Large Salem Sets Avoiding Configurations

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1   Introduction

Geometric measure theory explores the relationship between the geometry of subsets of Euclidean spaces, and regularity properties of the family of Borel measures supported on those subsets. From the perspective of harmonic analysis, it is interesting to explore what structural information can be gathered from the Fourier analytic properties of measures supported on a particular subset of Euclidean space. In this paper, we focus on the relationship between the Fourier analytic properties of a set and the existence of patterns on the set. In particular, given a pattern, we construct a family of sets which generically avoids this pattern, and which supports measures with fast Fourier decay.

A useful statistic associated with any Borel set $E \subset \mathbf{R}^d$ is it's *Fourier dimension*; given a finite Borel measure μ on $\mathbf{R}^d$, it's Fourier dimension, $\dim_{\mathbf{F}}(\mu)$, is the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ |\widehat{\mu}(\xi)||\xi|^{s/2} : \xi \in \mathbf{R}^d \right\} < \infty. \tag{1.1}$$

The Fourier dimension of a Borel set $E \subset \mathbf{R}^d$, denoted $\dim_{\mathbf{F}}(E)$, is then the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel probability measures μ supported on E. A particularly tractable family of sets in this scheme are *Salem sets*, those sets whose Fourier dimension agrees with their Hausdorff dimension. Most pattern avoiding sets constructed in the literature are not Salem, often having Fourier dimension zero. Nonetheless, the sets we construct in this paper are Salem. We prove two results, one avoiding 'rough' patterns, in the sense of [1], and the second avoiding patterns formed by a family of smooth surfaces.

Theorem 1.1. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{R}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $E \subset [0,1]^d$ with dimension*

$$\beta = \min \left(\frac{nd - \alpha}{n - 1/2}, d \right)$$

such that for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.

Remark 1.2. *Theorem 1.1 is an attempt to strengthen the main result of [1] to construct a set with a Fourier dimension bound rather than a Hausdorff dimension bound, albeit under a weaker dimension bound than that obtained in [1]. Unlike in [1], the case of Theorem 1.1 when $0 \leqslant \alpha < d$ is still interesting, since the trivial construction $[0,1]^d - \pi(Z)$ is not*

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necessarily a Salem set, where $\pi(x_1, \ldots, x_n) = x_1$ is projection onto the first coordinate. For instance, in Example 8 of [3] it is shown that there exists a compact set $E \subset [0,1]$ such that $\dim_{\mathbf{M}}(E) < 1$ and $\dim_{\mathbf{F}}([0,1] - E) < 1$. Setting $Z = E \times \{0\} \cup \{0\} \times E$ shows that neither subtracting projections onto the first nor the second coordinate gives the required Fourier dimension bounds.

In the case $d = 1$, under the stronger assumption that Z is a countable union of smooth hypersurfaces which are either curved, or satisfy a kind of finite-type condition (in the manner of [6], though slightly different from the definition there), we are able to fully recover the dimension bound originally proved in the Hausdorff dimension setting in [4] in the Fourier dimension instead.

Theorem 1.3. *Suppose $Z \subset \mathbf{R}^{dn}$ is a countable union of sets of the form*

$$\{x \in U \times V : x_i = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)\}$$

where U is an open set of $\mathbf{R}^d$, V is an open set of $\mathbf{R}^{d(n-1)}$, and $f : V \to \mathbf{R}^d$ is a smooth map such that for each unit vector $\eta \in \mathbf{R}^d$, either $\nabla(f \cdot \eta)(x) \neq 0$, or there exists $|\alpha| \leqslant 2(n-1)/d$ with $\partial^\alpha (f \cdot \eta)(x) \neq 0$. Then there exists a compact Salem set $E \subset [0,1]^d$ with dimension

$$\beta = \frac{d}{n-1}$$

such that for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.

Remark 1.4. *If $d = 1$, the conditions of this theorem are satisfied if Z is the countable union of manifolds which are projected submersively onto at least two coordinate axis, or have finite-type (as in [6]) of order at most $2(n-1)$.*

In [8], it is shown that for each $n > 0$, there exists a set $E \subset \mathbf{R}^{dn}$ with Fourier dimension $1/(n-1)$ such that for any $x_1, \ldots, x_n \in E$ and any integers $m_1, \ldots, m_n \in \mathbf{Z}$, not all zero, $m_1 x_1 + \cdots + m_n x_n \neq 0$. We note that for the set

$$Z = \{(x_1, \ldots, x_n) \in \mathbf{R}^d : m_1 x_1 + \cdots + m_n = 0 \text{ for some nonzero } m \in \mathbf{Z}^n\} \qquad (1.2)$$

is a countable union of hyperplanes in $\mathbf{R}^n$. Theorem 1.3 applies to the set

$$Z' = \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^d : \begin{array}{l} \text{there are integers } m_1, \ldots, m_n \in \mathbf{Z}, \text{ at least two of} \\ \text{which nonzero, such that } m_1 x_1 + \cdots + m_n x_n = 0. \end{array} \right\}.$$

Thus we can find a set E avoiding Z'.

$$Z - Z' = \{x \in \mathbf{R}^d : x_i = 0 \text{ for some } i\}$$

the set $E - \{0\}$ avoids Z and has the same Fourier dimension as E. Our arguments are heavily inspired by the techniques of [8], but augmented with novel applications of probabilistic concentration inequalities, which enables us to push the results of [8] to a much more general setting. In particular, our arguments show that the results of that paper do not depend on

the rich arithmetic structure of the pattern Z in (1.2), but rather on the dimension of the hypersurfaces of which Z is composed.

There are two key features of the patterns Z satisfying the hypothesis of Theorem 1.3 that enable us to obtain our result; firstly, they have thin projections onto a single 'coordinate' which enables us to obtain a concentration inequality, and secondly, they obey a nondegeneracy condition which enables us to obtain an appropriate decay on their Fourier transform. Though convienient to express in the terms of smooth surfaces, the techniques in the proof of Theorem 1.3 can be applied to more 'rough' situations which still exhibit the same properties required of the surfaces studied.

Theorem 1.5. *There exists a compact Salem set $E \subset [0,1]$ of dimension 1 such that for each non-zero $x \in E - E$ and each $\alpha > 0$, there are at most finitely many rational numbers $p/q \in \mathbf{Q}$ such that $|x - p/q| \leqslant q^{-(2+\alpha)}$.*

Because we are working with *compact* sets avoiding patterns, working in the domain $\mathbf{R}^d$ is not significantly different from working in a periodic domain $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, and working in $\mathbf{T}^d$ has several technical advantages over $\mathbf{R}^d$. For a finite Borel measure μ on $\mathbf{T}^d$, we can define it's Fourier dimension $\dim_{\mathbf{F}}(\mu)$ as the supremum of all $0 \leqslant s \leqslant d$ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.3}$$

We can then define the Fourier dimension $\dim_{\mathbf{F}}(E)$ of any Borel set $E \subset \mathbf{T}^d$ as the supremum of $\dim_{\mathbf{F}}(\mu)$, over all Borel measures μ supported on E. Since $\mathbf{T}^d$ has a natural metric space structure, we can also define the Hausdorff dimension of measures and sets on $\mathbf{T}^d$. It is a simple consequence of the Poisson summation formula that if μ is a compactly supported measure on $\mathbf{R}^d$, then (1.1) is equivalent to the more discrete condition

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2} < \infty. \tag{1.4}$$

A proof is given in Lemma 39 of [2]. In particular, if μ^* is the *periodization* of μ, i.e. the measure on $\mathbf{T}^d$ such that for any $f \in C(\mathbf{T}^d)$,

$$\int_{\mathbf{T}^d} f(x) \, d\mu^*(x) = \int_{\mathbf{R}^d} f(x) \, d\mu(x), \tag{1.5}$$

then by Poisson summation and (1.4) we see that $\dim_{\mathbf{F}}(\mu^*) = \dim_{\mathbf{F}}(\mu)$. Since μ is compactly supported, it is also simple to see that $\dim_{\mathbf{H}}(\mu^*) = \dim_{\mathbf{H}}(\mu)$ (for instance, by equating Frostman measure conditions). Thus Theorem 1.1 is clearly equivalent to Theorem 1.6, which is it's periodic variant.

Theorem 1.6. *Let $0 \leqslant \alpha < dn$, and let $Z \subset \mathbf{T}^{dn}$ be a countable union of compact sets, each with lower Minkowski dimension at most α. Then there exists a compact Salem set $E \subset \mathbf{T}^d$ with dimension*

$$\beta = \min\left(\frac{dn - \alpha}{n - 1/2}, d\right)$$

such that for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.

Noting that the periodization of any manifold is a countable union of manifolds with the same local properties as the original manifold, Theorem 1.7 also is equivalent to a periodic version.

Theorem 1.7. *Suppose $Z \subset \mathbf{T}^{dn}$ is a countable union of C^∞ surfaces of dimension $dn - d$, each of which having either non-vanishing curvature, or have finite type at most $2(n-1)/d$ at each point. Then there exists a compact Salem set $E \subset \mathbf{T}^d$ with dimension*

$$\beta = \frac{d}{n-1}$$

such that for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.

We construct the sets E in Theorems 1.6 and 1.7 by relying on a Baire-category type approach. Thus we consider a complete metric space $\mathcal{X}_\beta$, whose elements consist of pairs (E, μ), where E is a subset of $\mathbf{T}^d$, and μ is a probability measure supported on E. We then show that for *quasi-all* elements $(E, \mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β, and for distinct $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$. It thus follows that the consequences of Theorem 1.6 holds in a 'generic' sense for elements of $\mathcal{X}_\beta$.

