Salem Sets Avoiding Rough Configurations

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November 1, 2019

Recall that a set $X \subset \mathbf{R}^d$ is a *Salem set* of dimension t if it has Hausdorff dimension t, and for every $\varepsilon > 0$, there exists a probability measure μ_ε supported on X such that

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(\xi)| < \infty. \tag{0.1}$$

It will be useful to note that if μ_ε is compactly supported, then (0.1) is equivalent to the equation

$$\sup_{k \in \mathbf{Z}^d} |k|^{s-\varepsilon} |\widehat{\mu_\varepsilon}(k)| < \infty. \tag{0.2}$$

Our goal in these notes is to obtain high dimensional Salem sets avoiding rough configurations.

Theorem 1. *Let $Z \subset \mathbf{R}^{dn}$ be the countable union of bounded sets, each with lower Minkowski dimension at most s. Then there exists a Salem set $X \subset \mathbf{R}^d$ with dimension*

$$\frac{nd - s}{n},$$

such that for any n distinct elements $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$.

We rely on a random selection approach, like in our paper on rough configurations, to obtain such a result, since such random selections give high probability bounds on the Fourier transform of the measures we study.

1 Concentration Inequalities

Define a convex function $\psi_2 : [0,\infty) \to [0,\infty)$ by $\psi_2(t) = e^{t^2} - 1$, and a corresponding Orlicz norm on the family of scalar valued random variables X over a probability space by setting

$$\|X\|_{\psi_2(L)} = \inf \{A \in (0,\infty) : \mathbf{E}(\psi_2(|X|/A)) \leqslant 1\}.$$

The family of random variables $\psi_2(L)$ are known as *subgaussian random variables*. Here are some important properties:

- (Gaussian Tails): If $\|X\|_{\psi_2(L)} \leqslant A$, then for each $t \geqslant 0$,

$$\mathbf{P}(|X| \geqslant t) \leqslant 10 \exp\left(-t^2/10A^2\right).$$

- (Bounded Variables are Subgaussian): For any random X,

$$\|X\|_{\psi_2(L)} \leqslant 10\|X\|_{L^\infty}.$$

- (Hoeffding's Inequality): If $X_1, \dots, X_N$ are *independent* variables, then

$$\|X_1 + \dots + X_N\|_{\psi_2(L)} \leqslant 10\left(\|X_1\|^2_{\psi_2(L)} + \dots + \|X_N\|^2_{\psi_2(L)}\right)^{1/2}.$$

The Orlicz norm is a convenient notation to summarize calculations involving the principle of concentration of measure. Roughly speaking, we can think of a random variable X with $\|X\|_{\psi_2(L)} \leqslant A$ as vary rarely deviating outside the interval $[-A, A]$.

2 A Family of Cubes

Fix integer-valued sequences $\{K_m : m \geqslant 1\}$ and $\{M_m : m \geqslant 1\}$, and then set $N_m = K_m M_m$. We then define two sequences of real numbers $\{l_m : m \geqslant 0\}$ and $\{r_m : m \geqslant 0\}$, by

$$l_m = \frac{1}{N_1 \dots N_m} \quad \text{and} \quad r_m = \frac{1}{N_1 \dots N_{m-1} M_m}.$$

For each $m, d \geqslant 0$, we define two collections of strings

$$\Sigma_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \dots \times [M_m]^d \times [K_m]^d$$

and

$$\Pi_m^d = \mathbf{Z}^d \times [M_1]^d \times [K_1]^d \times \cdots \times [K_{m-1}]^d \times [M_m]^d.$$

For each string $i \in \Sigma_m^d$, we define a vector $a_i \in (l_m \mathbf{Z})^d$ by setting

$$a_i = i_0 + \sum_{k=1}^{m} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k$$

Then each string $i \in \Sigma_m^d$ can be identified with the sidelength l_m cube

$$Q_i = \prod_{j=1}^{d} [a_{ij}, a_{ij} + l_m].$$

centered at a_i. Similarly, for each string $i \in \Pi_m^d$, we define a vector $a \in (r_m \mathbf{Z})^d$ by setting, for each $1 \leqslant j \leqslant d$,

$$a_i = i_0 + \left(\sum_{k=1}^{m-1} i_{2k-1} \cdot r_k + i_{2k} \cdot l_k \right) + i_{2m-1} \cdot r_m,$$

and then define a sidelength r_m cube

$$R_i = \prod_{j=1}^{d} [a_{ij}, a_{ij} + r_m].$$

We let $\mathcal{Q}_m^d = \{Q_i : i \in \Sigma_m^d\}$, and $\mathcal{R}_m^d = \{R_i : i \in \Pi_m^d\}$. Here are some important properties of this collection of cubes:

- For each m, $\mathcal{Q}_m^d$ and $\mathcal{R}_m^d$ are covers of $\mathbf{R}^d$.
- If $Q_1, Q_2 \in \bigcup_{m=0}^{\infty} \mathcal{Q}_m^d$, then either Q_1 and Q_2 have disjoint interiors, or one cube is contained in the other. Similarly, if $R_1, R_2 \in \bigcup_{m=1}^{\infty} \mathcal{R}_m^d$, then either R_1 and R_2 have disjoint interiors, or one cube is contained in the other.
- For each cube $Q \in \mathcal{Q}_m$, there is a unique cube $Q^* \in \mathcal{R}_m$ with $Q \subset Q^*$. We refer to Q^* as the *parent cube* of Q. Similarly, if $R \in \mathcal{R}_m$, there is a unique cube in $R^* \in \mathcal{Q}_{m-1}$ with $R \subset R^*$, and we refer to R^* as the *parent cube* of R.

