Radial Multipliers

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Chapter 1

Papers / Books To Read In More Detail

- Sogge, L^p Estimates For the Wave Equation and Applications (1993).

  A survey of results on regularity results for the wave equation. In particular, reviews (without proof) the ideas of Mockenhaipt, Seeger, and Sogge which give local smoothing for Fourier integral operators satisfying the cone condition, as well as mixed norm estimates for non-homogeneous results on wave equations.

- In Sogge's Book, he mentions the main developments in harmonic / microlocal analysis he couldn't discuss in the book were the following:

  - Bennett, Carbery, Tao, On the Multilinear Restriction and Kakeya Conjecture (2006).

    Introduction to multilinear methods in harmonic analysis.

  - Bourgain, Guth, Bounds on Oscillatory Integral Operators Based on Multilinear Estimates (2010).

    Application of multilinear methods to bounding oscillatory integrals.

  - Bourgain, Demeter, The Proof of the l2 Decoupling Conjecture (2014).

    Introduction to Decoupling.

  - Peetre, New Thoughts on Besov-Spaces.

Characterizes boundedness of Fourier multipliers on homogeneous Besov spaces.

  - Johnson, Maximal Subspaces of Besov-Spaces Invariant Under Multiplication By Characters.

    Shows a Fourier multiplier operator is bounded in the L^p norm if and only if it's translates are all localizably bounded as in Seeger.

- For more background reading in microlocal analysis:
  - Hörmander, The Analysis of Linear Partial Differential Operators, Volumes I-IV.
  - Treves, Introduction to Pseudodifferential and Fourier Integral Operators, Volumes I-II.
  - Taylor.

- Hormander, The Spectral Function of an Elliptic Operator - Avakumovic, Uber die Eigenfunktionen auf Geschlossenen Riemannschen Mannigfaltigkeiten - Levitan, On the Asymptotic Behaviour of the Spectral Function of a Self-Adjoint Differential Equation of Second Order.

Chapter 2

General Introduction

The question of the regularity of translation-invariant operators on $\mathbf{R}^n$ has proved central to the development of modern harmonic analysis. Indeed, answers to these questions underpin any subtle understanding of the Fourier transform, since with essentially any such operator T, we can associate a tempered distribution $m : \mathbf{R}^n \to \mathbf{C}$, known as the *symbol* of T, such that for any Schwartz function $f \in \mathcal{S}(\mathbf{R}^d)$,

$$Tf(x) = \int m(\xi)\widehat{f}(\xi)e^{2\pi i \xi \cdot x}\, d\xi,$$

i.e. such that $\widehat{Tf} = m\widehat{f}$. This is why these operators are also called *Fourier multipliers*. Using the spectral calculus of unbounded operators, one can also write this operator as $m(D)$, where $D = (2\pi i)^{-1}\nabla$ is a self-adjoint normalization of the gradient. Thus the study of the boundedness of translation invariant operators is closely connected to the study of the interactions of the projection operators

$$E_\xi f(x) = \widehat{f}(\xi)e^{2\pi i \xi \cdot x}$$

onto the eigenspaces of the components of D, since $m(D) = \int m(\xi)E_\xi$.

The study of translation invariant operators emerges from many classical questions in analysis, like that of the convergence properties of Fourier series, or in mathematical physics, through the study of the heat, wave, and Schrödinger equation. These operators also have rotational symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely those represented by symbols $m : \mathbf{R}^n \to \mathbf{C}$ which are *radial*, i.e. such that

$$m(\xi) = h(|\xi|)$$

for some function $h : [0,\infty) \to \mathbf{C}$. This class of operators is therefore also called the class of *radial Fourier multipliers*. The spectral calculus again implies one can write $m(D) = h(\sqrt{-\Delta})$, where Δ is the Laplacian on $\mathbf{R}^n$. Thus the study of radial multipliers is closely connected to interactions between the spherical projection operators

$$E_\lambda f(x) = \int_{|\xi|=1} \widehat{f}(\xi) e^{2\pi i \xi \cdot x},$$

for $0 < \lambda < \infty$, which are the projections onto the eigenspaces of $\sqrt{-\Delta}$, since we have $h(\sqrt{-\Delta}) = \int h(\lambda) E_\lambda$.

Setup in this way, one can define a theory of radial multipliers on a general *geodesically complete* Riemannian manifold X. On such a manifold we have a Laplace-Beltrami operator Δ which is an essentially self-adjoint unbounded operator on $L^2(X)$, $\sqrt{-\Delta}$ will be a self-adjoint operator, and one can consider the study of operators of the form $h(\sqrt{-\Delta})$ for functions $h : [0,\infty) \to \mathbf{C}$. Understanding such operators is strongly connected to an understanding of the interactions between the projection operators E_λ onto the eigenspaces of $\sqrt{-\Delta}$ because of the representation $h(\sqrt{-\Delta}) = \int h(\lambda) E_\lambda$.

This research project studies necessary and sufficient conditions to guarantee the L^p boundedness of radial multiplier operators, both in the Euclidean setting, and also in the setting of Riemannian manifolds, stimulated by recent developments which indicate lines of attack for three related problems in the field.

2.1 Radial Multipliers on Euclidean Space

The general study of the boundedness of Fourier multipliers was initiated in the 1960s. It was quickly realized that the most fundamental estimates were those of the form

$$\|Tf\|_{L^q(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)},$$

for $1 \leqslant p \leqslant 2$, and $p \leqslant q$. It is therefore natural to introduce the spaces $M^{p,q}(\mathbf{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$. The space $M^{p,q}(\mathbf{R}^d)$ is then naturally a Banach space by taking the operator norm

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} = \sup \left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \right\}.$$

For notational convenience, $M^{p,p}(\mathbf{R}^d)$ is denoted by $M^p(\mathbf{R}^d)$.

It was very simple to characterize the spaces $M^{1,q}(\mathbf{R}^d)$, by virtue of the fact that the theory of boundedness of operators with domain $L^1(\mathbf{R}^d)$ is often trivial. For any symbol m, if $k = \widehat{m}$, then

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \begin{cases} \|k\|_{L^q(\mathbf{R}^d)} & : q > 1 \\ \|k\|_{M(\mathbf{R}^d)} & : q = 1, \end{cases}.$$

where $M(\mathbf{R}^d)$ is the space of finite signed Borel measures equipped with the total variation norm. The orthogonality of the Fourier transform also characterized $M^2(\mathbf{R}^d)$ by the fact that

$$\|m\|_{M^2(\mathbf{R}^d)} = \|m\|_{L^\infty(\mathbf{R}^d)}.$$

But for any other pair of exponents p and q, finding a characterization of the spaces $M^{p,q}(\mathbf{R}^d)$ proved to be an impenetrable problem. In the past 60 years there has been no tractable characterization of these spaces for any other value of p or q.

A major tool to understand multipliers outside this range is the theory of Littlewood-Paley decompositions, which gives us a kind of orthogonality property of the projection operators E_ξ when mapping into $L^q(\mathbf{R}^d)$, for $1 < q < \infty$. We fix a smooth bump function $\phi \in C_c^\infty(\mathbf{R}^d)$ such that $1 = \sum_j \mathrm{Dil}_{2^j}\phi$. Then given a symbol m, we define

$$m_t = (\mathrm{Dil}_{1/t} m) \cdot \phi.$$

Thus m_t describes the behaviour of the multiplier m on the annulus of frequencies $|\xi| \sim t$, rescaled so that this behaviour is now lying on the annulus $|\xi| \sim 1$. Littlewood-Paley theory implies that for $1 < p, q < \infty$,

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \sim_{p,q} \sup_{t>0} t^{d(1/p-1/q)} \|m_t\|_{M^{p,q}(\mathbf{R}^d)}.$$

Thus in this regime, it suffices to understand the behaviour of Fourier multipliers compactly supported on the unit annulus.

One common heuristic to this theory is that the regularity of the symbol m, or equivalently, the decay of the convolution kernel k away from the origin, implies some boundedness of the symbol, viewed as a multiplier. The most well known condition of this form for $1 < p < \infty$ is the Hörmander-Mikhlin multiplier theorem, which shows that for $1 < p < \infty$, and $\varepsilon > 0$,

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_{p,\varepsilon} \sup_{t>0} \|k_t\|_{L^1((1+|x|)^\varepsilon)}$$

This implies the slightly weaker inequality

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim \sup_{t>0} \|m_t\|_{L^2_{d/2+\varepsilon}}.$$

Conversely, some control over the singular nature of the convolution kernel k is necessary in order to conclude that $m \in M^{p,q}(\mathbf{R}^d)$ for some exponents p and q. This is because if k_t is the convolution kernel corresponding to the multiplier operator m_t, then

$$\|k_t\|_{L^q(\mathbf{R}^d)} = \|\mathrm{Dil}_t m(D)\{\mathrm{Dil}_{1/t}\check{\phi}\}\|_{L^q(\mathbf{R}^d)} \lesssim t^{-d(1/p-1/q)} \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

One can phrase this in terms of the homogeneous Besov spaces $\dot{B}^{p,q}_s(\mathbf{R}^d)$, the space consisting of all distributions f such that the norm

$$\|f\|_{\dot{B}^{p,q}_s(\mathbf{R}^d)} = \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|P_j f\|_{L^p(\mathbf{R}^d)} \right)^q \right)^{1/q} = \|2^{js} P_j f\|_{l^q(\mathbf{Z}) L^p(\mathbf{R}^d)},$$

is finite, where $P_j = \phi(D/2^j)$ is the Littlewood-Paley projection operator onto a dyadic frequency band of radius 2^j. The calculation above shows that

$$\|k\|_{\dot{B}^{q,\infty}_{-d/p^*}} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

Thus we conclude that k must satisfy some (admittedly weak) regularity assumptions in order to be the convolution kernel of a bounded Fourier multiplier.

For $p = 1$ and $1 < q < \infty$, this result says that

$$\|k\|_{\dot{B}^{q,\infty}_0} \lesssim \|m\|_{M^{1,q}(\mathbf{R}^d)}.$$

On the other hand, the Littlewood-Paley inequality says that

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \|k\|_{L^q(\mathbf{R}^d)} \sim_q \|k\|_{\dot{B}^{q,2}_0}$$

Thus we have the double sided inequality

$$\|k\|_{\dot{B}^{q,\infty}_0} \lesssim_d \|m\|_{M^{1,q}(\mathbf{R}^d)} \lesssim_q \|m\|_{\dot{B}^{q,2}_0},$$

which shows that this regularity assumption is tight in the choice of q and the regularity assumptions. For $p = q = 2$, the result says that

$$\|k\|_{\dot{B}^{2,\infty}_{-d/2}} \lesssim_d \|m\|_{M^{2,2}(\mathbf{R}^d)}.$$

TODO: Can one bound $M^{2,2}(\mathbf{R}^d) = L^\infty(\mathbf{R}^d)$ by a negative exponent Besov space, i.e. look up homogeneous Besov embeddings. TODO: Can we conjecture a larger range of p and s for the radial multiplier conjecture if we weaken the tightness of the second order Besov exponents? TODO: Look up counterexample which shows we cannot get such a result if m is not radial.

Despite the lack of a complete characterization of the classes $M^{p,q}(\mathbf{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^{p,q}(\mathbf{R}^d)$ for *radial symbols* in this class, for an appropriate range of exponents. This conjecture is best phrased in terms of the result of [3], which concerned radial multipliers m whose associated operator T is bounded from the L^p norm to the L^q norm *restricted to radial functions*, i.e. such that the norm

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} = \sup \left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \text{ and } f \text{ is radial} \right\}$$

is finite. The main result of [3] was that if $d > 1$, if $1 < p < 2d/(d+1)$, and if $p \leqslant q < 2$, then $M^{p,q}_{\text{rad}}(\mathbf{R}^d)$ is a subset of $L^1_{\text{loc}}(\mathbf{R}^d)$, and for any such locally integrable radial symbol m,

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} \sim_{p,q,d} \sup_{t>0} t^{d(1/p-1/q)} \|k_t\|_{L^q(\mathbf{R}^d)} = \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}.$$

It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs f are radial, i.e. that for radial symbols m, for $d > 1$, $1 < p < 2d/(d+1)$, and for $p \leqslant q < 2$,

$$\|m\|_{M^{p,q}} \sim_{p,q,d} \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}$$

In the sequel, we call this the *radial multiplier conjecture* in $\mathbf{R}^d$.