Once we have setup the appropriate metric space $\mathcal{X}_\beta$, our approach is quite similar to the construction in [1], relying on a random selection procedure, which is now exploited to give high probability bounds on the Fourier transform of the measures we study. The use of the Baire category approach in this paper, rather than an algorithmic, 'nested set' approach as used in [1], is mostly of an aesthetic nature, avoiding the complex queuing method and dyadic decomposition strategy required in the nested set approach; our approach can, with some care, be converted into a queuing procedure like in [1]. But the Baire category argument makes our proof much simpler to read, and has the advantage that it indicates that Salem sets of a specified dimension 'generically' avoid a given rough pattern.

2 Notation

- Given a metric space X, $x \in X$, and $\varepsilon > 0$, we shall let $B_\varepsilon(x)$ denote the open ball of radius ε around x. We let δ_x denote the Dirac delta measure at x. For a given set $E \subset \Omega$ and $\varepsilon > 0$, we let

$$E_\varepsilon = \bigcup_{x \in E} B_\varepsilon(x),$$

denote the *ε-thickening* of the set E. A subset of X is of *first category* in X if it is the countable union of closed sets with empty interior. We say a property holds *quasi-always*, or a property is *generic* in X if the set of points in X failing to satisfy that property form a set of first category.

- We let $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$. Given $x \in \mathbf{T}$, we let

$$|x| = \min\{|x + n| : n \in \mathbf{Z}\},$$

and for $x \in \mathbf{T}^d$, we let

$$|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}.$$

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The canonical metric on $\mathbf{T}^d$ is then given by $d(x,y) = |x - y|$, for $x, y \in \mathbf{T}^d$. For an axis-oriented cube Q in $\mathbf{T}^d$, we let $2Q$ be the cube in $\mathbf{T}^d$ with the same centre and twice the sidelength.

• Suppose $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. For $\alpha \in [0, d]$ and $\delta > 0$, we define the Hausdorff content of a Borel set $E \subset \mathbf{E}$ as

$$H_\delta^\alpha(E) = \inf \left\{ \sum_{i=1}^\infty \varepsilon_i^\alpha : E \subset \bigcup_{i=1}^\infty B_{\varepsilon_i}(x_i) \text{ and } 0 < \varepsilon_i \leqslant \delta \text{ for all } i \geqslant 1 \right\}.$$

The α dimensional Hausdorff measure of E is equal to

$$H^\alpha(E) = \lim_{\delta \to 0} H_\delta^\alpha(E).$$

The Hausdorff dimension $\dim_{\mathbf{H}}(E)$ of a Borel set E is then the infimum over all $s \in [0, d]$ such that $H^s(E) = \infty$, or alternatively, the supremum over all $s \in [0, d]$ such that $H^s(E) = 0$. Frostman's lemma says that if we define the Hausdorff dimension $\dim_{\mathbf{H}}(\mu)$ of a finite Borel measure μ as the supremum of all $s \in [0, d]$ such that

$$\sup \left\{ \mu(B_\varepsilon(x)) \cdot \varepsilon^{-\alpha} : x \in \mathbf{E}, \varepsilon > 0 \right\} < \infty, \tag{2.1}$$

then $\dim_{\mathbf{H}}(E)$ is the supremum of $\dim_{\mathbf{H}}(\mu)$, over all Borel probability measures μ supported on E. This is analogous to the definition of the Fourier dimension of a set E given in the introduction.

A family of sets $\{F_\alpha\}$ is a *strong cover* of a set E if each point in E is contained in infinitely many of the sets $\{F_\alpha\}$. In Lemma 7.5 of [5], it is proved that if E is a compact subset of $\mathbf{E}$ with $\dim_{\mathbf{H}}(E) \leqslant \alpha$, then there exists a family of sets $\{E_\delta\}$, for each δ ranges over all *hyperdyadic numbers* of the form $2^{-\lfloor (1+\varepsilon)^k \rfloor}$ for some k, where E_δ is the union of dyadic cubes with sidelength δ and $|E_\delta| \leqslant r^{d-\alpha-\varepsilon}$

For a measurable set $E \subset \mathbf{E}$, we let $|E|$ denote it's Lebesgue measure. We define the lower Minkowski dimension of a compact Borel set $E \subset \mathbf{E}$ as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{r \to 0} d - \log_r |E_r|.$$

Thus $\underline{\dim}_{\mathbf{M}}(E)$ is the largest number such that for $\alpha < \underline{\dim}_{\mathbf{M}}(E)$, there exists a decreasing sequence $\{r_i\}$ with $\lim_{i \to \infty} r_i = 0$ and $|E_{r_i}| \leqslant r_i^{d-\alpha}$ for each i.

• In this paper we will need to employ probabilistic concentration bounds several times. In particular, we use *McDiarmid's inequality*, trivially modified from the standard theorem to work with complex-valued functions. Let $\mathbf{E} = \mathbf{T}^d$ or $\mathbf{E} = \mathbf{R}^d$. Let $\{X_1, \ldots, X_N\}$ be an independant family of random variables, and consider a function $f : \mathbf{E}^N \to \mathbf{C}$. Suppose that for each $i \in \{1, \ldots, N\}$, there exists a constant $A_i > 0$ such that for any $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N \in \mathbf{E}$, and for each $x_i, x_i' \in \mathbf{E}$,

$$|f(x_1, \ldots, x_i, \ldots, x_N) - f(x_1, \ldots, x_i', \ldots, x_N)| \leqslant A_i.$$

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Then McDiarmid's inequality guarantees that for all $t \geqslant 0$,

$$\mathbf{P}\left(|f(X_1,\ldots,X_N) - \mathbf{E}(f(X_1,\ldots,X_N))| \geqslant t\right) \leqslant 4\exp\left(\frac{-2t^2}{A_1^2 + \cdots + A_N^2}\right).$$

The complex-valued extension we have just stated is proved easily from the real-valued case by taking a union bound to the inequality for the real and imaginary values of f. Proofs of McDiarmid's inequality are given in many probability texts, for instance, in Theorem 3.11 of [7].

A special case of McDiarmid's inequality is *Hoeffding's Inequality*. The version of Hoeffding's inequality we use states that if $\{X_1,\ldots,X_N\}$ is a family of independant random variables, such that for each i, there exists a constant $A_i \geqslant 0$ such that $|X_i| \leqslant A_i$ almost surely, then for each $t \geqslant 0$,

$$\mathbf{P}\left(|X_1 + \cdots + X_N - \mathbf{E}(X_1 + \cdots + X_N)| \geqslant t\right) \leqslant 4\exp\left(\frac{-t^2/2}{A_1^2 + \cdots + A_N^2}\right).$$

- Throughout this paper, we will need to consider a standard mollifier. So we fix a smooth, non-negative function $\phi \in C^\infty(\mathbf{T}^d)$ such that $\phi(x) = 0$ for $|x| \geqslant 2/5$ and

$$\int_{\mathbf{T}^d} \phi(x)\,dx = 1.$$

For each $r \in (0,1)$, we can then define $\phi_r \in C^\infty(\mathbf{T}^d)$ by writing

$$\phi_r(x) = \begin{cases} r^{-d}\phi(x/r) & : |x| < r, \\ 0 & : \text{otherwise.} \end{cases}$$

The following standard properties hold:

(1) For each $r \in (0,1)$, ϕ_r is a non-negative smooth function with

$$\int_{\mathbf{T}^d} \phi_r(x)\,dx = 1, \tag{2.2}$$

and $\phi_r(x) = 0$ for $|x| \geqslant r$.

(2) For any $r \in (0,1)$,
$$\|\widehat{\phi}_r\|_{L^\infty(\mathbf{Z}^d)} \leqslant 1. \tag{2.3}$$

(3) For each $\xi \in \mathbf{Z}^d$,
$$\lim_{r \to 0} \widehat{\phi}_r(\xi) = 1. \tag{2.4}$$

(4) For each $T > 0$, and for all $r > 0$ and any non-zero $\xi \in \mathbf{Z}^d$,
$$|\widehat{\phi}_r(\xi)| \lesssim_T r^{-T}|\xi|^{-T}. \tag{2.5}$$

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3   A Metric Space Controlling Fourier Dimension

In order to work with a Baire category type argument, we must construct an appropriate metric space appropriate for our task and establish a set of tools for obtaining convergence in this metric space. In later sections we will fix a specific choice of β to avoid a particular pattern. But in this section we let β be an arbitrary element of $(0, d]$. Our approach in this section is heavily influenced by [8]. However, we employ a Frechét space construction instead of the Banach space used in [8], which enables us to use softer estimates in our arguments:

- We let $\mathcal{E}$ denote the family of all compact subsets of $\mathbf{T}^d$. If, for two compact sets $E, F \in \mathcal{E}$, we consider their Hausdorff distance

$$d_{\mathbf{H}}(E, F) = \inf\{\varepsilon > 0 : E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\},$$

  then $(\mathcal{E}, d_{\mathbf{H}})$ forms a complete metric space.