We say a set $E \subset \mathbf{R}^d$ is $\mathcal{Q}_m$ discretized if it is a union of cubes in $\mathcal{Q}_m^d$, and we then let $\mathcal{Q}_m(E) = \{Q \in \mathcal{Q}_m^d : Q \subset E\}$. Similarily, we say a set $E \subset \mathbf{R}^d$ is $\mathcal{R}_m$ discretized if it is a union of cubes in $\mathcal{R}_m^d$, and we then let $\mathcal{R}_m(E) = \{R \in \mathcal{R}_m^d : R \subset E\}$. We set $\Sigma_m(E) = \{i \in \Sigma_m^d : Q_i \in \mathcal{Q}_m(E)\}$, and $\Pi_m(E) = \{i \in \Pi_m^d : R_i \in \mathcal{R}_m(E)\}$. We say a cube $Q_1 \times \cdots \times Q_n \in \mathcal{Q}_m^{dn}$ is *strongly non diagonal* if there does not exist two distinct indices i, j, and a third index $k \in \Pi_m^d$, such that $R_k \cap Q_i, R_k \cap Q_j \neq \varnothing$.

3 A Family of Mollifiers

We now consider a family of mollifiers, which we will use to smooth out the Fourier transform of the measures we study.

Lemma 2. *There exists a non-negative, C^∞ function ψ supported on $[-1,1]^d$ such that*

$$\int_{\mathbf{R}^d} \psi(x)\, dx = 1, \tag{3.1}$$

and for each $x \in \mathbf{R}^d$,

$$\sum_{n \in \mathbf{Z}^d} \psi(x+n) = 1. \tag{3.2}$$

Proof. Let α be a non-negative, C^∞ function compactly supported on $[0,1]$, such that $\alpha(1/2+x) = \alpha(1/2-x)$ for all $x \in \mathbf{R}$, $\alpha(x) = 1$ for $x \in [1/3, 2/3]$, and $0 \leqslant \alpha(x) \leqslant 1$ for all $x \in \mathbf{R}$. Then define β to be the non-negative, C^∞ function supported on $[-1/3, 1/3]$ defined for $x \in [-1/3, 1/3]$ by

$$\beta(x) = 1 - \alpha(|x|).$$

Symmetry considerations imply that $\int_{\mathbf{R}^d} \alpha(x) + \beta(x) = 1$, and for each $x \in \mathbf{R}$,

$$\sum_{m \in \mathbf{Z}} \alpha(x+m) + \beta(x+m) = 1. \tag{3.3}$$

If we set

$$\psi_0(x) = \alpha(x) + \beta(x),$$

then ψ_0 satisfies the required constraints, at least in the one dimensional case. In general, define $\psi(x_1, \dots, x_d) = \psi_0(x_1) \dots \psi_0(x_d)$. $\square$

Fix some choice of ψ given by Lemma 2. Since ψ is C^∞ and compactly supported, then for each $t \in [0,\infty)$, we conclude

$$\sup_{\xi \in \mathbf{R}^d} |\xi|^t |\widehat{\psi}(\xi)| < \infty. \tag{3.4}$$

Now we rescale the mollifier. For each $m > 0$, we let

$$\psi_m(x) = l_m^{-d} \psi(l_m \cdot x).$$

Then ψ_m is supported on $[-l_m, l_m]^d$. Equation (3.1) implies that for each $x \in \mathbf{R}^d$,

$$\int_{\mathbf{R}^d} \psi_m = 1. \tag{3.5}$$

Equation (3.2) implies

$$\sum_{n \in \mathbf{Z}^d} \psi(x + l_m \cdot n) = l_m^{-d}. \tag{3.6}$$

An important property of the rescaling in the frequency domain is that for each $\xi \in \mathbf{R}^d$,

$$\widehat{\psi_m}(\xi) = \widehat{\psi}(l_m \xi), \tag{3.7}$$

In particular, (3.7) implies that for each $t \geqslant 0$,

$$\sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi_m}(\xi)||\xi|^t = l_m^{-t} \sup_{|\xi| \in \mathbf{R}^d} |\widehat{\psi}(\xi)||\xi|^t. \tag{3.8}$$

Thus, uniformly in m, $\widehat{\psi_m}$ decays sharply outside of the box $[-l_m^{-1}, l_m^{-1}]^d$, a manifestation of the Heisenberg uncertainty principle.

4 Discrete Lemma

We now consider a discrete form of the Fourier bound argument, which we can apply iteratively to obtain a Salem set avoiding configurations.

