Rings and Modules

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Part I

Basic Ring Theory

Chapter 1

Basic Definitions

Rings are algebraic structures closed under addition, subtraction, and multiplication, but not necessarily under division. They can be noncommutative, such as the ring of square matrices of a fixed dimension, or commutative, like the ring of integers. To be precise, a *ring* is a set R together with addition and multiplication operations $\cdot : R \times R \to R$ and $+ : R \times R \to R$, giving R the structure of an abelian group and the structure of a (not necessarily commutative) monoid respectively. In addition, the operations must satisfy the *distributive law*

$$a(b+c) = ab + ac \quad \text{and} \quad (b+c)a = ba + ca,$$

for any $a, b, c \in R$. Note that one equation does not establish the other since the multiplication operation need not be commutative. We will often denote the multiplicative identity of a ring by 1, and the additive identity as zero.

Remark. One often encounters nonunital rings in practice, i.e. rings without a multiplicative identity. But there are many different techniques one can reduce the study of nonunital rings by adjoining an identity, so for simplicity of exposition we assume all rings in these notes have identity. D.D. Anderson has written a useful article on the correspondence between unital and non-unital rings, entitled *Commutative Rngs*.

Example. *The integers* $\mathbf{Z}$ *form the classical example of a ring, as are the integers* $\mathbf{Z}_n$ *modulo* n*, and we find they exhibit most of the basic properties of rings. They have a nontrivial divisibility theory, yet still possess the property*

of unique factorization into prime elements, an idea we will study in the more general situation of unique factorization domains.

Example. *All the number systems* $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{F}_p$, *and* $\mathbf{C}$ *are rings, in which case every nonzero element is invertible. Such rings are known as* division rings, *and if the multiplicative operation is commutative,* fields.

Example. *For any ring* A, *the family of* $n \times n$ *matrices* $M_n(A)$ *with entries in* A *forms a ring, extensively studied in linear algebra and representation theory. An important fact about these matrices is that* $M_n(M_m(A))$ *is isomorphic to* $M_{nm}(A)$, *because* block multiplication *works in these rings.*

Example. *A key way to analyze the algebraic structure of a ring* A *is to introduce encodings of the algebraic structure of* A *through the theory of polynomials* $A[X]$ *over that ring, formal sums of the form* $a_0 + a_1 X + \cdots + a_N X^N$, *where* $a_n \in A$. *We view two polynomials as being equal precisely when their coefficients are equal. More generally, we can discuss their multivariate counterparts* $A[X_1, \ldots, X_n]$, *the ring of formal sums in the monomials* $X_1^{m_1} \ldots X_n^{m_n}$. *The addition of two polynomials is defined by taking the sum over each monomial separately, and the product is obtained by expanding and multiplying monomials together in the obvious way. The polynomial ring is the 'most general' way to add a new family of 'commuting' elements to a particular ring; it has the universal property that for any homomorphism* $f : A \to B$ *of rings and any* $x_1, \ldots, x_n \in C_B(f(A))$, *there exists a unique extension of* f *to a homomorphism* $f : A[X_1, \ldots, X_n] \to B$ *with* $f(X_i) = x_i$ *for each* $i \in \{1, \ldots, n\}$.

Example. *If* A *is a ring, and* M *is a multiplicative semigroup, then we can consider a ring* $A[M]$, *known as the* monoid ring, *whose elements are finite formal sums of the form* $\sum a_n x_n$, *with* $x_n \in A$, $x_n \in M$, *with the obvious additive structure, and with multiplicative structure defined by multiplying elements of the semigroup termwise. We calculate that*

$$\left(\sum_{x \in M} a_x x \right) \left(\sum_{y \in M} b_y y \right) = \sum_{x,y \in M} a_x b_y (xy) = \sum_{z \in M} \left(\sum_{xy = z} a_x b_y \right) z.$$

Thus multiplication is given by a kind of convolution in the coefficients of the ring. In the case where $M = \mathbf{N}$ *or* $M = \mathbf{N}^n$, $A[M]$ *is the polynomial ring in one or more variables. There are other variants of this group:*

- *If M is a semigroup such that, for each $k \in M$, there are only finitely many pairs $g, h \in M$ such that $gh = k$, then we can consider the ring $A_\infty[M]$ of infinite formal sums $\sum a_n x_n$, with multiplication defined as in $A[M]$. For $M = \mathbf{N}$, the ring $A[[M]]$ can be viewed as the ring of formal power series with elements in A.*

- *If G is a locally compact group with Haar measure μ, then for $f, g \in L^1(G)$, we can define the* convolution function *$f * g$ by the formula*

$$(f * g)(x) = \int f(y) g(y^{-1}x)\, d\mu(y).$$

  Then $\|f * g\|_{L^1(M)} \leqslant \|f\|_{L^1(M)} \|g\|_{L^1(M)}$, so $L^1(M)$ can be viewed as a generalization of the ring $\mathbf{R}[G]$ to more analytical settings. However, if G is non-discrete, then $L^1(G)$ does not have an identity; this can be fixed by introducing the Dirac mass *δ at the identity to $L^1(G)$, which is no longer an integrable function, but can be defined as a finite measure on G. More generally, we can consider the measure algebra $M(G)$, upon which we have an identity.*

If $M = G$ is a multiplicative group, then $R[G]$ is known as the group ring *associated with G.*

Example. *The quaternion division ring* $\mathbf{H}$*, named after their creator, Hamilton, which we can informally describe as the family of formal quantities $a + bi + cj + dk$ with operations induced by the identities*

$$i^2 = j^2 = k^2 = ijk = -1$$

Formally, we can construct this ring by considering the group algebra $\mathbf{R}[Q]$, where Q is the quaternion group, whose elements we denote by

$$\{1, \overline{1}, i, \overline{i}, j, \overline{j}, k, \overline{k}\}.$$

This group algebra cannot be identified with $\mathbf{H}$; for instance, we would like $\overline{1}$ to be equal to -1 in this ring. To solve this problem, we quotient by the ideal $\mathfrak{a} = (\overline{1} + 1)$. For any $s \in \{1, i, j, k\}$, we therefore conclude that

$$\overline{s} = \overline{1} \cdot s = -s + (\overline{1} + 1) \cdot s \in -s + \mathfrak{a}.$$

Thus any element of $\mathbf{R}[Q]/\mathfrak{a}$ can be written as $a + bi + cj + dk$ for some real numbers $a, b, c, d \in \mathbf{R}$. We claim no nontrivial elements of $a + bi + cj + dk$ are

in $\mathfrak{a}$*, so any element of* $\mathbf{R}[Q]/\mathfrak{a}$ *can be* uniquely *expressed in this way. Indeed, if there is* $x \in \mathbf{R}[Q]$ *such that*

$$x(\overline{1}+1) = a + bi + cj + dk$$

and if we write $x[s]$ *for the coefficient of* x *corresponding to* $s \in Q$*, then we must have* $x[s] = -x[\overline{s}]$ *for each* $s \in \{1, i, j, k\}$*. But then*

$$x(\overline{1}+1) = \sum_{s\in\{1,i,j,k\}} (x[s] + x[\overline{s}])\, s = \sum_{s\in\{1,i,j,k\}} (x[s] - x[s])s = 0.$$

Thus $a + bi + cj + dk = 0$*. Thus the quotient* $\mathbf{R}[Q]/\mathfrak{a}$ *can be identified with a ring structure on the set* $\{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\}$*, which is the algebraic structure we want to study. Invented by the Irish mathematician Lord Hamilton in the mid 19th century to obtain algebraic characterization of rotation in three dimensional space, the quaternions have a special place in an algebraists heart, for they are the first truly strange algebraic structures obtained in the historical development of abstract algebra.*

Example. *George Boole began the modern study of logic by studying the algebraic notions of truth. He saw that the logical operations of conjunction and disjunction behaved very similarily to the algebraic operations of multiplication and addition. If we consider the set of all equivalence classes of logical statements (two statements being equivalent if they both imply each other), and consider conjunction as a multiplication, and exclusive disjunction as an additive structure, then we obtain a ring satisfying* $x^2 = x$ *for all statements* x*, where* 0 *is a statement that is always false, and* 1 *a statement the is always true. In his honour, we call a ring* Boolean *if this equation is satisfied. In any Boolean ring,* $x + x = 0$*, since*

$$x + x = (x + x)^2 = x^2 + x + x + x^2 = x + x + x + x.$$

We say the ring has characteristic two*. Moreover, any Boolean ring is commutative, since*

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$$

from which we obtain $xy + yx = 0$*, so* $yx = -xy = xy$*. For any set* X*, the set of subsets of* X *form a Boolean ring, such that for* $A, B \subset X$*,*

$$A + B = (A - B) \cup (B - A) \quad \textit{and} \quad A \cdot B = A \cap B.$$

The set X acts as the multiplicative identity, and the empty set $\emptyset$ is the additive identity. Boolean rings are essentially the same as Boolean algebras studied in logic and measure theory, and the exact correspondence is provided by the Stone representation theorem, employing tools from set theoretic topology!

Example. *The theory of rings arises very often in the study of functions. If A is a ring, and X is a set then one can make the set A^X of maps from X to A into a ring, by defininining addition and multiplication pointwise. Thus, for instance, the set $\mathbf{R}^{\mathbf{N}}$ of real valued sequences forms a ring, as does $\mathbf{R}^{\mathbf{R}}$. Subrings of these rings occur all the time in analysis. The ring $C_c(\mathbf{R})$ of compactly supported continuous functions on the real line provides our first natural example of a ring without identity, as does the ring $C_0(\mathbf{R})$ of continuous functions vanishing at infinity. However, one can often reduce the study of $C_c(\mathbf{R})$ to the ring of functions which are* eventually constant *outside of a compact set, and reduce the study of $C_0(\mathbf{R})$ to the ring of functions f such that $\lim_{x\to\infty} f(x)$ exists. These rings correspond to adding an identity to the ring is the freest possible way.*

Example. *If B is a commutative subring of a ring A, then for any subset S of A, we can consider the ring generated by B and S, denoted by $B[S]$. Interesting examples of these include the* Gaussian integers $\mathbf{Z}[i]$*, whose points form a lattice in the plane, and the* dyadic numbers $\mathbf{Z}[1/2]$*, which are the fractions expressible with a denominator a power of two, which form a dense subset of the real line. In algebraic number theory, one studies the ring $\mathbf{Z}[\sqrt{D}]$, where D is a squarefree integers. Such a ring can be described as the set of all numbers of the form $n + m\sqrt{D}$, where $n, m \in \mathbf{Z}$.*

Remark. There is only a single example of a ring with identity in which the multiplicative identity equals the additive identity. This is because for any a in such a ring,

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$$

From which we conclude $a \cdot 0 = 0$. But if 0 is also a multiplicative identity, we conclude that $a = a \cdot 0$. This means the only ring for which the additive and multiplicative identities correspond is the ring consisting of a single element: the number zero! We denote the zero ring by (0), to mirror the notation we will develop for ideals later on.

Remark. One might remark that one might wish to study algebraic structures $(R, +, \cdot)$ in which the *addition operation* is noncommutative. Such

structures do exist, but are exceedingly rare in applications (they are known as *near rings*). Perhaps such rings do not show up often is that if a near ring has a multiplicative identity, the addition is automatically commutative. To see why, we note that if R is a near ring with identity, then for any $a, b \in R$, the two different applications of the distributive law imply that

$$(1+1)\cdot(a+b) = (1+1)\cdot a + (1+1)\cdot b = a+a+b+b$$

and

$$(1+1)\cdot(a+b) = 1\cdot(a+b) + 1\cdot(a+b) = a+b+a+b.$$

Combining these two equations gives $a+b = b+a$.

Rings arise naturally when we start studying symmetries of preexisting algebraic structures, like those that arise in group theory. Matrices are symmetries of vector spaces, which themselves can be seen as shifting symmetries of space. Polynomials are symmetries over a field of numbers, which themselves are also very well behaved symmetries. In fact, like groups, all rings can be represented as symmetries of some abelian group.

Example. *Let G be an abelian group, and consider the set* $\mathrm{End}(G)$ *of all homomorphisms from G to itself. We define a ring structure on this group. Given $f, g \in \mathrm{End}(G)$, we define $f+g$ to be the endomorphism on G defined by $(f+g)(x) = f(x)+g(x)$, and where composition $f \circ g$ is the multiplicative structure. The fact that* $\mathrm{End}(G)$ *satisfies the laws of a ring are trivial, with the identity behaving as* 1*, and the trivial homomorphism acting as* 0*.*

Theorem 1.1. *Suppose A is a ring such that there does not exist nonzero $a \in A$ with $ax = 0$ for all $x \in A$. Then A is isomorphic to a subring of $\mathrm{End}(G)$ for some G.*

Proof. Given a ring A, let A_+ denote the additive group structure of A. Then $\mathrm{End}(A_+)$ is a ring. Consider the map $\varphi : A \to \mathrm{End}(A_+)$, where for $a \in A$, $\varphi(a)$ acts as the map $x \mapsto ax$. The distributive law implies that such a map is an endomorphism. What's more φ is a ring homomorphism, since for each $x, y, z \in A$,

$$\varphi(x+y)(z) = (x+y)(z) = xz + yz = [\varphi(x) + \varphi(y)](z)$$

and

$$\varphi(xy)(z) = (xy)z = x(yz) = (\varphi(x) \circ \varphi(y))(z).$$

The map φ is injective, since if $\varphi(a) = 0$, then $ax = 0$ for all $x \in A$. Thus $\mathrm{End}(A)$ naturally contains A as a subring. □

The problem with this proof is that the theorem doesn't really give a 'nice' answer to what a ring really is. Groups are already abstract, so we may not necessarily be able to visualize what a symmetry of an arbitrary abstract object is. Alas, most general theories in mathematics do not have natural correspondences with a single object of study, unlike the niceities of group theory. This is to be expected, since ring theory arose from many fields of study, like number theory, geometry, and logic. We will just have to accept this theorem as a little tidbit of intuition, and move on. We will return to this idea in the theory of modules, where one studies a ring 'acting' on an abelian group, just like Cayley's theorem hints at using group actions to understand the theory of group actions.

1.1 Units of a Ring

We begin with discussing an operation that seems left out of the definition of a ring – divisibility. In the ring of rational numbers, we can divide a rational number by any *non-zero* rational number, and still get a rational number. On the other hand, an integer divided by an integer is only in very special cases an integer. If A is a ring, the *units* are the elements x which possess a multiplicative inverse x^{-1} such that $xx^{-1} = 1 = x^{-1}x$; note both ends of the equation need to be satisfied since ab may not equal to ba. For example, when A is the ring of endomorphisms on a ring, $ab = 1$ implies b is injective, whereas $ba = 1$ implies b is surjective, and when the set is infinite injectivity is not equivalent to surjectivity. Thus there are subtle differences between *left invertibility* and *right invertibility*; however, as might be expected from the endomorphism model case, if an element $a \in A$ has both a left inverse and a right inverse, then both inverses agree, and a is a unit (since if $ba = ac = 1$, then $ab = abac = ac = 1$. We let $U(A)$ denote the set of all units in a ring. It forms a multiplicative group. Here are some examples:

- In the ring of integers, $U(\mathbf{Z}) = \{\pm 1\}$.

  Proof: For any $n, m \in \mathbf{Z}$, $|nm| \geqslant \max(n, m)$, so if $nm = 1$, $n = \pm 1$.

- In the ring of dyadic numbers, $U(\mathbf{Z}[1/2]) = \{\pm 2^n : n \in \mathbf{Z}\}$.

  Any element of $\mathbf{Z}[1/2]$ is of the form $n2^m$ for some integers n, m, where n is odd. But $n2^m \in U(\mathbf{Z}[1/2])$ if and only if $1/n \in \mathbf{Z}[1/2]$.

But if we can write $1/n = r2^k$ for integers r and k with r odd, then $nr = 2^k$, which is only possible if $n, r \in \{\pm 1\}$.

- In the ring of Gaussian integers, $U(\mathbf{Z}[i]) = \{\pm 1, \pm i\}$.

  Let $x \in U(\mathbf{Z}[i])$, and suppose $x = n + im$. Then there is $k, r \in \mathbf{Z}$ such that $(n + im)(k + ir) = 1$. But this implies that

  $$|(n + im)(k + ir)|^2 = (n^2 + m^2)(k^2 + r^2) = 1$$

  so $n^2 + m^2 = 1$, implying $n \in \{\pm 1, \pm i\}$.

  This technique can be generalized to calculate the group of units of $U(\mathbf{Z}[\sqrt{D}])$ for any squarefree integer D. We define $N : \mathbf{Z}[\sqrt{D}] \to \mathbf{Z}$ by setting

  $$N(n + m\sqrt{D}) = \left(n + m\sqrt{D}\right)\left(n - m\sqrt{D}\right) = n^2 - Dm^2$$

  A perhaps surprising fact, following from a short calculation, is that N is multiplicative, i.e. $N(xy) = N(x)N(y)$ for any $x, y \in \mathbf{Z}[\sqrt{D}]$. Thus if $x \in U(\mathbf{Z}[\sqrt{D}])$, then $1 = N(xx^{-1}) = N(x)N(x^{-1})$, so $N(x) \in \{\pm 1\}$. Conversely, if $N(x) = \pm 1$, then $x^{-1} = \pm(n - m\sqrt{D})$, so $x \in U(\mathbf{Z}[\sqrt{D}])$. Calculating the units of these rings thus reduces to counting integer solutions to the Diophantine equation $n^2 - Dm^2 = \pm 1$.

  The value $D = -1$ gives the Gaussian integers. For any integer $k > 1$, the equation $n^2 + km^2 = \pm 1$ only has the trivial solutions $n = \pm 1$, $m = 0$, so $U(\mathbf{Z}[\sqrt{D}]) = \{\pm 1\}$ for $D < -1$. For any $D \geqslant 2$, $U(\mathbf{Z}[\sqrt{D}])$ is infinite; for instance, $(1 + \sqrt{2})^n \in \mathbf{Z}[\sqrt{2}]$ for any $n \geqslant 0$.

- The group of units in $\mathbf{Z}_n$ is the set of equivalence classes of integers coprime to n. This is because if m is coprime to n, then Bezout's theorem implies that there exists integers $k, k' \in \mathbf{Z}$ such that $km + k'n = 1$, and then $km = 1$ in $\mathbf{Z}_n$. Thus $U(\mathbf{Z}_n)$ contains $\phi(n)$ elements.

- If A is a ring with no zero divisors, then the units of the polynomial ring $A[X_1, \ldots, \ldots, X_n]$ are precisely the units of A.

  One way to verify this is using the degree formula. If A has no zero divisors, then $\deg(fg) = \deg(f) + \deg(g)$ for any polynomials $f, g \in A[X_1, \ldots, X_n]$. Thus if $fg = 1$, then $\deg(f) = \deg(g) = 0$, which implies $f, g \in A$, and thus $f, g \in U(A)$. Similarily, we see that for two

power series f and g, the constant term of fg is the product of the constant terms of f and g. Thus if $fg = 1$, then the constant term of f is in $U(A)$. Conversely, if

$$f = \sum_{n=0}^{\infty} a_n X^n,$$

and a_0 is a unit, then we can define an inverse to f by defining

$$g = \sum_{n=0}^{\infty} b_n X^n,$$

by defining $b_0 = a_0^{-1}$, and then inductively defining

$$b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_{n-1} b_0).$$

Then $fg = 1$. But we can similarily define a left inverse to f, so f is invertible.

Similarily, if A is a ring without zero divisors, a power series in $A[[X]]$ is a unit in $A[[X]]$ if and only if the constant term in the power series is a unit in A. The trick is to recall the power series formula

$$\frac{1}{1-X} = \sum_{k=0}^{\infty} X^k$$

which *continues to hold* in any ring, in the sense that

$$(1-X)\sum_{k=0}^{\infty} X^k = 1.$$

Now if

$$f = a_0 + a_1 X + \ldots$$

Clearly if f is invertible, then a_0 is a unit. If a_0 is a unit of f, then we can certainly multiply by a_0^{-1}, so without loss of generality we only need to find the inverse of a power series f of the form

$$f = 1 + f_1 \cdot X.$$

for some power series f_1. But one can verify a suitable choice of inverse is

$$\sum_{k=0}^{\infty} (-1)^k f_1^k X^k.$$

- If A is a ring, then Cramer's rule tells us that the units of $M_n(A)$ are precisely those matrices whose determinant is a unit in A. In particular, if k is a field, then the group of units in $M_n(k)$ is the set of matrices with nonzero determinant, which gives the general linear group $GL_n(k)$.

- In number theory it is common to study the monoid ring $\mathbf{C}[[\mathbf{N}]]$ as the ring of functions $f : \mathbf{N} \to \mathbf{C}$, with addition given pointwise, and with multiplication given by *Dirichlet convolution*

$$(f * g)(k) = \sum_{nm=k)} f(n)g(m)$$

  Then $\mathbf{C}[[\mathbf{N}]]$ has a multiplicative identity

$$\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

  If μ is the *Möbius function*

$$\mu(n) = \begin{cases} 0 & p^2 \text{ divides } n \text{ for some prime } p \\ (-1)^k & n \text{ has } k \text{ distinct prime factors} \end{cases}$$

  and 1 is the constant function with $1(n) = 1$ for all $n \in \mathbf{N}$, then it is easy to see that $1 * \mu = \delta$. The Möbius inversion formula states that $f = g * \mu$ if and only if $f * 1 = g$, and this is nothing more than the fact than saying that μ is a unit in the convolution ring, with 1 as it's inverse.

- Let A be a commutative ring. The group of units in $M_n(A)$ is often denoted $GL_n(A)$. The determinant map $\det : M_n(A) \to A$ is a multiplicative map over the ring of matrices, i.e. for any $M, N \in M_n(A)$,

$$\det(MN) = \det(M)\det(N).$$

  In particular, if $M \in GL_n(A)$, we conclude $\det(M) \in U(A)$, since if $MN = I_n$, then $\det(M)\det(N) = 1$. Conversely, Cramer's rule shows that if $M \in M_n(A)$, then there is a matrix N such that $MN = \det(M) \cdot I_n$. In particular, we conclude that $GL_n(A)$ consists *precisely* of matrices with unit determinant.

As with groups, one may consider subrings of a ring, and homomorphisms between rings. After studying group theory, you should be able to figure out the definitions yourself, but for completeness, we now specify them. A *subring* of a ring is a subset of a ring which also possesses a ring structure. That is, a subring is closed under addition and multiplication.

Example. *The most classical chain of commutative subrings is*

$$\mathbf{Z} < \mathbf{Q} < \mathbf{R} < \mathbf{C} < \mathbf{H}$$

The other subrings in this chain are still actively researched today, most importantly in the theory of algebraic number theory.

Example. *If A is a ring, the center $Z(A)$ is defined to be the set of elements a such that, for all $b \in A$, $ab = ba$. Then $Z(A)$ is a commutative subring of A. Similarily, for any $a \in A$, the center $C(a) = \{x \in A : ax = xa\}$ is a subring of A, but is not necessarily commutative.*

A particularly important example of this occurs in the case when we study the group algebra $A[G]$. One can verify that if G is a finite group, and K is a conjugacy class of G containing elements $k_1, \ldots, k_n \in G$, then

$$k(K) = k_1 + \cdots + k_n$$

lies in $Z(A[G])$. This is because for any $a \in A$ and $g \in G$,

$$\begin{aligned} ag(k_1 + \cdots + k_n) &= a(gk_1g^{-1})g + \cdots + a(gk_ng^{-1})g \\ &= [(gk_1g^{-1}) + \cdots + (gk_ng^{-1})](ag) \\ &= (k_1 + \cdots + k_n)(ag), \end{aligned}$$

where the last equality follows because the map $k_i \mapsto gk_ig^{-1}$ permutes K. A similar calculation verifies that any elements of $Z(A[G])$ must be constant on any conjugacy class of G. Thus if $K_1, \ldots, K_N$ are the conjugacy classes of G, then

$$Z(A[G]) = \{a_1k(K_1) + \cdots + a_Nk(K_N) : a_1, \ldots, a_N \in Z(A)\}.$$

As verified in representation theory, if $A = \mathbf{C}$, then the irreducible characters of G form an orthonormal basis for $Z(A[G])$.

Example. *The continuous functions form a subring of $\mathbf{R}^{\mathbf{R}}$, as do the polynomial functions, or differentiable functions, and so on and so forth.*

Example. *We have a homomorphism from the semigroup ring $R[M]$ to R by setting*

$$\varphi\left(\sum a_i x_i\right) = \sum a_i.$$

The kernel of this map is called the augmented ideal *of the semigroup ring.*

Example. *For each complex number $z \in \mathbf{C}$, then, identifying $\mathbf{C}$ with $\mathbf{R}^2$, we obtain a linear transform $T_z : \mathbf{R}^2 \to \mathbf{R}^2$ given by multiplication on the left by z, which, if $z = u + iv$, we can identify with the 2×2 matrix*

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

The algebra of complex multiplication implies that the map $\varphi : \mathbf{C} \to M_2(\mathbf{R})$ obtained by setting

$$\varphi(u + iv) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$

is an injective homomorphism. Similarily, we can obtain an injective homomorphism $\varphi : \mathbf{H} \to M_4(\mathbf{R})$ by setting

$$\varphi(a + bi + cj + dk) = \begin{pmatrix} +a & -b & -c & -d \\ +b & +a & -d & +c \\ +c & +d & +a & -b \\ +d & -c & +b & +a \end{pmatrix}.$$

Thus we can view $\mathbf{C}$ and $\mathbf{H}$ as certain subrings of matrices.

The family $SO(2) \in M_2(\mathbf{R})$ can be identified precisely with the set of unitary *complex numbers, i.e. those $z \in \mathbf{C}$ with $|z| = 1$. In some sense, the complex numbers as we know them today were invented precisely to provide a coordinate system for rotations and dilations in space in this manner. The quaternions were invented to analyze rotations and dilations in three dimensional space, but things do not work out quite as nicely in this setting. We note that $\varphi(z) \in SO(4)$ for each unitary $z \in \mathbf{H}$ with $|z| = 1$. If we set G to be the group of all unit quaternions, then we obtain a group homomorphism $\psi : G \to SO(4)$ by setting $(\psi z)(w) = zwz^{-1}$, where $w \in \mathbf{H}$ is identified canonically with a unit vector in $\mathbf{R}^4$. Note that $(\psi z)(1) = 1$, so $\mathbf{R}$ is an invariant subspace of ψz. But since $\psi z \in SO(4)$, if we define $\mathbf{H}_0 = \{ai + bj + ck : a, b, c \in \mathbf{R}\}$ to be the set of* pure quaternions, *then ψz acts as an orthogonal transformation on $\mathbf{H}_0$. Thus we obtain an induced homomorphism from G to $SO(3)$. Geometrically,*

ψz can be described in the following way: Any element z of G can be written as $\cos\theta + \sin\theta z_0$, where z_0 is a pure unit quaternion. The rotation ψz is then a counterclockwise rotation of 2θ about z_0 in $\mathbf{H}_0$. Certainly z_0 is fixed by z, because

$$zz_0z^{-1} = (\cos\theta + \sin\theta z_0)z_0(\cos\theta - \sin\theta z_0) = z_0.$$

Thus z does rotate about z_0 at a certain angle. Let us see that this angle is 2θ is the angle in the special case where $z = \cos\theta + \sin\theta i$. Then

$$\begin{aligned} zjz^{-1} &= (\cos\theta + \sin\theta \cdot i)j(\cos\theta - \sin\theta \cdot i) \\ &= (\cos\theta \cdot j + \sin\theta \cdot k)(\cos\theta - \sin\theta \cdot i) \\ &= ((\cos^2\theta - \sin^2\theta) \cdot j + 2\sin(\theta)\cos(\theta) \cdot k) \\ &= (\cos(2\theta)j + \sin(2\theta)k). \end{aligned}$$

Thus we see the angle 2θ is correct here. But this claim is now true in general, because we can write any pure unit quaternion z_0 as $w_0iw_0^{-1}$ for some other unit quaternion w_0; the unit vector $w_0jw_0^{-1}$ is then orthogonal to z_0, and one can calculate that

$$z(w_0^{-1}jw_0)z^{-1} = (\cos(2\theta)(w_0^{-1}jw_0) + \sin(2\theta)(w_0^{-1}kw_0)).$$

This geometric description makes it easy to see that $\psi : G \to SO(3)$ is a double cover*, with kernel equal to -1. More generally, ψ extends to a map from the nonzero elements of $\mathbf{H}$ to the group of oriented rotations and dilations.*

1.2 Homomorphisms and Ideals

A *ring homomorphism* from a ring A to a ring B is a function $\phi : A \to B$ which is a homomorphism of abelian groups, and a homomorphism of the multiplicative monoid structure of the two spaces. In particular, this means for each $a_1, a_2 \in A$,

$$\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2),$$

$$\phi(a_1a_2) = \phi(a_1)\phi(a_2),$$

and

$$\phi(1) = 1.$$

As with groups, the kernel $\mathrm{Ker}(\phi)$ of the map ϕ is defined to be the set of all a such that $\phi(a) = 0$. As we found with groups, determining the algebraic structures of a kernel of a ring homomorphism will enable us to obtain a variant of the isomorphism theorems for rings, which we carry out in the next section.

We wish to establish a quotient structure on rings, and obtain analogies of the isomorphism theorems for groups. Let's consider $\mathfrak{a}$ as a subset of a ring A, and try to determine which properties allow the cosets $A/\mathfrak{a}$ of the form $x+\mathfrak{a}$ allow the operations on A to be well defined on the quotient. In order to even define these cosets, we first need $\mathfrak{a}$ to be an additive subgroup of the additive group structure on $\mathfrak{a}$. Since all subgroups of abelian groups are normal, this means the operation of addition on the quotient is well defined. In order for multiplication to be well defined, we need to conclude $(a+\mathfrak{a})(b+\mathfrak{a}) = (ab+\mathfrak{a})$. In terms of sets, this says

$$\{(a+x)(b+y) = ab + xb + ay + xy : x, y \in \mathfrak{a}\} = \{ab + x : x \in \mathfrak{a}\}$$

Thus we require $xb + ay + xy \in \mathfrak{a}$ for any $x, y \in \mathfrak{a}$. This implies that $\mathfrak{a}$ not *only* needs to be closed under multiplication, but also closed under multiplication by an element of A, both on the left and the right. We say $\mathfrak{a}$ is an *ideal* if it is an additive subgroup of R closed under multiplication on the left and the right. In a commutative ring, of course, we need only prove that an ideal is closed by multiplication on the left.

Example. *As should be expected, if $\phi : A \to B$ is a ring homomorphism, then the kernel $\mathrm{Ker}(\phi)$ is a double sided ideal of A. Conversely, if $\mathfrak{a}$ is a two-sided ideal, then $A/\mathfrak{a}$ is a ring, and the projection $\pi : A \to A/\mathfrak{a}$ is a homomorphism with kernel $\mathfrak{a}$. A ring homomorphism is an isomorphism if and only if the kernel is trivial.*

Example. *Every additive subgroup of $\mathbf{Z}$ is a set of multiples of some number n, which we denote by (n). It is also an ideal of $\mathbf{Z}$, and so an ideal in $\mathbf{Z}$ is in one correspondence with the set of integers. In general, if A is a commutative ring, and $x \in A$, we find $(x) = Ax = \{ax : a \in A\}$ is an ideal, known as a* principal ideal. *If A is a ring such that every ideal is principal, we say that A is a* principal ideal ring. *The integers are an example, as we have just argued. We can also consider the ideal $(x_1, \ldots, x_n)$, the smallest ideal containing the elements $x_1, \ldots, x_n$. An ideal of this form is known as a* finitely generated ideal.

Example. *Let A be a division ring, and suppose $\mathfrak{a}$ is an ideal containing some $x \neq 0$. Then for any $a \in \mathfrak{a}$, $aax^{-1}x \in \mathfrak{a}$. Thus $\mathfrak{a} = A$. Thus the only ideals of A are (0) and (1). In particular, we conclude that the only homomorphisms from A to any other ring B are injective, or identically zero.*

Example. *Suppose A is a ring. Then the only ideals of $M_n(A)$ are those of the form $M_n(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in A. Indeed, if I is an ideal in $M_n(A)$, then for each i, j,*

$$\mathfrak{a}_{ij} = \{M_{ij} : M \in I\}$$

forms an ideal in A. It is certainly an additive subgroup. Moreover, if $a \in A$, and $a_0 \in \mathfrak{a}_{ij}$, we can find $M \in I$ with $M_{ij} = a_0$. If I_n is the identity matrix in $M_n(A)$, then $[(aI_n)M]_{ij} = aa_0$ and $[M(aI_n)]_{ij} = a_0a$, so $aa_0, a_0a \in \mathfrak{a}_{ij}$. Multiplication by permutation matrices verifies that $\mathfrak{a}_{ij}$ does not really depend on i and j, so we denote it my $\mathfrak{a}$. And since

$$e_{ii}Me_{jj} = M_{ij}e_{ij},$$

it is clear that $I = M_n(\mathfrak{a})$. Note, in particular, that if A is a division ring, then $M_n(A)$ has no nontrivial ideals. *Thus any homomorphism with domain $M_n(A)$ is either identically zero, or injective.*

Example. *If k is a field and $M \in M_n(k)$, then there exists two invertible square matrices N and K such that*

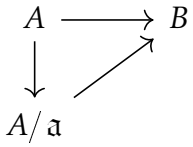
$$NMK = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

Then for any for $j \leqslant k$, $e_{ij}NMK = e_{ij}$. Similarily, if $i \leqslant k$, $NMKe_{ij} = e_{ij}$. In particular, we conclude that the two sided ideal generated by M over $M_n(k)$ contains all e_{ij}, so in particular $M_n(k)$ has only two ideals, (0) and (1). In particular, this means that every homomorphism from $M_n(k)$ to another ring A must be injective, or is identically zero. Nonzero rings with this property are called simple.

The phenomenon in the last example occurs rarely in the family of commutative rings. This is because A is commutative, nonzero, and simple if and only if it is a field. If A has these properties and $x \in A$ is nonzero, then $(x) = A$, so there is $y \in A$ such that $xy = 1$. Thus x is invertible.

Now we have defined quotient rings and homomorphisms, we obtain the isomorphism theorems for rings, which are direct analogues to the isomorphism theorems for groups.

Theorem 1.2 (First Isomorphism Theorem)**.** *Let $\phi : A \to B$ be a homomorphism of rings. If $\mathfrak{a}$ is an ideal contained in the kernel of ϕ, then there is a unique morphism from $A/\mathfrak{a} \to B$ satisfying the commutative diagram*



If $\mathfrak{a}$ is equal to the kernel of ϕ, then the morphism from $A/\mathfrak{a}$ to B is injective.

Theorem 1.3 (Second Isomorphism Theorem)**.** *Let B be a subring of A, and $\mathfrak{a}$ an ideal of A. Then $B + \mathfrak{a}$ is a subring of A, $\mathfrak{a}$ is an ideal in $B + \mathfrak{a}$, $B \cap \mathfrak{a}$ is an ideal in B, and*

$$B/(B \cap \mathfrak{a}) \cong (B + \mathfrak{a})/\mathfrak{a}.$$

Theorem 1.4 (Third Isomorphism Theorem)**.** *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A, with $\mathfrak{a} \subset \mathfrak{b}$. Then $\mathfrak{b}/\mathfrak{a}$ is an ideal of $A/\mathfrak{a}$, and*

$$(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \cong A/\mathfrak{b}.$$

Theorem 1.5 (Fourth Isomorphism Theorem)**.** *If $\phi : A \to B$ is a surjective homomorphism, there is a bijection correspondence with subrings of B and subrings of A containing ϕ, and a subring of B is an ideal if and only if the corresponding subring of A is an ideal.*

Example. *If A is a ring, we can consider the unique homomorphism from $\mathbf{Z}$ to A, whose kernel is of the form (n) for some positive integer n. We call n the* characteristic *of the ring A. The ring A therefore contains a subring isomorphic to $\mathbf{Z}_n$ for some integer n, known as the* prime subring *of A. The reason for the terminology is that, if A has no zero divisors, then n must necessarily be prime.*

Example. *The first isomorphism theorem implies the ring $\mathbf{Z}[i]$ of Gaussian integers is isomorphic to $\mathbf{Z}[X]/(X^2 + 1)$, where the isomorphism is induced by the $\mathbf{Z}$-algebra homomorphism $\phi : \mathbf{Z}[X] \to \mathbf{Z}[i]$ satisfying $\phi(X) = i$. Two applications of the third isomorphism theorem implies that*

$$\begin{aligned}\mathbf{Z}[i]/(i-2) &\cong \mathbf{Z}[X]/(X^2+1, X-2) \\ &= \mathbf{Z}[X]/(5, X-2) \cong \mathbf{Z}_5[X]/(X-2) \cong \mathbf{Z}_5.\end{aligned}$$

Given a ring A, we let $\mathcal{I}(A)$ denote the set of all ideals of A. A morphism $\phi : A \to B$ induces a map $\phi^{-1} : \mathcal{I}(B) \to \mathcal{I}(A)$. We can also consider a map $\phi_* : \mathcal{I}(A) \to \mathcal{I}(B)$ which associates with each ideal $\mathfrak{a} \in \mathcal{I}(A)$ the smallest ideal in $\mathcal{I}(B)$ containing $\phi(\mathfrak{a})$. If $\mathfrak{k}$ is the kernel of ϕ, we calculate that

$$\phi^{-1}(\phi_*(\mathfrak{a})) = \mathfrak{a} + \mathfrak{k} \quad \text{and} \quad \phi_*(\phi^{-1}(\mathfrak{b})) = \mathfrak{b}$$

for all ideals $\mathfrak{a} \in \mathcal{I}(\mathfrak{a})$ and $\mathfrak{b} \in \mathcal{I}(B)$. In particular, ϕ^{-1} is always an injective map. Studying the Galois connection between ϕ_* and ϕ^{-1} is useful to understand almost any homomorphism ϕ, in particular, determining which ideals in $\mathcal{I}(A)$ occur as images of ϕ^{-1}.

1.3 Properties of Ideals

Let A be a ring. For any family of ideals $S \subset \mathcal{I}(A)$, $\bigcap S \in \mathcal{I}(A)$. A consequence of this is we can talk about a generating set of an ideal. We say a set S *generates* $\mathfrak{a}$ if $\mathfrak{a}$ is the smallest ideal containing S, and we denote $\mathfrak{a}$ by (S). Using this fact, we can define algebraic operations on ideals. Given $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(A)$, the set $\mathfrak{a} + \mathfrak{b}$ is an ideal, and is the smallest ideal containing $\mathfrak{a}$ and $\mathfrak{b}$. More generally, we can take infinite sums of ideals, often using the notation $\bigoplus \mathfrak{a}_\alpha$, which is just the smallest ideal containing all the ideals in the sum. Together with intersection, we find that the family of ideals forms a complete lattice on the subsets of a ring, just like the family of subgroups of a group. More interestingly, we have a product structure. Given two ideals $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(A)$, we can consider the product $\mathfrak{a}\mathfrak{b}$, which is the ideal *generated* by products of the form ab, with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ (CAUTION: the set $\{ab : a \in \mathfrak{a}, b \in \mathfrak{b}\}$ may not be an ideal).