- We let $M(\beta/2)$ consist of the class of all finite Borel measures μ on $\mathbf{T}^d$ such that for each $\varepsilon \in (0, \beta/2]$, the quantity

$$\|\mu\|_{M(\beta/2-\varepsilon)} = \sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{\beta/2-\varepsilon}$$

  is finite. Then $\|\cdot\|_{M(\beta/2-\varepsilon)}$ is a seminorm on $M(\beta/2)$, and the collection of all such seminorms for $\varepsilon \in (0, \beta/2]$ gives $M(\beta/2)$ the structure of a Frechét space. Under this topology, a sequence of probability measures $\{\mu_k\}$ converges to a probability measure μ in $M(\beta/2)$ if and only if for any $\varepsilon > 0$, $\lim_{k \to \infty} \|\mu_k - \mu\|_{M(\beta/2-\varepsilon)} = 0$.

We now let $\mathcal{X}_\beta$ be the collection of all pairs $(E, \mu) \in \mathcal{E} \times M(\beta/2)$, where μ is a probability measure such that $\text{supp}(\mu) \subset E$. Then $\mathcal{X}_\beta$ is a closed subset of $\mathcal{E} \times M(\beta/2)$ under the product metric, and thus a complete metrizable space. We remark that for any $\varepsilon > 0$ and $(E, \mu) \in \mathcal{X}_\beta$,

$$\lim_{|\xi| \to \infty} |\xi|^{\beta/2-\varepsilon}|\widehat{\mu}(\xi)| = 0, \tag{3.1}$$

which follows because $\|\mu\|_{M(\beta/2-\varepsilon/2)}$ is finite. Thus $\dim_{\mathbf{F}}(\mu) \geqslant \beta$ for each $(E, \mu) \in \mathcal{X}_\beta$.

Lemma 3.1. *The set of all (E, μ) with $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Proof. Consider $(E, \mu) \in \mathcal{X}_\beta$. For each $r \in (0, 1)$, consider the convolved measure $\mu_r = \mu * \phi_r$. Then $\mu_r \in C^\infty(\mathbf{T}^d)$. We claim that $\lim_{r \to 0}(E_r, \mu_r) = (E, \mu)$. Since $\text{supp}(\mu_r) \subset E_r$, we find that $d_{\mathbf{H}}(E, E_r) \leqslant r$, and so $\lim_{r \to 0} E_r = E$. Now fix $\varepsilon_1 \in (0, \beta/2]$ and $\varepsilon > 0$. For each $\xi \in \mathbf{Z}^d$, $|\widehat{\mu_r}(\xi)| = |\widehat{\phi_r}(\xi)||\widehat{\mu}(\xi)|$, so

$$|\xi|^{\beta/2-\varepsilon_1}|\mu_r(\xi) - \mu(\xi)| = |\xi|^{\beta/2-\varepsilon_1}|\widehat{\phi_r}(\xi) - 1||\widehat{\mu}(\xi)|. \tag{3.2}$$

We control (3.2) over large frequencies using the term $|\widehat{\mu}(\xi)|$, and by small frequencies using the term $|\widehat{\phi_r}(\xi) - 1|$. Since $(E, \mu) \in \mathcal{X}_\beta$, we can apply (3.1) to find $R > 0$ such that for $|\xi| \geqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\widehat{\mu}(\xi)| \leqslant \varepsilon. \tag{3.3}$$

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Combining (3.2), (3.3), and (2.3), for $|\xi| \geqslant R$ we find that

$$|\xi|^{\beta/2-\varepsilon_1}|\mu_r(\xi) - \mu(\xi)| \leqslant 2\varepsilon. \tag{3.4}$$

On the other hand, (2.4) shows that there exists $r_0 > 0$ such that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon}|\widehat{\phi}_r(\xi) - 1| \leqslant \varepsilon. \tag{3.5}$$

The (L^1, L^∞) bound for the Fourier transform implies that $|\widehat{\mu}(\xi)| \leqslant \mu(\mathbf{T}^d) = 1$, which combined with (3.5) gives that for $r \leqslant r_0$ and $|\xi| \leqslant R$,

$$|\xi|^{\beta/2-\varepsilon_1}|\mu_r(\xi) - \mu(\xi)| \leqslant \varepsilon. \tag{3.6}$$

Putting together (3.4) and (3.6), shows that for $r \leqslant r_0$, $\|\mu_r - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant 2\varepsilon$. Since ε and ε_1 were arbitrary, $\lim_{r\to 0} \mu_r = \mu$, completing the proof. $\qquad\square$

Remark 3.2. *Let*

$$\tilde{\mathcal{X}}_\beta = \{(E, \mu) \in \mathcal{X}_\beta : supp(\mu) = E\}.$$

Suppose $(E_0, \mu_0) \in \tilde{\mathcal{X}}_\beta$. Then, in the proof above, one may let E_r be equal to $supp(\mu_r)$, since it follows from this that $d_\mathbf{H}(E_0, E_r) \leqslant r$. This means that the set of pairs $(E, \mu) \in \tilde{\mathcal{X}}_\beta$ with $\mu \in C^\infty(\mathbf{T}^d)$ are dense in $\tilde{\mathcal{X}}_\beta$.

The reason we must work with $\mathcal{X}_\beta$ rather than the smaller space $\tilde{\mathcal{X}}_\beta \subset \mathcal{X}_\beta$ is that $\tilde{\mathcal{X}}_\beta$ is not a closed subset of $\mathcal{X}_\beta$, and so is not a complete metric space. However, as a consolation, quasi-all elements of $\mathcal{X}_\beta$ belong to $\tilde{\mathcal{X}}_\beta$, so that one can think of $\mathcal{X}_\beta$ and $\tilde{\mathcal{X}}_\beta$ as being equal 'generically'.

Lemma 3.3. *For quasi-all $(E, \mu) \in \mathcal{X}_\beta$, $supp(\mu) = E$.*

Proof. For each closed cube $Q \subset \mathbf{T}^d$, let

$$A(Q) = \{(E, \mu) \in \mathbf{T}^d : (E \cap Q) = \varnothing \text{ or } \mu(Q) > 0\}.$$

Then $A(Q)$ is an open set. If $\{Q_k\}$ is a sequence enumerating all cubes with rational corners in $\mathbf{T}^d$, then

$$\bigcap_{k=1}^\infty A(Q_k) = \{(E, \mu) \in \mathcal{X}_\beta : \text{supp}(\mu) = E\}.$$

Thus it suffices to show that $A(Q)$ is dense in $\mathcal{X}_\beta$ for each closed cube Q. Thus we fix $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$, $\varepsilon_1 \in (0, \beta/2]$, and $\varepsilon > 0$, and try and find $(E, \mu) \in A(Q)$ with $d_\mathbf{H}(E, E_0) \leqslant \varepsilon$ and $\|\mu_0 - \mu\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$. Applying Lemma 3.1, we may assume without loss of generality that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Because $(E_0, \mu_0) \in \mathcal{X}_\beta - A(Q)$, we know $E \cap Q \neq \varnothing$ and $\mu(Q) = 0$. Find a smooth probability measure ν supported on $E_\varepsilon \cap Q$ and, for $t \in (0, 1)$, define $\mu_t = (1-t)\mu_0 + t\nu$. Then $\text{supp}(\mu_t) \subset E_\varepsilon$, so if we let $E = \text{supp}(\nu) \cup \text{supp}(\mu)$, then $d_\mathbf{H}(E, E_0) \leqslant \varepsilon$. Clearly $(E, \mu_t) \in A(Q)$ for $t > 0$. And

$$\|\mu_t - \mu_0\|_{M(\beta/2-\varepsilon)} \leqslant t \left(\|\mu_0\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)} \right),$$

so if we choose $t \leqslant \varepsilon(\|\mu\|_{M(\beta/2-\varepsilon)} + \|\nu\|_{M(\beta/2-\varepsilon)})^{-1}$ we find $\|\mu_t - \mu\|_{M(\beta/2-\varepsilon)} \leqslant \varepsilon$. Since ε was arbitrary, we conclude $A(Q)$ is dense in $\mathcal{X}_\beta$. $\qquad\square$

Combining Lemma 3.3 with Remark 3.2 gives the following simple corollary.

Corollary 3.4. *The family of (E, μ) with $\text{supp}(\mu) = E$ and $\mu \in C^\infty(\mathbf{T}^d)$ is dense in $\mathcal{X}_\beta$.*

Our main way of constructing approximations to $(E_0, \mu_0) \in \mathcal{X}_\beta$ is to multiply μ_0 by a smooth function $f \in C^\infty(\mathbf{T}^d)$. For instance, we might choose f in such a way as to remove certain points from the support of μ_0 which contribute to the formation of a pattern we are trying to avoid. As long as μ_0 is appropriately smooth, and the Fourier transform of f decays appropriately quickly, the next lemma shows that $f\mu_0 \approx \mu_0$.