Lemma 3. *Fix $s \in [1, dn)$ and $\varepsilon \in [0, (dn-s)/2)$. Let $T \subset [0,1]^d$ be a non-empty, $\mathcal{Q}_m$ discretized set, and let μ_T be a smooth probability measure compactly supported on T. Let $B \subset \mathbf{R}^{dn}$ be a non-empty, $\mathcal{Q}_{m+1}$ discretized set such that*

$$\#(\mathcal{Q}_{m+1}(B)) \leqslant (1/l_{m+1})^{s+\varepsilon}. \tag{4.1}$$

Then there exists a large constant $C(\mu_T, l_m, n, d, s, \varepsilon)$, such that if

$$K_{m+1}, M_{m+1} \geqslant C(\mu_T, l_m, n, d, s, \varepsilon, l_m), \tag{4.2}$$

and

$$M_{m+1}^{\frac{s}{dn-s}+c\varepsilon} \leqslant K_{m+1} \leqslant 2M_{m+1}^{\frac{s}{dn-s}+c\varepsilon}, \tag{4.3}$$

where

$$c = \frac{6dn}{(dn-s)^2},$$

then there exists a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$ together with a smooth probability measure supported on S such that

(A) For any strongly non-diagonal cube

$$Q = Q_1 \times \cdots \times Q_n \in \mathcal{Q}_{m+1}(B),$$

there exists i such that $Q_i \notin \mathcal{Q}_{m+1}(S)$.

(B) If $L = \sup_{k \in \mathbf{Z}^d} |k|^{\frac{dn-s}{2n} - a\varepsilon} |\widehat{\mu_T}(k)|$, then

$$\sup_{k \in \mathbf{Z}^d} |k|^{a\varepsilon - \frac{dn-s}{2n}} |\widehat{\mu_S}(k)| \leqslant (1 + 1/2^m)(L + 1/2^m),$$

where

$$a = \frac{3d + 2dn - 2s}{dn}.$$

Remark 4. *To make the statement of* (3) *more clean, we have hidden the explicit choice of constant $C(\mu_T, l_m, n, d, s, \varepsilon)$. But this constant can certainly be made explicit; such a choice can be made by ensuring that* (4.2) *implies* (4.5), (4.10), (4.16), (4.25), (4.26), (4.27), (4.36), *and* (4.37).

Proof of Lemma 3. First, we describe the construction of the set S, and the measure μ_S. For each $i \in \Pi_{m+1}^d$, let j_i be a random integer vector chosen from $[K_{m+1}]^d$, such that the family $\{j_i : i \in \Pi_{m+1}^d\}$ is an independent family of random variables. Then it is certainly true for any $j \in [K_{m+1}]^d$,

$$\mathbf{P}(j_i = j) = K_{m+1}^{-d}. \tag{4.4}$$

Define a measure ν_S such that, for each $x \in \mathbf{R}^d$,

$$d\nu_S(x) = r_{m+1}^d \sum\nolimits_{i \in \Pi_{m+1}^d} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

We then normalize, defining

$$\mu_S = \frac{\nu_S}{\nu_S(\mathbf{R}^d)}.$$

If we set

$$S = \bigcup \{Q \in \mathcal{Q}^d_{m+1} : \mu_S(Q) > 0\},$$

then S is $\mathcal{Q}_{m+1}$ discretized, μ_S is supported on S, and $S \subset T$. Our goal is to show that, with non-zero probability, some choice of $\{j_i\}$ yields a set S satisfying Properties (A) and (B) of Lemma 3.

In our calculations, it will help us to decompose the measure ν_S into components roughly supported on sidelength r^d_{m+1} cubes. For each $i \in \Pi_{m+1}(T)$, define a measure ν_i such that for each $x \in \mathbf{R}^d$,

$$d\nu_i(x) = r^d_{m+1} \psi_{m+1}(x - a_{ij_i}) d\mu_T(x).$$

Then $\nu_S = \sum_{i \in \Pi^d_{m+1}(T)} \nu_i$. We shall split the proof of Properties (A) and (B) of Lemma 3 into several, more managable lemmas.

Lemma 5. *If*

$$M_{m+1} \geqslant \left(3^d \sqrt{d} \cdot l_m \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}\right)^2, \tag{4.5}$$

then almost surely, $|\nu_S(\mathbf{R}^d) - 1| \leqslant M^{-1/2}_{m+1}$.

Proof. If $j_0, j_1 \in [K_{m+1}]^d$, then

$$|a_{ij_0} - a_{ij_1}| = |j_0 - j_1| \cdot l_{m+1} \leqslant (\sqrt{d} K_{m+1}) \cdot l_{m+1} = \sqrt{d} \cdot r_{m+1},$$

which, together with (3.5), implies

$$\begin{aligned} &\left| r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_0}) \mu_T(x) - r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x - a_{ij_1}) \mu_T(x) \right| \\ &\quad \leqslant r^d_{m+1} \int_{\mathbf{R}^d} \psi_{m+1}(x) \, |\mu_T(x + a_{ij_0}) - \mu_T(x + a_{ij_1})| \\ &\quad \leqslant \sqrt{d} \cdot r^{d+1}_{m+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)} \int_{\mathbf{R}^d} \psi_{m+1}(x) \\ &\quad = \sqrt{d} \cdot r^{d+1}_{m+1} \cdot \|\nabla \mu_T\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \tag{4.6}$$

Thus (4.6) implies that almost surely, for each i,

$$|\nu_i(\mathbf{R}^d) - \mathbf{E}(\nu_i(\mathbf{R}^d))| \leqslant \sqrt{d} \cdot r^{d+1}_{m+1} \|\nabla \mu\|_{L^\infty(\mathbf{R}^d)}. \tag{4.7}$$

