Fractals avoiding Fractal Sets

Jacob Denson

University of British Columbia

December 9, 2018

Pattern Avoidance Problems

- How large can the Hausdorff dimension of a $\mathbf{Q}$ vector subspace of $\mathbf{R}^n$ be not containing any points in $\mathbf{Q}^n$.

Pattern Avoidance Problems

- How large can the Hausdorff dimension of a $\mathbf{Q}$ vector subspace of $\mathbf{R}^n$ be not containing any points in $\mathbf{Q}^n$.
- What is the largest Hausdorff dimension of a subset of $\mathbf{R}^d$ such that the distances between any two points is irrational?

Pattern Avoidance Problems

- How large can the Hausdorff dimension of a $\mathbf{Q}$ vector subspace of $\mathbf{R}^n$ be not containing any points in $\mathbf{Q}^n$.
- What is the largest Hausdorff dimension of a subset of $\mathbf{R}^d$ such that the distances between any two points is irrational?
- What is the largest Hausdorff dimension of $\mathbf{R}^d$ such that the angles formed from any three points is irrational?

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

  - For any $a_1, \dots, a_n$, if $Z = \{x \in \mathbf{R}^{nd} : a_1 x_1 + \dots + a_n x_n = 0\}$, X avoids solutions to the equation $a_1 x_1 + \dots + a_n x_n = 0$ for *distinct* $x_1, \dots, x_n$ if and only if $X^n \cap Z \subset \Delta$.

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i,j \in \{1,\dots,n\}\}$$

  - For any $a_1,\dots,a_n$, if $Z = \{x \in \mathbf{R}^{nd} : a_1x_1 + \dots + a_nx_n = 0\}$, X avoids solutions to the equation $a_1x_1 + \dots + a_nx_n = 0$ for *distinct* $x_1,\dots,x_n$ if and only if $X^n \cap Z \subset \Delta$.
  - If $Z = \{(x,y) : d(x,y) \in \mathbf{Q}\} \subset \mathbf{R}^{2d}$, then points in X have irrational distances if and only if $X^2 \cap Z \subset \Delta$.

Two Observations

- Problems can be summarized as finding X such that X^n avoids a given set Z, except for 'repeated coordinate points'. Let

$$\Delta = \{x \in (\mathbf{R}^d)^n : x_i = x_j \text{ for distinct } i, j \in \{1, \dots, n\}\}$$

  - For any $a_1, \dots, a_n$, if $Z = \{x \in \mathbf{R}^{nd} : a_1 x_1 + \dots + a_n x_n = 0\}$, X avoids solutions to the equation $a_1 x_1 + \dots + a_n x_n = 0$ for *distinct* $x_1, \dots, x_n$ if and only if $X^n \cap Z \subset \Delta$.
  - If $Z = \{(x,y) : d(x,y) \in \mathbf{Q}\} \subset \mathbf{R}^{2d}$, then points in X have irrational distances if and only if $X^2 \cap Z \subset \Delta$.
  - If $Z = \{(x,y,z) : \frac{(x-z)\cdot(y-z)}{|x-z||y-z|} \in \cos(\mathbf{Q})\}$, X avoids rational angles if and only if $X^3 \cap Z \subset \Delta$.

The Generic Problem

- **Fractal Avoidance Problem**: Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that $X^n \cap Z \subset \Delta$.

The Generic Problem

- **Fractal Avoidance Problem**: Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that $X^n \cap Z \subset \Delta$.
- Mathé (2012): If $Z \subset \mathbf{R}^{nd}$ is an algebraic hypersurface specified by a degree r polynomial in nd variables with rational coefficients, then we can find X solving the fractal avoidance problem for Z with dimension d/r. This is independent of n.

The Generic Problem

- **Fractal Avoidance Problem**: Given $Z \subset \mathbf{R}^{nd}$, find $X \subset \mathbf{R}^d$ with large Hausdorff dimension such that $X^n \cap Z \subset \Delta$.
- Mathé (2012): If $Z \subset \mathbf{R}^{nd}$ is an algebraic hypersurface specified by a degree r polynomial in nd variables with rational coefficients, then we can find X solving the fractal avoidance problem for Z with dimension d/r. This is independent of n.
- Pramanik and Fraser (2016): If Z is a smooth hypersurface of dimension $nd - d$, we can find X with dimension $d/(n-1)$.