Lemma 3.5. *Consider a finite measure μ_0 on $\mathbf{T}^d$, as well as a smooth probability density function $f \in C^\infty(\mathbf{T}^d)$. If we define $\mu = f\mu_0$, then*

$$\|\mu - \mu_0\|_{M(\beta/2)} \lesssim_d \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}.$$

Proof. Since $\widehat{\mu} = \widehat{f} * \widehat{\mu}_0$, and $\widehat{f}(0) = 1$, for each $\xi \in \mathbf{Z}^d$ we have

$$|\xi|^{\beta/2} |\widehat{\mu}(\xi) - \widehat{\mu}_0(\xi)| = |\xi|^{\beta/2} \left| \sum_{\eta \neq \xi} \widehat{f}(\xi - \eta) \widehat{\mu}_0(\eta) \right|. \tag{3.7}$$

If $|\eta| \leqslant |\xi|/2$, then $|\xi|/2 \leqslant |\xi - \eta| \leqslant 2|\xi|$, so

$$|\xi|^{\beta/2} |\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)} |\xi|^{\beta/2} |\xi - \eta|^{-\beta} \leqslant 2^{\beta/2} \|f\|_{M(\beta/2)} \lesssim_d \|f\|_{M(\beta/2)}. \tag{3.8}$$

Thus the bound (3.8) implies

$$|\xi|^{\beta/2} \left| \sum_{0 \leqslant |\eta| \leqslant |\xi|/2} \widehat{f}(\xi - \eta) \widehat{\mu}_0(\eta) \right| \lesssim_{\mu_0, d} \|\mu_0\|_{M(d+1)} \|f\|_{M(\beta/2)} \left(1 + \sum_{0 < |\eta| \leqslant |\xi|/2} \frac{1}{|\eta|^{d+1}} \right) \tag{3.9}$$

$$\lesssim_d \|\mu_0\|_{M(d+1)} \|f\|_{M(\beta/2)} \leqslant \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)}.$$

On the other hand, for all $\eta \neq \xi$,

$$|\widehat{f}(\xi - \eta)| \leqslant \|f\|_{M(\beta/2)} |\xi - \eta|^{-\beta} \leqslant \|f\|_{M(\beta/2)}. \tag{3.10}$$

Thus we calculate that

$$|\xi|^{\beta/2} \left| \sum_{\substack{|\eta| > |\xi|/2 \\ \eta \neq \xi}} \widehat{f}(\xi - \eta) \widehat{\mu}_0(\eta) \right| \lesssim_{d, \mu_0} \|\mu_0\|_{M(3d/2)} \|f\|_{M(\beta/2)} \cdot |\xi|^{\beta/2} \sum_{|\eta| > |\xi|/2} \frac{1}{|\eta|^{3d/2}} \tag{3.11}$$

$$\lesssim_d \|\mu_0\|_{M(\beta/2)} \|f\|_{M(\beta/2)}.$$

Combining (3.7), (3.9) and (3.11) completes the proof. $\qquad\square$

Remark 3.6. *In particular, we note that this lemma implies that $1 - \mu(\mathbf{T}^d) \lesssim_{d, \mu_0} \|f\|_{M(0)}$.*

The bound in Lemma 3.5, if $\|f\|_{M(\beta/2)}$ is taken appropriately small, also implies that the Hausdorff distance between the supports of μ and μ_0 is small.

Lemma 3.7. *Fix a probability measure $\mu_0 \in C^\infty(\mathbf{T}^d)$. For any $\varepsilon > 0$, there exists $\delta > 0$ depending on μ_0, ε, and d, such that if $\mu \in C^\infty(\mathbf{T}^d)$, $supp(\mu) \subset supp(\mu_0)$, and $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, then $d_{\mathbf{H}}(supp(\mu), supp(\mu_0)) \leqslant \varepsilon$.*

Proof. Consider any cover of $supp(\mu_0)$ by a family of radius $\varepsilon/3$ balls $\{B_1, \ldots, B_N\}$, and for each $i \in \{1, \ldots, N\}$, consider a smooth function $f_i \in C_c^\infty(B_i)$ such that there is $s > 0$ with

$$\int f_i(x)d\mu_0(x) \geqslant s. \tag{3.12}$$

for each $i \in \{1, \ldots, N\}$. Fix $A > 0$ with

$$\sum_{\xi \neq 0} |\widehat{f_i}(\xi)| \leqslant A \tag{3.13}$$

for all $i \in \{1, \ldots, N\}$ as well. Set $\delta = s/2A$. If $\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant \delta$, we apply Plancherel's inequality together with (3.12) and (3.13) to conclude that

$$\left| \int f_i(x)d\mu(x)\,dx - \int f_i(x)d\mu_0(x) \right| = \left| \sum_{\xi \in \mathbf{Z}^d} \widehat{f_i}(\xi)\,(\widehat{\mu}(\xi) - \widehat{\mu_0}(\xi)) \right|$$
$$\leqslant A\|\mu_0 - \mu\|_{M(\beta/2)} \leqslant s/2. \tag{3.14}$$

Thus we conclude from (3.12) and (3.14) that

$$\int f_i(x)d\mu(x)\,dx \geqslant \int f_i(x)d\mu_0(x) - s/2 \geqslant s/2 > 0. \tag{3.15}$$

Since equation (3.15) holds for each $i \in \{1, \ldots, N\}$, the support of μ intersects every ball in $\{B_1, \ldots, B_N\}$. Combined with the assumption that $supp(\mu) \subset supp(\mu_0)$, this implies that $d_{\mathbf{H}}(\mu_0, \mu) \leqslant \varepsilon$. $\qquad\square$

To obtain the smooth function f to which we can apply Lemmas 3.5 and 3.7, we take a measure η, which is a linear combination of Dirac deltas, and let $f = \eta * \phi_r$. To obtain the appropriate control on $\widehat{f}$, it suffices to have a decay bound for $\widehat{\eta}$ for $|\xi| \leqslant 1/r$, and a weaker bound for $|\xi|$ slightly bigger than $1/r$, which is required before we can take complete advantage of the Fourier decay of the mollifier ϕ_r for $|\xi| \geqslant 1/r$.

Lemma 3.8. *Fix $C > 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, with $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ depending on all these quantities, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad for\ 0 < |\xi| \leqslant (1/r) \tag{3.16}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2} \log(1/r)^{1/2} \qquad for\ (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{3.17}$$

if we define $f(x) = (\eta * \phi_r)(x)$, then $\|f\|_{M(\beta/2 - \varepsilon_2)} \leqslant 2\varepsilon$.

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Proof. For each $\xi \in \mathbf{Z}^d$,

$$\widehat{f}(\xi) = \widehat{\eta}(\xi)\widehat{\phi}_r(\xi). \tag{3.18}$$

For $|\xi| \leqslant 1/r$ we combine (3.16), (3.18) and (2.3) to conclude that

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{3.19}$$

If $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$, (2.5) implies $|\widehat{\phi}_r(\xi)| \lesssim_\beta r^{-\beta/2}|\xi|^{-\beta/2}$, which together with (3.17), (3.18), and (2.3), show that for $r \leqslant r_1$,

$$\begin{aligned}
|\widehat{f}(\xi)| &\lesssim_\beta \left(Cr^{\beta/2}\log(1/r)^{1/2}\right)\left(r^{-\beta/2}|\xi|^{-\beta/2}\right) \\
&\leqslant C\log(1/r)^{1/2} \cdot |\xi|^{-\beta/2} \\
&\leqslant Cr^{\varepsilon_2}\log(1/r)^{1/2} \cdot |\xi|^{\varepsilon_2 - \beta/2}.
\end{aligned} \tag{3.20}$$

Since $Cr^{\varepsilon_2}\log(1/r)^{1/2} \to 0$ as $r \to 0$, so we conclude from (3.20) that there exists $r_1 > 0$ such that for $r \leqslant r_1$ and $(1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}$

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{3.21}$$

If $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$, we apply (2.5) for $T \geqslant \beta/2$ to conclude

$$\begin{aligned}
|\widehat{f}(\xi)| &\lesssim_T r^{-T}|\xi|^{-T} \\
&= r^{-T}|\xi|^{\beta/2 - T} \cdot |\xi|^{-\beta/2} \\
&\leqslant r^{-T}(1/r)^{(\beta/2 - T)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2} \\
&= r^{\varepsilon_1 T - (\beta/2)(1+\varepsilon_1)} \cdot |\xi|^{-\beta/2}.
\end{aligned} \tag{3.22}$$

If we choose $T > (\beta/2)(1 + 1/\varepsilon_1)$, then as $r \to 0$, $r^{\varepsilon_1 T - (\beta/2)(1+\varepsilon_1)} \to 0$. Thus we conclude from (3.22) that there exists $r_2 > 0$ satisfying such that for $0 < r \leqslant r_2$ and $|\xi| \geqslant (1/r)^{1+\varepsilon_1}$,

$$|\widehat{f}(\xi)| \leqslant \varepsilon \cdot |\xi|^{-\beta/2} \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \tag{3.23}$$

All that remains is to combine (3.19), (3.21), and (3.23), defining $r_0 = \min(r_1, r_2)$. $\qquad\square$

Corollary 3.9. *Fix $C > 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_2 \leqslant \beta/2$. Then there exists $r_0 > 0$ and $\delta > 0$ depending on these quantities, such that if $0 < r \leqslant r_0$, then for any Borel probability measure η on $\mathbf{T}^d$ satisfying*

$$|\widehat{\eta}(\xi)| \leqslant \delta \cdot |\xi|^{\varepsilon_2 - \beta/2} \qquad for\ 0 < |\xi| \leqslant (1/r) \tag{3.24}$$

and

$$|\widehat{\eta}(\xi)| \leqslant C \cdot r^{\beta/2}\log(1/r)^{1/2} \qquad for\ (1/r) \leqslant |\xi| \leqslant (1/r)^{1+\varepsilon_1}, \tag{3.25}$$

if we define $f(x) = (\eta * \phi_r)(x)$, and a probability measure

$$\mu = \frac{f\mu_0}{(f\mu_0)(\mathbf{T}^d)}$$

then $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_2)} \leqslant \varepsilon$.

