Fractals Avoiding Fractal Sets

Jacob Denson Malabika Pramanik Josh Zahl

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Abstract

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Consider a geometric point configuration, such as three points forming an isoceles triangle in the plane, or four points lying in a two dimensional plane in $\mathbf{R}^3$. A natural problem is to determine the minimum size a collection of points must be before we can guarantee a subselection lies in a specified configuration. This paper gives techniques to constructs large sets avoiding configurations, thus providing lower bounds for a general class of configuration avoidance problems. We call the construction of such sets a *configuration avoidance problem.*

It is often the case that we can obtain arbitrarily large finite sets not containing these configurations. On the other hand, every positive measure set contains the configuration. This is the case for the examples we obtained, or many other examples, especially when affine invariant. Thus we must use finer analytical measures of size for infinite sets, and the standard in this field is given by the Hausdorff dimension. More precisely, this paper gives techniques to construct sets with large Hausdorff dimension avoiding a very generic class of configurations. As a continuous analogue to Ramsey theory, what makes these problems interesting is seeing how the continuous setting differs from its discrete counterpart.

There are already generic pattern avoidance methods in the literature. We compare our method to them in section 6. But these rely on the non-singular nature of the configurations. The novel feature of our method is we can avoid configurations which have an *arbitrary* fractal quality to them. Meanwhile, the Hausdorff dimension obtained from the techniques holds up to previous methods.

A key idea to our method is the introduction of a novel geometric framework for pattern avoidance problems, described in section 1. We believe this new framework should help develop further methods in the field, some of which we are currently developing for publication in a later paper. The essence of our approach relies on a simple combinatorial argument, described in section 4, which can be applied once we have discretized the problem.

1 A Fractal Avoidance Framework

One way to think about generic pattern avoidance methods is to specify the pattern as the zero set of a function. For example,

- A set $X \subset \mathbf{R}^d$ contains the vertices of no isosceles triangles if and only if for any three distinct $x, y, z \in X$,

$$f(x,y,z) = d(x,y) - d(y,z) \neq 0$$

  If $f(x,y,z) = 0$, then the lines connecting x and y, and z and y form the legs of the isoceles triangle.

- A set $X \subset \mathbf{R}^d$ does not contain $d+1$ points in a lower dimensional hyperplane if and only if for any distinct $x_0, \dots, x_d \in X$,

$$f(x_0, \dots, x_d) = \det(x_1 - x_0, \dots, x_d - x_0) \neq 0$$

  since $\det(x_1 - x_0, \dots, x_d - x_0) = 0$ only when the vectors $x_1 - x_0, \dots, x_d - x_0$ do not form a basis, and thus span a plane of dimension smaller than d.

The pattern behind our description of these problems is summarized by a general framework.

The Configuration Avoidance Problem: Given a function $f : (\mathbf{R}^d)^n \to \mathbf{R}$ as input, find $X \subset \mathbf{R}^d$ such that for any *distinct* $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$, with as high a Hausdorff dimension as possible.

The functional framework is common in the literature. For instance, it is the viewpoint behind the methods of [2] and [3], who give results assuming various regularity conditions on the function f. For this paper, we take the perspect that f gives extraneous information irrelevant to the problem. It's real importance is suggesting geometric properties of it's zero set Z. Once Z is taken as the primary object to avoid, the configuration avoidance problem becomes equivalent to another framework. It is the viewpoint of this paper that this framework is more flexible to work with, and thinking in this framework leads to new general avoidance methods.

The Fractal Avoidance Problem: Given $Z \subset (\mathbf{R}^d)^n$, find a high dimensional set $X \subset \mathbf{R}^d$ such that if $x_1, \dots, x_n \in X$ are *distinct*, $(x_1, \dots, x_n) \notin Z$.

