SciencesPo Computational Economics
Spring 2017

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January 17, 2019

1 Numerical Dynamic Programming

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1.1 Intro

- Numerical Dynamic Programming (DP) is widely used to solve dynamic models.
- You are familiar with the technique from your core macro course.
- We will illustrate some ways to solve dynamic programs.

  1. Models with one discrete or continuous choice variable
  2. Models with several choice variables
  3. Models with a discrete-continuous choice combination

- We will go through:

  1. Value Function Iteration (VFI)
  2. Policy function iteration (PFI)
  3. Projection Methods
  4. Endogenous Grid Method (EGM)
  5. Discrete Choice Endogenous Grid Method (DCEGM)

1.2 Dynamic Programming Theory

- Payoffs over time are

$$U = \sum_{t=1}^{\infty} \beta^t u\left(s_t, c_t\right)$$

  where $\beta < 1$ is a discount factor, s_t is the state, c_t is the control.

- The state (vector) evolves as $s_{t+1} = h(s_t, c_t)$.
- All past decisions are contained in s_t.

1.2.1 Assumptions

- Let $c_t \in C(s_t), s_t \in S$ and assume u is bounded in $(c,s) \in C \times S$.
- Stationarity: neither payoff u nor transition h depend on time.
- Write the problem as

$$v(s) = \max_{s' \in \Gamma(s)} u(s,s') + \beta v(s')$$

- $\Gamma(s)$ is the constraint set (or feasible set) for s' when the current state is s

1.2.2 Existence

Theorem. Assume that $u(s,s')$ is real-valued, continuous, and bounded, that $\beta \in (0,1)$, and that the constraint set $\Gamma(s)$ is nonempty, compact, and continuous. Then there exists a unique function $v(s)$ that solves the above functional equation.

Proof. [@stokeylucas] [4] theoreom 4.6.

2 Solution Methods

2.1 Value Function Iteration (VFI)

- Find the fix point of the functional equation by iterating on it until the distance between consecutive iterations becomes small.
- Motivated by the Bellman Operator, and it's characterization in the Continuous Mapping Theorem.

2.2 Discrete DP VFI

- Represents and solves the functional problem in $\mathbb{R}$ on a finite set of grid points only.
- Widely used method.
  - Simple (+)
  - Robust (+)
  - Slow (-)
  - Imprecise (-)
- Precision depends on number of discretization points used.
- High-dimensional problems are difficult to tackle with this method because of the curse of dimensionality.

2.2.1 Deterministic growth model with Discrete VFI

- We have this theoretical model:

$$\begin{aligned} V(k) &= \max_{0<k'<f(k)} u(f(k)-k') + \beta V(k') \\ f(k) &= k^\alpha \\ & k_0 \text{given} \end{aligned}$$

- and we employ the followign numerical approximation:

$$V(k_i) = \max_{i'=1,2,\ldots,n} u(f(k_i) - k_{i'}) + \beta V(i')$$

- The iteration is then on successive iterates of V: The LHS gets updated in each iteration!

$$\begin{aligned} V^{r+1}(k_i) &= \max_{i'=1,2,\ldots,n} u(f(k_i) - k_{i'}) + \beta V^{r}(i') \\ V^{r+2}(k_i) &= \max_{i'=1,2,\ldots,n} u(f(k_i) - k_{i'}) + \beta V^{r+1}(i') \\ &\ldots \end{aligned}$$

- And it stops at iteration r if $d(V^r, V^{r-1}) < \text{tol}$
- You choose a measure of *distance*, $d(\cdot,\cdot)$, and a level of tolerance.
- V^r is usually an *array*. So d will be some kind of *norm*.
- maximal absolute distance
- mean squared distance

### Exercise 1: Implement discrete VFI

2.3 Checklist

1. Set parameter values
2. define a grid for state variable $k \in [0,2]$
3. initialize value function V
4. start iteration, repeatedly computing a new version of V.
5. stop if $d(V^r, V^{r-1}) < \text{tol}$.
6. plot value and policy function
7. report the maximum error of both wrt to analytic solution

```
In [1]: alpha      = 0.65
        beta       = 0.95
        grid_max   = 2  # upper bound of capital grid
        n          = 150  # number of grid points
        N_iter     = 3000  # number of iterations
        kgrid      = 1e-2:(grid_max-1e-2)/(n-1):grid_max  # equispaced grid
        f(x) = x^alpha  # defines the production function f(k)
        tol = 1e-9

Out[1]: 1.0e-9
```

2.4 Analytic Solution

- If we choose $u(x) = \ln(x)$, the problem has a closed form solution.
- We can use this to check accuracy of our solution.

```
In [2]: ab         = alpha * beta
        c1         = (log(1 - ab) + log(ab) * ab / (1 - ab)) / (1 - beta)
        c2        = alpha / (1 - ab)
        # optimal analytical values
        v_star(k) = c1 .+ c2 .* log.(k)
        k_star(k) = ab * k.^alpha
        c_star(k) = (1-ab) * k.^alpha
        ufun(x) = log.(x)
```

```
Out[2]: ufun (generic function with 1 method)
```

```
In [3]: kgrid[4]
```

