1, \dots, x_K \in \mathbf{T}^d$ such that if

$$\eta(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{X_k}(x),$$

then for each $|\xi| \leqslant K^{1+100/\beta}$,

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \tag{4.45}$$

Equation (4.45) shows η satisfies the hypotheses of Lemma 4.12, and the result of Lemma 4.12 can then be fed into Corollary 4.9 to conclude that for any $\varepsilon_1 > 0$, there exists $r_1 > 0$ such that if $r \leqslant r_1$, if

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k=1}^{K} \phi_r(x - x_k) \right) \mu_0(x),$$

and if

$$\mu = \mu_1/\mu_1(\mathbf{T}^d),$$

then

$$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \min(\varepsilon, \varepsilon_1). \tag{4.46}$$

If ε_1 is chosen appropriately, Lemma 4.7 implies

$$d_{\mathbf{H}}(\operatorname{supp}(\mu), \operatorname{supp}(\mu_0)) \leqslant \varepsilon. \tag{4.47}$$

Note that μ is supported on K balls of radius r. Thus for $r \leqslant \delta$,

$$H^\alpha_\delta(\operatorname{supp}(\mu)) \leqslant Kr^\alpha \leqslant (r^{-\beta} + 1) r^\alpha = r^{\alpha-\beta} + r^\alpha. \tag{4.48}$$

Since $\alpha > \beta$, (4.48) implies that there is $r_2 > 0$ depending on α, β, and s such that for $r \leqslant r_2$, $H^\alpha_\delta(\operatorname{supp}(\mu)) \leqslant s$. This means $(\operatorname{supp}(\mu), \mu) \in A(\alpha, \delta, s)$, and since $\varepsilon > 0$ was arbitrary, we see that we have proved what was required. $\square$

This concludes the setup to the proof of Theorems 1.1 and 1.2. All that remains is to show that quasi-all elements of $\mathcal{X}_\beta$ avoid the given set Z for a suitable parameter β; it then follows from Lemma 4.13 that quasi-all elements of $\mathcal{X}_\beta$ are *Salem* and avoid the given set Z. The advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales of the problem at once, which allows us to focus solely on the quantitative question at the heart of the problem.

5 Random Avoiding Sets for Rough Patterns

We begin by proving Theorem 1.1, which requires simpler calculations than Theorem 1.2. In the last section, our results held for an arbitrary $\beta \in (0, d]$. But in this section, we assume

$$\beta \leqslant \min\left(d, \frac{dn - \alpha}{n - 1/2}\right),$$

which will enable us to generically avoid the pattern Z described in Theorem 1.1. The construction here is very similar to the construction in [1], albeit in a Baire category setting, and with modified parameters to ensure a Fourier dimension bound rather than just a Hausdorff dimension bound.

Lemma 5.1. *Let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.*

Proof. The set $Z \subset \mathbf{R}^{dn}$ is the countable union of sets with lower Minkowski dimension at most α. For a closed set $W \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, and $s > 0$, consider the set

$$B(W, s) = \left\{ (E, \mu) \in \mathcal{X}_\beta : \begin{array}{c} \text{for all } x_1, \dots, x_n \in E \text{ such that} \\ |x_i - x_j| \geqslant s \text{ for } i \neq j,\ (x_1, \dots, x_n) \notin W \end{array} \right\},$$

which is open in $\mathcal{X}_\beta$. If Z is a countable union of closed sets $\{Z_k\}$ with lower Minkowski at most α, then clearly the set

$$\bigcap_{k=1}^\infty \bigcap_{n=1}^\infty B(Z_k, 1/n)$$

consists of the family of sets (E, μ) such that for distinct $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$. Thus it suffices to show that $B(W, s)$ is dense in $\mathcal{X}_\beta$ for any $s > 0$, and any closed set W with lower Minkowski dimension at most α.

Let us begin by fixing a set $W \subset \mathbf{T}^{dn}$ and a pair $(E_0, \mu_0) \in \mathcal{X}_\beta$. We will show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, we can find $(E, \mu) \in B(W, s)$ with $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. We may assume by Corollary 4.4 that $\operatorname{supp}(\mu_0) = E$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$. Since W has lower Minkowski dimension at most α, we can find arbitrarily small $r \in (0, 1)$ such that

$$|W_r| \leqslant r^{dn - \alpha - \varepsilon_1/4}. \tag{5.1}$$

Assume also that r is small enough that we can find an integer $K \geqslant 10$ with

$$r^{-(\beta - \varepsilon_1/2)} \leqslant K \leqslant r^{-(\beta - \varepsilon_1/2)} + 1. \tag{5.2}$$

Let $X_1, \dots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $k_1, \dots, k_n \in \{1, \dots, K\}$, the random vector $X_k = (X_{k_1}, \dots, X_{k_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (5.1) and (5.2) imply that

$$\mathbf{P}(d(X_k, W) \leqslant r) \leqslant |W_r| \leqslant r^{dn - \alpha - \varepsilon_1/4} \lesssim_{d,n,\beta} K^{\frac{-(dn - \alpha - \varepsilon_1/4)}{\beta - \varepsilon_1/2}} \leqslant K^{-(n-1/2)}, \tag{5.3}$$

where we used the calculation

$$\begin{aligned}\frac{dn-\alpha-\varepsilon_1/4}{\beta-\varepsilon_1/2} &= \frac{dn-\alpha}{\beta} + \frac{[(dn-\alpha)/\beta](\varepsilon_1/2)-\varepsilon_1/4}{\beta-\varepsilon_1/2} \\ &\geqslant \frac{dn-\alpha}{\beta} + \frac{(n-1/2)(\varepsilon_1/2)-\varepsilon_1/4}{\beta-\varepsilon_1/2} \\ &\geqslant \frac{dn-\alpha}{\beta} \geqslant n-1/2.\end{aligned} \tag{5.4}$$