With the operations of addition and multiplication, one might expect $\mathcal{I}(A)$ to have a ring structure, but this isn't true; nonetheless, $\mathcal{I}(A)$ does form a monoid under addition and multiplication; the multiplicative identity is (1), and the additive identity is (0). They are some useful algebraic relations here; for instance, we have the distributive laws

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c} \quad \text{and} \quad (\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c}.$$

If we place products with intersections, then we obtain the partial distributive law

$$\mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c} \subset \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}).$$

However, in the special case that $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{a} \subset \mathfrak{c}$, we do have equality here, a fact known as the *modular law*. Using unique factorization, we see that

$$(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) = \mathfrak{a}\,\mathfrak{b}$$

in principal ideal domains, but we only have

$$(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) \subset \mathfrak{a}\,\mathfrak{b}$$

in general rings. One trivially verifies that $\mathfrak{a}\,\mathfrak{b} \subset \mathfrak{a}\cap\mathfrak{b}$ if $\mathfrak{a},\mathfrak{b} \in \mathcal{I}(A)$. In a commutative ring, one verifies we have equality here provided $\mathfrak{a}$ and $\mathfrak{b}$ are '*relatively prime*'.

Lemma 1.6. *If A is a commutative ring, and $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ are ideals of A which are* pairwise relatively prime, *in then sense that $\mathfrak{a}_i+\mathfrak{a}_j = A$ for $i \neq j$, then*

$$\mathfrak{a}_1\cdots\mathfrak{a}_n = \mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_n.$$

Proof. Let us consider first the case $n = 2$. It suffices to prove $\mathfrak{a}_1\cap\mathfrak{a}_2 \subset \mathfrak{a}_1\,\mathfrak{a}_2$. But we may find $a_1 \in \mathfrak{a}_1$ and $a_2 \in \mathfrak{a}_2$ such that $a_1+a_2 = 1$ by assumption. Given $c \in \mathfrak{a}_1\cap\mathfrak{a}_2$, we can then write

$$c = c(a+b) = ca+cb \in \mathfrak{a}_1\,\mathfrak{a}_2.$$

This completes the proof in this case. In general, we apply induction to conclude that

$$\mathfrak{a}_1\ldots\mathfrak{a}_n = (\mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_{n-1})\,\mathfrak{a}_n.$$

To complete the proof, it suffices to verify that $\mathfrak{a}_1\ldots\mathfrak{a}_{n-1}$ and $\mathfrak{a}_n$ are relatively prime. If for each $i \in \{1,\ldots,n-1\}$ we find $x_i \in \mathfrak{a}_i$ and $y_i \in \mathfrak{a}_n$ such that $x_i+y_i = 1$, then

$$1 = (x_1+y_1)\ldots(x_{n-1}+y_{n-1}) \in x_1\ldots x_{n-1}+\mathfrak{a}_n \subset \mathfrak{a}_1\ldots\mathfrak{a}_{n-1}+\mathfrak{a}_n,$$

which verifies what was needed. □

1.4 Direct and Inverse Limits

A key step to constructing more complicated rings is to consider limits of certain families of rings. For instance, this is the key to constructing the ring of germs of functions around a point, or the ring of p-adic integers.

Let I be a directed index set, and consider a family of rings $\{A_i : i \in I\}$, and for each $i \leqslant j$ in I, a homomorphism $f_{ij} : A_i \to A_j$ such that $f_{jk} \circ f_{ij} = f_{ik}$. We can define a universal object from this construction, the *direct limit* $\lim_{i\to\infty} A_i$, together with morphisms $f_i : A_i \to \lim_{i\to\infty} A_i$, which has the universal property that any family of morphisms $\phi_i : A_i \to B$ such that $\phi_j \circ f_{ij} = \phi_i$ for each $i \leqslant j$, then there exists $\phi : \lim_{i\to\infty} A_i \to B$ such that $\phi_i = \phi \circ f_i$ for each i.

One can clearly consider the direct limit of the rings $\{A_i\}$ as Abelian groups, but this direct limit also has a ring structure, which can be defined as follows; given $x_1, x_2 \in \lim_i A_i$, there exists an index i and $a_1, a_2 \in A_i$ such that $f_i(a_1) = x_1$ and $f_i(a_2) = x_2$. We define $x_1 x_2 = f_i(a_1 a_2)$. This is independant of the choice of a_1 and a_2, since if $a_1', a_2' \in A_i$ are selected with $f_i(a_1) = f_i(a_1')$ and $f_i(a_2) = f_i(a_2')$, then there exists an index $k \geqslant i$ such that $f_{ik}(a_1) = f_{ik}(a_1')$, $f_{ik}(a_2) = f_{ik}(a_2')$, and then

$$\begin{aligned} f_i(a_1 a_2) &= f_k(f_{ik}(a_1 a_2)) \\ &= f_k(f_{ik}(a_1) f_{ik}(a_2)) \\ &= f_k(f_{ik}(a_1') f_{ik}(a_2')) \\ &= f_k(f_{ik}(a_1' a_2')) \\ &= f_i(a_1' a_2'). \end{aligned}$$

Moreover, the compatibility condition $f_j \circ f_{ij} = f_i$ ensure that this definition is independent of the index i selected. Thus this operation gives $\lim_i A_i$ a well defined ring structure such that each f_i is a ring homomorphism. If we consider a family of ring homomorphisms $\phi_i : A_i \to B$ compatible with the family of maps $\{f_{ij}\}$, then the induced group homomorphism $\phi : \lim_i A_i \to B$ is also a ring homomorphism, since given $x_1, x_2 \in \lim_i A_i$, there is an index i and $a_1, a_2 \in A_i$ such that $\phi_i(a_1) = x_1$, $\phi_i(a_2) = x_2$, so $\phi_i(a_1 a_2) = x_1 x_2$, and

$$\phi(x_1 x_2) = \phi_i(a_1 a_2) = \phi_i(a_1)\phi_i(a_2) = \phi(x_1)\phi(x_2).$$

Thus the direct limit of Abelian groups can be equipped with the structure of a ring in a way as to make this also the direct limit of rings.

Example. *Consider a topological space X, and fix a point $x \in X$. For each open set U contained in x, let $C(U)$ consist of the ring of scalar valued continuous functions on U. Given $V \subset U$, we have a map $R_{UV} : U \to V$, such that $R_{UV} f$*

is the restriction of f to V. Then the family $\{R_{UV}\}$ are ring homomorphisms. Thus we can consider the direct limit of this family, often denoted $\mathcal{O}_x$, which is the ring of germs *of functions defined in a neighbourhood of x. The ring $\mathcal{O}_x$ is* local*, in the sense that it has a unique maximal ideal $\mathfrak{m}_x$, which consists of the germs of continuous function f defined in a neighbourhood of x with $f(x) = 0$. This is because every element of $\mathcal{O}_x - \mathfrak{m}_x$ is invertible in $\mathcal{O}_x$, since if f is a continuous function defined in a neighbourhood of x with $f(x) \neq 0$, then f has no zeroes in a neighbourhood of x, so $g(x) = 1/f(x)$ is well defined in this negihbourhood and fg equals one in a neighbourhood of the origin.*

Dual to the construction of direct limits is the construction of inverse limits. Given a family of rings $\{A_i : i \in I\}$ with maps $f_{ji} : A_j \to A_i$ for each $i \leqslant j$, we can construct an inverse limit $\lim_i A_i$ together with morphisms $f_i : \lim_i A_i \to A_i$, which has the universal property that for each families of morphisms $\phi_i : B \to A_i$ such that $f_{ji} \circ \phi_j = \phi_i$ for each $i \leqslant j$, there is a unique morphism $\phi : B \to \lim_i A_i$ such that $\phi_i \circ f_i = f$. Inverse limits exist in the category of abelian groups, and just as with direct limits it is easy to equip the inverse limit with an additional multiplication structure since all the maps involved here are ring homomorphisms.

Example. *For each prime p, let $A_i = \mathbf{Z}/p^i\mathbf{Z}$. For $i \leqslant j$, $(p^j) \subset (p^i)$, so we can consider a morphism $\mathbf{Z}/p^j\mathbf{Z} \to \mathbf{Z}/p^i\mathbf{Z}$. These morphisms are compatible, so they give rise to an inverse limit $\lim_i \mathbf{Z}_{p^i}$, which we denote by $\mathbf{Z}_p$ and call the ring of* p-adic integers.

Chapter 2

Divisibility in Commutative Rings

In this chapter, all rings are commutative unless stated overwise. Our goal is to study the *divisibility theory* of a ring. Given a ring A and $a_1, a_2 \in A$, we say a_1 *divides* a_2, written $a_1 \mid a_2$ if there is $c \in A$ such that $a_2 = ca_1$. Our goal is to build a theory of divisibility in commutative rings. This is clearly related to the theory of ideals in a ring, since the set of factors of an element x in a ring is precisely the ideal (x), and a factorization $x = x_1 \dots x_n$ with $x_1, \dots, x_n$ is equivalent to the condition that $(x) = (x_1) \cdots (x_n)$.

There are two useful assumptions on the rings we study which can make our lives easier; a minor assumption is that one is working with an *integral domains*, i.e. a nonzero ring without zero divisors. Such rings are also called *entire*. A much more powerful assumption is that all ideals are principal; a principal ring which is also an interal domain is called a *principal ideal domain*, and then the divisor theory is incredibly closely related to the ideal theory of the principal ideal domain. Let us consider an example.

Example. *If A is a ring, then for a family of elements $a_1, \dots, a_n \in A$, we say $a \in A$ is a* greatest common divisor *if $a \mid a_1, \dots, a_n$, and if $x \in A$ is any element which divides each of $a_1, \dots, a_n$, then $x \mid a$. A* least common multiple *of $a_1, \dots, a_n \in A$ is $a \in A$ such that $a_1, \dots, a_n \mid a$, and if $x \in A$ is any element divisible by all of $a_1, \dots, a_n$, then $a \mid x$. Greatest common divisors and least common divisors exist in any principal ideal domain; given $a_1, \dots, a_n \in A$, if $(a) = (a_1, \dots, a_n)$, then a is the greatest common divisor of $a_1, \dots, a_n$, and if*

$(a) = (a_1) \cap \cdots \cap (a_n)$, *then* a *is the least common multiple of* $a_1, \ldots, a_n$.

An ideal $\mathfrak{p}$ of a ring A is *prime* if $\mathfrak{p} \neq A$, and if $a, b \in A$ and $ab \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. This mimics the definition of prime integers as those integers p such that if p divides the product of two integers nm, then p divides either n or m. It is clear from the definition that $\mathfrak{p}$ is a prime ideal if and only if $A/\mathfrak{p}$ is an integral domain.

Theorem 2.1. *If* $f : A \to B$ *is a non-zero homomorphism, and* $\mathfrak{p}$ *is a prime ideal of* B*, then* $f^{-1}(\mathfrak{p})$ *is a prime ideal of* A*. In particular, if* A *and* B *are rings and* f *is a homomorphism, then the inverse image of any prime ideal is prime.*

Proof. Suppose $a, b \in A$, and $ab \in f^{-1}(\mathfrak{p})$. Then $f(ab) = f(a)f(b) \in \mathfrak{p}$, which implies either $f(a) \in \mathfrak{p}$, so $a \in f^{-1}(\mathfrak{p})$, or $f(b) \in IP$, so $b \in f^{-1}(\mathfrak{p})$. Since $1 \notin f^{-1}(\mathfrak{p})$, this implies $f^{-1}(\mathfrak{p})$ is a proper ideal of A, and is therefore prime. □

Remark. This theorem implies that if A is a subring of a ring B, and $\mathfrak{p}$ is a prime ideal of B, then $A \cap \mathfrak{p}$ is a prime ideal of A.

2.1 Fields of Fractions

Given an integral domain A, it is often useful to embed A in a field K. We consider doing this is the simplest way possible. Given a ring A, we consider a field K formed from a quotient of $A \times A - \{0\}$, where $(a_1, a_2) \sim (a_3, a_4)$ if $a_1 a_4 = a_2 a_3$. The equivalcen class of (a_1, a_2) will be denoted by a_1/a_2. One can verify that the operations $a_1/a_2 + a_3/a_4 = (a_1 a_4 + a_2 a_3)/a_2 a_4$ and $(a_1/a_2)(a_3/a_4) = (a_1 a_3/a_2 a_4)$ are well defined, and give K the structure of a field, known as the *field of fractions* of A. For any $a \neq 0$, K has a multiplicative identity a/a. There is a natural embedding of $i : A \to K$ given by $i(a) = a/1$.

Example. *The field of fractions of* $\mathbf{Z}$ *is* $\mathbf{Q}$*. Similarily, the field of fractions of* $2\mathbf{Z}$ *is also* $\mathbf{Q}$*.*

Example. *If* A *is an integral domain, then* $A[x_1, \ldots, x_n]$ *is also an integral domain, then the field of fractions of* $A[x_1, \ldots, x_n]$ *is the field of* rational functions *in one variable with coefficients in* A*, often denoted* $A(x_1, \ldots, x_n)$*. If the field of fractions of* A *is* K*, then* $A(x_1, \ldots, x_n)$ *is equal to* $K(x_1, \ldots, x_n)$*.*

Example. *Let $A(D)$ denote the complex algebra of functions holomorphic in some connected open region D of $\mathbf{C}$. Then $A(D)$ is an integral domain, for if $fg = 0$, where $f, g \neq 0$, then $f^{-1}(0)$ and $g^{-1}(0)$ are two discrete sets whose union is D, which is impossible. We may therefore form the field of fractions of $A(D)$, which is precisely the set of meromorphic functions on D. These functions f/g are defined except for certain points upon which $g(z) = 0$, except in the case that z is a removable singularity of g, which means that we can write $f/g = f_1/g_1$, where $g_1(z) \neq 0$.*

Example. *Given a field k, consider the field of fractions of the ring $k[[X]]$ of formal power series in k. Then the field of fractions of $k[[X]]$ is isomorphic to the field $k((X))$ of* Laurent series *in k, i.e. the ring of formal power series of the form*

$$\sum_{n=-\infty}^{\infty} a_n X^n.$$

where $a_n = 0$ for sufficiently small n. To see this, we note that $k[[X]]$ embeds naturally in $k((X))$. Moreover, $k((X))$ is a field, because any element of $k[[X]]$ with nonzero constant term is invertible, and any nonzero element of $k((X))$ may be written as $X^n f$ for some $n \in \mathbf{Z}$ and some $f \in k[[X]]$ with nonzero constant term. Thus if K is the field of fractions of $k[[X]]$, then there is a homomorphism $F : K \to k((X))$. As a nonzero homomorphism between fields, f is injective. But F is also surjective, hence an isomorphism.

Theorem 2.2. *For any ring homomorphism $\phi : A \to B$ such that $\phi(a)$ is invertible for each $a \neq 0$, there is a unique homomorphism $\phi' : K \to B$ such that $\phi' \circ i = \phi$.*

Proof. Given $\phi : A \to B$, we are forced to define $\phi'(a_1/a_2) = \phi(a_1)/\phi(a_2)$. It is simple to verify this map is a ring homomorhpism. □

Thus every integral domain can be considered as a subring of a field. Given any subset S of $A - \{0\}$, we let $S^{-1}A$ denote the subring of K generated by elements of the form a/s, with $a \in A$ and $s \in S$. If S' is the multiplicative monoid generated by S, then every element of $S^{-1}A$ can be written as a/s with $a \in A$ and $s \in S'$.

2.2 Maximal Ideals

An ideal $\mathfrak{m}$ is *maximal* if $\mathfrak{m} \neq A$, and there is no ideal strictly containing $\mathfrak{m}$ except the entire ring. and there is no ideal containing it but the entire ring, and the ideal itself is not the entire ring. Using Zorn's lemma in the classical manner, one may verify that any proper ideal of a ring is contained in some maximal ideal. The most useful fact about maximal ideals to use in basic proofs is to use the fact that if $a \notin \mathfrak{a}$, then $(a) + \mathfrak{m} = A$. Thus (0) is a prime ideal if and only if A is entire to begin with.

Example. *The maximal ideals of* $\mathbf{Z}$ *are* $p\mathbf{Z}$*, where* p *is a prime number.*

Theorem 2.3. *If* A *is a commutative ring, then every maximal ideal is prime.*

Proof. Suppose $\mathfrak{m}$ is maximal, and let $ab \in \mathfrak{m}$. If $a \notin \mathfrak{m}$, then $(a) + \mathfrak{m} = A$, and so we can write $xa + m = 1$ for some $x \in A$, $m \in \mathfrak{m}$. But this implies that $b = 1 \cdot b = xab + mb \in \mathfrak{m}$. □

One of the tenants of commutative ring theory is that one can obtain powerful control over a ring by understanding it's prime ideals. One heuristic is that most 'maximal' ideals, which respect to certain properties, are prime. Consider the following example.

Theorem 2.4. *If all prime ideals of a ring* A *are principal, then* A *is principal.*

Proof. Let $\mathcal{I}$ be the set of ideals in A which are not principal. If we have an infinite chain of ideals $\{\mathfrak{a}_\alpha\}$ contained in $\mathcal{I}$, then $\bigcup \mathfrak{a}_\alpha$ is an ideal; if this ideal was principle it would be generated by some $a \in \mathfrak{a}_{\alpha_0}$ for some index α_0. But this would imply that the chain eventually terminated. Thus Zorn's lemma implies that there is a maximal ideal $\mathfrak{a}$ of $\mathcal{I}$. Since $\mathfrak{a}$ is not principal, it cannot be prime, so we can find $x, y \notin \mathfrak{a}$ such that $xy \in \mathfrak{a}$. Let $\mathfrak{a}_x = (x) + \mathfrak{a}$, $\mathfrak{a}_y = (y) + \mathfrak{a}$, and let $\mathfrak{b}$ be the set of all $a \in A$ such that $(a)\,\mathfrak{a}_x \subset \mathfrak{a}$. Then $\mathfrak{a}_y \subset \mathfrak{b}$, so $\mathfrak{a}_x$, $\mathfrak{a}_y$, and $\mathfrak{b}$ are all principal ideals, generated by some elements x_0, y_0, and z_0 in A respectively. Thus z_0 divides y_0, and $x_0 z_0 \in \mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{a}_x$, if $a \in \mathfrak{a}$, we can write $a = tx_0$ for some $t \in A$. But then $t\,\mathfrak{a}_x \subset \mathfrak{a}$, so $t \in \mathfrak{b}$. Thus $a = sx_0 z_0$ for some $s \in A$. Thus we conclude that $\mathfrak{a} = (x_0 z_0)$, which gives a contradiction. Thus $\mathcal{I}$ must be empty, which implies all ideals in A are principal. □

Recall that the nilradical of a commutative ring A is the ideal $\sqrt{A}$ of all *nilpotent* x, i.e. those elements with $x^n = 0$. The *Jacobson radical* $J(R)$ of a

(not necessarily commutative) ring R is the intersection of all prime ideals in the ring. In the commutative case, we find $J(R) = \sqrt{R}$.

Theorem 2.5. *If A is a commutative ring, $J(A) = \sqrt{A}$.*

Proof. If $\mathfrak{a}$ is a prime ideal, and $x^n = 0$, then $x^n \in \mathfrak{a}$, hence $x \in \mathfrak{a}$, showing $\sqrt{A} \subset J(A)$. Conversely, suppose $x \notin \sqrt{A}$. Consider the set S of all powers x^n. Let L be the set of all (not necessarily prime) ideals in A disjoint from S. Then L is nonempty, since (0) is in L, and L is inductively ordered, so we can consider some maximal element $\mathfrak{a}^*$. Given $a, b \notin \mathfrak{a}^*$, $\mathfrak{a}^* + (a)$ and $\mathfrak{a}^* + (b)$ are both strictly larger than $\mathfrak{a}^*$, and so there is x_1, y_1 and x_2, y_2 such that $x_1 + ay_1 = x^n$ and $x_2 + by_2 = x^m$. But then

$$x^{m+n} \in (\mathfrak{a}^* + (a))(\mathfrak{a}^* + (b)) = \mathfrak{a}^* + (a)\,\mathfrak{a}^* + (b)\,\mathfrak{a}^* + (ab)$$

And therefore $ab \notin \mathfrak{a}^*$, so $\mathfrak{a}^*$ is prime, not containing x, and so $J(A)$ does not contain x. □

This result comes up more often in more advanced contexts of commutative algebra, as do many more incidences where ideals maximal with respect to some property are prime. For instance, this is exploited in the standard proof that for any ideal $\mathfrak{a}$ in a Noetherian ring A, there are finitely many prime ideals minimal with respect to including $\mathfrak{a}$.

Remark. The statement that maximal ideals is prime is not true for non unital rings. For instance, one maximal ideal in the non unital ring $2\mathbf{Z}$ is $4\mathbf{Z}$, yet $2\mathbf{Z}/4\mathbf{Z}$ contains zero divisors, so $4\mathbf{Z}$ is not prime.

Example. *Every prime ideal in a Boolean ring is maximal. Let A be a Boolean ring, and $\mathfrak{a}$ a prime ideal. Then $A/\mathfrak{a}$ is a Boolean integral domain. Since $a^2 - a = 0$ for all $a \in A$, this implies that for each $a \in A$, either $a \in \mathfrak{a}$, or $a - 1 \in \mathfrak{a}$. Since $\mathfrak{a} \neq A$, $A/\mathfrak{a}$ is a nontrivial integral domain with two elements, and thus a field, so $\mathfrak{a}$ is maximal.*

Theorem 2.6. *If A is commutative, then an ideal $\mathfrak{m}$ is maximal iff $A/\mathfrak{m}$ is a field.*

Proof. Suppose $\mathfrak{m}$ is maximal, and $a \notin \mathfrak{m}$. Then $(a) + \mathfrak{m} = A$, and so we can write $xa + m = 1$, which implies that $xa \cong 1$ modulo $\mathfrak{m}$. This verifies that all nonzero residues in the quotient ring have inverses. On the other hand, the third isomorphism theorem says there is a one to one correspondence

between ideals in $A/\mathfrak{m}$ and ideals in A containing $\mathfrak{m}$. If $A/\mathfrak{m}$ is a field, then the only ideals are (0) and (1), implying that the only ideals containing $\mathfrak{m}$ are $\mathfrak{m}$ and A. Thus $\mathfrak{m}$ is maximal if $A/\mathfrak{m}$ is a field. □

Over non-commutative rings, this need not be the case, for instance, the only maximal ideal of the ring of $n \times n$ matrices $M_n(\mathbf{C})$ is (0), but $M_n(\mathbf{C})$ is not a field, nor even a division algebra.

Example. *For any prime integer p, the ideal (X, p) is a maximal ideal of $\mathbf{Z}[X]$, since*

$$\mathbf{Z}[X]/(X, p) \cong \mathbf{Z}_p,$$

and $\mathbf{Z}_p$ is a field.

As one may have noticed, over a principal ideal domain, *all* prime ideals are maximal. This is a general phenomenon.

Theorem 2.7. *Every prime ideal is maximal in a principal ideal domain.*

Proof. Let $p \in A$ and suppose (p) is a prime ideal of A. Then (p) is contained in some maximal ideal (m), and thus m divides p, so we can write $p = xm$ for some $x \in A$. But then either p divides x or p divides m. If p divides m, then $(p) = (m)$, and the theorem is proved. Otherwise, we can write $x = py$ for some $y \in A$. Thus $p = pym$, hence $1 = ym$, implying that m is a unit, which is impossible. □

2.3 Quotients and Radicals

In more advanced contexts, two additional operations on ideals are necessary for the analysis. Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a commutative ring A, the quotient is the ideal $(\mathfrak{a} : \mathfrak{b}) = \{a \in A : a\mathfrak{b} \subset \mathfrak{a}\}$. Special cases include the *annihilator* of an ideal $\mathfrak{a}$, which is $(0 : \mathfrak{a})$, often denoted $\mathrm{Ann}(\mathfrak{a})$. The following properties hold:

- $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$.
- $(\mathfrak{a} : \mathfrak{b})\,\mathfrak{b} \subset \mathfrak{a}$.
- $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.
- $(\bigcap \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$.

- $(\mathfrak{a} : \bigoplus \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Ideal quotients become especially useful in the calculus of primary decompositions.

Example. *In the integers,* $(n : m) = (k)$*, where* $k = m/gcd(n, m)$.

Another operation that proves very useful is forming the *radical* of an ideal. Given an ideal $\mathfrak{a}$ in a commutative ring A, we let

$$\mathrm{Rad}(\mathfrak{a}) = \{a \in A : \text{there is } n \text{ such that } a^n \in \mathfrak{a}\}.$$

An ideal $\mathfrak{a}$ is *radical* if $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{a}$. Taking radicals satisfies the following properties:

- $\mathrm{Rad}(\mathrm{Rad}(\mathfrak{a})) = \mathrm{Rad}(\mathfrak{a})$.
- $\mathrm{Rad}(\mathfrak{a}\mathfrak{b}) = \mathrm{Rad}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{Rad}(\mathfrak{a}) \cap \mathrm{Rad}(\mathfrak{b})$.
- $\mathrm{Rad}(\mathfrak{a}) = A$ if and only if $\mathfrak{a} = A$.
- $\mathrm{Rad}(\mathfrak{a} + \mathfrak{b}) = \mathrm{Rad}(\mathrm{Rad}(\mathfrak{a}) + \mathrm{Rad}(\mathfrak{b}))$.
- If $\mathfrak{p}$ is a prime ideal, $\mathrm{Rad}(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.

The Jacobson radical $J(A)$ is precisely $\mathrm{Rad}(0)$. We can write $J(A)$ as the intersection of all prime ideals in A. Similarily, it is easy to prove $\mathrm{Rad}(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$.

2.4 Euclidean Domains

If A is an integral domain, a *Euclidean function* is a positive integer valued function N on non-zero values of A such that if $a, b \neq 0$, then there is q, r such that $a = qb + r$, where $r = 0$ or $N(r) < N(q)$. A *Euclidean domain* is an integral domain possessing a Euclidean function. For convinience, we define $\mathrm{ord}(0) = -\infty$.

Example. *The integers* $\mathbf{Z}$ *is a Euclidean domain, with order function*

$$N(n) = |n|.$$

Indeed, given integers n, m*, which we both may assume without loss of generality to be positive, if* r *is the smallest positive integer such that there is* k *such that* $n = km + r$*, then* $0 \leqslant r < m$*, since if* $r > m$*, then* $n = (k + 1)m + (r - m)$*, and* $0 \leqslant r - m \leqslant r$.

Example. *If k is a field, then $k[X]$ is a Euclidean domain, with order function*

$$N(f) = \deg(f).$$

If $f(X) = a_0 + \cdots + a_n X^n$, $g(X) = b_0 + \cdots + b_m X^m$, and $n \geqslant m$, then we can write

$$f(X) = b_m^{-1} X^{n-m} g(X) + (f - b_m^{-1} X^{n-m} g(X))$$

and $N(f - b_m^{-1} X^{n-m} g(X)) < n$, so continuing this process inductively results in the required remainder.

Example. *The ring $\mathbf{Z}[i]$ of Gaussian integers of the form $n + im$, where $n, m \in \mathbf{Z}$, is a Euclidean domain if we define $N(z) = |z|^2$. To verify this, given nonzero $z, w \in \mathbf{Z}[i]$ with $|z| \geqslant |w|$, pick $u \in \{w, iw, -w, -iw\}$ with an angle of $\theta \leqslant \pi/4$ with z. Then*

$$\begin{aligned} |z-u|^2 &= |z|^2 + |u|^2 - 2\langle z, u\rangle \\ &\leqslant |z|^2 + |w|^2 - 2\cos(\theta)|z||w| \\ &\leqslant |z|^2 + |w|^2 - \sqrt{2}\cdot|z||w| \\ &\leqslant |z|^2 + (|w| - \sqrt{2}\cdot|z|)|w| \leqslant |z|^2 - \sqrt{2}|w|^2 \end{aligned}$$

and so $N(z-u) < N(z)$. Applying this process inductively thus gives the required decomposition.

Example. *If k is a field, a* discrete valuation *on k is a surjective homomorphism $\nu : U(k) \to \mathbf{Z}^+$ such that if $x + y \neq 0$, then $\nu(x+y) \geqslant \min(\nu(x), \nu(y))$. Then the set*

$$A = \{x \in k : \nu(x) \geqslant 0\}$$

is a subring of k, called a discrete valuation ring. *It forms a Euclidean domain under the map ν, for if $\nu(x) \leqslant \nu(y)$, then there exists $c \in k$ such that $y = cx$, and then $\nu(c) = \nu(y) - \nu(x) \geqslant 0$, so $c \in A$. This fact identifies A as a* local ring, *since it shows that A has a maximal ideal $\mathfrak{m} = \{x \in A : \nu(x) \geqslant 1\}$. More generally, the only nontrivial ideals in A are of the form $\mathfrak{m}^n$ for some $n \geqslant 1$.*

Theorem 2.8. *If A if a Euclidean domain, then A is principal.*

Proof. We mimic the proof that $\mathbf{Z}$ is principal. Let $\mathfrak{a}$ be a nonzero ideal in A, and let a be an element of smallest order. If $b \in \mathfrak{a}$, then we can write $b = qa + r$, where $N(r) < N(a)$. But $r = b - qa \in \mathfrak{a}$, so $r = 0$, and so a divides b. □

The fact that $\mathbf{Z}$ was a principal ideal domain was known since the time of the Greeks. Gauss was the first to realise that $\mathbf{Z}[i]$ was a principal ideal domain, and he used it to prove some interesting results about the ordinary integers, solving congruences modulo primes.

It is not true that every principal ideal domain is a Euclidean domain. However, principal ideal domains do exist a less powerful version of a norm, known as a *Dedekind Hasse* norm, which often means that working with Euclidean domains is not that much more powerful than working with principal ideal domains. If A is a ring, a Dedekind Hasse norm is a positive integer valued function N on non-zero values of A such that for any $x, y \in A$, either x divides y, or there is $s, t \in A$ such that $sx + ty \neq 0$ and $N(sx + ty) < N(x)$.

Theorem 2.9. *If A has a Dedekind-Hasse norm, then A is principal.*

Proof. Given an ideal $\mathfrak{a}$ in A, let x be a nonzero element of $\mathfrak{a}$ with smallest norm. Then for any $y \in \mathfrak{a}$, any element of $s, t \in A$, either $sx + ty \neq 0$ or $N(sx + ty) \geqslant x$. Since N is a Dedekind Hasse norm, this implies that x divides y, so that $\mathfrak{a} = (x)$. □

Example. *The ring $A = \mathbf{Z}[(1 + \sqrt{-19})/2]$ has a Dedekind-Hasse norm, from which it follows the ring is principal. We define*

$$N(z) = |z|^2.$$

Then $N(z)$ is a positive integer for each $z \in A$. Indeed,

$$\begin{aligned}\left(n + m\frac{1+\sqrt{-19}}{2}\right)\left(n + m\frac{1-\sqrt{-19}}{2}\right) &= (n + m/2)^2 + 19(m^2/4) \\ &= n^2 + nm + 5m^2.\end{aligned}$$

We claim that N is a Hasse Dedekind norm, hence A is principal. Suppose α, β are nonzero elements of A, but β does not divide α in A. We must show that there are $s, t \in A$ such that $0 < N(st + xy) < N(x)$. This is equivalent to showing

$$0 < N\left((\alpha/\beta)s - t\right) < 1.$$

Find $a, b, c \in \mathbf{Z}$ with no common divisor such that

$$\alpha/\beta = \frac{a + b\sqrt{-19}}{c}.$$

Then $c > 1$, for otherwise $\alpha/\beta \in A$ and so β divides α. Then we can find integers $x, y, z \in \mathbf{Z}$ such that $ax + by + cz = 1$. Write $ay - 19bx = cq + r$ for integers q, r with $|r| \leqslant c/2$, and let $s = y + x\sqrt{-19}$ and $t = q - z\sqrt{-19}$. Then

$$\begin{aligned}(\alpha/\beta)s - t &= \frac{(a + b\sqrt{-19})(y + x\sqrt{-19})}{c} - (q - z\sqrt{-19}) \\ &= \frac{ay - 19bx - cq}{c} + \frac{(ax + by + cz)\sqrt{-19}}{c} \\ &= \frac{r}{c} + \frac{\sqrt{-19}}{c}.\end{aligned}$$

This shows $(\alpha/\beta)s - t \neq 0$, and

$$N((\alpha/\beta)s - t) = \frac{r^2 + 19}{c^2} \leqslant 1/4 + 19/c^2.$$

Provided $c \geqslant 5$, this completes the calculation, and so we address the remaining cases on a case by case basis for $c \in \{2, 3, 4\}$.

Suppose $c = 2$. Then a and b cannot be both even or both odd, for then $\alpha/\beta \in A$ (A consists precisely of $(n + m\sqrt{-19})/2$ such that $n - m$ is even). But then

$$\frac{(a-1) + b\sqrt{-19}}{2} \in A$$

and

$$\frac{\alpha}{\beta} - \frac{(a-1) + b\sqrt{-19}}{2} = \frac{1}{2}$$

So we can set $s = 1$ and $t = [(a-1) + b\sqrt{-19}]/2$.

Now consider the case $c = 3$. Then $a^2 + 19b^2$ is not divisible by 3, for otherwise $a^2 + b^2$ is divisible by 3, which implies a and b are divisible by 3. Write $a^2 + 19b^2 = 3q + r$, where $r = 1$ or $r = 2$. Then

$$\frac{a + b\sqrt{-19}}{3}(a - b\sqrt{-19}) - q = r/3 < 1.$$

so we may pick $s = a - b\sqrt{-19}$ and $t = q$.

Finally, we consider the case $c = 4$. Then a and b are not both even. If a and b are both odd, then $a^2 + 19b^2$ is congruent to 4 modulo 8. Thus we can write $a^2 + 19b^2 = 8q + 4$. Then

$$\frac{a + b\sqrt{-19}}{4}\frac{a - b\sqrt{-19}}{2} - q = \frac{1}{2}$$

so we can set $s = (a - b\sqrt{-19})/2$ *and* $t = q$. *If one of* a *and* b *is odd, and the other is even, then* $a^2 + 19b^2$ *is odd, then we can write* $a^2 + 19b^2 = 4q + r$ *with* $0 < r < 4$. *Then*

$$\frac{a + b\sqrt{-19}}{2}(a - b\sqrt{-19}) - q = r/4.$$

Thus we can set $s = a - b\sqrt{-19}$ *and* $t = q$.

We claim $\mathbf{Z}[(1 + \sqrt{-19})/2]$ is a PID which is not a Euclidean domain. To begin with, we must find a property that Euclidean domains have but which PIDs do not necessarily have. Given a ring A, let $V(A) = U(A) \cup \{0\}$. An element $u \in A - V(A)$ is called a *universal side divisor* if for every $x \in A$ there exists $a \in V(A)$ such that u divides $x - a$.

Theorem 2.10. *If* A *is a Euclidean domain, it possesses universal side divisors.*

Proof. Let u be the element of $A - V(A)$ with smallest norm. Then u is a universal side divisor, because if $x \in A$, we can write $x = cu + r$, where $N(r) < N(u)$, so $r \in V(A)$. Thus u divides $x - r$. □

Example. *Recall the ring* $A = \mathbf{Z}[(1 + \sqrt{-19})/2]$ *is a principal ring. We now show it is not a Euclidean domain, because it does not possess universal side divisors. Recall the Hasse Dedekind norm* N *constructed earlier. Since* N *is multiplicative, it is easy to see that the units of* A *are precisely* ± 1, *since the only integer solutions to the Diophantine equation*

$$n^2 + nm + 5m^2 = 1$$

are given by setting $n = \pm 1$, $m = 0$. *However,* A *does not have a universal side divisor, from which it follows that* A *cannot be a Euclidean domain. Suppose* u *is a universal side divisor. Then* u *is a side divisor of* 2, *so* u *divides one of* $\{1, 2, 3\}$. *Since* u *is not a unit,* u *must either divide* 2 *or* 3. *If* $2 = uv$, *then* $N(uv) = N(u)N(v) = 4$, *which implies* $u, v \in \mathbf{Z}$ *since* $N(u) \geqslant 5$ *for any* $u \notin \mathbf{Z}$, *and so without loss of generality we find* $u = 2$. *Similarily, if* u *divides* 3, *with* $3 = uv$, *then* $N(uv) = 9$. *Since there are no* $u \in A$ *with* $N(u) = 3$, *we conclude* $N(u) = 9$. *Note that for any integers* n, m,

$$n^2 + nm + 5m^2 \geqslant \min(5m^2, n^2 + 4m^2) \geqslant 4m^2,$$

we can see that the only solutions to

$$n^2 + nm + 5m^2 = 9$$

have $|m| \leqslant 1$. *If* $m = 0$, *then* $n = 3$. *If* $m = 1$, *then* $n(n+1) = 4$, *which has no solutions. If* $m = -1$, *then* $n(n-1) = 4$, *which again has no solutions. Thus we conclude that if* u *is a side divisor of* A, *then* $u = \pm 3$ *or* $u = \pm 2$. *But none of*

$$\frac{-1+\sqrt{19}}{2}, \quad \frac{1+\sqrt{-19}}{2}, \quad \text{or} \quad \frac{3+\sqrt{-19}}{2}$$

are divisible by 2 *or* 3, *which shows* u *is not a side divisor. Thus* A *is not a Euclidean domain.*

Let us now show that all principal ideal domains have Dedekind-Hasse norms.

Theorem 2.11. *Any principal ideal domain has a Dedekind-Hasse norm.*

Proof. If A is a principal ideal domain, then we can define a norm N by setting $N(u) = 1$ if $u \in U(A)$, and if $p_1, \ldots, p_n$ are primes, then define

$$N(p_1^{k_1} \ldots p_n^{k_n}) = 2^{k_1 + \cdots + k_n}.$$

Then N is multiplicative and positive. Now suppose a and b are nonzero elements of A. Then $(a, b) = (x)$ for some x. If

$$a = p_1^{t_1} \ldots p_n^{t_n}$$

and

$$b = u p_1^{s_1} \ldots p_n^{s_n},$$

Then we can choose

$$x = p_1^{\min(t_1, s_1)} \ldots p_n^{(t_n, s_n)}.$$

If a does not divide b, then there must exist some i with $s_i < t_i$, and then $N(x) < N(a)$. Thus we have shown N is a Dedekind-Hasse norm. □

An integral domain A is a *quasi-Euclidean domain* if there exists a function N such that for any $r_{-1}, r_0 \in A$, there exists k and $q_0, \ldots, q_{k-1} \in A$ and $r_1, \ldots, r_k \in A$ such that for each $i \in \{-1, \ldots, k-2\}$, $r_i = q_{i+1} r_{i+1} + r_{i+2}$, and $N(r_k) < N(r_0)$. In the fraction field of A, we may write this sequence in terms of a continued fraction, i.e. writing

$$\frac{r_{-1}}{r_0} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \cdots + \frac{1}{q_{k-1} + r_k / r_{k-1}}}}.$$

We may write such a continued fraction as $[q_0,\dots,q_{k-1},r_{k-1}/r_k]$, where the bracket notation for continued fractions is defined inductively by setting $[a_1,a_2] = a_1 + 1/a_2$, and $[a_1,\dots,a_k] = [a_1,[a_2,\dots,a_k]]$. A ring is an n stage Euclidean domain if we can always choose $k \leqslant n$.

