Fractals Avoiding Fractal Sets

Jacob Denson Malabika Pramanik Josh Zahl

March 24, 2019

We now describe an avoidance technique applied at a single scale. The technique is repeated for an infinite sequence of scales to then get the general result. Given a dyadic length L, we let $\mathcal{B}(L,d)$ denote the partition of $\mathbf{R}^d$ into a family of half open cubes with corners lying on the lattice $(\mathbf{Z}/L)^d$. If the dimension is clear, we simplify denote $\mathcal{B}(L,d)$ as $\mathcal{B}(L)$.

Lemma 1. *Consider three dyadic scales $L \gg R \gg S$. If $I \subset \mathbf{R}^d$ is a union of $\mathcal{B}(L)$ cubes and $K \subset \mathbf{R}^{nd}$ a union of $\mathcal{B}(S)$ cubes, then we can find $J \subset I$ such that for distinct $\mathcal{B}(S)$ subcubes $J_1, \dots, J_n$ of J, $J_1 \times \cdots \times J_n$ is disjoint from K, and J contains a $\mathcal{B}(S)$ subcube from all but $|K|R^{-nd}$ of the $\mathcal{B}(R)$ subcubes of I.*

Proof. Form a random set U by selecting uniformly randomly, from each $\mathcal{B}(R)$ subcube of I, a single subcube in $\mathcal{B}(S)$. Thus the probability that any subcube is selected is $(S/R)^d$. Since any two $\mathcal{B}(S)$ subcubes of U lie in distinct elements of $\mathcal{B}(R)$, the only chance that a $\mathcal{B}(S)$ subcube I of K with distinct sides intersects U^n is if $I_1, \dots, I_n$ all lie in separate cubes in $\mathcal{B}(R)$. Then the chance that each occurs is independant of one another, and so

$$\mathbf{P}(I \in U^n) = \mathbf{P}(I_1 \in U) \dots \mathbf{P}(I_n \in U) = (S/R)^{nd}$$

If E denotes the number of $\mathcal{B}(S)$ subcubes I of K contained in U^n,

$$\mathbf{E}(E) = \sum_{I \subset K} \mathbf{P}(I \in U^n) = [|K|S^{-nd}][(S/R)^{nd}] = |K|R^{-nd}$$

If, for each $\mathcal{B}(S)$ subcube I of U^n, we remove I_i from U, for any index i, we obtain a set J with $J_1 \times \cdots \times J_n$ disjoint from K for any distinct $\mathcal{B}(R)$ subcubes J_i of J. The interval J contains an interval from all but E sidelength R cubes. In particular, we can select some nonrandom choice of U such that $E \leqslant |K|R^{-nd}$, which gives the required J. $\square$

We will be most interested in applying this lemma when L and S are consecutive hyperdyadic numbers. We fix a small δ, and define $H_N = 2^{-\lfloor (1+\delta)^N \rfloor}$ to be the N'th hyperdyadic number. Hyperdyadic numbers are useful scales for discussing Hausdorff dimension, because a weak-type bound allows us understand coverings of the set. The idea to use such scales was inspired by Katz and Tao's 2001 paper on Falconer's distance problem. We say a sequence $Y_1, Y_2, \dots$ *strongly* covers Y if $Y \subseteq \limsup Y_N$, so each $y \in Y$ is in infinitely many of the sets Y_N.

Lemma 2. *If Y has Hausdorff dimension α, then for each $\varepsilon > \delta$ we can find a strong cover of Y by a sequence of sets Y_N, where Y_N is a union of $O_\varepsilon(N/H_N^{\alpha+\varepsilon})$ hyperdyadic cubes with sidelength H_N.*

Proof. Since Y is α dimensional, then for each ε_0 and N we can find a covering of Y by dyadic cubes I_i, where I_i has sidelength $L_i \leqslant H_N$, and $\sum L_i^{\alpha+\varepsilon_0} \lesssim_{\varepsilon_0} 1$. But we can now apply a weak type bound to conclude that the number of L_i between H_{N+1} and H_N is $O_{\varepsilon_0}(1/H_{N+1}^{\alpha+\varepsilon_0})$. The calculation

$$1/H_{N+1}^{\alpha+\varepsilon_0} = (H_N/H_{N+1})^{\alpha+\varepsilon_0}(1/H_N^{\alpha+\varepsilon_0}) \lesssim 1/H_N^{\alpha+\varepsilon_0+\delta}$$

implies we have used $O_{\varepsilon_0}(1/H_N^{\alpha+\delta+\varepsilon_0})$ sidelength H_N cubes to cover Y. If we now perform this process for each index N, then collect all cubes together from all the covers to obtain a strong cover, and set $\varepsilon = \delta + \varepsilon_0$, we obtain the required result. $\square$

Corollary. *Suppose R is the closest dyadic number to S^β, where $\beta < 1$. Furthermore, suppose $|K| \lesssim NS^{nd-\gamma}$. Then provided*

$$\beta \leqslant \frac{nd - \gamma - \log_S(|I|N^{-1}) - O\left(\log(1/S)^{-1}\right)}{(n-1)d}$$

the set J obtained in the discrete avoidance lemma contains a portion of at least half of all the sidelength L_1 intervals contined in I.