What if the Patterns are Fractally Specified...

Main Result

Theorem

If Z is the countable union of sets with lower Minkowski dimension bounded by $\alpha \geq d$, we can find X with $X^n \cap Z \subset \Delta$ and

$$\dim_{\mathbf{H}}(X) = \frac{nd - \alpha}{n - 1} = \frac{codim(Z)}{n - 1}$$

Low Rank Avoidance

Theorem

If we have countably many sets $Z_i \subset \mathbf{R}^{n_i d}$ with linear transformations $T_i : \mathbf{R}^{n_i d} \to \mathbf{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension β_i, we can find X with $X^{n_i} \cap Z_i \subset \Delta$ for each i and

$$\dim_{\mathbf{H}}(X) = \sup_i \left(\frac{n_i k_i - \beta_i}{2k_i - 1} \right) = \sup_i \frac{codim(T_i(Z_i))}{2k_i - 1}$$

Low Rank Avoidance

Theorem

If we have countably many sets $Z_i \subset \mathbf{R}^{n_i d}$ *with linear transformations* $T_i : \mathbf{R}^{n_i d} \to \mathbf{R}^{k_i d}$ *with rational coordinates such that* $T_i(Z_i)$ *has lower Minkowski dimension* β_i*, we can find* X *with* $X^{n_i} \cap Z_i \subset \Delta$ *for each* i *and*

$$\dim_{\mathbf{H}}(X) = \sup_i \left(\frac{n_i k_i - \beta_i}{2k_i - 1} \right) = \sup_i \frac{codim(T_i(Z_i))}{2k_i - 1}$$

- The hypothesis says Z is coverable efficiently by lower dimensional thickened hyperplanes. Result should also extend when each Z_i is efficiently covered by thickened pencils of low degree algebraic surfaces, i.e. $f(Z)$ has low dimension where f is a polynomial map.

Low Rank Avoidance

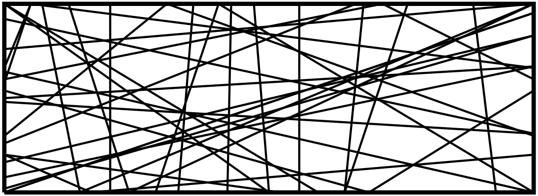
Theorem

If we have countably many sets $Z_i \subset \mathbf{R}^{n_i d}$ with linear transformations $T_i : \mathbf{R}^{n_i d} \to \mathbf{R}^{k_i d}$ with rational coordinates such that $T_i(Z_i)$ has lower Minkowski dimension β_i, we can find X with $X^{n_i} \cap Z_i \subset \Delta$ for each i and

$$\dim_{\mathbf{H}}(X) = \sup_i \left(\frac{n_i k_i - \beta_i}{2k_i - 1} \right) = \sup_i \frac{codim(T_i(Z_i))}{2k_i - 1}$$

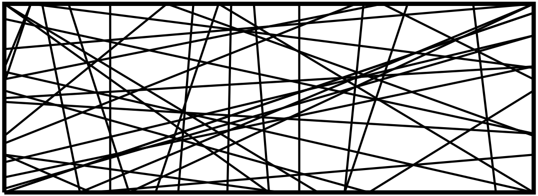
- The hypothesis says Z is coverable efficiently by lower dimensional thickened hyperplanes. Result should also extend when each Z_i is efficiently covered by thickened pencils of low degree algebraic surfaces, i.e. $f(Z)$ has low dimension where f is a polynomial map.
- Want to push the $2k-1$ to $k-1$, at least for $k \geq 2$. Know this is true for certain families of examples.

Applications



- More robust generalization of Pramanik and Fraser, showing that we can ‘thicken’ or ‘thin’ the zero set of our function with stable effects on the Hausdorff dimension of X.