Lemma 2.12. *Any quasi-Euclidean domain is a Bezout domain.*

Proof. Fix $x_0, y_0 \in A$, and suppose r is the smallest nonzero element of (x_0, y_0) which respect to N. Then there is k such that we can perform the k stage Euclidean algorithm with $r_{-1} = x_0$ and $r_0 = y_0$, constructing $r_1,\dots,r_k \in A$ and $q_1,\dots,q_{k-1}$ such that $r_i = q_{i+1}r_{i+1} + r_{i+2}$ for each i, and $N(r_k) < N(y)$. One verifies quite easily that $(r_i, r_{i+1}) = (r_{i+1}, r_{i+2})$. Thus $(x_0, y_0) = (x_1, y_1)$, where $x_1 = r_{k-1}$ and $y_1 = r_k$. Thus $N(y_1) < N(y_0)$ or $y_1 = 0$. If $y_1 = 0$, then $(x_0, y_0) = (x_1)$. Otherwise, we may repeat this process, writing $(x_1, y_1) = (x_2, y_2)$ with $y_2 = 0$ or $N(y_2) < N(y_1)$. Clearly this process must terminate eventually, so that (x_0, y_0) is a principal ideal. □

2.5 Bezout Domains

An integral domain A is a *Bezout Domain* if every finitely generated ideal is principal. An inductive construction shows that this is equivalent to showing every ideal generated by two elements is principal, which means precisely that for every pair of elements $a_1, a_2 \in A$, the greatest common divisor of a_1 and a_2 can be written as $sa_1 + ta_2$ for some $s, t \in A$.

Example. *Every Boolean ring is a Bezout domain. Given a Boolean ring A, and $x, y \in A$, we have*

$$x(x + y - xy) = x^2 + xy - x^2y = x + xy + xy = x$$

and

$$y(x + y - xy) = xy + y^2 - xy^2 = y + xy - xy = y.$$

Thus we conclude $(x, y) = (x + y - xy)$.

Example. *Let U be a connected, open subset of $\mathbf{C}$, and let $A(U)$ denote the family of all holomorphic functions on U. Then $A(U)$ is a Bezout domain. It is a consequence of the Weirstrass factorization theorem and the Mittag-Lefler theorem that if S is a discrete subset of U, and for each $s \in S$ we associate a natural*

number n_s and complex numbers $w_{s1}, \ldots, w_{sn_s}$, then there exists $f \in A(U)$ such that for all $s \in S$ and $k \leqslant n_s$, $f^{(k)}(s) = w_{sk}$, and the zeroes of f are contained in S. Moreover, if $f_1, f_2 \in A(U)$ are functions such that $ord_z(f_2) \geqslant ord_z(f_1)$ for all $z \in U$, then there is $g \in A(U)$ such that $f_2 = gf_1$. These analytical facts imply that $A(U)$ is a Bezout domain.

To begin with, assume that $f_1, f_2 \in A(U)$ are holomorphic functions sharing no common zeroes. Then we can find a function $g_2 \in A(U)$ with specialized values and derivatives on the zeroes of f_1 such that all zeroes of f_1 are zeroes of $1 - f_2 g_2$, and the order of this zero for $1 - f_2 g_2$ is greater than f_1. It follows that there exists $g_1 \in A(U)$ such that $f_1 g_1 = 1 - f_2 g_2$, and so $(f_1, f_2) = 1$.

More generally, given f_1, f_2, we can find a function f with

$$ord_z(f) = \min(ord_z(f_1), ord_z(f_2))$$

for each $z \in \mathbf{C}$. Then there are $g_1, g_2 \in A(U)$ such that $f_1 = g_1 f$, $f_2 = g_2 f$. Thus $(f_1, f_2) \subset (f)$. But since g_1 and g_2 share no common zeroes, there is $h_1, h_2 \in A(U)$ such that $h_1 g_1 + h_2 g_2 = 1$, and so $f = (h_1 g_1 + h_2 g_2) f = h_1 f_1 + h_2 f_2 \in (f_1, f_2)$. Thus $(f_1, f_2) = (f)$, and we have shown all finitely generated ideals are principal.

We note that $A(U)$ is not *a unique factorization domain, which we will later see implies that $A(U)$ is not a principal ring. The units of $A(U)$ are precisely the functions with no zeroes, and the Weirstrass factorization theorem irreducible elements of $A(U)$ are precisely those elements with a single, simple zero. In particular this implies that any element of $A(U)$ which can be written as a finite product of irreducibles has only finitely many zeroes. On the other hand, the Weirstrass factorization theorem essentially says that all elements of $A(U)$ can be written as an* infinite product *of irreducible elements of $A(U)$, which is not measured in the algebraic theory of factorization.*

Example. *Let A be the subring of $\mathbf{Q}[X]$ consisting of polynomials with integer constant term. The units of A are precisely $\{\pm 1\}$. This means that the only irreducible elements of A are prime numbers, or irreducible polynomials in $\mathbf{Q}[X]$ with constant term 1. It is easy to see such elements are irreducible in A. Any nonconstant polynomial with constant term not equal to 1 is reducible. Conversely, if $f \in A$, and we can write $f = gh$, where $g, h \in \mathbf{Q}[X]$ are nonconstant polynomials. If $x = g(0)$ and $y = h(0)$, then $xy = 1$, so we can write $f = [g/x][xh]$, and $g/x, xh \in A$ since both have constant coefficient equal to 1, so f is reducible over A.*

In particular, this means that the polynomial X is not irreducible in A. On the other hand, the only irreducible factors of X in A are prime integers. Thus X cannot be factored into irreducibles, and thus A is not *a unique factorization domain. This implies the ring A is not Noetherian, and we can see this more explicitly from the infinite chain*

$$(X) \subsetneq (X/2) \supsetneq (X/4) \supsetneq \dots$$

is an infinite increasing chain of ideals. However, A is *a Bezout domain, as we now prove, so that all finitely generated ideals are principal.*

Let $f, g \in \mathbf{Q}[X]$, and suppose either $f(0)$ or $g(0)$ is nonzero. Let $h \in \mathbf{Q}[X]$ be a greatest common divisor of f and g, and scale h so that $\mathbf{Z}h(0) = \mathbf{Z}f(0) + \mathbf{Z}g(0)$. If we write $f = hf_0$ and $g = hg_0$, then $f_0, g_0 \in A$, since $f(0)$ and $g(0)$ are integer multiplies of $h(0)$. Thus we conclude that $(f,g) \subset (h)$. On the other hand $f_0(0)$ and $g_0(0)$ must be relatively prime, because if $k_1, k_2 \in \mathbf{Z}$ are chosen such that $h(0) = k_1 f(0) + k_2 g(0)$, then this implies $1 = k_1 f_0(0) + k_2 g_0(0)$. We claim this implies that we can write $1 = af_0 + bg_0$ with $a, b \in A$, which would imply that $h = af + bg \in (f,g)$, so that $(h) \subset (f,g)$. Certainly we can write $1 = a_0 f_0 + b_0 g_0$ with $a_0, b_0 \in \mathbf{Q}[X]$. For any rational number m, if we write $a = a_0 + mg_0$ and $b = b_0 - mf_0$, then $1 = af_0 + bg_0$. And setting $m = (k_1 - a_0(0)/g_0(0)$ where k_1 is as in the last paragraph shows $a, b \in A$. Thus we conclude that $(f,g) = (h)$.

It is now easy to generalize this construction to the case where $f(0) = g(0) = 0$. For general $f, g \in A$, we can find an integer r such that $f = X^r f_0$, $g = X^r g_0$, and either $f_0(0)$ or $g_0(0)$ is nonzero. The previous techniques show the existence of $h_0 \in A$ such that $(f_0, g_0) = (h_0)$. If we let $h = X^r h_0$, we conclude that

$$(f,g) = (X^r)(f_0, g_0) = (X^r)(h_0) = (h),$$

which completes the general case.

Lemma 2.13. *Let k be the fraction field of a Bezout domain A. Then every element of k can be written as a/b where $a, b \in A$ and $(a) + (b) = A$.*

Proof. Any element of k can be written as x/y with $x, y \in A$, and suppose $(x,y) = (a)$ with a not a unit. Then we can find $x_1, y_1, t, s \in A$ such that

$$x = x_1 a, \quad y = y_1 a, \quad \text{and} \quad a = tx + sy.$$

Thus $a = a(tx_1 + sy_1)$, so $tx_1 + sy_1 = 1$. But then $(x_1, y_1) = (1)$, and $x/y = x_1/y_1$, completing the proof. □

Following the standard proof that PIDs are UFDs, we can conclude that any Bezout domain such that any element can be factored into irreducible elements is a factorial ring. In fact, this occurs if and only if the Bezout domain is a principal ideal domain.

Lemma 2.14. *Any factorial Bezout domain is principal.*

Proof. Let A be a factorial Bezout domain. For each $a \in A$, we can write a as $p_1^{k_1} \dots p_n^{k_n}$ for primes $p_1, \dots, p_n$. Define

$$N(a) = k_1 + \cdots + k_n.$$

We claim N is a Dedekind-Hasse norm. Indeed, if $a_1, a_2 \in A$, then there is $x \in A$ such that $(a_1, a_2) = (a)$. Then a is a greatest common divisor for a_1 and a_2. If

$$a_1 = p_1^{t_1} \dots p_n^{t_n} \quad \text{and} \quad a_2 = p_1^{s_1} \dots p_n^{s_n},$$

we therefore conclude that

$$a = p_1^{\min(t_1, s_1)} \dots p_n^{\min(t_n, s_n)}.$$

Thus either a_1 divides a_2, a_2 divides a_1, or $N(a) < \min(N(a_1), N(a_2))$. This verifies that N is a Dedekind-Hasse norm. □

2.6 Uniqueness of Congruences

In classical number theory, one takes a series of integers $k_1, \dots, k_m$ and values $a_1, \dots, a_m$, and asks to find an integer n such that $n \equiv a_i \pmod{k_i}$ for each i. The classical Chinese remainder theorem says that if the set of integers $\{k_i\}$ are pairwise coprime, such an element n can always be solved. These ideas can be extended to solve congruences over general rings. In the general setup, we are given a family of ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_N$ over a commutative ring A, and we consider the corresponding projection $\pi : A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_N$. The generalization of the Chinese remainder theorem is summarized in the next theorem. We saw two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are *coprime* if $\mathfrak{a} + \mathfrak{b} = A$.

Theorem 2.15. *Let A be a commutative ring, fix $n \geqslant 2$, and suppose $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are pairwise coprime. Then the map $\pi : A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$ is surjective, and is an isomorphism if $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = (0)$.*

Proof. Consider the case where we have two coprime ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Given any $x_1, x_2 \in A$, we wish to find $x \in A$ such that $x - x_1 \in \mathfrak{a}_1$ and $x - x_2 \in \mathfrak{a}_2$. Since $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are coprime, we can find $a_1 \in \mathfrak{a}_1$ and $a_2 \in \mathfrak{a}_2$ such that $a_1 + a_2 = x_2 - x_1$. Thus $\pi(a_1) = (0, x_2 - x_1)$ and $\pi(a_2) = (x_1 - x_2, 0)$. But this means that $\pi(x_1 + a_1) = (x_1, x_2)$. The kernel of the map π is $\mathfrak{a}_1 \cap \mathfrak{a}_2$, so the proof is completed in the case $n = 2$.

Let us now prove the remaining cases by induction. Suppose we could show that $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_{n-1}$ and $\mathfrak{a}_n$ are coprime. Then the Chinese remainder theorem for the case of two ideals, combined with the Chinese remainder theorem for $n-1$ ideals, would complete the proof. By an easy induction, it suffices to show that if $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{c}$ are pairwise coprime, then $\mathfrak{a} \cap \mathfrak{b}$ are coprime to $\mathfrak{c}$. Using the fact that $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, and that $\mathfrak{b}$ and $\mathfrak{c}$ are relatively prime, we find

$$\begin{aligned}(\mathfrak{a} \cap \mathfrak{b}) + \mathfrak{c} &= \mathfrak{a}\mathfrak{b} + \mathfrak{c} \\ &= \mathfrak{a}\mathfrak{b} + (\mathfrak{a} + \mathfrak{b})\mathfrak{c} \\ &= \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c} \\ &= \mathfrak{a}(\mathfrak{b} + \mathfrak{c}) + \mathfrak{b}\mathfrak{c} \\ &= \mathfrak{a} + \mathfrak{b}\mathfrak{c}.\end{aligned}$$

Thus the ideal contains $\mathfrak{a}$ and $\mathfrak{c}$. But since $\mathfrak{a}$ and $\mathfrak{c}$ are relatively prime, we conclude $(\mathfrak{a} \cap \mathfrak{b}) + \mathfrak{c} = A$. □

Example. *Given an integer n, the units of $\mathbf{Z}_n$ are in one to one correspondence with the set of integers $1 \leqslant m \leqslant n$ which are relatively prime to n. Thus $\varphi(n) = \#(U(\mathbf{Z}_n))$, where φ is the number of such integers. If n and m are relatively prime, then $(n) + (m) = \mathbf{Z}$, and $(n) \cap (m) = (nm)$. Thus the Chinese remainder theorem applies, and we conclude that $\mathbf{Z}_{nm}$ is isomorphic to $\mathbf{Z}_n \times \mathbf{Z}_m$. If any pair of rings A and B, $U(A \times B) = U(A) \times U(B)$, so we conclude*

$$\phi(nm) = \#(U(\mathbf{Z}_{nm})) = \#(U(\mathbf{Z}_n) \times U(\mathbf{Z}_m)) = \phi(n)\phi(m).$$

This statement enables us to calculate $\phi(n)$ for any integer n. We note first that if p is prime, then

$$\varphi(p^n) = p^{n-1}(p-1),$$

because there are p^n integers between 1 and p^n, and they are all relatively prime to p^n, except for the multiples of p. Thus

$$\varphi(p_1^{n_1} \cdots p_m^{n_m}) = \prod_{i=1}^{m} \varphi(p_i^{n_i}) = \prod_{i=1}^{m} p_i^{n_i - 1}(p_i - 1).$$

Example. *For each $n, m \in \mathbf{Z}$, the map $f_m : \mathbf{Z}_n \to \mathbf{Z}_n$ given by setting $f_m(n) = nm$ is an endomorphism of $\mathbf{Z}_n$. It is clear that*

$$f_{n_1} \circ f_{n_2} = f_{n_1 n_2} \quad \textit{and} \quad f_{n_1+n_2} = f_{n_1} + f_{n_2},$$

Thus we obtain a morphism from $\mathbf{Z}$ to $End(\mathbf{Z}_n)$, which is surjective since if $\varphi : \mathbf{Z}_n \to \mathbf{Z}_n$ is a morphism with $\varphi(1) = k$, then $\varphi = f_k$. The kernel of the morphism from $\mathbf{Z}$ to $End(\mathbf{Z}_n)$ is then clearly (m), so we have an isomorphism between $\mathbf{Z}_n$ and $End(\mathbf{Z}_n)$.

Example. *Let A be a nontrivial finite Boolean ring. Let $I_1, \ldots, I_n$ be the set of all prime ideals of A. Since all prime ideals are maximal in Boolean rings, the ideals $\{I_1, \ldots, I_n\}$ are all pairwise coprime. Since A is a reduced ring, $I_1 \cap \cdots \cap I_n = (0)$. For each i, A/I_i is isomorphic to $\mathbf{Z}_2$, so the Chinese remainder theorem implies $A \cong A/I_1 \times \cdots \times A/I_n \cong \mathbf{Z}_2^n$. In particular, A is isomorphic to a Boolean ring of sets.*

If $\mathfrak{a}$ is a finite ideal in a Boolean ring, then the direct sum $\mathbf{Z}_2 \oplus \mathfrak{a}$ has the natural structure of a Boolean ring, where we define

$$(x_1 \oplus a_1)(x_2 \oplus a_2) = (x_1 x_2) \oplus (x_1 a_2 + x_2 a_1 + a_1 a_2)$$

The ring axioms are easily verifiable, where $1 \oplus 0$ is the multiplicative identity. One then simply verifies that

$$(x \oplus a)^2 = (x^2 \oplus 2xa + a^2) = x \oplus a,$$

so the ring is Boolean. $\mathfrak{a}$ is clearly an ideal in $\mathbf{Z}_2 \oplus \mathfrak{a}$. It thus follows that $\mathfrak{a}$ is isomorphic to a maximal ideal in a Boolean ring of a finite set. In particular this actually implies that $\mathfrak{a}$ is a ring, *since $\mathfrak{a}$ has a maximum element with respect to inclusion and thus has a multiplicative identity. Note, however, that the multiplicative identity of $\mathfrak{a}$ is not the same as the multiplicative identity of $\mathbf{Z}_2 \oplus \mathfrak{a}$.*

2.7 Factorial Rings

Let A be a ring. An element $a \in A$ is *irreducible* if, given $x, y \in A$ such that $a = xy$, either x or y is a unit of A. A *factorial ring*, or *unique factorization domain*, is an integral domain A such that every nonzero $a \in A - U(A)$ can be written as $p_1 p_2 \ldots p_N$, for some irreducibles p_n, and such that if $p_1 \ldots p_N = q_1 \ldots q_M$, then $N = M$, and, after a permutation, each p_n differs from q_n by a unit.

Example. *The ring* $\mathbf{Z}$ *of integers is a factorial ring, as shown by the fundamental theorem of arithmetic. On the other hand, the ring* $\mathbf{Z}[2i]$ *is* not *factorial, since* $2i$, $-2i$, *and* 2 *are irreducible in* $\mathbf{Z}[2i]$, *so that*

$$4 = (2i) \cdot (-2i) = 2 \cdot 2$$

gives two distinct factorizations of 4. Similarily, $\mathbf{Z}[2\sqrt{2}]$ *is not a factorial ring, since* $8 = (2\sqrt{2})^2 = 2^3$.

Example. *Every field is trivially a unique factorization domain.*

We say a ring is *Noetherian* if every ideal is finitely generated (thus every principal ring is Noetherian); this is equivalent to saying the ring satisfies the *ascending chain condition*. That is, there do not exist an infinite increasing linear chain $\{\mathfrak{a}_\alpha\}$ of distinct ideals. Thus every infinite chain of ideals must eventually become constant. Noetherian rings have a factorization theory, but this factorization need not be unique. All principal ideals A are Noetherian, because for any linear increasing chain of ideals $\{\mathfrak{a}_\alpha\}$, $\bigcup \mathfrak{a}_\alpha$ is an ideal, hence generated by some $a \in A$. The element a must be an element of some $\mathfrak{a}_{\alpha_0}$, and it then follows that for $\alpha \geqslant \alpha_0$, $\mathfrak{a}_\alpha = \mathfrak{a}_{\alpha_0}$.

Theorem 2.16. *Every nonzero element of a Noetherian ring may be factored into irreducible elements.*

Proof. Fix some $x_0 \neq 0$. If x_0 is irreducible, we're done. Otherwise, we can write $x_0 = a_0 x_1$, where a and x_1 are both not units. If x_1 is not irreducible, we can write $x_1 = a_1 x_2$, where neither $a_1 x_2$ are not units. It is clear that if this process never stops, we can find a sequence $\{x_i\}$ with $x_{i+1} \mid x_i$ for each i, but such that x_i does not divide x_{i+1}. This corresponds to an infinite chain

$$(x_0) \subsetneq (x_1) \subsetneq (x_2) \subsetneq \dots$$

which is impossible since the ring is Noetherian. □

Theorem 2.17. *Every principal ideal domain is factorial.*

Proof. The fact that every principal ideal domain *has* a factorization is justified because it is Noetherian. It now suffices to prove such a factorization is unique. Let $p_1 \dots p_N = q_1 \dots q_M$. We proceed by induction on N. If $p = q_1 \dots q_M$, then p divides one of the quantities on the right, implying p

must divide one of the q_n, which, without loss of generality, we may assume is q_M. Then $q_M = ap$, so, dividing by p on both sides of the equation, we conclude that $1 = aq_1 \dots q_{M-1}$, so each q_n is a unit, which is a contradiction unless $M = 1$. Now in general, suppose $p_1 \dots p_{N+1} = q_1 \dots q_M$. Then p_{N+1} divides one of the quantities on the right, say q_M, so $q_M = ap_{N+1}$, hence, dividing out, we conclude $p_1 \dots p_N = aq_1 \dots q_{M-1}$, hence by induction, $N = M - 1$, and by permutation, we can assume $p_n = a_n q_n$. But then $p_1 \dots p_{N+1} = aa_1 \dots a_{M-1} p_1 \dots p_N q_M$, hence $p_{N+1} = aa_1 \dots a_{M-1} q_M$, so p_{N+1} differs from q_M by a unit. □

The *primes* of a factorial ring can be broken up into equivalence classes, where we identify two primes that differ by a unit. Thus if A is a factorial ring, and $\{p_\alpha\}$ is a family of representatives for the irreducible elements of A modulo $U(A)$, then any nonzero $a \in A$ may be *uniquely written as*

$$a = u p_{\alpha_1}^{k_1} \dots p_{\alpha_n}^{k_n}$$

where $u \in U(A)$.

Example. $\mathbf{Z}$ *is a principal ideal domain, hence factorial. The group of units are* 1 *and* -1*, so the equivalence class of irreducibles in* $\mathbf{Z}$ *consist of integers* p *and* $-p$*, where* p *is a non-negative prime. It is canonical to take the positive primes as representatives, and so we find every positive integer can be uniquely decomposed as a product of positive prime number, and every negative integer is the negation of a product of primes.*

If A is a ring of functions, then that ring being non-factorial normally indicates the presence of some singularity in the ring.

Example. *Consider the curve in* $\mathbf{A}^2 = k^2$ *defined as the locus of points satisfying the equation* $Y^2 = X^3$*. Then the curve has a singularity at the origin. The corresponding ring* $k[X, Y]/(Y^2 - X^3)$*, which is isomorphic to the ring of polynomial functions in two dimensions restricted to the curve, is not factorial, since* $X \neq Y$ *are both irreducible elements not differing by primes, yet* $X^3 = Y^2$*.*

Example. *The relation*

$$\sin^2 x = (1 + \cos x)(1 - \cos x)$$

indicates that the ring $\mathbf{R}[\sin x, \cos x]$ *of functions generated by* $\sin x$ *and* $\cos x$ *is not a factorial ring. Define the* trigonometric degree $\deg(f)$ *of a function*

$$f(x) = a + b_1 \cos(x) + \dots + b_N \cos(Nx) + c_1 \sin(x) + \dots + c_M \sin(Mx)$$

where $b_N, c_M \neq 0$, as $\max(N, M)$. One can show $\deg(fg) = \deg(f) + \deg(g)$ directly, but it is convenient to extend our calculations to the complex algebra $\mathbf{C}[e^{ix}]$, which contains $\mathbf{R}[\cos x, \sin x]$ as a subring. We do this in the next paragraph, but note that this implies that $\mathbf{R}[\sin x, \cos x]$ has no zero divisors, and all degree one trigonometric polynomials are irreducible, which includes $\sin x$, $1 + \cos x$, and $1 - \cos x$.

To see that the degree is well defined on $\mathbf{C}[e^{ix}]$, it suffices to note that e^{ix} is trancendental over $\mathbf{R}$, so $\mathbf{C}[e^{ix}]$ is isomorphic to $\mathbf{C}[X]$. To see that e^{ix} is trancendental, we note that

$$\int_{-\pi}^{\pi} e^{nix} e^{-mix} = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}$$

Thus, given $f(x) = \sum_{n=-N}^{N} a_n e^{nix}$,

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-nix}.$$

so if $f = 0$, then $a_n = 0$ for all n.

Lemma 2.18. *In an integral domain, every prime is irreducible.*

Proof. Suppose A is an integral domain, and suppose $p \in A$ is prime. Suppose $a, b \in A$ and $p = ab$. Since p is prime, either p divides a or p divides b. If we write $a = pa'$, and then $1 = a'b$, implying b is a unit. Thus p is irreducible. □

Conversely, in a factorial ring, every irreducible element is prime.

Theorem 2.19. *In a factorial ring, every irreducible element is prime.*

Proof. Suppose that A is a factorial ring with an equivalence class of irreducible elements $\{a_\alpha\}$, and fix some irreducible element a_{α_1} from this class. Suppose $x_1, x_2 \in A$ and a_{α_1} divides $x_1 x_2$. Then we can write $x_1 x_2 = x_3 a_{\alpha_1}$. Perform a factorization of each element, writing

$$x_1 = u_1 a_{\alpha_1}^{t_{11}} \dots a_{\alpha_n}^{t_{1n}},$$

$$x_2 = u_2 a_{\alpha_1}^{t_{21}} \dots a_{\alpha_n}^{t_{2n}},$$

and

$$x_3 = u_3 a_{\alpha_1}^{t_{31}} \dots a_{\alpha_n}^{t_{3n}}.$$

Thus we have

$$(u_1 u_2) a_{\alpha_1}^{t_{11}+t_{21}} \dots a_{\alpha_n}^{t_{1n}+t_{2n}} = u_3 a_{\alpha_1}^{1+t_{31}} \dots a_{\alpha_n}^{t_{3n}}.$$

Unique factorization implies that $t_{11} + t_{21} = 1 + t_{31}$, thus either $t_{11} > 0$ or $t_{21} > 0$. In particular, this implies that either a_{α_1} divides x_1 or a_{α_1} divides x_2. Thus a_{α_1} is prime. □

Remark. Suppose that A is a ring which every element can be uniquely factored (up to units) as a product of primes, but not necessarily uniquely as a product of irreducibles. If $a \in A$ is irreducible, and we write

$$a = u p_1^{k_1} \dots p_n^{k_n}$$

then we conclude that $n = 1$, $k_1 = 1$, and p_1 differs from a by a unit. Thus a is prime, and so A is really a factorial ring.

In a factorial domain A, the primes and units of the ring uniquely specify the multiplicative monoid structure of $A - \{0\}$. For instance, if $\mathbf{P} = \{2, 3, 5, \dots\}$ is the set of all primes in $\mathbf{Z}$, then for each permutation π of $\mathbf{P}$, we obtain a homomorphism $f : \mathbf{Z} - \{0\} \to \mathbf{Z} - \{0\}$ of the multiplicative monoid structure by setting

$$f(\pm p_1^{k_1} \dots p_n^{k_n}) = \pm \pi(p_1)^{k_1} \dots \pi(p_n)^{k_n}.$$

On the other hand, none of these permutations extend to ring homomorphisms, since the additive structure isn't reflected at all in these permutations.

Chapter 3

Polynomials

In a ring, we can add and multiply. It is natural then, to 'solve' equations of the form

$$5X^2 + 1 = 2 \qquad XYZ + 2Y = Z$$

Making an abstract concept into a precise mathematical object is often the key method to study mathematical phenomena. A polynomial is the static object representing the equations we can construct in a ring, which we can pin down and understand. In this Chapter, all rings will be assumed to be commutative unless stated otherwise.

3.1 Univariate Polynomials

We now provide a brief introduction to the ring of polynomials with coefficients in a ring. If A is a commutative ring, a *univariate polynomial* in the indeterminate X with coefficients in A is an abstract expression of the form $f(X) = a_0 + a_1X + \cdots + a_nX^n$, with $a_0, \ldots, a_n \in A$. The set of all univariate polynomials in X is denoted $A[X]$. We define a ring structure on $A[X]$ by letting

$$\sum a_k X^k + \sum b_k X^k = \sum (a_k + b_k) X^k$$

$$\left(\sum a_i X^i\right)\left(\sum b_j X^j\right) = \sum a_i b_j X^{i+j} = \sum_k \left(\sum a_i b_{k-i}\right) X^k$$

Since A embeds itself in $A[X]$ as the set of terms with no occurence of X, we can view $A[X]$ as an algebra over A.

If A is a subring of a ring B, then each polynomial

$$f = a_0 + a_1 X + \cdots + a_n X^n \in A[X]$$

gives rise to a function from B to itself, mapping $x \in B$ to

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in B.$$

This gives a homomorphism from $A[X]$ to the ring A^A. The dual of this is the *evaluation homomorphism*. Given $x \in B$, we obtain a homomorphism $\mathrm{ev}_x : A[X] \to B$ mapping f to $f(x)$. Thus we can interpret $A[X]$ as the *free commutative A-algebra* generated by X, i.e. the 'most general' way of adding an additional element to the ring A.

Remark. If A is *not* a commutative ring, we may still define $A[X]$ as in the commutative case. But the evaluation maps are now *not* necessarily homomorphisms. For instance, over the Hamiltonian ring $\mathbf{H}$, in $\mathbf{H}[X]$ we find

$$(x+i)(x-i) = x^2 + 1$$

yet

$$(j+i)(j-i) = 2k \quad \text{and} \quad j^2 + 1 = 0.$$

In fact, for $x \in A$, $\mathrm{ev}_x : A[X] \to A$ is a homomorphism if and only if $x \in Z(A)$, since if ev_x is a homomorphism, then for any $a \in A$, the polynomial X, times the constant a, is equal to aX. Thus

$$\mathrm{ev}_x(Xa) = \mathrm{ev}_x(aX) = ax$$

whereas

$$\mathrm{ev}_x(X)\mathrm{ev}_x(a) = xa.$$

Thus $ax = xa$ for any $a \in A$.

If A and B are rings, then each homomorphism $\varphi : A \to B$ extends uniquely to a *reduction* homomorphism from $A[X]$ to $B[X]$ such that for each $a \in A$, the diagram below commutes

$$\begin{array}{ccc} A[X] & \longrightarrow & B[X] \\ \downarrow{\scriptstyle \mathrm{ev}_a} & & \downarrow{\scriptstyle \mathrm{ev}_{\varphi(a)}} \\ A & \xrightarrow{\varphi} & B \end{array}$$

The diagram forces us to define the mapping as

$$a_0 + a_1 X + \cdots + a_n X^N \mapsto \varphi(a_0) + \varphi(a_1)X + \cdots + \varphi(a_n)X^n$$

The most important case of the reduction homomorphism we will consider is when we consider an ideal $\mathfrak{a}$ in A, and then obtain the reduction homomorphism $A[X] \to (A/\mathfrak{a})[X]$; for instance, we can reduce an integer polynomial in $\mathbf{Z}[X]$ modulo some prime p to obtain a polynomial in $\mathbf{Z}_p[X]$. If we abuse notation, also writing $\mathfrak{a}$ for the ideal in $A[X]$ generated by $\mathfrak{a}$, then $A[X]/\mathfrak{a}$ is actually isomorphic to $(A/\mathfrak{a})[X]$. This is because a polynomial $a_0 + \cdots + a_n X^n$ is in the kernel of the reduction map if and only if $a_0, \ldots, a_n \in \mathfrak{a}$, which occurs if and only if $a_0 + \cdots + a_n X^n \in \mathfrak{a}$.

Corollary 3.1. *If $\mathfrak{a} \subset A$ is a prime ideal, then $\mathfrak{a} \subset A[X]$ is a prime ideal.*

3.2 The Euclidean Algorithm

If $f = a_0 + \cdots + a_n X^n \in A[X]$ is a non-zero polynomial with $a_n \neq 0$, we define the *degree* of f, denoted $\deg(f)$, to be n - the largest index with a nonzero coefficient in A. If $f = 0$, we define the degree of f to be $-\infty$. One can think of the degree of a polynomial as a measure of complexity of the corresponding arithmetic structure. The simplest polynomials are the *linear* polynomials, which have degree one. Upping the difficulty gives us the *quadratic* polynomials of degree two, the *cubic* polynomials of degree three, and so on and so forth. Looking at the operations defining polynomial addition and multiplication, it is easy to see that for any $f, g \in A[X]$,

$$\deg(f + g) \leqslant \max(\deg(f), \deg(g)),$$

and provided one of the leading coefficients of f or g is not a zero divisor,

$$\deg(fg) = \deg(f) + \deg(g)$$

thus multiplication of two polynomials always bilinearly magnifies the complexity of the polynomial. The multiplicative identity shows that the degree gives a *filtration* turning $A[X]$ into a *graded algebra*.

Remark. Note that the reason why we define $\deg(0) = -\infty$ is precisely so that these equations continuous to hold even if we do not assume the polynomials involved are nonzero.

Lemma 3.2. *If A is an integral domain, then $A[X]$ is an integral domain, and the units of $A[X]$ are precisely the units of A.*

Proof. The degree formula guarantees that if $fg = 0$, then $\deg(fg) = \deg(f) + \deg(g) = -\infty$, so either $\deg(f) = -\infty$, or $\deg(g) = -\infty$, which implies either $f = 0$ or $g = 0$. Thus $A[X]$ is an integral domain. Now suppose $f, g \in A[X]$ and $fg = 1$. Then $\deg(fg) = \deg(f) + \deg(g) = 0$. Thus $\deg(f) = \deg(g) = 0$, so $f, g \in A$. Thus $U(A[X]) = U(A)$. □

Corollary 3.3. *If A is a commutative ring, then the units of $A[X]$ are precisely polynomials of the form*

$$f = a_0 + a_1 X + \cdots + a_n X^n,$$

where a_0 is a unit of A, and $a_1, \ldots, a_n$ are nilpotent elements of A.

Proof. If a is a nilpotent element of a ring B, and u is a unit of B, then $u + a$ is a unit in B. This is because for any n,

$$(u + a)(u - a)(u^2 + a^2)\ldots(u^{2^n} - a^{2^n}) = u^{2^{n+1}} - a^{2^{n+1}}.$$

For suitably large n, $a^{2^{n+1}} = 0$, so

$$(u + a)(u - a)(u^2 + a^2)\ldots(u^{2^n} - a^{2^n}) = u^{2^{n+1}}$$

is invertible, which implies u is invertible. Thus if we can show that $a_1 X + \cdots + a_n X^n$ is nilpotent in $A[X]$ if $a_1, \ldots, a_n$ are nilpotent, then this will show $a_0 + \cdots + a_n X^n$ is a unit. But the nilpotent elements of a ring form an ideal, and since it is easy to see $a_i X^i$ is nilpotent if a_i is nilpotent, it is obvious that $a_1 X + \cdots + a_n X^n$ is nilpotent. Now suppose f is a unit in $A[X]$. Then for any prime $\mathfrak{p}$, f is a unit in $(A/\mathfrak{p})[X]$, which implies from the previous theorem that $a_i \in \mathfrak{p}$ for all $i > 0$. Thus $a_1, \ldots, a_n$ lies in every prime ideal; but this means that $a_1, \ldots, a_n$ are all nilpotent elements of the ring A. □

One of the most important facts about the degree of a univariate polynomial is that we can perform the Euclidean algorithm on them, which gives the ring $A[X]$ properties analogous to the ring of integers.

Theorem 3.4. *If $f, g \in A[X]$ and the leading coefficient of g is a unit, then there exists polynomials $h, r \in A[X]$ such that*

$$f = gh + r,$$

and $deg(r) < deg(g)$.

Proof. We prove the theorem by induction. If $\deg(f) < \deg(g)$, the theorem is trivial. Otherwise, write

$$f = a_0 + a_1 X + \cdots + a_n X^n \qquad g = b_0 + b_1 X + \cdots + b_m X^m$$

Then

$$\deg(f - a_n b_m^{-1} X^{n-m} g) < \deg(f)$$

so by induction,

$$f - a_n b_m^{-1} X^{n-m} g = hg + r$$

where $\deg(r) < \deg(g)$. But this implies

$$f = (h + a_n b_m^{-1} X^{n-m})g + r$$

so we have found an expansion for f. □

We have found that every polynomial ring is 'almost' a Euclidean domain, except that the expansion properties of the domain only hold for polynomials whose leading term is invertible. In particular, this means that if A is a field k, then *any* nonzero polynomial $A[X] = k[X]$ satisfies this property, and so the general argument for Euclidean domains gives the following corollary.

Corollary 3.5. *If k is a field, then $k[X]$ is a principal ideal domain.*

Proof. Let $\mathfrak{a}$ be a nonzero ideal of $k[X]$, and let g be a nonzero element of $\mathfrak{a}$ with smallest degree. Given any $f \in \mathfrak{a}$, the Euclidean algorithm enables us to find h and r with $f = gh + r$, where $\deg(r) < \deg(g)$. Since $r = f - gh \in \mathfrak{a}$, we conclude that $r = 0$. Thus we conclude that $\mathfrak{a} = (f)$. □

Corollary 3.6. *If k is a field, then $k[X]$ is factorial.*

Remark. For any nonzero ideal $\mathfrak{a}$ in $k[X]$, then $\mathfrak{a}$ is a generated by any f in $\mathfrak{a}$ where f is nonzero and has the smallest possible degree in $\mathfrak{a}$. But we can choose a unique generator by requiring $\mathfrak{a}$ to be monic, since if f and g are monic polynomials of smallest degree in $\mathfrak{a}$, then $\deg(f - g) < \deg(f)$, so we conclude $f - g = 0$, hence $f = g$.

Theorem 3.7. *Let A be an integral domain, and fix $f \in A[X]$.*

- *If $f(a) = 0$, then $X - a$ divides f.*

- *If $f \neq 0$, then f can have at most $\deg(f)$ roots in F.*

Proof. Since $X-a$ has degree 1, we may use the Euclidean alogrithm to find a polynomial $g \in A[X]$ and $r \in A$ such that we may write $f = g(X-a) + r$. Since $f(a) = r$, we conclude $r = 0$. If we have n distinct roots $a_1, \ldots, a_n$, we may apply induction to write $f = r(X-a_1)\ldots(X-a_n)$. The degree of the left hand side is n, and the degree of the right hand side is $n + \deg(r)$, hence $\deg(r) = 0$, so r is a nonzero constant. If $b \neq a_i$ for any i, then

$$f(b) = r \cdot (b-a_1)\ldots(b-a_n) \neq 0$$

Thus f can have at most n roots. □

Corollary 3.8. *If A is an integral domain, $f \in A[X]$, and $f(a) = 0$ for infinitely many $a \in A$, then $f = 0$.*

For finite integral domains, non-zero polynomials may still induce the zero function. For instance, if k is a finite field of order n, then $x^n = x$ for all $x \in k$. Thus the polynomial $X^n - X$ induces the zero function on k, yet $X^n - X$ is not formally the zero polynomial. This causes problems in certain problems where we must find a nonzero polynomial of low degree vanishing over a set of points in some k^n in an interesting way. Fortunately, the next lemma shows that these techniques generalize provided we can bound the degree of the nonzero terms.

Lemma 3.9. *Let k be a finite field with n elements. If $f \in k[X]$ induces the zero function on k, and $\deg(f) < n$, then $f = 0$.*

Proof. If f is nonzero but induces the zero function on k, then we obtain a contradiction by factoring out the linear terms corresponding to each element of k, which contradicts the degree of f. □

Now suppose k is a finite field with n elements, and $f \in k[X]$. Given f, the *reduced form* of f is a polynomial $g \in k[X]$ with $\deg(g) < n$ and $f(x) = g(x)$ for all $x \in k$. Repeatedly using the identity $x^n - x = 0$ in k shows that reduced forms always exist, and the above lemma shows they are unique.