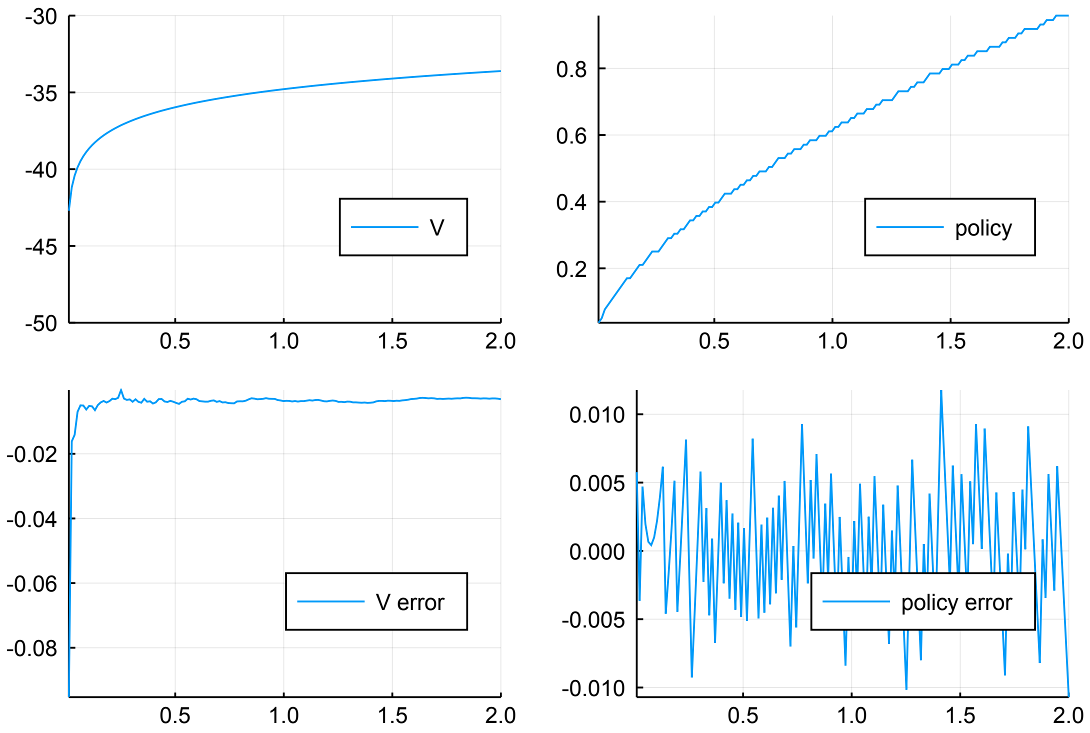
```
Out[3]: 0.04026943624161074
```

```
In [3]: # Bellman Operator
        # inputs
        # `grid`: grid of values of state variable
        # `v0`: current guess of value function

        # output
        # `v1`: next guess of value function
        # `pol`: corresponding policy function

        #takes a grid of state variables and computes the next iterate of the value function.
        function bellman_operator(grid,v0)

            v1  = zeros(n)     # next guess
            pol = zeros(Int,n)     # policy function
            w   = zeros(n)   # temporary vector

            # loop over current states
            # current capital
            for (i,k) in enumerate(grid)

                # loop over all possible kprime choices
                for (iprime,kprime) in enumerate(grid)
                    if f(k) - kprime < 0   #check for negative consumption
                        w[iprime] = -Inf
                    else
                        w[iprime] = ufun(f(k) - kprime) + beta * v0[iprime]
                    end
                end
                # find maximal choice
                v1[i], pol[i] = findmax(w)     # stores Value und policy (index of optimal cho
            end
            return (v1,pol)   # return both value and policy function
        end
```

```julia
# VFI iterator
#
## input
# `n`: number of grid points
# output
# `v_next`: tuple with value and policy functions after `n` iterations.
function VFI()
    v_init = zeros(n)     # initial guess
    for iter in 1:N_iter
        v_next = bellman_operator(kgrid,v_init)  # returns a tuple: (v1,pol)
        # check convergence
        if maximum(abs,v_init.-v_next[1]) < tol
            verrors = maximum(abs,v_next[1].-v_star(kgrid))
            perrors = maximum(abs,kgrid[v_next[2]].-k_star(kgrid))
            println("Found solution after $iter iterations")
            println("maximal value function error = $verrors")
            println("maximal policy function error = $perrors")
            return v_next
        elseif iter==N_iter
            warn("No solution found after $iter iterations")
            return v_next
        end
        v_init = v_next[1]  # update guess
    end
end

# plot
using Plots
function plotVFI()
    v = VFI()
    p = Any[]

    # value and policy functions
    push!(p,plot(kgrid,v[1],
            lab="V",
            ylim=(-50,-30),legend=:bottomright),
            plot(kgrid,kgrid[v[2]],
            lab="policy",legend=:bottomright))

    # errors of both
    push!(p,plot(kgrid,v[1].-v_star(kgrid),
        lab="V error",legend=:bottomright),
        plot(kgrid,kgrid[v[2]].-k_star(kgrid),
        lab="policy error",legend=:bottomright))

    plot(p...,layout=grid(2,2) )

end
```

```
plotVFI()
```

Out[3]:



2.4.1 Exercise 2: Discretizing only the state space (not control space)

- Same exercise, but now use a continuous solver for choice of k'.
- in other words, employ the following numerical approximation:

