Cartesian Products Avoiding Patterns

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Abstract

We construct subsets of $[0,1]^d$ with large Hausdorff dimension whose Cartesian product avoids a set with low Minkowski dimension. This generalizes the pattern avoidance problem found in the literature. Given $Z \subset (\mathbf{R}^d)^n$ covered by the countable union of compact sets with lower Minkowski dimension at most α, we construct a Cantor type set X with Hausdorff dimension $(nd-\alpha)/(n-1)$ such that for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. In particular, our result generalizes Fraser and Pramanik's result on pattern avoidance, which assume Z is smooth. We use the result to construct high dimensional sets whose sum set avoids a given set, as well as construct subsets avoiding isoceles triangles. General pattern avoidance methods in the literature can only construct subsets of Euclidean space avoiding configurations, which makes the latter result particularly surprising.

Can subsets of $\mathbf{R}^d$ with large fractional dimension be constructed avoiding patterns? For instance, is it possible to find a high dimensional set containing no colinear triple of points, or not containing any three points forming an isosceles triangle? If the pattern is specified as the zero set of a smooth function $f : (\mathbf{R}^d)^n \to \mathbf{R}$, then results in [2] and [5] give general methods for finding X such that for any distinct points $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) = 0$. Rather than avoiding the zeroes of a function, in this paper, we fix a set $Z \subset (\mathbf{R}^d)^n$, and construct sets X such that for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. Surprisingly, in this setting we can obtain results only assuming conditions on the fractional dimension of Z.

Theorem 1. *Suppose $Z \subset (\mathbf{R}^d)^n$ is the countable union of compact sets with lower Minkowski dimension at most α. Then there exists $X \subset [0,1]^d$ with*

$$\dim_{\mathbf{H}}(X) = \min\left(\frac{nd-\alpha}{n-1}, d\right),$$

such that if $x_1, \dots, x_n \in X$ are distinct, $(x_1, \dots, x_n) \notin Z$.

One advantage of the Cartesian product approach we have taken to the problem is that certain geometric features of Z are more explicit than when Z is expressed as the zero set of the function. In particular, studying the difficulty of the problem in terms of the fractional dimension of Z is completely non-obvious from the functional perspective.

Despite the generality of our theorem, we are still able to recover Theorems 1.1 and 1.2 of [2] as special cases when Z is formed from a countable collection of smooth manifolds. Meanwhile, our proof is less technical than their result. Furthermore, our result shows that the results remains robust under small changes in the fractional dimension of Z. Section 6 is devoted to a comparison of our method with [2], as well as other generic pattern avoidance methods.

Because our result can be applied to very general sets Z, we can apply the result to yield a great many interesting pattern avoiding sets. In particular, we can find large subsets of Euclidean space whose sums avoid a given set. Most interesting of the applications is a 'restricted' construction of a large set avoiding configurations, via use of a projection trick. In this scenario, in addition to a set Z, we are given an arbitrary set Y, and we must construct a high dimensional subset X of Y avoiding Z. None of the existing constructions are able to be applied to yield results of this form, because the patterns involved are highly irregular. The discussion of these applications is in Section 5.

The key idea to avoiding sparse configurations is a random mass equidistribution strategy. This is featured as the main technique in our solution to a discrete variant of the theorem in Section 2. This discrete problem is very difficult to solve optimally, but we use a random strategy which is optimal enough in expectation, and is likely tight for general inputs to the problem. By overlaying the solution to the discretized problem at a sequence of scales, in Section 3 we are able to obtain the required set X via a Cantor-type construction.

An important property of our discrete strategy is that it provides an 'equidistributed' mass selection strategy. Exploiting this, in Section 4 we are able to show the set X has the required Hausdorff dimension regardless of how fast our sequence of scales decay. The equidistribution technique occurs implicitly in at least one other Hausdorff dimension calculation, for example, in [2]. But we do not believe equidistribution has been explicitly identified in the literature as a method to maintain fractal dimension despite a rapid decay of scales used in the construction of the fractal.

Remark. *The difficult setting of Theorem 1 occurs when $\alpha \geqslant d$. If $\alpha < d$,*

$$X = \left\{x \in [0,1]^d : x \neq z_k \text{ for all } (z_1, \dots, z_n) \in Z \text{ and } 1 \leqslant k \leqslant d\right\}$$

gives a set with full Hausdorff dimension satisfying the properties of the theorem. In our proof, we will assume $d \leqslant \alpha < dn$, and so must find a set X with Hausdorff dimension $(dn - \alpha)/(n-1)$ avoiding the set Z.

1 Frequently Used Notation and Terminology

- For a length l, $\mathcal{B}_l^d$ denotes the family of all half open cubes in $\mathbf{R}^d$ with sidelength l and corners on the lattice $(l \cdot \mathbf{Z})^d$. That is,

$$\mathcal{B}_l^d = \{[a_1, a_1 + l) \times \cdots \times [a_d, a_d + l) : a_i \in l \cdot \mathbf{Z}\}.$$

If $E \subset \mathbf{R}^d$, $\mathcal{B}_l^d(E)$ is the family of cubes in $\mathcal{B}_l^d$ intersecting E, i.e.

$$\mathcal{B}_l^d(E) = \{I \in \mathcal{B}_l^d : I \cap E \neq \varnothing\}.$$

For instance, $\mathcal{B}_l^d(\mathbf{R}^d) = \mathcal{B}_l^d$.