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Proof. Let
$$\delta = \min\left(\frac{\varepsilon}{2}, \frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right).$$

Thus if we apply Lemmas 3.5 and 3.8, we conclude there exists r_0 such that for $r \leqslant r_0$,

$$\|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon_2)} \leqslant \varepsilon/2, \tag{3.26}$$

and

$$\|f\mu_0 - \mu_0\|_{M(0)} \leqslant \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right). \tag{3.27}$$

Equation (3.27) implies that

$$1 - \min\left(\frac{1}{2}, \frac{\varepsilon}{4\|\mu_0\|_{M(\beta/2-\varepsilon_2)}}\right) \leqslant (f\mu_0)(\mathbf{T}^d) \leqslant 1. \tag{3.28}$$

But now (3.26) and (3.28) show that

$$\begin{aligned}
\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_2)} &\leqslant \|f\mu_0 - \mu_0\|_{M(\beta/2-\varepsilon)} + \|\mu - f\mu_0\|_{M(\beta/2-\varepsilon)} \\
&\leqslant (\varepsilon/2) + \left(1 - \frac{1}{(f\mu_0)(\mathbf{T}^d)}\right)\|\mu_0\|_{M(\beta/2-\varepsilon)} \\
&\leqslant (\varepsilon/2) + (\varepsilon/2) \leqslant \varepsilon. \qquad \square
\end{aligned}$$

A useful technique to find functions with small Fourier coefficients is to apply some random techniques. This is because heuristically, a generic function has small Fourier coefficients.

Lemma 3.10. *Fix a positive integer K. Let $X_1, \ldots, X_K$ be independant random variables on $\mathbf{T}^d$, such that for each nonzero $\xi \in \mathbf{Z}^d$,*

$$\sum_{k=1}^{K} \mathbf{E}\left(e^{2\pi i \xi \cdot X_k}\right) = 0. \tag{3.29}$$

Set

$$\eta(x) = \frac{1}{K}\sum_{k=1}^{K}\delta_{X_k}(x)$$

and

$$B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| \leqslant K^{100/\beta}\}.$$

Then there exists a constant C depending on β and d, such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10.$$

Remark 3.11. *In particular, (3.29) holds if the $\{X_i\}$ are uniformly distributed on $\mathbf{T}^d$.*

Proof. For each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$, consider the random variable

$$Y(\xi, k) = K^{-1} e^{2\pi i (\xi \cdot X_k)}. \tag{3.30}$$

Then for each $\xi \in \mathbf{Z}^d$,

$$\sum_{k=1}^{K} Y(\xi, k) = \widehat{\eta}(\xi). \tag{3.31}$$

We also note that for each $\xi \in \mathbf{Z}^d$ and $k \in \{1, \ldots, K\}$,

$$|Y(\xi, k)| = K^{-1}. \tag{3.32}$$

Moreover,

$$\sum_{k=1}^{K} \mathbf{E}(Y(\xi, k)) = 0. \tag{3.33}$$

Since the family of random variables $\{Y(\xi, k)\}$ is independent for a fixed ξ, we can apply Hoeffding's inequality together with (3.31) and (3.32) to conclude that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\eta}(\xi)| \geqslant t\right) \leqslant 2 e^{-Kt^2/2}. \tag{3.34}$$

A union bound obtained by applying (3.34) over all $|\xi| \leqslant K^{100/\beta}$ shows that there exists a constant $C \geqslant 1$ depending on d and β such that

$$\mathbf{P}\left(\|\widehat{\eta}\|_{L^\infty(B)} \geqslant t\right) \leqslant \exp\left(C \log(K) - \frac{5Kt^2}{C}\right). \tag{3.35}$$

But then, setting $t = CK^{-1/2} \log(K)^{1/2}$ in (3.35) completes the proof. $\qquad\square$

Let us now consider the consequences of the square-root cancellation bound in Lemma 3.10. Given η as in Lemma 3.10, consider the smooth function $f = \eta * \phi_r$. The support of f consists of K radius r balls. Provided $K \approx r^{-\beta}$, the support therefore behaves like an r-thickening of a set with Minkowski dimension β. If

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2} \log(K)^{1/2}, \tag{3.36}$$

as is guaranteed with high probability in Lemma 3.10, we actually find that the support of f also behaves like an r-thickening of a set with *Fourier dimension* β as well. This is one reason why constructions involving some kind of square-root cancellation are often a viable tactic to construct Salem sets, with random examples being an important example of such a construction.

Lemma 3.12. *Fix $\varepsilon, \varepsilon_1 > 0$ and $C > 0$ with $\varepsilon_1 \leqslant \beta/2$. Then there exists $r_0 > 0$ and $C' > 0$ depending on these quantities such that for $r \leqslant r_0$, if $K \geqslant (1/C) r^{-\beta}$ and η is a Borel probability measure with*

$$\|\widehat{\eta}\|_{L^\infty(B)} \leqslant CK^{-1/2} \log(K)^{1/2},$$

then

$$|\widehat{\eta}(\xi)| \leqslant \varepsilon |\xi|^{\varepsilon_1 - \beta/2} \qquad \text{for } |\xi| \leqslant (1/r)$$

and

$$|\widehat{\eta}(\xi)| \leqslant C' r^{\beta/2} \log(1/r)^{1/2} \qquad \text{for } (1/r) \leqslant |\xi| \leqslant (1/r)^2.$$

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Proof. The function $x \mapsto x^{-1/2} \log(x)^{1/2}$ is decreasing for sufficiently large x. Thus if r_0 is chosen appropriately small, then for $|\xi| \leqslant (1/r)$ we find

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2} \leqslant Cr^{\varepsilon_2} \log((1/C)r^{-\beta})^{1/2} \cdot |\xi|^{\varepsilon_2 - \beta/2} \leqslant \varepsilon \cdot |\xi|^{\varepsilon_2 - \beta/2}. \qquad (3.37)$$

If r_0 is appropriately small, then

$$(1/r)^2 \leqslant C^{2/\beta} K^{2/\beta} \leqslant K^{100/\beta}$$

Thus for $(1/r) \leqslant |\xi| \leqslant (1/r)^2$, $\xi \in B$, and so we find there is $C' > 0$ such that

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2} \leqslant Cr^{\beta/2} \log((1/C)r^{-\beta})^{1/2} \leqslant C'r^{\beta/2} \log(1/r)^{1/2}. \qquad (3.38)$$

Together, (3.37) and (3.38) give what was required. $\qquad \square$

It is a general heuristic that quasi-all sets are as 'thin as possible' with respect to the Hausdorff metric. In particular, we should expect the Hausdorff dimension and Fourier dimension of a generic element of $\mathcal{X}_\beta$ to be as low as possible. For each $(E, \mu) \in \mathcal{X}_\beta$, the condition that $\mu \in M(\beta/2)$ implies that $\dim_{\mathbf{F}}(\mu) \geqslant \beta$, so $\dim_{\mathbf{F}}(E) \geqslant \beta$. Thus it is natural to expect that for quasi-all $(E, \mu) \in M(\beta/2)$, the set E has both Hausdorff dimension and Fourier dimension equal to β.

Lemma 3.13. *For quasi-all $(E, \mu) \in \mathcal{X}_\beta$, E is a Salem set of dimension β.*

Proof. We shall assume $\beta < d$ in the proof, since when $\beta = d$, E is a Salem set for any $(E, \mu) \in \mathcal{X}_\beta$, and thus the result is trivial. Since the Hausdorff dimension of a measure is an upper bound for the Fourier dimension, it suffices to show that for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, E has Hausdorff dimension at most β. For each $\alpha > \beta$ and $\delta, s > 0$, and let $A(\alpha, \delta, s) = \{(E, \mu) \in \mathcal{X} : H_\delta^\alpha(E) < s\}$. Then $A(\alpha, \delta, s)$ is an open subset of $\mathcal{X}_\beta$, and

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} A(\beta + 1/n, 1/m, 1/k)$$

is precisely the family of $(E, \mu) \in \mathcal{X}_\beta$ such that E has Hausdorff dimension at β. Thus it suffices to show that $A(\alpha, \delta, s)$ is dense in $\mathcal{X}_\beta$ for $\alpha \in (\beta, d)$ and $\delta, s > 0$. Fix $(E_0, \mu_0) \in \mathcal{X}_\beta$, $\alpha \in (\beta, d)$, $\delta > 0$, $s > 0$, and $\varepsilon_1 > 0$. We aim to show that for each $\varepsilon > 0$, there exists $(E, \mu) \in A(\alpha, \delta, s)$ such that $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. Without loss of generality, in light of Lemma 3.1, we may assume that $\mu_0 \in C^\infty(\mathbf{T}^d)$.