Applications



- More robust generalization of Pramanik and Fraser, showing that we can 'thicken' or 'thin' the zero set of our function with stable effects on the Hausdorff dimension of X.
- Uncountable unions of regular sets are allowed!

Applications

- Given a subset Y of $\mathbf{R}^d$ which is the countable union of sets with Minkowski dimension α, we can find a $\mathbf{Q}$ vector subspace X of $\mathbf{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from Y.

Applications

- Given a subset Y of $\mathbf{R}^d$ which is the countable union of sets with Minkowski dimension α, we can find a $\mathbf{Q}$ vector subspace X of $\mathbf{R}^d$ with Hausdorff dimension $d - \alpha$ disjoint from Y.
- We can find a full dimensional subset of $\mathbf{R}^d$ avoiding the zero sets of all polynomials with rational coefficients of the form $f(y \cdot x)$ with $y \in \mathbf{Q}$. No dependence on the degree of the polynomial. Complexity is measured by rank rather than degree.

The Method

Two key ideas:

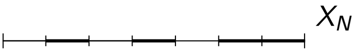
Two key ideas:

- Discretization of Scales.

Two key ideas:

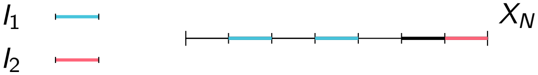
- Discretization of Scales.
- Random Disection.

Discretization of Scales



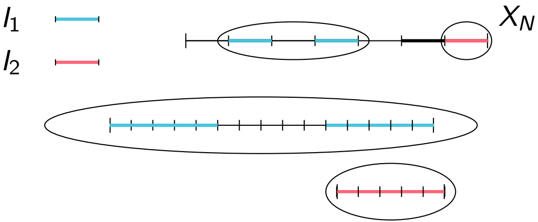
- Construct X as a Cantor set by interval dissection.

Discretization of Scales



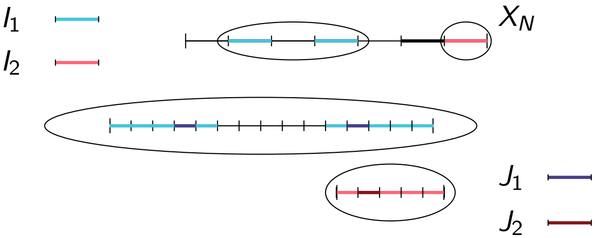
- Construct X as a Cantor set by interval dissection.

Discretization of Scales



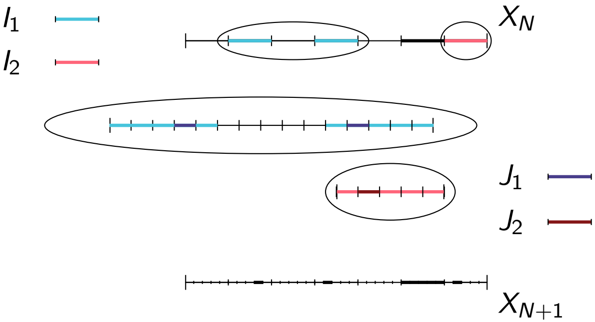
- Construct X as a Cantor set by interval dissection.

Discretization of Scales



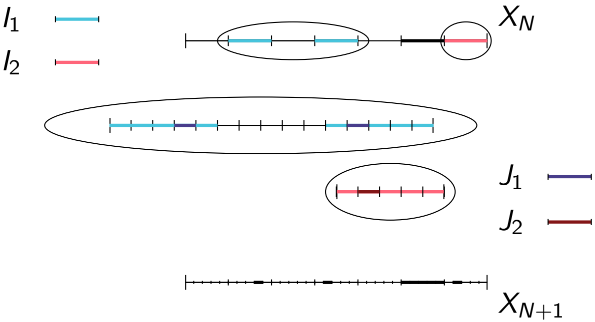
- Construct X as a Cantor set by interval dissection.

Discretization of Scales



- Construct X as a Cantor set by interval dissection.