- The *lower Minkowski dimension* of a compact set $E \subset \mathbf{R}^d$ is

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{l \to 0} \frac{\log(\#(\mathcal{B}_l^d(E)))}{\log(1/l)}.$$

  Thus there is $l_k \to 0$ with $\#(\mathcal{B}_{l_k}^d(E)) = (1/l_k)^{\underline{\dim}_{\mathbf{M}}(E)+o(1)}$.

- A *Frostman measure* of dimension α is a compactly supported finite Borel measure μ on $\mathbf{R}^d$ such that for any ball $B_r(x)$ of radius r centered at a point x, $\mu(B_r(x)) \lesssim r^\alpha$. The *Hausdorff dimension* of a set $X \subset \mathbf{R}^d$ is

$$\dim_{\mathbf{H}}(X) = \sup \left\{ \alpha : \begin{array}{c} \text{There is an } \alpha \text{ dimensional Frostman} \\ \text{measure supported on } X \end{array} \right\}$$

- Adopting the terminology of [3], we say a collection of sets $U_1, U_2, \dots$ is a *strong cover* of some set E if $E \subset \limsup U_k$, which means every element of E is contained in infinitely many of the sets U_k.

- Given $I \in \mathcal{B}_l^{dn}$, we can decompose I as $I_1 \times \cdots \times I_n$ for unique cubes $I_1, \dots, I_n \in \mathcal{B}_l^d$. We say I is *non diagonal* if the cubes $I_1, \dots, I_n$ are distinct.

2 Avoidance at Discrete Scales

We avoid Z by considering an infinite sequence of scales. At each scale, we solve a discretized version of the problem. Combining these solutions then solves the original problem. This section describes the discretized avoidance technique. This is the *core* part of our construction, and the Hausdorff dimension we achieve is a direct result of our success in the discrete setting.

Fix two dyadic scales $l > s$. In the discrete setting, Z is replaced by a union of sidelength s cubes, denoted Z'. Our goal is to take a set E, which is a union of sidelength l cubes, and carve out a union of sidelength s cubes F such that F^n is disjoint from the non-diagonal cubes of Z'.

In order to ensure the Hausdorff dimension calculations of X go smoothly, it is crucial that the mass of F is spread uniformly over E in the discrete setting. We can achieve this by trying to include a equal portion of mass in each sidelength r subcube of E, for some intermediary dyadic scale $r \in (s, l)$ with $l > r > s$. The next lemma shows that we can select a equal portion of mass from *almost* all of the sidelength r cubes.

Lemma 1. *Fix three dyadic lengths $l > r > s$. Let E be a union of cubes in $\mathcal{B}_l^d$, and Z' a union of cubes in $\mathcal{B}_s^{dn}$. Then there exists $F \subset E$, which is a union of cubes in $\mathcal{B}_s^d$, such that*

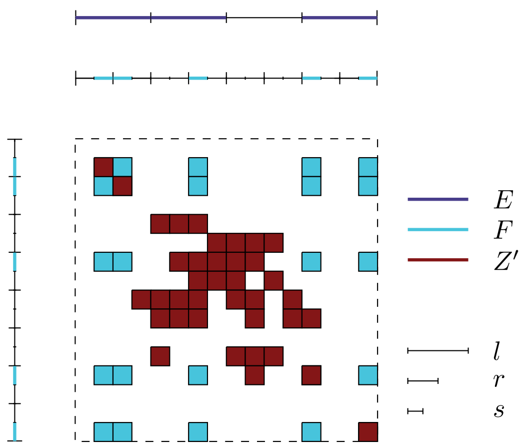

Figure 1: An example choice of F in Lemma 1 where $d = 1$ and $n = 2$. F satisfies the non-concentration and avoidance property, as well as containing an interval from all but 3 of the intervals in $\mathcal{B}_r^d(E)$.

- Avoidance*: For any distinct* $I_1, \dots, I_n \in \mathcal{B}_s^d(F)$*,* $I_1 \times \dots \times I_n \notin \mathcal{B}_s^{dn}(Z')$*.*
- Non Concentration*:* $\#(\mathcal{B}_s^d(F) \cap \mathcal{B}_s^d(I)) \leqslant 1$ *for* $I \in \mathcal{B}_s^d(E)$*.*
- Equidistribution*:* $\#(\mathcal{B}_s^d(F) \cap \mathcal{B}_s^d(I)) = 1$ *for all but at most* $|Z'|r^{-dn}$ *of the cubes* $I \in \mathcal{B}_s^d(E)$*.*

Proof. Form a random set U by selecting a sidelength s cube from each sidelength r cube uniformly at random. More precisely, set

$$U = \bigcup \{J_I : I \in \mathcal{B}_r^d(E)\},$$

where J_I is an element selected uniformly randomly from $\mathcal{B}_s^d(I)$. U certainly satisfies the equidistribution and non-concentration properties, but not the avoidance property. We will show that with non-zero probability, we can obtain the avoidance property by removing at most $|Z'|r^{-dn}$ cubes from U.