Theorem 3.10. *If A is an integral domain, every finite subgroup of A^* is cyclic.*

Proof. Let G be such a subgroup. Since G is abelian, we can write it as the product of p groups, and so it suffices to prove the theorem by proving that a p-subgroup of A^* is abelian. Let x be an element of G of maximal period p^r. Then all elements of G are roots of the polynomial $X^{p^r} - 1$. But we know that there can only be at most p^r roots, and so G consists precisely of these roots, which are $x, x^2, \ldots, x^{p^r}$. □

Example. *If k is a finite field, then k^* is cyclic. In particular, $\mathbf{Z}_p^*$ is cyclic for each prime p; however, the proof above is not constructive, so we do not actually have efficient ways of* finding *generators for $\mathbf{Z}_p^*$ when p is a large prime.*

Example. *For each n, the set μ_n of n'th roots of unity, i.e. solutions to the equation $X^n - 1$ over $\mathbf{C}$, forms a finite subgroup of $\mathbf{C}^*$, and is therefore cyclic. The set $\mu = \bigcup_{n=1}^{\infty} \mu_n$ is a group, the group of all roots of unity. More generally, over any algebraically complete field k, we can conisder the groups $\mu_n(k)$ and $\mu(k)$. If k is a finite field with n elements, then $k^* = \mu(k)$, since all elements of k^* are roots of the polynomial $X^{n-1} - 1$.*

3.3 Algebraic and Trancendental Elements

Given a ring B with a subring A, we say $b \in B$ is *algebraic* over A if there is a nonzero polynomial $f \in A[X]$ such that $f(b) = 0$. Otherwise, b is called *trancendental*. It is fairly easy to show a particular element of a ring is algebraic (e.g. $\sqrt{2}$ is algebraic over $\mathbf{Q}$, since we can set $f(X) = X^2 - 2$), but it is often very difficult to show that an element of a ring is trancendental. We know that π and e are trancendental over $\mathbf{Q}$, but the proof is a difficult analytical argument. It is still an open question whether $\pi + e$ and π/e are trancendental; it is not even known whether they are irrational! For multivariate polynomial rings, the situation is even less understood. We say $b_1, \ldots, b_n \in B$, we say these elements are *algebraically independent* over A if there is no polynomial $f \in A[X_1, \ldots, X_n]$ with $f(b_1, \ldots, b_n) = 0$.

3.4 Multivariate Polynomials

We can study multivariate expressions in a commutative ring by using multivariate polynomials. Given n variables $X_1, \ldots, X_n$, we can consider

expressions of the form

$$\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$$

such that only finitely many $a_{i_1,\ldots,i_n}$ are non-zero. The set of all such expressions forms a ring over A, denoted $A[X_1,\ldots,X_n]$. Let us list some commonly used properties of this ring.

- One can reduce multi-dimensional polynomial rings to univariate polynomial rings by noticing that

  $$A[X_1,\ldots,X_n] = A[X_1,\ldots,X_{n-1}][X_n],$$

  because every polynomial can be uniquely written as $\sum f_k X_n^k$ for some $f_k \in A[X_1,\ldots,X_{n-1}]$, formed by factoring out the right powers of f_k.

- Given a tuple $b = (b_1,\ldots,b_n) \in B^n$, where A is a subring of a ring B, we can consider an evaluation morphism $\mathrm{ev}_b : A[X_1,\ldots,X_n] \to B$, as in the one-dimensional case.

- Given a homomorphism $f : A \to B$, there is a unique homomorphism from $A[X_1,\ldots,X_n]$ to $B[X_1,\ldots,X_n]$ causing the evaluation diagrams to commute.

- The polynomials $X_1^{i_1}\ldots X_n^{i_n}$ are known as *primitive monomials*. We define the degree of this primitive polynomial to be $i_1+i_2+\cdots+i_n$, and we define the degree of a general polynomial to be the maximal degree of the primitive polynomials in the expansion of the polynomial which have non-zero coefficients.

- Alternatively, if we want to focus on a particular variable, we define the degree of f with respect to X_n to be the degree of f viewed as an element of $A[X_1,\ldots,X_{n-1}][X_n]$.

- A polynomial $f \in A[X_1,\ldots,X_n]$ is *homogenous* of degree m if the only monomials $X^{i_1}\ldots X^{i_n}$ occuring in f satisfy $i_1+\cdots+i_n = m$. If A is a subring of B, and $u, t_1,\ldots,t_n \in B$, then we find

  $$f(ut_1,\ldots,ut_n) = u^m f(t_1,\ldots,t_n).$$

Homogenous polynomials are precisely those polynomials satisfying this equation, provided that there exists algebraically independent $u, t_1, \ldots, t_n$ in B over A, because then the fact that $f(ut_1, \ldots, ut_n) = u^m f(t_1, \ldots, t_n)$ implies

$$f(YX_1, \ldots, YX_n) = Y^m f(X_1, \ldots, X_n)$$

and looking at the terms in this expansion shows all monomials must have the same degree.

Just as in the univariate case, a nonzero multivariate polynomial cannot have too many zeroes.

Corollary 3.11. *Let $f \in A[X_1, \ldots, X_n]$, where A is an integral domain. If there exists infinite sets $S_1, \ldots, S_n \subset A$ such that $f(a_1, \ldots, a_n) = 0$ for each $a_1 \in S_1, \ldots, a_n \in S_n$, then $f = 0$.*

Proof. We prove by induction on n, the case $n = 1$ having already been proven. For each $a \in A$, we have an evaluation homomorphism

$$\mathrm{ev}_a : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_{n-1}]$$

obtained by setting $X_n = a$. By induction, we know $\mathrm{ev}_a(f) = 0$ for each $a \in S_n$. Now write

$$f = \sum_{i_1, \ldots, i_{n-1}} \left(\sum_{i_n} a_{i_1 \ldots i_n} X_n^{i_n} \right) X_1^{i_1} \ldots X_{n-1}^{i_{n-1}}$$

Since $\mathrm{ev}_a(f) = 0$ for each $a \in S_n$, then for each $i_1, \ldots, i_{n-1}$, the polynomial $\sum_{i_n} a_{i_1 \ldots i_n} X_n^{i_n}$ has infinitely may zeroes, and thus vanishes identically. Thus $f = 0$, completing the induction. □

For finite fields we obtain a similar result after applying a reduction.

Lemma 3.12. *Suppose k is a finite field with m elements. If $f \in k[X_1, \ldots, X_n]$ induces the zero function on k^n and has degree less than m in each variable $\{X_1, \ldots, X_n\}$, then $f = 0$. The ideal of functions vanishing on k^n is precisely*

$$(X_1^m - X_1, \ldots, X_n^m - X_n).$$

Proof. We proceed by induction. Write

$$\mathfrak{a} = (X_1^m - X_1, \ldots, X_n^m - 1).$$

Suppose $f \in k[X_1, \ldots, X_n]$ has degree less than m in each variable and induces the zero function on k^n. Write

$$f = \sum_{i_1, \ldots, i_{n-1}} \left(\sum_{i_n} a_{i_1 \ldots i_n} X_n^{i_n} \right) X_1^{i_1} \ldots X_{n-1}^{i_{n-1}}$$

The inductive case applies that for each $i_1, \ldots, i_{n-1}$, the polynomial $\sum_{i_n} a_{i_1 \ldots i_n} X^{i_n}$ induces the zero function on k. Since the degree in i_n is less than m, $a_{i_1 \ldots i_n} = 0$ for all $i_1, \ldots, i_n$. This completes the proof of the first property of this lemma.

Now write

$$\mathfrak{a} = (X_1^m - X_1, \ldots, X_n^m - X_m)$$

If $f \in k[X_1, \ldots, X_n]$ induces the zero function on k^n, then it is certainly equivalent modulo $\mathfrak{a}$ to a polynomial $g \in k[X_1, \ldots, X_n]$ with degree less than m in each variable. But then g induces the zero function on k^n, hence $g = 0$, so $f \in \mathfrak{a}$. □

3.5 Polynomials over a Factorial Ring

Let A be an integral domain. If k is the field of fractions of A, then $k[X]$ is a principal ideal domain, hence factorial. One might naturally ask what the relation is between the divisibility theory of $A[X]$ and the divisibility theory of $k[X]$. Normally this can be obtained by 'cancelling denominators' of equations in $k[X]$ to obtain equations in $A[X]$. Clearly we cannot use this fact to conclude $A[X]$ is factorial in general, since if $A[X]$ is factorial, A is factorial.

However, we can use this technique to prove $A[X]$ is factorial if A is factorial, a process we now carry out. Let A be a factorial ring. Since A is an integral domain, we may consider the field of fractions k. We shall show that $f \in A[X]$ is irreducible over $k[X]$ if and only if f is irreducible over $A[X]$, and if the greatest common denominator of the coefficients of f is equal to zero. For each prime $p \in A$, and non-zero $x \in k$, we may

uniquely write $x = p^r u$, where $r \in \mathbf{Z}$, and $p \nmid u$. We define the *order* of x at p to be r, and denote it by $\mathrm{ord}_p(x)$. Just as with polynomials, we have

$$\mathrm{ord}_p(x+y) \geqslant \min\left(\mathrm{ord}_p(x), \mathrm{ord}_p(y)\right) \qquad \mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$$

If $x = 0$, we define $\mathrm{ord}_p(x) = \infty$ so that these identities continue to hold. Any prime $p \in A$ is also a prime in $A[X]$, so we can define $\mathrm{ord}_p(f)$ for each $f \in A[X]$; if $f = a_0 + \cdots + a_n X^n$, then

$$\mathrm{ord}_p(f) = \min\left(\mathrm{ord}_p(a_0), \ldots, \mathrm{ord}_p(a_n)\right).$$

If A is a factorial ring, we define the *content* $\mathrm{cont}(f) \in A$ of a non-zero $f \in A[X]$ to be the greatest common denominator of the coefficients of f (technically, we must interpret $\mathrm{cont}(f)$ as a coset of A modulo it's units, but we abuse notation here). If we pick a prime p from each coset of primes identified up to units, then

$$\mathrm{cont}(f) = \prod_p p^{\mathrm{ord}_p(f)}.$$

If $f = 0$, define $\mathrm{cont}(f) = 0$. Then the content is unique up to a unit in A. We may always write $f = \mathrm{cont}(f)g$, where g is a polynomial in $A[X]$ with unit content (such polynomials are known as *primitive*.

Lemma 3.13 (Gauss)**.** *Let A be a factorial ring, and k it's field of fractions. Then for $f, g \in k[X]$, $cont(fg) = cont(f) \cdot cont(g)$.*

Proof. Assume without loss of generality that f and g have unit content. Then it suffices to prove fg has unit content. Let $p \in A$ be a prime, denote $A/(p)$ by B, and consider the reduction homomorphism $\varphi : A \to B$, which extends to a map $\varphi : A[X] \to B[X]$. Since $\varphi(f)$ and $\varphi(g)$ are nonzero polynomials, and p is prime, $\varphi(fg)$ is a nonzero polynomial. □

Corollary 3.14. *Suppose A is a factorial ring, let $f \in A[X]$ be primitive, and let k be the field of fractions of A. Then f is irreducible in $A[X]$ if and only if it is irreducible in $k[X]$.*

Proof. Suppose $f \in A[X]$ is primitive and irreducible over $A[X]$, and suppose $f = gh$, where $g, h \in k[X]$. Then we can find primitive polynomials $g_0, h_0 \in A[X]$ and $a_0, b_0 \in A$ such that $a_0 g = g_0$, $b_0 h = h_0$. If $c = a_0 b_0$, then $cf = g_0 h_0$. But then since g_0 and h_0 is primitive, we conclude by Gauss'

lemma that c is a unit in A. Thus $f = (g_0/c)h_0$, where $g_0/c, h_0 \in A[X]$, and so we conclude that either g_0 or h_0 is a unit in $A[X]$, and thus either g or h is a unit in $k[X]$.

Conversely, if $f \in A[X]$ is primitive and irreducible over $k[X]$, and if $f = gh$ for $g, h \in A[X]$, then either g or h is a unit in $k[X]$,which implies that either g or h is a constant. Since $\text{cont}(g)\text{cont}(h) = 1$, this implies that either g or h is a unit in A. □

Corollary 3.15. *If A is factorial, then $A[X_1, \ldots, X_n]$ is factorial.*

Proof. We just prove that $A[X]$ is factorial if A is, from which the general theorem holds by induction. The existence of a factorization is quite easy to show. Let k be the field of fractions of A. If $f \in A[X]$, we may write

$$f = g_1 \cdots g_n$$

where g_n are irreducible elements of $k[X]$. Now write $g_i = a_i g_i'$, where g_i' is a primitive polynomial in $A[X]$. Thus $f = (a_1 \ldots a_n) g_1' \ldots g_n'$. Each g_i' is an element of $A[X]$ which is irreducible over $k[X]$ and has unit content, so it is irreducible over $A[X]$. We may write

$$a_1 \ldots a_n = p_1^{k_1} \ldots p_m^{k_m}$$

where each p_i is an irreducible element of A (and thus irreducible over $A[X]$). Thus

$$f = p_1^{k_1} \ldots p_m^{k_m} g_1' \cdots g_n'$$

has been written as a product of irreducible elements in $A[X]$. If we have two different factorizations

$$p_1^{k_1} \ldots p_m^{k_m} g_1 \ldots g_n = q_1^{l_1} \ldots q_r^{l_r} h_1 \ldots h_t$$

Then by unique factorization in $k[X]$, we must have $t = n$, and after some rearranging, $f_i = u_i g_i$, for some nonzero $u_i \in k$. But since f_i and g_i are primitive, we may assume without loss of generality that u_i is a unit in A. Cancelling out appropriate factors, we conclude that

$$p_1^{k_1} \ldots p_m^{k_m} = (u_1 q_1^{l_1}) \ldots (u_r q_r^{l_r}),$$

and we may now apply unique factorization in A. □

Note that for $n \geqslant 2$, the ring $k[X_1,\ldots,X_n]$ is not principal. Indeed (X,Y) is an ideal in $K[X,Y]$ which cannot be principal, for the greatest common divisor of X and Y is a unit. Thus the fact that these rings are factorial is truly a novel part of the proof above.

Corollary 3.16. *Let $f \in k[X_1,\ldots,X_n]$, and suppose we can find irreducibles $f_1,\ldots,f_N$ and integers $k_1,\ldots,k_N > 0$ such that $f = f_1^{k_1}\ldots f_N^{k_N}$, so*

$$k[X_1,\ldots,X_n]/(f) \cong k[X_1,\ldots,X_n]/(f_1)^{k_1} \times \cdots \times K[X_1,\ldots,X_n]/(f_N)^{k_N}.$$

Proof. The ideals $(f_i^{k_i})$ and $(f_j^{k_j})$ are coprime since they have no common factor. Thus

$$(f) = (f_1^{k_1})\ldots(f_N)^{k_N} = (f_1^{k_1}) \cap \cdots \cap (f_N^{k_N}),$$

and so we can apply the Chinese remainder theorem. □

3.6 Criterion for Irreducibility

It is actually quite tricky to determine whether a given polynomial is irreducible. For instance, X^4+4 does not have any roots in $\mathbf{Q}$, yet X^4+4 is reducible,

$$X^4+4 = (X^2+2X+2)(X^2-2X+2)$$

Some techniques can be used to determine when a polynomial is irreducible. We begin with a powerful result for polynomials over factorial rings.

Theorem 3.17 (Integral Root Test)**.** *Let A be a factorial ring, and k its quotient field. Let*

$$f = a_0 + a_1X + \cdots + a_nX^n$$

Suppose $f(b/d) = 0$, where b and d are relatively prime. Then b divides a_0, and d divides a_n. In particular, if $a_n = 1$, then the only roots of f are in A.

Proof. We have

$$a_0 + a_1(b/d) + \cdots + a_n(b/d)^n = 0$$

Then

$$d^n a_0 + a_1 b d^{n-1} + \cdots + a_n b^n = 0$$

which implies

$$b(a_1 d^{n-1} + \cdots + a_n b^{n-1}) = -d^n a_0$$

since b does not divide d, b does not divide d^n, and thus b divides a_0. Similarily, by factoring out d, we find d divides a_n. □

Example. *The polynomial $X^3 - 3X - 1$ is irreducible in $\mathbf{Z}[X]$. If the polynomial was reducible, it would have an integer root. But the integral root test implies that the only possible roots are either $+1$ or -1. Neither gives a root, completing the proof.*

Example. *For any prime $p \in \mathbf{Z}$, the polynomials $X^2 - p$ and $X^3 - p$ are irreducible in $\mathbf{Z}[X]$. To see this, they would only be reducible if they had an integer root. But the only possible integer roots are either p or $-p$ by the integral root test, completing the proof.*

Another way to prove a polynomial is irreducible is to reduce the polynomial's coefficients modulo an ideal, detailed in the next proposition. In the case $\mathbf{Z}[X]$, we reduce modulo a prime to obtain an element of $\mathbf{F}_p[X]$, and we can easily check this polynomial's properties since $\mathbf{F}_p$ is finite.

Theorem 3.18 (Reduction Criterion)**.** *Let A and B be integral domains and consider a surjective homomorphism $\varphi : A \to B$. If $f \in A[X]$, $\varphi(f)$ has the same degree in f, and $\varphi(f)$ cannot be factored into two polynomials of smaller degree in B, then f is irreducible.*

Example. *Consider the polynomial $X^2 + XY + 1 \in \mathbf{Z}[X, Y]$. View $\mathbf{Z}[X, Y]$ as $\mathbf{Z}[Y][X]$. Let $\phi : \mathbf{Z}[X, Y] \to \mathbf{Z}[X]$ be the homomorphism obtained by setting $Y = 0$. Then $\phi(X^2 + XY + 1) = X^2 + 1$ has the same degree in X. Moreover, $X^2 + 1$ is irreducible in $\mathbf{Z}[X]$, and so $X^2 + XY + 1$ is irreducible in $\mathbf{Z}[X, Y]$ by the reduction criterion.*

Theorem 3.19 (Eisenstein)**.** *Let A be an integral domain, and let $\mathfrak{a}$ be a prime ideal. Consider a polynomial*

$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in A[X],$$

with $a_0, \ldots, a_{n-1} \in \mathfrak{a}$, but $a_0 \notin \mathfrak{a}^2$. Then f is irreducible in $A[X]$.

Proof. Let $\phi : A \to A/\mathfrak{a}$ denote the reduction homomorphism. If $f = gh$ for $g, h \in A[X]$, then $X^n = \phi(f) = \phi(g)\phi(h)$. Since $(A/\mathfrak{a})$ is an integral domain,

the only divisors of X^n in $(A/\mathfrak{a})[X]$ are powers of X^n multiplied by a unit, so $\phi(g) = tX^m$, $\phi(h) = t^{-1}X^l$ for some $t \in U(A/\mathfrak{a})$, where $m + l = n$. If $m, l > 0$, this gives a contradiction, for it implies $g(0), h(0) \in \mathfrak{a}$, and thus $a_0 = g(0)h(0) \in \mathfrak{a}^2$. This we may assume without loss of generality that $m = 0$. But then $\deg(h) = n$, $\deg(g) = 0$. Thus g is a constant, and since f is monic, this implies that g is actually an element of $U(A)$. □

Example. *Eisenstein's criterion's can often be used to determine when the polynomial $X^n - a \in \mathbf{Z}[X]$ is irreducible, i.e. when some prime p divides a, but p^2 does not. This shows $X^n - 6$ and $X^n - 4$ are irreducible. On the other hand, this cannot detect that $X^3 - 8 = (X - 2)(X^2 + 2X + 4)$ is irreducible.*

Example. *The polynomial $X^{p-1} + \cdots + X + 1$ is irreducible in $\mathbf{Q}$ if p is prime. Consider the transformation $X = Y + 1$. The transformation preserves irreducibility, since it is really an isomorphism of $\mathbf{Q}[X]$. Then*

$$(Y+1)^{p-1} + \cdots + (Y+1) + 1 = \frac{(Y+1)^p - 1}{Y} = \sum_{k=0}^{p-1} \binom{p}{k+1} Y^k$$

All coefficients of this polynomial are divisible by p except for the higher order term, which is equal to one, and the lowest term is $\binom{p}{1} = p$, so Eisenstein's criterion tells us the polynomial is irreducible.

Example. *Let k be a field, and consider the field of rational functions $k(X)$. The polynomial $Y^n - X$ is irreducible in $k(X)[Y]$. Note first that $Y^n - X$ has content one with respect to $k[X]$, so $Y^n - X$ is irreducible over $k(X)[Y]$ if and only if it is irreducible over $k[X][Y]$. But over $k[X][Y]$ we may apply Eisenstein's criterion, since X is a prime in $k[X]$, to conclude that $Y^n - X$ is irreducible.*

3.7 Symmetric Functions

Let A be a commutative ring, and consider the polynomial ring $A[t_1, \ldots, t_n]$. Given a family of symmetries acting on $A[t_1, \ldots, t_n]$, the field of algebraic invariant theory tries to classify the polynomials invariant according to this symmetry. Here we study one particular family of invariant polynomials, namely those invariant under permutations of variables.

Each element $\pi \in S_n$ gives rise to an isomorphism of $A[t_1, \ldots, t_n]$ by mapping t_i to $t_{\pi(i)}$. Thus we obtain a left group action of S_n on $A[t_1, \ldots, t_n]$

(moreover, this action is A-linear, so we have a *representation* of S_n). A polynomial f is *symmetric* if $\pi f = f$ for all $\pi \in S_n$. The symmetric polynomials form a subring of $A[t_1,\dots,t_n]$, and our goal is classify this subring.

Consider a new variable X and consider the polynomials $s_1,\dots,s_n \in A[t_1,\dots,t_n]$ such that

$$(X-t_1)\dots(X-t_n) = X^n - s_1X^{n-1} + \cdots + (-1)^n s_n.$$

Then, for instance,

$$s_1 = t_1 + \cdots + t_n \quad \text{and} \quad s_n = t_1 \dots t_n.$$

These are the *elementary symmetric polynomials*. Our aim will be show that every symmetric polynomial can be uniquely written as a polynomial in $s_1,\dots,s_n$. As a temporary definition, define the *weight* of a monomial $x_1^{k_1}\dots x_n^{k_n}$ to be $k_1 + 2k_2 + \cdots + nk_n$, and the weight of a general polynomial $g \in A[x_1,\dots,x_n]$ to be the maximum weight of a monomial which has an associated nonzero coefficient.

Theorem 3.20. *For any symmetric polynomial f of degree d. Then there exists $g \in A[x_1,\dots,x_n]$ with weight less than or equal to n such that*

$$f(t_1,\dots,t_n) = g(s_1,\dots,s_n).$$

Proof. Let f be a symmetric polynomial of degree n. We prove the theorem by double induction on d and n. The theorem is clear for $d = 1$, since any symmetric linear polynomial is equal to $a_0 + a_1s_1$ for some $a_0, a_1 \in A$. The theorem is also obvious for $n = 1$. Thus we may now assume the theorem has been proved for $n-1$ variables, and n variables of degree less than d. Then $f(t_1,\dots,t_{n-1},0) \in A[t_1,\dots,t_{n-1}]$ is symmetric, so there exists a polynomial $g \in A[x_1,\dots,x_{n-1}]$ such that $f(t_1,\dots,t_{n-1},0) = g(s'_1,\dots,s'_{n-1})$, where $s'_1,\dots,s'_{n-1}$ are the elementary symmetric polynomials in $n-1$ variables and g has weight at most d. If we consider the polynomial

$$f_1(t) = f(t) - g(s_1(t),\dots,s_{n-1}(t))$$

then $f_1(t)$ has degree less than or equal to d, is symmetric, and

$$f_1(t_1,\dots,t_{n-1},0) = 0.$$

This implies that f_1 is divisible by t_n. But then by symmetry, f_1 is divisible by s_n. Thus we can write $f_1 = s_nf_2$, where f_2 is symmetric of degree $d-n$. But then we can apply induction to f_2. □

Remark. If f is a homogenous, symmetric polynomial of degree d, a very similar inductive argument shows that f can be written as a polynomial $g(s_1,\dots,s_n)$ with each monomial in g having weight d.

Using this claim, we can now show that $s_1,\dots,s_n$ are algebraically independent in $A[t_1,\dots,t_n]$. Indeed, if they weren't independant, there would be a non-zero polynomial $f \in A[x_1,\dots,x_n]$ of smallest degree such that $f(s_1,\dots,s_n) = 0$. Consider $f_i \in A[x_1,\dots,x_{n-1}]$ such that

$$f = f_0 + f_1 x_n + \cdots + f_m x_n^m.$$

If $t_n = 0$, then $f(s_1,\dots,s_n) = f_0(s_1,\dots,s_{n-1}) = 0$. Applying induction, we conclude $f_0 = 0$. But this means x_n divides f, hence $f = x_n g$. Since g has degree less than f, $g(s_1,\dots,s_n) \neq 0$, which implies

$$f(s_1,\dots,s_n) = s_n g(s_1,\dots,s_n) \neq 0,$$

giving a contradiction. Thus $s_1,\dots,s_n$ are algebraically independant.

Example. *Let* $f(X) = (X - t_1)\dots(X - t_n) \in A[t_1,\dots,t_n][X]$. *Set*

$$\delta = \prod_{i<j} (t_i - t_j),$$

and then take $\Delta = \delta^2$. *Since* $\Delta \in \mathbf{Z}[t_1,\dots,t_n]$. *Then* Δ *is a symmetric polynomial, the* discriminant *of* f, *and thus can be written as an element of* $\mathbf{Z}[s_1,\dots,s_n]$. *Let us consider* Δ *in the special case of lower degree polynomials. For instance, if* $f = X^2 + bX + c$, *then* $t_1 + t_2 = -b$ *and* $t_1 t_2 = c$, *so we conclude that*

$$\Delta = (t_1 - t_2)^2 = t_1^2 + t_2^2 - 2t_1 t_2 = b^2 - 4c.$$

Let us now consider the polynomial $f = X^3 + aX + b$. *Then the discriminant is a homogenous polynomial of weight* 6 *in* a *and* b, *and so there exists* $u, v \in \mathbf{Z}$ *such that* $\Delta = ua^3 + vb^2$. *To calculate* u *and* v, *it suffices to consider some particular, easily calculatable examples. If* $f = X^3 - X = X(X-1)(X+1)$, *since we find here that* $\Delta = 4$, $a = -1$ *and* $b = 0$, *so we find* $u = -4$. *Next, consider the polynomial* $f = X^3 - 1$. *Then* $a = 0$, $b = -1$, *and if* ω *is a primitive root of unity, then*

$$\Delta = (1-\omega)^2(1-\overline{\omega})^2(\omega - \overline{\omega})^2 = 3^2(-3)^2 = -27.$$

Thus we find $v = -27$, *and so* $\Delta = -4a^3 - 27b^2$.

Example. *Let $f, g \in A[x]$ be two polynomials of degree n and m respectively. For each i, let $A_i[x]$ be the polynomials with degree less than i. Then we have an A-linear map from $A_n[x] \times A_m[x]$ to $A_{n+m}[x]$ given by the map*

$$(\phi, \psi) \mapsto g\phi + f\psi.$$

Both $A_n[x] \times A_m[x]$ and $A_{n+m}[x]$ have the same dimension, so we can consider the determinant, called the resultant *of f and g, denoted $Res(f,g)$. One can also write $Res(f,g)$ as the determinant of a matrix whose entries are coefficients of f and g. Applying Cramer's rule, there exists (ϕ, ψ) such that $g\phi + f\psi = Res(f,g)$. Carrying out the expansion of the resultant, we see $Res(f,g)$ is an integer polynomial in the coefficients of f and g.*

We note that $Res(f,g)$ is independant of a field extension. In particular, if K is a field contained in a larger field L such that $f, g \in K[X]$ have a common root u in L, then for any (ϕ, ψ), $g\phi + f\psi$ has a zero at u, so that $Res(f,g) = 0$.

Since $Res(f,g)$ is a polynomial in the coefficients of f and g, it should be expressible as a symmetric polynomial in the zeroes of f and g. If we know $f = (X - t_1)\dots(X - t_n)$ and $g = (X - s_1)\dots(X - s_m)$, then applying the principle above repeatedly verifies that

$$Res(f,g) = \prod_{i=1}^{n}\prod_{j=1}^{m}(t_i - s_j).$$

One can also verify this by induction, combined with the homogeneity

$$Res(tf,g) = t^n Res(f,g) \quad \text{and} \quad Res(f,tg) = t^m Res(f,g).$$

which also verifies that if $f = a(X - t_1)\dots(X - t_n)$ and $g = b(X - s_1)\dots(X - s_m)$, then

$$Res(f,g) = a^n b^m \prod_{i=1}^{n}\prod_{j=1}^{m}(t_i - s_j).$$

We also obtain the formula

$$Res(f,g) = b^m \prod_{i=1}^{m} g(t_i) = (-1)^{nm} a^n \prod_{j=1}^{m} f(s_j).$$

Thus we can verify that

$$Res(f,f') = a^{n-1} \prod_{i=1}^{n} f'(t_i)$$

and

$$Res(f, f) = (-1)^{n(n-1)/2} a\Delta.$$

Thus the resultant is connected directly to the discriminant.

Chapter 4

Modules

All groups are really sets of bijective maps in disguise. Regardless of the complex nature that grants us a specific group, we can still relate it back to some symmetric group, by Cayley's theorem. This leads to the study of group actions. It turns out that all rings can be seen as a set of endomorphisms over an abelian group. The counterpart to a group action on a G-set is then a ring action on an A-module. A representation of a ring A on an abelian group M is a ring homomorphism from A into $\mathrm{Hom}(M)$. If such a representation is fixed, we can define a 'scalar multiplication' structure on M by elements of A, for $x \in M$ and $a \in A$, letting ax denote the action of a on x via the representation. We obtain the relations

$$a(x+y), \quad (ab)x = a(bx), \quad (a+b)x = ax + bx,$$

and

$$1 \cdot x = x.$$

Conversely, any multiplication map from $A \times M$ to M satisfying these properties induces a representation of A on $\mathrm{Hom}(M)$, and we call any such M with a fixed scalar multiplication a (left) A *module*.

A *homomorphism* $f : M \to N$ between two A modules M and N is a homomorphism of abelian groups satisfying the additional requirement that $f(ax) = af(x)$ for each $x \in M$ and $a \in A$. The family of all morphisms between M and N is denoted $\mathrm{Hom}_A(M,N)$, or just $\mathrm{Hom}(M,N)$ if the underlying ring is obvious. Thus for each ring A, we have a category Mod_A of A-modules. It is simple to verify that $\mathrm{Hom}(M,N)$ is an abelian group; if A is commutative, then $\mathrm{Hom}(M,N)$ is even an A module if we define

$(af)(x) = a \cdot f(x)$, since we then find that

$$(af)(sx) = af(sx) = asf(x) = saf(x) = s(af)(x).$$

It is easy to see from this calculation that if A is not commutative, then af is a homomorphism of abelian groups which may not be a homomorphism of A modules. On the other hand, $\mathrm{End}(M)$ is always a ring, which forms an A algebra when A is commutative.

A *submodule* of an A module M is a subset N of M such which forms an abelian subgroup and such that $ax \in N$ for each $x \in N$ and $a \in A$. Submodules are the natural objects to quotient by; if N is a submodule of an A module M, then the quotient group M/N naturally has the structure of an A module. The natural analogues of the isomorphism theorems for abelian groups remain true for modules. In particular, one can use the first isomorphism theorem to pass through a quotient by the kernel of a homomorphism. The second isomorphism theorem implies if N_1 and N_2 are submodules, then $N_1 + N_2$ and $N_1 \cap N_2$ are submodules, and $N_1/(N_1 \cap N_2)$ is isomorphic to $(N_1 + N_2)/N_2$. For a submodule N of a module M, the third isomorphism theorem gives a correspondence with submodules of M/N and submodules of M containing N.

Example. *If A is a ring, it is a module over itself. The same is true of A^n, known as the* free A module of rank n. *Submodules of A are precisely* left ideals *of A, i.e. subrings $\mathfrak{a}$ of A which are closed under left multiplication by elements of A. The homomorphisms in $Hom(A^n, A^m)$ can be identified with the family $M_{n,m}(A)$ of $n \times m$ matrices over A, where compositions of homomorphisms act the same way as matrix multiplication does.*

Example. *Any abelian group M has the structure of a $\mathbf{Z}$ module, for we may define $nx = x + \cdots + x$. A $\mathbf{Z}$ module homomorphism is just a homomorphism of abelian groups, so that the category* $\mathsf{Mod}_{\mathbf{Z}}$ *is really just the category* Ab *in disguise.*

Example. *A module over a field is called a* vector space, *and the study of such modules forms the field of* linear algebra. *But much of the basis theory remains true in the context of modules over division rings. Thus we refer more generally to a module over a division ring as a vector space.*

However, even in linear algebra one needs to study more general modules; given a vector space V over a field k and a linear transformation T, one usually

gains deeper information on the structure of T by studying V as a $k[X]$-module, under the action $f \cdot v = f(T)(v)$. The study of invariant subspaces of a vector space under a linear transformation is really a discussion of the submodules of V, viewed as a $k[X]$ module.

More generally, let A be a commutative ring. If M is an A-module, then we have a homomorphism $\phi : A \to End(M)$. If $T \in End(M)$ is fixed, then ϕ extends to a homomorphism from $A[X]$ to $End(M)$ mapping X to T. Thus M naturally has the structure of an $A[X]$ module, where multiplication by $f \in A[X]$ acts as the endomorphism $f(T)$. Submodules of $A[X]$ are precisely the T invariant submodules of M. A more general fact is that $A[X]$ module structures on an A module M are in one to one correspondence with elements of $End(M)$, so studying a general endomorphism of an A module, when A is commutative, is exactly the same as studying an $A[X]$ module.

Example. *Let us consider some deeper examples of modules over polynomial rings. If $C^\infty(\mathbf{R}^d)$ is the real vector space of real-valued infinitely differentiable functions, then for each $i \in \{1, \ldots, n\}$ we can consider the linear operator*

$$D_i f = \frac{\partial f}{\partial x^i}.$$

This induces a natural $\mathbf{R}[X_1, \ldots, X_d]$ module structure such that $C^\infty(\mathbf{R}^d)$, where a polynomial corresponds to a constant coefficient differential operator. Similarily, we can consider the linear operators

$$(M_i f)(x) = 2\pi x_i f(x)$$

which induce a separate $\mathbf{R}[X_1, \ldots, X_d]$ module structure. The set of Schwartz functions $\mathcal{S} \subset C^\infty(\mathbf{R}^d)$ forms a $\mathbf{R}[X_1, \ldots, X_d]$ submodule under both representations. Moreover, the Fourier transform gives an isomorphism between both representations of $\mathbf{R}[X_1, \ldots, X_d]$ on $\mathcal{S}$, since for $f \in \mathcal{S}$,

$$M_i \hat{f} = \widehat{D_i f}.$$

This is a powerful algebraic relation exploited in Harmonic anlaysis.

In operator theory, one studies bounded linear operators acting on Hilbert spaces H. If H is a complex Hilbert space and T is a bounded linear operator on H, then H naturally has the structure of a $\mathbf{C}[X]$ module, where each polynomial f acts as the bounded linear operator $f(T)$. The study of the spectral theory of such operators shows that we can actually extend this operation to give H the

natural structure of a $\mathcal{O}(\sigma(T))$ module, where $\mathcal{O}(\sigma(T))$ is the ring of functions analytic in a neighbourhood of the spectrum of T, each acting as a bounded linear operator on H. If T is self adjoint, then we can further extend this operation to give H the structure of a $C(\sigma(T))$ module, where each continuous function f on the spectrum of T acts as a bounded linear operator.

Both $k[X]$ and $\mathbf{Z}$ are principal ideal domains. Later on, we will develop a powerful theory which classifies finitely generated modules over principal ideal domains, which generalizes the classification of endomorphisms over a vector space by means of the Jordan normal form, and the classification of finite abelian groups.

Example. *Let A be a commutative ring. Then let $Tor(M)$ denote the submodule of all $x \in M$ such that there is a $a \neq 0$, which is* not *a zero divisor of A, such that $ax = 0$. Such elements are said to* have torsion. *A* torsion module *is a module M such that $Tor(M) = M$, and a module M is called* torsion free *if $Tor(M) = (0)$. If A is an integral domain, the ring $M/Tor(M)$ is always torsion free, and can be viewed as the most general torsion free quotient of M; if $\phi : M \to N$ is a homomorphism, then $\phi(Tor(M)) \subset Tor(N)$, so if A is an integral domain and N is torsion free, then ϕ automatically factors through $M/Tor(M)$ by the first isomorphism theorem.*

Example. *If N is a submodule of an A module M, we let $Ann(N)$ be the set of all $a \in A$ such that $aN = (0)$. Then $Ann(N)$ is a left ideal in A. Conversely, if $\mathfrak{a}$ is a left ideal of A, then we set $Ann(\mathfrak{a})$ to be the set of all $x \in M$ such that $\mathfrak{a}x = (0)$. We note that $Ann(N)$ is contained in $\mathfrak{a}$ if and only if N is contained $Ann(\mathfrak{a})$, so we obtain a Galois connection between left ideals of A and submodules of M. Note that if $\phi : M \to N$ is a homomorphism between two modules, then for any $N \subset M$, $Ann(N) \subset Ann(\phi(N))$, and $\phi(Ann_M(\mathfrak{a})) \subset Ann_N(\mathfrak{a})$.*

Example. *If $\mathfrak{a}$ is a left ideal of a ring A, and M is an A module, then the subgroup $\mathfrak{a}M$ of M generated by elements of the form ax, with $a \in \mathfrak{a}$ and $x \in M$, forms a submodule of M. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are left ideals in A, then we can consider the natural homomorphism*

$$M \to M/\mathfrak{a}_1 M \times \cdots \times \mathfrak{a}_n M$$

which has kernel $\mathfrak{a}_1 M \cap \cdots \cap \mathfrak{a}_n M$. In fact, if the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are coprime, *then this map is surjective, then $\mathfrak{a}_1 M \cap \cdots \cap \mathfrak{a}_n M = (\mathfrak{a}_1 \ldots \mathfrak{a}_n)M$, and so*

$$M/(\mathfrak{a}_1 \ldots \mathfrak{a}_n)M \cong M/\mathfrak{a}_1 M \times \cdots \times M/\mathfrak{a}_n M.$$

Thus we have a version of the Chinese remainder theorem for modules, and this more general version is proved essentially by the same techniques as in the case of the normal Chinese remainder theorem.

Example. *Let A be a ring, and fix $a \in Z(A)$. Given a module M, the set of all ax, for $x \in M$ forms a submodule of M, denoted aM, since $t(ax) = a(tx)$ for $t \in A$. This is no longer true if a is not an element of $Z(A)$. For instance, since*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

On the other hand, for any ring A, the set $e_{11}M_2(A)$ is not a submodule of $M_2(A)$ because

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin e_{11}M_2(A).$$

4.1 Algebras

It is often useful to discuss R-algebras, i.e. R-modules which also have the structure of a ring. To be more precise, if R is a *commutative ring*, then an *associative R-algebra A* is an R-module which is also a ring, and satisfies the identity

$$r(xy) = (rx)y = x(ry)$$

for all $x, y \in A$ and $r \in R$. Thus each element of R acts as a A-linear homomorphism. In particular, this structure induces a ring homomorphism $f : R \to A$ given by $f(r) = r \cdot 1$, since certainly $f(r+s) = f(r) + f(s)$, since

$$f(rs) = (rs) \cdot 1 = (rs) \cdot (1 \cdot 1) = (r \cdot 1) \cdot (s \cdot 1) = f(r)f(s),$$

and since $f(1) = 1 \cdot 1 = 1$. Note the definition implies that f maps R to $Z(A)$. Thus by an R-algebra we mean a ring A and a homomorphism $f : R \to Z(A)$. The most important examples of algebras are k-algebras, where k is a field.