Remark. Let $m(\xi) = \mathbf{I}(|\xi| \leqslant 1)$ be the ball multiplier. Then we can write

$$m(\xi) = m_0(\xi) + \sum_{j=1}^{\infty} m_j(\xi) = m_0(\xi) + m_{\geqslant 1}(\xi).$$

where $m_0(\xi)$ is smooth and compactly supported on $|\xi| \leqslant 1/2$, and where $m_j(\xi) = \phi(2^j(1-x))$ lives at the frequency scale $\approx 2^j$ and is supported on an annulus of width $1/2^j$ and radius ≈ 1. Now

$$\|m_0\|_{\dot{B}^{p,\infty}_s(\mathbf{R}^d)} \sim \sup_{j \leqslant 0} 2^{js} \|P_j m_0\|_{L^p(\mathbf{R}^d)} \sim \sup_{j \leqslant 0} 2^{j(s+d/p^*)}$$

which is $O(1)$ for $s + d/p^* \geqslant 0$. On the other hand,

$$\|m_{\geqslant 1}\|_{\cdot B^{p,\infty}_s(\mathbf{R}^d)} \sim \sup_{j > 0} 2^{js} \|m_j\|_{L^p(\mathbf{R}^d)} \sim \sup_{j > 0} 2^{j(s-1/p)}.$$

This is $O(1)$ provided that $s \leqslant 1/p$. Since

$$\|m\|_{\cdot B^{p,\infty}_s(\mathbf{R}^d)} \sim_{p,q,s} \|m_0\|_{\cdot B^{p,\infty}_s(\mathbf{R}^d)} + \|m_{\geqslant 1}\|_{\cdot B^{p,\infty}_s(\mathbf{R}^d)},$$

we find $m \in \cdot B^{p,\infty}_s(\mathbf{R}^d)$ for $-d/p^* \leqslant s \leqslant 1/p$. Now it is a result of Fefferman (TODO: CITE?) that m does not lie in any of the spaces $M^{p,q}(\mathbf{R}^d)$ except for when $p = q = 2$. Looking at the results of the radial multiplier conjecture, this example shows that the generalization of the radial multiplier conjecture to all exponents cannot possibly be true for $-d/p^* \leqslant 1/p$,

Remark. If $m(\xi) = h(|\xi|)$ is radial, the condition that

$$\sup_{t > 0} t^{d(1/p - 1/q)} \|k_t\|_{L^q(\mathbf{R}^d)} < \infty$$

can be rephrased in terms involving the Fourier transform of h. Namely, we have

$$\sup_{t > 0} t^{d(1/p-1/q)} \|k_t\|_{L^q(\mathbf{R}^d)} \\ \sim \sup_{t>0} t^{d(1/p-1/q)} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|^q (1+|s|)^{(d-1)(1-q/2)} \, dt \right)^{1/q}.$$

The weight inside the norm prevents us from easily converting this condition into a homogeneous Besov condition on the function w, but roughly speaking, we have $|\widehat{h}(s)| \lesssim s^{d(1/p-1/q)}\langle s\rangle^{-(d-1)(1-q/2)}$ for *most* inputs x, TODO: What does this imply about the intuitive smoothness of h?

We now know, by the results of [6] and [2], that the radial multiplier conjecture is true when $n > 4$ and $1 < p < 2-4/(n+1)$, and when $n = 4$ and $1 < p < 2 - 3.79/(n+1)$. We also know [2] the criterion in the conjecture is sufficient to obtain a *restricted weak type* bound

$$\|Tf\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbf{R}^n)}$$

when $n = 3$ and $1 < p < 2 - 3.66/(n+1)$. But the radial multiplier conjecture has not yet been completely resolved in any dimension n, we do not have any strong type L^p bounds when $n = 3$, and we don't have any bounds whatsoever when $n = 2$. One goal of this research project is to investigate whether one can use modern research techniques to improve upon these bounds.

The full proof of the radial multiplier is likely far beyond current research techniques. Indeed, it remains a major open problem in harmonic analysis to determine the range of exponents for which *specific* radial Fourier multipliers are bounded in the range where the conjecture would apply, such as the Fourier multiplier on $\mathbf{R}^d$ with symbol $m_\lambda(\xi) = (1-|\xi|)_+^\lambda$, the family of *Bochner-Riesz multipliers*. The radial multiplier conjecture characterizes the range of the Bochner-Riesz multipliers, and thus the conjecture would also imply the Kakeya and restriction conjectures. All three of these results are major unsolved problems in harmonic analysis. On the other hand, the Bochner Riesz conjecture is completely resolved when $n = 2$, while in contrast, no results related to the radial multiplier conjecture are known in this dimension at all. And in any dimension $n > 2$, the range under which the Bochner-Riesz multiplier is known to hold [4] is strictly larger than the range under which the radial multiplier conjecture is known to hold, even for the restricted weak-type bounds obtained in [2]. Thus it still seems within hope that the techniques recently applied to improve results for Bochner-Riesz problem, such as broad-narrow analysis [1], the polynomial Wolff axioms [7], and methods of incidence geometry and polynomial partitioning [11] can be applied to give improvements to current results characterizing the boundedness of general radial Fourier multipliers.

Our hopes are further emboldened when we consult the proofs in [6] and [2], which reduce the radial multiplier conjecture to the study of upper bounds of quantities of the form

$$\left\| \sum_{(y,r)\in\mathcal{E}} F_{y,r} \right\|_{L^p(\mathbf{R}^n)},$$

where $\mathcal{E} \subset \mathbf{R}^n \times (0,\infty)$ is a finite collection of pairs, and $F_{y,r}$ is an oscillating function supported on a $O(1)$ neighborhood of a sphere of radius r centered at a point y. The L^p norm of this sum is closely related to the study of the tangential intersections of these spheres, a problem successfully studied in more combinatorial settings using incidence geometry and polynomial partitioning methods [12], which provides further estimates that these methods might yield further estimates on the radial multiplier conjecture.

When $n = 3$, the results of [2] are only able to obtain bounds on the L^p sums in the last paragraph when $\mathcal{E}$ is a Cartesian product of two subsets of $(0,\infty)$ and $\mathbf{R}^n$. This is why only restricted weak-type bounds have been obtained in this dimension. It is therefore an interesting question whether different techniques enable one to extend the L^p bounds of these sums when the set $\mathcal{E}$ is *not* a Cartesian product, which would allow us to upgrade the result of [2] in $n = 3$ to give strong L^p bounds. This question also has independent interest, because it would imply new results for the 'endpoint' local smoothing conjecture, which concerns the regularity of solutions to the wave equation in $\mathbf{R}^n$. Incidence geometry has been recently applied to yield results on the 'non-endpoint' local smoothing conjecture [5], which again suggests these techniques might be applied to yield the estimates needed to upgrade the result of [2] to give strong L^p-type bounds.

2.2 Multipliers on Riemannian Manifolds

Fix a geodesically complete Riemannian manifold X. We can then define operators $h(\sqrt{-\Delta})$, which are analogues to the radial multipliers studied in the Euclidean setting. Just like multiplier operators on $\mathbf{R}^n$ are crucial to an understanding of the interactions between the functions $e_\xi(x) = e^{2\pi i \xi \cdot x}$ on $\mathbf{R}^n$, understanding the operators $h(\sqrt{-\Delta})$ is crucial to understanding the interactions of eigenfunctions of the Laplace-Beltrami operator on X. We let $M^{p,q}(X, \sqrt{-\Delta})$ denote the family of all symbols $h : \mathbf{R} \to \mathbf{C}$ such that the operator $T_h = h(\sqrt{-\Delta})$ is bounded from $L^p(X)$ to $L^q(X)$, with the analogous operator norm, though, when there is no ambiguity, we will overload notation and write this space as $M^{p,q}(X)$.

TODO: Is the study of $M^{1,q}(X)$ and $M^{2,2}(X)$ trivial like in $\mathbf{R}^d$?

For compact Riemannian manifolds X, there is a problem which prevents a direct generalization of the radial multiplier conjecture. On such a manifold, $\left(\varepsilon + \sqrt{-\Delta}\right)^{-1}$ is a compact, bounded operator from $L^2(X)$ to itself for any $\varepsilon > 0$, and so there exists $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots$ with $\lambda_i \to \infty$, and an orthonormal family of eigenfunctions $e_n \in C^\infty(X)$ which diagonalize $L^2(X)$ such that

$$\sqrt{-\Delta} f = \sum \lambda_n \langle f, e_n \rangle e_n.$$

Thus for any function $h : [0, \infty) \to \mathbf{C}$,

$$h\left(\sqrt{-\Delta}\right) f = \sum h(\lambda_n) \langle f, e_n \rangle e_n.$$

If h has compact support, this sum will be finite, and thus by the triangle inequality, trivially bounded from $L^p(X)$ to $L^q(X)$ for any exponents p and q. Thus $M^{p,q}(X)$ contains all compactly supported radial multipliers.

This trivializes the study of compactly supported radial multipliers in some sense. To avoid trivializing the problem, we instead determine what conditions ensure that we have a bound of the form

$$\sup_{t > 0} t^{d(1/q - 1/p)} \|\mathrm{Dil}_t h\|_{M^{p,q}(X)} < \infty.$$

We let $M^{p,q}_{\mathrm{Dil}}(X)$ denote the family of all multipliers for which the inequality above holds, and give it the norm induced by the quantity on the left hand side. A transference principle of Mitjagin [9] shows that if X is a compact Riemannian manifold, and $m : \mathbf{R}^d \to \mathbf{C}$ is radial, with $m(\xi) = h(|\xi|)$, then

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \lesssim_{X,p,q} \|h\|_{M^{p,q}_{\mathrm{Dil}}(X)}.$$

Thus, in some sense, the dilation invariant Fourier multiplier problem on a compact manifold X is at least as hard as it is on $\mathbf{R}^n$. Another goal of this research project is to try and extend the radial multiplier conjecture to this setting.

Directly translating the assumptions of this conjecture to this setting yields the following statement: If $h : [0,\infty) \to \mathbf{R}$ is a function, and we define

$$A_{p,q}(h) = \sup_{t>0} t^{d(1/p-1/q)} \left(\int_{t/2\leqslant|s|\leqslant 2t} |\widehat{h}(s)|^q (1+|s|)^{(d-1)(1-q/2)}\, ds \right)^{1/q},$$

then for what values of p and q is is true that

$$\|h\|_{M^{p,q}_{\mathrm{Dil}}(X)} \lesssim A_{p,q}(h)?$$

Mitjagin's result implies that we require $1 < p < 2d/(d+1)$ and $p \leqslant q < 2$, and we conjecture that, perhaps under appropriate assumptions on X, we can achieve similar ranges of exponents as have been obtained for the Euclidean radial multiplier conjecture.

On general compact manifolds, there are difficulties arising from a generalization of the radial multiplier conjecture, connected to the fact that analogues of the Kakeya / Nikodym conjecture are false in this general setting [8]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [3] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [6] or even [2].

2.3 Summary

In conclusion, the results of [6] and [2] indicate three lines of questioning about radial Fourier multiplier operators, which current research techniques place us in reach of resolving. The first question is whether we can extend the range of exponents upon which the conjecture of [3] is true, at least in the case $n = 2$ where Bochner-Riesz has been solved. The second is whether we can use more sophisticated arguments to prove the L^p sum bounds obtained in [2] when $n = 3$ when the sums are no longer Cartesian products, thus obtaining strong L^p characterizations in this settiong, as well as new results about the endpoint local smoothing conjecture.

The third question is whether we can generalize these bounds obtained in these two papers to study radial Fourier multipliers on the sphere.

Chapter 3

Notes on Bochner-Riesz

The goal of this section is to compare and contrast approaches to understanding the Bochner-Riesz conjecture on Euclidean space and on compact Riemannian manifolds, in order to reflect on the differences in understanding multipliers on $\mathbf{R}^d$ vs on a compact manifold X before we attack the more general multiplier problem in this setting. We define the Riesz multipliers via symbols $r^\delta_\rho : [0,\infty] \to [0,\infty)$, defined for $\rho > 0$ and a real number δ by setting, for $\tau > 0$,

$$r^\delta_\rho(\tau) = (1 - \tau/\rho)^\delta_+.$$

Here $s_+ = \max(s, 0)$. The resulting radial multipliers on $\mathbf{R}^n$, and on a compact Riemannian manifold X, will be denoted by

$$B^\delta_\rho = r^\delta_\rho\left(\sqrt{-\Delta}\right).$$

The goal of the Bochner-Riesz conjecture is to determine bounds on the operators $\{B^\delta_\rho\}$ invariant under dilation of the symbol.

3.1 Bochner Riesz Bounds Via Tomas-Stein

3.1.1 Euclidean Case

Let's review a reduction of Bochner-Riesz to Tomas Stein:

- First, we can *rescale the problem*. If $r^\delta = r^\delta_1$, then

$$r^\delta_\rho(\lambda) = r^\delta(\lambda/\rho).$$

Thus if $B^\delta = B^\delta_1$, then $B^\delta_\rho = B^\delta \circ \mathrm{Dil}_{1/\rho}$, and so the operators $\{B^\delta_\rho\}$ are uniformly bounded from L^p to L^p for all ρ if and only if B^δ is bounded from L^p to L^p.

- We can perform a *spatial decomposition*. Let k^δ be the convolution kernel corresponding to the operator B^δ. We break up the effects of the operator spatially into dyadic annuli, i.e. writing

$$k^\delta(x) = \sum_{j=0}^\infty k^\delta_j(2^j x),$$

where k^δ_0 is supported on $|x| \leqslant 2$, and all of the other kernels k^δ_j are supported on the annuli $\{1/2 \leqslant |x| \leqslant 1\}$, and can be written as

$$k^\delta_j(x) = \psi \cdot \mathrm{Dil}_{1/2^j} k^\delta$$

for some $\psi \in C^\infty_c$ is supported on the annulus $\{1/2 \leqslant |x| \leqslant 1\}$. We analyze each of the convolution kernels separately and then collect up each of the bounds we obtain by applying the triangle inequality. Thus we set B^δ_j to be the operator corresponding to the convolution kernel k^δ_j. Provided we can bound the L^p operator norm of the operators B^δ_j by $O(2^{-\varepsilon j})$ we can sum up the bounds using the triangle inequality to bound B^δ.

- Spatial localization means that the operators $\{B^\delta_j\}$ are *local*, i.e. the support of $B^\delta_j f$ is contained in a $O(1)$ neighborhood of the support of f. A decomposition argument, thus implies that a general bound of the form $\|B^\delta_j f\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$ follows from establishing the bound restricted to functions f which are supported on balls of radius 1.