$$V(k_i) = \max_{k' \in [0,\bar{k}]} \ln(f(k_i) - k') + \beta V(k')$$

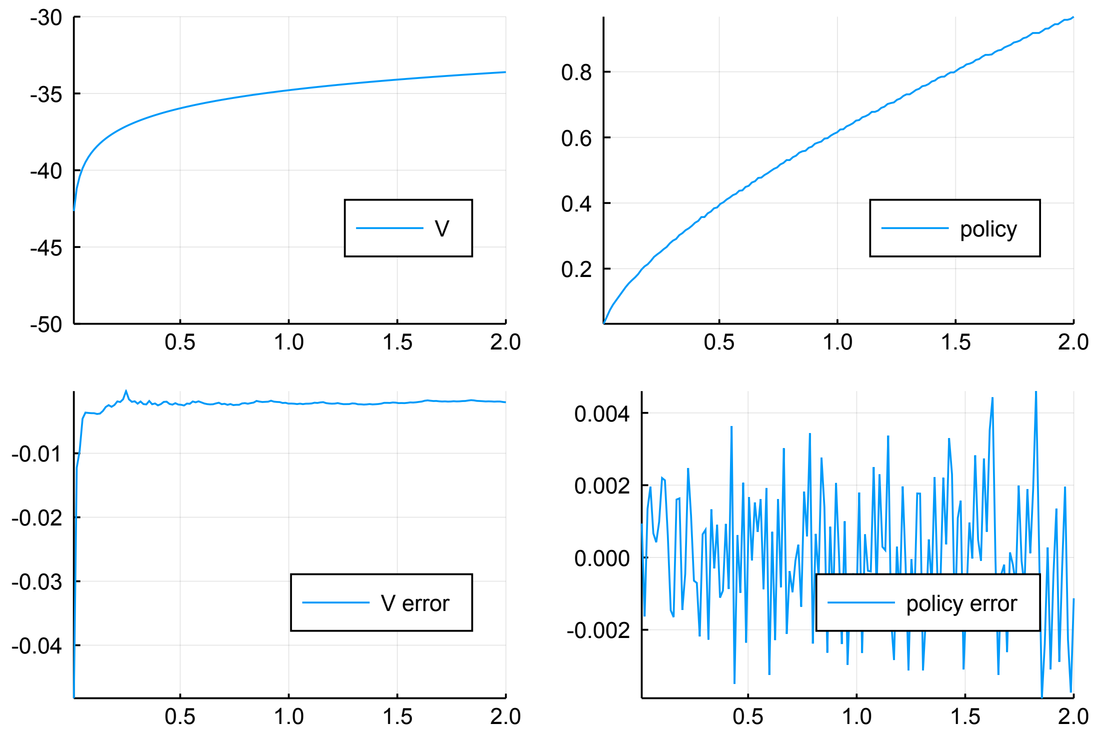
- To do this, you need to be able to evaluate $V(k')$ where k' is potentially off the `kgrid`.
- use `Interpolations.jl` to linearly interpolate V.
  - the relevant object is setup with function `interpolate((grid,),v,Gridded(Linear()))`
- use `Optim::optimize()` to perform the maximization.
  - you have to define an ojbective function for each k_i
  - do something like `optimize(objective, lb,ub)`

In [4]:
```julia
using Interpolations
using Optim
function bellman_operator2(grid,v0)
```

```
    v1  = zeros(n)      # next guess
    pol = zeros(n)      # consumption policy function

    Interp = interpolate((collect(grid),), v0, Gridded(Linear()) )

    # loop over current states
    # of current capital
    for (i,k) in enumerate(grid)

        objective(c) = - (log.(c) + beta * Interp[f(k) - c])
        # find max of ojbective between [0,k^alpha]
        res = optimize(objective, 1e-6, f(k))  # Optim.jl
        pol[i] = f(k) - res.minimizer   # k'
        v1[i] = -res.minimum
    end
    return (v1,pol)   # return both value and policy function
end

function VFI2()
    v_init = zeros(n)     # initial guess
    for iter in 1:N_iter
        v_next = bellman_operator2(kgrid,v_init)  # returns a tuple: (v1,pol)
        # check convergence
        if maximum(abs,v_init.-v_next[1]) < tol
            verrors = maximum(abs,v_next[1].-v_star(kgrid))
            perrors = maximum(abs,v_next[2].-k_star(kgrid))
            println("continuous VFI:")
            println("Found solution after $iter iterations")
            println("maximal value function error = $verrors")
            println("maximal policy function error = $perrors")
            return v_next
        elseif iter==N_iter
            warn("No solution found after $iter iterations")
            return v_next
        end
        v_init = v_next[1]  # update guess
    end
    return nothing
end

function plotVFI2()
    v = VFI2()
    p = Any[]

    # value and policy functions
    push!(p,plot(kgrid,v[1],
            lab="V",
            ylim=(-50,-30),legend=:bottomright),
            plot(kgrid,v[2],
            lab="policy",legend=:bottomright))

    # errors of both
    push!(p,plot(kgrid,v[1].-v_star(kgrid),
        lab="V error",legend=:bottomright),
        plot(kgrid,v[2].-k_star(kgrid),
        lab="policy error",legend=:bottomright))

    plot(p...,layout=grid(2,2) )

end

plotVFI2()
```

```
continuous VFI:
Found solution after 418 iterations
maximal value function error = 0.048284533681723474
maximal policy function error = 0.004602668698483803
```

Out[4]:



2.5 Policy Function Iteration

- This is similar to VFI but we now guess successive *policy* functions
- The idea is to choose a new policy p^* in each iteration so as to satisfy an optimality condition. In our example, that would be the Euler Equation.
- We know that the solution to the above problem is a function $c^*(k)$ such that

$$c^*(k) = \arg\max_z u(z) + \beta V(f(k) - z)\ \forall k \in [0, \infty]$$

- We **don't** directly solve the maximiation problem outlined above, but it's first order condition:

$$\begin{aligned} u'(c^*(k_t)) &= \beta u'(c^*(k_{t+1}))f'(k_{t+1}) \\ &= \beta u'[c^*(f(k_t) - c^*(k_t))]f'(f(k_t) - c^*(k_t)) \end{aligned}$$