Fix a small value r, and then find an integer K such that $r^{-\beta} \leqslant K \leqslant r^{-\beta} + 1$. Lemma 3.10 shows that there exists a constant C depending on β and d, as well as K points $x_1, \ldots, x_K \in \mathbf{T}^d$ such that if

$$\eta(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{X_k}(x),$$

then for each $|\xi| \leqslant (1/r)^{1+1/\beta} \leqslant K^{1/\beta + 1}$,

$$|\widehat{\eta}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \qquad (3.39)$$

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Equation (3.39) shows η satisfies the hypotheses of Lemma 3.12, and the result of Lemma 3.12 can then be fed into Corollary 3.9 to conclude that for any $\delta > 0$, there exists $r_1 > 0$ such that if $r \leqslant r_1$, if

$$\mu_1(x) = \frac{1}{K} \left(\sum_{k=1}^{K} \phi_r(x - x_k) \right) \mu_0(x),$$

and if

$$\mu = \mu_1/\mu_1(\mathbf{T}^d),$$

then

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{3.40}$$

If δ is chosen appropriately, Lemma 3.7 implies

$$d_{\mathbf{H}}(\mathrm{supp}(\mu), \mathrm{supp}(\mu_0)) \leqslant \varepsilon. \tag{3.41}$$

Note that μ is supported on K balls of radius r. Thus for $r \leqslant \delta$,

$$H_\delta^\alpha(\mathrm{supp}(\mu)) \leqslant Kr^\alpha \leqslant (r^{-\beta} + 1)r^\alpha = r^{\alpha - \beta} + r^\alpha. \tag{3.42}$$

Since $\alpha > \beta$, (3.42) implies that there is $r_2 > 0$ depending on α, β, and s such that for $r \leqslant r_2$, $H_\delta^\alpha(\mathrm{supp}(\mu)) \leqslant s$. This means $(\mathrm{supp}(\mu), \mu) \in A(\alpha, \delta, s)$, and since $\varepsilon > 0$ was arbitrary, we see that we have proved what was required. $\qquad\square$

This concludes the setup to the proof of Theorems 1.6 and 1.7. All that remains is to show that quasi-all elements of $\mathcal{X}_\beta$ avoid the given set Z; just as with some of the proofs we have given in this section, the advantage of the Baire category approach is that we can reduce our calculations to discussing only a couple scales at once, which allows us to focus solely on the discrete, quantitative question at the heart of the problem.

4   Random Avoiding Sets for Rough Patterns

We begin by proving Theorem 1.6, which requires simpler calculations than Theorem 1.7. In the last section, our results held for an arbitrary $\beta \in (0, d]$. But in this section, we assume

$$\beta = \min\left(d, \frac{dn - \alpha}{n - 1/2} \right),$$

which will enable us to generically avoid the pattern Z described in Theorem 1.6. The construction here is very similar to the construction in [1], albeit in a Baire category setting and with modified parameters to ensure a Fourier dimension bound rather than just a Hausdorff dimension bound.

Lemma 4.1. *Suppose $Z \subset \mathbf{T}^{dn}$ is a countable union of compact sets, each with lower Minkowski dimension at most α. Then for quasi-all $(E, \mu) \in \mathcal{X}_\beta$, for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.*

Proof. The set $Z \subset \mathbf{R}^{dn}$ is the countable union of sets with lower Minkowski dimension at most α. For a closed set $W \subset \mathbf{T}^{dn}$ with lower Minkowski dimension at most α, and $s > 0$, consider the set

$$B(W,s) = \left\{ (E,\mu) \in \mathcal{X}_\beta : \begin{array}{c} \text{for all } x_1, \ldots, x_n \in E \text{ such that} \\ |x_i - x_j| \geqslant s \text{ for } i \neq j, \ (x_1, \ldots, x_n) \notin W \end{array} \right\}.$$

If $(E_0, \mu_0) \in B(W,s)$, then because E_0 is compact, so too is the set

$$F = \{(x_1, \ldots, x_n) \in E_0^n : |x_i - x_j| \geqslant s \text{ for } i \neq j\}$$

Since W is also closed, hence compact, there exists $\varepsilon > 0$ such that if $(x_1, \ldots, x_n) \in F$, then $d((x_1, \ldots, x_n), W) > \varepsilon$. It follows that if $d_{\mathbf{H}}(E_0, E) \leqslant \varepsilon$, then for any measure μ supported on E, $(E, \mu) \in B(W,s)$. Thus $B(W,s)$ is an open subset of $\mathcal{X}_\beta$. If Z is a countable union of closed sets $\{Z_k\}$ with lower Minkowski at most α, then clearly the set

$$\bigcup_{k=1}^\infty \bigcup_{n=1}^\infty B(Z_k, 1/n)$$

consists of the family of sets (E, μ) such that for distinct $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$. Thus it suffices to show that $B(W,s)$ is dense in $\mathcal{X}_\beta$ for any $s > 0$, and any closed set W with lower Minkowski dimension at most α.

Let us begin by fixing a set $W \subset \mathbf{T}^{dn}$ and a pair $(E_0, \mu_0) \in \mathcal{X}_\beta$. We will show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, we can find $(E, \mu) \in B(W,s)$ with $d_{\mathbf{H}}(E, E_0) \leqslant \varepsilon$ and $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$. We may assume by Corollary 3.4 that $\operatorname{supp}(\mu_0) = E$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$. Since W has lower Minkowski dimension at most α, we can find arbitrarily small $r \in (0,1)$ such that

$$|W_r| \leqslant r^{dn - \alpha - \varepsilon_1/4}. \tag{4.1}$$

Assume also that r is small enough that we can find an integer $K \geqslant 10$ with

$$r^{-(\beta - \varepsilon_1/2)} \leqslant K \leqslant r^{-(\beta - \varepsilon_1/2)} + 1. \tag{4.2}$$

Let $X_1, \ldots, X_K$ be independent and uniformly distributed on $\mathbf{T}^d$. For each distinct set of indices $k_1, \ldots, k_n \in \{1, \ldots, K\}$, the random vector $X_k = (X_{k_1}, \ldots, X_{k_n})$ is uniformly distributed on $\mathbf{T}^{nd}$, and so (4.1) and (4.2) imply that

$$\mathbf{P}(d(X_k, W) \leqslant r) \leqslant |W_r| \leqslant r^{dn - \alpha - \varepsilon_1/4} \lesssim_{d,n,\beta} K^{\frac{-(dn - \alpha - \varepsilon_1/4)}{\beta - \varepsilon_1/2}} \leqslant K^{-(n - 1/2)}, \tag{4.3}$$

where we used the calculation

$$\begin{aligned}
\frac{dn - \alpha - \varepsilon_1/4}{\beta - \varepsilon_1/2} &= \frac{dn - \alpha}{\beta} + \frac{[(dn - \alpha)/\beta](\varepsilon_1/2) - \varepsilon_1/4}{\beta - \varepsilon_1/2} \\
&\geqslant \frac{dn - \alpha}{\beta} + \frac{(n - 1/2)(\varepsilon_1/2) - \varepsilon_1/4}{\beta - \varepsilon_1/2} \\
&\geqslant \frac{dn - \alpha}{\beta} \geqslant n - 1/2.
\end{aligned} \tag{4.4}$$

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If M_0 denotes the number of indices i such that $d(X_i, W) \leqslant r$, then by linearity of expectation we conclude from (4.3) that there is a constant C depending only on d, n, and β such that

$$\mathbf{E}(M_0) \leqslant (C/10)K^{1/2}. \tag{4.5}$$

Applying Markov's inequality to (4.5), we conclude that

$$\mathbf{P}(M_0 \geqslant CK^{1/2}) \leqslant 1/10. \tag{4.6}$$

Taking a union bound to (4.6) and the results of Lemma 3.10, we conclude that there exists K points $x_1, \ldots, x_K \in \mathbf{T}^d$ and a constant C depending only on d, n, and β such that the following two statements hold:

(1) Let S be the set of indices $k_1 \in \{1, \ldots, K\}$ with the property that we can find distinct indices $k_2, \ldots, k_n \in \{1, \ldots, K\}$ such that if $X = (X_{k_1}, \ldots, X_{k_n})$, then $d(X, W) \leqslant r$. Then

$$\#(S) \leqslant CK^{1/2}. \tag{4.7}$$

(2) If we define

$$\eta_0(x) = \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k}(x)$$

then for $0 < |\xi| \leqslant K^{100/\beta}$,

$$|\widehat{\eta_0}(\xi)| \leqslant CK^{-1/2} \log(K)^{1/2}. \tag{4.8}$$

Thus (4.7) and (4.8) imply that if

$$\eta_1(x) = \sum_{k \notin S} \delta_{x_k}(x),$$

then for each $|\xi| \leqslant K^{1/\beta_0 + 1}$,

$$|\widehat{\eta_1}(\xi)| \leqslant 2CK^{-1/2} \log(K). \tag{4.9}$$

Since $K \geqslant r^{\varepsilon_1/2 - \beta}$, we can apply Lemma 3.12 to η_1, and then apply Corollary 3.9 to the result of Lemma 3.12 to conclude that for any $\delta > 0$, there is $r_0 > 0$ such that if $r \leqslant r_0$ and we define

$$\mu'(x) = \left(\sum_{k \notin S} \phi_{(r/2n^{1/2})}(x - X_k) \right) \mu_0(x),$$

and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, then

$$\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \min(\delta, \varepsilon). \tag{4.10}$$

Choosing δ in accordance with Lemma 3.7, (4.10) also implies that $d_{\mathbf{H}}(\text{supp}(\mu), \text{supp}(\mu_0)) \leqslant \varepsilon$. Since ε and ε_1 are arbitrary, our proof would therefore be complete if we could show that $(\text{supp}(\mu), \mu) \in B(W, s)$.