We are free to choose X anywhere, and because dimension is a local property of X, we can always fix a bounded region of space to work over, and then force X to lie in this bounded region. Thus we may assume Z lies in a bounded region, enabling us to use compactness arguments. This is necessary for our method, so we work with a local version of the fractal avoidance framework.

The Local Fractal Avoidance Problem: Given a set $Z \subset ([0,1]^d)^n$, find a high dimensional set $X \subset [0,1]^d$ such that if $x_1, \dots, x_n \in X$ are *distinct*, $(x_1, \dots, x_n) \notin Z$.

Because we are the first to introduce the fractal avoidance problem, a natural goal is to solve the generic problem with minimal assumptions on Z. Thus we let Z take the form of an arbitrary fractal, and the only assumption we place on Z is it's fractal dimension. To avoid too much technicality, we use the lower Minkowski dimension, which for a compact set E is defined as

$$\underline{\dim}_{\mathbf{M}}(E) = \liminf_{L \to 0} \frac{\log N(L,E)}{\log(1/L)}$$

where $N(L,E)$ is the minimal number of sidelength L boxes required to cover E.

Theorem 1. *If Z is the countable union of sets Y_N with $\underline{\dim}_{\mathbf{M}}(Y_N) \leqslant \alpha$ for each N, and $\alpha \geqslant d$, then there is X solving the local fractal avoidance problem for Z with*

$$\dim_{\mathbf{H}}(X) = \frac{dn - \alpha}{n-1} = \frac{\operatorname{codim}_{\mathbf{H}}(Z)}{n-1}$$

Remark. *If Z has dimension $\alpha < d$, the set obtained from $[0,1]^d$ by removing the projections of Z onto each coordinate has full Hausdorff dimension and trivially solves the fractal avoidance problem. Thus we need not consider these parameters in our theorem.*

Due to the lack of any rigid geometric information about the set Z, we are led to avoid Z by discretization. At the discrete scale, we can efficiently avoid Z by randomly choosing X. Exploiting this technique repeatedly at all scales gives a complete avoiding set.

2 Notation

Here we provide a concise summary of all the non-standard notation and terminology we use to describe our configuration avoidance method.

- For a length L, we let $\mathcal{B}(L,d)$ denote the partition of $\mathbf{R}^d$ into the family of all half open cubes with corners on the lattice $(\mathbf{Z}/L)^d$, i.e.

  $$\mathbf{B}(L,d) = \{[a_1,b_1) \times \dots \times [a_d,b_d) : a_i, b_i \in \mathbf{Z}/L\}$$

  If the dimension d is clear, or it's emphasis is unnecessary, we abbreviate $\mathcal{B}(L,d)$ as $\mathcal{B}(L)$.

- Families of cubes are denoted using calligraphic font, and the non-calligraphic version of the same character denotes the union of the family. For instance, we might have $\mathcal{I}, \mathcal{J} \subset \mathcal{B}(L)$, and then I is the union of all cubes in $\mathcal{I}$, and J the union of cubes in $\mathcal{J}$.

- Adopting the terminology of [8], we say a collection U_N is a *strong cover* of some set E if $E \subset \limsup U_N$, which means every element of E is contained in infinitely many of the sets U_N.

- Given a cube $I \in \mathcal{B}(L, dn)$, we can consider it's d dimensional sides $I_k \in \mathcal{B}(L,d)$, which decompose I as a Cartesian product $I_1 \times \dots \times I_n$. We say the cube is *non diagonal* if $I_1, \dots, I_n$ are distinct from one another.

3 Fractal Discretization

To construct a set X solving the fractal avoidance problem, we fix a decreasing sequence of dyadic scales L_N, which enable us to discretize the problem. The fact that Z is the countable union of sets with Minkowski dimension α implies that we can find an efficient strong cover of Z by cubes restricted to lie at the dyadic scales L_N. The next lemma says we can choose these scales to decrease at an arbitrary rate, which will be useful later on.

