Salem Sets Avoiding Fractal Sets

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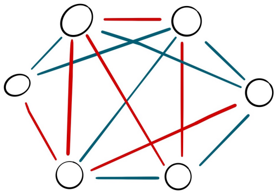
General Research Question



- Phenomenon: Structure appears in suitably large objects.
- Like Ramsey Theory, but with a more analytical foundation, e.g. Geometric Measure Theory / Harmonic Analysis.

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- What does 'largeness' mean?

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- *Hausdorff dimension* $\approx$ Minkowski dimension for compact X.

The Generic Problem

- **Avoidance Problem**: Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large dimension such that for distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We say X *avoids* Z.

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- e.g. Z is a degree 2 algebraic hypersurface in last example.

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- What if we use less rigid geometric information, i.e. the fractal dimension of the set Z?

Our Results

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- (Proved in Msc Thesis, but want to find higher dimensional result before full publication).

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$$Z_m = \{(x_1, \dots, x_n) \in \mathbf{R}^{nd} : m_1 x_1 + \cdots + m_n x_n \in H\}.$$

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  - We would hope that whatever higher dimensional generalization would construct $G \subset \mathbf{R}^d$ with Hausdorff dimension $d - s$ for any H of Minkowski dimension s.

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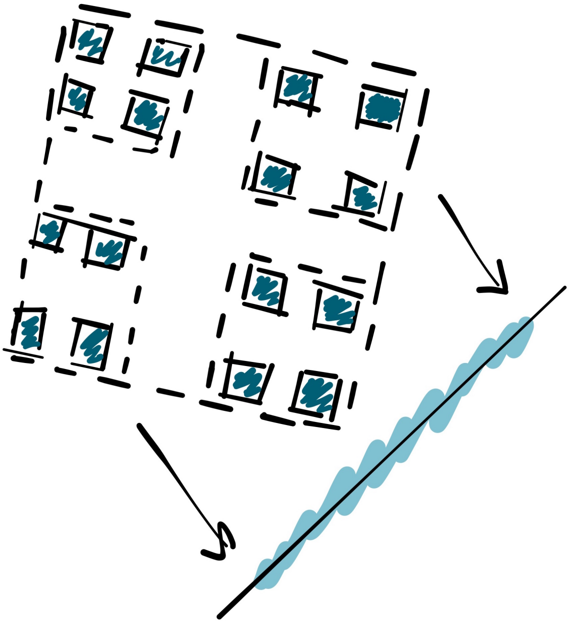
- We have also used this technique to bound the existence of isosceles triangles on Lipschitz curves.



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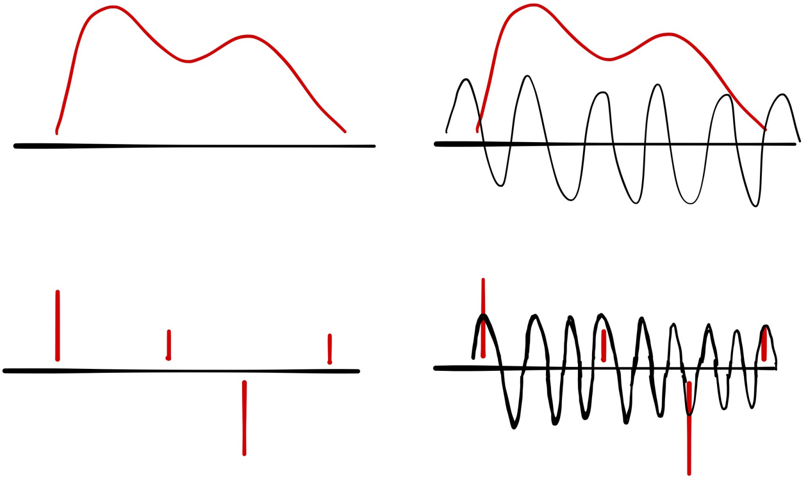
- (Rudin, 1960) If X has Fourier dimension greater than $1/n$, then there exists some $m \in \mathbf{Z}^n$ and some $x_1, \dots, x_n \in X$ such that $m_1 x_1 + \dots + m_n x_n = 0$.



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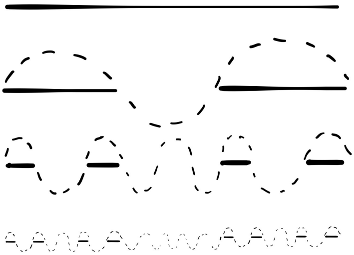

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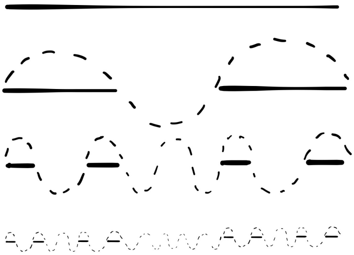

- Heuristic: Typically need 'square root cancellation' to obtain optimal Fourier decay, e.g. by using randomness.

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Theorem (2020, Denson)

If Z has lower Minkowski dimension bounded by s, we can find X avoiding Z with

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Conjecture: If Z is 'suitably smooth', then we can find X with $dim_{\mathbf{H}}(X) = dim_{\mathbf{F}}(X) = (nd - s)/(n - 1)$.

Thanks for listening!