For any $J \in \mathcal{B}_s^d(E)$, there is a unique $I \in \mathcal{B}_l^d(E)$ such that $J \subset I$. Then

$$\mathbf{P}(J \subset U) = \mathbf{P}(J_I = J) = (s/r)^d.$$

Since any two elements of $\mathcal{B}_s^d(U)$ lie in distinct cubes of $\mathcal{B}_r^d$, the only chance that a *non-diagonal* cube $K = J_1 \times \dots \times J_n$ in $\mathcal{B}_s^{dn}(Z')$ is a subset of U^n is if $J_1, \dots J_n$ all lie in separate cubes of $\mathcal{B}_r^d$. They each have an independent chance of being added to U, and so

$$\mathbf{P}(K \subset U^n) = \mathbf{P}(J_1 \subset U) \cdots \mathbf{P}(J_n \subset U) = (s/r)^{dn}.$$

If $\mathcal{K}(U)$ denotes the family of all non-diagonal cubes $K \in \mathcal{B}_s^{dn}(Z')$ contained in U^n, then, letting K range over the non-diagonal cubes of $\mathcal{B}_s^{dn}(Z')$, we find

$$\mathbf{E}(\#(\mathcal{K}(U))) = \sum\nolimits_K \mathbf{P}(K \subset U^n) \leqslant |\mathcal{B}_s^{dn}(Z')|(s/r)^{dn} = |Z'|r^{-dn}.$$

In particular, this means that out of all possible outcomes for the set U, there is at least one *particular* U_0 we can choose for which

$$\#(\mathcal{K}(U_0)) \leqslant \mathbf{E}(\mathcal{K}(U)) = |\mathcal{B}_s^{dn}(Z')|(s/r)^{dn} = |Z'|r^{-dn}.$$

Thus we have selected a section of mass with very few intersections with Z_0.

We now define $F = U_0 - \{J_1 : K = J_1 \times \cdots \times J_n \in \mathcal{K}(U_0)\}$. As a subset of U_0, F inherits the non-concentration property. We have removed at most $|Z'|r^{-dn}$ cubes from U_0, and since U_0 contains an cube from *every* cube in $\mathcal{B}_r^d(E)$, F satisfies the equidistribution property. Finally, since we have removed a single side from every non-diagonal cube in U_0^n intersecting Z', F satisfies the avoidance property. So our construction is complete. □

Remark. *The existence of U_0 was justified using a randomized selection process. Nonetheless, its existence can be made constructive: We simply interate through all possible outcomes of U and pick one minimizing the cardinality of $\mathcal{K}$. As a result, the set X in our theorem is obtained by explicit, constructive means.*

If the original set Z has dimension α, we will later show its discretization Z' will satisfy bounds of the form $|Z'| \leqslant 2^{dn}s^{dn-\gamma}$, with γ converging to α as $s \to 0$. For convenience, we will also set r to be the closest power of two to s^λ, for some $\lambda \in (0,1)$. The size of λ is directly related to the Hausdorff dimension of the set X we will obtain. The next corollary calculates how large λ can be if F must be equidistributed over a constant fraction of cubes in $\mathcal{B}_r^d(E)$. The error term $5A\log_s|E|$ will be made insignificant by the rapid decay of the values s used in our construction.

Corollary 1. *Consider the last lemma's setup, in addition to three parameters $\lambda \in (0,1]$, $\gamma \in [d,dn)$, and $m > 0$. Suppose $r = 2^{-\lfloor \lambda \log_2(1/s)\rfloor}$, $|E| \leqslant 1/2$, and $|Z'| \leqslant 2^{dn}s^{dn-\gamma}$. If*

$$0 < \lambda \leqslant \frac{dn-\gamma}{d(n-1)} - 5m\log_s|E|\,,$$

then F is equidistributed over all but a fraction $1/2^m$ of the cubes in $\mathcal{B}_r^d(E)$.

Proof. The inequality for λ implies

$$dn - \gamma - \lambda d(n-1) \geqslant 5d(n-1)m\log_s|E|.$$

Since r is within a factor of two from s^λ, we compute

$$\begin{aligned}\frac{|\{I \in \mathcal{B}_r^d(E) : \mathcal{B}_s^d(I) \cap \mathcal{B}_s^d(F) = \varnothing\}|}{|\mathcal{B}_s^d(E)|} &\leqslant \frac{|Z'|r^{-dn}}{|E|r^{-d}} \leqslant \frac{(2^{dn}s^{dn-\gamma})}{|E|r^{d(n-1)}} \\ &\leqslant (2^{dn}s^{dn-\gamma})(s/2)^{-\lambda d(n-1)}|E|^{-1} \leqslant 2^{dn+\lambda d(n-1)}s^{dn-\gamma-\lambda d(n-1)}|E|^{-1} \\ &\leqslant 2^{dn+\lambda d(n-1)}|E|^{5d(n-1)m-1} \leqslant 2^{dn+d(n-1)-(5d(n-1)m-1)} \leqslant 1/2^m.\end{aligned}$$

The last inequality was obtained because

$$\begin{aligned}[dn+&d(n-1)-(5d(n-1)m-1)]+m\\ &\leqslant 2dn+1-d+(1-5d(n-1))m\\ &\leqslant 2dn+(1-5d(n-1))\\ &\leqslant 5d-3dn+1\leqslant 0.\end{aligned}$$