Discretization of Scales



- Construct X as a Cantor set by interval dissection.
- **Discrete Problem**: Given disjoint unions of length L intervals $I_1, \ldots, I_n$, find J_i for each i containing a part of each interval in I_i such that $J_1 \times \cdots \times J_n$ is disjoint from Z.

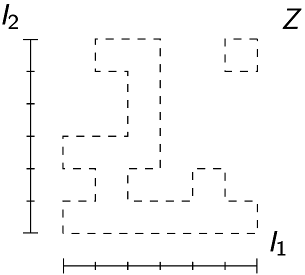
Queueing

- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set X gives a fractal avoiding set X.

Queueing

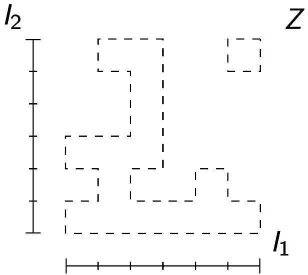
- Using a queueing procedure, and performing this single scale procedure over all arbitrarily fine covers $I_1, \dots, I_n$ of the set X gives a fractal avoiding set X.
- **Reason**: If we consider distinct $x_1, \dots, x_n \in X$, there are intervals $I_1, \dots, I_n$ with $x_1 \in I_1, \dots, x_n \in I_n$ considered at some scale. Then $x_1 \in J_1, \dots, x_n \in J_n$, and so $(x_1, \dots, x_n) \in J_1 \times \dots \times J_n$ cannot be contained in Z.

Exploiting Randomness



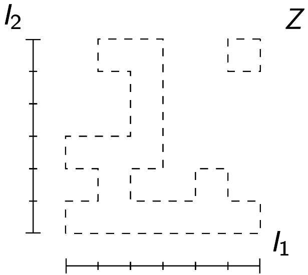
- How do we prove the discrete scale argument?

Exploiting Randomness



- How do we prove the discrete scale argument?
- Aside from Z's dimension, we have little structural knowledge.

Exploiting Randomness



- How do we prove the discrete scale argument?
- Aside from Z's dimension, we have little structural knowledge.
- Random choices of the J_k avoid Z effectively.
- We obtain for all but $o(1)$ of the length L intervals in I_k, J_k contains a length L^β section of each, where $\beta = d(nd - \alpha)/(n-1)$. This ratio gives the Hausdorff dimension bound $(nd - \alpha)/(n-1)$ for X.

Conclusion

So What's Next?

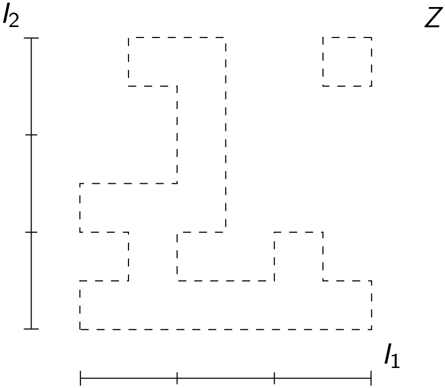
Extension to Hausdorff Dimension

- There is no obvious reason why our techniques should fail when Z has *Hausdorff dimension* α rather than a Minkowski dimension bound.

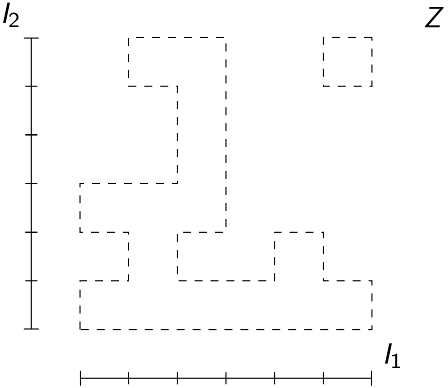
Extension to Hausdorff Dimension

- There is no obvious reason why our techniques should fail when Z has *Hausdorff dimension* α rather than a Minkowski dimension bound.
- We are trying to use hyperdyadic coverings rather than coverings at a single scale to achieve this.