- We *reduce to L^2 bounds*: Since our inputs are supported on balls of radius 1, we have

$$\|B^\delta_j f\|_{L^p(\mathbf{R}^d)} \lesssim \|B^\delta_j f\|_{L^2(\mathbf{R}^d)}.$$

Thus it suffices to obtain a bound of the form $\|B^\delta_j f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$. This is the most inefficient part of the proof, but enables us to apply more powerful results which only hold in $L^2(\mathbf{R}^d)$. Getting around this reduction is key to improving the currently known Bochner-Riesz bounds.

- We reduce the problem to Tomas-Stein. Since we are now in $L^2(\mathbf{R}^d)$, we can apply Plancherel, obtaining that

$$\|B_j^\delta f\|_{L^2(\mathbf{R}^d)} = \|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)}.$$

  A calculation shows that ψ_j^δ has the majority of its mass on an annulus of radius 2^j and width $O(1)$, and has magnitude $O(2^{-j\delta})$ there. Obtaining a bound of the form

$$\|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$$

  is thus comparable to the Tomas-Stein result, which proves the boundedness of Bochner-Riesz in the Tomas-Stein range.

Let us expand on the computations that yield the required bounds on ψ_j^δ. We have

$$\psi_j^\delta = \widehat{\eta} * \mathrm{Dil}_{2^j} m^\delta.$$

The function is thus defined as the average of $\widehat{\eta}$ over a ball of radius $O(2^j)$ so we immediately obtain a bound by bringing in absolute values and using the rapid decay of $\widehat{\eta}$, thus obtaining that

$$|\psi_j^\delta(\xi)| \lesssim_N \langle |\xi| - 2^j \rangle^{-N}.$$

We can do much better than this using *cancellation*. The function η is supported away from the origin, so it's Fourier transform is oscillatory. Thus, as j increases, ψ_j^δ is defined by averaging an oscillating quantity over larger and larger regions of frequency space, and so we should expect some cancellation above and beyond the trivial bound. For instance, we could perform an integration by parts if we were able to replace m^δ with something that was more smooth near the boundary. Since $\widehat{\eta}$ oscillates at frequencies ~ 1, we should expect integration by parts to give us useful decay of a quantity $\widehat{\eta} * \mathrm{Dil}_{2^k} f$ provided that $|\nabla^k f| \lesssim 2^{kj}$ for large k. This is true of m^δ if we perform a cutoff so it is supported a distance $O(2^{-j})$

from the boundary of the unit ball. Thus we are motivated to define $m^\delta = a_j^\delta + b_j^\delta$, where

$$a_j^\delta(\xi) = m^\delta(\xi)\left(\phi(2^j(|\xi|-1))\right) \quad \text{and} \quad b_j^\delta(\xi) = m_j^\delta(\xi)\left(1-\phi(2^j(|\xi|-1))\right)$$

for some $\phi \in C_c^\infty(\mathbf{R}^d)$ supported on $|t| \leqslant 1$. Then for all $k > 0$,

$$|\nabla^k b_j^\delta(\xi)| \lesssim (1-|\xi|)^{\delta-k} \cdot \mathbf{I}(1-|\xi| \lesssim 1/2^j) + 2^{j(k-\delta)}\mathbf{I}(1-|\xi| \sim 1/2^j).$$

Since η is supported away from the origin, we may antidifferentiate $\widehat{\eta}$ arbitrarily many times without any singular behaviour emerging, thus we find that

$$\begin{aligned} |(\widehat{\eta} * \mathrm{Dil}_{2^j} b_j^\delta)(\xi)| &\lesssim_N \left(\int_0^{1-1/2^j} r^{d-1}\langle \min(|\xi|-r, |\xi|+r)\rangle^{-N}(1-r)^{\delta-k}\right) + 2^{jk-j\delta-j}|\xi|^{-N} \\ &\lesssim 2^{(d-\delta)j}\langle \max(|\xi|-2^j, 0)\rangle^{-N}. \end{aligned}$$

TODO: FInish this calculation. The multiplier $m_{j,1}^\delta$ is less well behaved, since it is singular near the boundary, which might interefere cancellation. But the multiplier is supported on a very thin annulus, of radius 1 and thickness $O(2^{-j})$, and $m_{j,1}^\delta$ has magnitude $O(2^{-j\delta})$ on this annulus. Thus it's dilation is supported on an annulus of radius 2^j and thickness $O(1)$, and the rapid decay of $\widehat{\eta}$ implies that

$$|(\widehat{\eta} * \mathrm{Dil}_{2^j} a_j^\delta)(\xi)| \leqslant 2^{j(d-1)-j\delta}\langle |\xi| - 2^j\rangle^{-N}.$$

Putting these bounds together gives that

$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-j\delta}\langle ||\xi|-2^j|\rangle^{-N}.$$

This is the geometric decay we need in order to sum up the required operator norm bounds.

The bounds we need thus reduce to something of the form

$$\int \frac{|\widehat{f}(\xi)|^2}{\langle 2^j ||\xi|-1|\rangle^N}\, d\xi \lesssim \|f\|_{L^p(\mathbf{R}^n)}.$$

for suitably large N. The fact that f is compactly supported allows us to use the pointwise bound $|\widehat{f}(\xi)| \lesssim 2^{O(j)}\|f\|_{L^p(\mathbf{R}^d)}$, which gives the estimate for $||\xi| - 1| \geqslant 1/2$. The remaining integrand then follows by applying the restriction theorem in the Tomas-Stein range.

How about a version of the proof that involves the wave operator?

- We write

$$B^\delta f(x) = \int r^\delta(|\xi|) E_\xi$$

3.1.2 Manifold Case

The analogue of the Tomas Stein theorem on a compact Riemannian manifold X is a result due to Sogge, so let's see if we can obtain a result for compact manifolds using similar techniques:

- The first problem is that we cannot necessarily rescale the multiplier, so that we must analyze a general multiplier of the form B^δ_ρ; we cannot remove the ρ parameter by applying a scaling symmetry.

- Let us try and perform an analogue of the spatial decomposition. We note that, without the rescaling symmetry, the spatial decomposition would be on annuli of width and radius $2^j/\rho$ rather than annuli of width 2^j. If B^δ_ρ has kernel K^δ_ρ, we consider the operators $B^\delta_{\rho,j}$ which for $j > 0$ have kernels of the form

$$K^\delta_{\rho,j}(x_1, x_2) = \chi\left((\rho/2^j) d(x,y)\right) K^\delta_\rho(x_1, x_2)$$

  for some $\chi \in C_c^\infty(\mathbf{R}^d)$. One feature of the compact manifold setting differing from the Euclidean setting is that we only need to consider $2^j \lesssim \rho$, since $d(x_1, x_2) \lesssim 1$ for all $x_1, x_2 \in X$.

- Since the operators $K^\delta_{\rho,j}$ are local, we can localize, only needing to prove a bound of the form

$$\|K^\delta_{\rho,j} f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$$

  for functions f supported on a metric ball of radius $O(2^j/\rho)$.

- We perform an L^2 reduction: The metric ball of radius $O(2^j/\rho)$ has measure $O((2^j/\rho)^d)$. Thus we have

$$\|K^\delta_{\rho,j} f\|_{L^p(X)} \lesssim (2^j/\rho)^{d(1/p-1/2)} \|K^\delta_{\rho,j} f\|_{L^2(X)}.$$

- Now we try and reduce Sogge's spectral cluster bounds, which are analogous to the Tomas-Stein bounds in $\mathbf{R}^d$. If we are able to justify that $K^\delta_{\rho,j}$ behaves like a spectral band projection operator, as in the Euclidean setting, we'd be able to apply this bound. Plancherel does not quite have an analogy to the L^2 setting on a manifold. But we can instead use the wave operator and it's parametrices, i.e. that

$$\begin{aligned} B^\delta_\rho &= \sum_\lambda r^\delta(\lambda/\rho) E_\lambda \\ &= \rho \int_0^\infty \widehat{r^\delta}(\rho t) e^{2\pi i t \sqrt{-\Delta}}\, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty e^{2\pi i \rho t} (t + i0)^{-\delta-1} e^{2\pi i t \sqrt{-\Delta}}\, dt. \end{aligned}$$

The singularity in the definition of this integral occurs at $t = 0$, so the operator should, for large t, be relatively well behaved.

Let's start by analyzing the large order terms. Fix $\alpha \in C_c^\infty(\mathbf{R})$ equal to one in a neighborhood of zero, and consider the behaviour of B^δ_ρ for large t, i.e. the operator

$$R^\delta_\rho = c_\delta \cdot \rho^{-\delta} \int_0^\infty (1 - \alpha(t)) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i t \sqrt{-\Delta}}\, dt.$$

If ψ is the inverse Fourier transform of $c_\delta t^{-\delta-1}(1 - \alpha(t))$, then ψ is bounded and rapidly decreasing because all of the derivatives of it's Fourier transform are smooth and integrable. We thus can revert back to the multiplier setting and write

$$R^\delta_\rho = \rho^{-\delta} \sum_\lambda \psi(\lambda - \rho) E_\lambda.$$

We can now be fairly lazy in understanding this operator, for instance, employing the Sobolev embedding bound

$$\|E_\lambda f\|_{L^2(X)} \lesssim \langle \lambda \rangle^{d(1/p-1/2)-1/2} \|f\|_{L^p(X)}$$

and the triangle inequality, using the rapid decay to obtain that

$$\|R_\rho^\delta f\|_{L^p(X)} \lesssim \langle\rho\rangle^{-[\delta - d(1/p-1/2)+1/2]} \|f\|_{L^p(X)},$$

which is better than what we need.

For small times, we perform a decomposition about the singularity, writing

$$B_\rho^\delta = \sum_{j=0}^{\infty} B_{\rho,j}^\delta$$

where

$$B_\rho^\delta = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)\beta(\rho t/2^j) e^{-2\pi i\rho t} t^{-\delta-1} e^{2\pi i\sqrt{-\Delta}}\, dt$$

For $j=0$, there is a symbol trick which automatically implies boundedness.

TODO: Symbol trick for $k=0$.

$$A_\rho^\delta = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) e^{-2\pi i\rho t} t^{-\delta-1} \tilde{A}(t)\, dt$$

is appropriately bounded. To understand this bound, we do a decomposition about the singularity, writing $A_\rho^\delta = \sum_{j=0}^\infty A_{\rho,j}^\delta$ where

$$A_{\rho,j}^\delta = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)\beta(\rho t/2^j) e^{-2\pi i\rho t} t^{-\delta-1} \tilde{A}(t)\, dt.$$

$$\sum_\lambda r^\delta(\lambda/\rho)$$

We now employ the small time parametrices for the wave operator, i.e. we consider $\alpha \in C_c^\infty(\mathbf{R})$, and then write

$$\begin{aligned} e^{2\pi it\sqrt{-\Delta}} &= \alpha(t)e^{2\pi it\sqrt{-\Delta}} + (1-\alpha(t))e^{2\pi it\sqrt{-\Delta}} \\ &= \alpha(t)P(t) + \alpha(t)R(t) + (1-\alpha(t))e^{2\pi it\sqrt{-\Delta}}. \end{aligned}$$

where $P(t)$ is the parametrices for the operator, and $R(t)$ is a smoothing operator.

Thus

$$B_{\rho,j}^\delta = \rho \int_0^\infty \chi\left((\rho/2^j)\cdot d(x,y)\right) \widehat{r^\delta}(\rho\lambda) e^{2\pi it\sqrt{-\Delta}}$$

3.2 Proof Involving Carleson-Sjölin

On $\mathbf{R}^d$, we can take Fourier transforms, applying stationary phase to determine that if $K_{\rho,\delta}$ is the convolution kernel corresponding to B^δ_ρ, then

$$\begin{aligned} K_\delta(x) &= \int_0^1 \lambda^{n-1}(1-\lambda)^\delta e^{2\pi i \xi \cdot x}\, d\lambda \\ &= a_1(x)\frac{e^{2\pi i|x|}}{\langle x\rangle^{\frac{n+1}{2}+\delta}} + a_2(x)\frac{e^{-2\pi i|x|}}{\langle x\rangle^{\frac{n+1}{2}+\delta}} + O\left(\frac{1}{\langle x\rangle^{n+1}}\right), \end{aligned}$$

where a_1 and a_2 are symbols of order zero, with $|a_1(x)|, |a_2(x)| \gtrsim 1$ for $|x| \gtrsim 1$. TODO: Necessity of the $\delta(p)$ values.

Lemma 3.1. *If $a \in C_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and $a(x,y) = 0$ unless $1/2 \leqslant |x-y| \leqslant 2$, then*

$$\left\| \int e^{2\pi i \rho|x-y|} a(x,y) f(y) \right\|_{L^q(\mathbf{R}^n)} \lesssim \rho^{n/q} \|f\|_{L^p(\mathbf{R}^n)}$$

if $q = [(n+1)/(n-1)]p^*$ and $1 \leqslant p \leqslant 2$.

Proof. This is a non homogeneous oscillatory integral operator with wavefront set

$$\left\{ \left(x, y; \frac{x}{|x-y|}, \frac{y}{|x-y|} \right) \right\}$$

which, because of the assumption of the support of a, satisfies the Carleson Sjölin conditions, and thus the result follows. $\square$

The required bounds now follows by applying a dyadic spatial decomposition, rescaling, and applying the result above, hich can be applied because of our explicit computation of the kernel K_δ above.