Furthermore, (3.6) implies

$$\begin{aligned}\sum_{i\in\Pi_{m+1}^d}\mathbf{E}(\nu_i(\mathbf{R}^d)) &= r_{m+1}^d\sum\nolimits_{(i,j)\in\Sigma_{m+1}^d}\mathbf{P}(j_i=j)\int_{\mathbf{R}^d}\psi_{m+1}(x-a_{ij})\mu_T(x)\,dx\\ &= \frac{r_{m+1}^d}{K_{m+1}^d}\int_{\mathbf{R}^d}\left(\sum\nolimits_{(i,j)\in\Sigma_{m+1}^d}\psi_{m+1}(x-a_{ij})\right)\mu_T(x)\,dx\\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d}=1.\end{aligned}\tag{4.8}$$

For all but at most $3^d\cdot r_{m+1}^{-d}$ indices i, $\nu_i=0$ almost surely. Thus we can apply the triangle inequality together with (4.7) and (4.8) to conclude that almost surely,

$$\begin{aligned}|\nu_S(\mathbf{R}^d)-1| &= \left\|\sum\nolimits_{i\in\Pi_{m+1}^d}\left[\nu_i(\mathbf{R}^d)-\mathbf{E}(\nu_i(\mathbf{R}^d))\right]\right\|_{L^\infty}\\ &\leqslant \sum\nolimits_{i\in\Pi_{m+1}^d}\left\|\nu_i(\mathbf{R}^d)-\mathbf{E}(\nu_i(\mathbf{R}^d))\right\|_{L^\infty}\\ &\leqslant 3^d\sqrt{d}\cdot r_{m+1}^{-d}r_{m+1}^{d+1}\|\nabla\mu\|_{L^\infty(\mathbf{R}^d)}\\ &= 3^d\sqrt{d}\cdot r_{m+1}\|\nabla\mu\|_{L^\infty(\mathbf{R}^d)}\\ &= \frac{3^d\sqrt{d}\cdot l_m\|\nabla\mu\|_{L^\infty(\mathbf{R}^d)}}{M_{m+1}}.\end{aligned}\tag{4.9}$$

Thus (4.5) and (4.9) imply that almost surely, $|\nu_S(\mathbf{R}^d)-1|\leqslant M_{m+1}^{-1/2}$. □

Lemma 6. *If*

$$M_{m+1}\geqslant\left(10\cdot 3^{dn}\cdot l_m^{-(s+\varepsilon)}\right)^{1/\varepsilon},\tag{4.10}$$

then

$$\mathbf{P}\left(S\text{ does not satisfies Property (A)}\right)\leqslant 1/10.$$

Proof. For any cube $Q_{ij}\in\Sigma_{m+1}^d$, there are at most 3^d pairs $(i_0,j_0)\in\Sigma_{m+1}^d$ such that $Q_{i_0j_0}\cap Q_{ij}\neq\varnothing$, and so a union bound together with (4.4) gives

$$\mathbf{P}(Q_{ij}\in\mathcal{Q}_{m+1}(S))\leqslant\sum\nolimits_{Q_{i_0j_0}\cap Q_{ij}\neq\varnothing}\mathbf{P}(j_{i'}=j')\leqslant 3^dK_{m+1}^{-d}.\tag{4.11}$$

Without loss of generality, removing cubes from B if necessary, we may assume all cubes in B are strongly non-diagonal. Let $Q=Q_{i_1j_1}\times\cdots\times Q_{i_nj_n}\in$

$\mathcal{Q}_{m+1}(B)$ be such a cube. Since Q is strongly diagonal, the events $\{Q_{i_k j_k} \in S\}$ are independent from one another, which together with (4.11) implies that

$$\mathbf{P}(Q \in \mathcal{Q}_{m+1}(S^n)) = \mathbf{P}(Q_{i_1 j_1} \in S) \dots \mathbf{P}(Q_{i_n j_n} \in S) \leqslant 3^{dn} K_{m+1}^{-dn}. \tag{4.12}$$

Taking expectations over all cubes in B, and applying (4.1) and (4.12) gives

$$\begin{aligned} \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) &\leqslant \#(\mathcal{Q}_{m+1}(B)) \cdot (3^{dn} K_{m+1}^{-dn}) \\ &\leqslant l_{m+1}^{-(s+\varepsilon)} (3^{dn} K_{m+1}^{-dn}) \\ &= \frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}}. \end{aligned} \tag{4.13}$$

Since $\varepsilon \leqslant (dn-s)/2$, we conclude

$$\begin{aligned} (dn-s-\varepsilon)\left(\frac{s}{dn-s} + c\varepsilon\right) &= s + \varepsilon\left(c(dn-s-\varepsilon) - \frac{s}{dn-s}\right) \\ &\geqslant s + \varepsilon\left(\frac{c(dn-s)}{2} - \frac{s}{dn-s}\right) \\ &= s + \varepsilon\frac{3dn-s}{dn-s} \geqslant s + 2\varepsilon. \end{aligned}$$

Applying (4.3) together with this bound, we conclude that

$$K_{m+1}^{dn-s-\varepsilon} \geqslant M_{m+1}^{(dn-s-\varepsilon)\left(\frac{s}{dn-s}+c\varepsilon\right)} \geqslant M_{m+1}^{s+2\varepsilon}.$$