If M_0 denotes the number of indices i such that $d(X_i, W) \leqslant r$, then by linearity of expectation we conclude from (5.3) that there is a constant C depending only on d, n, and β such that

$$\mathbf{E}(M_0) \leqslant (C/10)K^{1/2}. \tag{5.5}$$

Applying Markov's inequality to (5.5), we conclude that

$$\mathbf{P}(M_0 \geqslant CK^{1/2}) \leqslant 1/10. \tag{5.6}$$

Taking a union bound to (5.6) and the results of Lemma 4.10, we conclude that there exists K points $x_1, \dots, x_K \in \mathbf{T}^d$ and a constant C depending only on d, n, and β such that the following two statements hold:

(1) Let S be the set of indices $k_1 \in \{1, \dots, K\}$ with the property that we can find distinct indices $k_2, \dots, k_n \in \{1, \dots, K\}$ such that if $X = (X_{k_1}, \dots, X_{k_n})$, then $d(X, W) \leqslant r$. Then

$$\#(S) \leqslant CK^{1/2}. \tag{5.7}$$

(2) If we define

$$\eta_0(x) = \frac{1}{K}\sum_{k=1}^{K} \delta_{x_k}(x)$$

then for $0 < |\xi| \leqslant K^{100/\beta}$,

$$|\widehat{\eta_0}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2}. \tag{5.8}$$

Thus (5.7) and (5.8) imply that if

$$\eta_1(x) = \sum_{k \notin S} \delta_{x_k}(x),$$

then for each $|\xi| \leqslant K^{1/\beta_0+1}$,

$$|\widehat{\eta_1}(\xi)| \leqslant 2CK^{-1/2}\log(K). \tag{5.9}$$

Since $K \geqslant r^{\varepsilon_1/2-\beta}$, we can apply Lemma 4.12 to η_1, and then apply Corollary 4.9 to the result of Lemma 4.12 to conclude that for any $\delta > 0$, there is $r_0 > 0$ such that if $r \leqslant r_0$ and we define

$$\mu'(x) = \left(\sum_{k \notin S} \phi_{(r/2n^{1/2})}(x - X_k)\right)\mu_0(x),$$

and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, then

$$\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{5.10}$$

Choosing δ in accordance with Lemma 4.7, (5.10) also implies that $d_{\mathbf{H}}(\mathrm{supp}(\mu), \mathrm{supp}(\mu_0)) \leqslant \varepsilon$. Since ε and ε_1 are arbitrary, our proof would therefore be complete if we could show that $(\mathrm{supp}(\mu), \mu) \in B(W, s)$.

Consider n points $y_1, \dots, y_n \in \mathrm{supp}(\mu)$, with $|y_i - y_j| \geqslant r$ for any two indices $i \neq j$. We can therefore find distinct indices $k_1, \dots, k_n \in \{1, \dots, K\}$ such that for each $i \in \{1, \dots, n\}$, $|x_{k_i} - y_i| \leqslant (n^{-1/2}/2) \cdot r$, which means if we set $x = (x_{k_1}, \dots, x_{k_n})$ and $y = (y_1, \dots, y_n)$, then

$$|x - y| \leqslant (r/2). \tag{5.11}$$

Since $i_1 \notin S$, $d(x, W) \geqslant r$, which combined with (5.11) implies

$$d(y, W) \geqslant d(x, W) - |x - y| \geqslant r/2. \tag{5.12}$$

Thus in particular we conclude $y \notin W$, which shows $(E, \mu) \in B(W, s)$. □

The Baire category theorem, applied to Lemma 5.1, completes the proof of Theorem 1.1. Before we move onto the proof of Theorem 1.2, let us discuss where the loss in Theorem 1.1 occurs in our proof, as compared to the Hausdorff dimension bound of [1]. In the proof of Lemma 5.1, in order to obtain the bound (5.9), we were forced to choose the parameter r such that $\#(S) \leqslant K^{1/2}$, so that the trivial bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \leqslant \#(S) \tag{5.13}$$

obtained by the triangle inequality is viable to control the Fourier dimension of the resulting pair (E, μ). On the other hand, if we were able to justify that with high probability, we could obtain a square root cancellation bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \lesssim \#(S)^{1/2}, \tag{5.14}$$

then we would only need to choose the parameter r such that $\#(S) \lesssim K$, which leads to a set with larger Fourier dimension, matching with the Hausdorff dimension bound obtained in [1]. Under stronger assumptions on the pattern we are trying to avoid, which form the hypotheses of Theorem 1.2, we are able to obtain some such square root cancellation, though with an additional term that decays fast as $|\xi| \to \infty$, which enables us to obtain the improved dimension bound in the conclusions of that theorem.

6 Concentration Bounds for Smooth Surfaces

In this section we prove Theorem 1.2 using some more robust probability concentration calculations. We set

$$\beta \leqslant \begin{cases} d & : n = 2 \\ d/(n - 3/4) & : n \geqslant 3 \end{cases}.$$

For this choice of β we will be able to generically avoid the pattern Z described in Theorem 1.2.

Lemma 6.1. *Let $Z \subset \mathbf{T}^{dn}$ satisfy the hypothesis of Theorem 1.2. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, and for any distinct points $x_1, \dots, x_n \in E$, $(x_1, \dots, x_n) \notin Z$.*

Proof. Fix a set

$$W = \{(x_1, \dots, x_n) \in U \times V : x_n = f(x_1, \dots, x_{n-1})\}$$

as in the statement of Theorem 1.2, where $f \in C^\infty(V)$. Given any family of disjoint, closed cubes $R_1, \dots, R_n \subset \mathbf{T}^d$ such that $(R_1 \times \cdots \times R_n) \cap W$ is a closed set, we let

$$H(W; R_1, \dots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : (R_1 \times \cdots \times R_n) \cap W \cap E^n = \varnothing\}. \tag{6.1}$$

Then $H(W; R_1, \dots, R_n)$ is an open subset of $\mathcal{X}_\beta$. For the purpose of a Baire category argument, the Lemma we are proving will follow by showing $H(W; R_1, \dots, R_n)$ is dense in $\mathcal{X}_\beta$ for any family of disjoint cubes $\{R_1, \dots, R_n\}$, each having common sidelength s for some $s > 0$, such that if $Q_1 = 2R_1, \dots, Q_n = 2R_n$, then $Q_1 \subset U$ and $Q_2 \times \cdots \times Q_n \subset V$, and $d(R_i, R_j) \geqslant 10s$ for each $i \neq j$. Since f is smooth, we can fix a constant $L \geqslant 0$ such that for any $x, y \in Q_1 \times \cdots \times Q_{n-1}$,

$$|f(x) - f(y)| \leqslant L|x - y|.$$

As with the proof of Lemma 5.1, we fix $(E_0, \mu_0) \in \mathcal{X}_\beta$ with $\mathrm{supp}(\mu_0) = E_0$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$, and show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, there is $(E, \mu) \in H(W; Q_1, \dots, Q_n)$ with

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon, \tag{6.2}$$

from which it follows that $H(W; R_1, \dots, R_n)$ is dense.