Chapter 4

Seeger: Singular Convolution Operators in L^p Spaces

Let $m : \mathbf{R}^d \to \mathbf{C}$ be the symbol for a Fourier multiplier operator $m(D)$. If the resulting operator $m(D)$ was bounded from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ with operator norm A, then the operator would also be bounded 'at all scales'. That is, if we consider a littlewood Paley decomposition, i.e. taking

$$f = \sum_{i=0}^{\infty} f_i$$

where $\widehat{f_i} = \eta_i \widehat{f}$ is supported on $2^i \leqslant |\xi| \leqslant 2^{i+1}$ for $i \geqslant 1$, and $|\xi| \leqslant 2$ for $i = 0$, then we would have estimates of the form

$$\|m(D) f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}, \tag{4.1}$$

where the implicit constant is uniform in i. The main focus of the paper in question is to determine whether a uniform bound of the form (4.1) implies $m(D)$ is bounded. More precisely, is it true that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_p \sup_{i \geqslant 0} \|m_i\|_{M^p(\mathbf{R}^d)}, \tag{4.2}$$

where $m_i = \eta_i m$.

The Hilbert transform H is a Fourier multiplier with symbol $m(\xi) = \text{sgn}(\xi)$. For each $i > 0$, $m_i(\xi) = \eta_i \text{sgn}(\xi)$, so that

$$K_i(x) = \widehat{\eta_i \text{sgn}(\xi)} = 2^i H\eta(2^i x).$$

Thus

$$\|K_i\|_{L^1(\mathbf{R})} = \|H\eta\|_{L^1(\mathbf{R})}.$$

TODO

It is clear that (4.2) is true for $p = 2$, since in this case the bound is equivalent to an inequality of the form

$$\|m\|_{L^\infty(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|m_i\|_{L^\infty(\mathbf{R}^d)},$$

which is true because the supports of the symbols $\{m_i\}$ are almost all pairwise disjoint. On the other hand, (4.2) does not hold when $p = 1$ or $p = \infty$, which makes sense, since Littlewood-Paley runs into all kinds of problems for these values of p. Arguing more precisely, the condition would be equivalent to showing that for any $K : \mathbf{R}^d \to \mathbf{C}$,

$$\|K\|_{L^1(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|K * \widehat{\eta}_i\|_{L^1(\mathbf{R}^d)}.$$

If

$$K_N(x) = \int_{|\xi| \leqslant 2^N} e^{2\pi i \xi \cdot x}\, d\xi$$

is the Dirichlet kernel, then $\|K_N\|_{L^1(\mathbf{R})} \sim N$. On the other hand, for $i \leqslant N - 1$, we have $K_N * \widehat{\eta}_i = \widehat{\eta}_i$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = \|\widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1.$$

For $i \geqslant N + 1$, we have $K_N * \widehat{\eta}_i = 0$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = 0 \lesssim 1.$$

For $i = N$, we have

$$(K_N * \widehat{\eta_N})(x) = 2^N \int_0^1 \eta(\xi) e^{2\pi i 2^N (\xi \cdot x)} + \int_1^2 \eta(-\xi) e^{-2\pi i 2^N (\xi \cdot x)}\, d\xi$$

$$\int |K_N * \widehat{\eta}_i|$$

whereas one

$$K_N * \widehat{\eta}_i = \begin{cases} \widehat{\eta}_i & : i \lesssim N \\ 0 & : i \gtrsim N \end{cases},$$

and so $\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1$ uniformly in N and i. We can then use Baire category techniques to find a kernel K not in $L^1(\mathbf{R})$, but such that $\|K * \eta_i\|_{L^1(\mathbf{R})} \lesssim 1$, uniformly in i.

The result actually fails for $2 < p < \infty$, due to an examples of Triebel. For simplicity, let's work in $\mathbf{R}$. If we fix a bump function $\phi \in C_c^\infty(\mathbf{R})$ supported in $[-1,1]$, and set

$$m_N(\xi) = \sum_{k=N}^{2N} e^{2\pi i(2^k\xi)}\phi(\xi - 2^k),$$

then $m_N(\xi)\eta_i(\xi) = m_{N,i}(\xi)$, where $m_{N,i}(\xi) = e^{2\pi i(2^k\xi)}\phi(\xi - 2^k)$, and so $K_{N,i}(x) = \widehat{m_{N,i}}(x) = e^{2\pi i 2^k(x-2^k)}\widehat{\phi}(x - 2^k)$, hence

$$\|m_{N,i}(D)f\|_{L^p(\mathbf{R}^d)} = \|K_{N,i} * f\|_{L^p(\mathbf{R}^d)} \leqslant \|\widehat{\phi}\|_{L^1(\mathbf{R})}\|f\|_{L^p(\mathbf{R})} \lesssim \|f\|_{L^p(\mathbf{R})}.$$

On the other hand, the operator norm of $m_N(D)$ from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$ is actually $\gtrsim_p N^{|1/p-1/2|}$, and thus not bounded uniformly in N, so Baire category shows things don't work so well here.

This paper shows that one *can* get uniform bounds assuming an additional, very weak smoothness condition, which rules out the example m_N above. Under the most simple assumptions, if (4.1) holds, and $\|m_i\|_{\Lambda^\varepsilon} \lesssim 2^{-ik}$, where Λ^ε is the ε-Lipschitz norm, then $\|m(D)f\|_{L^r(\mathbf{R}^d)} \lesssim \|f\|_{L^r(\mathbf{R}^d)}$ whenever $|1/r - 1/2| < |1/p - 1/2|$. Under slightly stronger smoothness assumptions, we can actually conclude $\|m(D)f\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$.

To prove the result, we rely on Littlewood-Paley theory and the Fefferman-Stein sharp maximal function. Without loss of generality we may assume that $2 < p < \infty$. We will actually show that if for all i and $\omega \geqslant 0$,

$$\int_{|x|\geqslant\omega} |K_i(x)|\, dx \leqslant B(1 + 2^i\omega)^{-\varepsilon},$$

consistent with the fact that, if m_i was smooth, the uncertainty principle would say that K_i would live on a ball of radius $1/2^i$. We will then prove that $\|m(D)f\|_{L^p(\mathbf{R}^d)} \leqslant A\widetilde{\log}(B/A)^{|1/2-1/p|}$. Our goal is to show that if

$$S^\# f(x) = \sup_{x\in Q} ⨍_Q \left(\sum_{i=0}^\infty \left| m_i(D)f(y) - ⨍_Q m_i(D)f(z)\, dz \right|^2 \right)^{1/2} dy,$$

then $\|S^\# f\|_{L^p(\mathbf{R}^d)} \lesssim A\widetilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbf{R}^d)}$. It then follows by Littlewood-Paley theory implies

$$\begin{aligned}
\|m(D)f\|_{L^p(\mathbf{R}^d)} &\lesssim_p \left\| \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\leqslant \left\| M \left[\left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right] \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim \left\| S^\# \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim A\widetilde{\log}(B/A)^{1/2-1/p}.
\end{aligned}$$

To bound $S^\#$, we linearize using duality, picking Q_x for each x, and a family of functions $\chi_i(x,y)$ such that $\left(\sum |\chi_i(x,y)|^2\right)^{1/2} \leqslant 1$, such that

$$S^\# f(x) \approx \fint_{Q_x} \sum_{i=0}^\infty \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy.$$

Thus $S^\# f = S_1 f + S_2 f$, where if Q_x has sidelength $2^{l(x)}$,

$$S_1 f(x) = \fint_{Q_x} \sum_{|i+l(x)| \leqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy$$

and

$$S_2 f(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy.$$

If $|i + l(x)| \lesssim 1$, then the uncertainty principle tells us that $m_i(D)f$ is roughly constant on squares on radius Q_x, up to some small error, so that we should expect

$$\left| m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\,dz \right| \lesssim \left| \fint_{Q_x} m_i(D)f(z)\,dz \right|.$$

Thus it is natural to use the bound, $|S_1 f(x)| \lesssim M(\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2}$, which implies

$$\begin{aligned}
\|S_1 f\|_{L^2(\mathbf{R}^d)} &\lesssim \|M(\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2}\|_{L^2(\mathbf{R}^d)} \\
&\lesssim \left\| (\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2} \right\|_{L^2(\mathbf{R}^d)} \\
&= \left(\sum_{i=0}^\infty \|m_i(D)f\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\|S_1 f\|_{L^\infty(\mathbf{R}^d)} &\leqslant \|M(\sum_{|i+l(x)|\leqslant \tilde{\log}(B/A)}^\infty |m_i(D)f|^2)^{1/2}\|_{L^\infty(\mathbf{R}^d)} \\
&\leqslant \left\| \left(\sum_{|i+l(x)|\leqslant \tilde{\log}(B/A)} |m_i(D)f|^2 \right)^{1/2} \right\|_{L^\infty(\mathbf{R}^d)} \\
&\lesssim \tilde{\log}(B/A)^{1/2} \sup_i \|m_i(D)f\|_{L^\infty(\mathbf{R}^d)}
\end{aligned}$$

Interpolation gives $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim \tilde{\log}(B/A)^{1/2-1/p}\|m_i(D)f\|_{L^p_x(l^p_i)}$. But now Littlewood-Paley theory shows that

$$\|m_i(D)f\|_{L^p_x(l^p_i)} \leqslant A\left(\sum_{i=0}^\infty \|P_i f\|_{L^p(\mathbf{R}^d)}\right)^{1/p} \leqslant A\left(\sum_{i=0}^\infty \|P_i f\|_{L^p(\mathbf{R}^d)}^2\right)^{1/2} \lesssim A\|f\|_{L^p}.$$

Thus $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim A\tilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbf{R}^d)}$.

On the other hand, if i is much smaller than $l(x)$, we should expect the error between $m_i(D)f(y)$ and $⨍_{Q_x} m_i(D)f(z)\,dz$ to be even smaller, and if i is much bigger, then $m_i(D)f$ is no longer constant at this scale, and so the averages should be small, so $m_i(D)f(x)$ should dominate $⨍_{Q_x} m_i(D)f(z)$. Now since our assumption implues that $\|m(D)f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^2(\mathbf{R}^d)}$, it is not so difficult to prove that

$$\|S_2 f\|_{L^2(\mathbf{R}^d)} \lesssim A\|f\|_{L^2(\mathbf{R}^d)} \sim A\left\| \left(\sum |P_i f|^2\right)^{1/2} \right\|_{L^2(\mathbf{R}^d)}.$$

The difficulty is proving $\|S_2 f\|_{L^\infty(\mathbf{R}^d)} \lesssim A\| \left(\sum |P_i f|^{1/2}\right) \|_{L^\infty(\mathbf{R}^d)}$, which we can interpolate into an inequality like above where we can apply Littlewood-Paley theory. To do this we perform another decomposition, writing

$$S_2 f = If + IIf$$

where

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \tilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{2Q_x} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{2Q_x} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

and

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \tilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

Now

$$\|If\|_{L^\infty} \leqslant \sup_x \fint_{Q_x} \left(\sum |m_i(D)(\mathbf{I}_{2Q_x} f)|^2 \right)^{1/2} dy \leqslant \sup_x |Q_x|^{-1/2} \left(\sum \|m_i(D)(\mathbf{I}_{2Q_x} f)\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim A|Q_x$$

Chapter 5

Heo, Nazarov, and Seeger

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's paper 2011 *Radial Fourier Multipliers in High Dimensions* [10]. Recall that if $m \in L^\infty(\mathbf{R}^d)$ is the symbol of a Fourier multiplier operator $T_m : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$. We let $\|m\|_{M^p(\mathbf{R}^d)}$ denote the operator norm of T_m from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. The goal of this paper is to show that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero, then

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2(d-1)}{d+1}\right),$$

where the implicit constant depends on p and η. Since

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\mathrm{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}$$

we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family $\{\mathrm{Dil}_t \eta\}$.

Note that the assumption of this result, if true, for m is compactly supported is equivalent to the assumption that $\widehat{m}$ is in $L^p(\mathbf{R}^n)$ (See Theorem 9.3 of this paper).

Note that, applying Littlewood-Paley theory, the assumption of this theorem is equivalent to the fact that

$$F^0_{p,2} = t^d \| \left(\sum_{n=0}^{\infty} |m \widehat{\mathrm{Dil}_{1/t}}\,\widehat{}\ \psi \eta_n|^2 \right)^{1/2} \|_{L^p(\mathbf{R}^n)}$$

In Garrigós and Seeger's 2007 paper *Characterizations of Hankel Multipliers*, it was proved that if $\eta \in \mathcal{S}(\mathbf{R}^d)$ is a nonzero, radial Schwartz function, then

$$\|m\|_{M^p_{\text{rad}}(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2d}{d+1}\right),$$

where $M^p_{\text{rad}}(\mathbf{R}^d)$ is the operator norm of T_m from $L^p_{\text{rad}}(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$. Thus, at least in the range $p \in \left(1, 2\frac{d-1}{d+1}\right)$, boundedness of T_m on radial functions is equivalent to boundedness on all functions.

Another consequence of the techniques of this paper is that an 'endpoint' result for local smoothing is proved for the wave equation. TODO: State this result.

5.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a convolution inequality, which can also be used to prove local smoothing results for the wave equation.

Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbf{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $f_{xr} = \text{Trans}_x(\sigma_r * \psi)$, which we view as a smoothened indicator function on a thickness ≈ 1 annulus of radius r centered at x. Our goal is to prove the following inequality.

Lemma 5.1. *For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} \, dx \, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

Why is Lemma 5.1 useful? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier given by some function $\tilde{m} : [1,\infty) \to \mathbf{C}$, and we set $a_r(x) = \tilde{m}(r) f(x)$ for some $f : \mathbf{R}^d \to \mathbf{C}$. Then it is simple to check that

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr}\, dx\, dr = K * \psi * f$$

where $K(x) = |x|^{1-d} m(x)$. In this setting, Lemma 5.1 says that

$$\|K * \psi * f\|_{L^p(\mathbf{R}^d)} \lesssim \|m\|_{L^p(\mathbf{R}^d)} \|f\|_{L^p(\mathbf{R}^d)},$$

which is clearly related to the convolution bound we want to show if $\psi = \widehat{\eta}$, provided that we are dealing with a multiplier supported away from the origin. To understand Lemma 5.1 it suffices to prove the following discretized estimate.