Thus $dn+d(n-1)-(5d(n-1)m-1)\leqslant -m$. □

Remark. *We reemphasize that the discrete method is the core of our avoidance technique. The remaining argument is modular. Indeed, this part of our paper was based on the construction method of [2]. If for a special case of Z, one can improve the lemma so fewer cubes are discarded, then the remaining parts of our paper can likely be applied near verbatim to yield a set X with a larger Hausdorff dimension. For instance, a variation on the argument in [5] shows that if Z is a degree m algebraic hypersurface, and $Z'=\mathcal{B}_l^{dn}(Z)$, then a different selection strategy at the discrete scale allows us to set $\lambda\approx 1/m$. Following through the remainder of our proof replicates the main result of the paper.*

3 Fractal Discretization

Now we apply the discrete result at many scales. The fact that Z is the countable union of compact sets with Minkowski dimension α implies that we can find an efficient *strong cover* of Z by cubes restricted to lie at a sequence of dyadic scales l_k converging to zero arbitrarily fast.

Lemma 2. *Let $Z\subset\mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α, and consider any decreasing sequence ε_k converging to zero with $\alpha+\varepsilon_k\leqslant dn$. Then there is a decreasing sequence of lengths $l_1,l_2,\dots,$ and $\mathcal{B}_{l_k}^{dn}$ sets Z_k such that Z is strongly covered by the sets Z_k and $|\mathcal{B}_{l_k}(Z_k)|\leqslant 2^{dn}/l_k^{\alpha+\varepsilon_k}$.*

Proof. Let Z be the union of sets Y_i with $\underline{\dim}_{\mathbf{M}}(Y_i)\leqslant\alpha$ for each i. Consider any sequence of integers $m_1,m_2,\dots$ which repeats each integer infinitely often. Given k, there are infinitely many lengths l with $\#(\mathcal{B}_l^{dn}(Y_{m_k}))\leqslant 1/l^{\alpha+\varepsilon_k}$. Replacing l with a dyadic number at most twice the size of l, there are infinitely many *dyadic* lengths l with $\#(\mathcal{B}_l^{dn}(Y_{m_k}))\leqslant 1/(l/2)^{\alpha+\varepsilon_k}\leqslant 2^{dn}/l^{\alpha+\varepsilon_k}$. In particular, we may fix a length l_k smaller than $l_1,\dots,l_{k-1}$. Then the union of the cubes in $\mathcal{B}_{l_k}^{dn}(Y_{m_k})$ forms the set Z_k. □

Remark. *In the proof, we are free to make l_k arbitrarily small in relation to the previous parameters $l_1,\dots,l_{k-1}$ we have chosen. For instance, later on when calculating the Hausdorff dimension, we will assume that $l_{k+1}\leqslant l_k^{k^2}$, and the argument above can be easily modified to incorporate this inequality. We will also find that setting $\varepsilon_k=c\cdot k^{-1}$ suffices to give the results we need, where c is a sufficiently small constant such that $\alpha+c\leqslant dn$.*

We can now construct X by avoiding the various discretizations of Z at each scale. The aim is to find a nested decreasing family of discretized sets X_k with $X = \lim X_k$. One condition guaranteeing that X avoids Z is that X_k^n is disjoint from *non diagonal* cubes in Z_k.

Lemma 3. *If for each k, X_k^n avoids non-diagonal cubes in Z_k, $(x_1, \dots, x_n) \notin Z$ for any distinct $x_1, \dots, x_n \in X$.*

Proof. Let $z \in Z$ be given with $z_1, \dots, z_n$ are distinct. Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i \neq j \text{ such that } w_i = w_j\}.$$

Then $d(\Delta, z) > 0$. The point z is covered by cubes in infinitely many of collections Z_{k_m}. For suitably large N, the cube I in $\mathcal{B}^{dn}_{l_{k_N}}$ containing z is disjoint from Δ. But this means that I is non diagonal, and so $z \notin X_N^d$. In particular, z is not an element of X^n. □

It is now simple to see how we iteratively apply our discrete scale argument to construct X. First, we set $X_0 = [0, 1/2]^d$, so that $|X_0| \leqslant 1/2$. To obtain X_{k+1} from X_k, we set

$$E = X_k, \quad Z' = Z_{k+1}, \quad l = l_k, \quad s = l_{k+1}, \quad \text{and} \quad \gamma = \alpha + \varepsilon_k = \alpha + c \cdot \varepsilon_k,$$

We set $r = r_{k+1}$, where r_{k+1} is the closest power of two to l_{k+1}^λ, and

$$\lambda = \beta_{k+1} := \frac{dn - \alpha}{d(n-1)} - \frac{\varepsilon_{k+1}}{d(n-1)} - 10(k+1) \log_{L_{k+1}} |X_k|.$$

We can now apply Corollary 1 to constructs a set F with F^n avoiding non diagonal cubes in Z_{k+1}, and containing a $\mathcal{B}^d_{l_{k+1}}$ subcube from all but a fraction $1/2^{2k+2}$ of the $\mathcal{B}^d_{r_{k+1}}$ cubes in I. We set $X_{k+1} = F$. Repeatedly doing this builds an infinite sequence of the X_k. Since X_k^n avoids Z_k, for any distinct $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin X$.