Fix a family of non-negative bump functions $\psi_0, \psi_1, \dots, \psi_n \in C^\infty(\mathbf{T}^d)$, such that for $i \in \{1, \dots, n\}$, $\psi_i(x) = 1$ for $x \in R_i$, $\psi_i(x) = 0$ for $x \notin Q_i$, and $\psi_0(x) + \cdots + \psi_n(x) = 1$ for $x \in \mathbf{T}^d$. For $i \in \{0, \dots, n\}$, let $A_i = \int \psi_i(x)\, dx$ denote the total mass of ψ_i. Now fix a large integer $K > 0$, and consider a family of independent random variables $\{X_i(k) : 0 \leqslant i \leqslant n, 1 \leqslant k \leqslant K\}$, where the random variable $X_i(k)$ is chosen with probability density function $A_i^{-1}\psi_i$. Choose $r > 0$ such that $K = r^{\varepsilon_1/2 - \beta}$, and then let S be the set of indices $k_n \in \{1, \dots, K\}$ such that there are indices $k_1, \dots, k_{n-1} \in \{1, \dots, K\}$ with the property that

$$|X_n(k_n) - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leqslant (L+1)r. \tag{6.3}$$

A simple argument following from (6.3) shows that if $k_n \notin S$, then for any $k_1, \dots, k_{n-1} \in \{1, \dots, K\}$, if $X = (X_1(k_1), \dots, X_n(k_n))$, then $d(X, W) \geqslant r$. Thus if we define

$$\eta = \frac{1}{K} \cdot \left(\sum_{i \in \{0, \dots, n-1\}} A_i \sum_{k=1}^K \delta_{X_i(k)} + A_1 \sum_{k \notin S} \delta_{X_n(k)} \right),$$

if we define $\mu' = (\eta * \phi_{(r/2n^{1/2})}) \cdot \mu_0$, then define $\mu = \mu'/\mu'(\mathbf{T}^d)$ and finally set $E = \mathrm{supp}(\mu)$, then $(E, \mu) \in H(W; R_1, \dots, R_n)$. The remainder of our argument consists of obtaining control on the exponential sum

$$\widehat{\eta}(\xi) = \frac{1}{K} \cdot \sum_{i \in \{0, 2, \dots, n\}} A_i \sum_{k=1}^K e^{-2\pi i \xi \cdot X_i(k)} + \frac{1}{K} \cdot A_1 \sum_{k \notin S} e^{-2\pi i \xi \cdot X_1(k)}, \tag{6.4}$$

for nonzero $\xi \in \mathbf{Z}^d$, so that we may apply Corollary 4.9 to bound $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)}$.

To analyze $\widehat{\eta}$, introduce the measures

$$\nu = \frac{1}{K} \sum_{i=0}^{n} A_i \sum_{k=1}^{K} \delta_{X_i(k)}$$

and

$$\sigma = \nu - \eta = \frac{A_n}{K} \sum_{k \in S} \delta_{X_n(k)}.$$

Obtaining a bound on $\widehat{\nu}$ is simple. For non-zero $\xi \in \mathbf{Z}^d$,

$$\mathbf{E}(\widehat{\nu}(\xi)) = \sum_{i=0}^{n} \int \psi_i(x) e^{2\pi i \xi \cdot x}\, dx = \int_{\mathbf{T}^d} e^{2\pi i \xi \cdot x}\, dx = 0. \tag{6.5}$$

Applying Lemma 4.10, we conclude that if $B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| < K^{100/\beta}\}$, then there is $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\nu}\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10. \tag{6.6}$$

Analyzing $\widehat{\sigma}(\xi)$ requires a more subtle concentration bound, which we delegate to a series of lemmas following this proof. In Lemma 6.2, we will show that

$$\mathbf{P}\left(\|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10. \tag{6.7}$$

In Lemma 6.4 we show that for any $\delta > 0$, there exists $r_1 > 0$ such that for $r \leqslant r_1$ and any nonzero $\xi \in \mathbf{Z}^d$,

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \leqslant \delta|\xi|^{-\beta/2} + O(K^{-1/2}). \tag{6.8}$$

Combining (6.6), (6.7), and (6.8), we conclude that there exists a choice of random variables $\{X_i(k)\}$ such that for any $\xi \in B$,

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2}\log(K)^{1/2} + \delta|\xi|^{-\beta/2}. \tag{6.9}$$

Applying Lemma 4.12 and then Corollary 4.9, we conclude that if δ and r_1 is chosen appropriately small, then $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$, which completes the proof. $\square$

All that remains is to prove (6.7) and (6.8).

Lemma 6.2. *Let σ be the random measure described in Lemma 6.1. Then*

$$\mathbf{P}\left(\|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10$$

for some universal constant $C > 0$.