Theorem 5.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1,\infty)$, which is* discretized *in the sense that $|(x_1,r_1) - (x_2,r_2)| \geqslant 1$ for each distinct pair (x_1,r_1) and (x_2,r_2) in $\mathcal{E}$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r)\in\mathcal{E}} |a_r(x)|^p r^{p-1} \right)^{1/p},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independant of $\mathcal{E}$.

Proof of Lemma 5.1 from Lemma 5.2. For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$,

$$\int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} = \int_{[0,1)^d} \int_0^1 \sum_{n\in\mathbf{Z}^d} \sum_{m\in\mathbf{Z}} \mathrm{Trans}_{n,m}(a f_{rx})\, dr\, dx$$

Minkowski's inequality thus implies that

$$
\begin{aligned}
\left\| \int_{\mathbf{R}^d} \int_1^\infty a_r(x) f_{xr} \right\|_{L^p(\mathbf{R}^d)} &\leqslant \int_{[0,1)^d} \int_0^1 \| \sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} \mathrm{Trans}_{n,m}(a f_{rx}) \|_{L^p(\mathbf{R}^d)} \, dr \, dx \\
&\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} |a_r(x)|^p r^{p-1} \right)^{1/p} dr \, dx \\
&\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum_{n \in \mathbf{Z}^d} \sum_{m \in \mathbf{Z}} |a_r(x)|^p r^{p-1} \, dr \, dx \right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} \int_1^\infty |a_r(x)|^p r^{d-1} dr dx \right)^{1/p}. \qquad \square
\end{aligned}
$$

Lemma 5.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r) \in \mathcal{E}} a_r(x) f_{xr}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 5.2 is implied by the following Lemma.

Lemma 5.3. *For any* $1 \leqslant p < 2(d-1)/(d+1)$ *and* $k \geqslant 1$,

$$
\left\| \sum_{(x,r) \in \mathcal{E}_k} f_{xr} \right\|_{L^p(\mathbf{R}^d)} \lesssim 2^{k(d-1)} \#(\mathcal{E}_k)^{1/p} = 2^k \cdot (2^{k(d-p-1)} \#(\mathcal{E}_k))^{1/p}.
$$

Proof of Lemma 5.2 from Lemma 5.3. Applying a dyadic interpolation result (Lemma 2.2 of the paper), Lemma 5.3 implies that

$$
\| \sum_{(x,r) \in \mathcal{E}} f_{xr} \|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum 2^{kp} 2^{k(d-p-1)} \#(\mathcal{E}_k) \right)^{1/p} = \left(\sum 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}
$$

This is a restricted strong type bound for Lemma 5.2, which we can then interpolate. $\square$

If ψ is compactly supported, and r is sufficiently large depending on the size of this support, then f_{xr} is supported on an annulus with centre

x, radius r, and thickness $O(1)$. Thus $\|f_{xr}\|_{L^p(\mathbf{R}^d)} \sim r^{(d-1)/p}$, which implies that

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr}\|_{L^p(\mathbf{R}^d)} \gtrsim 2^{k(d-1)/p} \#(\mathcal{E}_k)^{1/p}.$$

Thus this bound can only be true if $p \geqslant 1$, and becomes tight when $p = 1$, where we actually have

$$\| \sum_{(x,r)\in\mathcal{E}_k} f_{xr}\|_{L^1(\mathbf{R}^d)} \sim 2^{k(d-1)} \#(\mathcal{E}_k)$$

because there can be no constructive interference in the L^1 norm. Understanding the sum in Lemma 5.3 for $1 < p < 2(d-1)/(d+1)$ will require an understanding of the interference patterns of annuli with comparable radius. We will use almost orthogonality principles to understand these interference patterns.

Lemma 5.4. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ and $r_1, r_2 \geqslant 1$,*

$$|\langle f_{x_1 r_1}, f_{x_2 r_2}\rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

In particular,

$$|\langle f_{x_1 r_1}, f_{x_2 r_2}\rangle| \lesssim \left(\frac{r_1 r_2}{|(x_1, r_1) - (x_2, r_2)|} \right)^{(d-1)/2}$$

Remark. Suppose $r_1 \leqslant r_2$. Then Lemma 5.4 implies that $f_{x_1 r_1}$ and $f_{x_2 r_2}$ are roughly uncorrelated, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. In particular, Laura Cladek's paper exploits this tangency information.

Proof. We write

$$\begin{aligned}
\langle f_{x_1r_1}, f_{x_2r_2}\rangle &= \left\langle \widehat{f}_{x_1r_1}, \widehat{f}_{x_2r_2}\right\rangle \\
&= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1)\cdot \xi}\, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbf{R}^d} \widehat{\sigma}(r_1\xi)\overline{\widehat{\sigma}(r_2\xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1)\cdot \xi}\, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle f_{x_1r_1}, f_{x_2r_2}\rangle = C_d (r_1r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s)\, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now substituting in, assuming $A(s)$ vanishes to order $100N$ at the origin, we conclude that

$$\begin{aligned}
\langle f_{x_1r_1}, f_{x_2r_2}\rangle &= C_d \left(\frac{r_1r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\
&\quad \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s}\, ds \right\} \\
&\lesssim_N \left(\frac{r_1r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1||)^{-5N} \\
&\lesssim_N \left(\frac{r_1r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1\tau_3 r_1 + \tau_2\tau_3 r_2 + |x_2 - x_1||)^{-5N}.
\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geqslant |r_1 - r_2|/10$ and $|x_1 - x_2| \geqslant 1$. If $1 + |x_1 - x_2| \leqslant |r_1 - r_2|/10$, then the supports of $f_{x_1r_1}$ and $f_{x_2r_2}$ are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leqslant 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leqslant 10$, and in this case the inequality reduces to the simple inequality

$$\langle f_{x_1 r_1}, f_{x_2 r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. $\square$

The exponent $(d-1)/2$ in Lemma 5.4 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r)\in\mathcal{E}_k} f_{xr}$. To fix this, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in $\mathbf{R}^d \times [R, 2R)$ is of *density type* (u, R) if

$$\#(B \cap \mathcal{E}) \leqslant u \cdot \mathrm{diam}(B)$$

for each ball B in $\mathbf{R}^{d+1}$ with diameter $\leqslant R$. A covering argument then shows that for any ball B,

$$\#(B \cap \mathcal{E}) \lesssim_d u \cdot \left(1 + \frac{\mathrm{diam}(B)}{R}\right)^d \cdot \mathrm{diam}(B).$$

(NOTE: WE MIGHT BE ABLE TO DO BETTER USING THE FACT THAT $\mathcal{E} \subset \mathbf{R}^d \times [R, 2R)$, USING THE VALUE R).

Theorem 5.5. *For any 1-separated set $\mathcal{E} \subset \mathbf{R}^d \times [R, 2R)$, we can consider a disjoint union $\mathcal{E} = \bigcup_{k=1}^\infty \bigcup_{m=1}^\infty \mathcal{E}_k(2^m)$ with the following properties:*

- *For each k and m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $\leqslant 2^m$ containing at least $2^m rad(B)$ points of $\mathcal{E}_k$, then*
$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'}).$$
- *For each k and m, there are disjoint balls $\{B_i\}$ of radius at most 2^k, such that*
$$\sum_i rad(B) \leqslant \frac{\#(\mathcal{E}_k)}{u}$$
such that $\bigcup B_i^*$ covers $\bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Vitali Covering. □

Given a sum $F = \sum_{(x,r)\in\mathcal{E}} f_{xr}$, decompose $\mathcal{E}$ as $\mathcal{E}_k(2^m)$, and define F_{km} to be the sum over $\mathcal{E}_k(2^m)$. It follows from the convering argument above that measure of the support of F_{km} is $O(2^{k(d-1)-m}\#(\mathcal{E}_k))$. We define $F_m = \sum_k F_{km}$. To Prove Lemma 5.3, it will suffice to prove the following L^2 estimate on F_m.

Lemma 5.6. *Suppose $\mathcal{E}$ is a set with density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} f_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim 2^{m/(d-1)}\sqrt{\log(2+2^m)}2^{k(d-1)/2}\cdot\#(\mathcal{E}_k)^{1/2}.$$

Proof of Lemma 5.3 from Lemma 5.6. We have

$$\|F_{km}\|_{L^2(\mathbf{R}^d)} \lesssim 2^{m/(d-1)}\sqrt{\log(2+2^m)}2^{k(d-1)/2}\#(\mathcal{E}_k)^{1/2}.$$

If we interpolate this bound with the support bound for F_{km}, we conclude that for $0 < p \leqslant 2$,

$$\begin{aligned}
\|F_{km}\|_{L^p(\mathbf{R}^d)} &\leqslant |\mathrm{Supp}(F_{km})|^{1/p-1/2}\|F_{km}\|_{L^2(\mathbf{R}^d)} \\
&\lesssim (2^{(k(d-1)-m)})^{1/p-1/2}2^{m/(d-1)}\sqrt{\log(2+2^m)}2^{k(d-1)/2}\#(\mathcal{E}_k)^{1/2} \\
&\lesssim 2^{m(1/p_d-1/p)}\sqrt{\log(2+2^m)}\cdot 2^{k(d-1)/p}\#(\mathcal{E}_k)^{1/2}.
\end{aligned}$$

where $p_d = 2(d-1)/(d+1)$. This bound is summable in m for $p < p_d$, which enables us to conclude that

$$\|F_k\|_{L^p(\mathbf{R}^d)} \lesssim_p 2^{k(d-1)/p}\#(\mathcal{E}_k)^{1/2}.$$

TODO: THIS SEEMS LIKE A TYPO. Thus for $1 \leqslant p < p_d$, we obtain the bound stated in Lemma 5.3. □

Proving 5.6 is where the weak-orthogonality bounds from Lemma 5.4 come into play.

Proof of Lemma 5.6. Split the interval $[2^k, 2^{k+1}]$ into $\lesssim 2^{(1-\alpha)k}$ intervals of length $2^{\alpha k}$, for some α to be optimized later. For appropriate integers a, let $I_a = [2^k + (a-1)2^{\alpha k}, 2^k + a2^{\alpha k}]$. Let $\mathcal{E}_a = \{(x,r)\in\mathcal{E} : r \in I_a\}$, and write

$F = \sum f_{xr}$, and $F_a = \sum_{(x,r)\in\mathcal{E}_a} f_{xr}$. Without loss of generality, splitting up the sum appropriately, we may assume that the set of a such that $\mathcal{E}_a$ is nonempty is 10-separated. We calculate that

$$\|F\|_{L^2(\mathbf{R}^d)}^2 = \sum_a \|F_a\|_{L^2(\mathbf{R}^d)}^2 + 2\sum_{a_1<a_2} |\langle F_{a_1}, F_{a_2}\rangle|$$

Given $a_1 < a_2$, $(x_1, r_1) \in \mathcal{E}_{a_1}$, and (x_2, r_2) such that $\langle f_{x_1 r_1}, f_{x_2 r_2}\rangle \neq 0$, then $|x_1 - x_2| \leqslant 2^{k+2}$. Since $|r_1 - r_2| \leqslant 2^{k+1}$ follows because $r_1, r_2 \in [2^k, 2^{k+1}]$, it follows that $|(x_1, r_1) - (x_2, r_2)| \leqslant 3 \cdot 2^{k+1}$. For each such pair, since we may assume that $a_2 - a_1 \geqslant 10$ without loss of generality, it follows that $|r_1 - r_2| \geqslant 2^{\alpha k}$, and so applying Lemma 5.4 together with the density property, we conclude that for $d \geqslant 4$,

$$\begin{aligned}
|\langle f_{x_1 r_1}, F_{a_2}\rangle| &\leqslant \sum_{l=1}^{(1-\alpha)k+1} \sum_{2^l 2^{\alpha k} \leqslant |(x_1,r_1)-(x_2,r_2)| \leqslant 2^{l+1} 2^{\alpha k}} \langle f_{x_1 r_1}, f_{x_2 r_2}| \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} (2^m 2^l 2^{\alpha k}) \left(\frac{2^{2k}}{2^l 2^{\alpha k}} \right)^{(d-1)/2} \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} 2^m (2^k)^{(d-1)-(d-3)/2\alpha} 2^{-(d-3)/2\cdot l} \\
&\lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha}.
\end{aligned}$$

Summing over all choices of x_1 and r_1, we conclude that

$$2\sum_{a_1<a_2} |\langle F_{a_1}, F_{a_2}\rangle| \lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha} \#(\mathcal{E}).$$

On the other hand, TODO $\square$

Chapter 6

Cladek: Improvements to Radial Multiplier Problem Using Incidence Geometry

Chapter 7

Mockenhaupt, Seeger, and Sogge: Exploiting Wave-Equation Periodicity

The main goal of the paper *Local Smoothing of Fourier Integral Operators and Carleson-Sjölin Estimates* is to prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; \mathcal{C})$, where Y is a manifold of dimension $n \geqslant 2$, and Z is a manifold of dimension $n+1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is δ depending on p and n, such that if $T : Y \to Y \times \mathbf{R}$ is the solution operator to the wave equation, and Y is a compact manifold whose geodesics are periodic, then T is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t,x) = T_t f(x)$, then the operator T_t is, for each t, a Fourier integral operator of order zero, with canonical relation

$$\mathcal{C}_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators T_t is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus T is *smoothing* in the t variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of δ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation $\mathcal{C}$ is a conic Lagrangian manifold of dimension $2n+1$. The fact that $\mathcal{C}$ is Lagrangian implies $\mathcal{C}$ is locally parameterized by $(\nabla_\zeta H(\zeta,\eta), \nabla_\eta H(\zeta,\eta), \zeta, \eta)$, where H is a smooth, real homogeneous function of order one. If we assume $\mathcal{C} \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta,\eta)]$ has full rank, which implies $D_\eta[\nabla \xi H(\zeta,\eta)] = (D_\xi[\nabla_\eta H(\zeta,\eta)])^T$ has full rank, and thus the projection $\mathcal{C} \to T^*Z$ is an immersion. We make the further assumption that the projection $\mathcal{C} \to Z$ is a submersion, from which it follows that for each z in the image of this projection, the projection of points in $\mathcal{C}$ onto T_z^*Z is a conic hypersurface Γ_z of dimension n. The final assumption we make is that all principal curvatures of Γ_z are non-vanishing.

