Salem Sets Avoiding Nonlinear Patterns

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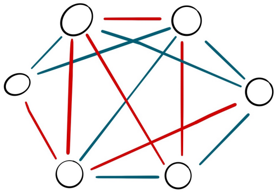
General Research Question



- Phenomenon: Structure appears in suitably large objects.
- Like Ramsey Theory, but with a more analytical foundation, e.g. Geometric Measure Theory / Harmonic Analysis.

Examples

- How large can a subset X of $\mathbf{R}^d$ be such that there does not exist four distinct points $x_1, x_2, x_3, x_4 \in X$ which form a parallelogram, i.e. satisfy $x_2 - x_1 = x_4 - x_3$.
- How large can a subset X of $\mathbf{R}^d$ be such that no three distinct points $x_1, x_2, x_3 \in X$ form a right angle, i.e satisfy $(x_2 - x_1) \cdot (x_3 - x_1) = 0$.
- How large can a subset of $\mathbf{R}^d$ be, such that the distances between any two points is irrational?

- The problem isn't well posed for these patterns
  - If $S \subset \mathbf{R}^d$ has positive measure, it cannot avoid these patterns.
  - We can find discrete sets of arbitrarily large cardinality avoiding these patterns.
  - Need a measure of size 'between' cardinality and Lebesgue measure.

Fractional Dimension

- *Fractional dimensions* measure largeness / thickness of sets. Standard fractional dimension are defined in terms of coverings.
  - Roughly speaking, a set $X \subset \mathbf{R}^d$ has *Minkowski dimension* s if it can be covered by at most r^{-s} balls of radius r, for arbitrarily small $r > 0$.
- If $|X| > 0$, $\dim_{\mathbf{H}}(X) = \underline{\dim}_{\mathbf{M}}(X) = d$.
- If $\#(X) < \infty$, $\dim_{\mathbf{H}}(X) = \underline{\dim}_{\mathbf{M}}(X) = 0$.

Fourier Dimension

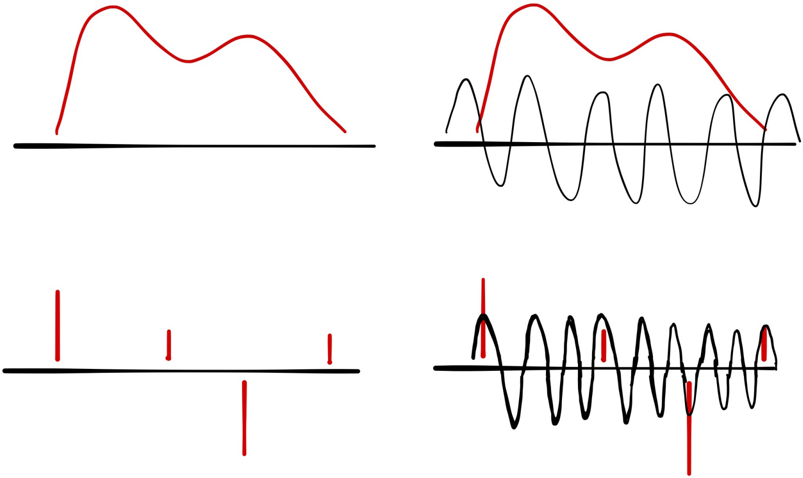
- A compact set X has *Fourier dimension* at least s if there exists a Borel probability measure μ supported on X such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$$

  for $\xi \in \mathbf{R}^d$. Then $\dim_{\mathbf{F}}(X)$ is the supremum of such values s.
- If $s < \dim_{\mathbf{H}}(X)$, then $|\widehat{\mu}(\xi)||\xi|^{s/2}$ is small for *most* ξ, i.e.

$$\frac{|\{\xi \in B_R : |\widehat{\mu}(\xi)| \geq |\xi|^{-s/2}\}|}{|B_R|} = o(1).$$

  But a uniform bound is not always possible.
- In general $\dim_{\mathbf{F}}(X) \leq \dim_{\mathbf{H}}(X) \leq \dim_{\mathbf{M}}(X)$.