Lemma 1. *Let Z be a countable union of sets Y_N with $\underline{\dim}_{\mathbf{M}}(Y_N) \leqslant \alpha$, let f be a positive function on finite sequences of lengths, and consider any sequence ε_N converging to zero. Then there is a decreasing sequence of lengths L_N, with $L_{N+1} \leqslant f(L_1, \dots, L_N)$, and collections $\mathcal{Z}_N \subset \mathcal{B}(L_N)$ such that Z is strongly covered by the sets Z_N, and as $N \to \infty$, $|\mathcal{Z}_N| \leqslant 1/L_N^{\alpha+\varepsilon_N}$.*

Proof. Consider any sequence $k_1, k_2, \dots$ of integers which repeats each integer infinitely often. We define lengths L_N inductively. Given $L_1, \dots, L_N$, there are arbitrary small lengths L_{N+1} with $N(L_{N+1}, Y_{k_{N+1}}) \leqslant 1/L^{\alpha+\varepsilon_{N+1}}$. In particular, we can choose $L_{N+1} \leqslant f(L_1, \dots, L_N)$ and a set $\mathcal{Z}_{N+1} \subset \mathcal{B}(L_{N+1})$ covering $Y_{k_{N+1}}$ such that $|\mathcal{Z}_{N+1}| \leqslant 1/L^{\alpha+\varepsilon_{N+1}}$. It is easy to see this choice of Z_N gives a strong cover. $\square$

We can now construct X by avoiding the various discretizations of Z at each scale. We assume Z has been discretized into a strong cover Z_N. We then aim to construct a decreasing discretized version of X avoiding Z_N at each scale. More precisely, we construct a nested family of discretized sets X_N with $X = \lim X_N$. One condition that guarantees that X solves the fractal avoidance problem is that X_N^n is disjoint from *non diagonal* cubes in Z_N.

Lemma 2. *If for each N, X_N^n avoids non-diagonal cubes in Z_N, X solves the fractal avoidance problem for Z.*

Proof. Let $z \in Z$ be given with $z_1, \dots, z_n$ are distinct. Set

$$\Delta = \{w \in (\mathbf{R}^d)^n : \text{there exists } i,j \text{ such that } w_i = w_j\}$$

Then $d(\Delta, z) > 0$. The point z is covered by cubes in infinitely many of collections $\mathcal{Z}_{k_N}$. For suitably large N, the cube I in $\mathcal{B}(L_{k_N})$ containing z is disjoint from Δ. But this means that I is non diagonal. Thus $z \notin X_N^d$, and so in particular, is not an element of X^n. $\square$

4 Avoidance at Discretized Scales

Our goal now is quite simple. Given a set X_N formed from a union of sidelength L_N cubes, we must form a refinement $X_{N+1} \subset X_N$ formed from a union of sidelength L_{N+1} cubes, such that X_{N+1} avoids all non-diagonal cubes in $\mathcal{Z}_{N+1}$. For technical reasons in the calculation of the Hausdorff dimension of X, we strongly prefer that the cubes used to form X_{N+1} be evenly distributed throughout X_N. Thus we fix an intermediate dyadic scale R between L_N and L_{N+1}, and try and ensure that X_{N+1} contains a single sidelength L_{N+1} cube from each sidelength R cube. A random choice of this cube is a very effective strategy to do this.

Lemma 3. *Let $\mathcal{X}_N \subset \mathcal{B}(L_N, d)$, $\mathcal{Z}_{N+1} \subset \mathcal{B}(L_{N+1}, dn)$. Then there exists $\mathcal{X}_{N+1} \subset \mathcal{B}(L_{N+1}, d)$ with $X_{N+1} \subset X_N$, such that for any distinct $I_1, \dots, I_n \in \mathcal{X}_{N+1}$, $I_1 \times \dots \times I_n \notin \mathcal{Z}_{N+1}$. Furthermore, for all but at most $|Z_N| R^{-dn}$ of the $\mathcal{B}(R, d)$ cubes contained in X_N, X_{N+1} contains a single $\mathcal{B}(L_{N+1}, d)$ cube contained in I.*