Remark. The projection properties of $\mathcal{C}$ imply that, in $T^*(Z \times Y)$, there exists a smooth phase ϕ defined on an open subset of $Z \times T^*Y$, homogeneous in T^*Y, such that locally we can write $\mathcal{C}$ as $(z, \nabla_z \phi(z,\eta), \nabla_\eta \phi(z,\eta), \eta)$ for $\eta \neq 0$. Then, working locally on conic sets,

$$\Gamma_z = \{(\nabla_z \phi(z,\eta))\},$$

and the curvature condition becomes that the Hessian $H_{\eta\eta}\langle \nabla_z \phi, \nu \rangle$ has constant rank $n-1$, where ν is the normal vector to Γ_z. This is a natural homogeneous analogue of the Carleson-Sjölin condition for non-homogeneous oscillatory integral operators, i.e. the Carleson-Sjölin condition is allowed to assume $H_{\eta\eta}\phi$ has rank n, which cannot be possible in our case, since ϕ is homogeneous here. An approach using the analytic interpolation method of Stein or the Strichartz / Fractional Integral approach generalizes the Carleson-Sjölin theorem to show that for any smooth, non-homogeneous phase function $\Phi : \mathbf{R}^{n+1} \times \mathbf{R}^n \to \mathbf{R}$, and any compactly supported smooth amplitude a on $\mathbf{R}^{n+1} \times \mathbf{R}^n$. Consider the operators

$$T_\lambda f(z) = \int a(z,y) e^{2\pi i \lambda \Phi(z,y)} f(y)\, dy.$$

If the associated canonical relation $\mathcal{C}$, if $\mathcal{C}$ projects submersively onto $T^*\mathbf{R}^n$, so that for each $z \in \mathbf{R}^{n+1}$ in the image of the projection map $\mathcal{C}$, the set $S_z \subset \mathbf{R}^{n+1}$ obtained from the inverse image of the projection of $\mathcal{C} \to Z$ at z is a n dimensional hypersurface with k non-vanishing curvatures. Then for $1 \leqslant p \leqslant 2$,

$$\|T_\lambda f\|_{L^q(\mathbf{R}^{n+1})} \lesssim \lambda^{-(n+1)/q} \|f\|_{L^p(\mathbf{R}^n)}.$$

where $q = p^*(1 + 2/k)$.

Remark. We can also see these assumptions as analogues in the framework of cinematic curvature, splitting the z coordinates into 'time-like' and 'space-like' parts. Working locally, because $\mathcal{C} \to T^*Y$ is a submersion, we can consider coordinates $z = (x,t)$ so that, with the phase ϕ introduced above, $D_x(\nabla_\eta \phi)$ has full rank n, and that $\partial_t \phi(x,t,\eta) \neq 0$. Then for each $z = (x,t)$, we can locally write $\partial_t \phi(x,t,\eta) = q(x,t,\nabla_x \phi(x,t,\eta))$, homogeneous in η, and then

$$\mathcal{C} = \{(x,t,y;\xi,\tau,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},$$

where χ_t is a canonical transformation. Our curvature conditions becomes that $H_{\xi\xi} q$ has full rank $n-1$. This is the cinematic curvature condition introduced by Sogge.

Under these assumptions, the paper proves that any Fourier integral operator T in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ maps $L^2_c(Y)$ to $L^q_{\text{loc}}(Z)$ if

$$2\left(\frac{n+1}{n-1}\right) \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/q.$$

and maps $L^p_c(Y)$ to $L^p_{\text{loc}}(Z)$ if

$$p > 2 \quad \text{and} \quad \mu \leqslant -(n-1)(1/2 - 1/p) + \delta(p,n).$$

If we introduce time and space variables locally as in the remark above, any operator in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ can be written locally as a finite sum of operators of the form

$$Tf(x) = \int_{-\infty}^{\infty} T_t f(x),$$

where

$$T_t f(x) = \int a(t,x,\eta) e^{2\pi i \phi(x,t,y,\eta)} f(y)\, dy\, d\eta.$$

is a Fourier integral operator whose canonical relation is a locally a canonical graph, then the general theory implies that each of the maps T_t maps $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$ if

$$2 \leqslant q \leqslant \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q)$$

so that here we get local smoothing of order $1/q$, and also maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if

$$1 < p < \infty \quad \text{and} \quad \mu \leqslant -(n-1)|1/p - 1/2|$$

so we get $\delta(p,n)$ smoothing. A consequence of the smoothing, via Sobolev embedding, is a maximal theorem result for the operator T_t, i.e. that for any finite interval I, the operator

$$Mf = \sup_{t \in I} |T_t f|$$

maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if $\mu < -(n-1)(1/2 - 1/p) - (1/p - \delta(p,n))$. If the local smoothing conjecture held, we would conclude that, except at the endpoint T^* has the same $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ mapping properties as each of the operators T_t. We also get square function estimates, such that for any finite interval I, if we consider

$$Sf(x) = \left(\int_I |T_t f(x)|^2 \, dt \right)^{1/2},$$

then for

$$2\frac{n+1}{n-1} \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/2,$$

the operator S is bounded from $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$.

Our main reason to focus on this paper is the results of the latter half of the paper applying these techniques to radial multipliers on compact manifolds with periodic geodesics. Thus we consider a compact Riemannian manifold M, such that the geodesic flow is periodic with minimal period $2\pi \cdot \Pi$. We consider $m \in L^\infty(\mathbf{R})$, such that $\sup_{s>0} \|\beta \cdot \text{Dil}_s m\|_{L^2_\alpha(\mathbf{R})} = A_\alpha$ is finite for some $\alpha > 1/2$ and some $\beta \in C^\infty_c(\mathbf{R})$. We define a 'radial multiplier' operator

$$Tf = \sum_\lambda m(\lambda) E_\lambda f$$

where E_λ is the projection of f onto the space of eigenfunctions for the operator $\sqrt{-\Delta}$ on M with eigenvalue λ. We can also write this operator as $m(\sqrt{-\Delta})$. Then the wave propogation operator $e^{2\pi i t\sqrt{-\Delta}}$ is periodic of period Π. The Weyl formula tells us that the number of eigenvalues of $\sqrt{-\Delta}$ which are smaller than λ is equal to $V(M) \cdot \lambda^n + O(\lambda^{n-1})$.

Theorem 7.1. *Let $m \in L^2_\alpha(\mathbf{R})$ be supported on $(1,2)$, and assume $\alpha > 1/2$, then for $2 \leqslant p \leqslant 4$, $f \in L^p(M)$, and for any integer k,*

$$\left\| \sup_{2^k \leqslant \tau \leqslant 2^{k+1}} |Dil_\tau m(\sqrt{-\Delta}) f| \right\|_{L^p(M)} \lesssim_\alpha \|m\|_{L^2_\alpha(M)} \|f\|_{L^p(M)}.$$

Proof. To understand the radial multipliers we apply the Fourier transform, writing

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = m(\sqrt{-\Delta}/\tau) f = \int_{-\infty}^{\infty} \tau \widehat{m}(t\tau) e^{2\pi i t \sqrt{-\Delta}} f \, dt.$$

If we define $\beta \in C_c^\infty((1/2, 8))$ such that $\beta(s) = 1$ for $1 \leqslant s \leqslant 4$, and set $L_k f = \mathrm{Dil}_{2^k} \beta(\sqrt{-\Delta}) f$, then for $2^k \leqslant \tau \leqslant 2^{k+1}$

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = (\mathrm{Dil}_\tau m \cdot \mathrm{Dil}_{2^k} \beta)(\sqrt{-\Delta}) = T_\tau L_k f.$$

so Cauchy-Schwartz implies that

$$\begin{aligned}
|T_\tau f(x)| &= \left| \int_{-\infty}^{\infty} \tau \widehat{m}(\tau) e^{2\pi i t \sqrt{-\Delta}} L_k f(x) \, dt \right| \\
&\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{\tau}{(1 + |t\tau|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \\
&\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{2^k}{(1 + |2^k t|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2}
\end{aligned}$$

Because of periodicity, if we set $w_k(t) = 2^k/(1 + |2^k t|^2)^\alpha$, it suffices to prove that for $\alpha > 1/2$,

$$\left\| \left(\int_0^\Pi w_k(t) |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}.$$

This is a weighted combination of the wave propogators, roughly speaking, assigning weight 2^k for $t \lesssim 1/2^k$, and assigning weight $1/t$ to values $t \gtrsim 1/2^k$.

For a fixed $0 < \delta$, we can split this using a partition of unity into a region where $t \gtrsim \delta$ and a region where $t \lesssim \delta$, where δ is independent of k.

For each t, the wave propogation $e^{2\pi it\sqrt{-\Delta}}$ is a Fourier integral operator of order zero (we have an explicit formula for small t, and the composition calculus for Fourier integral operators can then be used to give a representation of the propogation operators for all times t, such that the symbols of these operators are locally uniformly bounded in S^0). Thus the square function estimate above can be applied in the region where $t \gtrsim \delta$, because the weighted square integral above has weight $O_\delta(1)$ uniformly in k.

Next, we move onto the region $t \lesssim 1/2^k$. The symbol of the operator $e^{2\pi it\sqrt{-\Delta}}$

Finally we move onto the region $1/2^k \lesssim t \lesssim \delta$. On this region we have $w_k(t) \sim 1/t$, which hints we should try using dyadic estimates. In particular, suppose that for $\gamma \leqslant \delta$, we have a family of dyadic estimates of the form

$$\left\| \left(\int_\gamma^{2\gamma} |e^{2\pi it\sqrt{-\Delta}} L_k f|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim \gamma^{1/2} (1 + \gamma 2^k)^\varepsilon \cdot \|f\|_{L^p(M)}.$$

Summing over the $O(k)$ dyadic numbers between $1/2^k$ and δ gives

$$\left\| \left(\int_{1/2^k \lesssim t \lesssim \delta} |e^{2\pi it\sqrt{-\Delta}} L_k f|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{L^p(M)} \lesssim 2^{\varepsilon k} \|f\|_{L^p(M)}$$

If we were able to obtain this inequality for some $\varepsilon > 0$, then we could bound

that for all $0 < \gamma < \Pi/2$

If we localize near $t \lesssim 1/2^k$ by multiplying by $\phi(2^k t)$ for some compactly supported smooth ϕ supported on $|t| \lesssim 1$, then for t on the support of $\phi(2^k t)$ we have a weight proportional to 2^k, and rescaling shows that it suffices to bound the quantities

$$\left\| \left(\int \phi(t) |e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|$$

the family of functions

$$\left\| \left(\int |\phi(t) e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, Dt \right)^{1/2} \right\|_{L^p_x} \lesssim \sup \|e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f\|_{L^p_x}$$

$a_k(t) = 2^{-k/2}\widehat{\phi}(t/2^k)\beta(\tau/2^k)$
it suffices to uniformly bound quantities of the form

$$\left\| \left(\int 2^k \phi(2^k t) |e^{2\pi i \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}$$

We now apply a dyadic decomposition to deal with the smaller values of t. Let us assume for simplicity of notation that $\delta < 1$, and then consider a partition of unity $1 = \sum_{j=1}^{\infty} \phi(2^j t)$ for $0 \leqslant t \leqslant 1$, and such that ϕ is localized near $1/4 \leqslant t \leqslant 2$, then our goal is to bound the quantities

$$\left\| \left(\int_{-\infty}^{\infty} \phi(2^j t) \frac{2^k}{(1 + |2^k t|^2)^{\alpha}} |A_t L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)},$$

which are each proportional to

s

□

Chapter 8

Lee and Seeger: Decomposition Arguments For Estimating Fourier Integral Operators

In the paper *Lebesgue Space Estimates For a Class of Fourier Integral Operators Associated With Wave Propogation*, Lee and Seeger prove a variable coefficient version of the result of Heo, Nazarov, and Seeger, i.e. generalizing their result from proving results about the boundedness of radial Fourier multipliers on $\mathbf{R}^n$ to certain Fourier integral operators satisfying the cinematic curvature condition.