4 Dimension Bounds

All that remains in our argument is showing X has the right Hausdorff dimension. At the discrete scale l_k, X looks like a $d\beta_k$ dimensional set. If the lengths l_k rapidly converge to zero, then we can ensure $\beta_k \to \beta$, where

$$\beta = \frac{dn - \alpha}{d(n-1)}.$$

Thus X looks $d\beta = (dn - \alpha)/(n-1)$ dimensional at the discrete scales l_k, which is the Hausdorff dimension we want. To obtain the complete dimension bound, it then suffices to interpolate to get a $d\beta$ dimensional behaviour at all intermediary scales. We won't be penalized here by making the gaps between discrete scales too large, because the uniform way that we have selected cubes in consecutive scales implies that between the scales l_k and l_{k+1}^β, X behaves like a full dimensional set. This section fills in the details to this argument.

Lemma 4. $\beta_k = \beta - O(1/k)$.

Proof. We must show

$$\beta - \beta_k = \frac{\varepsilon_{k+1}}{d(n-1)} + 10(k+1)\log_{l_{k+1}}|X_k| = O(1/k).$$

Since $\varepsilon_k = c \cdot k^{-1}$, the first term is easily seen to be $O(1/k)$. On the other hand, we need the lengths to tend to zero rapidly to make the other error term decay to zero. Since $l_{k+1} \leqslant l_k^{k^2}$, we find

$$(k+1)\log_{l_{k+1}}|X_k| \leqslant \frac{(k+1)\log l_k}{\log l_{k+1}} \leqslant \frac{(k+1)\log l_k}{k^2 \log l_k} = \frac{k+1}{k^2} = O(1/k).$$

Thus both components of the error term are $O(1/k)$. □

The most convenient way to look at X's dimension at various scales is to use Frostman's lemma. We construct a non-zero measure μ supported on X such that for all $\varepsilon > 0$, for all lengths l, and for all $I \in \mathcal{B}_l^d$, $\mu(I) \lesssim_\varepsilon l^{d\beta - \varepsilon}$. We can then understand the behaviour of X at a scale l by looking at the behaviour of μ restricted to cubes at the scale l.

To construct μ, we take a sequence of measures μ_k, supported on X_k, and then take a weak limit. We initialize this construction by setting μ_0 to be the uniform probability measure on $X_0 = [0,1/2]^d$. We then define μ_{k+1}, supported on X_{k+1}, by modifying the distribution of μ_k. First, we throw away the mass of the $\mathcal{B}_{l_k}^d$ cubes I for which half of the elements of $\mathcal{B}_{r_{k+1}}^d(I)$ fail to contain a part of X_{k+1}. For the cubes I with more than half of the cubes $\mathcal{B}_{r_{k+1}}^d(I)$ containing a part of X_{k+1}, we distribute the mass $\mu_k(I)$ uniformly over the subcubes of I in X_{k+1}, giving the distribution of μ_{k+1}.

A glance at the cumulative distribution functions of the μ_k shows these measures converge weakly to a function μ. For any $I \in \mathcal{B}_{l_k}^d$, we find $\mu(I) \leqslant \mu_k(I)$, which will be useful for passing from bounds on the discrete measures to bounds on the final measure. This occupies our attention for the remainder of this section.

Lemma 5. *If $I \in \mathcal{B}_{l_k}^d$, then*

$$\mu(I) \leqslant \mu_k(I) \leqslant 2^k \left[\frac{r_k r_{k-1} \dots r_1}{l_{k-1} \dots l_1}\right]^d.$$

Proof. Consider $I \in \mathcal{B}_{l_{k+1}}^d$, $J \in \mathcal{B}_{l_k}^d$. If $\mu_k(I) > 0$, J contains an element of $\mathcal{B}_{l_k}^d$ in at least half of the cubes in $\mathcal{B}_{r_k}^d(J)$. Thus the mass of J distributes itself evenly over at least $2^{-1}(l_{k-1}/r_k)^d$ cubes, which gives that $\mu_k(I) \leqslant 2(r_k/l_{k-1})^d \mu_{k-1}(J)$. Expanding this recursive inequality, using that μ_0 has total mass one as a base case, we obtain exactly the result we need. □

Corollary 2. *The measure μ is positive.*

Proof. To prove this result, it suffices to show that the total mass of μ_k is bounded below, independently of k. At each stage k,

$$\#(\mathcal{B}^d_{l_k}(X_k)) \leqslant \left[\frac{l_{k-1}\dots l_1}{r_k\dots r_1}\right]^d.$$

Since only a fraction $1/2^{2k+2}$ of the cubes in $\mathcal{B}^d_{r_k}(X_k)$ do not contain an cube in X_{k+1}, it is only for at most a fraction $1/2^{2k+1}$ of the cubes in $\mathcal{B}^d_{r_k}(X_k)$ cubes that X_{k+1} fails to equidistribute over more than half of the cubes. But this means that we discard a total mass of at most

$$\left(\frac{1}{2^{2k+1}}\left[\frac{l_{k-1}\dots l_1}{r_k\dots r_1}\right]^d\right)\left(2^k\left[\frac{r_k\dots r_1}{l_{k-1}\dots l_1}\right]^d\right) \leqslant 1/2^{k+1}.$$

Thus

$$\mu_k(\mathbf{R}^d) \geqslant 1 - \sum_{i=0}^k \frac{1}{2^{i+1}} \geqslant 1/2.$$

This implies $\mu(\mathbf{R}^d) \geqslant 1/2$, and in particular, $\mu \neq 0$. ☐

Ignoring all parameters in the inequality for I which depend on indices smaller than k, we 'conclude' that $\mu_k(I) \lesssim r_k^d \lesssim l_k^{\beta d - O(1/k)}$. The equation $l_{k+1} \leqslant l_k^{k^2}$ implies l_k decays very rapidly, which enables us to ignore quantities depending on previous indices, and obtain a true inequality.

