Sets, Patterns, and Fourier Decay

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Fourier Analysis and Patterns in Sets

- What can one learn about the geometry of a compact set $E \subset \mathbb{T}^d$ via analytical properties of probability measures μ supported on E?
- A set E has *Hausdorff dimension* s if for any $t < s$, E supports a probability measure μ_t with

$$\sum_{k \neq 0} |\widehat{\mu_t}(k)|^2 |k|^{t-d} < \infty.$$

- A set has *Fourier Dimension* s if it supports μ_t with $|\widehat{\mu_t}(k)| \lesssim |k|^{-t/2}$ for *all* k.
- $\dim_{\mathbb{F}}(E) \leq \dim_{\mathbb{H}}(E)$.

Pattern Avoidance

- If $\dim(E)$ is large, does E 'contain patterns'.
- Basic Example: If $\dim(E)$ is large, are there $m_1, \dots, m_n \in \mathbb{Z}$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$? (Can large sets be linearly independent over $\mathbb{Q}$)
- (Keleti, 1999) There is $E \subset \mathbb{T}$ with $\dim_{\mathbb{H}}(E) = 1$ such that for any $m_1, \dots, m_n$ and distinct $x_1, \dots, x_n \in E$, $m_1 x_1 + \dots + m_n x_n \neq 0$.
- If $\dim_{\mathbb{F}}(E) > 0$, there is n, $m_1, \dots, m_n \in \mathbb{Z}$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$.
- If $\dim_{\mathbb{F}}(E) > 2/n$, then there are $m_1, \dots, m_n$ and distinct $x_1, \dots, x_n \in E$ such that $m_1 x_1 + \dots + m_n x_n = 0$.

Independent Sets

- (Rudin, 1960): There exists $E \subset \mathbb{T}$ and a finite Borel measure μ with $\mathrm{supp}(\mu) \subset E$ such that E is independent but $|\widehat{\mu}(k)| \to 0$ as $|k| \to \infty$.
- (Körner, 2007): There exists independent E supporting measures converging to zero as 'fast as possible'.
- (Körner, 2009): There exists $E \subset \mathbb{T}$ with $\dim_{\mathbb{F}}(E) = 1/(n-1)$ such that E avoids solutions to all n-term linear equations.

Arithmetic Progressions ($x_1 - 2x_2 + x_3 = 0$)

- (Łaba and Pramanik, 2007): For some small $\varepsilon > 0$, if $|\widehat{\mu}(k)| \leq C_1|k|^{-(1-\varepsilon)/2}$ and $\mu((x, x+r)) \leq C_2 r^{\alpha}$ for appropriate C_1, C_2, and α, $\text{supp}(\mu)$ contains arithmetic progressions.
- (Schmerkin, 2015): There is $E \subset \mathbb{T}$ avoiding arithmetic progressions with $\dim_{\mathbb{F}}(E) = 1$.
- (Liang and Pramanik, 2020): Generalized Schmerkin's construction to all translation-invariant patterns.

Fourier Dimension and Nonlinear Patterns

- (Henriot and Łaba and Pramanik, 2015): For certain linear maps $A_1, \ldots, A_n$ and polynomials Q, there is $\varepsilon > 0$ such that if $E \subset \mathbb{T}$ and $\dim_{\mathbb{F}}(E) \geq 1 - \varepsilon$, E contains a family of points of the form

$$\{x, x + A_1 y, \ldots, x + A_{n-1} y, x + A_n y + Q(y)\}.$$

  The pattern $\{x, x+t, x+t^2\}$ is *not* covered.
- (Fraser and Guo and Pramanik, 2019): If $\deg(f) > 1$ and $f(0) = 0$, then patterns of the form $\{x, x+t, x+f(t)\}$ exist in $\operatorname{supp}(\mu)$ if μ satisfies explicit estimates ala Łaba and Pramanik.
- (Kuca, Orponen, Sahlsten, Preprint 2021): If $E \subset \mathbb{T}^2$ and $\dim_{\mathbb{H}}(E) \geq 2 - \varepsilon$, then E contains solutions to $y_2 - x_2 = (y_1 - x_1)^2$ for distinct $x, y \in E$.

Sets Avoiding Nonlinear Patterns for Hausdorff Dimension

- Find large $E \subset \mathbb{T}^d$ such that for distinct $x_1, \dots, x_n \in E$,

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

Author	**Property of f**	$\dim_{\mathbb{H}}(X)$
Mathé (2017)	A degree r polynomial	d/r
Fraser Pramanik (2019)	f is C^1	$1/(n-1)$
D. Pramanik Zahl (2020)	f Lipschitz	$1/(n-1)$
D. (2020)	$f = g \circ \pi$ where the linear map $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^{m-1}$ is is surjective	$1/(m-1)$

- Can we modify these constructions to obtain Salem sets?

Main Result

Theorem

Suppose $f(x_1, \ldots, x_{n-1})$ *is* C^{d+1}*, and for each* $1 \le i \le n-1$*,*

$$D_{x_k} f = \left(\frac{\partial f_i}{\partial x_{kj}} \right)$$

is invertible. Then there exists $E \subset \mathbb{T}^d$ *with*

$$dim_{\mathbb{F}}(E) = \beta = \frac{d}{n - 3/4}$$

avoiding solutions to $x_n = f(x_1, \ldots, x_{n-1})$*.*

- (Fraser and Pramanik, 2016) obtains a set $E \subset \mathbb{R}$ with

$$\dim_{\mathbb{H}}(E) = \frac{d}{n-1}.$$

Aside: Baire Categories vs Probabilities

- Result actually show that a *generic* set E with Fourier dimension $s \leq \beta$ avoids solutions to the equation $x_n = f(x_1, \dots, x_{n-1})$.
- (Schmerkin, 2017): For any $y_1, \dots, y_n \in \mathbb{T}^2$, a *random* set of dimension $s > (dn - (d+1))/n$ will almost surely contain a translated, dilated copy of $\{y_1, \dots, y_n\}$.
- A generic set of dimension