We consider a localized Fourier integral operator $T : \mathcal{D}(Y) \to \mathcal{D}^*(Z)$ of order $\mu - 1/4$, where $\dim(Y) = d$ and $\dim(Z) = d + 1$, with a canonical relation $\mathcal{C}$, which must be $2d + 1$ dimensional, satisfying the following properties:

- The projection map $\pi_{T^*Y} : \mathcal{C} \to T^*Y$ is a submersion. It follows that around any point $(z_0, y_0; \zeta_0, \eta_0)$ we can choose coordinate systems y on Y and (x, t) on Z centered at z_0 and y_0 such that $\zeta_0 = dx_1$, $\eta_0 = dy_1$, and the tangent plane to $\mathcal{C}$ at this point is given by

  $$dx = dy \quad \text{and} \quad d\xi = d\eta \quad \text{and} \quad d\tau = 0.$$

  In particular, it follows that $\pi_Z : \mathcal{C} \to Z$ is a submersion, and we can locally find a function $\phi(z, \eta)$, homogeneous in η, such that, locally,

  $$\mathcal{C} = \{(z, \nabla_\eta \phi(z, \eta); \nabla_z \phi(z, \eta), \eta)\}.$$

By assumption on the tangent space of $\mathcal{C}$,

$$\nabla_\eta \phi(0, e_1) = 0 \quad \text{and} \quad \nabla_z \phi(0, e_1) = e_1.$$

The equivalence of phase theorem implies we can find a symbol $a(x, t, y, \eta)$ of order μ such that, after appropriately localizing the operator T, we have

$$Tf(x) = \int a(x, t, y, \eta) e^{2\pi i [\phi(x,t,\eta) - y\cdot\eta]} f(y)\, d\eta\, dy.$$

- The last assumption implies that for each z_0, $\Sigma_{z_0} \pi_Z^{-1}(z_0)$ is a d dimensional submanifold of $\mathcal{C}$. Moreover, our choice of coordinates makes it easy to see that the natural map $\Sigma_{z_0} \to T^*_{z_0} Z$ is an immersion, whose image is the immersed hypersurface Γ_{z_0} of $T^*_{z_0}$. Indeed, the tangent plane to Σ_{z_0} at the point above is given in coordinates by

  $$dx = dy = dt = d\tau = 0 \quad \text{and} \quad d\xi = d\eta.$$

  And this is projected injectively to the plane defined by $d\tau = 0$ in $T^*_{z_0} Z$. Our other assumption we make about $\mathcal{C}$ is an assumption on *cinematic curvature*. We assume that for each z_0, the hypersurface Σ_{z_0} is a cone with l nonvanishing principal curvatures, for some $1 \leqslant l \leqslant d-1$. Since

  $$\Sigma_{z_0} = \{(z_0; \nabla_z \phi(z_0, \eta_0)\}.$$

  The projection assumptions imply that the $(d+1) \times d$ matrix $D_\eta \nabla_z \phi$ has full rank, and the curvature assumptions imply that the Hessian matrix $H_\eta\{\partial\phi/\partial t\}$ has rank at least l in a neighborhood of our initial point. TODO: Why is it cinematic curvature?

Given these assumptions, the following result is obtained.

Theorem 8.1. *If $l \geqslant 3$, and $2 + 4/(l-2) < q < \infty$, and $\mu \leqslant d/q - (d-1)/2$, then T maps $L^q(Y)$ to $L^q(Z)$.*

If we take $l = d-1$, we get the full assumption of 'cinematic curvature' and we can use this to get results about local smoothing of the wave equation on compact Riemannian manifolds, which recovers the local smoothing result of Heo, Nazarov, and Seeger obtained in their paper on radial Fourier multipliers.

Theorem 8.2. *Let $d \geqslant 4$, $2+4/(d-3) < q < \infty$, and I a compact time interval. Then if M is a compact Riemannian manifold, and $\alpha = (d-1)/2 - d/q$, then*

$$\left(\|e^{it\sqrt{-\Delta}} f\|_{L^q(M)}^q \, dt \right)^{1/q} \lesssim_I \|f\|_{L^q_\alpha(M)}.$$

Proof. For any compact time interval I, the Lax parametrix construction allows one to, modulo a smoothing operator, find a phase function $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ and a symbol a of order zero supported on a neighborhood of the diagonal and on large frequencies, such that if $\phi_0(t,x,y,\eta) = \phi(x,y,\eta) + t|\eta|_g$, then, modulo smoothing operators,

$$e^{it\sqrt{-\Delta}} f = \int a(x,t,y,\eta) e^{2\pi i \phi_0(t,x,y,\eta)} f(y) \, d\eta \, dy.$$

We define

$$Tf(x,t) = e^{it\sqrt{-\Delta}} f = \int a(x,t,y,\xi) e^{2\pi i \phi_0(t,x,y,\xi)} f(y) \, d\xi \, dy,$$

which is a Fourier integral operator with canonical relation

$$\mathcal{C} = \{(\exp_y(t\xi/|\xi|), t, y; \eta, |\eta|_g, \eta)\}.$$

One immediately sees that the projection condition is satisfied, and if we are working on a coordinate system localized smaller than the injectivity radius of M, for each $z_0 = (x_0,t_0)$, Γ_{z_0} is a spherical cone, and thus has $d-1$ nonvanishing principal curvatures. The required result then immediately follows from the main result. $\square$

8.1 Frequency Localization and Discretization

Let us describe the idea of the proof. Write $K(z,y)$ for the kernel of T, and perform a frequency decomposition, writing

$$K(z,y) = \sum_{i=1}^{\infty} 2^{i\mu} K_i(z,y)$$

where

$$K_i(z,y) = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]}$$

where $\{\chi_i\}$ are supported on a common compact subset of $Z \times Y \times \Xi$ and satisfy estimates of the form We can set

$$\chi_i(z,y,\eta) = 2^{-i\mu} a(z,y,2^i\eta)\chi(\eta),$$

since then χ_i is supported on $|\eta| \sim 1$ and we have

$$(\partial_z^\alpha \partial_y^\beta \partial_\eta^\kappa \chi_i)(z,y,\eta) \lesssim_{\alpha,\beta,\kappa} 1.$$

By performing another decompsition, we may assume Ξ is an arbitrarily small neighborhood of e_1, such that for $z \in Z$ and $\eta \in \Xi$,

$$\nabla_z \phi(z,\eta) \approx e_1 \quad \text{and} \quad D_z \nabla_\eta \phi(z,\eta) \approx \begin{pmatrix} I_d \\ 0 \end{pmatrix}$$

and $H_\eta\{\partial\phi/\partial t\}$ has rank at least l. In this section, we analyze each of these operators separately. If we write T_i for the operator with kernel K_i, then here we will prove that

$$\|T_i f\|_{L^p} \lesssim 2^{i\left(\frac{d-1}{2} - \frac{d}{q}\right)} \|f\|_{L^p}.$$

for $q > 2l/(l-2)$. In Seeger, Sogge, and Stein, it is proved that

$$\|T_i f\|_{L^\infty} \lesssim 2^{i\frac{d-1}{2}} \|f\|_{L^\infty}.$$

By interpolation, it thus suffices to prove a restricted weak type inequality of the form

$$\|T_i \chi_E\|_{L^{q_l,\infty}} \lesssim 2^{i(d/l-1/2)} |E|^{1/q_l}$$

where $q_l = 2 + 4/(l-2)$. By duality, it suffices to show that for $p_l = 2 - 4/(l+2)$,

$$\|T_i^* \chi_E\|_{L^{p_l,\infty}} \lesssim 2^{i(2d-l)/2l} |E|^{1/p_l},$$

which is equivalent to prove that for $t > 0$, the measure of the set

$$\{y : |T_i^* \chi_E(y)| \geqslant t\}$$

is bounded by $O(t^{-p_l} 2^{i(2d-l)/(l+2)} |E|)$. The operator T_i^* has kernel

$$K^*(y,z) = \overline{K(z,y)} = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i (y\cdot\eta - \phi(z,\eta))}$$

so we still have a Fourier integral operator, but with a flipped canonical relation. We will obtain these bounds by proving an analogous discretized result at a scale $1/2^i$.

We consider $Z_k = 2^{-k}\mathbf{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$, for a small constant $\varepsilon > 0$. For each $z \in Z_k$ we consider a function a_z supported on $|\eta| \sim 2^k$ so that

$$|\partial_\eta^\alpha a_z(\eta)| \leqslant 2^{-i|\alpha|}$$

for $|\alpha| \lesssim 1$. Set

$$S_z(y) = \int a_z(\eta) e^{2\pi i (y\cdot\eta - \phi(z,\eta))}\, d\eta.$$

Our job is to understand the sums $\sum S_z$.

Lemma 8.3. *For each $\mathcal{E} \subset Z_k$, the measure of the set of y such that*

$$|\sum S_z(y)| \geqslant t$$

is $\lesssim 2^{i((d+1)l/(l+2)-1)}) t^{-p_l} \cdot \#(\mathcal{E})$.

TODO: How do we recover the continuous version of the result.

8.2 L^1 Estimates

To understand the individual pieces S_z, we consider a maximal $2^{-i/2}$ separated set Θ covering the unit sphere, and perform a further decomposition

$$a_z(\eta) = \sum_{\theta\in\Theta} a_{z,\theta}(\eta),$$

where $a_{z,\theta}$ is supported in a cone with aperture $O(2^{-i/2})$ centered at θ. Then $a_{z,\theta}$ is roughly speaking, supported on a set with length $2^{i/2}$ tangent to the radial direction, and with length 2^i in the radial direction. Thus differentiating in the radial direction no longer leads to quite as good derivative estimates, namely, if $u_1, \dots, u_M$ are unit vectors tangent to θ, and $M + N \lesssim 1$, then

$$(\theta \cdot \nabla_\eta)^N \prod_{i=1}^{M} (u_i \cdot \nabla_\eta)\{a_{z,\theta}\} \lesssim 2^{-kN-kM/2}.$$

The decomposition of a_z of course leads to a decomposition $S_z = \sum S_{z,\theta}$.

Now because each component of $\nabla_\eta \phi$ is homogeneous of degree 0, Euler's homogeneous function theorem says that

$$H_\eta \phi(x,\eta) \cdot \eta = 0.$$

Integration by parts (TODO: How? Also is there a typo?) yields that

$$|S_{z,\theta}(y)| \lesssim 2^{i\frac{d+1}{2}} \left(1 + 2^i |(\nabla_\xi \phi(z,\theta) - y) \cdot \theta| + 2^{k/2} |\Pi_{\theta^\perp}(\nabla_\xi \phi(z,\theta) - y)|\right)^{-O(1)}.$$

Roughly speaking, this inequality says that, roughly speaking, $S_{z,\theta}$ has magnitude $2^{i\frac{d+1}{2}}$, and is concentrated on a tube centered at $\nabla_\xi \phi(z,\theta)$, with thickness 2^{-i} in the radial direction, and thickness $2^{-i/2}$ in the tangential direction to θ. In particular, we find that

$$\|S_{z,\theta}\|_{L^1} \lesssim 1.$$

The triangle inequality (probably the best we can do in general in the L^1 setting) implies that

$$\|S_z\|_{L^1} \lesssim 2^{i(d-1)/2}.$$

This is the bound we will use in L^1.

To get more interesting bounds in other L^p spaces, we look at the orthogonality of the functions $\{S_z\}$. On the Fourier side of things, we have

$$\widehat{S_z}(\eta) = a_z(\eta) e^{-2\pi i \phi(z,\eta)}.$$

Thus by Parsevel, we have

$$\langle S_z, S_w \rangle = \langle \widehat{S_z}, \widehat{S_w} \rangle = \int a_z(\eta) a_w(\eta) e^{2\pi i [\phi(z,\eta) - \phi(w,\eta)]}.$$

TODO: Expand on rest of argument.

8.3 Adapting the Argument to Fourier Multipliers

Let $T = m(-\sqrt{\Delta})$ be a radial multiplier on $\mathbf{R}^n$, i.e. such that

$$Tf(x) = \int m(|\xi|) e^{2\pi i \xi \cdot (x-y)} f(y) \, d\xi \, dy.$$

If m is a symbol, then we can interpret T directly as a Pseudodifferential Operator. But Heo, Nazarov, and Seeger's result discuss families of multipliers m that are not even necessarily smooth, but do satisfy certain integrability conditions. To fix this, we assume a priori that we have applyied a decomposition argument, so we may assume m is compactly supported away from the origin. Then (by Paley-Wiener) $\widehat{m}$ is a smooth symbol of some finite order satisfying some integrability properties, which indicates how we might apply the theory of Fourier integral operators, i.e. by taking the Fourier transform of m, we get that

$$Tf(x) = \int \widehat{m}(\rho) e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\rho\, d\xi\, dy.$$

This is 'almost' a Fourier integral operator, except the phase is not smooth unless $\widehat{m}$ is supported away from the origin (fixed by a decomposition argument), and the phase is non-homogeneous. To fix the non-homogeneity, we just isolate the operator in ρ, writing

$$Tf(x) = \int_{-\infty}^{\infty} \widehat{m}(\rho) T_\rho f(x)\, d\rho,$$

where

$$T_\rho f(x) = e^{2\pi i \rho \sqrt{-\Delta}} f(x) = \int e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\xi\, dy$$

is the propogation operator for the half-wave equation $\partial_t u = \sqrt{-\Delta} \cdot u$. It has phase $\phi(x, y, \xi) = \rho|\xi| + \xi \cdot (x - y)$, and thus we have a stationary frequency value when $x = y - \rho\widehat{\xi}$, where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. This has canonical relation

Chapter 9

Relations to Local Smoothing

Let us now try and prove certain special cases of the radial multiplier conjecture on the sphere S^n. Thus we fix a symbol h, and study operators of the form

$$T_R = h\left(\sqrt{-\Delta}/R\right) = \sum h(\lambda/R)E_\lambda,$$

where E_λ is the projection operator onto the eigenspace corresponding to the eigenvalue λ. In particular, we wish to characterize the boundedness properties of the operators $T_{h,R}$, in terms of appropriate control of the Fourier transform of the function h. More precisely, we fix an exponent p, and assume that the quantity

$$A_p(h) = \sup_{t>0} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|^p (1+|s|)^{(d-1)(1-p/2)}\, dt \right)^{1/q}$$

is finite, which is a necessary condition for the multiplier $h(\sqrt{-\Delta})$ to be bounded on $L^p(\mathbf{R}^d)$ or $L^{p^*}(\mathbf{R}^d)$, and thus by the result of Mityagin, necessary for the family of operators $\{T_R : R > 0\}$ to be uniformly bounded in R on $L^p(S^n)$ or $L^{p^*}(S^n)$. For simplicity, let us assume that $\text{supp}(h)$ is contained in $1/2 \leqslant \lambda \leqslant 2$.