Corollary 3. *For all $I \in \mathcal{B}^d_{l_k}$, $\mu(I) \leqslant \mu_k(I) \lesssim l_k^{d\beta - O(1/k)}$.*

Proof. Given ε, we find

$$\begin{aligned}\mu_k(I) &\leqslant 2^k\left[\frac{r_k\dots r_1}{l_{k-1}\dots l_1}\right]^d \leqslant \left(\frac{2^{d+k}}{l_{k-1}^{d(1-\beta_{k-1})}\dots l_1^{d(1-\beta_1)}}\right) l_k^{d\beta_k}\\ &\leqslant \left(2^{d+k} l_k^{\varepsilon}/l_{k-1}^{d(k-1)}\right) l_k^{d\beta_k - \varepsilon} \leqslant \left(2^{d+k} l_{k-1}^{\varepsilon k^2 - d(k-1)}\right) l_k^{d\beta_k - \varepsilon}.\end{aligned}$$

The open bracket term decays as $k \to \infty$ so fast that it still tends to zero if ε is not fixed, but is instead equal to $1/k$, which gives the result. ☐

This is the cleanest expression of the $d\beta$ dimensional behaviour at discrete scales we will need. To get a general inequality of this form, we use the fact that our construction distributes uniformly across all cubes.

Lemma 6. *If $l \leqslant l_k$ is dyadic and $I \in \mathcal{B}^d_l$, then $\mu(I) \lesssim l^{d\beta - O(1/k)}$.*

Proof. We use a covering argument, which breaks into cases depending on the size of l in proportion to l_k and r_k:

- If $r_{k+1} \leqslant l \leqslant l_k$, we can cover I by $(l/r_{k+1})^d$ cubes in $\mathcal{B}^d_{r_{k+1}}$. For each of these cubes, because the mass is equidistributed over r_{k+1} cubes, we know the mass is bounded by at most $2(r_{k+1}/l_{k+1})^d$ times the mass of a $\mathcal{B}^d_{l_k}$ cube. Thus

  $$\mu(I) \lesssim (l/r_{k+1})^d (2(r_{k+1}/l_k)^d) l_k^{d\beta - O(1/k)} \leqslant 2l^d / l_k^{d+O(1/k)-d\beta} \leqslant 2l^{d\beta - O(1/k)}.$$

  where we used the fact that $d + O(1/k) - d\beta \geqslant 0$.

- If $l_{k+1} \leqslant l \leqslant r_{k+1}$, we can cover I by a single cube in $\mathcal{B}^d_{r_{k+1}}$. Each cube in $\mathcal{B}^d_{r_{k+1}}$ contains at most one cube in $\mathcal{B}^d_{l_{k+1}}$ which is also contained in X_{k+1}, so

  $$\mu(I) \lesssim l_{k+1}^{d\beta - O(1/k)} \leqslant l^{d\beta - O(1/k)}.$$

- If $l \leqslant l_{k+1}$, there certainly exists M such that $l_{M+1} \leqslant l \leqslant l_M$, and one of the previous cases yields that

  $$\mu(I) \lesssim 2l^{d\beta - O(1/M)} \leqslant 2l^{d\beta - O(1/k)}. \qquad \square$$

To use Frostman's lemma, we need the result $\mu(I) \lesssim l^{d\beta - O(1/k)}$ for an *arbitrary* dyadic cube, not just one with $l \leqslant l_k$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. If $l \geqslant l_k$, then $\mu(I)/l^{d\beta - O(1/k)} \leqslant 1/l_k^{d\beta - O(1/k)} \lesssim_k 1$, so $\mu(I) \lesssim_k l^{d\beta - O(1/k)}$ holds automatically for all sufficiently large cubes. Thus $\dim_{\mathbf{H}}(X) \geqslant d\beta - O(1/k)$, and letting $k \to \infty$ gives $\dim_{\mathbf{H}}(X) \geqslant d\beta = (dn - \alpha)/(n-1)$.

Lemma 7. $\dim_{\mathbf{H}}(X) = (dn - \alpha)/(n-1)$.

Proof. X_k is covered by at most

$$\left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} \right]^d$$

sidelength l_k cubes. It follows that if $\gamma > \beta_k$, then

$$H^{d\gamma}_{l_k}(X) \leqslant \left[\frac{l_{k-1} \dots l_1}{r_k \dots r_1} l_k^\gamma \right]^d \lesssim \left[\frac{l_{k-1} \dots l_1}{r_{k-1} \dots r_1} l_k^{\gamma - \beta_k} \right]^d \leqslant l_k^{d(\gamma - \beta_k)}.$$

Since $l_k \to 0$ as $k \to \infty$, $H^{d\gamma}(X) = 0$. Since γ was arbitrary, $\dim_{\mathbf{H}}(X) \leqslant d\beta_k$, and since k was arbitrary, $\dim_{\mathbf{H}}(X) \leqslant d\beta$. $\square$

5 Applications

Our result already generalizes methods with interesting applications. But the most novel applications of our method occur when the configurations truly form a fractal set.