- In practice, we have to find the zeros of

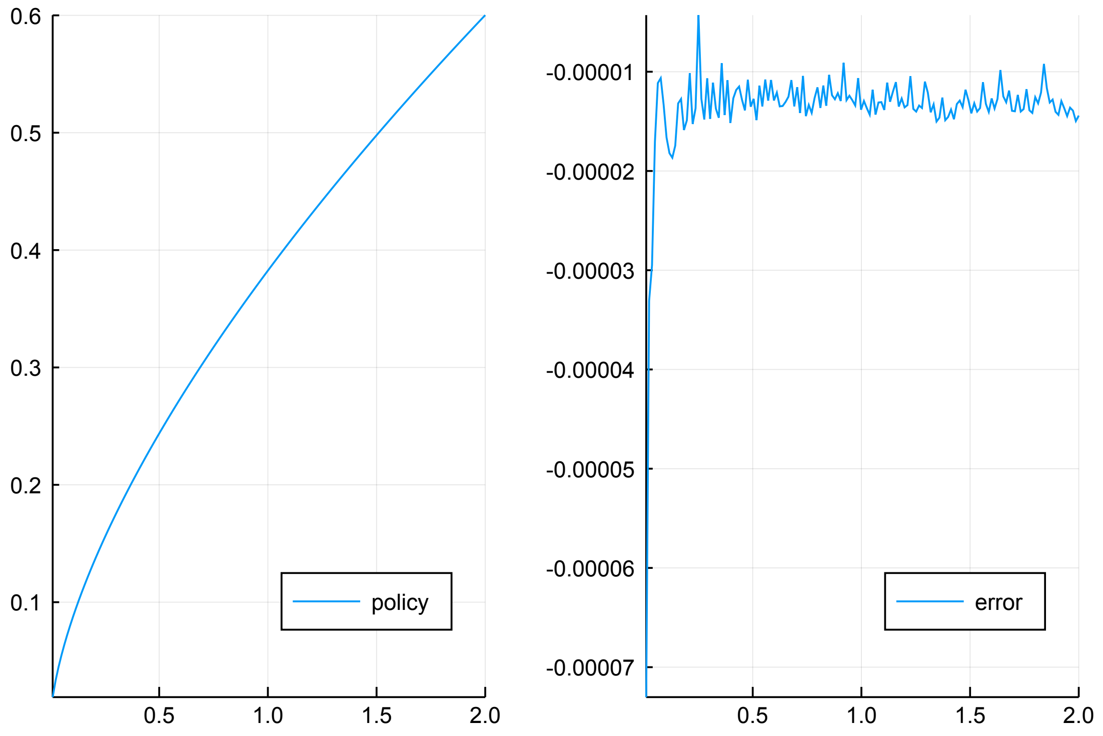
$$g(k_t) = u'(c^*(k_t)) - \beta u'[c^*(f(k_t) - c^*(k_t))]f'(f(k_t) - c^*(k_t))$$

```
In [5]: # Your turn!


        using Roots
        function policy_iter(grid,c0,u_prime,f_prime)

            c1  = zeros(length(grid))     # next guess
            pol_fun = interpolate((collect(grid),), c0, Gridded(Linear()) )

            # loop over current states
            # of current capital
            for (i,k) in enumerate(grid)
                objective(c) = u_prime(c) - beta * u_prime(pol_fun[f(k)-c]) * f_prime(f(k)-c)
                c1[i] = fzero(objective, 1e-10, f(k)-1e-10)
            end
            return c1
        end

        uprime(x) = 1.0 ./ x
        fprime(x) = alpha * x.^(alpha-1)

        function PFI()
            c_init = kgrid
            for iter in 1:N_iter
                c_next = policy_iter(kgrid,c_init,uprime,fprime)
                # check convergence
                if maximum(abs,c_init.-c_next) < tol
```

```
                    perrors =  maximum(abs,c_next.-c_star(kgrid))
                    println("PFI:")
                    println("Found solution after $iter iterations")
                    println("max policy function error = $perrors")
                    return c_next
                elseif iter==N_iter
                    warn("No solution found after $iter iterations")
                    return c_next
                end
                c_init = c_next  # update guess
            end
        end
        function plotPFI()
            v = PFI()
            plot(kgrid,[v v.-c_star(kgrid)],
                    lab=["policy" "error"],
                    legend=:bottomright,
                    layout = 2)
        end
        plotPFI()
```

```
INFO: Recompiling stale cache file /Users/florian.oswald/.julia/lib/v0.6/Roots.ji for module Ro

PFI:
Found solution after 39 iterations
max policy function error = 7.301895796647459e-5
```

Out[5]:



3 Projection Methods

- Many applications require us to solve for an *unknown function*
  - ODEs, PDEs
  - Pricing functions in asset pricing models
  - Consumption/Investment policy functions
- Projection methods find approximations to those functions that set an error function close to zero.

3.1 Example: Growth, again

- We stick to our working example from above.
- We encountered the Euler Equation g for optimality.
- At the true consumption function c^*, $g(k) = 0$.
- We define the following function operator:

$$\begin{aligned} 0 &= u'(c^*(k)) - \beta u'[c^*(f(k) - c^*(k))]f'(f(k) - c^*(k)) \\ &\equiv (\mathcal{N}(\rfloor^*))(k) \end{aligned}$$

- The Equilibrium solves the operator equation

$$0 = \mathcal{N}(c^*)$$

3.1.1 Projection Method example

1. create an approximation to c^*: find
$$\bar{c} \equiv \sum_{i=0}^{n} a_i k^i$$
which nearly solves
$$\mathcal{N}(c^*) = 0$$
2. Compute Euler equation error function:
$$g(k;a) = u'(\bar{c}(k)) - \beta u'[\bar{c}(f(k) - \bar{c}(k))]f'(f(k) - \bar{c}(k))$$
3. Choose a to make $g(k;a)$ small in some sense

What's *small in some sense*?

- Least-squares: minimize sum of squared errors
$$\min_a \int g(k;a)^2 dk$$
- Galerkin: zero out weighted averages of Euler errors
- Collocation: zero out Euler equation errors at grid $k \in \{k_1, \ldots, k_n\}$:
$$P_i(a) \equiv g(k_i;a) = 0, i = 1, \ldots, n$$

3.1.2 General Projection Method

1. Express solution in terms of unknown function
$$\mathcal{N}(h) = 0$$
where $h(x)$ is the equilibrium function at state x
2. Choose a space for appximation
3. Find $\bar{h}$ which nearly solves
$$\mathcal{N}(\bar{h}) = 0$$

3.1.3 Projection method exercise

- suppose we want to find effective supply of an oligopolistic firm in cournot competition.
- We want to know $q = S(p)$, how much is supplied at each price p.
- This function is characterized as
$$p + \frac{S(p)}{D'(p)} - MC(S(p)) = 0, \forall p > 0$$
- Take $D(p) = p^{-\eta}$ and $MC(q) = \alpha\sqrt{q} + q^2$.
- Our task is to solve for $S(p)$ in
$$p - \frac{S(p)p^{\eta+1}}{\eta} - \alpha\sqrt{S(p)} - S(p)^2 = 0, \forall p > 0$$
- No closed form solution. But collocation works!