Remark 6.3. *Before we begin the proof of this lemma, let us describe the idea of the proof. As a random quantity, $\widehat{\sigma}(\xi)$ is a function of the independent random quantities $\{X_i(k)\}$, and so McDiarmid's inequality presents itself as a useful concentration bound. However, a naive application of McDiarmid's inequality fails here, because changing a single random variable $X_i(k)$ for $1 \leqslant i \leqslant n-1$ while fixing all other random variables can change $\widehat{\sigma}(\xi)$ by as much as*

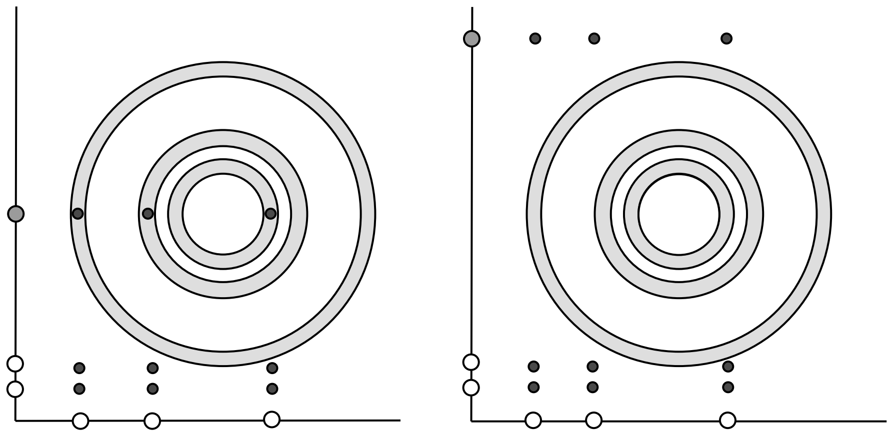

Figure 1: The two diagrams displayed indicate two instances of the process of defining the set S for the function $f(x_1, x_2) = x_1^2 + x_2^2$. Here $K = 3$, $n = 3$, the values on the x-axis represent the values $X_1(1), X_1(2)$, and $X_1(3)$, the values on the y-axis represent the values $X_2(1), X_2(2)$, and $X_2(3)$, the dark points represent the family of all pairs $(X_1(k_1), X_2(k_2))$, and the annuli represent the r-neighborhoods of $f^{-1}(X_3(1)), f^{-1}(X_3(2))$, and $f^{-1}(X_3(3))$. In the left diagram, $S = \{1, 2, 3\}$, whereas in the right diagram $S = \varnothing$. Thus S can be completely changed simply by adjusting the position of the shaded point on the y-axis (adjusting one of the variables $X_2(k)$), which completely changes the value of $\widehat{\sigma}$. Nonetheless, this only occurs if the other values are arranged in a highly particular manner.

$O(1)$ (see Figure 1), which is far too much to obtain the square root cancellation bounds like we obtained in (6.6). On the other hand, it seems that a single variable $X_i(k)$ only changes $\widehat{\sigma}(\xi)$ by $O(1)$ if the random variables $\{X_1(k)\}$ are configured in a very particular way, which is unlikely to happen. Thus we should expect that adjusting a single random variable $X_i(k)$ does not influence the value of $\widehat{\sigma}(\xi)$ very much if the quantity $\widehat{\sigma}(\xi)$ is averaged *over the possible choices of $\{X_1(k)\}$, and then we can apply McDiarmid's inequality.*

Proof. Consider the random set Ω of values $x_n \in Q_n$ such that there are $k_1, \dots, k_{n-1} \in \{1, \dots, K\}$ with

$$|x_n - f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))| \leqslant (L+1)r. \tag{6.10}$$

Then

$$\widehat{\sigma}(\xi) = \frac{1}{K} \sum_{k=1}^{K} Z(k). \tag{6.11}$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_n(k)} & : X_n(k) \in \Omega, \\ 0 & : X_n(k) \notin \Omega \end{cases}.$$

If Σ is the σ algebra generated by the random variables

$$\{X_i(k) : i \in \{1, \dots, n-1\}, k \in \{1, \dots, K\}\},$$

then the random variables $\{Z(k)\}$ are *conditionally independent* given Σ. Since we have $|Z(k)| \leqslant 1$ almost surely, Hoeffding's inequality thus implies that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2}\right). \tag{6.12}$$

It is simple to see that

$$\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) = \int_\Omega \psi_n(x) e^{2\pi i \xi \cdot x}\, dx. \tag{6.13}$$

Since

$$\Omega = \bigcup \left\{B_r(f(X_1(k_1), \dots, X_{n-1}(k_{n-1}))) : 1 \leqslant k_1, \dots, k_{n-1} \leqslant K\right\}. \tag{6.14}$$

we see that varying each random variable $X_i(k)$, for $1 \leqslant i \leqslant n-1$ which fixing the other random variables adjusts at most K^{n-2} of the radius r balls forming Ω, and thus varying $X_i(k)$ independently of the other random variables changes $\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)$ by at most

$$2 \cdot (2r)^d \cdot K^{n-2} = 2^{d+1} \cdot r^d \cdot K^{n-2} \leqslant \frac{2^{d+1}}{K} \leqslant \frac{2^{d+1}}{K}. \tag{6.15}$$

Thus McDiarmid's inequality shows that for any $t \geqslant 0$,

$$\mathbf{P}\left(|\,\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{6.16}$$

Combining (6.12) and (6.16), we conclude that for each $\xi \in \mathbf{Z}^d$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 8 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{6.17}$$

Applying a union bound to (6.17) over all $\xi \in B$ shows that there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10. \qquad \square$$

The analysis of (6.8) requires a more technical calculation.

Lemma 6.4. *Let σ be the random measure described in Lemma 6.1. Then there exists $C > 0$ such that for any $\delta > 0$, there exists $r_0 > 0$ such that for $r \leqslant r_0$,*

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \leqslant CK^{-1/2} + \delta|\xi|^{-\beta/2}.$$

Proof. We break the analysis of $\mathbf{E}(\widehat{\sigma}(\xi))$ into two cases, depending on whether $n = 2$ or $n > 2$. The major difference here is that when $n > 2$, $\beta < d$, whereas when $n = 2$, $\beta = d$, i.e. we are constructing a full dimensional set, so that some argument that work for the case $n > 2$ fail when $n = 2$. On the other hand, the analysis of patterns when $n = 2$ is more trivial than the analysis for $n > 2$, which makes this argument more simple in other respects.