Proof. The inequality here is equivalent to

$$nd - \gamma - \beta(n-1)d \geqslant \log_S(|I|N^{-1}) + O(1/\log(1/S))$$

We therefore find

$$\begin{aligned}
\frac{\#(\mathcal{B}(R) \text{ subcubes not selected from})}{\#(\text{all } \mathcal{B}(R) \text{ subcubes})} &= \frac{|K|R^{-nd}}{|I|R^{-d}} \\
&\lesssim \frac{[NS^{nd-\gamma}][S^{-\beta nd}]}{|I|S^{-\beta d}} = N|I|^{-1}S^{nd-\gamma-\beta(n-1)d} \\
&\leqslant N|I|^{-1}S^{\log_S(|I|N^{-1})+O(1/\log(1/S))} = S^{O(1/\log(1/S))}
\end{aligned}$$

By choosing the constant in the exponent appropriately, we can make the fraction we began with not exceed 1/2, so less than half of the intervals are missed in the selection process. $\square$

Remark. *If $\log(|I|N^{-1}) = o((1+\delta)^{N+1})$, then $\log_S(|I|N^{-1}) = o(1)$, which enables us to take $\beta = (nd-\gamma)/d(n-1) - o(1)$. Now if I is an interval obtained from repeatedly dissecting intervals at hyperdyadic scales according to the corollary above, throwing out at most half the intervals at each stage, we know*

$$\begin{aligned}
|I| &\geqslant 2^{-N}(1/H_1^{\beta_1})^d(H_1/H_2^{\beta_2})^d \dots (H_{N-1}/H_N^{\beta_N})^d \\
&= 2^{-N}H_1^{d(1-\beta_1)} \dots H_{N-1}^{d(1-\beta_N)}H_N^{-d\beta_N}
\end{aligned}$$

Thus

$$|\log_2 |I|N^{-1}| \approx \left|\beta(1+\delta)^N - N - \log N - (1-\beta)\sum_{k=1}^{N-1}(1+\delta)^k\right| \lesssim (1+\delta)^N$$

This means we don't have $\log|I|N^{-1} = o((1+\delta)^{N+1})$*, only* $|\log|I|N^{-1}| = O((1+\delta)^{N+1})$*. This prevents us from choosing our required* β*, but, only barely. It certainly means that we can lower bound* β*, so we can obtain some Hausdorff dimension output result given a Hausdorff dimension input. Nonetheless, it's not the right output dimension we would expect (CHANGE OUR HAUSDORFF DIMENSION TO MAKE THIS 'JUST' WORK?).*

We now construct X as a limit of discrete nested sets X_N, where X_N is a union of cubes in $\mathcal{B}(L_N)$, and X_N^n is disjoint from all *non-diagonal* cubes with length L_N in Y_N. By a non-diagonal cube, we mean $I \subset \mathcal{B}(L_N)$ such that $I_i \neq I_j$ for $i \neq j$, where I_i and I_j are projections onto the i'th and j'th coordinate set of $\mathbf{R}^{nd} = (\mathbf{R}^d)^n$. In particular, this means that X^n avoids all cubes at all lengths in the cover of Y, and in particular, avoids non-diagonal elements of Y.

Lemma 3. *If* X_N^n *avoids non-diagonal cubes in* Y_N*, for large* N*,* $X^n \cap Y \subset \Delta$*.*

Proof. Let $y \in Y$ be a point not contained in Δ. Because Y_N forms a strong cover, we can find an infinite sequence of indices N_k with $y \in Y_{N_k}$. For a suitably large choice of K, $\sqrt{n} \cdot L_{N_k} < d(y, \Delta)$ for $k \geqslant K$. But this means that the cube of $\mathcal{B}(L_{N_k})$ containing y is disjoint from Δ, and therefore cannot be a non-diagonal cube. This means that $X_{N_k}^n$ is disjoint from this cube, and therefore does not contain y. But then $X^n = \lim X_{N_k}^n$ cannot contain y. Taking contrapositives to our argument, we have shown every point in $X^n \cap Y$ must lie in Δ. □