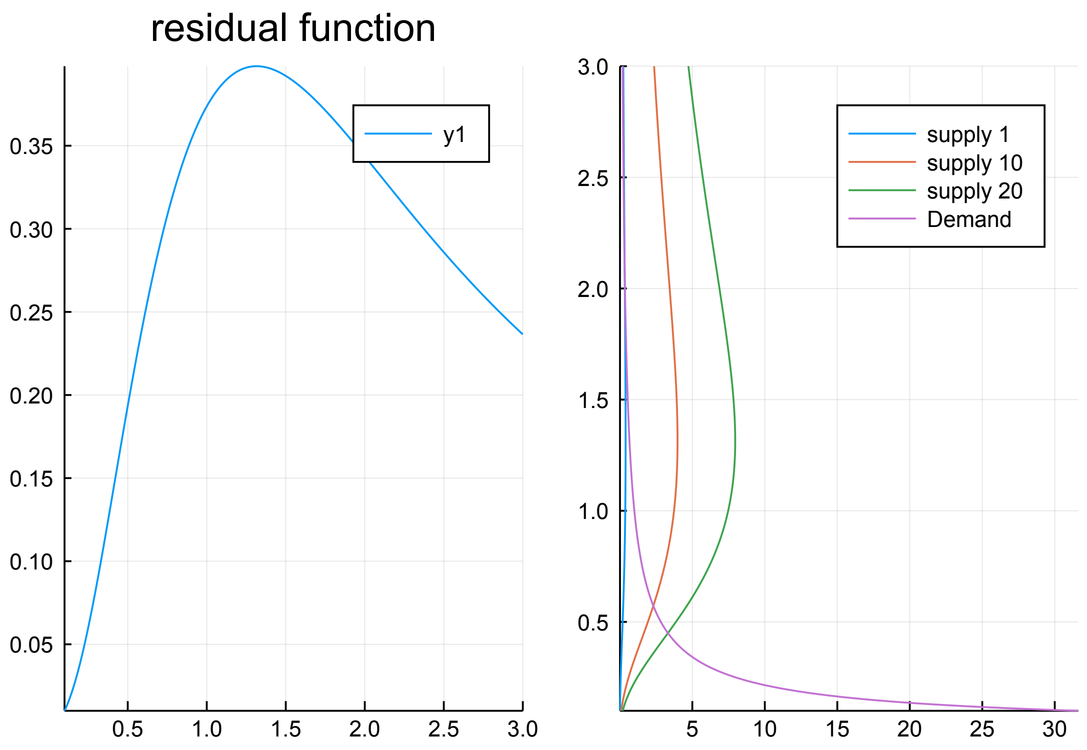
TASK

1. solve for $S(p)$ by collocation
2. Plot residual function
3. Plot resulting $mS(p)$ together with market demand and $m = 1, 10, 20$ for market size.

```
In [45]: using CompEcon
         using NLsolve
         function proj(n=25)

             alpha = 1.0
             eta   = 1.5
             a     = 0.1
             b     = 3.0
             basis = fundefn(:cheb,n,a,b)
             p     = funnode(basis)[1]    # collocation points

             c0 = ones(n)*0.3
             function resid!(c::Vector,result::Vector,p,basis,alpha,eta)
                 # your turn!
                 q = funeval(c,basis,p)[1]
                 q2 = zeros(q)
                 for i in eachindex(q2)
                     if q[i] < 0
                         q2[i] = -20.0
                     else
                         q2[i] = sqrt(q[i])
                     end
                 end
                 result[:] = p.+ q .*((-1/eta)*p.^(eta+1)) .- alpha*q2 .- q.^2

             end
             f_closure(r::Vector,x::Vector) = resid!(x,r,p,basis,alpha,eta)
             res = nlsolve(f_closure,c0)
             println(res)

             # plot residual function

             # plot supply functions at levels 1,10,20

             # plot demand function


         end
         proj()
```

```
Results of Nonlinear Solver Algorithm
 * Algorithm: Trust-region with dogleg and autoscaling
 * Starting Point: [0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3,
 * Zero: [0.248768, 0.0838916, -0.13965, 0.0447411, 0.00701804, -0.0135233, 0.00715223, -0.002
 * Inf-norm of residuals: 0.000000
 * Iterations: 9
 * Convergence: true
   * |x - x'| < 0.0e+00: false
   * |f(x)| < 1.0e-08: true
 * Function Calls (f): 8
 * Jacobian Calls (df/dx): 7
```

Out[45]:



4 Endogenous Grid Method (EGM)

- Fast, elegant and precise method to solve consumption/savings problems
- One continuous state variable
- One continuous control variable

$$V(M_t) = \max_{0<c<M_t} u(c) + \beta E V_{t+1}(R(M_t - c) + y_{t+1})$$

- Here, M_t is cash in hand, all available resources at the start of period t
  - For example, assets plus income.

- $A_t = M_t - c_t$ is end of period assets
- y_{t+1} is stochastic next period income.
- R is the gross return on savings, i.e. $R = 1 + r$.
- utility function can be of many forms, we only require twice differentiable and concave.