Let's start with the case $n = 2$. Using the fact that the family $\{X_n(k) : 1 \leqslant k \leqslant K\}$ are uniformly distributed, we calculate that

$$\begin{aligned} \mathbf{E}(\widehat{\sigma}(\xi)) &= \mathbf{E}(A_2 \cdot e^{2\pi i \xi \cdot X_2(1)} \cdot \mathbf{I}(1 \in S)) \\ &= A_2 \int_{\mathbf{T}^d} \psi_2(x)\, \mathbf{P}\left(1 \in S | X_2(1) = x\right) e^{2\pi i \xi \cdot x}\, dx. \end{aligned} \tag{6.18}$$

For each $x \in \mathbf{T}^d$, a change of variables formula implies that

$$\begin{aligned}\mathbf{P}(1 \in S | X_1(1) = x) &= 1 - \left(1 - \int_{f^{-1}(B_{(L+1)r}(x))} \psi_1(x_1)\, dx_1\right)^K \\ &= 1 - \left(1 - \int_{B_{(L+1)r}(x)} \frac{(\psi_1 \circ f^{-1})(x_2)}{|\det(Df)(f^{-1}(x_2))|}\, dx_2\right)^K \\ &= 1 - \left(1 - \int_{B_{(L+1)r}(x)} \tilde{\psi}_1(x_2)\, dx_2\right)^K,\end{aligned} \tag{6.19}$$

where we have introduced $\tilde{\psi}_1$ for notational convenience. If we define

$$g(x) = \mathbf{P}(1 \in S | X_1(1) = x),$$

then $\mathbf{E}(\widehat{\sigma}(\xi)) = A_2 \cdot \widehat{\psi_2 g}(\xi)$. We can obtain a bound on this quantity by bounding the partial derivatives of $\psi_2 g$. Bernoulli's inequality implies that

$$g(x) = 1 - \left(1 - \int_{B_{(L+1)r}} \tilde{\psi}_1(x_2)\, dx_2\right)^K \lesssim_L Kr^d \leqslant r^{\varepsilon_1/2}. \tag{6.20}$$

On the other hand, if $\alpha > 0$, then $\partial_\alpha g(x)$ is a sum of terms of the form

$$(-1)^m \frac{K!}{(K-m)!} \left(1 - \int_{B_{(L+1)r}(x)} \tilde{\psi}_1(x_2)\, dx_2\right)^{K-m} \left(\prod_{i=1}^m \int_{B_{(L+1)r}(x)} \partial_{\alpha_i} \tilde{\psi}_1(x_2)\, dx_2\right), \tag{6.21}$$

where $\alpha_i \neq 0$ for any i and $\alpha = \alpha_1 + \cdots + \alpha_m$. This implies $0 < m \leqslant |\alpha|$ for any terms in the sum. Now the bound $|\partial_{\alpha_i} \tilde{\psi}_1(x_2)| \lesssim_{\alpha_i} 1$ implies that

$$\left| \int_{B_{(L+1)r}(x)} \partial_{\alpha_i} \tilde{\psi}_1(x_2)\, dx_2 \right| \lesssim_{\alpha_i} r^d. \tag{6.22}$$

Applying (6.22) to (6.21) enables us to conclude that

$$|\partial_\alpha g(x)| \lesssim_\alpha \max_{m \leqslant |\alpha|} K^m r^{md} \leqslant r^{\varepsilon_1/2}, \tag{6.23}$$

Since the fact that $\psi_2 \in C^\infty(\mathbf{T}^d)$ implies that $\|\partial_\alpha \psi_2\|_{L^\infty(\mathbf{T}^d)} \lesssim_\alpha 1$ for any multi-index α, the product rule applied to (6.23) implies that $\|\partial_\alpha(\psi_2 g)\|_{L^\infty(\mathbf{T}^d)} \lesssim_\alpha r^{\varepsilon_1/2}$ for all $\alpha > 0$, which means that for any $N > 0$ and $\xi \neq 0$,

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \lesssim_N r^{\varepsilon_1/2} |\xi|^{-N}. \tag{6.24}$$

In particular, setting $N = \beta/2$, fixing $\delta > 0$, and then choosing r_0 appropriately, (6.24) shows that for $r \leqslant r_0$,

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \leqslant \delta |\xi|^{-\beta/2}. \tag{6.25}$$

This completes the proof in the case $n = 2$.

Now we move on to the case $n \geqslant 3$. A version of equation (6.18) continues to hold in this setting, namely that

$$\mathbf{E}(\widehat{\sigma}(\xi)) = A_n \int_{\mathbf{T}^d} \psi_n(x)\, \mathbf{P}(1 \in S | X_n(1) = x_n) e^{2\pi i \xi \cdot x_n}\, dx_n. \tag{6.26}$$

However, the analysis of this equation is made more complicated by the lack of an explicit formula for $\mathbf{P}(1 \in S | X_n(1) = x)$. For a set $E \subset \mathbf{T}^{d(n-1)}$, let $A(E)$ denote the event that there exists $k_1, \dots, k_{n-1}$ such that $(X_1(k_1), \dots, X_{n-1}(k_{n-1})) \in E$. Then

$$\mathbf{P}(1 \in S | X_1(1) = x) = \mathbf{P}(A(f^{-1}(B_{(L+1)r}(x_n)))). \tag{6.27}$$

For any cube $Q \in \mathbf{T}^{d(n-1)}$ and any indices $1 \leqslant k_1, \dots, k_{n-1} \leqslant K$, set $k = (k_1, \dots, k_{n-1})$ and let $A(Q; k)$ denote the event that $(X_1(k_1), \dots, X_{n-1}(k_{n-1})) \in Q$. Then

$$A(Q) = \bigcup_k A(Q; k).$$

For any cube Q and index k,

$$\mathbf{P}(A(Q; k)) = \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1})\, dx_1 \dots dx_{n-1}, \tag{6.28}$$

and so

$$\sum_k \mathbf{P}(A(Q; k)) = K^{n-1} \int_Q \psi_1(x_1) \cdots \psi_{n-1}(x_{n-1})\, dx_1 \dots dx_{n-1}. \tag{6.29}$$

An application of inclusion exclusion to (6.29) thus shows that

$$\begin{aligned} &\left| \mathbf{P}(A(Q)) - K^{n-1} \int_Q \psi_1(x_1) \cdots \psi_{n-1}(x_{n-1})\, dx_1 \dots dx_{n-1} \right| \\ &\qquad \leqslant \sum_{k \neq k'} \mathbf{P}(A(Q; k) \cap A(Q; k')). \end{aligned} \tag{6.30}$$