Proof. Form a random $\mathcal{U} \subset \mathcal{B}(L_{N+1}, d)$ by selecting uniformly randomly, from each $\mathcal{B}(R)$ subcube of I, a single subcube in $\mathcal{B}(L_{N+1})$. The probability that any $\mathcal{B}(R)$ subcube is selected is $(L_{N+1}/R)^d$. Since any two $\mathcal{B}(L_{N+1})$ subcubes of $\mathcal{U}$ lie in distinct elements of $\mathcal{B}(R)$, the only chance that a $\mathcal{B}(L_{N+1})$ subcube I of $\mathcal{Z}_{N+1}$ with distinct sides intersects U^n is if $I_1, \dots, I_n$ all lie in separate cubes in $\mathcal{B}(R)$. Then the chance that each occurs is independant of one another, and so

$$\mathbf{P}(I \subset U^n) = \mathbf{P}(I_1 \in \mathcal{U}) \dots \mathbf{P}(I_n \in \mathcal{U}) = (L_{N+1}/R)^{dn}$$

If E denotes the number of $\mathcal{B}(L_{N+1})$ subcubes I of $\mathcal{Z}_{N+1}$ contained in U^n,

$$\begin{aligned}\mathbf{E}(E) &= \sum_{I \in \mathcal{Z}_N} \mathbf{P}(I \subset U^n) \\ &= (|Z_{N+1}| L_{N+1}^{-dn})((L_{N+1}/R)^{dn}) = |Z_{N+1}| R^{-dn}\end{aligned}$$

If, for each cube $I \in \mathcal{Z}_{N+1}$ contained in U^n, we remove I_1 from $\mathcal{U}$, we obtain a set $\mathcal{X}_{N+1}$ with $I_1 \times \dots \times I_n$ disjoint from $\mathcal{Z}_{N+1}$ for any distinct $\mathcal{B}(L_{N+1})$ subcubes I_k selected from $\mathcal{X}_{N+1}$. The set $\mathcal{X}_{N+1}$ contains an cube from all but E sidelength R cubes. In particular, we can select some nonrandom choice of U such that $E \leqslant |Z_{N+1}| R^{-dn}$, which gives the required set $\mathcal{X}_{N+1}$. $\square$

Corollary. *Fix*

$$0 < \beta \leqslant \min\left(1, \frac{dn - \gamma - \log_{L_{N+1}} |X_N| - \log_{L_{N+1}}(O(A))}{d(n-1)}\right)$$

and suppose R is the closest dyadic number to L_{N+1}^{β}. Furthermore, suppose that $|\mathcal{X}_{N+1}| \leqslant L_{N+1}^{-\gamma}$. Then $\mathcal{X}_{N+1}$ contains a portion of all but a fraction A of all $\mathcal{B}(R)$ subcubes of I.

Proof. The inequality for β implies

$$dn - \gamma - \beta(n-1)d \geqslant \log_{L_{N+1}} |X_N| + \log_{L_{N+1}}(O(A))$$

Since R is within a factor of two from S^{β}, we compute

$$\begin{aligned}\frac{\#(\mathcal{B}(R) \text{ subcubes not selected from})}{\#(\text{all } \mathcal{B}(R) \text{ subcubes})} &= \frac{|Z_{N+1}| R^{-dn}}{|X_N| R^{-d}} \\ &\leqslant |X_N|^{-1} L_{N+1}^{dn-\gamma} R^{-(n-1)d} \leqslant |X_N|^{-1} L_{N+1}^{dn-\gamma} (L_{N+1}/2)^{-d(n-1)} \\ &\leqslant 2^{\beta d} |X_N|^{-1} L_{N+1}^{\log_{L_{N+1}} |X_N| + \log_{L_{N+1}}(O(A))} = 2^{\beta d} O(A) \leqslant A.\end{aligned}$$

At the end, we are free to chose the constant in the $O(A)$ term to be on the order of $2^{-\beta d}$, which gives the required inequality. $\square$

In the next section, we show that for a sufficiently fast decaying set of lengths L_N, X has the Hausdorff dimension we need. This completes the description of our method.