4.1 EGM after [@carroll2006method]

- [@carroll2006method] [1] introduced this method.
- The idea is as follows:
  - Instead of using non-linear root finding for optimal c (see above)
  - fix a grid of possible end-of-period asset levels A_t
  - use structure of model to find implied beginning of period cash in hand.
  - We use euler equation and envelope condition to connect M_{t+1} with c_t

4.1.1 Recall Traditional Methods: VFI and Euler Equation

- Just to be clear, let us repeat what we did in the beginning of this lecture, using the M_t notation.

$$\begin{aligned} V(M_t) &= \max_{0<c<M_t} u(c) + \beta E V_{t+1}(R(M_t - c) + y_{t+1}) \\ M_{t+1} &= R(M_t - c) + y_{t+1} \end{aligned}$$

4.1.2 VFI

1. Define a grid over M_t.
2. In the final period, compute
$$V_T(M_T) = \max_{0<c<M_t} u(c)$$
3. In all preceding periods t, do
$$V_t(M_t) = \max_{0<c_t<M_t} u(c_t) + \beta E V_{t+1}(R(M_t - c_t) + y_{t+1})$$
4. where optimal consumption is
$$c_t^*(M_t) = \arg\max_{0<c_t<M_t} u(c_t) + \beta E V_{t+1}(R(M_t - c_t) + y_{t+1})$$

4.1.3 Euler Equation

- The first order condition of the Bellman Equation is

$$\begin{aligned} \frac{\partial V_t}{\partial c_t} &= 0 \\ u'(c_t) &= \beta E \left[\frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \right] \quad (FOC) \end{aligned}$$

- By the Envelope Theorem, we have that

$$\frac{\partial V_t}{\partial M_t} = \beta E\left[\frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}}\right]$$

by FOC

$$\frac{\partial V_t}{\partial M_t} = u'(c_t)$$

true in every period:

$$\frac{\partial V_{t+1}}{\partial M_{t+1}} = u'(c_{t+1})$$

- Summing up, we get the Euler Equation:

$$u'(c_t) = \beta E\left[u'(c_{t+1})R\right]$$

4.1.4 Euler Equation Algorithm

1. Fix grid over M_t
2. In the final period, compute

$$c_T^*(M_T) = \arg\max_{0<cT<M_t} u(c_T)$$

3. With optimal $c_{t+1}^*(M_{t+1})$ in hand, backward recurse to find c_t from

$$u'(c_t) = \beta E\left[u'(c_{t+1}^*(R(M_t - c_t) + y_{t+1}))R\right]$$

4. Notice that if M_t is small, the euler equation does not hold.
   - In fact, the euler equation would prescribe to *borrow*, i.e. set $M_t < 0$. This is ruled out.
   - So, one needs to tweak this algorithm to check for this possibility
5. Homework.

4.2 The EGM Algorithm

Starts in period T with $c_T^* = M_T$. For all preceding periods:

1. Fix a grid of *end-of-period* assets A_t
2. Compute all possible next period cash-in-hand holdings M_{t+1}

$$M_{t+1} = R * A_t + y_{t+1}$$

   - for example, if there are n values in A_t and m values for y_{t+1}, we have $dim(M_{t+1}) = (n, m)$
3. Given that we know optimal policy in $t+1$, use it to get consumption at each M_{t+1}

$$c_{t+1}^*(M_{t+1})$$

4. Invert the Euler Equation to get current consumption compliant with an expected level of cash-on-hand, given A_t

$$c_t = (u')^{-1}\left(\beta E\left[u'(c_{t+1}^*(M_{t+1}))R|A_t\right]\right)$$

5. Current period *endogenous* cash on hand just obeys the accounting relation

$$M_t = c_t + A_t$$

```
#ăCore of a simple implementation


type iidModel <: Model

    # computation grids
    avec::Vector{Float64}
    yvec::Vector{Float64}   # income support
    ywgt::Vector{Float64}   # income weights

    # intermediate objects (na,ny)
    m1::Array{Float64,2}    # next period cash on hand (na,ny)
    c1::Array{Float64,2}    #ănext period consumption
    ev::Array{Float64,2}

    # result objects
    C::Array{Float64,2}     # consumption function on (na,nT)
    S::Array{Float64,2}     # savings function on (na,nT)
    M::Array{Float64,2}     # endogenous cash on hand on (na,nT)
    V::Array{Float64,2}     # value function on (na,nT). Optional.
    Vzero::Array{Float64,1} # value of saving zero
end

function EGM!(m::iidModel,p::Param)

    # final period: consume everything.
    m.M[:,p.nT] = m.avec
    m.C[:,p.nT] = m.avec
    m.C[m.C[:,p.nT].<p.cfloor,p.nT] = p.cfloor

    # preceding periods
    for it in (p.nT-1):-1:1

        # interpolate optimal consumption from next period on all cash-on-hand states
        # using C[:,it+1] and M[:,it+1], find c(m,it)

        tmpx = [0.0; m.M[:,it+1] ]
        tmpy = [0.0; m.C[:,it+1] ]
        for ia in 1:p.na
            for iy in 1:p.ny
                m.c1[ia+p.na*(iy-1)] = linearapprox(tmpx,tmpy,m.m1[ia+p.na*(iy-1)],1,p.na)
                # m.c1[ia,iy] = linearapprox(tmpx,tmpy,m.m1[ia,iy],1,p.na)  # equivalent
            end
        end
        # get expected marginal value of saving: RHS of euler equation
        # beta * R * E[ u'(c_{t+1}) ]
        Eu = p.R * p.beta .* up(m.c1,p) * m.ywgt

        # get optimal consumption today from euler equation: invert marginal utility
        m.C[:,it] = iup(Eu,p)

        # floor consumption
        m.C[m.C[:,it].<p.cfloor,it] = p.cfloor

        # get endogenous grid today
        m.M[:,it] = m.C[:,it] .+ m.avec
    end
end
```

4.3 Discrete Choice EGM

- This is a method developed by Fedor Iskhakov, Thomas Jorgensen, John Rust and Bertel Schjerning.
- Reference: [@iskhakovRust2014] [3]
- Suppose we have several discrete choices (like "work/retire"), combined with a continuous choice in each case (like "how much to consume given work/retire").
- Let $d = 0$ mean to retire.
- Write the problem of a worker as

$$\begin{aligned}
V_t(M_t) &= \max\left[v_t(M_t|d_t=0), v_t(M_t|d_t=1)\right] \\
&\text{with} \\
v_t(M_t|d_t=0) &= \max_{0<c_t<M_t} u(c_t) + \beta E W_{t+1}(R(M_t-c_t)) \\
v_t(M_t|d_t=1) &= \max_{0<c_t<M_t} u(c_t) - 1 + \beta E V_{t+1}(R(M_t-c_t)+y_{t+1})
\end{aligned}$$