9.1 Attempt # 1: Exploiting Local Smoothing

Our goal is to show that, uniformly in R, we have

$$\|T_R f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Since T_R is a multiplier with symbol supported on $R/2 \leqslant \lambda \leqslant 2R$, we may assume that f is also supported on this frequency range, i.e. is in the span of eigenfunctions to $\sqrt{-\Delta}$ with eigenvalue $R/2 \leqslant \lambda \leqslant 2R$. In particular, this implies that $\|f\|_{L^p_\alpha} \lesssim (1+R)^\alpha \|f\|_{L^p}$.

To exploit the fact that $A_p(h)$ is finite, we apply the Fourier transform to the sum defining T_R, writing $T_R f$ as the vector valued integral

$$T_R f = \int_{-\infty}^{\infty} R\widehat{h}(Rt) e^{2\pi i t\sqrt{-\Delta}} f.$$

where the wave propogators $\{e^{2\pi i t\sqrt{-\Delta}}\}$ give solution operators to the half wave equation

$$\frac{\partial}{\partial t} = 2\pi i \sqrt{-\Delta}.$$

We break up $T_R f = \sum_{k=0}^{\infty} T_{R,k} f$, where

$$T_{R,0} f = \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho_0(Rt) e^{2\pi i t\sqrt{-\Delta}} f$$

and for $k \geqslant 1$,

$$T_{R,k} f = \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho(Rt/2^k) e^{2\pi i t\sqrt{-\Delta}} f.$$

If m is the inverse Fourier transform of $\widehat{h} \cdot \rho_0$ then $T_{R,0} = m\left(\sqrt{-\Delta}/R\right)$. This is probably easy to bound so I'll do this one later (e.g. reducing to ΨDO theory since m is smooth and rapidly decaying). Next, Hölder's inequality, combined with the trick of first multiplying and dividing by $(1+|Rt|)^{(d-1)(1/p-1/2)}$ implies that

$$\begin{aligned}
|T_{R,k} f| &= \left| \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho(Rt/2^k) e^{2\pi i t\sqrt{-\Delta}} f \; dt \right| \\
&\leqslant R \left(\int_{|t|\sim 2^k/R} |\widehat{h}(Rt)|^p (1+|Rt|)^{(d-1)(1-p/2)} \; dt \right)^{1/p} \\
&\qquad \left(\int_{-\infty}^{\infty} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \rho(Rt/2^k)(1+|Rt|)^{-(d-1)(p^*/2-1)} \; dt \right)^{1/p^*} \\
&\lesssim R^{1-1/p} 2^{-k(d-1)(1/2-1/p^*)} A_p(h) \left(\int_{|t|\sim 2^k/R} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \; dt \right)^{1/p^*}.
\end{aligned}$$

Applying the periodicity of the wave equation, for $2^k \geqslant R$ we have

$$\left(\int_{|t| \sim 2^k/R} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*} \lesssim (2^k/R)^{1/p^*} \left(\int_{|t| \lesssim 1} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*}.$$

Now if h is compactly supported, then we can also replace f with the spectral projection $P_{\lambda \sim R} f$ (do this before Littlewood-Paley). If the endpoint local smoothing conjecture held, then we would have

$$\left(\int_{|t| \lesssim 1} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*} \lesssim \|f\|_{L^{p^*}_{\alpha_p}},$$

where

$$\alpha_p = d(1/2 - 1/p) - 1/2.$$

Since f is a sum of well behaved eigenfunctions, Sobolev embedding should give us a bound of the form

$$\|f\|_{L^{p^*}_{\alpha_p}} \lesssim R^T \|f\|_{L^p}$$

for an appropriate power of R. If we choose T such that

$$1/p^* - \alpha_p/d = 1/p - T/d,$$

i.e. we pick

$$T = d(1/p - 1/2) - 1/2$$

TODO: Check assumptions of Sobolev embedding are true. Then we conclude that

$$\|f\|_{L^{p^*}_{\alpha_p}} \lesssim R^T \|f\|_{L^p}.$$

Putting all the bounds together, we conclude that

$$\begin{aligned} \|T_{R,k} f\|_{L^p} &\lesssim R^{1-1/p} 2^{-k(d-1)(1/2-1/p^*)} A_p(h) (2^k/R)^{1/p^*} R^T \|f\|_{L^p} \\ &\lesssim R^{d(1/p-1/2)-1/2} 2^{-k[(d-1)/2 - d/p^*]} A_p(h) \|f\|_{L^p}. \end{aligned}$$

Provided that $p < 2d/(d+1)$, this bound is summable in k. And provided that $p \geqslant 2d/(d+1)$, the bound is uniform in R. This indicates we're 'precisely' at the endpoint. In particular, local smoothing implies that for any $\varepsilon > 0$, there exists a range of p such that for each such p, there exists δ with

$$\|T_{R,k} f\|_{L^p} \lesssim 2^{-\delta k} A_{p,\varepsilon}(h) \|f\|_{L^p}.$$

where

$$A_{p,\varepsilon}(h) = \sup_{t>0} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|(1+|s|)^{(d-1)(1-p/2)+\varepsilon} \right).$$

Even in this case, we still need to deal with $2^k \leqslant R$, i.e. where there's no periodicity, but probably Euclidean techniques apply here since there's no overlap. But let's forget about that for now.

The small time parameterix for the half-wave operator, combined with the composition calculus of Fourier integral operators, allows us to write, for $|t| \leqslant 1$,

$$e^{2\pi i t\sqrt{-\Delta}} f = T_t f + S_t^\infty f,$$

where S_t^∞ is a *smoothing operator*, i.e. an integral operator with

$$S_t^\infty f(x) = \int K(t,x,y) f(y)\, dy,$$

where $K \in C^\infty([-1,1] \times S^n \times S^n)$, and where we can locally write

$$T_t f(x) = \int_{\mathbf{R}^n} a(t,x,y,\xi) e^{2\pi i \Phi(x,y,\xi)} f(y)\, d\xi\, dy$$

for some symbol $a \in S^0$, and some symbol $\Phi \in S^1$ satisfying

$$\Phi(x,y,t,\xi) = (x-y)\cdot\xi + t g_y(\xi,\xi) + O(|x-y|^2|\xi|).$$

To calculate the canonical relation of this operator, we look at the principal symbol of the operator $\sqrt{-\Delta}$. If g is the metric of S^n, then the principal symbol will be

$$p(x,\xi) = C\left(\sum g_{ij}(x)\xi^i\xi^j\right)^{1/2} = C|\xi|$$

for an appropriate constant C (TODO: Do this calculation more precisely). One can calculate (see Remark in Section 4.1 Sogge's book) that the canonical relation of the family of operators $\{T_t\}$ is given by

$$\mathcal{C} \subset \{(x,t,\xi,\tau,y,\eta) : (y,\eta) = \phi_t(x,\xi), \tau = p(x,\xi)\},$$

where $t \mapsto \phi_t(x,\xi)$ is the geodesic travelling at a velocity of 2π which starts at x, and travels in the direction given by ξ. We claim this canonical relation satisfies the cinematic curvature condition. Indeed, the projections

$\Pi_{y,\eta} : \mathcal{C} \to T^*S^n$ and $\Pi_{x,t} : \mathcal{C} \to T^*S^n$ are both submersions, so the Fourier integral operator is nondegenerate. For each pair (x_0, t_0), the cone

$$\mathcal{C}_{x_0,t_0} = \{(\xi, \tau) : |\xi| = C^{-1}\tau\}$$

is an n dimensional hypersurface in $T^*_{x_0,t_0}((-1,1) \times S^n)$, and it is easy to see this hypersurface is curved for all t. The endpoint local smoothing conjecture claims that if $f \in L^{p^*}(S^{n-1})$ for precisely the range of p we care about in the radial multiplier conjecture, then $Tf \in L^{p^*}_{-\alpha_p}((-1,1) \times S^{n-1})$, where $\alpha_p = n(1/2 - 1/p^*) - 1/2 = n(1/p - 1/2) - 1/2$. In particular, if we assume that Sobolev norms work on S^n the same way they work on $\mathbf{R}^n$, this means that if $f \in L^{p^*}(S^{n-1})$ has frequency supported on $|\xi| \leqslant L$, then

$$\|T_R f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim \|f\|_{L^{p^*}(\mathbf{R}^d)_{\alpha_p}} \lesssim L^{\alpha_p} \|f\|_{L^{p^*}(\mathbf{R}^d)}.$$

Thus we see that local smoothing is pretty hopeless in proving the result we need to prove for general f.

The only non optimal inequality we applied here was Hölder's inequality, which would be tight if there exists a non-negative function γ such that

$$|e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} (1 + |Rt|)^{-(d-1)\frac{(2-p)}{2(p-1)}} = \gamma(x)^{p^*} |\widehat{h}(Rt)|^p (1 + |Rt|)^{(d-1)(1-p/2)},$$

i.e. where

$$|e^{2\pi i t\sqrt{-\Delta}} f(x)| = \gamma(x)(1 + |Rt|)^{(d-1)\frac{(2-p)}{2}} |\widehat{h}(Rt)|^{1/p^*}.$$

TODO: Think about why local smoothing is useless. Is the theorem trivial if Hölder's inequality is applied?

9.2 Junk

Rescaling and applying Hölder's inequality, we have

$$
\begin{aligned}
R\int_0^{2\pi}\int_{\mathbf{R}^d}\sum_{k=0}^{\infty} w(Rs+(2\pi k)R)a(s,x,y,\xi)e^{2\pi i\Phi(x,y,s/R,\xi)}\,d\xi\,ds \\
&= \sum_{k=0}^{\infty}\int_0^{2\pi R} w(s+(2\pi k)R)\int_{\mathbf{R}^d} a(s/R,x,y,\xi)e^{2\ pii\Phi(x,y,s/R,\xi)}\,d\xi\,ds \\
&\leqslant \sum_{k=0}^{\infty}\left(\int_0^{2\pi R}|w(s+(2\pi k)R)|^q\,ds\right)^{1/q}(s)
\end{aligned}
$$

Now suppose that $\|w\|_{L^q(\mathbf{R}^d,(1+|x|)^{(d-1)(1-q/2)})}<\infty$

$$
\int_0^{2\pi}\int_{\mathbf{R}^d}\sum_{k=0}^{\infty} b_t(Rs+(2\pi k)R)a(s,x,y,\xi)e^{2\pi i\Phi(x,y,\xi)}\,d\xi\,ds,
$$

Let us begin with the qualitative assumption that h is compactly supported. Then, by breaking things up into a finite sum, we may assume that h is supported on $[1/2,2]$. Fix a function $\chi\in C_c^\infty(\mathbf{R})$ equal to one on $[1/2,2]$, and vanishing outside of $[1/4,4]$. Write

$$
P_R=\chi\left(\sqrt{-\Delta}/R\right)=\sum\chi(\lambda/R)E_\lambda.
$$

Then for any function $f\in C^\infty(S^n)$, $T_Rf=T_R\{P_Rf\}$. Thus when bounding the behaviour of the operator T_R, we may assume inputs are linear combinations of eigenfunctions to $\sqrt{-\Delta}$ with eigenvalues $\lambda\sim R$.

9.3 Adapting The Proof

The proof of Heo, Nazarov, and Seeger controls discrete sums of the form

$$
\sum_{(y,r)\in\mathcal{E}} c(y,r)F_{y,r}
$$

where $F_{y,r}=\mathrm{Trans}_y(\sigma_r*\psi)$, where ψ is radial, and it's Fourier transform is non-negative, and vanishes to high order at the origin, i.e. so it has some oscillation. A natural question to ask is whether on a compact Riemannian manifolds, there are functions analogous to the $F_{y,r}$ which we can study and control.

One option that might be comparable to the operators σ_r is the operators $e^{2\pi i r\sqrt{-\Delta}}$ which has a singularities supported near geodesic spheres, which corresponds to the multiplier $\lambda \mapsto e^{2\pi i \lambda r}$. The convolution thus corresponds to the multiplier operator

$$m_r(\xi) = e^{2\pi i \lambda r}\psi^\vee(\lambda),$$

for which

$$m_r(\sqrt{-\Delta}) = \int_{-\infty}^{\infty} \psi(t-r)e^{2\pi i t\sqrt{-\Delta}}.$$

Let K_r be the kernel of $m_r(\sqrt{-\Delta})$. Then

$$(\psi m)(\sqrt{-\Delta})f = \int_{\infty}^{\infty} \psi(t)m(t)e^{2\pi i t\sqrt{-\Delta}}f$$

A comparable option for the operators σ_r are the operators $e^{2\pi i r\sqrt{-\Delta}}$ which have singularities suported near s.

But how do we add the oscillating term

$$e^{2\pi i t\sqrt{-\Delta}}f(x) = \int e^{2\pi i(x\cdot\xi+|\xi|t)}\widehat{f}(\xi)\, d\xi.$$

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