Fractals Avoiding Fractal Sets

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1 A Discrete Building Block

We now develop a discrete technique used to construct solutions to the fractal avoidance problem. It depends very little on the Euclidean structure of the plane. As such, we rephrase the construction as a combinatorial problem on graphs.

An n *uniform hypergraph* is a collection of *vertices* and *hyperedges*, where a hyperedge is a set of n distinct vertices. An *independent set* is a subset of vertices containing no complete set of vertices in any hyperedge of the graph. A *colouring* is a partition of the vertex set into finitely many independent sets, each of which we call a *colour*. Such a colouring is K *uniform* if each colour class has K elements.

The next lemma is a variant of Turán's theorem on independent sets. For technical reasons, we need an extra restriction on the independant set so it is 'uniformly' chosen over the graph. This is why colorings are introduced.

Lemma 1. *Let G be an n uniform hypergraph with a K uniform coloring. Then there exists an independent set W containing all but $|E|/K^n$ colors.*

Proof. Let U be a random vertex set chosen by selecting a vertex of each color uniformly randomly. Every vertex occurs in U with probability $1/K$. For any edge $e = (v_1, \dots, v_n)$, the vertices v_i all have different colors. Thus they have an independent chance of being added to U, and we calculate

$$\begin{aligned}&\mathbf{P}(v_1 \in U, \dots, v_n \in U)\\ &\quad = \mathbf{P}(v_1 \in U) \dots \mathbf{P}(v_n \in U) = 1/K^n\end{aligned}$$

If we let E' denote the set of all edges $e = (u_1, \dots, u_n)$ with $u_1, \dots, u_n \in U$, then

$$\mathbf{E}|E'| = \sum_{e \in E} \mathbf{P}(e \in E') = \sum_{e \in E} 1/K^n = \frac{|E|}{K^n}$$

This means we may choose a *particular*, nonrandom U for which $|E'| \leqslant |E|/K^n$. If we form a vertex set $W \subset U$ by removing, for each $e \in E'$, a vertex in U adjacent to the edge, then W is an independent set containing all but $|E'| \leqslant |E|/K^n$ colors. $\square$

Corollary. *If $|V| \gtrsim N^a$, $|E| \lesssim N^b$, and $K \gtrsim N^c$, where $b < a + c(n-1)$, then as $N \to \infty$ we can find an independent set containing all but a fraction $o(1)$ of the colors.*

Proof. A simple calculation on the quantities of the previous lemma yields

$$\begin{aligned}\frac{\#(\text{colors removed})}{\#(\text{all colors})} &= \frac{|E|/K^n}{|V|/K}\\ &= \frac{|E|}{|V|K^{n-1}} \lesssim \frac{N^b}{N^{a+c(n-1)}}\end{aligned}$$

This is $o(1)$ if $b < a + c(n-1)$. $\square$

We now apply these constructions to a problem clearly related to the fractal avoidance problem. It will form our key method to construct fractal avoidance solutions. Given an integer N, we subdivide $\mathbf{R}^d$ into a lattice of sidelength $1/N$ cubes with corners on $\mathbf{Z}^d/N$, the collection of such cubes we will denote by $\mathcal{B}_N$. This grid is used to granularize configuration avoidance.

Theorem 1. *Suppose $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint collections of cubes in $\mathcal{B}_N$, with $|\mathcal{I}_i| \gtrsim N^d$. We assume the lower Minkowski dimension of Y is bounded above by α, and $\beta > d(n-1)/(n-\alpha)$. Then there exists arbitrarily large N and collections of cubes $\mathcal{J}_1, \dots, \mathcal{J}_n \in \mathcal{B}_{N^\beta}$ with each cube in $\mathcal{J}_1 \times \dots \times \mathcal{J}_n$ disjoint from Y, and as $N \to \infty$, each $\mathcal{J}_i$ contains cubes in all but a fraction $o(1)$ of cubes in $\mathcal{I}_i$.*

Proof. If $\mathcal{K} \subset \mathcal{B}_{N^\beta}^n$ is the collection of all cubes in a sidelength $1/N^\beta$ lattice intersecting Y, then $|\mathcal{K}| \lesssim N^{\alpha\beta}$. We then let $\mathcal{I}_i'$ be all cubes in $\mathcal{B}_{N^\beta}$ contained in $\mathcal{I}_i$. Considering these cubes as vertices gives us an n uniform hypergraph G with a hyperedge between $I_1 \in \mathcal{I}_1', \dots, I_n \in \mathcal{I}_n'$ if $I_1 \times \dots \times I_n \in \mathcal{K}$. We say two cubes in G are the same color if they are contained in a common cube in $\mathcal{I}_i$.

Using the fact that a sidelength $1/N$ cube contains $N^{d(\beta-1)}$ sidelength $1/N^\beta$ cubes, we conclude that G has $\sum |\mathcal{I}_i| = N^{d(\beta-1)} \sum |\mathcal{I}_i| \gtrsim N^{d\beta}$ vertices. The number of edges in G is bounded by $|\mathcal{K}| \lesssim N^{\alpha\beta}$.