$$\frac{dn - (d+1)}{n - 1/2}$$

  avoids translated, dilated copies of $\{y_1, \dots, y_n\}$.
- Thus Baire category and probabilistic notions of 'almost everywhere' do not correspond for sets of dimension

$$\frac{dn - (d+1)}{n} < s \leq \frac{dn - (d+1)}{n - 1/2}$$

Technical Reduction

Theorem holds if we can prove that for any disjoint intervals $I_1, \dots, I_n \subset \mathbb{T}^d$, there exists arbitrarily large sets $S = \{x_1, \dots, x_N\}$ such that if $\delta = N^{-1/\beta}$,

- If $x_i \in I_i \cap N_\delta(S)$ for $1 \leq i \leq n$, $x_n \neq f(x_1, \dots, x_{n-1})$.
- For any $|\xi| \leq 1/\delta$,

$$\left| \frac{1}{N} \sum_i e^{2\pi i \xi \cdot x_i} \right| \ll N^{-1/2}$$

The Construction

- For $1 \leq i \leq n$, let $\{X_{i1}, \ldots, X_{iM}\}$ be independently selected from I_i.
- Let

$$\mathcal{B} = \left\{ j_n : \begin{array}{c} |X_{nj_n} - f(X_{1j_1}, \ldots, X_{n-1,j_{n-1}})| \leq \delta \\ \text{for some } j_1, \ldots, j_{n-1} \end{array} \right\}.$$

- Take $S = \{X_{ij}\} - \{X_{nj} : j \in \mathcal{B}\}$.
- Set $N = \#(S)$.
- Then $\mathcal{N}_\delta(S)$ avoids solutions to $x_n = f(x_1, \ldots, x_{n-1})$.

Goal

- If
$$Y_\xi = \frac{1}{N}\sum_{x\in S} e^{2\pi i \xi\cdot x},$$
then $|Y_\xi| \ll N^{-1/2}$.
  - Step 1: $|Y_\xi - \mathbb{E}\, Y_\xi| \ll N^{-1/2}$ with large probability.
  - Step 2: $|\mathbb{E}\, Y_\xi| \ll N^{-1/2}$.
- Step 1 is obtained through *concentration inequalities*.
- Step 2 is obtained through *oscillatory integral estimates*.

Concentration Inequalities

- If $Y = f(X_1, \dots, X_N)$, where $X_1, \dots, X_N$ are independant, and have 'equal influence' on Y, then $|Y - \mathbb{E}\, Y| \lesssim \sqrt{N}$ with high probability.
- **Hoeffding's Inequality** If $|X_i| \leq A_i$ for each i, and

$$Y + X_1 + \cdots + X_N,$$

  then $|X_1 + \cdots + X_N| \lesssim (A_1^2 + \cdots + A_N^2)^{1/2}$ with high probability.
- **McDiarmid's Inequality** If

$$|f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i', \dots, x_n)| \leq A_i$$

  then $|Y - \mathbb{E}\, Y| \lesssim (A_1^2 + \cdots + A_N^2)^{1/2}$ with high probability.

Bounding $\mathbb{E}\, Y_\xi$

- We have
$$\mathbb{E}(Y_\xi) = \int_{\mathbb{T}^d} \psi(x_n) e^{2\pi i \xi \cdot x_n}\, d\,\mathbb{P}(x_n),$$
where $d\,\mathbb{P}(x) = \mathbb{P}(j \in \mathcal{B} | X_{nj} = x)$ and $\psi \in C^\infty(\mathbb{T}^d)$.
- If $\beta \leq d/(n-3/4)$,
$$d\,\mathbb{P}(x) = M^{n-1} |f^{-1}(B_\delta(x))| + O(N^{-1/2}),$$
so
$$\mathbb{E}(Y_\xi) = M^{n-1} \int_{B_r(0)} \int_{\mathbb{T}^{d(n-1)}} \psi(y, v) e^{2\pi i \xi \cdot (f(y) - v)}\, dy\, dv + O(N^{-1/2}).$$

Linear Result

- Difficulties of estimating $\mathbb{E}(Y_\xi)$ can be eliminated if $d\mathbf{P}(x)$ is independant of x (so $\mathbb{E}(Y_\xi) = 0$ for all $\xi \neq 0$). This is true, for instance, if there is some linearity in the equation.

Theorem

Suppose f is Lipschitz. Then there exists $E \subset \mathbb{T}^d$ with

$$dim_{\mathbb{F}}(E) = \frac{d}{n-1}$$

avoiding solutions to the equation

$$x_n - x_{n-1} = f(x_1, \dots, x_{n-2}).$$

- One application of this is a higher dimensional generalization of Körner's result on sets avoiding n term linear equations.

In Progress: Subsets of Curves Avoiding Isosceles Triangles

- If $\gamma : [0,1] \to \mathbb{T}^d$ is a smooth curve with γ' non-vanishing, then the result we discussed shows there is $E \subset [0,1]$ with $\dim_{\mathbb{F}}(E) = 4/9$ such that $\gamma(E)$ does not contain the vertices of an isosceles triangle.
- (Fraser Pramanik 2019) show there exists $E \subset [0,1]$ with $\dim_{\mathbb{H}}(E) = 1/2$ such that $\gamma(E)$ does not contain the vertices of an isosceles triangle.
- If γ has non-vanishing curvature, can one find E such that $\dim_{\mathbb{F}}(\gamma(E)) = 4/9$?