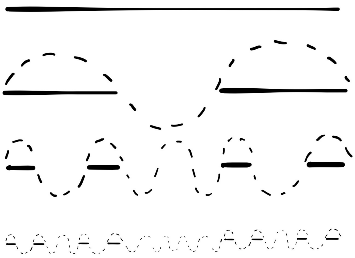


An Example

- Let C be the middle thirds Cantor set.
- For each n, C is covered by 2^n intervals of length $1/3^n$.
- Recall a set has Minkowski dimension s if it can be covered by r^{-s} intervals of length r. Here $r = 1/3^n$, and

$$2^n = 3^{n \log_3 2} = r^{-\log_3 2}.$$

  This suggests that $\dim_{\mathbf{H}}(C) = \dim_{\mathbf{M}}(C) = \log_3 2 \approx 0.63$.
- On the other hand, $\dim_{\mathbf{F}}(C) = 0$, since C is highly correlated with waves of frequency 3^n.



Salem Sets

- If, at each stage of the Cantor set construction, instead of taking the middle third J from each length l interval I, we remove $l \cdot t_I + J$, where $t_I \in [-1/6, 1/6]$ is selected uniformly at random, then we find that

$$\dim_{\mathbf{F}}(X) = \dim_{\mathbf{H}}(X) = \dim_{\mathbf{M}}(X) = \log_3 2.$$

- A set is *Salem* if $\dim_{\mathbf{F}}(X) = \dim_{\mathbf{H}}(X)$.
- Main Focus of Talk: To construct Salem sets, the more probabilistic tools we can develop (especially concentration of measure / square root cancellation results) the better.

Now Let's Return to Pattern Avoidance

The General Problem

- **Avoidance Problem**: Given a set $Z \subset \mathrm{R}^{nd}$, find $X \subset \mathrm{R}^d$ with large *Fourier* dimension such that for distinct points $x_1, \dots, x_n \in X$, $(x_1, \dots, x_n) \notin Z$. We say X *avoids* Z.
- Let $Z = \{(x, y, z) \in (\mathbf{R}^d)^3 : (x - z) \cdot (y - z) = 0\}$.
  - $X \subset \mathbf{R}^d$ avoids Z iff X does not contain any right angles.
- For each $m \in \mathbf{Z}^n - \{0\}$ and $a \in \mathbf{Z}$, define

$$Z(m, a) = \{(x_1, \dots, x_n) \in (\mathbf{R}^d)^n : m_1 x_1 + \dots + m_n x_n = a\}.$$

  If $Z_n = \bigcup_{m \in \mathbf{Z}^n - \{0\}} \bigcup_{a \in \mathbf{Z}} Z(m, a)$, then $X \subset \mathbf{R}^d$ avoids Z_n for all $n > 0$ if and only if X generates a subgroup of $\mathbf{R}^d$ disjoint from $\mathbf{Q}^d - \{0\}$.

- Fourier Dimension often gives much more structural information about a set than Minkowski dimension does.
- (Keleti, 1998) There exist an 'independant' set $X \subset \mathbf{R}$ with $\dim_{\mathbf{H}}(X) = 1$ such that there exists no nontrivial solutions to $m_1 x_1 + \cdots + m_n x_n = 0$ for any $m \in \mathbf{Z}^n$ and $x_1, \ldots, x_n \in X$.
- (Rudin, 1960) If $\dim_{\mathbf{F}}(X) \geq 1/n$, then there exists some $m \in \mathbf{Z}^n$ and some $x_1, \ldots, x_n \in X$ such that $m_1 x_1 + \cdots + m_n x_n = 0$.

- (Körner, 2009) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1/(n-1)$ that contains no solutions to $m_1 x_1 + \cdots + m_n x_n = 0$ for any $m \in \mathbf{Z}^n$.
- (Schmerkin, 2015) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1$ that contains no three term arithmetic progressions, i.e. no nontrivial solutions to the equation $x_2 - x_1 = x_3 - x_2$.
- (Liang and Pramanik, 2019) There exists a Salem set X with $\dim_{\mathbf{F}}(X) = 1$ that contains no solutions to a 'translation invariant' equation of the form $m_1 x_1 + \cdots + m_n x_n = m_0 x_0$, where $m_0, \dots, m_n \geq 0$ and $m_1 + \cdots + m_n = m_0$.