Finally, the coloring is $N^{d(\beta-1)}$ uniform. Thus in the terminology of the previous corollary, $a = d\beta$, $b = \alpha\beta$, and $c = d(\beta - 1)$, and the inequality in the hypothesis of this theorem is then equivalent to the inequality in the hypothesis of the corollary. Applying the corollary gives the required result. □

The value $d(n-1)/(n-\alpha)$ in the theorem is directly related to the dimension $(n-\alpha)/(n-1)$ we obtain in our main result. Any improvement on this bound for special cases of the fractal avoidance problem immediately leads to improvements on the Hausdorff dimension of the set constructed. The fact that our hypergraph result is tight indicates that for the general fractal avoidance problem, our construction gives tight bounds.

2 A Fractal Avoidance Set

We construct our solution X to the fractal avoidance problem by breaking down the problem into a sequence of discrete configuration problems on disecting cubes which lead to the complete fractal avoidance problem in the limit. The central idea of this construction was first used by Pramanik and Fraser (TODO: Insert Citation) in their general constructions to configuration avoidance problems. We construct $X = \lim X_N$, where each X_N is a union of cubes of a fixed length, and X_{N+1} is obtained from X_N by taking a certain subset of cubes in X_N, and dissecting this subset, subdividing the cube into cubes of a smaller sidelength and removing a portion of them. We will associate with each N a disjoint collection of sidelength L_N cubes $\mathcal{I}_1(N), \dots, \mathcal{I}_n(N)$, with all such cubes contained in X_N. The previous section immediately allows us to find a collection of sidelength $L_N^{\beta_n}$ cubes $\mathcal{J}_1(N) \subset \mathcal{I}_1(N), \dots, \mathcal{J}_n(N) \subset \mathcal{I}_n(N)$ with $\mathcal{J}_1(N) \times \dots \times \mathcal{J}_n(N)$ disjoint from Y, with β_n converging to $d(n-1)/(n-\alpha)$ from above. We then form X_{N+1} from X_N by removing each part of an cubes in $\mathcal{I}_i(N)$ which is not contained in an cubes in $\mathcal{J}_i(N)$, for each index i. We choose $X_0 = [0,1]$ as an initial to start off our construction.

There is only a simple constraint required on the parameters to this construction to ensure that X is a solution to the fractal avoidance problem: For any choice of distinct $x_1, \dots, x_n \in X$, there exists N such that for each i, x_i is contained in a cube in $\mathcal{I}_i(N)$. Since we surely know $x_1, \dots, x_n \in X_{N+1}$, it then follows that $x_1 \in \mathcal{J}_1(N), \dots, x_n \in \mathcal{J}_n(N)$, and so the tuple $(x_1, \dots, x_n)$ are contained in a cube in $\mathcal{J}_1(N) \times \mathcal{J}_n(N)$, which is disjoint from Y.

We achieve the constraint to the construction by choosing our parameters subject to a dynamically changing queue consisting $(I_1, \dots, I_n)$, where $I_1, \dots, I_n$ are disjoint cubes. To get the process tarted, we can initialize the queue to begin with the tuple $([0,1/n],[1/n,2/n],\dots,[(n-1)/n])$. At each step N of our process, we take off the front tuple $(I_1, \dots, I_n)$, subdivide X_N into a grid of length L_N cubes, and for each i, set $\mathcal{I}_i(N)$ to be the set of all such length L_N cubes which are contained in I_i. After this is done, we have a subdivision of X_{N+1} into length L_N^β cubes, and we add each choice of n length L_N^β disjoint intervals in X_{N+1} in this subdivision to the end of the queue. The queue grows inconcievably fast over time, but in the limit, every subdivision is processed. Provided that $L_N \to 0$, for any distinct $x_1, \dots, x_n \in X$ there is L_N with $|x_i - x_j| \geqslant 2L_N$, and so on the step N, we will add intervals $I_1, \dots, I_n$ with $x_1 \in I_1, \dots, x_n \in I_n$ to the end of the queue, and so eventually considered much further on in the construction. Thus we conclude that X is a solution to the fractal avoidance problem.

3 Dimension Bounds

To complete the proof, it suffices to choose the parameters L_N and β_N which lead to the correct Hausdorff dimension bound on X. The 'uniformity' result present in our discrete construction will aid us in eliminating the superexponentially increasing constants which emerge from the exponentially decrasing values of L_N we are forced to pick to eliminate the inherent multiplicative constants which occur in our construction.

To prove the dimension bounds on X, we rely on the mass distribution principle to construction a probability measure μ on X from which we can apply Frostman's lemma. We begin by putting the uniform probability measure μ_0 on $X_0 = [0,1]$. Then, at each stage of the construction, we construction μ_{N+1} from the measure μ_N supported on X_N by taking the mass of μ_N supported on a certain length L_{N-1} interval in X_N, and uniformly distributing it over the length L_N intervals contained with this interval which remain in X_{N+1}. Then we just use the weak compactness of the unit ball in $L^1(\mathbf{R}^d)^*$ to construct a weak limit $\mu = \lim \mu_n$, for which μ is supported on X. It should be intuitive that the mass will be distributed more thinly at each stage the fatter the intervals that are kept at each stage, and thus Frostman's lemma will obtain a higher Hausdorff dimension bound.

Lemma 2. *If I is an interval of length L_N in X_N, and $J \subset I$ is an interval of length L_{N+1} kept at the next stage in X_{N+1}, then*

References

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