5 Dimension Bounds

Now we show that the set X obtained is βd dimensional, where $\beta = (dn - \alpha)/d(n-1)$. To offset implicit obstructions to this dimension, we must choose the lengths L_N to decay suitably rapidly. The constraints on L_N will emerge naturally from our arguments, but for the impatient, one such choice is to set L_N to be the closest dyadic number to $2^{-N^2} (L_1 \dots L_{N-1})^{Nd(1-\beta)}$. The construction we considered looks like a $d\beta_N$ dimensional set at the discrete scales L_N. Another very useful fact is that the construction looks *full* dimensional between the scales L_{N-1} and R_N because of the relative uniformity by which we have taken intervals. This enables us to let the scales L_N to decrease arbitrarily rapidly, without penalizing us for initially looking at the Hausdorff dimension at discrete scales.

An initial requirement to get the βd dimensional result is to show that $\beta_N \to \beta$ as $N \to \infty$. Otherwise, we do not even get a βd dimensional result at the discrete scales. To obtain this, we just require the L_N decrease rapidly in proportion to the exponential 2^{-2N} and $|X_{N-1}|$. We know $\varepsilon_N = o(1)$ provided $L_N \to 0$, which is easy to obtain. If $L_N \lesssim_C 2^{-CN}$ for all large $C > 0$, then $\log_{L_N}(O(1/2^{2N+2})) = o(1)$. Given the bound $L_N \lesssim_C |X_{N-1}|^C$, we conclude $\log_{L_N} |X_{N-1}| = o(1)$. These two constraints thus imply that $\beta_N \to \beta$ as $N \to \infty$. Thus X behaves like a βd dimensional set at discrete scales. We first quantify this behaviour by working with a measure supported on X, then interpolate to obtain the behaviour at all scales.

Using Frostman's lemma, to prove X has dimension βd, it suffices to find a non-zero measure μ supported on X such that for all $\varepsilon > 0$, for all lengths L, and for all sidelength L cube I, $\mu(I) \lesssim_{\varepsilon} L^{\beta d - \varepsilon}$. To construct μ, we rely on a variant of the mass distribution principle, i.e. as the weak limit of measures μ_N supported on the discrete sets X_N. Initially, we put the uniform probability measure μ_0 on $X_0 = [0,1]^d$. We then define μ_N, supported on X_N, by modifying the distribution of μ_{N-1}. First, we throw away the mass of the $\mathcal{B}(L_{N-1})$ cubes I in X_{N-1} for which X_N fails to contain a $\mathcal{B}(L_N)$ cube in more than half of the $\mathcal{B}(R_N)$ subcubes of I. For the remaining cubes I, we uniformly distribute the mass $\mu_{N-1}(I)$ over the cubes in X_N contained in I. Throwing away mass is necessary to avoid the mass of μ_N clumping in undesirable places. It is easy to see from the cumulative distribution functions of the μ_N that μ_N converges weakly to a limit μ. The measure μ has the property that for any $I \in \mathcal{B}(L_N)$, $\mu(I) \leqslant \mu_N(I)$, which is useful from passing from discrete results about our construction to properties of the final measure. The measure μ is the measure for which we will ultimately obtain a Frostman type inequality.