Example. *Let R be a commutative ring. Then the ring of matrices $M_n(R)$ is an R-algebra. Note that in this case the induced morphism $f : R \to M_n(R)$ is injective, since $f(r) = r \cdot I$, where $I \in M_n(R)$ is the identity matrix.*

Example. *If R is a commutative ring and G is a finite group, then $R[G]$ is an R-algebra. In this case R can be identified with the subring of $R[G]$ consisting of $r \cdot e$ with $r \in R$, and where e is the identity of G.*

Example. *If M is a R-module, then we can consider the* tensor algebra

$$T(M) = R \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \dots$$

with multiplicative given by the tensor product. If we consider the quotient of $T(M)$ by the ideal generated from elements of the form $x \otimes y - y \otimes x$, for all $x, y \in M$, then we obtain the symmetric algebra $S(M)$. *If we instead quotient by the ideal generated by elements of the form $x \otimes x$, for $x \in M$, we get the* alternating algebra $A(M)$.

Given two R-algebras A and B, the natural homomorphisms are ring homomorphisms that are also R-module homomorphisms. Note that any ideal in an algebra A is automatically closed under multiplication by R, i.e. any ideal is automatically a submodule. Thus we have a quotient theory in this setting which is equivalent to the setting of ring theories. Of course, the standard isomorphism theorems generalize to this setting. Associative algebras occur naturally in more advanced contexts of algebra, but it is useful to introduce them in this part of the theory for the purpose of examples.

An *A-module* on an R-algebra A is a module M equipped with both A-module and R-module structures, such that

$$(ra)x = r(ax) = a(rx).$$

If A has unity, then the more complicated structure collapses because the action of R embeds in the action of A; that is, if A has unity, an A-module is the same as an A-module viewing A solely as a ring.

4.2 Generators of Modules

Given an A-module M and a set $S \subset M$, we can consider the smallest submodule generated by S, which can be written as

$$AS = \{a_1 s_1 + \dots + a_n s_n : a_1, \dots, a_n \in A, s_1, \dots, s_n \in S\}$$

We say S *spans* the set AS. A module is *finitely generated* if it is generated by a finite set, and *cyclic* if it is generated by a single element.

Remark. If A is non unital things can go very wrong here. For instance, S might not even be a subset of AS. Even weirder things can happen; as a module, A might not even be finitely generated over itself; this happens, for instance, if A is the ring of compactly supported continuous functions on $\mathbf{R}^d$.

It shall be useful to introduce some general constructions. The category of modules is closed under direct products and direct sums. We can construct a direct product for a family of modules $\{M_\alpha : \alpha \in I\}$ by consider the module M of all sequences $\{x_\alpha\}$ such that $x_\alpha \in M_\alpha$ for each α, with addition and multiplication taken componentwise. We must be slightly more careful when constructing direct sums; we take the submodule of $\prod M_\alpha$ consisting of all sequences $\{x_\alpha\}$ such that only finitely many x_α are nonzero. One writes the direct product and direct sum as

$$\bigtimes_\alpha M_\alpha \quad \text{and} \quad \bigoplus_\alpha M_\alpha$$

respectively, and for finite families $M_1, \ldots, M_n$ of modules, we write the direct product and sum as $M_1 \times \cdots \times M_n$ and $M_1 \oplus \cdots \oplus M_n$ (note that in this case, the direct product is equal to the direct sum). One can also produce direct and inverse limits, just as in the case of abelian groups, but we leave this for the reader to construct.

A set S is a *basis* for a module M if every element of M can be *uniquely* written as $a_1 s_1 + \cdots + a_n s_n$ for some $a_1, \ldots, a_n \in A$ and distinct $s_1, \ldots, s_n \in S$, which is equivalent to the condition that for any distinct $s_1, \ldots, s_n$ and $a_1, \ldots, a_n \in A$, $a_1 s_1 + \cdots + a_n s_n = 0$ if and only if $a_1 = \cdots = a_n = 0$. If S is finite, containing n elements, then this implies precisely that M is isomorphic to A^n. More generally, M is isomorphic to the direct sum $\oplus_{s \in S} A$, since elements of M can be written freely as a finite sum of elements from the set S. Over a field, all modules are free, but this is not true for general rings.

Lemma 4.1. *Let V be a vector space over a division ring k. If S generates V, then there exists $T \subset S$ such that T is a basis for V. In particular, every vector space is free.*

Proof. Let $\mathcal{S}$ be the set of all subsets of S which are linearly independant. Then $\mathcal{S}$ is closed under limits; namely if $\{S_i\}$ is a linearly independant

chain, then $\bigcup S_i$ is linearly independant. We conclude from Zorn's lemma that there is a maximal element T in $\mathcal{S}$. For each $x \in V$, we can write

$$x = a_1 s_1 + \cdots + a_n s_n$$

for some $s_1, \ldots, s_n \in S$ and $a_1, \ldots, a_n \in k$. We claim that s_i lies in the vector space generated by T for each i, from which it follows that T generates V and is therefore a basis. If s_i is not an element of T, then $T \cup \{s_i\}$ is linearly dependant. This means that there is a relation in $T \cup \{s_i\}$, i.e. there exists nonzero $u \in k$, and $a_1, \ldots, a_n \in k$ and $t_1, \ldots, t_n \in T$ such that $us_i + a_1 t_1 + \cdots + a_n t_n = 0$. But then $s_i = -u^{-1}(a_1 t_1 - \cdots - a_n t_n)$, so s_i is in the vector space generated by T, which gives a contradiction. Thus T spans V and is therefore a basis for V. □

Remark. A similar argument applying Zorn's lemma shows that every linearly independant subset of a vector space over a division ring can be extended to a basis.

if $x_1, \ldots, x_n$ generate a module M, but are not a basis, we can still use the module A^n to understand M, because there is a natural surjective morphism from A^n to M, and so M is isomorphic to a quotient of A^n by some submodule. In particular, a cyclic module is isomorphic to $A/\mathfrak{a}$ for some left ideal $\mathfrak{a}$. A module M is *irreducible* if it contains no nontrivial submodules, and it is easy to see that such modules must be isomorphic to $A/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$, or equal to (0). Irreducible modules play a crucial role in representation theory; note non-zero homomorphisms between irreducible modules must be isomorphisms, so in particular, if M is irreducible then $\mathrm{End}(M)$ is a division ring.

Even though not all modules over a ring are free, one might still like to ascribe a notion of *dimension* to the family of free modules. However, over certain exotic rings, even this may not be established, since we might have A^n isomorphic to A^m for $n \neq m$.

Example. *Let A be the ring of endomorphisms over a vector space over a field k with a countable basis $\{e_k\}$. Then as A modules, we claim that $A \oplus A$ is isomorphic to A, so in particular, any module isomorphic to a free module of rank n is automatically isomorphic to a free module of rank 1. Divide the basis $\{e_i\}$ into two infinite sets $\{f_i\}$ and $\{g_i\}$, and define two endomorphisms F and G by setting $F(f_n) = G(g_n) = e_n$, and $F(g_n) = G(f_n) = 0$. If $TF + SG = 0$, then*

$$0 = (TF)(f_n) + (SG)(f_n) = T(e_n) + S(0) = T(e_n)$$

Thus $T = 0$. Conversely,

$$0 = (TF)(g_n) + (SG)(g_n) = T(0) + S(e_n) = S(e_n),$$

so that $S = 0$. Thus F and G are linearly independant over A. If T is an arbitrary endomorphism in A, we can find two endomorphisms T_0 and T_1 such that $T_0(e_n) = T(f_n)$ and $T_1(e_n) = T(g_n)$. Then $T = T_0F + T_1G$. Thus the map $(T_0, T_1) \mapsto T_0F + T_1G$ induces an isomorphism between $A \oplus A$ and A.

A ring A has the *invariant basis property* if A^n is not isomorphic to A^m for $n \neq m$. This is only a problem in the finite dimensional case. If a free module has two bases S_1 and S_2 with $\#(S_1) \leqslant \#(S_2)$ and $\#(S_2) \geqslant \infty$, then each element of S_1 can be written as a finite combination of elements of S_2. Each element of S_2 must be involved in one of these equations; if S_1 is infinite then

$$\#(S_2) \leqslant \#(S_1 \times \mathbf{N}) = \#(S_1),$$

and if S_1 is finite then each finite combination has at most N elements for some N, and so we conclude that

$$\#(S_2) \leqslant \#(S_1 \times \{1, \ldots, N\}) < \infty,$$

which gives a contradiction. Every vector space over a division ring has the invariant basis property.

Theorem 4.2. *Any two bases of a vector space over a division ring have the same cardinality.*

Proof. Suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are two bases generating a vector space V over a division ring k. We prove this statement by induction on the cardinality of $\{v_1, \ldots, v_n\} \cap \{w_1, \ldots, w_m\}$, working *backwards* from n and assuming without loss of generality that $n \leqslant m$. If the intersection has size n, then

$$\{v_1, \ldots, v_n\} \subset \{w_1, \ldots, w_m\}$$

Reorder elements so that $w_i = v_i$ for $1 \leqslant i \leqslant n$. If $m > n$, then there is $a_1, \ldots, a_n \in k$ such that we can write

$$w = a_1 v_1 + \cdots + a_n v_n$$

but then

$$w - a_1 v_1 - \cdots - a_n v_n = 0$$

contradicting the fact that $w_1, \ldots, w_m$ is a basis. To carry out the inductive case, suppose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ share i elements, which we may assume by reordering to be $v_1, \ldots, v_i$ and $w_1, \ldots, w_i$. There must exist some v_j not in the span of the elements $w_1, \ldots, w_i, w_{i+2}, \ldots, w_n$, since otherwise w_{i+1} is in the span of $w_1, \ldots, w_i, w_{i+2}, \ldots, w_n$, which gives a contradiction. This implies that $w_1, \ldots, w_i, v_j, w_{i+2}, \ldots, w_n$ is linearly independant. But it is also spanning; to see this, it suffices to show w_{i+1} is in the span of these elements. But we can write

$$v_j = c_1 w_1 + \cdots + c_m w_m,$$

and $c_{i+1} \neq 0$, from which it follows that

$$w_{i+1} = c_{i+1}^{-1}(v_j - c_1 w_1 - \cdots - c_i w_i - c_{i+2} w_{i+2} - \cdots - c_m w_m).$$

But since $\{w_1, \ldots, w_i, v_j, w_{i+2}, \ldots, w_m\}$ is a basis sharing one more element with $\{v_1, \ldots, v_n\}$ than the previous base, we can apply induction to conclude $n = m$. □

If M is a finitely generated free module over a ring A with the invariant basis property, it follows that $\dim_A(M)$ is a well defined quantity. All commutative rings have the invariant basis property. To see this, if A^n is isomorphic to A^m, and $\mathfrak{m}$ is a maximal ideal of A, then we induce an isomorphism between $(A/\mathfrak{m})^n$ and $(A/\mathfrak{m})^m$ which is also an isomorphism of $A/\mathfrak{m}$ modules; since $A/\mathfrak{m}$ is a field, we conclude $n = m$. If A is an algebra over a field k, which is a finitely generated module over M, then A also has the invariant basis property; if A^n is isomorphic to A^m, and A is isomorphic to k^l as a vector space, then we conclude that k^{nl} is isomorphic to k^{ml}, hence $nl = ml$ and so $n = m$. This covers most of the rings one studies in the fundamentals of abstract algebra.

One verifies that for two free modules M and N,

$$\dim_A(M \oplus N) = \dim_A(M) + \dim_A(N).$$

If M, N, and L are finitely generated free modules for which there is an exact sequence

$$0 \to M \to N \to L \to 0$$

then $N \cong M \oplus L$, from which it follows that $\dim_A(M) + \dim_A(L) = \dim_A(N)$. Thus the *rank nullity theorem* holds for free modules over a ring with the

invariant basis property. This also holds for infinite dimensional vector spaces over a field k, where we can define $\dim_k(V)$ to be the cardinality of a basis, by the same proof.

A simple consequence of the rank nullity theorem is that if W is a subspace of a vector space V, then $\dim(W) \leqslant \dim(V)$, with equality if and only if $V = W$. This is true in for general commutative rings when stated correctly, but the proof is more technical.

Lemma 4.3. *Let M be a finitely generated free module over a commutative ring, and let N be a free submodule. Then $\dim(M) \leqslant \dim(N)$, with equality if and only if $N = M$.*

Proof. It suffices to show there is no injective R-linear map from R^{n+1} to R^n for any integer n. So let M be any $n \times (n+1)$ matrix. Let $D_i(M)$ be the ideal of R generated by the $i \times i$ minors of M, where $D_t(M) = 0$ for $t > n$. Let r be the largest integer such that $D_r(M)$ has no annihilator. Pick $a \in R$ annihilating $D_{r+1}(M)$, but not annihilating some $r \times r$ minor, which we may assume to be the top left minor. Let $t_1, \ldots, t_{r+1}$ be the cofactors obtained by expanding along the bottom row of the top left $(r+1) \times (r+1)$ matrix (well defined even if $r = n$). In particular $a \cdot t_{r+1} \neq 0$. Consider the nonzero vector

$$x = (at_1, \ldots, at_{r+1}, 0, \ldots, 0).$$

But then $Mx = 0$ by Cramer's rule. □

4.3 Exact Sequences and Homomorphisms

For any two R-modules M and N, the family of all R-linear maps $\mathrm{Hom}_R(M,N)$ naturally has the structure of an abelian group, where for $f, g \in \mathrm{Hom}_R(M,N)$, we define $(f+g)(x) = f(x) + g(x)$. If R is *commutative*, then $\mathrm{Hom}_R(M,N)$ even has the natural structure of an R-module, since we can define $(rf)(x) = r \cdot f(x)$. The reason this only works if R is commutative is that to guarantee rf is a homomorphism, we must have $(rf)(sx) = s(rf)(x)$, which only holds if $(rs) \cdot f(x) = (sr) \cdot f(x)$ holds for all $x \in M$, which we can expect to be rarely true unless R is commutative.

For a fixed R-module M, the map $N \mapsto \mathrm{Hom}_R(M,N)$ is a covariant functor from the category of modules to the category of abelian groups / modules, in the following way. Given a morphism $f : N \to L$, we obtain a

morphism f_* from $\mathrm{Hom}_R(M,N)$ to $\mathrm{Hom}_R(M,L)$ by setting $f_*(\phi) = f \circ \phi$. On the other hand, if N is fixed, then the map $M \mapsto \mathrm{Hom}_R(M,N)$ for a fixed N is a contravariant functor, since given a m ap $f : M \to L$, we obtain $f^* : \mathrm{Hom}(L,N) \to \mathrm{Hom}(M,N)$ by setting $f^*(\phi) = f \circ \phi$.

Almost all the structure of the modules over a ring can be seen through the structure of the homomorphism groups; in category theory this would be a consequence of the Yoneda lemma, but in this particular we have an even more powerful theory since the set of morphisms have algebraic structure.

Lemma 4.4. *Let N, L, and K be R-modules. A sequence*

$$0 \to N \to L \to K$$

is exact if and only if for any R-module M, the induced sequence of abelian groups

$$0 \to Hom_R(M,N) \to Hom_R(M,L) \to Hom_R(M,K)$$

is exact, which is exact if and only if for any R-module M, the induced sequence of abelian groups

$$0 \to Hom_R(K,M) \to Hom_R(L,M) \to Hom_R(N,M)$$

is exact.

Proof. Let $f : N \to L$ and $g : L \to K$ be maps in the sequence above. Suppose the sequence is exact, so f is an isomorphism from N to the kernel of g. But then f_* is injective since if there is $\phi : M \to N$ such that $f \circ \phi = 0$, then $\phi = 0$ since f is injective. If $\phi : M \to L$ and $g_*(\phi) = 0$, then $g \circ \phi = 0$, which implies that $\phi(M)$ is contained in the kernel of L. But then the fact that f is anisomorphism implies that there is $\psi : M \to N$ such that $f \circ \psi = \phi$. Thus the sequence of homomorphisms is exact. Conversely, if we set $M = R$ in the bottom diagrams we obtain the original sequence, so if the bottom sequence is always exact we find the top sequence is exact. The argument for the contravariant hom functor is similar, and left to the reader. □

Let Mod_R and Mod_S be the category of modules over two rings R and S. We say a covariant functor $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is *exact* if, for any sequence

$$0 \to M \to N \to L \to 0$$

the induced sequence

$$0 \to F(M) \to F(N) \to F(L) \to 0$$

is exact. Similarily, a contravariant functor is exact if the induced sequence

$$0 \to F(L) \to F(N) \to F(M) \to 0$$

is exact. What we have shown is that $\mathrm{Hom}(M, \cdot)$ is a *left exact* functor, and $\mathrm{Hom}(\cdot, N)$ is a *right exact* contravariant functor, that is, they preserve the exactness of the left and right sides of the induced exact sequence.

We now consider short exact sequences of modules

$$0 \to M \to N \to L \to 0$$

given by maps $f : M \to N$ and $g : N \to L$. We say such a diagram *splits* if the following of two equivalent conditions hold:

- There exists a section $\psi : L \to N$ such that $g \circ \psi$ is the identity.
- There exists a section $\eta : N \to M$ such that $\eta \circ f$ is the identity.

We now prove that if either of these conditions hold, then the other must hold, and then N decomposes as $\mathrm{Ker}(g) \oplus \mathrm{Im}(\psi)$ and as $\mathrm{Im}(f) \oplus \mathrm{Ker}(\eta)$, and is therefore isomorphic to $M \oplus L$. Surely the existence of ψ implies the direct sum decomposition, because $g(\psi(x)) = x$, so $\mathrm{Ker}(g)$ is disjoint from $\mathrm{Im}(\psi)$, and $x - \psi(g(x)) \in \mathrm{Ker}(g)$. The second condition implies the second decomposition in a similar manner. To prove the equivalence of the two splitting conditions, we note that if $N = \mathrm{Ker}(g) \oplus \mathrm{Im}(\psi) = \mathrm{Im}(f) \oplus \mathrm{Im}(\psi)$, we can define $\eta(f(x) + \psi(y)) = x$, since f is injective. If $N = \mathrm{Im}(f) \oplus \mathrm{Ker}(\eta) = \mathrm{Ker}(g) \oplus \mathrm{Ker}(\eta)$, since g is surjective, setting $\psi(x)$ to be the unique element of $\mathrm{Ker}(\eta)$ with $g(\psi(x)) = x$. Such an element exists because g is surjective, and such an element is unique since if $\eta(x) = \eta(y) = 0$ and $g(x) = g(y)$, then $x - y = f(z)$, and so $0 = \eta(x) - \eta(y) = z$, so $x = y$.

Remark. If we are considering the short exact sequence

$$0 \to M \to M \oplus L \to L \to 0$$

Then the existence of the splitting maps is obvious. The splitting argument above shows that this situation is essentially the only case where a short exact sequence can split.

Any short exact sequence of vector spaces splits because every module is a free module, but there are short exact sequences which do not split. Let us consider an example.

Example. *Let $M = \mathbf{Z}$, let $L = (\mathbf{Z}/2\mathbf{Z})^\infty$ be the $\mathbf{Z}$-module of sequences in $\mathbf{Z}/2\mathbf{Z}$, and let $N = M \oplus L$. Let us construct a short exact sequence*

$$0 \to M \to N \to L \to 0$$

which does not split. Define $f : M \to N$ by setting $f(x) = (2x, 0)$, and define $g : N \to L$ by setting $g(x, y) = (x + 2\mathbf{Z}, y_1, y_2, \dots)$. Then f is injective, g is surjective, and the kernel of g is equal to the image of f. However, this sequence does not split. If there was a map $\psi : L \to N$ such that $g \circ \psi$ is the identity, then we would obtain two morphisms $\psi_1 : L \to M$ and $\psi_2 : L \to L$ such that $\psi(y) = (\psi_1(y), \psi_2(y))$. But

$$(2) = Ann(L) \subset Ann(\psi_1(L)),$$

which implies that $\psi_1(L) = 0$ since $\mathbf{Z}$ is an integral domain so no nonzero element can annihilate any other nonzero element of itself. But this is clearly impossible since we must have $(\psi_1(y), y_2, \dots) = (y_1, y_2, \dots)$, which cannot hold if $y_1 = 1 + 2\mathbf{Z}$.

4.4 Projective Modules

A module P such that any short exact sequence

$$0 \to M \to N \to P \to 0$$

splits is known as a *projective module*. These are the modules that are easy to define maps out of.

Theorem 4.5. *Fix a module P. Then the following are equivalent.*

- *For any map $f : P \to N$ and a surjective map $g : M \to N$, there exists a map $h : P \to M$ such that $g \circ h = f$.*
- *Any short exact sequence $0 \to M \to N \to P \to 0$ splits.*
- *There is a module N such that $N \oplus P$ is a free module.*

- *The functor $Hom(P,\cdot)$ is exact.*

If any of these conditions are satisfied, we say P is a projective module.

Proof. Consider the first condition. Then given any $0 \to M \to N \to P \to 0$, we can take $N = P$ in the triangle condition to obtain the splitting map, showing P is projective. If any short exact sequence terminating at P splits, there certainly exists a free module M with a surjective map $M \to P \to 0$, and then M is isomorphic to the direct sum of P and the kernel of the homomorphism. If N is a free module, then the functor $M \mapsto \mathrm{Hom}(N,M)$ is obviously exact, and since $\mathrm{Hom}(N \oplus P, M) \cong \mathrm{Hom}(N,M) \oplus \mathrm{Hom}(P,M)$. Thus if we have $M_0 \to M_1 \to M_2$ is exact, then

$$\mathrm{Hom}(N \oplus P, M_0) \to \mathrm{Hom}(N \oplus P, M_1) \to \mathrm{Hom}(N \oplus P, M_2)$$

is exact. Restricting the domain of our homomorphisms gives an exact sequence

$$\mathrm{Hom}(P, M_0) \to \mathrm{Hom}(P, M_1) \to \mathrm{Hom}(P, M_2)$$

Finally, assume that the functor is exact. Then we have an exact sequence $M \to N \to 0$, so we have an exact sequence $\mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$. In particular, for any homomorphism from P to N, there exists a homomorphism from $P \to M$ which completes the triangle, so the first property is proved. □

Since projective modules are those for which it is easy to define maps out of, all free modules are projective. Assuming certain circumstances on the ring underlying the module, we can prove that all projective modules are free. In particular, at the end of this chapter we shall show that all projective modules over a principal ideal domain is free. Later on we shall show that all projective modules over a local ring are free. But let us now consider some example of projective modules which are not free.

Example. *If R and S are rings, consider the product ring $R \times S$. If M is an R-module and N is an S-module, then $M \times N$ is naturally an $R \times S$ module, where*

$$(r,s) \cdot (x,y) = (rx, sy).$$

If R and S are rings, and L is a module over $R \times S$, then multiplication by $(1,0)$ and $(0,1)$ identifies two submodules M and N of L. Clearly L is isomorphic to $M \times N$. Thus all $R \times S$ modules arise out of a product of an R-module and an

S-module. In particular, the finitely generated free modules of $R \times S$ are of the form $R^n \times S^n$. The module $R^n \times 0$ is not a free $R \times S$ module since it has torsion. On the other hand, $(R^n \times 0) \oplus (0 \times S^n) = R^n \times S^n$, so $R^n \times 0$ is a projective module on $R \times S$.

Example. *Let S be a circle in $\mathbf{R}^3$, and consider the normal bundle $N(S)$ to S. If M is the family of all continuous sections of $N(S)$, then M is naturally a module over $C(S)$. In fact, M is actually a free module over $C(S)$ of dimension two, since we can find two continuous vector fields $X : S \to N(S)$ and $Y : S \to N(S)$ such that at each $p \in S$, X_p and Y_p span $N(S)$. Thus if $Z : S \to N(S)$ is any other continuous vector field, there are unique functions a_1, a_2 such that $Z_p = a_1(p)X_p + a_2(p)Y_p$, and it is simple to verify these functions are continuous.*

Now consider a Möbius strip Σ in $\mathbf{R}^3$ containing S. Then $N(S) \cap T\Sigma$ and $N(S) \cap N(\Sigma)$ are both vector bundles, and we can consider the two submodules N_1 and N_2 of M which map into either of these vector bundles. Clearly $N_1 \oplus N_2 = M$, so N_1 and N_2 are projective modules over $C(S)$. On the other hand, N_1 and N_2 are not free modules over $C(S)$. If they were free, they would have to have dimension one, since if X_1, X_2 are two sections of $N(S) \cap T\Sigma$, then they are linearly dependant at each point $p \in S$ so there exists $a_1, a_2 \in C(S)$ such that $a_1X_1 + a_2X_2 = 0$. But neither module is generated by a single element, because every continuous section of $N(S) \cap T(\Sigma)$ and $N(S) \cap N(\Sigma)$ must vanish at some point. Thus neither module is free. Now this is certainly a very analytical example, involving rings with many zero divisors. However, this example can be made into an algebraic example over the ring

$$R = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1) \cong \mathbf{R}[\cos\theta, \sin\theta],$$

and considering only algebraic vector fields.

Example. *Let R be the ring of continuous functions on the sphere Σ. Let M be the module of all vector fields on Σ, not necessarily tangent to Σ. Let N_1 be the submodule of all vector fields tangent to Σ, and N_2 be the usbmodules of all vector fields perpendicular to Σ. Then $N_1 \cup N_2$ is a free module, but N_1 is not a free module. To see this, we note that it would have to have dimension two if it was free. But the hairy ball theorem says that each vector field in N_1 vanishes at some point, which one can use, as in the last theorem, to show N_1 cannot have a basis. Switching from R to the coordinate ring of the sphere and using algebraic vector fields gives a more algebraic example.*

Example. *Let R be a ring, and let A be the ring of all $n \times n$ matrices in R. If M is the family of column vectors with entries in R, then M is naturally an A module. Viewing A as a left module over itself, we see that A is isomorphic as an A-module to M^n, since multiplication by a matrix on the left distributes over the columns of a matrix. Thus M is a projective module. But M is certainly not free. We can view it as a 'free module of dimension $1/n$', in some senses.*

The dual of a projective module is a *injective module*, i.e. a module I such that the functor $\mathrm{Hom}(\cdot, I)$ is exact. This is equivalent to saying any short exact sequence

$$0 \to I \to M \to N \to 0$$

splits. Free modules are not usually injective. For instance, if $I = \mathbf{Z}$ and $M = \mathbf{Z}/2$, then the inclusion map does not split, since there is no subgroup N of M such that $\mathbf{Z}/2 = \mathbf{Z} \oplus N$. Similarily, every vector space is injective.

4.5 Dual Modules

For any R-module M, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ is naturally a ring with composition as multiplication. Thus if R is commutative then $\mathrm{End}_R(M)$ is an R-algebra, since

$$r(f \circ g) = (rf) \circ g = f \circ (rg).$$

For any module M, we have an abelian group $M^* = \mathrm{Hom}_R(M, R)$, known as the *dual* to M. But M^* is also naturally *right* R-module, i.e. if we define

$$(fr)(x) = f(x)r$$

Thus M^* is the *dual module* to M. The map $M \mapsto M^*$ is thus a contravariant functor from the category of left R-modules to the category of right R-modules. We have a natural bilinear map $\langle \cdot, \cdot \rangle : M \times M^* \to R$ given by $\langle x, f \rangle = f(x)$. Dual modules show up in a variety of places in mathematics.

Example. *Coordinates on a finite dimensional vector space V over a field k correspond to a sequence of coordinate functions $x_1, \ldots, x_n : V \to k$. These coordinate functions are linear functionals in V^*.*

Example. *The family of continuous, compactly supported functions $C_{00}(\mathbf{R})$ is naturally a module over $\mathbf{R}$, or $\mathbf{C}$, depending on whether we are discussing*

real-valued or complex-valued functions. For each finite measure μ, we have a linear functional

$$f \mapsto \int_{-\infty}^{\infty} f(x)\, d\mu(x).$$

Thus μ can be viewed as an element of $C_{00}(\mathbf{R})^*$. The Riesz representation theorem *says that every element of $C_{00}(\mathbf{R})^*$ is induced by some finite, signed measure.*

If R is a commutative ring and M is a free module, then M^* is free. If M is finite dimensional, then M^* is finite dimensional, and $\dim_R(M^*) = \dim_R(M)$. This is simple to see because if $M \cong \bigoplus_i R$, then

$$M^* = \mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_R\left(\bigoplus_i R, R\right) \cong \bigoplus_i \mathrm{Hom}_R(R,R) \cong \bigoplus_i R.$$

We can do this more concretely in the finite dimensional case. Given a basis $e_1,\dots,e_n$ for M, a natural basis for M^* is given by $e_1^*,\dots,e_n^*$, defined such that $\langle e_i, e_j^* \rangle = \delta_{ij}$. Because M is free, such functionals $e_1^*,\dots,e_n^*$ certainly exist, and they span M^*; given a functional $\lambda : M \to R$, let $a_1,\dots,a_n \in K$ be such that $\lambda(e_i) = a_i$. Then $\lambda = a_1 e_1^* + \cdots = a_n e_n^*$, because both functionals agree on $e_1,\dots,e_n$. We call the basis $e_1^*,\dots,e_n^*$ the *dual basis* to $e_1,\dots,e_n$.

We have a natural transformation from M to it's *double dual* M^{**}, given by mapping $x \in M$ to the R-linear morphism $x^{**} : M^* \to R$ given by setting $x^{**}(\lambda) = \lambda(x)$. If M is a free module, it is easy to see this natural transformation is a monomorphism. If M is *finite dimensional* and free, then this map is actually an isomorphism; if $\{e_1,\dots,e_n\}$ is a basis for M, we consider the dual basis $\{e_1^*,\dots,e_n^*\}$ for M^* and the induced double dual basis $\{e_1^{**},\dots,e_n^{**}\}$. It is easy to verify that the double dual of e_i is e_i^{**}, so the double dual operator maps a basis onto a basis.

Let M and N be modules over a commutative ring R, and consider a bilinear map $f : M \times N \to R$. Then each $x \in M$ corresponds to a linear functional $\lambda_x \in N^*$ given by $\lambda_x(y) = f(x,y)$. The kernel consists precisely of the submodule M_0 of M containing the points x_0 such that $f(x_0,y) = 0$ for all $y \in N$, thus we obtain an injective homomorphism from $M/M_0 \to N^*$. If N_0 is the submodule of N consisting of values y_0 such that $f(x,y_0) = 0$ for all $x \in N$, then the morphism descends from $M/M_0 \to (N/N_0)^*$, and remains injective. If R is a field, so that M and N are vector spaces, we conclude that $\dim(M/M_0)$

Lemma 4.6. *Let k be a field. Then a bilinear map $M \times N \to K$ with associated submodules M_0 and N_0. If M/M_0 or N/N_0 is finite dimensional, then the morphism induced by the bilinear map $M \times N \to R$ is an isomorphism.*

Proof. By symmetry, we may assume N/N_0 is finite dimension. The injectivity of $M/M_0 \to (N/N_0)^*$ implies that

$$\dim(M/M_0) \leqslant \dim((N/N_0)^*) = \dim(N/N_0).$$

In particular, M/M_0 is finite dimensional. Conversely, we can consider the injective map $N/N_0 \to (M/M_0)^*$ which implies

$$\dim(N/N_0) \leqslant \dim((M/M_0)^*) = \dim(M/M_0).$$

Thus the two spaces have the same dimension, which together with injectivity implies the isomorphism. □

4.6 Tensor Products

Given two A modules M and N, the *tensor product* $M \otimes_A N$, or $M \otimes N$ if the module A is implicit (or over $\mathbf{Z}$, if M and N are just abelian groups), is the most general way we can form a 'bilinear space' corresponding to M and N. More specifically, $M \otimes N$ is the initial object in the category of bilinear maps $f : M \times N \to L$ into a module L. We can construct $M \otimes N$ by consider the quotient of the free module with basis elements $M \times N$, subject to the submodule generated by $(x+y,z) - (x,z) - (y,z)$, $a(x,y) - (ax,y)$, and $a(x,y) - (x,ay)$. We let the image of (x,y) in the quotient be denoted by $x \otimes y$, so that $(x+y) \otimes z = x \otimes z + y \otimes z$, $a(x \otimes y) = (ax) \otimes y = x \otimes (ay)$. The embedding of $M \times N$ in the free abelian group obviously descends to a bilinear map from $M \times N$ to $M \otimes N$, which is bilinear precisely because of the quotients defining $M \otimes N$. If $f : M \times N \to L$ is bilinear, then f extends uniquely to a map on the free group generated by $M \times N$. Furthermore, the relations which make f bilinear precisely mean that f descends to a map from $M \otimes N$ to L, so we get a unique morphism from $M \otimes N$ to L which represents f. However, this definition is the 'wrong' definition to use in most cases when understanding the tensor product, because it's quite a strange definition to work with.

Example. *Given the abelian groups* $\mathbf{Z}_{10}$ *and* $\mathbf{Z}_{12}$*, we find that* $\mathbf{Z}_{10} \otimes \mathbf{Z}_{12}$ *is isomorphic to* $\mathbf{Z}_2$*. We find that for any integers* n, m*,*

$$n \otimes m = (11n) \otimes m = n \otimes (11m) = -n \otimes m$$

Thus 2 annihilates all of $\mathbf{Z}_{10} \otimes \mathbf{Z}_{12}$*, and so we get the natural structure of* $\mathbf{Z}_{10} \otimes \mathbf{Z}_{12}$ *as a vector space over* $\mathbf{Z}_2$*. Yet*

$$n \otimes m = n(1 \otimes m) = (nm)(1 \otimes 1)$$

so the vector space is generated by a single element $1 \otimes 1$*. The element* $1 \otimes 1$ *doesn't equal to zero in* $\mathbf{Z}_{10} \otimes \mathbf{Z}_{12}$*. We have a bilinear map* $f : \mathbf{Z}_{10} \times \mathbf{Z}_{12} \to \mathbf{Z}_2$ *given by* $f(x, y) = xy$*, which is well defined because* $(x+10)y = x(y+12) = xy$ *modulo 2. Thus we have an induced map* $f_* : \mathbf{Z}_{10} \otimes \mathbf{Z}_{12} \to \mathbf{Z}_2$ *where* $f_*(1 \otimes 1) = f(1, 1) = 1$*, which is different from* $f_*(0 \otimes 0) = 0$*, so* $1 \otimes 1 \neq 0$*. In particular, our calculation shows that for any bilinear map* $f : \mathbf{Z}_{10} \times \mathbf{Z}_{12} \to M$*, there exists a unique morphism* $g : \mathbf{Z}_2 \to M$ *such that* $f(x, y) = g(xy)$*.*

The tensor product is a covariant bifunctor on the category of modules, since if $f : M_0 \times M_1$ and $g : N_0 \times N_1$, then we have a unique morphism $(f \otimes g) : (M_0 \otimes N_0) \to (M_1 \times N_1)$ obtained by $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$. If we fix a module N on the right, then we obtain a covariant functor which is known as *right exact*. More specifically, given an exact sequence

$$M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2 \longrightarrow 0$$

The induced exact sequence

$$M_0 \otimes N \xrightarrow{f \otimes \mathrm{id}} M_1 \otimes N \xrightarrow{g \otimes \mathrm{id}} M_2 \otimes N \longrightarrow 0$$

is also exact. The surjectivity is easy to prove. Since g is surjective, given for any $y \in M_2$, there is $x \in M_1$ with $g(x) = y$, so $(g \otimes \mathrm{id})(x \otimes z) = y \otimes z$ for any $z \in N$. Now we prove that the image of $(f \otimes \mathrm{id})$ is the kernel of $(g \otimes \mathrm{id})$. Certainly the image, which we denote by L, is a subset of the kernel, which we denote by K. So we get an induced surjective map from $(M_1 \otimes N)/L \to (M_2 \otimes N)$. We claim it is an isomorphism, which would show that $L = K$. To define a left inverse, given $y \otimes z \in M_2 \otimes N$, choose $x \in M_1$ such that $g(x) = y$. The map $h(y \otimes z) = x \otimes z + L$ is a well defined map into the quotient, because if $x, x' \in M_1$ are such that $f(x) = f(x') = y$, then $x - x'$

is in the kernel of g, so $(x - x') \otimes z$ is in L. The map is clearly bilinear, and thus extends to a complete map h on $M_2 \otimes N$, and it is easy to check this is a left inverse on a generating set, hence everywhere. A module M is called *flat* if tensoring by M is an exact functor, or equivalently, tensoring by M preservings injective maps.

Given a commutative ring R and any two R-algebras A and B, the tensor product $A \otimes_R B$ is an R-algebra, since for any $a_0 \in A$ and $b_0 \in B$, the map $(a, b) \mapsto (aa_0, bb_0)$ is R-bilinear, so the map $(a \otimes b)(a_0 \otimes b_0) = (aa_0) \otimes (bb_0)$ is a well-defined R-bilinear map. If A and B are commutative, the tensor product $A \otimes_R B$ is the 'free product' of commutative R-algebras, because given any pair of R-algebra homomorphisms $\phi : A \to C$ and $\psi : B \to C$, where C is a commutative R-algebra, the map $\phi \otimes \psi : A \otimes_R B \to C$ is the unique algebra homomorphism from $A \otimes_R B$ to C such that $a \otimes b = \phi(a)\psi(b)$. Similarily, given any ring homomorphism $f : R \to S$, and any R-module M, $S \otimes_R M$ is naturally an S-module where $s_1(s \otimes m) = (s_1 s) \otimes m$. We often denote $S \otimes_R M$ by $f_*(M)$. Similarily, for any S-module N we can naturally give N an R-module structure by defining $r \cdot n = f(r)n$. As an R-module N is often denoted $f^*(N)$. The map f^* is known as *restriction*, and f_* is a *base change*, and they form an adjoint pair.

Tensoring commutes with the direct sum operation, a fact easy to prove by the universal property. That is, we have $M \otimes \bigoplus N_\alpha$ isomorphic to $\bigoplus (M \otimes N_\alpha)$. Any bilinear map f from $M \times \bigoplus N_\alpha$ to L corresponds to a unique family of bilinear maps f_α from $M \times N_\alpha$ to L, inducing a map from $M \otimes N_\alpha$ to L, which can be put together to form a unique map from $\bigoplus (M \otimes N_\alpha)$ to L. Thus multiplication and addition of modules is 'distributive'. Considering the tensor product of modules as a multiplication operation, and addition as a direct sum, the family of modules over an abelian group is given an algebraic structure. This structure is studied in detail in the field of K-theory.

4.7 Modules over Principal Ideal Domains

The structure of prinicpal ideals is rich enough that we can obtain a complete classification of finitely generated modules over these rings. This has many important applications, for instance, to the theory of finitely generated abelian groups, and the classification of operators on finite dimensional vector spaces (i.e. finitely generated modules over $\mathbf{Z}$ and $k[X]$). Our

goal will be to show all modules over a principal ideal domain R can be written as a direct sum of cyclic modules, in particular, and in particular, each cyclic module will be isomorphic to $R/(r)$ for some $r \in R$.

