Putnum and Beyond Solution Manual

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Chapter 1

Basic Concepts

1.1 Preliminaries

Exercise 1.1. *Basic Vector Space Terminology.*

(a) Show that if A is an absorbing set or a nonempty balanced set, then $0 \in A$.

Proof. If A is absorbing, there is $\lambda > 0$ for which $0 \in \lambda A$. But then

$$0 = \lambda^{-1} 0 \in \lambda^{-1} \lambda A = A$$

If A is a non-empty balanced set, then $0 \in 0A \subset A$, since $|0| < |1|$. □

(b) Show that if A is balanced, then $\alpha A = A$ whenever $|\alpha| = 1$.

Proof. It is obvious that $\alpha A \subset A$. Conversely, since $|\alpha^{-1}| = 1$, $\alpha^{-1} A \subset A$. Given $a \in A$, $\alpha^{-1} a \in \alpha^{-1} A \subset A$, but then $a = \alpha(\alpha^{-1} a) \in \alpha A$. □

(c) Suppose that $\mathcal{B}$ is a collection of balanced subsets of X. Show that $\bigcup\{S : S \in \mathcal{B}\}$ and $\bigcap\{S : S \in \mathcal{B}\}$ are both balanced.

Proof. For any $|\alpha| \leq 1$, $B \in \mathcal{B}$, $\alpha B \subset B$, so that

$$\alpha \bigcap_{B \in \mathcal{B}} B = \bigcap_{B \in \mathcal{B}} \alpha B \subset \bigcap_{B \in \mathcal{B}} B$$

$$\alpha \bigcup_{B \in \mathcal{B}} B = \bigcup_{B \in \mathcal{B}} \alpha B \subset \bigcup_{B \in \mathcal{B}} B$$

and therefore the union and intersection of balanced sets is balanced. ☐

(d) *Suppose that $\mathcal{C}$ is a collection of convex subsets of X. Show that $\bigcap\{S : S \in \mathcal{C}\}$ is convex.*

Proof. If $C \in \mathcal{C}$ $a, b \in C$, $\lambda \in [0,1]$, then $\lambda a + (1-\lambda)B \in C$. By putting $\forall C \in \mathcal{C}$ in the front of these statements, we obtain the statement for the intersection. ☐

(e) *Show that if A is convex, then $x + A$ and αA are convex.*

Proof. If $x + a, x + b \in x + A$, $\lambda \in [0,1]$, then

$$\lambda(x+a) + (1-\lambda)(x+b) = x + (\lambda a + (1-\lambda)b) \in x + A$$

and therefore $x + A$ is convex. ☐

Exercise 1.2. (a) *Show that the "addition" and "multiplication by scalars" defined for sets obey the commutative and associative laws for vector spaces. That is, show that $A+B = B+A$, that $A+(B+C) = (A+B)+C$, and that $\alpha(\beta A) = (\alpha\beta)A$. Show also that $(x+A)+(y+B) = (x+y)+(A+B)$.*

(b) *Show that $\alpha(A+B) = \alpha A + \alpha B$.*

(c) *Show that $(\alpha+\beta)A \subset \alpha A + \beta A$, but that equality does not always hold.*

Proof. The equations can be verified pointwise. If the equations is satisfied on the left side by a point, it holds on the right side, and vice versa. This is not true of the third question, since ☐

Exercise 1.3. (a) *Prove that A is convex if and only if $sA + tA = (s+t)A$ for all positive s and t. (Consider the special case in which $s + t = 1$).*

Proof. s ☐

1.2 Norms

1.3 First Properties of Norm Spaces

Exercise 1.4. *Let K be a compact Hausdorff space and let X be a normed space. By Corollary 1.3.4, the collection of all continuous functions from K into X is a vector spacewhen functions are added and multiplied by scalars in the usual way. Define a norm on this vector space by the formula*

$$\|f\|_\infty = \begin{cases} \max\{\|f(x)\| : x \in K\} & \textit{if } K \neq \emptyset \\ 0 & K = \emptyset \end{cases}$$

The resulting normed space is denoted $C(K, X)$.

(a) Show that $\|\cdot\|_\infty$ is in fact a norm on $C(K, X)$.

Proof. $\|f + g\| \leq \|f\| + \|g\|$, since $\sup(A + B) \leq \sup A + \sup B$ for any A and B. $\|\alpha f\| = |\alpha|\|f\|$, since

$$\max\{\|\alpha f(x)\| : x \in K\} = |\alpha| \max\{\|f(x)\| : x \in K\}$$

And if $\|f\| = 0$, then $\|f(x)\| = 0$ for all x, so that $f(x) = 0$ for all x. □

(b) Show that if X is a Banach space, then so is $C(K, X)$.

Proof. Let $f_1, f_2, \dots$ be a Cauchy sequence in $C(K, X)$, so that $\|f_i - f_j\| \to 0$. □