Combined with (4.10), we conclude that

$$\frac{3^{dn} l_m^{-(s+\varepsilon)} M_{m+1}^{s+\varepsilon}}{K_{m+1}^{dn-s-\varepsilon}} \leqslant \frac{3^{dn} l_m^{-(s+\varepsilon)}}{M_{m+1}^{\varepsilon}} \leqslant 1/10. \tag{4.14}$$

We can then apply Markov's inequality with (4.13) and (4.14) to conclude

$$\begin{aligned} \mathbf{P}(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n) \neq \varnothing) &= \mathbf{P}(\#(\mathcal{Q}_{k+1}(B) \cap \mathcal{Q}_{k+1}(S^n)) \geqslant 1) \\ &\leqslant \mathbf{E}(\#(\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n))) \\ &\leqslant 1/10. \qquad \square \end{aligned}$$

Lemma 7. *Set $D = \{k \in \mathbf{Z}^d : |k| \leqslant 10 l_{m+1}^{-1}\}$. Then if*

$$K_{m+1} \leqslant M_{m+1}^{\frac{2dn}{dn-s}}, \tag{4.15}$$

and

$$M_{m+1} \geqslant \exp\left(\frac{10^7(3dn-s)d^2}{dn-s}\right), \tag{4.16}$$

then

$$\mathbf{P}\left(\|\widehat{\nu_S} - \widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10$$

Proof. For each $i \in \Pi_{m+1}^d$, and $k \in \mathbf{Z}$, define $X_{ik} = \widehat{\nu_i}(k) - \widehat{\mathbf{E}(\nu_i)}(k)$. Applying (3.2) gives

$$\begin{aligned}\sum_{i \in \Pi_{m+1}^d} \widehat{\mathbf{E}(\nu_i)}(k) &= \sum_{i \in \Pi_{m+1}^d} l_{m+1}^d \sum_{j \in [K_{m+1}]^d} \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} \psi_{m+1}(x - a_{ij}) d\mu_T(x) \\ &= \int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) = \widehat{\mu_T}(k).\end{aligned} \tag{4.17}$$

For each i and k, the standard (L^1, L^∞) bound on the Fourier transform, combined with (4.7), shows

$$\begin{aligned}\|X_{ik}\|_{\psi_2(L)} &\leqslant 10\|X_{ik}\|_{L^\infty} \\ &\leqslant 10[\|\nu_i(\mathbf{R}^d)\|_{L^\infty} + \mathbf{E}(\nu_i)(\mathbf{R}^d)] \\ &\leqslant 10^2\left(\mathbf{E}(\nu_i)(\mathbf{R}^d) + \sqrt{d} \cdot r_{m+1}^{d+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{4.18}$$

For a fixed k, the family of random variables $\{X_{ik}\}$ are independent. Furthermore, $\sum X_{ik} = \widehat{\nu_S}(k) - \widehat{\mathbf{E}(\nu_S)}(k)$. Equations (3.6) and (4.4) imply that

$$\begin{aligned}\mathbf{E}(\widehat{\nu_S}(k)) &= \frac{r_{m+1}^d}{K_{m+1}^d}\int_{\mathbf{R}^d} e^{-2\pi i k \cdot x}\left(\sum_{(i,j) \in \Sigma_{m+1}^d} \psi_{m+1}(x - a_{ij})\right) d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d}\int_{\mathbf{R}^d} e^{-2\pi i k \cdot x} d\mu_T(x) \\ &= \frac{r_{m+1}^d l_{m+1}^{-d}}{K_{m+1}^d}\widehat{\mu_T}(k) = \widehat{\mu_T}(k).\end{aligned} \tag{4.19}$$

Hoeffding's inequality, together with (4.18) and (4.19), imply that

$$\begin{aligned}&\|\widehat{\nu}(k) - \widehat{\mu_T}(k)\|_{\psi_2(L)} \\ &\quad \leqslant 10^3\sqrt{d}\left(\left(\sum \mathbf{E}(\nu_i)(\mathbf{R}^d)^2\right)^{1/2} + r_{m+1}^{d/2+1}\|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right).\end{aligned} \tag{4.20}$$

Equation (3.5) shows

$$\begin{aligned}\mathbf{E}(\nu_i)(\mathbf{R}^d) &= l_{m+1}^d \sum_{j\in[K_{m+1}]^d} \int \psi_{m+1}(x-a_{ij})d\mu_T(x) \\ &\leqslant r_{m+1}^d \|\mu_T\|_{L^\infty(\mathbf{R}^d)}.\end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21) gives

$$\|\widehat{\nu}(k)-\widehat{\mu_T}(k)\|_{\psi_2(L)} \leqslant 10^3\sqrt{d}\left[\|\mu_T\|_{L^\infty(\mathbf{R}^d)} + \|\nabla\mu_T\|_{L^\infty(\mathbf{R}^d)}\right] r_{m+1}^{d/2}. \tag{4.22}$$

We can then apply a union bound over the set D, which has cardinality at most $10^{d+1}l_{m+1}^{-d}$, together with (4.22) to conclude that

$$\begin{aligned}&\mathbf{P}\left(\|\widehat{\nu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \\ &\qquad\leqslant 10^{d+2}\cdot l_{m+1}^{-d}\exp\left(-\frac{\log(M_{m+1})^2}{10^7 d}\right) \\ &\qquad= 10^{d+2}l_m^{-d}\exp\left(d\log(M_{m+1}K_{m+1}) - \frac{\log(M_{m+1})^2}{10^7 d}\right).\end{aligned} \tag{4.23}$$