Lemma 4.7. *Let M be a free module over a principal ideal domain R, and let N be a submodule. Then N is free and* $\dim_R(N) \leqslant \dim_R(M)$.

Proof. Let us begin with the case where M is finite dimensional, with some basis $e_1, \ldots, e_n$. Let $N_r = N \cap \operatorname{span}(e_1, \ldots, e_r)$. Now the map $a \mapsto ae_1$ induces a surjective R-linear morphism from R to $\operatorname{span}(e_1)$. Thus $\operatorname{span}(e_1)$ is isomorphic to $R/\operatorname{Ann}(e_1)$. Thus N_1 corresponds to an ideal of $R/\operatorname{Ann}(e_1)$, and since $R/\operatorname{Ann}(e_1)$ is principal, there is $x \in R$ which generates N_1. Since M is free, either $x = 0$ and so $N_1 = R^0$, or $x \neq 0$ and $N_1 \cong R$. Now assume for the purpose of induction that N_r is free. Let $\mathfrak{a}$ be the ideal of all $a \in R$ for which there is $a_1, \ldots, a_r \in R$ and $x \in M$ such that

$$x = a_1 e_1 + \cdots + a_r e_r + a e_{r+1}.$$

Then $\mathfrak{a}$ is principal, generated by some a_{r+1}. If $a_{r+1} = 0$, then $N_{r+1} = N_r$, and the induction is complete. On the other hand if $a_{r+1} \neq 0$, pick $w \in N$ and $a_1, \ldots, a_r \in R$ such that

$$w = a_1 e_1 + \cdots + a_r e_r + a_{r+1} e_{r+1}.$$

If $x \in N_{r+1}$, there is c such that $x - cw \in N_r$. Thus $N_{r+1} = N_r + (w)$. Since $N_r \cap (w) = 0$, $N_{r+1} = N_r \oplus (w)$, and is therefore free. □

Remark. A modification of this proof easily extends the proof to general rings by well ordering a basis and proceeding inductively as here, with the limit ordinal cases being easily shown to be benign.

Remark. Any commutative ring such that every submodule of a free module is free *must* be a principal ring. If If R is a commutative ring, then an ideal $\mathfrak{a}$ of R is a submodule of R. If $a_1, a_2 \in \mathfrak{a}$, then $a_2 a_1 - a_1 a_2 = 0$, so $\{a_1, a_2\}$ is not linearly independant over R. In particular, if $\mathfrak{a}$ is a free module over R, it must have a basis consisting only of a single element. But this means $\mathfrak{a}$ is principal.

The next corollary follows from general properties of Noetherian rings but the proof follows easily enough in this situation from the last claim.

Corollary 4.8. *Let M be a finitely generated module over a principal ideal domain R. Then every submodule of M is finitely generated, and the optimal number of generators is less than or equal to the optimal number of generators as M.*

Proof. If M is a finitely generated module over R generated by n elements, then there is a surjective morphism $\phi : R^n \to M$. If N is a submodule of M, then $\phi^{-1}(N)$ is a submodule of M, and therefore free, with $\dim_A(\phi^{-1}(N)) \leqslant \dim_A(M)$. If $x_1, \ldots, x_m$ are a basis for $\phi^{-1}(N)$, then

$$\phi(x_1), \ldots, \phi(x_m)$$

generate N, where $m \leqslant n$. □

Lemma 4.9. *Every finitely generated torsion free module over a principal ideal domain is free.*

Proof. Let M be a finitely generated torsion free module, and let $\{x_1, \ldots, x_n\}$ be generators of M. Let $\{x_{i_1}, \ldots, x_{i_m}\}$ be a maximally linearly independent subset. Given any i, there is $a_{i_1}, \ldots, a_{i_m} \in R$ and $c_i \neq 0$ such that

$$c_i x_i + a_{i_1} x_1 + \cdots + a_{i_m} x_m = 0$$

Let $c = c_1 \ldots c_n$. Then cM is contained in $\{x_{i_1}, \ldots, x_{i_m}\}$. The map $x \mapsto cx$ is injective because M is torsion free. But this means we may identify M with a submodule of R^m, and thus M is free. □

Let us now reduce our analysis to torsion groups.

Lemma 4.10. *Let M be a finitely generated module over a principal ideal domain. Then there exists a unique integer n such that*

$$M \cong Tor(M) \oplus R^n.$$

Proof. We have already shown $M/\mathrm{Tor}(M)$ is torsion free, and is therefore free since it is finitely generated. We have an exact sequence

$$0 \to \mathrm{Tor}(M) \to M \to M/\mathrm{Tor}(M) \to 0.$$

Since $M/\mathrm{Tor}(M)$ is free, it is projective, so $M \cong \mathrm{Tor}(M) \oplus M/\mathrm{Tor}(M)$, which gives the existence of some n such that $M \cong \mathrm{Tor}(M) \oplus R^n$, where $n = \dim_A(M/\mathrm{Tor}(M))$. But note that if

$$M \cong \mathrm{Tor}(M) \oplus R^m,$$

then $\mathrm{Tor}(\mathrm{Tor}(M) \oplus R^m) = \mathrm{Tor}(M)$, so

$$M/\mathrm{Tor}(M) \cong (\mathrm{Tor}(M) \oplus R^m)/\mathrm{Tor}(M) \cong R^m$$

so $n = m$. □

The dimension of $M/\mathrm{Tor}(M)$ is called the *rank* of M. This determines the 'free part' of any module M over a principal ideal domain. Thus it now suffices to determine the structure of a *torsion* module M over a principal ideal domain. This is just a matter of introducing some notation.

Let M be a module over R. Given each $x \in M$, there exists $r \in R$ such that $\mathrm{Ann}(x) = (r)$. We say r is the *period* of x, and is uniquely determined up to a unit. An element of $\mathrm{Ann}(M)$ is called an *exponent* for M. Given a prime $p \in R$, we let $M(p) = \{x \in M : p^n x = 0 \text{ for some } n\}$. A *p-submodule* of M is a submodule contained in $M(p)$. For $r \in R$, let $M_r = \mathrm{Ann}(r)$. For a fixed prime $p \in R$ and $n_1, \ldots, n_r$, a p-module M is of type $(n_1, \ldots, n_r)$ if it is isomorphic to $R/(p^{n_1}) \oplus \cdots \oplus R/(p^{n_r})$.

Lemma 4.11. *Let M be a torsion module over a principal ideal domain R. Then*

$$M \cong \bigoplus_p M(p),$$

where p ranges over a representation of each equivalence class of primes modulo units in R.

Proof. Let $x \in M$. Then $\mathrm{Ann}(x)$ is nonzero, generated by some non-zero element $r = p_1^{k_1} \ldots p_n^{k_n}$. If

$$x_i = p_1^{k_1} \ldots \widehat{p_i}^{k_i} \ldots p_n^{k_n} x$$

then $x_i \in M(p)$. Since the greatest common denominator of the coefficients above is equal to one, there exists $a_1, \ldots, a_n \in R$ such that $a_1 x_1 + \cdots + a_n x_n = x$. If $p_1, \ldots, p_n$ are fixed primes, we pick $x_1 \in M(p_1), \ldots, x_n \in M(p_n)$, and

$$x_1 + \cdots + x_n = 0,$$

then we claim $x_i = 0$ for each i. For each i, there is k_i such that $p^{k_i} x_i = 0$. Then

$$0 = p_2^{k_2} \ldots p_n^{k_n}(x_1 + \cdots + x_n) = p_2^{k_2} \ldots p_n^{k_n} x_1,$$

Since nonzero elements of $M(p_1)$ are only annihilated by elements divisible by a power of p_1, we conclude $x_1 = 0$. Thus the direct decomposition occurs. □

Remark. If M is a finitely generated module, then only finitely many primes occur in this decomposition, each of which being finitely generated, which makes understanding the decomposition feasible.

Our goal will be to slowly crunch away cyclic groups until we have completely decomposed the module into irreducible parts. We can then show this factorization is unique. Say that elements $x_1, \ldots, x_n$ in a module are *independent* if for any $r_1, \ldots, r_n \in R$, if $r_1 x_1 + \cdots + r_n x_n = 0$, then $r_i x_i = 0$ for each i. This is true if and only if

$$(x_1, \ldots, x_n) = (x_1) \oplus \cdots \oplus (x_n).$$

Note this is *not* the same thing as linear independence, which implies the existence of a free structure which can never occur in a torsion module.

Lemma 4.12. *Suppose M is a torsion module of exponent p^n, and let x have period p^r. If $\tilde{y_1}, \ldots, \tilde{y_n}$ are independant elements of $M/(x)$, we may find $y_1, \ldots, y_n \in M$ such that $y_i + (x) = \tilde{y_i}$, y_i has the period of $\tilde{y_i}$, and $y_1, \ldots, y_n$ are independant.*

Proof. Fix $y \in M$ such that $\tilde{y} = y + (x_0)$ has period p^r. We claim there is $x \in (x_0)$ such that $y + x$ has period p^r. The period of y is equal to p^s for some s, where $s \geqslant r$. Now $p^r y \in (x_0)$, so we may find $0 \leqslant t \leqslant n$ and $c \in R$ such that p does not divide c and $p^r y = cp^t x_0$. If $t = n$, then $p^r y = 0$, so $s = r$. If $t < n$, then $0 = p^s y = cp^{t+s-r} x_0$, which implies $t + s - r \geqslant n$, and $0 = cp^n x_0 = p^{r+n-t} y$, so $r + n - t \geqslant s$. Together these equations imply that $s = n + r - t$. Since M has exponent p^n, this implies $n + r - t \leqslant n$, so $r \leqslant t$. But then we can set $x = -cp^{s-r} x_0$, since if $p^k(y + x) = 0$, then $p^k y = cp^{k+s-r} x_0$, hence $k \geqslant r$. But $p^r(y + x) = p^r y - cp^s x_0 = 0$ so $y + x$ has period equal to p^r.

Now suppose $\tilde{y_1}, \ldots, \tilde{y_n}$ are independant elements of $M/(x)$, and we find representations $y_1, \ldots, y_n$ of $\tilde{y_1}, \ldots, \tilde{y_n}$ in M with the same periods. Suppose $r_0, r_1, \ldots, r_n \in R$, and $r_0 x_0 + r_1 y_1 + \cdots + r_n y_n = 0$. Then certainly $r_1 \tilde{y_1} + \cdots + r_n \tilde{y_n} = 0$, so $r_i \tilde{y_i} = 0$ for each i. But this implies that the period of $\tilde{y_i}$ divides r_i, which in partiuclar, implies that the period of y_i divides r_i. Thus $r_i y_i = 0$, and so $r_0 x_0 = 0$ follows as a result. Thus we have proved independence. □

All that remains is to apply one final trick to obtain the decomposition into cyclic submodules.

Lemma 4.13. *If M is a finitely generated p-module, then M is a direct sum of cyclic submodules.*

Proof. We note that M is finitely generated. Then M_p is finitely generated, and can be viewed as a vector space over the field R/pR. Thus M_p is a finite dimensional vector space. Let $x_0 \in M$ have maximal period p^n, so M has exponent p^n. If $M' = M/(x_0)$, we claim that $\dim(M'_p) < \dim(M_p)$, which will enable us to carry out an inductive procedure. If $\tilde{y_1},\ldots,\tilde{y_n}$ are linearly independent elements of M'_p, then we may find $y_1,\ldots,y_n \in M_p$ such that $y_i + (x_0) = \tilde{y_i}$ and $x_0, y_1,\ldots,y_n$ are independant. But then $p^{n-1}x_0, y_1,\ldots,y_n$ are linearly independant elements of M_p, since if $r_0 p^{n-1}x_0 + r_1 y_1 + \cdots + r_n y_n = 0$, then $r_i y_i = 0$ for each i, and $r_0 p^{n-1}x_0 = 0$, which implies that p divides each of the $r_0,\ldots,r_n$, and thus $r_0 = \cdots = r_n = 0$ in R/pR. Thus $\dim(M_p) \geqslant \dim(M'_p)+1$. This gives us a way to carry out an inductive procedure. If $\dim(M_p) = 0$, then $M = 0$, so is certainly a direct sum of cyclic modules. Now suppose that $M/(x_0)$ is a direct sum of cyclic modules. Then we may find an independant generating set $\tilde{y_1},\ldots,\tilde{y_n}$ for $M/(x_0)$. But by finding representatives $y_1,\ldots,y_n$ with the same period in M, we find $x_0, y_1,\ldots,y_n$ is linearly independant, so $M \cong M/(x_0) \oplus (x_0)$. Since $M/(x_0)$ is a direct sum of cyclic modules this completes the proof. □

Now all that remains is to prove that this decomposition is unique.

Lemma 4.14. *Let M be a finitely generated p-module. Then there are unique integers $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant 1$ such that $M \cong R/p^{k_1}R \oplus \cdots \oplus R/p^{k_n}R$.*

Proof. We have shown such integers exist. Consider a particular isomorphism

$$M \cong R/p^{k_1}R \oplus \cdots \oplus R/p^{k_n}R.$$

Then M_p has dimension n as a vector space over R/pR (M_p splits into a direct sum, each one dimensional over R/pR) so in particular any two expansions of M must have the same number of terms. We calculate that

$$pM \cong R/p^{k_1-1}R \oplus \cdots \oplus R/p^{k_n-1}R.$$

Thus if

$$R/p^{k_1}R \oplus \cdots \oplus R/p^{k_n}R \cong R/p^{k'_1}R \oplus \cdots \oplus R/p^{k'_n}R,$$

then

$$R/p^{k_1-1}R \oplus \cdots \oplus R/p^{k_n-1}R \cong R/p^{k'_1-1}R \oplus \cdots \oplus R/p^{k'_n-1}R.$$

Thus we can perform an induction on $k_1 + \cdots + k_n$ to conclude that for each i, $k_i - 1 = k'_i - 1$, and thus $k_i = k'_i$ for each i. The base case, where $k_1 + \cdots + k_n = 0$, occurs when $M = 0$, and we get the empty product. □

Recall the Chinese remainder theorem, which implies that for distinct primes $p_1, \ldots, p_n$,

$$R/p_1^{k_1} \ldots p_n^{k_n} R \cong R/p_1^{k_1} R \oplus \cdots \oplus R/p_n^{k_n} R.$$

Given any finitely generated R module M, we can find two integers n and m and a unique family of integers k_{ij}, with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ and with $0 \leqslant k_{i1} \leqslant \cdots \leqslant k_{im}$ for each i, such that

$$M = R^n \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} R/p_i^{k_{ij}} R.$$

For $j \in \{1, \ldots, m\}$ define

$$q_j = p_1^{k_{1j}} \ldots p_n^{k_{nj}}.$$

Then the Chinese remainder theorem implies

$$M \cong R^n \oplus R/q_1 R \oplus \cdots \oplus R/q_m R,$$

and $q_1 \mid q_2 \mid \cdots \mid q_n$. These elements of R are also a unique expression of M up to multiplication by units. Thus they are known as the *invariants* of M.

Example. *Let G be a finitely generated abelian group. Then we may apply the results we have proved, since $\mathbf{Z}$ is principal, we can apply the results above to G. In particular, there exists an integer n, known as the* rank *of G, and positive integers $n_1 \mid n_2 \mid \cdots \mid n_m$ such that*

$$G \cong \mathbf{Z}^n \oplus \mathbf{Z}/n_1 \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/n_m \mathbf{Z}.$$

In particular, every finite abelian group is a product of finite cyclic groups.

Example. *Let T be an endomorphism of a vector space V over a field k. Then V has the structure of a $k[X]$ module. The Cayley Hamilton theorem implies that $Ann(V)$ contains the characteristic polynomial $f(X) = \det(X - T)$. In particular, V is a torsion module over $k[X]$. It therefore follows that there are irreducible polynomials $f_1, \ldots, f_n \in k[X]$ and integers $\{k_{ij}\}$ such that*

$$V \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} k[X]/f_i^{k_{ij}} k[X].$$

In particular, we conclude that we can write V as a direct sum of T-invariant subspaces W_{ij} which are cyclic with respect to the opeartion of T. If k is algebraically closed, we may assume without loss of generality that there exists $\lambda_1,\dots,\lambda_n \in k$ such that $f_i = X - \lambda_i$, since up to units these are the only irreducible polynomials. In the subspace

$$W_{ij} \cong k[X]/(X-\lambda_i)^{k_{ij}}$$

we have a basis $\{v_1,\dots,v_{k_{ij}}\}$ of V corresponding to the polynomials $\{1, X-\lambda_i,\dots,(X-\lambda_i)^{k_{ij}-1}\}$, which for $r \in \{1,\dots,k_{ij}-1\}$ satisfies

$$T(v_r) = v_{r+1} + \lambda_i v_r \quad \textit{and} \quad T(v_{k_{ij}}) = \lambda v_{k_{ij}}.$$

Taking the basis over all the spaces W_{ij} gives a basis of V, and with respect to this basis, the matrix representation of T gives a Jordan normal form *of the operator T. The submatrix corresponding to each T-invariant subspace W_{ij} is called a* Jordan block *of T, and takes the following form*

$$\begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

. Over the real numbers, things are slightly more complex. The only irreducible polynomials are linear or quadratic. The T-invariant subspaces isomorphic to $\mathbf{R}[X]/(X-\lambda_i)^k$ correspond to Jordan blocks as above. Thus it suffices to study how the module $W = \mathbf{R}[X]/f^k$ behaves, where f is a quadratic polynomial with two complex roots $\lambda = a + ib$ and $\lambda^* = a - ib$, where we may assume $b > 0$. Consider the complexification *of W, i.e. the $\mathbf{C}[X]$ module*

$$U = \mathbf{C}[X]/f^k.$$

Then W embeds in U in the natural way. We have

$$U \cong \mathbf{C}[X]/(X-\lambda)^k \oplus \mathbf{C}[X]/(X-\lambda^*)^k,$$

Thus U has a basis formed from two disjoint sets $\{u_1,\dots,u_k\}$ and $\{r_1,\dots,r_k\}$ which forms Jordan blocks corresponding to the polynomials $(X-\lambda)^i$ and $(X-\lambda^*)^j$, i.e. with

$$Xu_j = \lambda u_j + u_{j+1} \quad Xu_k = \lambda u_k,$$

and

$$Xr_j = \lambda^* r_j + r_{j+1} \quad Xr_k = \lambda^* r_k.$$

Let $v_j, w_j \in W$ such that $v_j + iw_j$ corresponds to $u_j \oplus 0$. Then for $1 \leqslant j < k$, $X(v_j + iw_j)$ corresponds to $(\lambda u_j + u_{j+1}) \oplus 0$. It therefore follows that

$$X(v_j + iw_j) = \lambda(v_j + iw_j) + (v_{j+1} + iw_{j+1}).$$

Expanding, we find

$$Xv_j = av_j - bw_j + v_{j+1} \quad \textit{and} \quad Xw_j = bv_j + aw_j + w_{j+1}.$$

Similarily, we verify that

$$Xv_k = av_k - bw_k \quad \textit{and} \quad Xw_k = bw_k + av_k.$$

We also note that $v_j - iw_j$ corresponds to $0 \oplus r_j$ since conjugation in U corresponds to flipping basis in the quotient given by the Chinese remainder theorem. It thus follows that $\{v_1 + iw_1, \ldots, v_k + iw_k, v_1 - iw_1, \ldots, v_k - iw_k\}$ constitute a basis for U as a vector space over $\mathbf{C}$. In particular, this implies that $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ constitute a basis for W over $\mathbf{R}$, for if there are $t_1, \ldots, t_k, s_1, \ldots, s_k \in \mathbf{R}$ such that

$$\sum_{j=1}^{k} t_j v_j + s_j w_j = 0,$$

then

$$\sum_{j=1}^{k} (t_j - is_j)(v_j + iw_j) + (t_j + is_j)(v_j - iw_j) = 2\sum_{j=1}^{k} t_j v_j + s_j w_j = 0$$

Thus $t_j - is_j = t_j + is_j = 0$, so $t_j = s_j = 0$. If we pair up the basis as $\{(v_1, w_1), \ldots, (v_k, w_k)\}$, then the matrix representation is

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

where λ *and* 1 *are* matrices, *i.e.*

$$\lambda = \begin{pmatrix} +a & +b \\ -b & +a \end{pmatrix} \quad \textit{and} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix λ *corresponds to a composition of rotation and dilation, and so we find* W *is isomorphic to* $W_1 \oplus \cdots \oplus W_k$*, where each* W_i *is a copy of* $\mathbf{R}^2$ *given by the image of some map* $\pi_i : \mathbf{R}^2 \to W$*, for each* $1 \leqslant i < k$*,*

$$X(\pi_i(v)) = \lambda\pi_i(v) + \pi_{i+1}(v) \quad \textit{and} \quad X(\pi_k(v)) = \lambda\pi_k(v).$$

Thus by passing to complex-variables, we can return to obtain a purely real-variable characterization of finite dimensional operators.

Theorem 4.15. *Let* M *be a free module over a principal ideal domain* R*, and let* N *be a non-zero finitely generated submodule. Then there is a basis for* M *containing a finite subset* $\{e_1, \ldots, e_n\}$*, and unique nonzero elements* $r_1, \ldots, r_n \in R$ *such that* $r_1 \mid \cdots \mid r_n$ *and* $r_1 e_1, \ldots, r_n e_n$ *is a basis of* M*.*

Proof. We note that N is a submodule of a summand of M which is finitely generated (just take all elements of a basis for M which are used to expand a generating set for N). Thus without loss of generality we may assume that M is finitely generated. Let n be the rank of M.

To prove uniqueness, consider some basis $\{e_1, \ldots, e_n\}$ for M as in the statement of the theorem with nonzero elements $a_1, \ldots, a_n$. There is $0 \leqslant s \leqslant n$ such that $a_1, \ldots, a_s$ are units, and since we can always multiply out units we can assume $a_1 = \cdots = a_s = 1$. If $s + r = n$, then we can set $q_i = a_{s+i}$. The module M/N is finitely generated, and

$$M/N \cong R^s \oplus R/q_1R \oplus \cdots \oplus R/q_rR.$$

The uniqueness statement for modules over a principal ideal domain thus gives the uniqueness statement in this theorem.

Now let $\lambda \in M^*$ be a functional. Then $\lambda(N)$ is an ideal in R. Since R is principal, we can choose λ such that $\lambda(N)$ is a maximal proper ideal among ideals of this form. Suppose $\lambda(N) = (r_1)$. Then $r_1 \neq 0$, because there do exist functionals from M to R which are non-zero on N precisely because N is nonzero. Pick $x_0 \in M$ with $\lambda(x_0) = r_1$. Then for any $\gamma \in M^*$, $\gamma(x_0)$ is divisible by r_1, because otherwise we can find $a_1, a_2 \in R$ such that $(a_1\lambda + a_2\gamma)(x_0)$ is the greatest common denominator of r_1 and $\gamma(x_0)$, so

$(a_1\lambda + a_2\gamma)(N)$ contains $\lambda(N)$ as a proper subset. In particular, this implies that for any basis of M, all coefficients in the expansion of N must be divisible by r_1. Thus we can find $e_1 \in M$ such that $x_1 = r_1 e_1$. Then if K is the kernel of M, then $M = K \oplus (e_1)$. Then $N \cap K$ is a submodule of M with dimension one less than N. Continuing this argument by induction completes the proof. □

The following, very concrete result follows.

Theorem 4.16. *Let R be a principal ideal domain, and suppose the elementary matrices generate $GL_n(R)$. Then by applying row and column operations, any nonzero matrix in $M_n(R)$ can be reduced to the form*

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_m \end{pmatrix}.$$

where $a_1, \ldots, a_m$ are nonzero and $a_1 \mid a_2 \mid \cdots \mid a_m$.

In particular, this result gives us a reduce any finite family of linear equations to a canonical form. If $\mathfrak{o}$ is a complete discrete valuation ring, then this result can be applied in $\mathfrak{o}[[X]]$. But there are principal rings in which one cannot use this result, since the elementary matrices do not generate $GL_n(R)$.

Part II

Commutative Algebra

In this part of the notes we study more advanced techniques in the theory of commutative rings, with applications dealing mainly with algebraic geometry and number theory. Most of our techniques utilize the powerful assumption that our rings are Noetherian, which make certain finite methods more tractable.

Chapter 5

Noetherian Rings

A ring A is *Noetherian* if every ideal of A is finitely generated, or equivalently, if every ascending chain of ideals terminates. The Noetherian property was used first by Hilbert, but it's importance as an axiom in algebraic geometry was discovered by Emmy Noether. We have previously seen the important of Noetherian rings in the context of factorization theory, showing that elements of Noetherian rings can always be factorized into irreducible elements. Most of the rings we encounter in basic algebra are Noetherian. This is because the class is closed under adjoining finitely many new elements to a ring, taking quotients, and taking power series, which are most operations we care about in basic commutative algebra.

Lemma 5.1. *Every quotient ring over a Noetherian ring is Noetherian.*

Proof. Every ideal in the quotient ring corresponds to an ideal in the covering ring, which is finitely generated, and these generators project onto generators for the original idea. □

Thus, provided we can prove that $A[x_1,\dots,x_n]$ is Noetherian if A is Noetherian, we can prove that all finitely generated algebras over Noetherian rings are Noetherian. This is provided by Hilbert's basis theorem.

Theorem 5.2. *If A is Noetherian, then $A[x]$ is Noetherian.*

Proof. Let $\mathfrak{a}$ be an ideal of $A[x]$. For each $n \geqslant 0$, let $\mathfrak{a}_n$ be the ideal in A consisting of $a \in A$ such that there is $f \in \mathfrak{a}$ with $\deg(f) = n$ and with leading coefficient a. Since $A[x]$ is Noetherian, the increasing sequence $\{\mathfrak{a}_n\}$ must terminate, so there is N such that $\mathfrak{a}_{N+n} = \mathfrak{a}_N$ for all $n \geqslant 0$. For

each $i \in \{1,\dots,N\}$, pick a_{ij} for $j \in \{1,\dots,m_i\}$ such that $\mathfrak{a}_i = (a_{i1},\dots,a_{im_i})$, and then pick f_{ij} with $\deg(f_{ij}) = i$ such that f_{ij} has leading coefficient a_{ij}. We claim the family f_{ij} generates $\mathfrak{a}$, which shows $\mathfrak{a}$ is finitely generated, so that $A[x]$ is Noetherian. We can verify this by induction. If $a \in \mathfrak{a}$, then there exists $t_1,\dots,t_{m_0}$ such that $a = t_1 f_{01} + \cdots + t_{m_0} f_{0m_0}$. If $i \leqslant N$, and f has degree i, then there exists $t_j \in A$ such that $f - t_1 f_{i1} - \cdots - t_{m_i} f_{in}$ has degree less than i. If f has degree K greater than N, then we can find $t_j \in A$ such that $f - X^{K-N}(t_1 f_{N1} + \cdots + t_{m_N} f_{Nm_N})$ has degree less than K. Thus an inductive procedure verifies $\mathfrak{a}$ is generated by the set $\{f_{ij}\}$. □

Corollary 5.3. *Any finitely generated algebra over a Noetherian ring is Noetherian.*

Modifying the proof that $A[X]$ is Noetherian gives that $A[[X]]$ is Noetherian.

Theorem 5.4. *If A is a Noetherian ring, then $A[[X]]$ is Noetherian.*

Proof. We modify the proof that $A[X]$ is Noetherian if $A[[X]]$. Let $\mathfrak{a}$ be an ideal in $A[[X]]$, and for each i, we let $\mathfrak{a}_i$ be the set of $a \in A$ such that there is $f \in A[[X]]$ such that $X^i a + X^{i+1} f \in \mathfrak{a}$. Then $\mathfrak{a}_i$ is an increasing sequence of ideals in A. Since A is Noetherian, there is n such that $\lim_{i\to\infty} \mathfrak{a}_i = \mathfrak{a}_n$. For each $i \in \{1,\dots,n\}$, let $a_{i1},\dots,a_{im_i} \in A$ be generators for $\mathfrak{a}_i$. Pick $f_{ij} = a_{ij}X^i + \cdots \in \mathfrak{a}$. We now proceed by induction on the order of $f \in \mathfrak{a}$ to show that $\mathfrak{a}$ is generated by the power series f_{ij}. If f has degree $i \leqslant n$, then we can write $f = aX^i + gX^{i+1}$ for some $g \in A[[X]]$. Then we can write $a = t_1 a_{i1} + \cdots + t_{m_i} a_{im_i}$. Thus $f - t_1 f_{i1} - \cdots - t_{m_i} f_{im_i}$ has degree greater than i. Proceeding inductively, for any $f \in \mathfrak{a}$, there exists t_{ij} such that

$$f - \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} f_{ij}$$

has order greater than n, and it suffices to write such an element as an element of $(f_{n1},\dots,f_{nm_n})$. If $f \in \mathfrak{a}$ has order $k \geqslant n$, then we can find $t_{k1},\dots,t_{km_n}$ such that

$$f - \sum_{i=1}^{m_n} X^{k-n} t_{ki} f_{ni}.$$

But this means we can write

$$f = \sum_{i=1}^{m_n} \left(\sum_{k=n}^{\infty} X^{k-n} t_{ki} \right) f_{ni} \in (f_{n1},\dots,f_{nm_n}).$$ □

A more general definition is often useful in commutative algebra. A module M is *Noetherian* if every submodule is finitely generated. This is equivalent to an ascending chain condition for submodules. A quotient of a Noetherian module is Noetherian, as is the direct sum $M \oplus N$ of two Noetherian modules. To see the latter property, we note that if we have an exact sequence

$$0 \to K \to M \to N \to 0$$

Then M is Noetherian if and only if N and K are. If M is Noetherian, K can be identified as a submodule, and N as a quotient, so this is trivial. Conversely, if K and N are Noetherian, and M_0 is a submodule of M, then $M_0 \cap K$ is finitely generated by $x_1, \dots, x_n$ and $M_0/N \cap M_0$, viewed as a submodule of N, is finitely generated by $y_1 + N, \dots, y_m + N \in M_0$, with $y_n \in M_0$. Thus given any $x \in M_0$, there is a_n such that $x - \sum a_n y_n \in N = K$, so $x - \sum a_n y_n = \sum b_n x_n$, showing that $x \in (x_1, \dots, x_n, y_1, \dots, y_m)$.

Theorem 5.5. *Every finitely generated module over a Noetherian ring is Noetherian.*

Proof. Let A be a Noetherian ring, and M a finitely generated module over A. We let M be generated by $x_1, \dots, x_n$, and we prove the theorem by induction on n. For $n = 1$, M is isomorphic to an ideal in A, which is finitely generated by assumption. In general, we have a commutative diagram $0 \to Ax_n \to M \to M/Ax_n \to 0$ induced by multiplication, and Ax_n and M/Ax_n are Noetherian by induction, so M is also Noetherian. □

In geometric terms, the next theorem says that every variety is the unique union of finitely many maximal irreducible subvarieties, exploting the heuristic that maximal ideals are often prime.

Theorem 5.6. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring. Then among all prime ideals containing $\mathfrak{a}$, there are only finitely many which are minimal.*

Proof. Since A is Noetherian, if this proposition fails, then their must be a maximal ideal $\mathfrak{a}$ in which it fails. We know that $\mathfrak{a}$ cannot be prime, so there is $a, b \notin \mathfrak{a}$ such that $ab \in \mathfrak{a}$. The ideals $\mathfrak{a} + Aa$ and $\mathfrak{a} + Ab$ are strictly larger than A, so there are only finitely many prime ideals containing $\mathfrak{a} + Aa$ and $\mathfrak{a} + Ab$ subject to inclusion. But if $\mathfrak{a} \subset \mathfrak{p}$, then since $ab \in \mathfrak{p}$, either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, implying that $\mathfrak{a} + Aa \subset \mathfrak{p}$ or $\mathfrak{a} + Ab \in \mathfrak{p}$. Thus we reach a contradiction. □

Chapter 6

Localization

In many situations, we study a commutative ring A and wish to embed A in a larger ring B in which certain elements $a \in A$ have an inverse in B. Unfortunately, this is not always possible. For instance, if $a^2 = 0$, then we cannot possibly embed A in such a way that makes a invertible. More generally, if $f : A \to B$ is a homomorphism in which $f(a)$ is invertible, and $ab = 0$, then we must have $f(b) = 0$. This implies that if we desire f to be injective, then the ring A cannot have any zero divisors. Nonetheless, if we remove this condition, then the only condition that prevents $f(a)$ from having an inverse is if $a = 0$. For a given set $S \subset A$, constructing the most general map $f : A \to B$ such that $f(A) \subset U(B)$ is a process known as *localization*.

6.1 Fields of Fractions

The classical situation where we can localize is in the case where A is an integral domain, in which case the problems of zero divisors disappear completely. In this case, we can embed A into it's *field of fractions* B, which consist of formal quotients a/b, with $b \neq 0$, where a/b is identified with c/d if $ad - bc = 0$. After identification, we can define a multiplication and addition operation by setting $(a/b)(c/d) = ac/bd$, and by setting $(a/b) + (c/d) = (ad+bc)/bd$. It is simple to check these operations are well defined on B. Then B is given the structure of a commutative ring in which every nonzero element has an inverse. Thus B is not only a ring, but a field, and we can embed A in B by mapping a to the formal quotient $a/1$.

Example. *The localization of* $\mathbf{Z}$ *produces a field of fractions which is just the rational numbers* $\mathbf{Q}$ *in disguise. Constructing the field of fractions over an integral domain is essentially just a generalization of this process.*

Example. *If* $A[X_1,\ldots,X_n]$ *is a polynomial ring with coefficients in some integral domain* A*, then the polynomial ring is an integral domain, and performing localization gives the field* $k(X_1,\ldots,X_n)$ *of rational functions over the field of fractions* A*, which consists of all finitary expressions of the form*

$$\frac{\sum a_\alpha X^\alpha}{\sum b_\beta X^\beta}$$

These can be considered as functions mapping certain 'nonsingular' elements of A *into it's field of fractions* k*. In particular,* f/g *is defined at* $x \in k$ *if* $g(a) \neq 0$*, because then the quotient* $f(a)/g(a) = f(a)g(a)^{-1}$ *is well defined. As an example, the field of fractions of* $\mathbf{Z}[X_1,\ldots,X_n]$ *is the field* $\mathbf{Q}(X_1,\ldots,X_n)$ *of rational functions over the rationals.*

Example. *Let* $A(D)$ *denote the complex algebra of functions holomorphic in some connected open region* D *of* $\mathbf{C}$*. Then* $A(D)$ *is an integral domain, for if* $fg = 0$*, where* $f, g \neq 0$*, then* $f^{-1}(0)$ *and* $g^{-1}(0)$ *are two discrete sets whose union is* D*, which is impossible. We may therefore form the field of fractions of* $A(D)$*, which is precisely the set of meromorphic functions on* D*. These functions* f/g *are defined except for certain points upon which* $g(z) = 0$*, except in the case that* z *is a removable singularity of* g*, which means that we can write* $f/g = f_1/g_1$*, where* $g_1(z) \neq 0$*.*

More generally, given any multiplicatively closed subset S of $A - \{0\}$, we let $S^{-1}A$ be the subring of the field of fractions of A, consisting of all fractions of the form a/s, with $s \in S$. This is called the *localization* of A at S, for reasons we will get to later. The ring $S^{-1}A$ satisfies the following universal property; any ring homomorphism $f : A \to B$ such that $f(s) \in U(B)$ for each $s \in S$ extends to a unique morphism from $S^{-1}A$ to B. Thus f is the 'most general way' to construct inverses of S in a consistent way.

Example. *Let* A *be an integral domain, and* $\mathfrak{p}$ *a prime ideal in* A*. Then* $A - \mathfrak{p}$ *is a multiplicatively closed subset of* A*. Thus we can consider the localization at* $A - \mathfrak{p}$*, which we will denote by* $A_\mathfrak{p}$*. For instance,* $\mathbf{Z}_{(2)} = \mathbf{Z}[1/2]$*. The ring* $A_\mathfrak{p}$ *is a* local ring*, in the sense that it has a unique maximal ideal*

$$\mathfrak{p}_\mathfrak{p} = \{r/s : r \in \mathfrak{p}, s \notin \mathfrak{p}\}.$$

Often, the way we obtain results in commutative algebra is by reducing studies to local rings, which makes localizing at a prime ideal very useful.

6.2 Factorization in Localizations

Let us show that for any multiplicatively closed set S in a unique factorization domain A, the localization $S^{-1}A$ is a unique factorization domain.

Lemma 6.1. *If A is an integral domain containing some element x, then x becomes a unit in $S^{-1}A$ if and only if $(x) \cap S \neq \varnothing$.*

Proof. If $x(m/n) = 1$, $xm = n \in S$. If $xm \in S$, then $x(m/xm) = 1$. □

Lemma 6.2. *If A is an integral domain, and p is prime in A, then p is either a unit in $S^{-1}A$ or irreducible in $S^{-1}A$.*

Proof. Suppose $p = (m/n)(x/y)$ for $m, n, x, y \in A$. Then $nyp = mx$. Thus p divides mx, so it follows that either p divides m or p divides x. If p divides m, then we can write $m = m_0 p$, and it then follows that $1 = (m_0/n)(x/y)$, so that x/y is a unit. Similarily, if p divides x, then m/n is a unit. □

Lemma 6.3. *Let A be factorial. For any $a, b \in A$, with b nonzero, a/b is irreducible in $S^{-1}A$ if and only if $a/b = up$, where $u \in U(S^{-1}A)$, and p is irreducible in A and $S^{-1}A$.*

Proof. Suppose $a, b \in A$, and a/b is irreducible in $S^{-1}A$. Write $a = p_1 \dots p_n$ and $b = q_1 \dots q_n$, where p_i and q_i are irreducible in A. Then exactly one p_i is irreducible in $S^{-1}A$, the others being units. But this means $a/b = up_i$ for some $u \in U(S^{-1}A)$. The converse is obvious. □

Lemma 6.4. *If y differs from x by a unit, and y is uniquely factorizable, then x is uniquely factorizable.*

Proof. Write $x = yu$, where y is factorizable, $y = p_1 \dots p_n$, then $x = up_1 \dots p_n$. Now suppose that x can be factorized in two ways

$$x = p_1 \dots p_n = q_1 \dots q_m$$

Then,

$$ux = (up_1)p_2 \dots p_n = p_1' \dots p_n' = (uq_1)q_2 \dots q_m = q_1' \dots q_n'$$

so, up to a permutation, $p_i' = u_i q_{\pi(i)}'$. But one verifies, by taking the vary cases, that this implies that $p_i = v_i q_{\pi(i)}$, where v_i is a unit. □

Theorem 6.5. *If A is factorial, and $S \subset A - \{0\}$ is multiplicative, then $S^{-1}A$ is factorial.*

Proof. Let a/b be given. We need only verify that a/b differs from a uniquely factorizable element by a unit. Write $a = p_1 \dots p_n$, where p_i is irreducible in A. We know that each p_i is either still irreducible, or a unit, so without loss of generality we may as well assume all p_i are irreducible in $S^{-1}A$. Suppose

$$p_1 \dots p_n = (u_1 q_1) \dots (u_m q_m) = (u_1 \dots u_m q_1) q_2 \dots q_m$$

Let $u_1 \dots u_m = x/y$. If $u_1 \dots u_m$ can be written as the quotient of two units in A, then we are done, for then the p_i and q_i differ by units in A, and thus the p_i differs from $u_i q_i$ by a unit. We show this is the only case that could happen, since we assume the p_i are irreducible in $S^{-1}A$.