Theorem 2 (Sum-Sets Avoiding Fractals). *If $Y \subset \mathbf{R}^d$ is a countable union of sets with lower Minkowski dimension upper at most α, then there exists a set X with Hausdorff dimension $d - \alpha$ such that $X + X$ is disjoint from Y.*

Proof. Consider the set $Z_0 := \{(x, y) : x + y \in Y\}$, which is the countable union of sets with lower Minkowski dimension upper bounded by $d + \alpha$. Our main theorem gives X with dimension $d - \alpha$ such that if $x_1, x_2 \in X$ are distinct, $(x_1, x_2) \notin Z_0$, so $x_1 + x_2 \notin Y$, which *almost* gives the correct answer to the theorem. But there may be $x \in X$ such that $x + x = 2x \in Y$. Fortunately, $Z_1 := Z_0 \cup (Y/2 \times \mathbf{R}^d)$ is also the countable union of sets with lower Minkowski dimension upper bounded by $d + \alpha$, and the resultant X with $(x_1, x_2) \notin Z_1$ for distinct $x_1, x_2 \in X$ *does* satisfy the requirements of the theorem. □

Remark. *One problem with our result is that as the number of variables n increases, the dimension of X tends to zero. Thus if we try and make the n-fold sum $X + \cdots + X$ be disjoint from Y, current techniques only yield a dimension $(d-\alpha)/(n-1)$ set. We have ideas on how to improve our main result when Z is 'flat', in addition to being sparse, which will enable us to remove the dependence of $\dim_{\mathbf{H}}(X)$ on n, which we plan to publish in a later paper. This will enable us to still obtain us to consider sums of arbitrary length. In particular, we expect to be able to construct a set X disjoint from Y with the same dimension $d - \alpha$, such that X is closed under addition, and multiplication by rational numbers. In particular, given a $\mathbf{Q}$ subspace V of $\mathbf{R}^d$ with dimension α, we can always find a 'complementary' $\mathbf{Q}$ vector space W with complementary fractal dimension $d-\alpha$ such that $V \cap W = (0)$.*

In [2], Hausdorff dimension 1/2 subsets of smooth curves with non-vanishing curvature are constructed avoiding isoceles triangles. Our method improves this to find subsets of fractal sets avoiding isoceles triangles, with the curvature condition replaced with a hypothesis more fitting geometric measure theory. We are unaware of methods in the literature which enable one to construct 'subfractals' avoiding configurations, which makes this result particularly interesting.

Theorem 3 (Isoceles Triangle Avoiding Subfractals). *Suppose we are given $Y \subset \mathbf{R}^2$ together with an orthogonal projection $\pi : \mathbf{R}^2 \to \mathbf{R}$ such that $\pi(Y)$ has non-empty interior. Provided that the set*

$$Z_0 = \{(y_1, y_2, y_3) \in Y^3 : d(y_1, y_2) = d(y_1, y_3)\}$$

is the countable union of sets with lower Minkowski dimension bounded by 2, there exists a dimension 1/2 subset $X \subset Y$ with no triple $(x_1, x_2, x_3) \in X^3$ forming the vertices of an isoceles triangle.

Proof. If we form the set

$$Z = \pi(Z_0) = \{(\pi(y_1), \pi(y_2), \pi(y_3)) : y \in Z_0\},$$

then Z is also the countable union of sets with lower Minkowski dimension bounded by 2. Our main method enables us to construct a Hausdorff dimension

1/2 set $X_0 \subset \pi(Y)$ such that for any distinct $x_1, x_2, x_3 \in X_0$, $(x_1, x_2, x_3) \notin Z$. Thus if we form a set X by picking, from each $x \in X_0$, a single element of $\pi^{-1}(x)$, then X avoids isoceles triangles, and has Hausdorff dimension at least as large as X_0. $\square$

For any fixed points P and Q, the points R with $d(P, R) = d(P, R)$ form a line L_{PQ} bisecting the plane between P and Q. Understanding the dimension of $L_{PQ} \cap Y$ for $P, Q \in Y$ is therefore key to prove that the set Z_0 in the hypothesis of the theorem has small dimension. If Y is a compact portion of a smooth curve with non-vanishing curvature, then $Y \cap L_{PQ}$ consists of finitely many points, bounded independantly of any choice of P, Q in the plane. This is the implicit condition in [2] which leads to their isoceles triangle avoiding result.

Results about slices of measures, e.g. in Chapter 6 of [6] indicate that for any one dimensional set Y, for almost every line l, $l \cap Y$ is a finite collection of points. This suggests that if Y is a generic set with fractal dimension one, then Z_0 has dimension at most 2, leading to a general result finding dimension 1/2 isoceles-avoiding subsets X of 'projectable' sets Y. But one difficulty in applying this theory is showing that the dimension of $L \cap Y$ is not too high on exceptional lines L. Thus this statement remains a conjecture, and we do not prove it here.

6 Relation to Literature, and Future Work

Our result is part of a growing body of work finding general methods to find sets avoiding patterns. The main focus of this section is comparing our method to the two other major results in the literature. [2] constructs sets with dimension $k/(n-1)$ avoiding the zero sets of rank k C^1 functions. In [5], sets of dimension d/l are constructed avoiding a degree l algebraic hypersurface specified by a polynomial with rational coefficients.