For each k, k', the quantity $\mathbf{P}(A(Q; k) \cap A(Q; k'))$ depends on the number of indices i such that $k_i = k_i'$. In particular, if $I \subset \{1, \dots, n-1\}$ is the set of indices where the quantity agrees, then

$$\mathbf{P}(A(Q; k) \cap A(Q; k')) = \left(\prod_{i \in I} \int_{Q_i} \psi_i(x)\, dx \right) \cdot \left(\prod_{i \notin I} \left(\int_{Q_i} \psi_i(x)\, dx \right)^2 \right). \tag{6.31}$$

In particular, if Q has sidelength l and $\#(I) = m$, then $\mathbf{P}(A(Q; k) \cap A(Q; k')) \lesssim l^{d(2n-m-2)}$. For each m, there are at most K^{2n-m-2} pairs k and k' with $\#(I) = m$. And so provided $l^d \leqslant 1/K$,

$$\sum_{k \neq k'} \mathbf{P}(A(Q; k) \cap A(Q; k')) \lesssim \sum_{m=0}^{n-2} (K l^d)^{2n-m-2} \lesssim K^n l^{dn}. \tag{6.32}$$

Thus we conclude from (6.30) and (6.32) that

$$\mathbf{P}(A(Q)) = K^{n-1} \int_Q \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) dx_1 \dots dx_{n-1} + O(K^n l^{dn}). \tag{6.33}$$

For a particular $x_n \in \mathbf{T}^d$, let $E = f^{-1}(B_{(L+1)r}(x_n))$. Since f is a submersion, E is contained in a $O(r)$-thickening of a $d(n-2)$ dimensional surface in $\mathbf{T}^{d(n-1)}$. Applying the Whitney covering lemma, we can find a family of almost disjoint dyadic cubes $\{Q_{ij} : j \geqslant 0\}$ such that

$$E = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{n_i} Q_{ij}, \tag{6.34}$$

where for each $i \geqslant 0$, Q_{ij} is a sidelength $r/2^i$ cube, and $n_i \lesssim (r/2^i)^{-d(n-2)}$. It follows from (6.34) that

$$A(E) = \bigcup_{i,j} A(Q_{ij}). \tag{6.35}$$

Since $n \geqslant 3$, we can use (6.33) to calculate that

$$\begin{aligned} &\left| \sum_{i,j} \mathbf{P}(A(Q_{ij})) - K^{n-1} \int_E \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) \; dx \right| \\ &\quad\lesssim \sum_{i=0}^{\infty} (r/2^i)^{-d(n-2)} \cdot \left(K^n (r/2^i)^{dn} \right) \\ &\quad\lesssim r^{2d} K^n \leqslant K^{-1/2}. \end{aligned} \tag{6.36}$$

Thus an inclusion exclusion bound together with (6.35) and (6.36) implies that

$$\begin{aligned} &\left| \mathbf{P}(A(E)) - K^{n-1} \int_E \psi_1(x_1) \dots \psi_{n-1}(x_{n-1}) \; dx \right| \\ &\quad\lesssim K^{-1/2} + \sum_{(i_1,j_1) \neq (i_2,j_2)} \mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})). \end{aligned} \tag{6.37}$$

The quantity $\mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2}))$ depends on the relation between the various sides of $Q_{i_1 j_1}$ and $Q_{i_2 j_2}$. Without loss of generality, we may assume that $i_1 \geqslant i_2$. If $I(Q_{i_1 j_1}, Q_{i_2 j_2})$ is the set of indices $1 \leqslant k \leqslant n-1$ where $Q_{i_1 j_1 k} \subset Q_{i_2 j_2 k}$, and $\#(I(Q_{i_1 j_1}, Q_{i_2 j_2})) = m$, then

$$\begin{aligned} \mathbf{P}(A(Q_{i_1 j_1}) \cap A(Q_{i_2 j_2})) &\lesssim (K(r/2^{i_1})^d)^m \cdot (K(r/2^{i_1})^d \cdot K(r/2^{i_2})^d)^{n-m-1} \\ &= 2^{-d[(n-1)i_1 + (n-m-1)i_2]} (Kr^d)^{2n-m-2}. \end{aligned} \tag{6.38}$$

The condition that $D_{x_k} f$ is invertible for all k on the domain of f implies that any axis-oriented plane in $\mathbf{T}^{dn}$ intersects transversally with the level sets of f. In particular, this means that the intersection of a $O(r/2^{i_1})$ thickening of a codimension dm axis-oriented hyperplane intersects a $O(r/2^{i_1})$ thickening of ∂E (which has codimension d) in a set with volume $O\left((r/2^{i_1})^d (r/2^{i_1})^{dm}\right)$, and intersects a $O(r/2^{i_2})$ thickening of ∂E in a set with volume

$O\left((r/2^{i_2})^d(r/2^{i_1})^{dm}\right)$. As a particular example of this, for any distinct indices $j_1, \ldots, j_m \in \{1, \ldots, n-1\}$, and any family of integers $0 \leqslant n_{11}, \ldots, n_{md} \leqslant 2^{i_1}/r$, the set

$$\left\{x \in E : \frac{n_{11}}{2^{i_1}} \leqslant x_{j_1 1} \leqslant \frac{(n_{11}+1)}{2^{i_1}}, \ldots, \frac{n_{md}}{2^{i_1}} \leqslant x_{j_m d} \leqslant \frac{n_{md}+1}{2^{i_1}}\right\} \tag{6.39}$$

contains at most

$$O\left((r/2^{i_1})^d(r/2^{i_1})^{dm}(r/2^{i_1})^{-d(n-1)}\right) = O\left(2^{d(n-m-2)i_1} r^{-d(n-m-2)}\right) \tag{6.40}$$

sidelength $r/2^{i_1}$ dyadic cubes in the decomposition of E, and at most

$$O\left((r/2^{i_2})^d(r/2^{i_1})^{dm}(r/2^{i_2})^{-d(n-1)}\right) = O\left(2^{d(n-2)i_2-(dm)i_1} r^{-d(n-m-2)}\right) \tag{6.41}$$