Results in Literature

- How does the geometry of Z help us?

Author	Geometry of Z	$\dim_{\mathbf{H}}(X)$
Mathé (2012)	A degree r algebraic hypersurface in $\mathbf{R}^{dn}$	d/r
Fraser and Pramanik (2016)	An $nd - m$ dimensional surface in $\mathbf{R}^{dn}$	$\frac{m}{n-1}$
Denson, Pramanik, and Zahl (2019)	A subset of $\mathbf{R}^{dn}$ with (lower) Minkowski dimension s	$\frac{dn-s}{n-1}$
Denson (2019)	A subset of $\mathbf{R}^{n}$ such that there exists a full rank linear map $\pi : \mathbf{R}^n \to \mathbf{R}^m$ where $\pi(Z)$ is s dimensional	$\frac{m-s}{m}$

- Can we modify these constructions to obtain Salem sets?

Salem Set Result

Theorem

If Z is a countable union of sets with (lower) Minkowski dimension bounded by s, we can find a Salem set X avoiding Z with

$$dim_{\mathsf{F}}(X) = \frac{nd - s}{n - 1/2}.$$

- The previous results find a set X with

$$\dim_{\mathbf{H}}(X) = \frac{nd - s}{n - 1}.$$

Salem Set Result

Theorem

If Z is a countable union of sets of the form

$$\{(x_1, \dots, x_n) \in \mathrm{R}^{dn} : x_n = f(x_1, \dots, x_{n-1})\}$$

where $f : \mathrm{R}^{d(n-1)} \to \mathrm{R}^d$ is smooth, and the matrix $D_{x_k} f(x_1, \dots, x_{n-1}) = \left(\frac{\partial f^i}{\partial x_{kj}} \right)$ is invertible for each k and distinct $x_1, \dots, x_n \in \mathrm{R}^d$, then we can find a Salem set X avoiding Z with

$$dim_{\mathrm{F}}(X) = \frac{d}{n - 3/4}.$$

- The previous results find a set X with $\dim_{\mathbf{H}}(X) = \frac{d}{n-1}$.
- We will focus on the ideas behind this proof in this talk.

Applications

TODO

Isolating a Single Scale

- We apply Baire category techniques to isolate a 'single scale' of the problem at a time.
- We consider a complete metric space $\mathcal{X}_s$ which consists of measures μ such that for each $\varepsilon > 0$,

$$\sup_{\xi \in \mathbf{R}^d} |\widehat{\mu}(\xi)||\xi|^{s/2-\varepsilon} < \infty.$$

  Thus $\text{supp}(\mu)$ is a set with Fourier dimension at least s.
- Our goal is to show that the set of measures μ such that $\text{supp}(\mu)$ avoids the pattern Z is a set of first category in $\mathcal{X}_s$, where $s = d/(n - 3/4)$.

- This means we must show that for any disjoint closed cubes $Q_1, \dots, Q_n$ in $[0,1]^d$ with common sidelength s, the family

$$\mathcal{Y}_{Q_1,\dots,Q_n} = \left\{ \mu \in \mathcal{X}_s : \begin{array}{c} \text{If } x_1 \in Q_1 \cap \text{supp}(\mu), \dots, \\ x_n \in Q_n \cap \text{supp}(\mu), x_n \neq f(x_1, \dots, x_{n-1}) \end{array} \right\}.$$

is dense in $\mathcal{X}_s$.
- It suffices to show that for any disjoint family of closed cubes $Q_1, \dots, Q_n \subset [0,1]^d$, and $\varepsilon_1, \varepsilon_2 > 0$, there exists a compactly supported measure μ such that

$$\sup_{\xi \in \mathbf{Z}^d} |\widehat{\mu}(\xi)||\xi|^{s/2-\varepsilon_1} \leq \varepsilon_2.$$

and if $x_1 \in Q_1 \cap \text{supp}(\mu), \dots, x_n \in Q_n \cap \text{supp}(\mu)$, then

$$x_n \neq f(x_1, \dots, x_{n-1}).$$

(The uncertainty principle implies we only need to look at integer frequencies).