Consider n points $y_1, \ldots, y_n \in \text{supp}(\mu)$, with $|y_i - y_j| \geqslant r$ for any two indices $i \neq j$. We can therefore find distinct indices $k_1, \ldots, k_n \in \{1, \ldots, K\}$ such that for each $i \in \{1, \ldots, n\}$, $|x_{k_i} - y_i| \leqslant (n^{-1/2}/2) \cdot r$, which means if we set $x = (x_{k_1}, \ldots, x_{k_n})$ and $y = (y_1, \ldots, y_n)$, then

$$|x - y| \leqslant (r/2). \tag{4.11}$$

Since $i_1 \notin S$, $d(x, W) \geqslant r$, which combined with (4.11) implies

$$d(y, W) \geqslant d(x, W) - |x - y| \geqslant r/2. \tag{4.12}$$

Thus in particular we conclude $y \notin W$, which shows $(E, \mu) \in B(W, s)$. $\qquad\square$

The Baire category theorem, applied to Lemma 4.1 implies Theorem 1.6, and thus concludes the proof of Theorem 1.1. Before we move onto the next proof, let us discuss where the loss in Theorem 1.1 occurs in our proof, as compared to the Hausdorff dimension bound of [1]. In the proof of Lemma 4.1, in order to obtain the bound (4.9), we were forced to choose the parameter r such that $\#(S) \leqslant K^{1/2}$, so that the trivial bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \leqslant \#(S) \tag{4.13}$$

obtained by the triangle inequality is viable to obtain a Fourier dimension bound. On the other hand, if we were able to justify that with high probability, we could obtain a square root cancellation bound

$$\left| \sum_{k \in S} e^{2\pi i (\xi \cdot X_k)} \right| \lesssim \#(S)^{1/2}, \tag{4.14}$$

then we would only need to choose the parameter r such that $\#(S) \lesssim K$, which leads to a set with larger Fourier dimension, matching with the Hausdorff dimension bound obtained in [1]. Under stronger assumptions on the pattern we are trying to avoid (the hypothesis of Theorem 1.7, we are roughly able to obtain such a result in the next section (though with an additional oscillatory term which decays fast as $\xi \to \infty$), enabling us to obtain this improved dimension bound.

5   Concentration Bounds for Smooth Surfaces

In this section we prove Theorem 1.7 using some more robust probability concentration calculations. We set

$$\beta = \frac{d}{n-1}.$$

For this value we will be able to generically avoid the pattern Z described in Theorem 1.7.

Lemma 5.1. *Let $Z \subset \mathbf{T}^{dn}$ satisfy the hypothesis of Theorem 1.7. Then for quasi-all $(E, \mu) \in \chi_\beta$, and for any distinct points $x_1, \ldots, x_n \in E$, $(x_1, \ldots, x_n) \notin Z$.*

Proof. Fix a set

$$W = \{(x_1, \ldots, x_d) \in U \times V : x_i = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\}$$

as in the statement of Theorem 1.7, where $f \in C^\infty(V)$. For notational convenience, we assume $i = 1$. Given any family of disjoint, sidelength s closed cubes $R_1, \ldots, R_n \subset \mathbf{T}^d$ such that $d(R_i, R_j) \geqslant 10s$ and $R_1 \times \cdots \times R_n \cap W$ is a closed set, we let

$$H(W; R_1, \ldots, R_n) = \{(E, \mu) \in \mathcal{X}_\beta : (R_1 \times \cdots \times R_n) \cap E^n \cap W = \varnothing\}. \tag{5.1}$$

Then $H(W; R_1, \ldots, R_n)$ is an open subset of $\mathcal{X}_\beta$. For the purpose of a Baire category argument, our result will follow by showing $H(W; R_1, \ldots, R_n)$ is dense in $\mathcal{X}_\beta$ for any family of cubes $\{R_1, \ldots, R_n\}$, such that if $Q_1 = 2R_1, \ldots, Q_n = 2R_n$, then $Q_2 \times \cdots \times Q_n \subset V$ and $Q_1 \subset U$. Fix a constant $L \geqslant 0$ such that for any $x_1, x_2 \in Q_2 \times \cdots \times Q_n$,

$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|. \tag{5.2}$$

As with the previous proof, we fix $(E_0, \mu_0) \in \mathcal{X}_\beta$ with $\mathrm{supp}(\mu_0) = E_0$ and $\mu_0 \in C^\infty(\mathbf{T}^d)$, and show that for any $\varepsilon_1 \in (0, \beta/100]$ and $\varepsilon > 0$, there is $(E, \mu) \in H(W; Q_1, \ldots, Q_n)$ with $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)} \leqslant \varepsilon$.

Fix a family of non-negative bump functions $\psi_0, \psi_1, \ldots, \psi_n \in C^\infty(\mathbf{T}^d)$, such that for $i \in \{1, \ldots, n\}$, $\psi_i(x) = 1$ for $x \in R_i$, $\psi_i(x) = 0$ for $x \notin Q_i$, and $\psi_0(x) + \cdots + \psi_n(x) = 1$ for $x \in \mathbf{T}^d$. For $i \in \{0, \ldots, n\}$, let $A_i = \int \psi_i(x)\,dx$ denote the total mass of ψ_i. Now consider a family of independant random variables $\{X_i(k) : 0 \leqslant i \leqslant n, 1 \leqslant k \leqslant K\}$, where the random variable $X_i(k)$ is continuous with distribution function $A_i^{-1}\psi_i$. Let $r = K^{\varepsilon_1/2 - 1/\beta}$ and then let S be the set of indices $k_1 \in \{1, \ldots, K\}$ such that there are indices $k_2, \ldots, k_n \in \{1, \ldots, K\}$ with the property that

$$|X_1(k_1) - f(X_2(k_2), \ldots, X_n(k_n))| \leqslant (L + 1)r. \tag{5.3}$$

A simple argument following from (5.3) shows that if $k_1 \notin S$, then for any $k_2, \ldots, k_n \in \{1, \ldots, K\}$, if $X = (X_1(k_1), \ldots, X_n(k_n))$, then $d(X, W) \geqslant r$. Thus if we define

$$\eta = \frac{1}{K}\left(\sum_{i \in \{0, 2, \ldots, n\}} A_i \sum_{k=1}^K \delta_{X_i(k)} + A_1 \sum_{k \notin S} \delta_{X_1(k)}\right),$$

if we define $\mu' = (\eta * \phi_{(r/2n^{1/2})}) \cdot \mu_0$, and then set $\mu = \mu'/\mu'(\mathbf{T}^d)$, $E = \mathrm{supp}(\mu)$, then $(E, \mu) \in H(W; R_1, \ldots, R_n)$. The remainder of our argument consists of obtaining control on the exponential sum

$$\hat{\eta}(\xi) = \frac{1}{K}\sum_{i \in \{0, 2, \ldots, n\}} A_i \sum_{k=1}^K e^{-2\pi i \xi \cdot X_i(k)} + \frac{1}{K}A_1 \sum_{k \notin S} e^{-2\pi i \xi \cdot X_1(k)}, \tag{5.4}$$

for nonzero $\xi \in \mathbf{Z}^d$, so that we may apply Corollary 3.9 to bound $\|\mu - \mu_0\|_{M(\beta/2 - \varepsilon_1)}$.

To analyze $\hat{\eta}$, introduce the measures

$$\nu = \frac{1}{K}\sum_{i \in \{0, \ldots, n\}} A_i \sum_{k=1}^K \delta_{X_i(k)}$$

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and
$$\sigma = \nu - \eta = \frac{A_1}{K} \sum_{k \in S} \delta_{X_1(k)}.$$

For non-zero $\xi \in \mathbf{Z}^d$,

$$\mathbf{E}(\widehat{\nu}(\xi)) = \sum_{i=0}^{n} \int \psi_i(x) e^{2\pi i \xi \cdot x} \, dx = \int_{\mathbf{T}^d} e^{2\pi i \xi \cdot x} \, dx = 0. \tag{5.5}$$

Applying Lemma 3.10, we conclude that if $B = \{\xi \in \mathbf{Z}^d : 0 < |\xi| < K^{100/\beta}\}$, then there is $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\nu}\|_{L^\infty(B)} \geqslant CK^{-1/2} \log(K)^{1/2}\right) \leqslant 1/10. \tag{5.6}$$

Analyzing $\widehat{\sigma}(\xi)$ requires a more subtle concentration bound. As a random quantity, $\widehat{\sigma}(\xi)$ is a function of the independant random quantities $\{X_i(k)\}$, and so McDiarmid's inequality presents itself as a useful concentration bound. However, a naive application of McDiarmid's inequality fails, because changing a single random vairable $X_i(k)$ for $i \geqslant 2$ while fixing other random variables can change $\widehat{\sigma}(\xi)$ by as much as $\Omega(1)$, which is far too much to obtain the square root cancellation bounds like we obtained in (5.6). On the other hand, it seems that a single variable $X_i(k)$ only changes $\widehat{\sigma}(\xi)$ by $\Omega(1)$ if the random variabels $\{X_1(k)\}$ are fixed in a very precise region of space, which is unlikely to happen (TODO: Diagram). Thuswe should expect that adjusting a single random variable $X_i(k)$ does not influence the value of $Y(\xi)$ very much if the quantity $Y(\xi)$ is *averaged* over the possible choices of $\{X_1(k)\}$.