If y is not a unit in A, write $y = y_1 \dots y_k$. If x is a unit in A, then when we apply unique factorization in A, we see y_1 differs from some p_i by a unit in A. But y_1 is a unit in $S^{-1}A$, so that p_i is a unit in $S^{-1}A$. If x is not a unit, then we may consider $x = x_1 \dots x_l$, and may assume no x_i and y_j differ by a unit (by cancelling like terms), so that when we apply unique factorization, y_1 is mapped to p_i again, contradicting the irreducibility of p_i. Thus y must be a unit in A, and when we expand x as we have already done, and write

$$(p_1/y) \dots p_n = x_1 \dots x_l q_1 \dots q_m.$$

But then some x_i differs from a p_j by a unit in A, hence p_j is a unit in $S^{-1}A$. □

6.3 Partial Fractions

In the field of fractions of a factorial ring A, it is often useful to perform a partial fraction decomposition to isolate primes in the denominators of fractions. This is useful even in basic integration theory, where a partial fraction decomposition enables us to find simple antiderivatives for rational functions.

Lemma 6.6. *Let R be a factorial ring, and let k be the field of fractions. Then for each $x \in R$, there exists primes $q_1, \dots, q_m$, integers $r_1, \dots, r_m$, and $a_0, a_1, \dots, a_m \in R$ where a_i is not divisible by q_i for each i, such that*

$$x = a_0 + \frac{a_1}{q_1^{r_1}} + \dots + \frac{a_n}{q_n^{r_n}}.$$

Moreover, the integers $r_1,\dots,r_n$ are uniquely determined, and so too are the values $a_1,\dots,a_n$ if they are interpreted modulo $q_i^{r_i}$.

Proof. Any element x of the quotient field k can be written uniquely (modulo units) as

$$u\frac{p_1^{k_1}\dots p_n^{k_n}}{q_1^{r_1}\dots q_m^{r_m}}$$

where u is a unit in R and $p_1,\dots,p_n$ and $q_1,\dots,q_m$ are distinct primes in A. To show the existence statement, we note there is a_1, a_2 such that

$$a_1(q_1^{r_1}\dots q_{m-1}^{r_{m-1}}) + a_2 q_m^{r_m} = 1.$$

Then

$$\begin{aligned} u\frac{p_1^{k_1}\dots p_n^{k_n}}{q_1^{r_1}\dots q_m^{r_m}} &= u\frac{p_1^{k_1}\dots p_n^{k_n}(a_1(q_1^{r_1}\dots q_{m-1}^{r_{m-1}}) + a_2 q_m^{r_m})}{q_1^{r_1}\dots q_m^{r_m}} \\ &= u\frac{a_1 p_1^{k_1}\dots p_n^{k_n}}{q_m^{r_m}} + u\frac{a_2 p_1^{k_1}\dots p_n^{k_n}}{q_1^{r_1}\dots q_{m-1}^{r_{m-1}}}. \end{aligned}$$

Continuing this process inductively gives an expansion $x = a_1/q_1^{r_1} + \dots + a_m/q_m^{r_m}$. Now suppose that

$$a_0 + a_1/q_1^{r_1} + \dots + a_m/q_m^{r_m} = a_0' + a_1'/q_1^{s_1}\dots a_m'/q_m^{r_m'}.$$

Then for each i,

$$\frac{a_i}{q_i^{r_i}} - \frac{a_i'}{q_i^{s_i}} = (a_0' - a_0) - \sum_{j\neq i}\left(\frac{a_i}{q_j^{r_j}} - \frac{a_i'}{q_j^{s_j}}\right).$$

Assume without loss of generality that $s_i \leqslant r_i$. Let q be the lowest common denominator of the denominators on the right hand side. Then there exists $a \in A$ such that

$$q(a_i - a_i' q_i^{r_i - s_i}) = q_i^{r_i} a.$$

If $r_i > s_i$ this leads to a contradiction since we assume a_i is not divisible by q_i. Thus $r_i = s_i$. Thus $q(a_i - a_i') = q_i^{r_i} a$. Since q_i does not divide q, we conclude $q_i^{r_i}$ divides $a_i - a_i'$, which gives the congruence statement. □

Suppose R is a graded integral domain $\bigoplus_{n=0}^{\infty} R_n$ such that the function

$$N(x) = \max\{k \geqslant 0 : x_k \neq 0\}$$

gives R the structure of a Euclidean domain. Such a function N is automatically multicative. If $q \in R$ is prime, n is an integer, and $N(q) \geqslant 1$, then we can apply the Euclidean algorithm for any $r \in A$ to find unique values $r_i \in R$ for each integer $i \geqslant -n$ such that

$$r/q^n = \sum_{i=-n}^{\infty} r_i/q^i,$$

where $r_i = 0$ for sufficiently large i and $N(r_i) < N(q)$ for each $k \in \{0, \ldots, n\}$. The existence of such an expansion follows from the Euclidean algorithm. To prove uniqueness, suppose

$$\sum_{i=-n}^{\infty} r_i/q^i = 0,$$

where $N(r_i) < N(q)$ for each i. Then

$$\sum_{i=-n}^{\infty} r_i q^{n+i} = 0$$

and for each i with $r_i \neq 0$,

$$(n+i) \cdot N(q) \leqslant N(r_i q^{n+i}) < (n+i+1)N(q).$$

Since the ring is graded, this implies all $r_i = 0$, which proves uniqueness. One can also see from this calculation that the largest nonzero coefficient i such that $r_i \neq 0$ corresponds to the largest i with $N(r) \geqslant (n+i) \cdot N(q)$.

6.4 General Localization

Considering this problem in a more general viewpoint, we consider a set $S \subset A$, and try to find the 'most general' homomorphism $f : A \to B$ such that $f(s)$ is invertible for each $s \in S$. If $f(s)$ and $f(t)$ are invertible, then $f(st) = f(s)f(t)$ is invertible, so we may assume from the outset that S is

closed under multiplication. We may also assume that $1 \in S$, because $f(1)$ is always invertible. In this case, S is a multiplicative submonoid of A, which we call a *multiplicative set*. By 'localizing' S, we mean extending A to a space B in which all elements of S have an inverse. By a localization of A by S, we mean a ring $S^{-1}A$ together with a map $i : A \to S^{-1}A$ such that for any homomorphism $f : A \to B$ such that $f(s)$ is invertible for each $s \in S$, there is a unique homomorphism $S^{-1}f : S^{-1}A \to B$ for which $f = S^{-1}f \circ i$. This is an initial object in a certain category, and is therefore unique up to isomorphism.

More generally, suppose that a commutative ring A has zero divisors. Then forming the field of fractions is impossible – we cannot give every element of A an inverse simultaneously. More generally, we might hope to find the 'most general' homomorphism $i : A \to S^{-1}A$ such that $i(s)$ is invertible for each element s in some multiplicative set S. In particular, we hope to find an object i and $S^{-1}A$ such that for *any* homomorphism $f : A \to B$ into a commutative ring B such that $f(s)$ is invertible for each $s \in S$, there is a homomorphism $S^{-1}f : S^{-1}A \to B$ such that $f = S^{-1}f \circ i$. This is an initial object in the category of homomorphisms from A into some other ring B which map S to units, which means it is unique up to isomorphism.

Often, the correct technique to finding a universal object is to determine what properties the object must have, and then trying to form a formal structure based on these properties. Given what we know, this object will either fail to be constructed in general, in which case we must try and find more properties of the object, or the formal object we construct will often be the required universal object. Let us try and derive what our initial object $S^{-1}A$ should be 'forced to have'. Note that if $f : A \to S^{-1}A$ is the required morphism, then the set B of elements of $S^{-1}A$ of the form $i(a)i(s)^{-1}$, for $a \in A$ and $s \in S$ is a subring of $S^{-1}A$ (an easy calculation left to the reader). This means that $i : A \to B$ is a map in which each $f(s)$ is invertible, and so there must be a map $S^{-1}i : S^{-1}A \to B$ such that $i = S^{-1}i \circ i$. Clearly $S^{-1}i$ must be the identity map, which implies $B = S^{-1}A$. Now, let us determine when $i(a)i(s)^{-1} = i(b)i(t)^{-1}$. If this is true, then $i(at - bs) = 0$. One condition guaranteeing this to be true is if there is $u \in S$ for which $u(at - bs) = 0$, because then $f(u)f(at - bs) = 0$, and multiplying by $f(u)^{-1}$ gives the required property. It turns out that these properties are sufficient to formally define $S^{-1}A$.

Consider the set $S^{-1}A$ whose objects are fractions a/s, as in the field of

fractions of an integral domain, but where $a \in A$ and $s \in S$. We identify two fractions a/s and b/t if there is an element $u \in S$ such that $u(at - bs) = 0$. We define multiplication by setting $(a/s)(b/t) = (ab/st)$, and addition by $a/s + b/t = (at + bs)/ts$. This gives $S^{-1}A$ a ring structure, and we have a map $i : A \to S^{-1}A$ given by $i(a) = a/1$, and then $i(s)^{-1} = 1/s$. If $f : A \to B$ is any ring homomorphism in which $f(s)$ is invertible for each $s \in S$, then we can define $S^{-1}f : S^{-1}A \to B$ by $S^{-1}f(a/s) = f(a)f(s)^{-1}$, and then it is a simple procedure to verify that the required diagram commutes, and that f is unique. Thus $S^{-1}A$ is exactly the initial object we required.

Example. *Let X be a topological space, and let $C(X)$ denote the ring of all (real/complex valued) continuous functions defined on X. If $p \in X$, then set the set S of all functions f with $f(p) \neq 0$ is a multiplicative set closed under multiplication. Thus we can consider the localization $S^{-1}[C(X)]$, which we denote by $C(X)_p$. Since $C(X)$ is almost never an integral domain, the map $C(X) \to C(X)_p$ will likely not be injective. Indeed, two functions f and g will be identified in $C(X)_p$ if there is a function h with $h(p) \neq 0$, and with $h(f - g) = 0$. Since $h(p) \neq 0$, the set of points q where $h(q) \neq 0$ contains an open neighbourhood of zero, and this implies that $(f - g)(q) = 0$ on this neighbourhood. Conversely, it suitably nice topological spaces (where Urysohn's theorem applies), if f agrees with g in a neighbourhood of p, we can find a function h such that h vanishes outside this neighbourhood, and then $h(f - g) = 0$. Thus functions are identified in $C(X)_p$ precisely when they are locally equal around p, and this is the context in which the term localization emerged, because localization takes a ring of functions, and identifies those functions which locally agree. More generally, if we set S to be the set of all functions with $f(p) \neq 0$ for all p in some $Y \subset X$, then $C(X)_Y$ consists of the equivalence class of all functions which agree on a neighbourhood of Y, provided we can construct functions vanishing outside of a neighbourhood of Y, with no zeroes on Y.*

Example. *Similarily, if M is a differentiable manifold, then the space $C^\infty(M)$ of differentiable functions on M forms a ring. For a fixed $p \in M$, the space of functions not vanishing at p forms a multiplicative set, and the corresponding localization corresponds to the equivalence class of differentiable functions which agree in a neighbourhood of p, known as the space of germs of differentiable functions at p. Viewed as a vector space over the real numbers, the dual space of germs of differentiable functions is used to construct the tangent space of a manifold at a point. A similar process is used to construct the germ of analytic functions on an analytic/holomorphic manifold, where we replace $C^\infty(M)$*

with $C^\omega(M)$.

Example. *Let X be an algebraic set over an algebraically closed field. Let U be a Zariski open subset of X. A 'natural' coordinate ring on U is the ring of rational functions on X which do not vanish at any point on U. In particular, we can consider the set*

$$S = \{f \in k[X] : f(x) \neq 0 \text{ for all } x \in U\},$$

which is multiplicatively closed and does not contain zero, and then consider the localization $k[U] = S^{-1}(k[X])$. For each $x \in U$, the evaluation morphism $ev_x : k[X] \to k$ induces an evaluation morphism $ev_x : k[U] \to k$. Of course, if X is irreducible, then $k[X]$ is an integral domain, so $k[U]$ corresponds precisely to a rational function on X. But things can become more tricky if X is reducible, so that $k[X]$ has zero divisors. For instance, in the simple example where $X = \{0,1\}$ in $\mathbf{A}^1$ and $U = \{0\}$, $k[U]$ is isomorphic to k, since x becomes zero in $k[U]$, so that f/g becomes equal to $f(0)/g(0)$. Thus $k[U]$ also describes the 'local behaviour' of $k[X]$ at p. For $V \subset U$, we have natural restriction morphisms from $k[U]$ to $k[V]$, so for each $p \in k[X]$ we can consider the ring $\mathcal{O}_p = \lim_{p \in U} k[U]$, which can also be described as the localization of $k[X]$ by the set $S = \{f \in k[X] : f(p) \neq 0\}$. Alternatively, we can write $\mathcal{O}_p = k[X]_{\mathfrak{m}_p}$, where $\mathfrak{m}_p = \{f \in k[X] : f(p) = 0\}$. By the Nullstellensatz, since $k[U]$ is Noetherian and reduced, there exists an algebraic set Y such that $k[U]$ is isomorphic to $k[Y]$. Thus we can view all open subsets of an algebraic set as an algebraic set, in some sense.

Perhaps this formal approach is not so intuitive from a more geometric perspective. But there is a more 'natural' approach to forming $S^{-1}A$, but it is much more messy. When learning fractions for the first time, you viewed them as ways to 'divide' certain integers into other integers. If you have 6 apples, you can 'apply' the fraction $1/2$ to divide the apples into two sets of three apples, the fraction $1/3$ to divide the 6 apples into three sets of two, but one cannot apply the fraction $1/5$. In other words, we can view a fraction $1/n$ as a partial function on $\mathbf{Z}$ (defined on $n\mathbf{Z}$, to be precise), which outputs m when given input nm. Similarily, n/m is the partial function defined on the set of integers k such that nk is divisible by m, in which case applying n/m to k results in nk/m. It seems reasonable to set fractions equal if they agree on the common input upon which they are defined. That is, we should set $1/2 = 2/4$, because they have the same domain,

and are equal to one another on this domain. To abstract these ideas to form $S^{-1}A$, we let Φ denote the set of all A-module homomorphisms from $(s) \to A$, for some $s \in S$. We then form a family of equivalence classes on Φ by identifying $f : (s) \to A$ and $g : (t) \to A$ if f and g agree on (st). On these equivalence classes, we can define addition between $f : (s) \to A$ and $g : (t) \to A$ by letting $f + g$ be the addition of the functions as morphisms from (st) to A. Similarily, we define fg to be $f \circ g$, once f and g are restricted to the proper ideals. We then embed A in Φ by mapping $a \in A$ to the 'multiplication by a' homomorphism from A to itself. Given $s \in S$, the inverse of s is the homomorphism with domain (s) mapping sa to a. Unfortunately, if A has zero divisors, then this approach does not work, in which case one must first quotient A by the ideal of all elements of A which are annihilated by elements of S.

Remark. Localization can be done in noncommutative rings. However, the resulting rings $S^{-1}A$ are extremely nontrivial to analyze, and as such we do not consider them. This follows because expressions of the form $rs^{-1}t + uv^{-1}w$ cannot in general be reduced to having a single common denominator. Thus one may have to repeat the process of localization many times to obtain inverses for all elements of S, and even if we repeat the process finitely many times we may still not end up with all the right inverses. What's more, even if A has no zero divisors, it can still be difficult to determine if the localization of A is nontrivial. However, one can in certain situations achieve success, by generalizing the 'partial homomorphism' technique of the last paragraph. The general technique is known as Ore localization, and is left for another time.

It is important to localize not only rings R, but also R-modules. Given a multiplicative subset S of a ring R and an R-module M, the module $S^{-1}M$ together with $i : M \to S^{-1}M$ is the initial R-module homomorphism in the category of all morphisms $f : M \to N$ such that for any $s \in S$, the map $x \mapsto sx$ is an isomorphism of N. To construct this initial object, we consider elements of the form x/s, with $x \in M$ and $s \in S$, and we identify x/s and y/s' if there is $s'' \in S$ such that $s''(xs' - ys) = 0$. The module $S^{-1}M$ is also naturally a $S^{-1}R$ module. Given this structure, the map $M \mapsto S^{-1}M$ is a functor from the category of R modules to $S^{-1}R$ modules, since if $f : M \to N$, then given the map $i : N \to S^{-1}N$, the map $f \circ i$ induces a $S^{-1}M$-linear morphism of $S^{-1}M$ with $S^{-1}N$.

6.5 General Properties

It is very useful to note the relation between the localization of a module and the operation of extension of scalars. Indeed, if we let $i : R \to S^{-1}R$ denote the inclusion map, then for any R module M, the localization $S^{-1}M$ is precisely the same as the push forward module $i_*(M) = S^{-1}R \otimes_R M$. This can be verified by noting that the two universal properties defining these modules are equivalent, or getting your hands dirty and computing.

In particular, this implies that the functor $M \mapsto S^{-1}M$ is a left adjoint to the forgetful functor $N \mapsto i^*N$ from $S^{-1}R$ modules to R modules. In particular, this automatically implies that the map $M \mapsto S^{-1}M$ preserves direct sums, and is right exact, in the sense that if

$$0 \to M \to N \to L \to 0$$

is a short exact sequence of R-modules, then the induced sequence

$$S^{-1}M \to S^{-1}N \to S^{-1}L \to 0$$

is exact. This follows because L is the cokernel of the map from $M \to N$, and from preservation of colimits it follows that $S^{-1}L$ is the cokernel of the map from $S^{-1}M \to S^{-1}N$. But the functor is actually exact entirely, because if $f : M \to N$ is an injective morphism, and there exists $m \in M$ and $s \in S$ such that $(S^{-1}f)(m/s) = f(m)/s = 0$, then there is $s' \in S$ suhc that $sf(m) = f(sm) = 0$, so that $sm = 0$, implying $m = 0$ in $S^{-1}M$. Thus the functor S^{-1} is exact. In particular, since $S^{-1}M$ is equivalent to $S^{-1}R \otimes_R M$, we conclude that for any commutative ring R and any multiplicative set S, the ring $S^{-1}R$, viewed as an R-module, is flat.

Remark. Unfortunately localization is not a right adjoint however, and as such does not preserve limits. If we consider the fraction field of the integers generated by $S = \mathbf{Z} - \{0\}$, with $M_1 = M_2 = \cdots = \mathbf{Z}$ then the two rings we get from the direct product above are $S^{-1}(\mathbf{Z}^\infty)$ and $\mathbf{Q}^\infty$. The inclusion of $\mathbf{Z}^\infty$ in $\mathbf{Q}^\infty$ certainly identifies $S^{-1}(\mathbf{Z}^\infty)$ with a subspace of $\mathbf{Q}^\infty$, but this subspace is proper; it consists of all infinite sequences of rational numbers with bounded denominator. Since $S^{-1}(\mathbf{Z}^\infty)$ is countable, whereas $\mathbf{Q}^\infty$ is uncountable, these spaces cannot be isomorphic.

There is a close correspondence between ideals of $S^{-1}R$ and R. More generally, there is a close correspondence between submodules of $S^{-1}M$

and M. We have two maps between submodules; if $i : M \to S^{-1}M$ is the inclusion map then the map $N \mapsto i^{-1}(N)$ maps $S^{-1}R$ submodules of $S^{-1}M$ to R submodules of M. Conversely, if N is an R-submodule of M, then $S^{-1}N$ can be naturally identified with an $S^{-1}R$-submodule of $S^{-1}M$ using the exactness of $S^{-1}N$. More concretely,

$$S^{-1}N = \{n/s : n \in N, s \in S\}.$$

This Galois connection gives a tight correspondence between the submodules of each ring.

Lemma 6.7. *The map i^{-1} is a bijective correspondence between $S^{-1}R$ submodules of $S^{-1}M$ and R-submodules N of M such that if $sm \in N$, then $m \in N$.*

Proof. We note that $S^{-1}(i^{-1}(L)) = L$ for any $S^{-1}R$-submodule L of $S^{-1}M$, because if $m/s \in L$, then $m \in i^{-1}(L)$, so $m \in S^{-1}(i^{-1}(L))$ and thus $m/s \in S^{-1}(i^{-1}(L))$. This implies that the map $L \mapsto i^{-1}L$ is injective. If $sm \in i^{-1}L$, then $m \in S^{-1}i^{-1}L = L$, so $m \in i^{-1}N$. Conversely, if N is an R-submodule of M such that if $sm \in N$, then $m \in N$, we claim that $N = i^{-1}S^{-1}N$. Certainly $N \subset i^{-1}S^{-1}N$. Conversely, if $m \in i^{-1}S^{-1}N$, then $m \in S^{-1}N$, so $m = n/s$ for some $n \in N$ and $s \in S$. Thus $n = sm$, which implies that $m \in N$ by assumption. Thus we have verified the correspondence. □

Remark. In the special case of the ideals of the ring R, the lemma above implies as a special case that there is a bijection between prime ideals of $S^{-1}R$ and prime ideals of R not intersecting S. We also note the additional algebraic relation

$$(S^{-1}\mathfrak{a})(S^{-1}\mathfrak{b}) = S^{-1}(\mathfrak{a}\mathfrak{b}).$$

The left hand side is generated by elements of the form $(a/s)(b/s') = ab/ss'$ with $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, and $s, s' \in S$, and the right hand side is generated by elements of the form $(ab)/s$ with $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, and $s \in S$. Since $1 \in S$ the two sides are generated by the same family of elements.

The following two lemmas are easy corollaries.

Proposition 6.8. *If A is principal, then $S^{-1}A$ is principal.*

Proof. This follows because all ideals in $S^{-1}A$ are of the form $S^{-1}\mathfrak{a}$, and if $\mathfrak{a} = (a)$, then $S^{-1}\mathfrak{a} = (a)$. □

Proposition 6.9. *If M is a Noetherian R-module, then $S^{-1}M$ is a Noetherian $S^{-1}R$ module.*

Proof. We note that the localization of a finitely generated module is finitely generated. If L is a submodule of $S^{-1}M$, then $N = i^{-1}(L)$ is a submodule of M, and thus finitely generated. But then $L = S^{-1}(N)$ is finitely generated. □

Remark. A special case of this proposition is that the localization of a Noetherian ring is Noetherian.

Proposition 6.10. *Let $M_1, \ldots, M_n$ are R-modules. and let S be a multiplicative subset of R. Then $S^{-1}(M_1 \cap \cdots \cap M_n) = S^{-1}(M_1) \cap \cdots \cap S^{-1}(M_n)$.*

Proof. We consider the exact sequence

$$0 \to M_1 \cap \cdots \cap M_n \to M \to M/M_1 \oplus \cdots \oplus M/M_n.$$

which gives an exact sequence

$$0 \to S^{-1}(M_1 \cap \cdots \cap M_n) \to S^{-1}(M) \to S^{-1}(M/M_1) \oplus \cdots \oplus S^{-1}(M/M_n).$$

We have a natural isomorphism

$$S^{-1}(M/M_1) \oplus \cdots \oplus S^{-1}(M/M_n) \cong S^{-1}(M)/S^{-1}(M_1) \oplus \cdots \oplus S^{-1}(M)/S^{-1}(M_n),$$

which follow because a quotient is a colimit. This implies the kernel of the map above is equal to $S^{-1}(M_1) \cap \cdots \cap S^{-1}(M_n)$. But this gives the equality above. □

On the other hand, this need not be true for infinite intersections. The reason the proof above fails is because the map into the direct sum no longer holds for infinitely many submodules (one would need to use the direct product instead, which is not preserved under localization).

Example. *Let k be an infinite field, and let $S = k[x] - \{0\}$. For each $a \in k$, let $M_a = (x - a)$ be an ideal in $k[x]$. Then $\bigcap M_a = (0)$, so $S^{-1}(\bigcap M_a) = (0)$. On the other hand, if $S = k[x] - \{0\}$, then $S^{-1}(x - a) = k(x)$ for each a, so $\bigcap S^{-1}(x - a) = k(x)$. Thus we do not have $S^{-1}(\bigcap M_a) = \bigcap S^{-1}(M_a)$.*

Since the quotient of a module M by a submodule N can be identified with the coequalizer of the zero map $0 : N \to M$ and the inclusion map $i : N \to M$, it follows that localization preserves quotients, namely $S^{-1}M/S^{-1}N$ is naturally isomorphic to $S^{-1}(M/N)$. As a special case, if $\mathfrak{a}$ is an ideal in R, then $S^{-1}R/S^{-1}\mathfrak{a}$ is isomorphic to $(S/\mathfrak{a})^{-1}(R/\mathfrak{a})$. Since localization is exact, it also preserves kernels even though they are not colimits. It follows from this that if $M_1,\ldots,M_n$ are submodules of a module M, then $S^{-1}(M_1 \cap \cdots \cap M_n) = S^{-1}(M_1) \cap \cdots \cap S^{-1}(M_n)$. To see this we apply functoriality to the exact sequence

$$0 \to M_1 \cap \cdots \cap M_n \to M \to M/M_1 \oplus \cdots \oplus M/M_n.$$

This technique fails for infinite families of modules, since we have to use the direct product instead of the direct sum in the exact sequence above, which is not preserved by localization.

Example. *Let k be an infinite field, let $M = k[x]$, and for each $a \in K$ define $M_a = (x-1) \in k[x]$. Then each M_a is a $k[x]$ module of M. Let $S = k[x] - \{0\}$. Now $\bigcap_a M_a = (0)$, so $S^{-1}(\bigcap_a M_a) = 0$. On the other hand, $S^{-1}(M_a) = k(x)$ for each a, so $\bigcap S^{-1}(M_a) = k(x)$. Thus localization and infinite intersections does not commute.*

6.6 Local Properties of Rings

Let M be an R-module. Then for each prime ideal $\mathfrak{p}$ of R, we can consider the localization $M_\mathfrak{p}$, and for each $m \in M$, the element $m_\mathfrak{p} \in \mathfrak{p}$. For $m \in M$, the *support* of m is the set of prime ideals $\mathfrak{p}$ where $m_\mathfrak{p} \neq 0$.

Lemma 6.11. *If $x \in M$ and $x_\mathfrak{m} = 0$ for any maximal ideal $\mathfrak{m}$, then $x = 0$.*

Proof. Suppose $m \in M$, and $m_\mathfrak{m} = 0$ for each prime ideal $\mathfrak{m}$. This means that for each prime ideal $\mathfrak{m}$, there is $r \notin \mathfrak{m}$ such that $rm = 0$. But this means that $\mathrm{Ann}(m) = R$ since otherwise it is contained in a maximal ideal, which is prime. But then $1 \in \mathrm{Ann}(m)$ so $m = 1 \cdot m = 0$. □

Corollary 6.12. *$M = 0$ if and only if $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$.*

This statement often allows us to reduce statements about modules to localizations of modules.

Corollary 6.13. *An R-linear homomorphism $\phi : M \to N$ is injective/surjective if and only if $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is for any maximal ideal $\mathfrak{m}$.*

Proof. This follows because $\mathrm{Ker}(\phi_{\mathfrak{m}}) = \mathrm{Ker}(\phi)_{\mathfrak{m}}$ and $\mathrm{Coker}(\phi_{\mathfrak{m}}) = \mathrm{Coker}(\phi)_{\mathfrak{m}}$. □

A property of R-modules is said to be *local* if it is true if and only if it holds under localization by any prime ideal. Thus we have shown injectivity and surjectivity is a local property of modules.

6.7 Preservation of Homomorphisms

On suitably nice rings, it is very simple to relate homomorphisms between M and N and the localizations $S^{-1}(M)$ and $S^{-1}(N)$. We discuss this in a more general context. Let $f : R \to S$ be a homomorphism. For any R-modules M and N, we have an R bilinear map from $f^*(\mathrm{Hom}_R(M,N))$ to $\mathrm{Hom}_S(f_*M, f_*N)$ given by $(s,\phi) \mapsto s \otimes \phi$, which is S-linear. Thus we obtain a homomorphism

$$\alpha : f^*(\mathrm{Hom}_R(M,N)) \to \mathrm{Hom}_S(f^*M, f^*N)$$

from one set to the other.

Theorem 6.14. *Let $f : R \to S$ be a homomorphism such that S is flat as an R-module. If M is a finitely presented R-module, then for any R-module N, α is an isomorphism.*

Proof. TODO (See Proposition 2.10 of Eisenbud). □

6.8 Local Rings

Originally, localization was used to construct the field of fractions of an integral domain. However, it has been studied in more detail to understand the *local rings*, which occur in areas such as complex analysis and algebraic geometry. A commutative ring A is *local* if it has a unique, maximal ideal. This condition is equivalent to saying that the set $A - U(A)$ of non-invertible elements in A forms an ideal, because if A has a unique maximal ideal $\mathfrak{m}$, then for any $a \in A - U(A)$, (a) is an ideal not equal to

A (because if $1 \in (a)$ then a is a unit), so $a \in (a) \subset \mathfrak{m}$. Another equivalent condition is that there exists a maximal ideal $\mathfrak{m}$ such that $1 + \mathfrak{m} \subset U(A)$, because if $x \notin \mathfrak{m}$, then there is y such that $xy \equiv 1$ modulo $\mathfrak{m}$, hence xy is invertible and in particular, x is invertible, so $\mathfrak{m} = U(A)^c$. Conversely, if, in a local ring, $1 + x$ is not invertible, where $x \in \mathfrak{m}$, then $1 + x \in \mathfrak{m}$, so $1 \in \mathfrak{m}$, which is absurd.

Recalling our intuition that maximal ideals in a ring of functions corresponds to a 'point' that the functions operate over, we see that a local ring can be seen as a ring of functions taking values in a unique ring, concentrated at a single point – this is the reason why local rings are called 'local', because they represent the properties of a ring of functions locally around a single point. Indeed, this means that, up to isomorphism, there is a unique field k, and a unique homomorphism from A into k. If a homomorphism $f : A \to k$ corresponds to some 'evaluation map' over elements of A, where k is some field, then we find that A has only a single evaluation map. The main context in which local rings occur is in the study of the localization of certain rings. If $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}^c$ is certainly a multiplicative subset of A containing 1, so we can form the localization with respect to $\mathfrak{p}^c$, which we denote by $A_\mathfrak{p}$, and call the local ring at $\mathfrak{p}$.

Theorem 6.15. *If $\mathfrak{p}$ is a prime ideal, then $A_\mathfrak{p}$ is a local ring.*

Proof. Since $\mathfrak{p}$ is an ideal, $U(A) \subset \mathfrak{p}^c$. If $i : A \to A_\mathfrak{p}$ is the inclusion map, then $i^{-1}(U(A_\mathfrak{p})) = \mathfrak{p}^c$. If $p \in \mathfrak{p}$ is invertible in $A_\mathfrak{p}$, then there is $a, b \in A$ such that $p(a/b) = 1$, which means there is $s \notin \mathfrak{p}$ such that $s(ap - b) = 0$. In particular, this means that $ap - b \in \mathfrak{p}$, so that $b \in \mathfrak{p}$. But this is clearly impossible by assumption. We claim that $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$. Indeed, if $\mathfrak{a}$ is a proper ideal of $A_\mathfrak{p}$, then it contains no element of $\mathfrak{p}^c$. Thus $i^{-1}(\mathfrak{a}) \subset \mathfrak{p}$, which implies $i^{-1}(\mathfrak{a})_\mathfrak{p} \subset \mathfrak{p}_\mathfrak{p}$. Our proof is complete if we can show $\mathfrak{p}_\mathfrak{p}$ is a proper ideal of $A_\mathfrak{p}$. But this follows because $\mathfrak{p}$ is prime, so that if $s \notin \mathfrak{p}$ and $sa \in \mathfrak{p}$, then $a \in \mathfrak{p}$. Thus $i^{-1}(\mathfrak{p}_\mathfrak{p}) = \mathfrak{p}$, and in particular, $\mathfrak{p}_\mathfrak{p} \neq A_\mathfrak{p}$. □

Example. *If $A(D)$ is the set of analytic functions on some open set D, then the set of functions $f \in A(D)$ such that $f(p) = 0$ forms a prime ideal, so we can form the local ring on this ideal, which is commonly denoted $\mathcal{O}_p(D)$. The invertible elements of $\mathcal{O}_p(D)$ are exactly those functions which are nonzero at p (or, viewing the functions as direct quotients, have a nonzero removable singularity at p). This ring is isomorphic to the subring of the ring $\mathbf{C}[[X -$*

$p]]$ *of power series in* $X - p$*, consisting of elements which are convergent in a neighbourhood of* p*.*

Example. *On* $\mathbf{Z}$*, we can view elements* $a \in \mathbf{Z}$ *as functions on the set of prime integers, mapping a prime* p *to the congruence class of* a *modulo* p *in* $\mathbf{F}_p$*. Thus the integer* $1984 = 2^6 \cdot 31$ *is a function on the primes which has two zeros at* 2 *and* 31*, where* 2 *to a 'zero of multiplicity six'. This corresponds to the fact that* 1984 *is invertible in* $\mathbf{Z}_{(p)}$ *except for* $p = 2$ *and* $p = 31$*, where* $1984/31$ *is invertible in* $\mathbf{Z}_{(1984)}$*, and* $1984/2^6$ *is invertible in* $\mathbf{Z}_{(2)}$*.*

In modern commutative algebra, one takes the set of prime ideals in a space and views them as points, through which the elements of the ring act as functions mapping into integral domains.

Theorem 6.16. *If* S *is multiplicative, and* $\mathfrak{p}$ *is a maximal ideal not containing elements of* S*, then* $\mathfrak{p}$ *is prime.*

Proof. We claim $S^{-1}\mathfrak{p}$ is a maximal ideal. If $S^{-1}\mathfrak{p} \subsetneq S^{-1}\mathfrak{a}$, then $\mathfrak{p} \subsetneq \mathfrak{a}$, implying $\mathfrak{a}$ contains element of S, so $S^{-1}\mathfrak{a} = S^{-1}A$. Now we claim $i^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$. If $b = a/s$, where $a \in \mathfrak{p}$, $s \in S$, and $b \notin \mathfrak{p}$, then $(b) + \mathfrak{p}$ contains elements in S, hence $xb + y = t$, for $y \in \mathfrak{p}$, $t \in s$. But then $xa + ys = ts$, with the left hand in $\mathfrak{p}$, and the right hand side in S, contradicting the construction of $\mathfrak{p}$. Thus we conclude $\mathfrak{p}$ is prime. □

Proposition 6.17. *If* A *is local, and* $f : A \to B$ *a surjective homomorphism, then* B *is local.*

Proof. If $\mathfrak{m}$ is a maximal ideal in B, then $f^{-1}(\mathfrak{m})$ is an ideal, and the isomorphism theorem guarantees that $A/f^{-1}(\mathfrak{m}) \cong B/\mathfrak{m}$, and since $B/\mathfrak{m}$ is a field, we conclude $f^{-1}(\mathfrak{m}) = U(A)^c$ is the unique maximal ideal in A. If $\mathfrak{n}$ is another maximal ideal in B, then $f^{-1}(\mathfrak{m}) = f^{-1}(\mathfrak{n})$, implying $\mathfrak{m} = \mathfrak{n}$ because f is surjective. □

6.9 Discrete Valuation Rings

Local rings were originally designed to analyze rings of functions, such as the ring $\mathcal{O}_p(D)$ of meromorphic functions on an open, connected subset of D, defined at the point p. As discovered in single variable complex

analysis, it is in this ring that the concept of orders of poles and zeroes occur. In particular, if f is a meromorphic function holomorphic in a neighbourhood of p, and if $f(p) = 0$, then we can write $f = (X - p)g$ for some meromorphic function g. Since $f \in \mathcal{O}_p(D)$ is non-invertible precisely when $f(p) = 0$, we conclude that the maximal ideal of non-invertible elements is principal, of the form $(X - p)$. More generally, we know that if f is a meromorphic function holomorphic in a neighbourhood of p, then there is a non-negative integer n such that we can write $f = (X - p)^n g$ for some meromorphic function g with $g(p) \neq 0$, and we call n the order of the zero at g. This implies that if $\mathfrak{a}$ is any proper ideal in $\mathcal{O}_p(D)$, then it is of the form $((X-p)^n)$ for some integer n, so $\mathcal{O}_p(D)$ is principal. Thus the smallest ideal in $A_\mathfrak{p}$ containing a function corresponds to it's order at the point p. Here's another example.

Example. *Let A be a factorial ring, and (p) a principal ideal, where p is prime. Then the ring A_p is principal, and also has the properties that $\mathcal{O}_p(D)$ has. Every principal ideal in A_p is of the form (p^N), because if $a = p^n q$, where $p \nmid q$, then $q \in U(A_p)$ and so $(a) = (p^n)$. But now if $\mathfrak{a}$ is any ideal, and we define the order of a to be the integer $ord(a)$ such that $(a) = (p^n)$, then*

$$\mathfrak{a} = \bigoplus_{a \in \mathfrak{a}} (a) = \bigoplus_{a \in \mathfrak{a}} (p^{ord(a)}) = (p^{\min ord(a)})$$

so every ideal is principal, and in particular, generated by a power of p. Thus the order of an element of the ring measures it's place in the linear heirarchy

$$(1) \supset (p) \supset (p^2) \supset \cdots \supset (0)$$

which consists of all ideals.

We want to consider rings where we can discuss the phenomenon of 'multiplicities of zeroes'. Since we are focusing on a ring, such a ring shuold be localized at the point where we want to measure zeroes, so our ring should be local. If the ring is Noetherian domain, but not a field, which maximal ideal is principal, we call the ring a *discrete valuation ring*. These are the rings having the properties we wish.

Proposition 6.18. *If A is a discrete valuation ring, then there exists an element $t \in A$ such that every nonzero element of A can be uniquely written as ut^n, where u is a unit in A.*

Proof. Let (t) be the maximal ideal of A. Suppose that $ut^n = vt^m$. If $n = m$, then $u = v$. Otherwise, if $n > m$, then $u = vt^{m-n}$, and this implies that (t) contains a unit, hence is not a maximal ideal. Thus it suffices to prove that every element of A has a required expansion of the form above. If $a \in A$ is a unit, we can write $a = at^0$, and we are done. If a is not a unit, then (a) is an ideal contained in (t), so we can write $a = a_1 t$ for some $a_1 \in A$. Then (a) is a proper subideal of (a_1), because if $a_1 = ba$, then $a = bat$, hence $1 = bt$, so t is invertible. If a_1 is a unit, we are done, otherwise we can write $a_1 = a_2 t$. Continuing this process, if this process does not terminate, we end up with an infinite ascending chain

$$(a) \subsetneq (a_1) \subsetneq (a_2) \subsetneq \dots$$

and this is impossible in a Noetherian ring. □

If A is a domain, the condition that we have a unique expansion of the form ut^n for each element of A is exactly the condition which guarantees that the ring is a discrete valuation ring. If this is true, then (t) is certainly a unique maximal ideal in A, so A is a local ring whose maximal ideal is principal. To prove that A is Noetherian, it suffices to notice that the proper ideals of A are exactly $(0), (t), (t^2), (t^3)$, and so on and so forth, so that the ring is actually principal. The element t in the theorem is known as a *uniformizing parameter* for A. Any other uniformizing parameter for A differs from t by a unit, so if $s = ut$ is another uniformizing parameter, then if $a = vt^n = rs^m$, then $rs^m = ru^m t^m$, so $v = ru^m$ and $n = m$. Since this value is invariant of the uniformizing parameter, it depends only on the element a, and we call this the *order of* a. We define the order of 0 to be ∞. If we consider the field B of fractions of A, then every nonzero element b of B can be written as ut^n for a unique integer $n \in \mathbf{Z}$, and we define this to be the order of b. If $n < 0$, we say that b has a pole of order $-n$.