The Importance of Square Root Cancellation

- Fix $K > 0$ and $r > 0$. Let $x_1, \dots, x_K$ be points such that for $|\xi| \lesssim 1/r$, $|e^{2\pi i \xi \cdot x_1} + \dots + e^{2\pi i \xi \cdot x_K}| \lesssim K^{1/2}$. A trivial bound (triangle inequality) is $O(K)$, so we have 'square root cancellation'.
- Fix a mollifier $\phi \in C_c^\infty(\mathbf{R}^d)$, let $\phi_r(x) = r^{-d}\phi(x/r)$ and define

$$\mu(x) = \frac{\phi_r(x - x_1) + \dots + \phi_r(x - x_K)}{K}.$$

  Then $\text{supp}(\mu)$ is covered by K radius r balls.
- Then

$$\widehat{\mu}(\xi) = K^{-1} \left(e^{2\pi i \xi \cdot x_1} + \dots + e^{2\pi i \xi \cdot x_K} \right) \widehat{\phi}(r\xi).$$

  If $K = r^{-s}$ and r is sufficiently small, then

$$|\widehat{\mu}(\xi)| \leq K^{-1/2} |\widehat{\phi}(r\xi)| \leq r^{s/2} |\widehat{\phi}(r\xi)|$$

  So if $|\xi| \leq 1/r$, $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$, and if $|\xi| \geq 1/r$, $\widehat{\phi}(r\xi)$ decays fast.
- $K^{-1/2}$ error (or even $K^{-1/2} \log(K)^{100}$) is perfectly fine.

An Interlude on Concentration Inequalities

Concentration Bounds

- Heuristic: A function of *many* independant random variables is tightly concentrated about it's mean (plus or minus it's variance).
- Where this is true: A sum $X_1 + \cdots + X_K$ of i.i.d. random variables, where K is large.
- Where this fails: $\sum_{k=1}^{\infty} X_k/2^k$, where $\{X_k\}$ are independent and uniformly distributed $\{0,1\}$ valued Bernoulli random variables.
- The distribution of this sum is uniform on $[0,1]$, so not tightly concentrated at all despite involving *infinitely many* random variables because X_k has much more influence on the overall result for small k vs for large k.

Concentration Bounds

Theorem (Hoeffding's Inequality)

Suppose $X_1, \ldots, X_K$ *are independant random variables with* $|X_i| \leq A$ *for each* i *and* $\sum \mathrm{E}(X_i) = 0$*, then*

$$\mathrm{P}\left(|X_1 + \cdots + X_K| \geq t\right) \leq 4 \exp\left(-2t^2/KA^2\right).$$

Thus $|X_1 + \cdots + X_K| \lesssim AK^{1/2}$ *with high probability.*

Concentration Bounds

Theorem (McDiarmid's Inequality)

Suppose $f : \mathrm{R}^K \to \mathrm{R}$ *is a function. Suppose that for each* $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_K \in \mathrm{R}$, *and any* $x_i, x_i' \in \mathrm{R}$,

$$|f(x_1, \dots, x_i, \dots, x_K) - f(x_1, \dots, x_i, \dots, x_K)| \leq A$$

Then if $X_1, \dots, X_K$ *are a family of independant random variables,*

$$\mathrm{P}(|f(X_1, \dots, X_K)) - \mathrm{E}(f(X_1, \dots, X_N))|) \leq 4 \exp\left(-t^2/2A^2K\right).$$

Thus $|f(X_1, \dots, X_K)) - \mathrm{E}(f(X_1, \dots, X_N))| \lesssim AK^{1/2}$ *with high probability.*

The Avoidance Method

Thanks for listening!