Lemma 4. *If $N \geqslant 1$, and $I \in \mathcal{B}(L_N)$,*

$$\mu(I) \leqslant \mu_N(I) \leqslant 2^N \left[\frac{R_N R_{N-1} \dots R_1}{L_{N-1} \dots L_1} \right]^d$$

Proof. Consider $I \in \mathcal{B}(L_N)$, $J \in \mathcal{B}(L_{N-1})$. If $\mu_N(I) > 0$, this means that J contains a $\mathcal{B}(L_N)$ cube in at least half of the $\mathcal{B}(R_N)$ cubes it contains. Thus the mass of J distributes itself evenly over at least $2^{-1}(L_{N-1}/R_N)^d$ cubes, which gives that $\mu_N(I) \leqslant 2(R_N/L_{N-1})^d \mu_{N-1}(J)$. But then expanding this recursive inequality, we obtain exactly the result we need. □

Corollary. *The measure μ is positive.*

Proof. To prove this result, it suffices to show that the total mass of μ_N is bounded below, independantly of N. At each stage N, X_N consists of at most

$$\left[\frac{L_{N-1} \dots L_1}{R_N \dots R_1} \right]^d$$

$\mathcal{B}(L_N)$ cubes. Since only a fraction $1/2^{2N+2}$ of the $\mathcal{B}(R_N)$ cubes do not contain an interval in X_{N+1}, it is only for at most a fraction $1/2^{2N+1}$ of the $\mathcal{B}(L_N)$ cubes that X_{N+1} fails to contain a $\mathcal{B}(L_{N+1})$ cube from more than half of the $\mathcal{B}(R_{N+1})$ cubes. Using the last lemma, this means that in the passage from μ_N to μ_{N+1}, we discard a mass of at most $1/2^{N+1}$. Thus

$$\mu_N(\mathbf{R}^d) \geqslant 1 - \sum_{k=0}^{N} \frac{1}{2^{k+1}} \geqslant 1/2$$

Thus $\mu(\mathbf{R}^d) \geqslant 1/2$, and in particular, $\mu \neq 0$. □

We fix an increasing sequence λ_N with $\lambda_N < \beta_N$, and $\lambda_N \to \beta$ as $N \to \infty$. This gives us slightly more room to bound mass when obtaining the Frostman's lemma result. We set $\lambda_N - \beta_N = 1/N$ to obtain the choice of L_N given as an example.

Corollary. *If $L_N \ll 1$, $\mu(I) \leqslant L_N^{d\lambda_N}$ for $I \in \mathcal{B}(L_N)$.*

Proof. We can rewrite the inequality in the last problem as

$$\mu(I) \leqslant \left[2^N \left(\frac{R_{N-1} \dots R_1}{L_{N-1} \dots L_1} \right)^d R_N^d L_N^{-d\lambda_N} \right] L_N^{d\lambda_N}$$

Now $R_N^d L_N^{-d\lambda_N} \leqslant (2L_N^{\beta_N})^d L_N^{-d\lambda_N} \leqslant 2^d L_N^{d(\beta_N - \lambda_N)}$, which tends to zero as $L_N \to \infty$, while the remaining parameters are fixed. Thus if L_N is sufficiently small, we can bound the constant in the square brackets by 1, which is sufficient to obtain the inequality. □

This is the cleanest expression of the $d\beta$ dimensional behaviour at discrete scales. To get a general inequality of this form, we use the fact that our construction distributes uniformly across all intervals.

Theorem 2. *Assume the last corollary holds. If $L \leqslant L_N$ is dyadic and $I \in \mathcal{B}(L)$, then $\mu(I) \leqslant 2L^{\lambda_N}$.*

Proof. We break our analysis into three cases, depending on the size of L in proportion to L_N and R_N:

- If $R_{N+1} \leqslant L \leqslant L_N$, we can cover I by $(L/R_{N+1})^d$ cubes in $\mathcal{B}(R_{N+1})$. For each of these cubes, we know the mass is bounded by at most $2(R_{N+1}/L_{N+1})^d$ times the mass of a $\mathcal{B}(L_N)$ cube. Thus

$$\begin{aligned} \mu(I) &\leqslant [(L/R_{N+1})^d][2(R_{N+1}/L_N)^d][L_N^{\lambda_N}] \\ &\leqslant 2L^d L_N^{d-\lambda_N} \leqslant 2L^{\lambda_N} \end{aligned}$$