Example. *Consider the ring $k[X] = k[\mathbf{A}^1]$. Then for any $a \in \mathbf{A}^1$, the ring $\mathcal{O}_a(\mathbf{A}^1)$ of rational functions defined at a (those polynomials f/g with $g(a) \neq 0$) is a discrete valuation ring. If we consider any function f/g with $g(a) \neq 0$, then $f = (X-a)^n h(X)$ for some $n \geqslant 0$ and since h with $h(a) \neq 0$. This gives us a decomposition $f/g = (h/g)(X-a)^n$, so $X - a$ is a uniformizing parameter, and $\mathcal{O}_a(\mathbf{A}^1)$ is a discrete valuation domain.*

Example. *Consider the ring $\mathcal{O}_\infty(\mathbf{A}^1)$ of rational functions of the form $f/g \in k(X)$, with $\deg g \geqslant \deg f$. This rings models the set of rational functions*

which converges to a well defined quantity 'near infinity'. The only invertible functions in this ring are those with $\deg g = \deg f$*, and so the noninvertible functions are generated by* $(1/X)$*, because if* $\deg g - \deg f = n$*, then* $X^n(f/g) = (X^n f/g)$ *is invertible, and contained in* $\mathcal{O}_\infty(\mathbf{A}^1)$*.*

Example. *If* p *is a prime number, then the local ring* $\mathbf{Z}_{(p)}$ *is a discrete valuation ring, because if* $a/b \in \mathbf{Z}_{(p)}$*, with* $b \notin (p)$*, we can write* $a = p^n c$ *with* c *and* p *relatively prime, and then* $a/b = p^n(c/b)$ *has* c/b *invertible. This gives an order function on* $\mathbf{Q}$ *defined by taking the order of a number* $m = p^n(a/b)$ *with respect to* p *to be* n*. This can be used to define a metric on* $\mathbf{Q}$*, and the completion is the field of p-adic numbers.*

The order function on the resulting field of fractions of a discrete valuation domain satisfies useful algebraic properties.

- $\operatorname{ord}(x) = 0$ if and only if $x = 0$.
- $\operatorname{ord}(xy) = \operatorname{ord}(x) + \operatorname{ord}(y)$.
- $\operatorname{ord}(x + y) \geqslant \min(\operatorname{ord}(x), \operatorname{ord}(y))$.

We will show that these properties are essentially the defining properties of a discrete valuation domain. Given any field k, an order function is a $\mathbf{Z} \cup \{\infty\}$ valued function φ on k with the properties above, and with $\varphi(x) = \infty$ if and only if $x = 0$.

Proposition 6.19. *For any order function* φ *on a field* k*, the ring* A *of elements* $x \in k$ *with* $\varphi(x) \geqslant 0$ *forms a discrete valuation domain, with* k *it's field of fractions.*

Proof. A is certainly closed under multiplication and addition. Since $\varphi(x) = \varphi(1 \cdot x) = \varphi(1) + \varphi(x)$, we conclude that $\varphi(1) = 0$. We use this to conclude that $\varphi(xx^{-1}) = \varphi(x) + \varphi(x)^{-1} = 0$, so an element $x \in A$ is invertible if and only if $\varphi(x) = 0$. This shows that the set of noninvertible elements forms an ideal, hence the ring A is local. The ring is certainly a domain. We may assume that there is $x \in k$ with $\varphi(x) = 1$, because otherwise every noninfinite value of the order function is a multiple of some integer, and we obtain another order function by dividing by this integer. If $\varphi(x) = 0$, then for every $x \in A$, there is n such that $\varphi(xt^{-n}) = 0$, hence $xt^{-n} = u$ is a unit, and $x = ut^n$. We have justified that this proves A is a discrete valuation domain, and since $\varphi(x^{-1}) = -\varphi(x)$, every element of k is either an

element of A, or of the form $1/x$ for some $x \in A$, showing that k is the field of fractions of A. □

Proposition 6.20. *If $ord(a) < ord(b)$, then $ord(a+b) = ord(a)$.*

Proof. $a = t^n u$, $b = t^m s$, then $a+b = t^n(u+t^{m-n}s)$, and $u+t^{m-n}s$ is invertible because it is congruent to u in the maximal ideal. This is analogous to the addition law for polynomials in $k[X]$. □

Often, a discrete valuation ring models the germ of functions around a point, and the evaluation map at this points gives us the maximal ideal, as well as an isomorphism between the ring of constant functions and the field upon which the functions are defined. In this situation, we can obtain some useful properties of the ring of constant functions, related to the Taylor expansion of functions around a point.

Proposition 6.21. *Suppose that a discrete valuation ring A contains a subfield k, such that if $\mathfrak{m}$ is the maximal ideal of A, then $k \to A \to A/\mathfrak{m}$ gives an isomorphism of fields. If t is a uniformizing parameter for A, then for any $n \geqslant 0$, every $x \in A$ has a unique expansion as*

$$x = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + z_n t^{n+1},$$

where $\lambda_0, \ldots, \lambda_n \in k$, and $z_n \in A$.

Proof. For any $x \in A$, there is $\lambda \in k$ such that x is congruent to λ modulo $\mathfrak{m} = (t)$, so $x = \lambda + z_0 t$. This gives the proposition for the case $n = 0$. For the inductive case, we write $x = \sum \lambda_i t^i + z_n t^{n+1}$. Then using the $n = 0$ case we can write $z_n = \lambda_{n+1} + z_{n+1} t$, and this gives the expansion for x one degree higher. To prove uniqueness, we note that if $\sum \lambda_i t^i + z_n t^{n+1} = 0$, then $\sum \lambda_i t^i = -z_n t^{n+1}$, and if $z_n \neq 0$, the right side has order greater than or equal to $n+1$, whereas the right side has order equal to the minimum index i such that $\lambda_i \neq 0$, and these two values cannot be equal. □

Example. *The ring of formal power series over a field k is written $k[[X]]$, and is the ring of 'infinite power series' $\sum_{k=0}^{\infty} a_k X^k$, with $a_k \in k$. Then $k[[X]]$ is a discrete valuation ring. A power series $\sum a_i X^i \in k[[X]]$ is invertible precisely when $a_0 \neq 0$. This is certainly a sufficient condition. Conversely, if $a_0 \neq 0$, then we can use the formula*

$$\frac{1}{a_0 + Xf} = \frac{1}{a_0} \sum_{k=0}^{\infty} (-1)^k X^k f^k / a_0^k.$$

to show that $a_0 + Xf$ is invertible in $k[[X]]$. This shows that the set of non-unital elements of $k[[X]]$ is precisely the ideal (X), so that $k[[X]]$ is local. We can write any power series as $X^n u$ for some $u \in k[[X]]$ with non-zero coefficient, so that the ring $k[[X]]$ is a discrete valuation domain, where the order function is precisely the degree corresponding to the smallest non-zero coefficient. The quotient field of $k[[X]]$ is denoted $k((X))$, which is the field of formal Laurent series.

For any discrete valuation domain A to which the last proposition holds, we have an isomorphism

$$k \to A \to A/\mathfrak{m},$$

the previous proposition induces a natural injective homomorphism from A to $k[[X]]$. This shows that the class of discrete valuation domains which contain a field corresponding to the quotient by their maximal ideal are precisely the rings where we can consider 'power series' of elements. This is similar to the fact that all holomorphic functions can be expanded in power series, but in this sense we also have an analytic relationship between the power series converging around a point.

Example. *In complex analysis, one memorizes the power series expansion*

$$(1-X)^{-1} = (1 + X + X^2 + \dots)$$

This equation holds in the ring $k[[X]]$ of power series over any field, because of the telescoping series properties of $(1-X)(1+X+X^2+\dots)$. Similarily,

$$\begin{aligned}(1-X)(1+X^2)^{-1} &= (1-X)(1+iX)^{-1}(1-iX)^{-1} \\ &= (1-X)\left(\sum(-i)^k X^k\right)\left(\sum i^k X^k\right) \\ &= (1-X)\left(\sum(-1)^k X^{2k}\right) \\ &= (1 - X - X^2 + X^3 + X^4 - X^5 - X^6 + \dots)\end{aligned}$$

Proposition 6.22. *Suppose that A is a discrete valuation ring, with quotient field k. Then there are no local rings B with $A \subsetneq B \subset k$, such that the maximal ideal of B contains the maximal ideal of A.*

Proof. If a nonzero x is in k, but not in A, then x has some order $-n < 0$, so x^{-1} has order n, and is consequently in A. This means that $x^{-1} \in A$

for each $x \in A$. If the maximal ideal $\mathfrak{m}$ of B contains the maximal ideal $\mathfrak{n}$ of A, we claim that $\mathfrak{m} = \mathfrak{n}$. Otherwise, we can pick $x \in \mathfrak{m} - \mathfrak{n}$, and then $x^{-1} \in A$, so $1 = xx^{-1} \in \mathfrak{m}$, contradicting the fact that $B \neq k$. Now let t be a uniformizing parameter for A. Every element of k, and in particular B, can be written as xt^n, where x is a unit in A. In particular, if $B - A$ is nonempty, it contains some element ut^{-n}, where $n > 0$, and u is a unit in A. But then B contains t^{-n}, and hence all elements of the form $t^{k-mn} = t^k(t^{-n})^m$, so $B = k$, which is impossible. □

Example. *Using this theorem, we can classify the discrete valuation rings with quotient field $k(X)$ which contain k, where k is algebraically closed. Let A be a discrete valuation ring, and suppose the uniformizing parameter is some irreducible $t \in A$. If A contains X, then A contains $k[X]$, and the set of elements of $k[X]$ which are not invertible in A forms a prime ideal, which is therefore of the form (f) for some irreducible monic polynomial f. Since k is algebraically closed, $f(X) = X - a$, for some $a \in k$, and so A contains $\mathcal{O}_a(\mathbf{A}^1)$, implying the two are equal to one another. If A does not contain X, then A contains X^{-1}. Since the order of any nonzero $a \in k$ is zero, and the order of X^{-1} is greater than zero because it is not invertible, $a_0 + a_1X^{-1} + \cdots + a_nX^{-n} = (a_0X^n + \cdots + a_n)/X^n$ is invertible in A, hence $X^n/(a_0X^n + \cdots + a_n) \in A$. Multiplying by $b_0 + b_1X^{-1} + \cdots + b_nX^{-n}$, we conclude that $(b_0X^n + \cdots + b_n)/(a_0X^n + \cdots + a_n) \in A$ for any $a_0 \neq 0$. This shows that A contains $\mathcal{O}_\infty(\mathbf{A}^1)$, and if $f(X)/g(X)$ has $\deg g > \deg f$, then g/f is not in A, for otherwise we may write $g = (X - a_1)\ldots(X - a_m)$, $f = (X - b_1)\ldots(X - b_l)$, and then $h = (X - a_1)\ldots(X - a_{m-1})/(X - b_1)\ldots(X - b_l) \in A$, so $hg/f = X - a_m \in A$, implying $X \in A$, contradicting our assumption. Thus the maximal ideal of A contains the maximal ideal of $\mathcal{O}_\infty(\mathbf{A}^1)$, and this implies that A is in fact equal to $\mathcal{O}_\infty(\mathbf{A}^1)$.*

Example. *The only discrete valuation rings with quotient field $\mathbf{Q}$ are the local rings $\mathbf{Z}_{(p)}$. If A is any such discrete valuation ring, then A contains all the integers $\mathbf{Z}$. Because A is a local ring, the set of non-invertible integers in A forms a prime ideal in $\mathbf{Z}$, and hence is of the form (p) for some prime integer. But then A contains $\mathbf{Z}_{(p)}$, which implies $A = \mathbf{Z}_{(p)}$.*

Similar techniques to the classifications above allow us to classify the set of all discrete valuation rings which are obtained from extensions of principal ideal domains. These valuation rings are exactly of the form A_p, where (p) is a prime ideal in the principal ideal domain.

Chapter 7

Dedekind Rings

In the understanding of integral solutions to polynomial equations such as $X^n - Y^n$ can be factored over $\mathbf{Z}[\zeta_n]$, where ζ_n is a primitive n th root of unity. In 1847 Gabriel Lumé used the fact that $\mathbf{Z}[\zeta_n]$ is a unique factorization domain to provide a proof of Fermat's last theorem, with one catch; $\mathbf{Z}[\zeta_n]$ is not always a unique factorization domain, and so his proof only works for certain values of n for which the ring is such a domain; in 1844 Ernst Kummer showed that $\mathbf{Z}[\zeta_{23}]$ is *not* a unique factorization domain. However, Ernst Kummer also showed that there are certain techniques which allow us to extend UFD type arguments to more general rings, including the rings $\mathbf{Z}[\zeta_{23}]$; rather than factorizing individual elements of a ring, we can factor ideals in the ring into prime ideal components.

Example. *Consider the ring* $\mathbf{Z}[\sqrt{-5}]$, *in which*

$$2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

all of 2, 3, $1 + \sqrt{-5}$, *and* $1 - \sqrt{-5}$ *are irreducible in* $\mathbf{Z}[\sqrt{-5}]$, *because we know* $|a + b\sqrt{-5}|^2 = a^2 + 5b^2$, *and there are no solutions in* $\mathbf{Z}^2 + 5\mathbf{Z}^2$ *to the equations* $XY = 4$, 9, *or* 6, *except for the trivial ones corresponding to a unit multiplied by a constant. Thus* $\mathbf{Z}[\sqrt{-5}]$ *is not a unique factorization domain. However, consider the corresponding relationship between the ideals, i.e.*

$$(2)(3) = (6) = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

Even though these numbers are irreducible element of the ring, they are not prime elements, since, for instance, 2 *divides* $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, *but*

can't divide $1+\sqrt{-5}$ *or* $1-\sqrt{-5}$. *On the other hand,* $(2,1+\sqrt{-5})$ *is a prime ideal, because* $\mathbf{Z}[\sqrt{-5}]/(2,1+\sqrt{-5})$ *is isomorphic to* $\mathbf{Z}_2$*, which is obtained from the fact that the embedding of* $\mathbf{Z}$ *into* $\mathbf{Z}[\sqrt{-5}]/(2,1+\sqrt{-5})$ *is surjective, with kernel* (2)*, as is* $(3,1-\sqrt{-5})$*, and we have*

$$(6) = (2,1+\sqrt{-5})(3,1-\sqrt{-5})$$

which is a unique factorization of ideals.

A *Dedekind ring* is precisely a domain where one can factor ideals uniquely into products of prime ideals. An equivalent definition, more interesting, occurs in the theory of ideal class groups in algebraic number theory. If A if a domain with a field of fractions k, we say an A submodule $\mathfrak{a}$ of k is a *fractional ideal* if there is $x \in A$ with $x\mathfrak{a} \subset A$, so that $\mathfrak{a}$ has 'bounded denominator'. The family of fractional ideals forms a monoid, with A as the identity element, if we take products just as in the case of normal ideals, $\mathfrak{a}\mathfrak{b}$ is the subgroup of $K - \{0\}$ generated by elements of the form ab, for $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. If the family of fractional ideals forms a group under this product, we will find that the ideals have a unique factorization theory.

To see this, let's explore some consequences of the group property. If $\mathfrak{a}$ is an ideal of A, this means there is a fractional ideal $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b} = A$, so that there are $x_1,\ldots,x_n \in \mathfrak{a}$, $y_1,\ldots,y_n \in \mathfrak{b}$ with $x_1y_1 + \cdots + x_ny_n = 1$. If $x \in \mathfrak{a}$ is arbitrary, then $x = x_1(y_1x) + \cdots + x_n(y_nx)$, and we know because of the product formula $\mathfrak{a}\mathfrak{b} = A$ that $y_kx \in A$, hence we have found $\mathfrak{a} = (x_1,\ldots,x_n)$. We conclude that any ring whose fractional ideals form a group is Noetherian.

Chapter 8

Completion

8.1 The p-adic integers

Let $\mathbf{Z}_p$ be the ring of p-adic integers, which is the limit of the rings $\mathbf{Z}/p^n\mathbf{Z}$. Thus for each i we have a ring homomorphism $\pi_i : \mathbf{Z}_p \to \mathbf{Z}/p^n\mathbf{Z}$ which is compatible with the projections of the rings $\mathbf{Z}/p^n\mathbf{Z}$ onto one another. For each integer n, we have a homomorphism $F_n : \mathbf{Z}[[X]] \to \mathbf{Z}/p^n\mathbf{Z}$, where

$$F_n\left(\sum_{k=0}^{\infty} a_k X^k\right) = \sum_{k=0}^{n-1} a_k p^k.$$

These homomorphisms commute with the inclusions of the various $\mathbf{Z}/p^n\mathbf{Z}$ with one another, and therefore induce a homomorphism $F : \mathbf{Z}[[X]] \to \mathbf{Z}_p$. For each $x \in \mathbf{Z}_p$, we can uniquely associate a formal series

$$x = b_0 + b_1 p + b_2 p^2 + \dots$$

with $b_n \in \{0, \dots, p-1\}$ for each n, in the sense that for each n,

$$\pi_n(x) = b_0 + \cdots + b_{n-1} p^{n-1}.$$

But then

$$x = F\left(b_0 + b_1 X + b_2 X^2 + \dots\right)$$

which shows F is surjective. We claim the kernel of F is $X - p$. Certainly $F(X-p) = 0$. If for each $k \geqslant 0$ we choose $a_k \in \{0, \dots, p-1\}$ and set

$$f = \sum_{k=0}^{\infty} a_k X^k,$$

then $F(f) = 0$ if and only if $a_k = 0$ for each k. On the other hand, given any $f \in \mathbf{Z}[[X]]$, there exists $g \in \mathbf{Z}[[X]]$ such that

$$f = g(X - p) + \sum_{k=0}^{\infty} a_k X^k$$

such that $a_k \in \{0, \ldots, p-1\}$ for each k. Then $F(f) = 0$ if and only if $a_k = 0$ for each k. Thus $\mathrm{Ker}(f) = (X - p)$, and so we conclude $\mathbf{Z}_p$ is isomorphic to $\mathbf{Z}[[X]]/(X - p)$. Thus p-adic integers operate as power series with additional relationships.

The ideal $(X-p)$ is prime in $\mathbf{Z}[[X]]$. To see this, since $\mathbf{Z}[[X]]$ is a unique factorization domain (implied by the fact that $\mathbf{Z}$ is principal), it suffices to show $X - p$ is irreducible. So suppose

$$X - p = \left(\sum_{k=0}^{\infty} a_k X^k \right) \left(\sum_{k=0}^{\infty} b_k X^k \right).$$

Then $-p = a_0 b_0$, which implies either a_0 or b_0 is a unit since p is prime. But if a_0 is a unit, then $\sum_{k=0}^{\infty} a_k X^k$ is a unit in $\mathbf{Z}[[X]]$. Thus $X - p$ is irreducible, and so $\mathbf{Z}_p$ is an integral domain.

If we consider some element x of $\mathbf{Z}_p$ represented as the power series

$$\sum_{k=0}^{\infty} a_k p^k,$$

with $a_0 \neq 0$, then there exists some integer n such that na_0 is congruent to 1 modulo p, and so nx corresponds to a unit in $\mathbf{Z}[[X]]$. In particular, nx is invertible, so x is invertible in A. Thus we conclude that $\mathbf{Z}_p$ contains a unique maximal ideal of the form

$$\mathfrak{m}_p = \left\{ \sum_{k=0}^{\infty} a_k p^k : a_0 = 0 \right\}.$$

Thus $\mathbf{Z}_p$ is a local ring.

Now consider the equation $X^{p-1} - 1$ in $\mathbf{Z}_p$. Suppose a_0 is an integer not divisible by p. Then $a_0^{p-1} \equiv 1$ (modulo p). We now construct a sequence of integers $\{a_i\}$ such that for each i,

$$\left(\sum_{k=0}^{i-1} a_k p^k \right)^{p-1} \equiv 1 \pmod{p^i} \tag{8.1}$$

We construct this sequence inductively. Suppose we have chosen $a_0, \ldots, a_{i-1}$ such that (8.1) holds. Then we must find an integer a_i such that

$$\left(\sum_{k=0}^{i} a_k p^k\right)^{p-1} \equiv 1 \quad (\text{modulo } p^{i+1}). \tag{8.2}$$

Now

$$\left(\sum_{k=0}^{i} a_k p^k\right)^{p-1} = \sum_{j=0}^{p-1} \binom{p-1}{j} \left(a_i p^i\right)^j \left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-1-j}$$

which is equivalent modulo p^{i+1} to

$$\left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-1} + a_i(p-1)p^i \left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-2}.$$

We can find an integer n such that

$$\left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-1} = 1 + np^i.$$

The inductive case guarantees that $\sum_{k=0}^{i-1} a_k p^k$ is relatively prime to p, hence this quantity has an inverse m modulo p^{i+1}, so (8.2) is equivalent to the equation

$$p^i \left(n + a_i(p-1)\left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-2}\right) \equiv 0 \quad (\text{modulo } p^{i+1}).$$

Thus it clearly suffices to show that we can pick $a_i \in \{0, \ldots, p-1\}$ such that

$$n + a_i(p-1)\left(\sum_{k=0}^{i-1} a_k p^k\right)^{p-2} \equiv 0 \quad (\text{modulo } p),$$

and this equation is solvable since $p-1$ and $\sum_{k=0}^{i-1} a_k p^k$ are relatively prime to p. This completes the construction. Performing this construction for all $a_1 \in \{1, \ldots, p-1\}$ constructs $p-1$ distinct values $x_1, \ldots, x_{p-1} \in \mathbf{Z}_p$ such that $x_i^{p-1} = 1$ for each i. Thus the polynomial equation $X^{p-1} - 1$ splits over $\mathbf{Z}_p$.

8.2 Complete Local Rings

Let A be a local ring with maximal ideal $\mathfrak{m}$. We say A is a *complete local ring* if

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$$

and A is complete with respect to the $\mathfrak{m}$-adic topology.

Example. *For any field k, the ring $k[[X_1,\ldots,X_n]]$ is complete and local. The maximal ideal of this ring is $(X_1,\ldots,X_n)$. If $\{f_i\}$ is a Cauchy sequence in $k[[X_1,\ldots,X_n]]$, and we write $f_i = \sum_{j=0}^{\infty} f_{ij}$, where f_{ij} is homogenous of degree j, then for each j, there is f_j and i_0 such that $f_{ij} = f_j$ for $i \geqslant i_0$. If we define $f = \sum_{j=0}^{\infty} f_j$ then f_i converges to f.*

Example. *If A is a complete local ring, then $A[[X]]$ is a complete local ring. If $\mathfrak{m}$ is the maximal ideal of A, then the maximal ideal of $A[[X]]$ is $\mathfrak{m}+(X)$. If $\{f_i\}$ is a Cauchy sequence in $A[[X]]$, and we write*

$$f_i = \sum_{k=0}^{\infty} a_{ik} X^k$$

then for each k and n, there exists i_0 such that for $i, j \geqslant i_0$,

$$f_i - f_j \in (\mathfrak{m}+(X))^{k+n} = (\mathfrak{m}^{k+n}, \mathfrak{m}^{k+n-1} X, \ldots, \mathfrak{m} X^{k+n-1}, X^{k+n}).$$

This implies that $a_{ik} - a_{jk} \in \mathfrak{m}^n$. In particular for each fixed k, the sequence $\{a_{ik}\}$ is Cauchy in A. Thus there exists $a_k \in A$ such that $\lim_{i\to\infty} a_{ik} = a_k$. If we set

$$f = \sum_{k=0}^{\infty} a_k X^k,$$

then we claim $\lim_{i\to\infty} f_i = f$. If n and m are fixed, then there exists i_0 such that for $i \geqslant i_0$ and $k \leqslant m$, $a_{ik} - a_k \in \mathfrak{m}^n$. But this means that $f_i - f \in \mathfrak{m}^n + X^m$. Taking $n, m \to \infty$ completes the proof.

The next theorem is a kind of euclidean algorithm for complete local rings.

Theorem 8.1. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$. Set*

$$f = \sum_{k=0}^{\infty} a_k X^k,$$

where $a_0, \ldots, a_{n-1} \in \mathfrak{m}$ and $a_n \notin \mathfrak{m}$. Then for any $g \in A[[X]]$, we can find unique elements $q \in A[[X]]$ and $r \in A[X]$ with $\deg(r) \leqslant n-1$ such that $g = qf + r$.

Proof. Let $\alpha : A[[X]] \to A[X]$ and $\tau : A[[X]] \to A[[X]]$ be the A-linear maps with

$$\alpha\left(\sum_{k=0}^{\infty} a_k X^k\right) = \sum_{k=0}^{n-1} a_k X^k$$

and

$$\tau\left(\sum_{k=0}^{\infty} a_k X^k\right) = \sum_{k=0}^{\infty} a_{n+k} X^k.$$

Note that $\tau(fX^n) = f$ for any $f \in A[[X]]$. It suffices to show we can find a unique $q \in A[[X]]$ such that $\tau(g) = \tau(qf)$. Note that

$$\begin{aligned}\tau(qf) &= \tau(q(\alpha(f) + \tau(f)X^n)) \\ &= \tau(q\alpha(f)) + \tau(q\tau(f)X^n) \\ &= \tau(q\alpha(f)) + q\tau(f),\end{aligned}$$

and that $\tau(f)$ is invertible in $A[[X]]$. Put $Z = q\tau(f)$. Then the equation $\tau(g) = \tau(qf)$ is equivalent to

$$\tau(g) = \tau(q\alpha(f)) + q\tau(f) = \tau\left(Z \cdot \frac{\alpha(f)}{\tau(f)}\right) + Z$$

Thus it suffices to show the A-linear endomorphism $1 + T : A[[X]] \to A[[X]]$, where

$$T(W) = \tau\left(W \cdot \frac{\alpha(f)}{\tau(f)}\right),$$

is an isomorphism. But this follows because $\alpha(f)/\tau(f) \in \mathfrak{m} \cdot A[[X]]$, so that for any $n \geqslant 0$, $T(\mathfrak{m}^n \cdot A[[X]]) \subset \mathfrak{m}^{n+1} \cdot A[[X]]$. Thus the operator

$$S(W) = \lim_{N \to \infty} \left(1 - T(W) + T^2(W) - \cdots + (-1)^N T^N(W)\right)$$

is well defined (due to the fact that A is *complete*), and it is simple to verify that S is the inverse of $1 + T$. □

A simple result of this calculation is the *Weirstrass Preparation Theorem*.

Theorem 8.2. *Let A be a complete local ring, let $f = \sum_{k=0}^{\infty} a_k X^k$, and suppose $a_0, \ldots, a_{n-1} \in \mathfrak{m}$, $a_n \notin \mathfrak{m}$. Then f can be written uniquely as*

$$(X^n + b_{n-1}X^{n-1} + \cdots + b_0)g,$$

where g is a unit in $A[[X]]$, and $b_{n-1}, \ldots, b_0 \in \mathfrak{m}$.

Proof. Use the last result to uniquely write

$$X^n = qf + r$$

Then the constant term of q is a unit in A, hence q is invertible in $A[[X]]$. This means that $f = q^{-1}(X^n - r)$, which proves the uniqueness and existence of the required power series. □

One application of this result is the proof that power series over a field forms a factorial ring.

Theorem 8.3. *Suppose k is a field. Then $k[[X_1, \ldots, X_n]]$ is a unique factorization domain.*

Proof. We prove the theorem by induction on n. The case $n = 0$ is trivial. But then the Weirstrass factorization theorem enables us to reduce the analysis of irreducible elements of $k[[X_1, \ldots, X_n]]$ to $k[[X_1, \ldots, X_{n-1}]][X_n]$, to which we can apply the standard result that if A is a unique factorization domain, then $A[X]$ is a unique factorization domain. □

Chapter 9

Graded Modules

A *graded ring* is a ring A which has a direct sum decomposition as

$$A = A_0 \oplus A_1 \oplus \ldots,$$

such that for each n and m, $A_n \cdot A_m \subset A_{n+m}$. Elements of A_n are known as *homogenous elements* of degree n. An ideal $\mathfrak{a}$ in a graded ring is *homogenous* if it is generated by homogenous elements, or, alternatively, if whenever $a \in \mathfrak{a}$, with $a = a_0 + a_1 + \ldots$ for $a_i \in A_i$, then $a_i \in \mathfrak{a}$. If $\mathfrak{a}_n = \mathfrak{a} \cap A_n$, then $A/\mathfrak{a}$ has a natural gradation as

$$A/\mathfrak{a} = A_0/\mathfrak{a}_0 \oplus A_1/\mathfrak{a}_1 \oplus \cdots.$$

Thus the quotient of a graded ring by a homogenous ideal also is naturally a graded ring. For any graded ring A, the set $A_+ = A_1 \oplus A_1 \oplus \ldots$ known as the *irrelevant ideal*, and the quotient A/A_+ is isomorphic to A_0, which is trivially graded. An obvious, though incredibly important fact about homogenous ideals is the following.

Example. *The fundamental example of a graded ring is the polynomial ring $k[x_1, \ldots, x_n]$, which is graded by degree. If $S = k[x_1, \ldots, x_n]$, then we can write*

$$S = S_1 \oplus S_2 \oplus \cdots$$

where for each k, S_k consists of homogenous polynomials of degree k. More generally, if V is a projective variety in $\mathbf{P}^n$ then it's homogenous coordinate ring has a natural gradation, since $I(V)$ is a homogenous ideal in $k[x_1, \ldots, x_n]$.

The natural modules to study over a graded rings A are the family of *graded modules*, an A module M with a decomposition as

$$M = M_0 \oplus M_1 \oplus \cdots,$$

where for each n and m, $A_n M_m \subset M_{n+m}$. A graded, or homogenous submodule N of a graded module M is then just a submodule that can be decomposed as a direct sum over the graded submodules, i.e. we can write

$$N = N_0 \oplus N_1 \oplus \cdots$$

where $N_i \subset M_i$ for each i. The quotient module M/N is then naturally graded as

$$M/N = (M_0/N_0) \oplus (M_1/N_1) \oplus \cdots.$$

Graded modules naturally occur when studying vector bundles over geometric spaces.

A simple, but powerful remark is useful in the analysis of graded modules. If M Is a finitely generated graded module, then it is certainly generated by finitely many homogenous elements of the module. If M is generated by homogenous elements $a_1, \ldots, a_N \in M$, with $a_i \in M_{n_i}$ for each i, then for each k and $a \in M_k$, we can write

$$a = \sum_{m=1}^{N} b_m a_m,$$

where $b_m \in A_{k-n_m}$ for each m. This is because we can certainly write this decomposition with $b_m \in A$ for each m, and then decompose b_m into a direct sum of gradations; any part of b_m that isn't in A_{k-n_m} will be cancelled out in the overall sum anyway.

Theorem 9.1. *If A is a graded ring with identity, then the following properties are equivalent to one another:*

1. *A is Noetherian.*
2. *A_0 is Noetherian, and A_+ is a finitely generated ideal of A.*
3. *A_0 is Noetherian, and A is a finitely generated A_0 algebra.*

Proof. Let us prove each implication separately:

- $(1 \Rightarrow 2)$ If A is Noetherian, then A_0 is Noetherian as a subring of A. Moreover, A_+ is an ideal of A, hence a finitely generated ideal of A since A is Noetherian.

- $(2 \Rightarrow 3)$ If A_0 is Noetherian, and A_+ is a finitely generated ideal of A with generators $(a_1, \ldots, a_N)$, with $a_i \in A_{n_i}$ for each i. We then claim that A is generated as an A_0 algebra by $\{1, a_1, \ldots, a_N\}$. We prove $A_n \subset A_0[1, a_1, \ldots, a_N]$ for each n by induction on n. The case $n = 0$ is trivial. If $n > 0$ and $a \in A_n$, then we can write

$$a = \sum_{i=1}^{N} c_i a_i$$

  where $a_0 \in A_0$, and $c_i \in A_{n-n_i}$ for each i. Since $n - n_i < n$, the inductive hypothesis implies $c_i \in A_0[1, a_1, \ldots, a_N]$ for each i, which implies that $a \in A_0[1, a_1, \ldots, a_N]$. This completes the inductive case and shows A is a finitely generated A_0 algebra.

- $(3 \Rightarrow 1)$ If A_0 is Noetherian, and A is a finitely generated A_0 algebra, then A is isomorphic to a quotient of the polynomial ring $A_0[x_1, \ldots, x_n]$; the ring $A_0[x_1, \ldots, x_n]$ is Noetherian, and so A is then Noetherian as well, as the quotient of a Noetherian ring. $\square$

Remark. Any finitely generated algebra over a field is Noetherian. This Lemma shows that if we work over the family of all graded rings A with A_0 a field, then such rings are Noetherian if and only if they are finitely generated over A_0.

The ideas behind graded modules were first developed by Hilbert, in the contexts of invariant theory. Here one studies an algebraic group G acting rationally on a variety V, inducing an action on the coordinate ring $k[V]$. We wish to study the ring $k[V]^G$ of invariants of G. Since constant functions are fixed by G, $k[V]^G$ is a k subalgebra of $k[V]$. Moreover, the group structure gives an 'averaging map' $\varphi : k[V] \to k[V]^G$, which is a graded $k[V]^G$ module homomorphism fixing $k[V]^G$. For finite groups G, we can choose

$$(\varphi f)(x) = \frac{1}{\#(G)} \sum_{a \in G} f(ax).$$

For larger groups, we must use integration against the Haar measure in G in a careful manner. If we define

$$k[V]k[V]^G_+ = \{\sum_i b_i a_i : b_i \in k[V], a_i \in k[V]^G_+\},$$

then $k[V]k[V]^G_+$ is a homogenous ideal in $k[V]$. Since $k[V]$ is Noetherian, $k[V]k[V]^G_+$ is generated by finitely many homogenous elements $a_1, \ldots, a_n \in A_+$. If $a \in k[V]^G_+$, we may thus write

$$a = \sum_i b_i a_i,$$

and then $a = \varphi(a) = \sum_i \varphi(b_i)a_i$. Since $\varphi(b_i) \in k[V]^G$ for each i, this shows that $k[V]^G_+$ is a finitely generated ideal of $k[V]^G$. Since $k[V]^G_0 = k$ is a field, and hence trivially Noetherian, we have seen this implies that $k[V]^G$ is a finitely generated algebra over k.

9.1 The Hilbert Function

Chapter 10

K Theory

Theorem 10.1 (Steinitz)**.** *Let R be a Dedekind domain. Let $P = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$, and let $Q = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ be finitely generated projective modules, where the ideals is the direct sum are nonzero. Then P is isomorphic to Q if and only if $r = s$ and $\mathfrak{a}_1 \dots \mathfrak{a}_r = \mathfrak{b}_1 \dots \mathfrak{b}_s$.*

Proof. The last result implies P is isomorphic to $\mathbf{R}^{r-1} \oplus \mathfrak{a}_1 \dots \mathfrak{a}_r$, and Q is isomorphic to $\mathfrak{r}^{s-1} \oplus \mathfrak{b}_1 \dots \mathfrak{b}_s$. Just because P is isomorphic to Q does not imply that $\mathfrak{a}_1 \dots \mathfrak{a}_r$ is isomorphic to $\mathfrak{b}_1 \dots \mathfrak{b}_s$ over *general rings*, but we find that such is true when working over Dedekind domains. First, we remark that for an R linear map $\phi : \mathfrak{a} \to \mathfrak{b}$, there is an element q in the fraction field k such that $\phi(a) = qa$ for all $a \in \mathfrak{a}$. To see this take any nonzero $a_0 \in \mathfrak{a}$. Then, in k,

$$\phi(a) = \frac{a_0 \phi(a)}{a_0} = \frac{\phi(a_0 a)}{a_0} = a \frac{\phi(a_0)}{a_0}$$

Therefore, associated to any R linear map ϕ there is an $r \times s$ matrix M with entries in K. If ϕ is an isomorphism, then M^{-1} exists, and so $r = s$. We now claim that $\det(M)\, \mathfrak{a}_1 \dots \mathfrak{a}_r$ □

Corollary 10.2. *If R is a Dedekind domain, then $K_0(R) \cong \mathbf{Z} \oplus \widetilde{K_0}(R)$, and as a group, $\widetilde{K_0}(R)$ is isomorphic to the class group of R. Moreover, the product of any two elements of the reduced group is zero.*

Proof. The group $\widetilde{K_0}(R)$ is the kernel of the map from $K_0(R)$ to $\mathbf{Z}$. There is a correspondence $[\mathfrak{a}_1 \oplus \dots \mathfrak{a}_r] \mapsto r$ (taking the rank of the module) which extends to an isomorphism from $K_0(R)$ to $\mathbf{Z} \oplus \mathrm{Cl}(R)$. To prove the product

of any two elements of $\tilde{K_0}(R)$ is zero, we consider $[\mathfrak{a}]-1$ and $[\mathfrak{b}]-1$ in $\tilde{K_0}(R)$. Then

$$([\mathfrak{a}-1])([\mathfrak{b}]-1) = [\mathfrak{a}\otimes\mathfrak{b}]-[\mathfrak{a}]-[\mathfrak{b}]+1$$

Since $\mathfrak{a}\oplus\mathfrak{b} \cong R\oplus\mathfrak{a}\,\mathfrak{b} \cong R\oplus(\mathfrak{a}\otimes\mathfrak{b})$, this is zero. Because the elements $[\alpha]-1$ generate $\tilde{K_0}(R)$. □

10.1 Invertible Modules

A finitely generated module M over an commutative ring is invertible if there exists some module N such that $M\otimes N$ is isomorphic to R. We have a canonical homomorphism from $M\otimes M^*$ to R. Whenever M is invertible, this is precisely an isomorphism.

Lemma 10.3. *If P is a finitely generated projective module then P^* is finitely generated and projective, and $(P^*)^* \cong P$.*

Proof. If P is finitely generated, we have an exact diagram

$$0 \to Q \to R^n \to P \to 0$$

which induces an exact diagram

$$0 \to P^* \to (R^n)^* \to Q^* \to 0$$

and if the first diagram splits, the second one splits. Thus $R^n = P\oplus Q$, and $(R^n)^* \cong P^*\oplus Q^*$, and $(R^n)^*$ is isomorphic to R^n, showing P^* is projective. The double dual map $\nu : M \to M^{**}$ is an isomorphism if $M = R^n$. If P is finitely generated and projective, then $\nu : P \to P^{**}$. If $\mathfrak{p}$ is a prime ideal of R, then $R_\mathfrak{p}\otimes P \to R_\mathfrak{p}\otimes P^{**}$. □

M is an invertible R module if and only if M is finitely generated and projective of rank 1. TODO: ADD PROOF.

The Picard group of a commutative ring R is the group of isomorphism classes of invertible R modules, with the operation being the tensor product. We have an inclusion from the Picard group of R to $K_0(R)$, which is a morphism of multiplicative monoids. It is an inclusion of groups, but not necessarily an isomorphism.