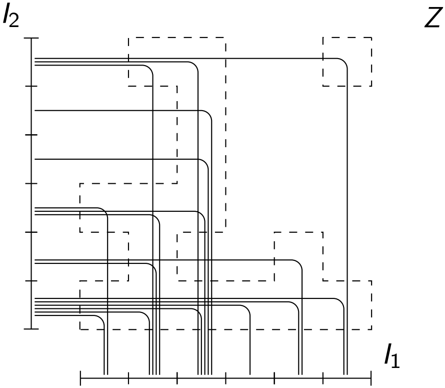
Analogies with Hypergraphs



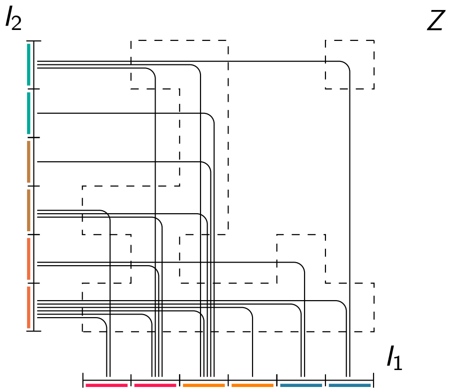
Analogies with Hypergraphs



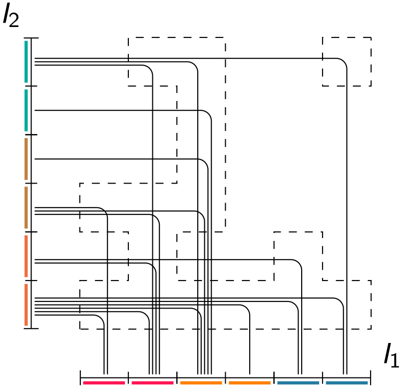
Analogies with Hypergraphs

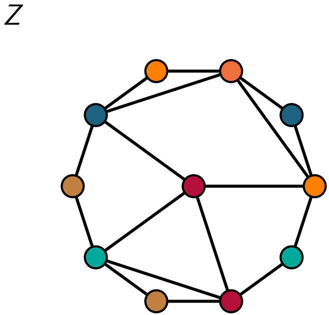


Analogies with Hypergraphs



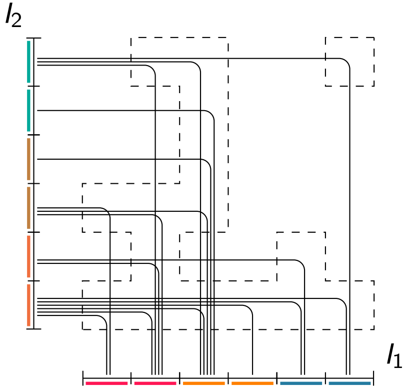
Analogies with Hypergraphs

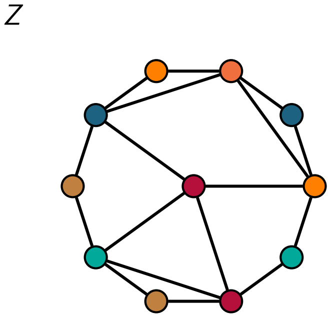




- The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.

Analogies with Hypergraphs





- The Discrete Scale problem can be viewed as finding an independent set in a hypergraph containing each color.
- We are looking to using other methods on hypergraphs to improve the bound when Z has certain structural properties (like for the low rank result).

Interested in Learning More?

- Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.

Interested in Learning More?

- Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.
- Read Máthé (2012) for a construction where f is assumed to be a polynomial of bounded degree.

Interested in Learning More?

- Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.
- Read Máthé (2012) for a construction where f is assumed to be a polynomial of bounded degree.
- Read Pramanik and Fraser (2016) for a general construction where f is smooth and nonsingular. We generalize their work.

Interested in Learning More?

- Read Keleti (1999) for a simple use of scale discretization on a particular configuration avoidance problem.
- Read Máthé (2012) for a construction where f is assumed to be a polynomial of bounded degree.
- Read Pramanik and Fraser (2016) for a general construction where f is smooth and nonsingular. We generalize their work.
- Read Josh's, Malabika, and my paper if it's ever published...

Thanks for listening!