We can view our result as a robust version of Pramanik and Fraser's result. Indeed, if we try and avoid the zero set of a C^1 rank k function, then we are really avoiding a dimension $dn - k$ dimensional manifold. Our method gives a dimension

$$\frac{dn - (dn - k)}{n - 1} = \frac{k}{n - 1}$$

set, which is exactly the result obtained in [2].

That our result generalizes [2] should be expected because the technical skeleton of our construction is heavily modelled after their construction technique. Their result also reduces the problem to a discrete avoidance problem. But they *deterministically* select a particular sidelength S cube in every sidelength R cube. For arbitrary Z, this selection procedure can easily be exploited for a particularly nasty Z, so their method must rely on smoothness in order to ensure some cubes are selected at each stage. Our discrete avoidance technique was motivated by other combinatorial optimization problems, where adding a random quantity prevents inefficient selections from being made in expectation.

This allows us to rely purely on combinatorial calculations, rather than employing smoothness, and greatly increases the applicability of the sets Z we can apply our method to. Furthermore, it shows that the underlying problem is robust to changes in dimension; slightly 'thickening' Z only slightly perturbs the dimension of X.

One useful technique in [2], and its predecessor [4], is the use of a Cantor set construction 'with memory'; a queue in the construction algorithm for their sets allows storage of particular discrete versions of the problem to be stored, and then retrieved at a much later stage of the construction process. This enables them to 'separate' variables in the discrete version of the problem, i.e. instead of forming a single set F from a set E, they from n sets $F_1, \dots, F_n$ from disjoint sets $E_1, \dots, E_n$. The fact that our result is more general, yet does not rely on this technique is an interesting anomoly. An obvious advantage is that the description of the technique is much more simple. But an additional advantage is that we can attack 'one scale' of the problem at a time, rather than having to rely on stored memory from a vast number of steps before the current one. We believe that we can exploit the single scale approach to the problem to generalize our theorem to a much wider family of 'dimension α' sets Z, which we plan to discuss in a later paper.

As a generalization of the result in [2], our result has the same issues when compared to the result of [5]. When the parameter n is large, the dimension of our result suffers greatly, as with the n fold sum application in the last section. Furthermore, our result can't even beat trivial results if Z is almost full dimensional, as the next example shows.

Example. *Consider an α dimensional fractal set of angles Y, and try and find $X \subset \mathbf{R}^2$ such that the angle formed from any three points in X aoids Y. If we form the set*

$$Z = \left\{ (x,y,z) : \textit{There is } \theta \in Y \textit{ such that } \frac{(x-y)\cdot(x-z)}{|x-y||x-z|} = \cos\theta \right\}$$

Then we can find X avoiding Z. But one calculates that Z has dimension $3d + \alpha - 1$, which means X has dimension $(1-\alpha)/2$. Provided the set of angles does not contain π, the trivial example of a straight line beats our result.

Nonetheless, we still believe our method is a useful inspiration for new techniques in the 'high dimensional' setting. Most prior literature studies sets Z only implicitly as zero sets of some regular function f. The features of the function f imply geometric features of Z, which are exploited by these results. But some geometric features are not obvious from the functional perspective; in particular, the fractal dimension of the zero set of f is not an obvious property to study. We believe obtaining methods by looking at the explicit geometric structure of Z should lead to new techniques in the field, and we already have several ideas in mind when Z has geometric structure in addition to a fractal dimension bound, which we plan to publish in a later paper.

We can compare our randomized selection technique to a discrete phenomenon that has been recently noticed, for instance in [1]. There, certain combinatorial problems can be rephrased as abstract problems on hypergraphs, and one can then generalize the solutions of these problems using some strategy to improve the result where the hypergraph is sparse. Our result is a continuous analogue of this phenomenon, where sparsity is represented by the dimension of the set Z we are trying to avoid. One can even view Lemma 1 as a solution to a problem about independent sets in hypergraphs. In particular, we can form a hypergraph by taking the cubes $\mathcal{B}_s^d(E)$ as vertices, and adding an edge $(I_1, \dots, I_n)$ between n distinct cubes $I_k \in \mathcal{B}_s^d(E)$ if $I_1 \times \cdots \times I_n$ intersects Z'. An independent set of cubes in this hypergraph corresponds precisely to a set F with F^n disjoint except on a discretization of the diagonal. And so Lemma 1 really just finds a 'uniformly chosen' independent set in a sparse graph. Thus we really just applied the discrete phenomenon at many scales to obtain a continuous version of the phenomenon.

References

[1] József Balogh, Robert Morris, and Wojciech Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. **28** (2015), 669–709.

[2] Robert Fraser and Malabika Pramanik, *Large sets avoiding patterns*, Anal. PDE **11** (2018), 1083–1111.

[3] Nets Hawk Katz and Terrence Tao, *Some connections between Falconer's distance set conjecture, and sets of Furstenburg type*, New York J. Math. **7** (2001), 149–187.

[4] Tamás Keleti, *A 1-dimensional subset of the reals that intersects each of its translates in at most a single point*, Real Anal. Exchange **24** (1999), 843–845.

[5] András Máthé, *Sets of large dimension not containing polynomial configurations*, Adv. Math. **316** (2017), 691–709.

[6] Pertti Mattila, *Fourier analysis and Hausdorff dimension*, Cambridge University Press, 2015.