Combined with (4.15) and (4.16), (4.23) implies

$$\mathbf{P}\left(\|\widehat{\nu_S}-\widehat{\mu_T}\|_{L^\infty(D)} \geqslant r_{m+1}^{d/2}\log(M_{m+1})\right) \leqslant 1/10. \tag{4.24}$$

Thus $\widehat{\nu_S}$ and $\widehat{\mu_T}$ are highly likely to differ only by a negligible amount over small frequencies. $\square$

Since μ_T is compactly supported, we can define, for each $t > 0$,

$$A(t) = \sup|\widehat{\mu_T}(\xi)||\xi|^t < \infty.$$

In light of (3.7), if we define, for each $t > 0$,

$$B(t) = \sup|\widehat{\psi}(\xi)||\xi|^t,$$

then

$$\sup|\widehat{\psi_{m+1}}(\xi)||\xi|^t = l_{m+1}^{-t}B(t).$$

Lemma 8. *Fix $r > 0$. If*

$$N_{m+1}^d \geqslant \frac{10 \cdot 2^{d+1+r/2}}{L} A(d+1+r/2), \tag{4.25}$$

$$N_{m+1}^d \geqslant \frac{10 \cdot 2^{3d}}{(1+r/2)L} A(d+1+r/2), \tag{4.26}$$

and

$$N_{m+1}^d \geqslant \frac{10 \cdot 2^{3d+r/2+1}}{L} B(d+r/2+1), \tag{4.27}$$

then almost surely, if $|\eta| \geqslant 10 l_{m+1}^{-1}$,

$$|\widehat{\nu_S}(\eta)| \leqslant \frac{L}{|\eta|^{r/2}}.$$

Proof. Define a random measure

$$\alpha = r_{m+1}^d \sum_{\substack{i \in \Pi_{m+1}^d \\ d(a_i, T) \leqslant 2r_{m+1}^{-1}}} \delta_{a_{ij_i}}.$$

Then $\nu_S = (\alpha * \psi_{m+1})\mu_T$. Thus we have $\widehat{\nu_S} = (\widehat{\alpha} \cdot \widehat{\psi_{m+1}}) * \widehat{\mu_T}$. The measure α is the sum of at most $2^d r_{m+1}^{-d}$ delta functions, scaled by r_{m+1}^d, so $\|\widehat{\alpha}\|_{L^\infty(\mathbf{R}^d)} \leqslant \alpha(\mathbf{R}^d) \leqslant 2^d$. Thus

$$|\widehat{\nu_S}(\eta)| \leqslant 2^d \int |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| \, d\xi. \tag{4.28}$$

If $|\xi| \leqslant |\eta|/2$, $|\eta - \xi| \geqslant |\eta|/2$, so for all $t > 0$, and since (3.5) implies $\|\widehat{\psi_{m+1}}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$, we find

$$\int_{0 \leqslant |\xi| \leqslant |\eta|/2} |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| \, d\xi \leqslant \frac{A(t) 2^{t-d}}{|\eta|^{t-d}}. \tag{4.29}$$

Set $t = d + 1 + r/2$. Equation (4.29), together with (4.25), implies

$$\begin{aligned} \int_{0 \leqslant |\xi| \leqslant |\eta|/2} & |\widehat{\mu_T}(\eta - \xi)| |\widehat{\psi_{m+1}}(\xi)| \, d\xi \\ &\leqslant \frac{A(d+1+r/2) 2^{1+r/2} |\eta|^{-1}}{|\eta|^{r/2}} \\ &\leqslant \frac{A(d+1+r/2) 2^{1+r/2} l_{m+1}}{|\eta|^{r/2}} \\ &\leqslant \frac{L}{10 \cdot 2^d \cdot |\eta|^{r/2}}. \end{aligned} \tag{4.30}$$

Conversely, if $|\xi| \geqslant 2|\eta|$, then $|\eta - \xi| \geqslant |\xi|/2$, so for each $t > d$,

$$\begin{aligned}
\int_{|\xi|\geqslant 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi}_{m+1}(\xi)|\, d\xi &\leqslant \int_{|\xi|\geqslant 2|\eta|} \frac{A(t)2^t}{|\xi|^t} \\
&\leqslant 2^d \int_{2|\eta|}^{\infty} r^{d-1-t} A(t) 2^t \qquad (4.31) \\
&\leqslant \frac{4^d A(t)}{t-d} |\eta|^{d-t}.
\end{aligned}$$

Set $t = d + 1 + r/2$. Equation (4.26), applied to (4.31), allows us to conclude

$$\int_{|\xi|\geqslant 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi}_{m+1}(\xi)|\, d\xi \leqslant \frac{L}{10 \cdot 2^d \cdot |\eta|^{s/2}}. \tag{4.32}$$

Finally, if $t > 0$, we use the fact that $\|\widehat{\mu_T}\|_{L^\infty(\mathbf{R}^d)} \leqslant 1$ to conclude that

$$\int_{|\eta|/2\leqslant|\xi|\leqslant 2|\eta|} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leqslant \frac{2^{d+t} B(t)}{|\eta|^{t-d}}. \tag{4.33}$$

