

Fractals avoiding Fractal Sets

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Pattern Avoidance Problems

- ▶ How large can the Hausdorff dimension of a \mathbf{Q} vector subspace of \mathbf{R}^n be not containing any points in \mathbf{Q}^n .
- ▶ What is the largest Hausdorff dimension of a subset of \mathbf{R}^d such that the distances between any two points is irrational?
- ▶ What is the largest Hausdorff dimension of \mathbf{R}^d such that the angles formed from any three points is irrational?

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z , except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

- For any a_1, \dots, a_n , if $Z = \{x \in \mathbf{R}^{nd} : a_1x_1 + \dots + a_nx_n = 0\}$, X avoids solutions to the equation $a_1x_1 + \dots + a_nx_n = 0$ if and only if $X^n \cap Z \subset \Delta$.
- If $Z = \{(x, y) : d(x, y) \in \mathbf{Q}\} \subset \mathbf{R}^{2d}$, then points in X have irrational distances if and only if $X^2 \cap Z \subset \Delta$.
- If $Z = \{(x, y, z) : \frac{(x-z) \cdot (y-z)}{|x-z||y-z|} \in \cos(\mathbf{Q})\}$, X avoids 120° angles if and only if $X^3 \cap Z \subset \Delta$.

The Generic Problem

- ▶ **Fractal Avoidance Problem:** Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that $X^n \cap Z \subset \Delta$.
- ▶ Mathé (2012): If $Z \subset \mathbf{R}^{nd}$ is an algebraic hypersurface specified by a degree r polynomial in nd variables with rational coefficients, then we can find X solving the fractal avoidance problem for Z with dimension d/r . This is independent of n .
- ▶ Pramanik and Fraser (2016): If Z is a smooth hypersurface of dimension $nd - d$, we can find X with dimension $d/(n - 1)$.

Increasing the Difficulty...

*What if the Patterns are
Fractally Specified...*

Main Result

Theorem

If Z is the countable union of sets with lower Minkowski dimension bounded by $\alpha \geq d$, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathbf{H}}(X) = \frac{nd - \alpha}{n - 1} = \frac{\text{codim}(Z)}{n - 1}$$

Low Rank Avoidance

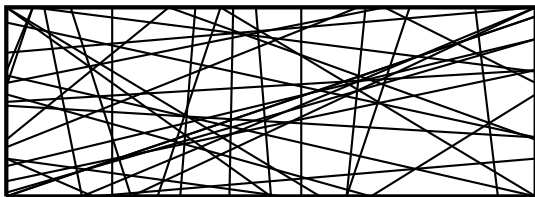
Theorem

If we have countably many sets $Z_i \subset \mathbf{R}^{n_i d}$ with linear transformations $T_i : \mathbf{R}^{n_i d} \rightarrow \mathbf{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension β_i , we can find X with $X^{n_i} \cap Z_i \subset \Delta$ for each i and

$$\dim_{\mathbf{H}}(X) = \sup_i \left(\frac{n_i k_i - \beta_i}{2k_i - 1} \right) = \sup_i \frac{\text{codim}(T_i(Z_i))}{2k_i - 1}$$

- ▶ The hypothesis says Z is coverable efficiently by lower dimensional thickened hyperplanes, rather than by boxes. Result should also extend when each Z_i is efficiently covered by other thickened pencils of low degree algebraic surfaces specified by polynomials.
- ▶ Want to push the $2k - 1$ to $k - 1$, at least for $k \geq 2$. Know this is true for certain families of examples.

Applications



- ▶ More robust generalization of Pramanik and Fraser, showing that we can 'thicken' or 'thin' the zero set of our function with stable effects on the Hausdorff dimension of X .
- ▶ Uncountable unions of regular sets are allowed!

Applications

- ▶ Given a subset Y of \mathbf{R}^d which is the countable union of sets with Minkowski dimension α , we can find a \mathbf{Q} vector subspace X of \mathbf{R}^d with Hausdorff dimension $d - \alpha$ disjoint from Y .
- ▶ We can find a full dimensional subset of \mathbf{R}^d avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $y \in \mathbf{Q}$. No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.
- ▶ Given a continuous $\gamma : [0, 1] \rightarrow \mathbf{R}^d$, finding $X \subset [0, 1]$ such that $\gamma(x), \gamma(y), \gamma(z)$ do not form an 'arithmetic progression' on the curve, in the sense that