Consider the random set $\Omega \subset Q_1$, which is the set of values $x \in Q_1$ such that there are $k_2, \ldots, k_n \in \{1, \ldots, K\}$ with

$$|x - f(X_2(k_2), \ldots, X_n(k_n))| \leqslant r. \tag{5.7}$$

Then

$$\widehat{\sigma}(\xi) = \frac{1}{K} \sum_{k=1}^{K} Z(k). \tag{5.8}$$

where

$$Z(k) = \begin{cases} e^{2\pi i \xi \cdot X_1(k)} & : X_1(k) \in \Omega, \\ 0 & : X_1(k) \notin \Omega \end{cases}. \tag{5.9}$$

If Σ is the σ algebra generated by the random variables $\{X_i(k) : i \geqslant 2, k \in \{1, \ldots, K\}\}$, then the random variables $\{Z(k)\}$ are *conditionally independant* given Σ. Since we have $|Z(k)| \leqslant 1$ almost surely, Hoeffding's inequality thus implies that for all $t \geqslant 0$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2}\right). \tag{5.10}$$

It is simple to see that

$$\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) = \int_\Omega \psi_1(x) e^{2\pi i \xi \cdot x} \, dx. \tag{5.11}$$

Since

$$\Omega = \bigcup \{B_r(f(X_2(k_2), \ldots, X_n(k_n))) : 1 \leqslant k_2, \ldots, k_n \leqslant K\}. \tag{5.12}$$

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we see that varying each random variable $X_i(k)$ adjusts at most K^{n-2} of the radius r balls forming Ω, and thus varying $X_i(k)$ independantly of the other random variables changes $\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma)$ by at most

$$2 \cdot (2r)^d \cdot K^{n-2} = 2^{d+1} \cdot r^d \cdot K^{n-2} = \frac{2^{d+1} \cdot r^{(n-1)\varepsilon_1/2}}{K} \leqslant \frac{2^{d+1}}{K}. \tag{5.13}$$

Thus McDiarmid's inequality shows that for any $t \geqslant 0$,

$$\mathbf{P}\left(|\mathbf{E}(\widehat{\sigma}(\xi)|\Sigma) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 4 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{5.14}$$

Combining (5.10) and (5.14), we conclude that for each $\xi \in \mathbf{Z}^d$,

$$\mathbf{P}\left(|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))| \geqslant t\right) \leqslant 8 \exp\left(\frac{-Kt^2}{2^{2d+1}}\right). \tag{5.15}$$

Applying a union bound to (5.15) over all $\xi \in B$ shows that there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\sigma}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/10. \tag{5.16}$$

Combining (5.6) and (5.16), we conclude that there exists a constant $C > 0$ such that

$$\mathbf{P}\left(\|\widehat{\eta}(\xi) - \mathbf{E}(\widehat{\sigma}(\xi))\|_{L^\infty(B)} \geqslant CK^{-1/2}\log(K)^{1/2}\right) \leqslant 1/2. \tag{5.17}$$

We have thus reduced the study of $\widehat{\eta}$ to the nonrandom quantity $\mathbf{E}(\widehat{\sigma}(\xi))$.

We start by using the fact that $\{X_1(k)\}$ are uniformly distributed to write

$$\mathbf{E}(\widehat{\sigma}(\xi)) = \mathbf{E}(e^{2\pi i \xi \cdot X_1(1)}\mathbf{I}(1 \in S)).$$

To understand this expectation it clearly suffices to understand the distribution of $X_1(1)$ given that $1 \in S$.

For each s, and $i \in \{2, \ldots, n\}$, let $N(i,s)$ denote the number of indices k_i such that there exists indices $k_2, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n$ such that

$$|s - f(X_2(k_2), \ldots, X_i(k_i), \ldots, X_n(k_n))| \leqslant (L+1)r. \tag{5.18}$$

Suppose that there exists a function g_i such that (5.18) holds if and only if there is s' with $|s - s'| \leqslant (L+1)r$ and

$$X_i(k_i) = f(s', X_2(k_2), \ldots, X_n(k_n)) \tag{5.19}$$

Let Σ be the sigma algebra generated by $\{X_j(k) : j \neq i, 1 \leqslant k \leqslant K\}$. Then $N(i,s)$ is the sum of K events which are conditionally independent given Σ. For each fixed set of values $\{X_j(k)\}$, each of these events occur with probability $O(K^{n-2}r^d) = O(1/K)$. It follows from Chernoff's inequality that

$$\mathbf{P}(N(i,s) \geqslant t) \lesssim \left(\frac{e}{tK^\varepsilon}\right)^t$$

If there are Y_s intersection points, then there are $Y_s^{1/(n-1)}$ intersection points in some projection so

$$\sum_{k=2}^{K^{n-1}} k\,\mathbf{P}(Y_s = k) \leqslant \sum_{k=2}^{K^{n-1}} k\left(\frac{e}{k^{1/(n-1)}K^\varepsilon}\right)^{k^{1/(n-1)}} \lesssim K^{-\varepsilon}$$

$\mathrm{k} = (1/2\ \varepsilon)^{n-1} K^{-\varepsilon k^{1/(n-1)}}$

Since the events are positively correlated,

$$
\begin{aligned}
\mathbf{E}\left(1 - \prod_{k_2,\dots,k_n} \mathbf{I}((X_2(k_2),\dots,X_n(k_n)) \notin E)\right) &= 1 - \mathbf{E}\left(\prod_{k_2,\dots,k_n} \mathbf{I}((X_2(k_2),\dots,X_n(k_n) \notin E))\right) \\
&\leqslant 1 - \prod_{k_2,\dots,k_n} \mathbf{P}\left((X_2(k_2),\dots,X_n(k_n) \notin E)\right) \\
&= 1 - (1-|E|)^{K^{n-1}}
\end{aligned}
$$

For each non-zero $\xi \in \mathbf{Z}^d$, let η_W be the surface measure on W obtained from the pullback of the Lebesgue measure via the projection onto $(x_2,\dots,x_d)$, let σ_r be the measure of the ball of radius r in $\mathbf{T}^d \times \{0\}^{n-1}$, and let $\psi = \psi_1 \otimes \cdots \otimes \psi_n$. Since the random variables $X = (X_1(k_1),\dots,X_n(k_n))$ are uniformly distributed for $1 \leqslant k_1,\dots,k_n \leqslant K$, we calculate that

$$
\begin{aligned}
\mathbf{E}(\widehat{\sigma}(\xi)) &= K^n \cdot \mathbf{E}\left(\frac{1}{K} \cdot \mathbf{I}\left(|X_1 - f(X_2,\dots,X_n)| \leqslant (L+1)r\right) e^{2\pi i \xi \cdot X_1}\right) \\
&= \frac{K^{n-1}}{A_1 \cdots A_n} \int_W \int_{B_{(L+1)r}(0)} \psi(x_1 + y, x_2,\dots,x_n) e^{2\pi i \xi \cdot (x_1 + y)}\, dy\, dx \\
&= \frac{K^{n-1}}{A_1 \cdots A_n} \left((\psi \cdot \eta_W) * \sigma_{(L+1)r}\right)^{\wedge}(-\xi, 0,\dots,0) \\
&= \frac{K^{n-1}}{A_1 \dots A_n} \widehat{\psi \cdot \eta_W}(-\xi, 0,\dots,0) \cdot \widehat{\sigma_{(L+1)r}}(-\xi, 0,\dots,0) \\
&= \frac{(L+1)^d}{A_1 \dots A_n} \cdot r^d K^{n-1} \cdot \widehat{\psi \cdot \eta_W}(-\xi, 0,\dots,0) \cdot \widehat{\sigma_1}(-(L+1)r\xi, 0,\dots,0) \\
&= \frac{(L+1)^d}{A_1 \dots A_n} \cdot r^{(n-1)\varepsilon_1/2} \cdot \widehat{\psi \cdot \eta_W}(-\xi, 0,\dots,0) \cdot \widehat{\sigma_1}(-(L+1)r\xi, 0,\dots,0).
\end{aligned}
$$
(5.20)

Since $\sigma_1(\mathbf{T}^d) = O_d(1)$, $\|\widehat{\sigma_1}\|_{L^\infty(\mathbf{Z}^d)} \lesssim_d 1$ (this is the best bound possible for $|\xi| \leqslant 1/r$). If we define $\widetilde{\psi}(x_2,\dots,x_n) = \psi(f(x_2,\dots,x_n),x_2,\dots,x_n)$, then

$$\widehat{\psi \cdot \eta_W}(-\xi, 0,\dots,0) = \int_W \widetilde{\psi}(x_2,\dots,x_n) e^{2\pi i \xi \cdot f(x_2,\dots,x_n)}\, dx_2 \dots dx_n$$

The conditions we placed on the surface W (proved, for instance, in [6]) imply that

$$|\widehat{\psi \cdot \eta_W}(-\xi, 0,\dots,0)| \lesssim |\xi|^{-\beta/2}.$$
(5.21)

Thus for any $\delta > 0$, there exists r_1 such that for any $r \leqslant r_1$, and for any nonzero $\xi \in \mathbf{Z}^d$,

$$| \mathbf{E}(Y(\xi))| \leqslant \delta|\xi|^{-\beta/2}. \tag{5.22}$$

From (5.17) and (5.22), we conclude there exists a choice of values $\{X_i(k)\}$ such that for any $\xi \in B$,

$$|\widehat{\eta}(\xi)| = |U(-\xi)| \leqslant CK^{-1/2}\log(K)^{1/2} + \delta|\xi|^{-\beta/2}. \tag{5.23}$$

Applying Lemma 3.12 and then Corollary 3.9, we conclude that if δ is chosen appropriately, then $\|\mu - \mu_0\|_{M(\beta/2-\varepsilon_1)} \leqslant \varepsilon$, which completes the proof. $\qquad\square$

6   Salem Sets with non-approximable difference Sets

Analyzing the last proof, you may have noticed that to

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