Set $t = d + 1 + r/2$. Then (4.33) and (4.27) imply

$$\int_{|\eta|/2\leqslant|\xi|\leqslant 2|\eta} |\widehat{\mu_T}(\eta-\xi)||\widehat{\psi_{m+1}}(\xi)|\, d\xi \leqslant \frac{L}{10 \cdot 2^d \cdot |\eta|^{r/2}}. \tag{4.34}$$

It then suffices to sum up (4.30), (4.32), and (4.34), and apply (4.28). $\square$

Proof of Lemma 3, Continued. Let us now put all our calculations together. In light of Lemma 6 and Lemma 7, there exists some choice of j_i for each i, and a resultant non-random pair (ν_S, S) such that S satisfies Property (A) of the Lemma, and for any $k \in \mathbf{Z}^d$ with $|k| \leqslant 10 l_{m+1}^{-1}$,

$$|\widehat{\nu_S}(k) - \widehat{\mu_T}(k)| \leqslant r_{m+1}^{d/2} \log(M_{m+1}). \tag{4.35}$$

Now

$$r_{m+1}^{d/2} \log(M_{m+1}) = \left(l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \right) l_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}.$$

Equation (4.3) implies

$$\begin{aligned}
&l_{m+1}^{a\varepsilon - \frac{dn-s}{2n}} \cdot r_{m+1}^{d/2} \log(M_{m+1}) \\
&\quad = \frac{l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} \log(M_{m+1}) K_{m+1}^{\frac{dn-s}{2n} - a\varepsilon}}{M_{m+1}^{a\varepsilon + \frac{s}{2n}}} \\
&\quad \leqslant \left[l_m^{a\varepsilon - \frac{dn-s}{2n} + d/2} 2^{\frac{dn-s}{2n} - a\varepsilon} \right] \log(M_{m+1}) M_{m+1}^{\left(\frac{s}{dn-s} + c\varepsilon\right)\left(\frac{dn-s}{2n} - a\varepsilon\right) - a\varepsilon - \frac{s}{2n}}.
\end{aligned}$$

Now

$$\left(\frac{s}{dn-s}+c\varepsilon\right)\left(\frac{dn-s}{2n}-a\varepsilon\right)-a\varepsilon \leqslant \left[\frac{(dn-s)c}{2n}-\left(\frac{s}{dn-s}+1\right)a\right]\varepsilon$$
$$= \left[\frac{d(3-na)}{(dn-s)}\right]\varepsilon$$
$$\leqslant -2\varepsilon.$$

Since $\log(M_{m+1}) \leqslant (2/\varepsilon)M_{m+1}^{\varepsilon/2}$, if we assume that

$$M_{m+1} \geqslant \left(l_m^{a\varepsilon-\frac{dn-s}{2n}+d/2}2^{\frac{dn-s}{2n}-a\varepsilon}(2/\varepsilon)\right)^{2/\varepsilon}, \tag{4.36}$$

then we conclude

$$r_{m+1}^{d/2}\log(M_{m+1}) \leqslant \left[l_m^{a\varepsilon-\frac{dn-s}{2n}+d/2}2^{\frac{dn-s}{2n}-a\varepsilon}\log(M_{m+1})M_{m+1}^{-\varepsilon}\right]M_{m+1}^{-\varepsilon}l_{m+1}^{\frac{dn-s}{2n}-a\varepsilon}$$
$$\leqslant M_{m+1}^{-\varepsilon}l_{m+1}^{\frac{dn-s}{2n}-a\varepsilon}$$

Applying Lemma 7, we conclude that if $|k| \leqslant 10l_{m+1}^{-1}$,

$$|\widehat{\nu_S}(k)| \leqslant |\widehat{\nu_S}(k)-\widehat{\mu_T}(k)|+|\widehat{\mu_T}(k)|$$
$$\leqslant r_{m+1}^{d/2}\log(M_{m+1})+L|k|^{c\varepsilon-\frac{dn-s}{2n}}$$
$$\leqslant l_{m+1}^{-\frac{dn-s}{2n}-c\varepsilon+\varepsilon}+L|k|^{c\varepsilon-\frac{dn-s}{2n}}$$
$$\leqslant \left[L+10^dM_{m+1}^{-\varepsilon}\right]|k|^{c\varepsilon-\frac{dn-s}{2n}}.$$

Applying Lemma 8 with $r = 2c\varepsilon-(dn-s)/n$ implies that for $|k| \geqslant 10l_{m+1}^{-1}$,

$$|\widehat{\nu_S}(k)| \leqslant L|k|^{c\varepsilon-\frac{dn-s}{2n}} \leqslant [L+10^dM_{M+1}^{-\varepsilon}]|k|^{c\varepsilon-\frac{dn-s}{2n}},$$

Thus we conclude that for all $k \in \mathbf{Z}^d$,

$$|\widehat{\nu_S}(k)| \leqslant (L+10^dM_{m+1}^{-\varepsilon})|k|^{c\varepsilon-\frac{dn-s}{2n}}.$$