- If $L_{N+1} \leqslant L \leqslant R_{N+1}$, we can cover I by a single cube in $\mathcal{B}(R_{N+1})$. Each cube in $\mathcal{B}(R_{N+1}, d)$ contains at most one cube in $\mathcal{B}(L_{N+1}, d)$ which is also contained in X_{N+1}, so $\mu(I) \leqslant L_{N+1}^{d\lambda_{N+1}} \leqslant L^{d\lambda_N}$.
- If $L \leqslant L_{N+1}$, there certainly exists M such that $L_{M+1} \leqslant L \leqslant L_M$, and one of the previous cases yields that $\mu(I) \leqslant 2L^{\lambda_M} \leqslant 2L^{\lambda_N}$.

This addresses all cases considered in the theorem. □

To use Frostman's lemma, we need the result $\mu(I) \lesssim L^{\lambda_N}$ for an *arbitrary* interval, not just one with $L \leqslant L_N$. But this is no trouble; it is only the behavior of the measure on arbitrarily small scales that matters. This is because if $L \geqslant L_N$, then $\mu(I)/L^{\lambda_N} \leqslant 1/L_N^{\lambda_N} \lesssim_N 1$, so $\mu(I) \lesssim_N L^{\lambda_N}$ holds automatically for all sufficiently large intervals. Thus the general bound is complete, and we have proven that there is a choice of parameters which constructs a set X with Hausdorff dimension no less than $(dn - \alpha)/(n-1)$. It is also easy to see X has *precisely* this dimension.

Theorem 3. $\dim_{\mathbf{H}}(X) \leqslant (dn - \alpha)/(n-1)$.

Proof. X_N is covered by at most

$$\left[\frac{L_{N-1} \dots L_1}{R_N \dots R_1} \right]^d$$

sidelength L_N cubes. It follows that if $\gamma > \beta$, then

$$H^{d\gamma}_{L_N}(X) \leqslant \left[\frac{L_{N-1}\dots L_1}{R_N\dots R_1} L_N^\gamma\right]^d \lesssim \left[\frac{L_{N-1}\dots L_1}{R_{N-1}\dots R_1} L_N^{\gamma-\beta}\right]^d$$

Thus if L_N is suitably small depending on previous constants, which we know to be true from the last corollary, we conclude that as $N \to \infty$, $H^\gamma(X)$ is finite. Since γ was arbitrary, taking it to β allows us to conclude that $\dim_{\mathbf{H}}(X) \leqslant d\beta$. $\square$

6 Applications

7 Comparison with Other Generic Avoidance Schemes

In the past few years in the discrete setting it has been noticed that rephrasing particular questions in terms of abstract problems on hypergraphs allows one to extend various results into sparse analogues [4]. In this paper we consider a continuous analogue, where sparsity is represented in terms of the dimension of the set Z we are trying to avoid.

8 Concluding Remarks

Another goal of our current research programme is to show an example of a fractal avoidance problem where extra geometric conditions on Z leads to constructions with a higher Hausdorff dimension. This means that our framework isn't designed for a single method, but naturally incorporates further methods in the field. We consider a condition where Z is efficiently coverable by parallel hyperplanes of a fixed dimension.

Theorem 4. *If there is $k \geqslant 2$ and a linear $T : \mathbf{R}^{dn} \to \mathbf{R}^{kd}$ such that $T(Z)$ is α dimensional, with $\alpha \leqslant (d-1)k$, then there exists X with*

$$\dim_{\mathbf{H}}(X) = \frac{dk-\alpha}{2k-1}$$

solving the fractal avoidance problem for Z.

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