sidelength $r/2^{i_2}$ dyadic cubes in the decomposition of E. Letting the integers $\{n_{kl}\}$ vary over all possible choices we conclude from (6.40) and (6.41) that for each i_1 and i_2 there are at most

$$\begin{aligned} &O\left((2^{i_1}/r)^{dm}\left(2^{d(n-m-2)i_1} r^{-d(n-m-2)}\right)\left(2^{d(n-2)i_2-(dm)i_1} r^{-d(n-m-2)}\right)\right) \\ &\quad = O\left(2^{d(n-m-2)i_1+d(n-2)i_2} r^{-d(2n-m-4)}\right) \end{aligned} \tag{6.42}$$

pairs $Q_{i_1 j_1}$ and $Q_{i_2 j_2}$ with $I(Q_{i_1 j_1}, Q_{i_2 j_2}) = m$. Thus we conclude from (6.38) and (6.42) that

$$\begin{aligned} &\sum_{(i,j)\neq(i',j')} \mathbf{P}(A(Q_{ij}) \cap A(Q_{i'j'})) \\ &\qquad \lesssim \sum_{m=0}^{n-2} \sum_{i_1 \geqslant i_2} \left(2^{d(n-m-2)i_1+d(n-2)i_2} r^{-d(2n-m-4)}\right) \\ &\qquad\qquad\qquad \left(2^{-d((n-1)i_1+(n-m-1)i_2)} (Kr^d)^{2n-m-2}\right) \\ &\qquad \lesssim r^{2d} \sum_{m=0}^{n-2} K^{2n-m-2} \sum_{i_1 \geqslant i_2} 2^{-d(m+1)i_1+d(m-1)i_2} \\ &\qquad \lesssim \sum_{m=0}^{n-2} K^{2n-m-2} r^{2d} \\ &\qquad \lesssim K^{2(n-1)} r^{2d} \lesssim K^{-1/2}. \end{aligned} \tag{6.43}$$

Returning to the bound in (6.37), (6.43) implies that

$$\left|\mathbf{P}(A(E)) - K^{n-1} \int_E \psi_1(x_1) \ldots \psi_{n-1}(x_{n-1})\, dx_1 \ldots dx_{n-1}\right| \lesssim K^{-1/2}. \tag{6.44}$$

Returning even further back to (6.26), recalling that $E = f^{-1}(B_r(x_n))$, (6.44) implies

$$\left|\mathbf{E}(\widehat{\sigma}(\xi)) - A_n \cdot K^{n-1} \int_{\mathbf{T}^d} \psi_n(x_n) \int_{f^{-1}(B_r(x_n))} \psi_1(x_1) \ldots \psi_{n-1}(x_{n-1}) e^{2\pi i \xi \cdot x_n}\, dx_1 \; \ldots \; dx_n\right| \lesssim K^{-1/2}. \tag{6.45}$$

Applying the co-area formula, writing $\psi(x) = \psi_1(x_1) \dots \psi_n(x_n)$, we find

$$\begin{aligned}
&\int_{\mathbf{T}^d} \int_{f^{-1}(B_r(x_n))} \psi(x) e^{2\pi i \xi \cdot x_n} \; dx_1 \; \dots \; dx_n \\
&\quad = \int_{B_r(0)} \int_{\mathbf{T}^d} \int_{f^{-1}(x+v)} \psi(x) e^{2\pi i \xi \cdot x_n} \; dH^{n-2}(x_1, \dots, x_{n-1}) \; dx_n \; dv \\
&\quad = \int_{B_r(0)} \int_{\mathbf{T}^{d(n-1)}} \psi(x, f(x) - v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} |Jf(x)| \; dx \; dv \\
&\quad = \int_{B_r(0)} \int_{\mathbf{T}^{d(n-1)}} \tilde{\psi}(x, v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} \; dx \; dv.
\end{aligned} \tag{6.46}$$

where $\tilde{\psi}(x,v) = \psi(x, f(x) - v) \cdot |Jf(x)|$, and Jf is the rank-d Jacobian of f. A consequence of (6.46) in light of (6.45) is that it reduces the study of $\mathbf{E}(\widehat{\sigma}(\xi))$ to a standard oscillatory integral. In particular, noting that Df is surjective on the domain of f, which implies the oscillatory integral has no stationary points. Applying Proposition 4 of [9], we conclude that for all $|v| \leqslant 1$ and $N > 0$,

$$\left| \int_{\mathbf{T}^{d(n-1)}} \tilde{\psi}(x, v) \cdot e^{2\pi i \xi \cdot (f(x) - v)} \; dx \right| \lesssim_N |\xi|^{-N}. \tag{6.47}$$

Now the bound in (6.47) can be applied with (6.46) to conclude that

$$\left| \int_{\mathbf{T}^d} \int_{f^{-1}(B_r(x))} \psi(x) e^{2\pi i \xi \cdot x_n} \; dx_2 \; \dots \; dx_n \; dx_1 \right| \lesssim_N r^d |\xi|^{-N}. \tag{6.48}$$

In particular, combined with (6.45), (6.48) shows that

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \lesssim K^{-1/2} + K^{n-1} r^d |\xi|^{-\beta/2} \lesssim K^{-1/2} + K^{-0.25} |\xi|^{-\beta/2}. \tag{6.49}$$

Thus for any $\delta > 0$, there exists $r_1 > 0$ such that for $r \leqslant r_1$, and any nonzero $\xi \in \mathbf{Z}^d$,

$$|\,\mathbf{E}(\widehat{\sigma}(\xi))| \leqslant \delta |\xi|^{-\beta/2}. \qquad \square$$

The proof of Lemma 6.4 is the only obstacle preventing us from constructing a Salem set X avoiding the pattern defined by Z with

$$\dim_{\mathbf{F}}(X) = \frac{d}{n-1}.$$

The problem here is that if $n \geqslant 3$ and $\beta > d/(n - 3/4)$, there is too much 'overlap' between the various cubes we use in our covering argument; thus the inclusion-exclusion argument found in this proof cannot be used to control $\mathbf{E}(\widehat{\sigma}(\xi))$. We believe our method can construct Salem sets with Fourier dimension $d/(n-1)$, but new tools are required to improve the estimates used in Lemma 6.4 to prove that $\mathbf{E}(\widehat{\sigma}(\xi))$ decays on the order of $|\xi|^{-\beta/2}$. In the next section, we are able to avoid these tools by reducing the calculation of $\mathbf{E}(\widehat{\sigma}(\xi))$ to a triviality.