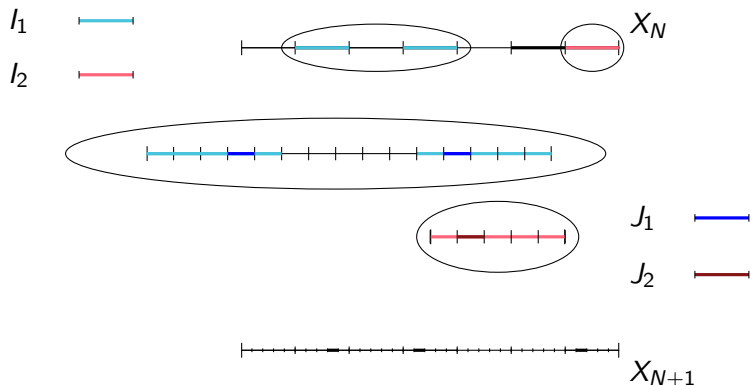
$$(\gamma(x) - \gamma(y)) - (\gamma(y) - \gamma(z)) = 0$$

Pramanik and Fraser can avoid smooth curves. We can now avoid *any* curve given a Hausdorff dimension calculation. Potentially even a Hilbert curve.

Two key ideas:

- ▶ Discretization of Scales by Queueing.
- ▶ Random Dissection.

Discretization of Scales

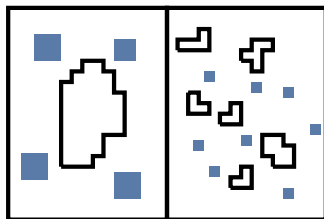


- **Discrete Problem:** Given disjoint unions of length L intervals l_1, \dots, l_n , find J_i for each i containing a part of each interval in l_i such that $J_1 \times \dots \times J_n$ is disjoint from Z .

Queueing

- ▶ Using a queueing procedure, and performing this single scale procedure over all possible choices of sets I_1, \dots, I_n gives a fractal avoiding set X .
- ▶ **Reason:** If we consider distinct $x_1, \dots, x_n \in X$, there are intervals I_1, \dots, I_n with $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \dots \times J_n$ cannot be contained in Z .

Exploiting Randomness



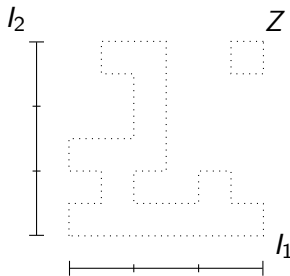
- ▶ How do we prove the discrete scale argument?
- ▶ Aside from Z 's dimension, we have little structural knowledge.
- ▶ Random choices of the J_k avoid Z effectively.
- ▶ We obtain for all but $o(1)$ of the length L intervals in I_k , J_k contains a length L^β section of each, where $\beta = d(nd - \alpha)/(n - 1)$. This ratio gives the Hausdorff dimension bound $(nd - \alpha)/(n - 1)$ for X .

So What's Next?

Extension to Hausdorff Dimension

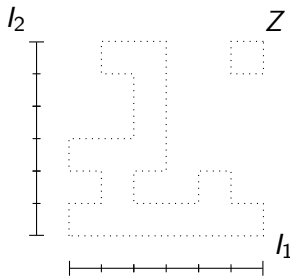
- ▶ There is no obvious reason why our techniques should fail when Z has *Hausdorff dimension* α rather than a Minkowski dimension bound.
- ▶ We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

Analogies with Hypergraphs



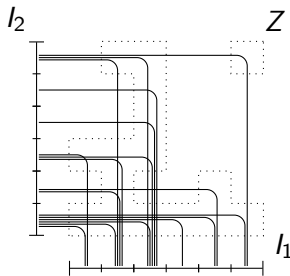
- ▶ The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- ▶ We are looking to using other methods on hypergraphs to improve the bound when Z has certain structural properties (like for the low rank result).

Analogies with Hypergraphs



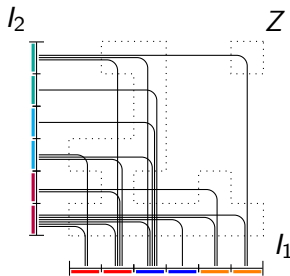
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You're Interested in Learning More?

Read these to find out more about configuration problems:

- ▶ Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.
- ▶ Read Máthé (2012) for a construction where f is assumed to be a polynomial of bounded degree.
- ▶ Read Pramanik and Fraser (2016) for a general construction where f is smooth and nonsingular. We generalize their work.
- ▶ Read Josh's, Malabika, and my papers if it's ever published...

Thanks for listening!