Applying Lemma 5, as well as a valid assumption that

$$M_{m+1} \geqslant 10^{d/\varepsilon}\cdot 4^{m/\varepsilon}, \tag{4.37}$$

we conclude that for all $k \in \mathbf{Z}^d$,

$$\begin{aligned} |\widehat{\mu_S}(k)| &\leqslant \frac{L + 10^d M_{m+1}^{-\varepsilon}}{1 - M_{m+1}^{-1/2}} |m|^{c\varepsilon - \frac{dn-s}{2n}} \\ &\leqslant (1 + M_{m+1}^{-1/2})[L + 10^d M_{m+1}^{-\varepsilon}]|k|^{c\varepsilon - \frac{dn-s}{2n}} \\ &\leqslant (1 + 1/2^m)[L + 10^d M_{m+1}^{-\varepsilon}]|k|^{c\varepsilon - \frac{dn-s}{2n}} \\ &\leqslant (1 + 1/2^m)[L + 1/2^m]|k|^{c\varepsilon - \frac{dn-s}{2n}}. \end{aligned}$$

□

5 Construction of the Salem Set

Let us now construct our configuration avoiding set. First, we fix some preliminary parameters. Write $Z \subset \bigcup_{i=1}^\infty Z_i$, where Z_i has lower Minkowski dimension at most s for each i. Then choose an infinite sequence $\{i_m : m \geqslant 1\}$ which repeats every positive integer infinitely many times. Also, choose an arbitrary, decreasing sequence of positive numbers $\{\varepsilon_m : m \geqslant 1\}$, with $\varepsilon_m < (dn - s)/2$ for each m.

We choose our parameters $\{M_m\}$ and $\{K_k\}$ inductively. First, set $X_0 = [0,1]^d$, and μ_0 an arbitrary smooth probability measure supported on X_0. At the m'th step of our construction, we have found a set X_{m-1} and a measure μ_{m-1}. We then choose K_m and M_m such that

$$K_m, M_m \geqslant C(\mu_{m-1}, n, d, s, \varepsilon_m),$$

such that

$$M_m \geqslant 2^{m/\varepsilon_m} \tag{5.1}$$

such that

$$M_m^{\frac{s}{dn-s} + c\varepsilon_m} \leqslant K_m \leqslant 2M_m^{\frac{s}{dn-s} + c\varepsilon},$$

and such that the set Z_{i_m} is covered by at most $l_m^{-(s+\varepsilon_m)}$ cubes in $\mathcal{Q}_m$, the union of which, we define to be equal to B_m. We can then apply Lemma 3 with $\varepsilon = \varepsilon_m$, $T = X_{m-1}$, $\mu_T = \mu_{m-1}$, and $B = B_m$. This produces a $\mathcal{Q}_{m+1}$ discretized set $S \subset T$, and a measure μ_S supported on S. We define $X_m = S$, and $\mu_m = \mu_S$.

The preceding paragraph recursively generates an infinite sequence $\{X_m\}$. We set $X = \bigcap X_m$. Just as in our previous paper, it is easy to see X must be a configuration avoiding set. We then find a measure μ, and some subsequence

μ_{m_k}, such that $\mu_{m_k} \to \mu$ weakly. It then follows from pointwise convergence of the Fourier transform that for each $\varepsilon > 0$,

$$\sup_{k \in \mathbf{Z}^d} |\widehat{\mu}(k)||k|^{\frac{dn-s}{2n} - \varepsilon} \leqslant \sup_{m > 0} \sup_{k \in \mathbf{Z}^d} |\widehat{\mu_m}(k)||k|^{\frac{dn-s}{2n} - \varepsilon}.$$

Fix $\varepsilon > 0$. For each m, define

$$A_{m,\varepsilon} = \sup_{k \in \mathbf{Z}^d} |\widehat{\mu_m}(k)||k|^{\frac{dn-s}{2n} - \varepsilon}.$$

Since each measure μ_m is smooth, all these quantities are finite. Since $\varepsilon_m \to 0$, there is M such that if $m \geqslant M$, then $a\varepsilon_m \leqslant \varepsilon$. Property (B) of Lemma (3) thus implies that for each $m \geqslant M$,

$$A_{m+1,\varepsilon} \leqslant (1 + 1/2^m)(A_{m,\varepsilon} + 1/2^m).$$

Since $\prod_{m=1}^{\infty}(1 + 2^{-m}) < \infty$ and $\sum 1/2^m < \infty$, we conclude that

$$\sup_{k \in \mathbf{Z}^d} |\widehat{\mu}(k)||k|^{\frac{dn-s}{2n} - \varepsilon} \leqslant \sup_{m \to \infty} A_{m,\varepsilon} < \infty.$$

Since ε was arbitrary, we conclude X has Fourier dimension $(dn-s)/n$. Since X_m is the union of $(M_1 \dots M_m)^d$ sidelength l_m cubes, one can easily show that the lower Minkowski dimension is upper bounded by $(dn - s)/n$. Thus X has Hausdorff dimension $(dn - s)/n$ as well, and so X is Salem. This concludes the proof of Theorem 1.

6 Körner's Work

The last sections gives a first, positive result finding Salem sets avoiding rough configurations. However, given the same assumptions, one can find a set X with *Hausdorff dimension* $(dn - s)/(n - 1)$ avoiding configurations. Improving the dimension to obtain this result requires a deeper knowledge of the stochastic behaviour of the set $\mathcal{Q}_{m+1}(B) \cap \mathcal{Q}_{m+1}(S^n)$, when N_{m+1} is chosen significantly smaller relative to M_{m+1}. In the next section, we provide a summary of an argument due to Körner, which deals with a very similar situation.