7 Expectation Bounds for Translational Patterns

The proof of Theorem 1.3 uses very similar arguments to Theorem 1.2. The concentration bound arguments will be very similar to those applied in the last section. The difference here is that the translation-invariance of the pattern can be used to bypass estimating the expectated values like those which caused us the most difficulty in Theorem 1.3. We can therefore construct Salem sets avoiding the pattern with dimension exactly matching the Hausdorff dimension of the sets which would be constructed using the method of [1]. In this section, let

$$\beta \leqslant \min\left(\frac{d(n-1)-\alpha}{n-1}, d\right).$$

We then show that generic elements of $\mathcal{X}_\beta$ avoid patterns satisfying the assumptions of Theorem 1.3.

Lemma 7.1. *Fix $a \in \mathbf{Q}-\{0\}$, and let $T : V \to \mathcal{E}$ satisfy the assumptions of Theorem 1.3. Then for quasi-all $(E,\mu) \in \mathcal{X}_\beta$, and any distinct points $(x_1,\dots,x_n) \in E$, $x_n - ax_{n-1} \notin S(x_1,\dots,x_{n-2})$.*

Proof. Set

$$W = \{(x_1,\dots,x_n) \in \mathbf{T}^{2d} \times V : x_n - ax_{n-1} \in S(x_1,\dots,x_{n-2})\}.$$

The assumption that T is a locally Lipschitz map, and thus continuous, implies that for any disjoint, closed cubes $R_1,\dots,R_n \subset \mathbf{T}^d$ such that $R_1 \times \cdots \times R_{n-2} \subset V$, $(R_1 \times \cdots \times R_n) \cap W$ will be a closed set. It follows that if

$$H(W;R_1,\dots,R_n) = \{(E,\mu) \in \mathcal{X}_\beta : (R_1 \times \cdots \times R_n) \cap W \cap E^n = \varnothing\},$$

then $H(W;R_1,\dots,R_n)$ is an open subset of $\mathcal{X}_\beta$. The Lemma will be proved that for each positive integer m, and any choice of cubes $R_1,\dots,R_n$ with common sidelength $1/am < 1/n$, the set $H(W;R_1,\dots,R_n)$ is dense in $\mathcal{X}_\beta$. Like in Lemma 5.1 and 6.1, we fix $(E_0,\mu_0) \in \mathcal{X}_\beta$ with $\text{supp}(\mu_0) = E_0$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$, as well as $\varepsilon_1 \in (0,\beta/100]$ and $\varepsilon > 0$, and show there is $(E,\mu) \in H(W;R_1,\dots,R_n)$ with $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. This implies $H(W;R_1,\dots,R_n)$ is dense.

To prove $H(W;R_1,\dots,R_n)$ is dense, we may assume without loss of generality that each set in the image of T is a/m periodic, i.e. for any $x \in V$ and $b \in \mathbf{Z}^d$, $S(x) + (a/m)\cdot b = S(x)$. To see why this is true, if $a = a_1/a_2$ for integers a_1, a_2, with $a_2 > 0$, we note that the set-valued function

$$\tilde{S}(x_1,\dots,x_{n-2}) = \bigcup_{0 \leqslant b_1,\dots,b_d < ma_2} S(x_1,\dots,x_{n-2}) + (a/m)\cdot b$$

satisfies the same continuity and Minkowski dimension bounds that were required for T in the assumptions of Theorem 1.3. Moreover, the set $\tilde{S}$ will then be a a/m periodic subset of $\mathbf{T}^d$. If we define

$$\tilde{W} = \{(x_1,\dots,x_n) \in \mathbf{T}^{2d} \times V : x_n - ax_{n-1} \in \tilde{S}(x_1,\dots,x_{n-2})\}.$$

and

$$\tilde{H}(W;R_1,\dots,R_n) = \{(E,\mu) \in \mathcal{X}_\beta : (R_1 \times \cdots \times R_n) \cap W \cap E^n = \varnothing\},$$

then $\tilde{H}(W; R_1, \dots, R_n)$ is an open subset of $H(W; R_1, \dots, R_n)$. Showing $\tilde{H}(W; R_1, \dots, R_n)$ is dense thus implies that $H(W; R_1, \dots, R_n)$ is dense. Thus we need only concentrate on the proof that $\tilde{H}(W; R_1, \dots, R_n)$ is dense.

Fix a large integer $K > 0$ such that both $(1 - n(a/m)) \cdot K$ and $(a/m) \cdot K$ are integers. Since S is Lipschitz, we may consider $L > 0$ such that for $x_1, x_2 \in R_1 \times \dots \times R_{n-2}$,

$$d_{\mathbf{H}}(S(x_1), S(x_2)) \leqslant L|x_1 - x_2|.$$

For $1 \leqslant i \leqslant n$, consider a family of independent random variables $\{X_i(k) : 1 \leqslant k \leqslant a \cdot K\}$ uniformly distributed on R_i, as well as another independant family of random variables $\{X_0(k) : 1 \leqslant k \leqslant (1 - n(a/m)) \cdot K\}$ uniformly distributed on $\mathbf{T}^d - (R_1 \cup \dots \cup R_n)$. Let S be the set of indices $k_n \in \{1, \dots, a \cdot K\}$ such that there are indices $k_1, \dots, k_{n-1} \in \{1, \dots, K\}$ with the property that

$$|X_n(k_n) - aX_{n-1}(k_{n-1}) - S(X_1(k_1), \dots, X_{n-1}(k_{n-2}))| \leqslant 10(a + \sqrt{n})L \tag{7.1}$$

□

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