- The problem of a retiree is

$$W_t(M_t) = \max_{0<c_t<M_t} u(c_t) + \beta E W_{t+1}(R(M_t-c_t))$$

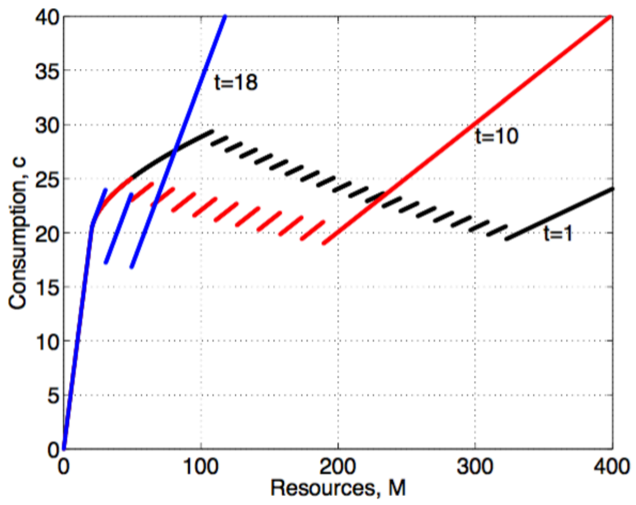
- Our task is to compute the optimal consumption functions $c_t^*(M_t|d_t=0), c_t^*(M_t|d_t=1)$

4.3.1 Problems with Discrete-Continuous Choice

- Even if all conditional value functions v are concave, the *envelope* over them, V, is in general not.
- [@clausenenvelope] [2]show that there will be a kink point $\bar{M}$ such that

$$v_t(\bar{M}|d_t=0) = v_t(\bar{M}|d_t=1)$$

  - We call any such point a **primary kink** (because it refers to a discrete choice in the **current period**)



[@iskhakovRust2014] figure 1

- V is not differentiable at $\bar{M}$.
- However, it can be shown that both left and right derivatives exist, with

$$V^{-}(\bar{M}) < V^{+}(\bar{M})$$

- Given that the value of the derivative changes discretely at $\bar{M}_t$, the value function in $t-1$ will exhibit a discontinuity as well:
  - v_{t-1} depends on V_t.
  - Tracing out the optimal choice of c_{t-1} implies next period cash on hand M_t, and as that hits $\bar{M}_t$, the derivative jumps.
  - The derivative of the value function determines optimal behaviour via the Euler Equation.
  - We call a discontinuity in v_{t-1} arising from a kink in V_t a **secondary kink**.
- The kinks propagate backwards.

- [@iskhakovRust2014] [3] provide an analytic example where one can compute the actual number of kinks in period 1 of T.
- Figure 1 in [@clausenenvelope]:

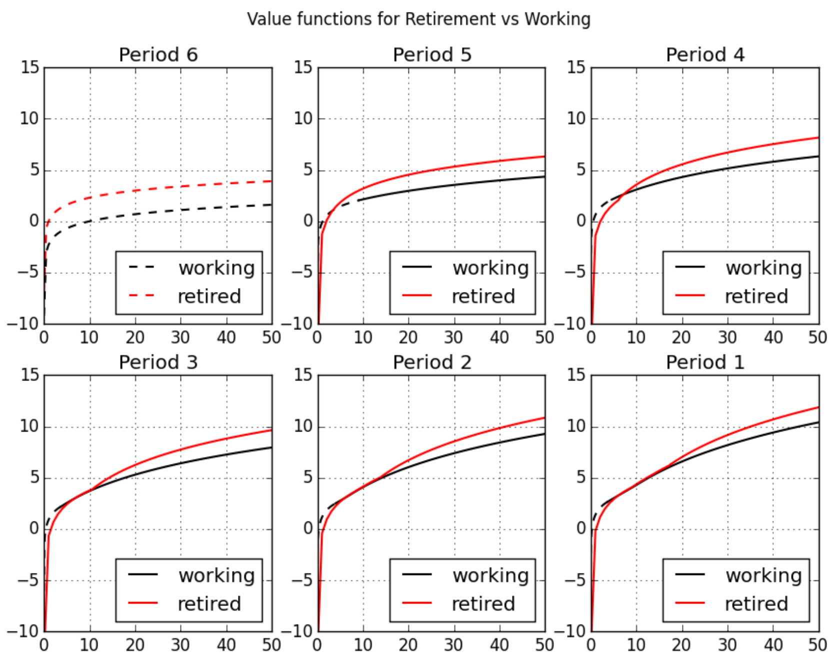
4.3.2 Kinks

- Refer back to the work/retirement model from before.
- 6 period implementation of the DC-EGM method:

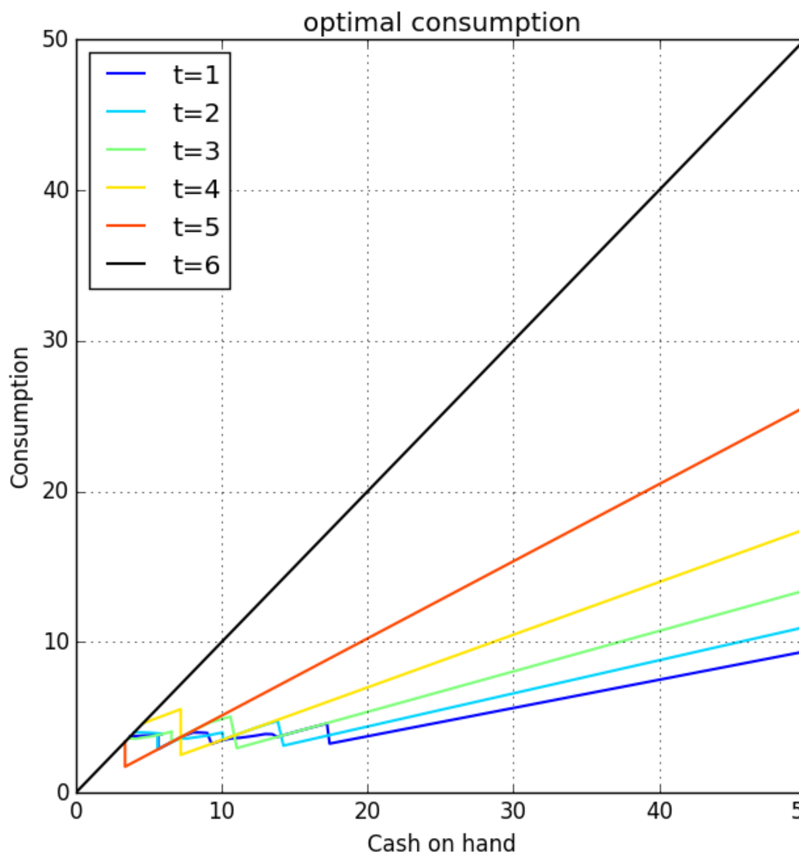
- Iskhakov @ cemmap 2015: Value functions in T-1

- Iskhakov @ cemmap 2015: Value functions in T-2

- Iskhakov @ cemmap 2015: Consumption function in T-2



github/floswald



- Optimal consumption in 6 period model:

4.3.3 The Problem with Kinks

- Relying on fast methods that rely on first order conditions (like euler equation) will fail.
- There are multiple zeros in the Euler Equation, and a standard Euler Equation approach is not guaranteed to find the right one.
- picture from Fedor Iskhakov's master class at cemmap 2015:

4.3.4 DC-EGM Algorithm

1. Do the EGM step for each discrete choice d
2. Compute d-specific consumption and value functions
3. compare d-specific value functions to find optimal switch points
4. Build envelope over d-specific consumption functions with knowledge of which optimal d applies where.

4.3.5 But EGM relies on the Euler Equation?!

- Yes.
- An important result in [@clausenenvelope] is that the Euler Equation is still the necessary condition for optimal consumption
  - Intuition: marginal utility differs greatly at $\epsilon + \bar{M}$.
  - No economic agent would ever locate **at** $\bar{M}$.
- This is different from saying that a proceedure that tries to find the zeros of the Euler Equation would still work.
  - this will pick the wrong solution some times.
- EGM finds **all** solutions.
  - There is a proceedure to discard the "wrong ones". Proof in [@iskhakovRust2014]

4.3.6 Adding Shocks

- This problem is hard to solve with standard methods.
- It is hard, because the only reliable method is VFI, and this is not feasible in large problems.
- Adding shocks to non-smooth problems is a widely used remedy.
  - think of "convexifying" in game theoretic models
  - (Add a lottery)
  - Also used a lot in macro
- Adding shocks does indeed help in the current model.
  - We add idiosyncratic taste shocks: Type 1 EV.
  - Income uncertainty:
  - In general, the more shocks, the more smoothing.
- The problem becomes

$$V_t(M_t) = \max\left[v_t(M_t|d_t = 0) + \sigma_\epsilon \epsilon_t(0), v_t(M_t|d_t = 1) + \sigma_\epsilon \epsilon_t(1)\right]$$
$$v_t(M_t|d_t = 1) = \max_{0<c_t<M_t} \log(c_t) - 1 + \beta \int EV_{t+1}(R(M_t - c_t) + y\eta_{t+1}) f(d\eta_{t+1})$$

where the value for retirees stays the same.

4.3.7 Adding Shocks

4.3.8 Full DC-EGM

- Needs to discard *false* solutions.
- Criterion:
  - grid in A_t is **increasing**
  - Assuming concave utility function, the function

    $$A(M|d) = M - c(M|d)$$

    is **monotone non-decreasing**
  - This means that, if you go through A_i, and find that

    $$M_t(A^j) < M_t(A^{j-1})$$

    you know you entered a non-concave region
- The Algorithm goes through the upper envelope and *prunes* the *inferior* points M from the endogenous grids.
- Precise details of Algorithm in paper.
- Julia implementation on floswald/ConsProb.jl

References

[1] Christopher D Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics letters*, 91(3):312–320, 2006.

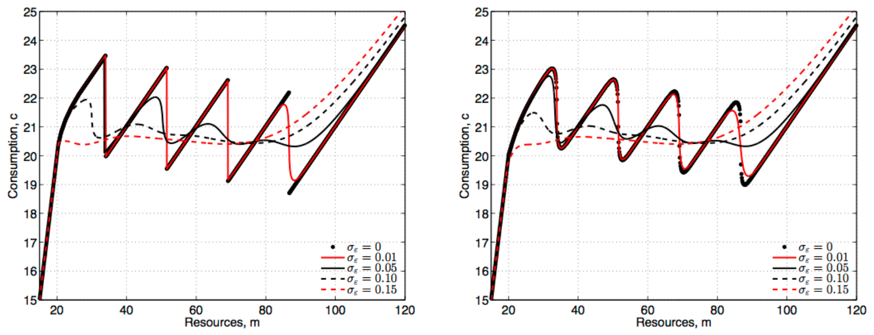
[2] A. Clausen and C. Strub. Envelope theorems for non-smooth and non-concave optimization. *https://andrewclausen.net/research.html*, 2013.

[3] Fedor Iskhakov, John Rust, Bertel Schjerning, and Thomas Jorgensen. Estimating Discrete-Continuous Choice Models: Endogenous Grid Method with Taste Shocks. *SSRN working paper*, 2014.

[4] Nancy Stokey and R Lucas. *Recursive Methods in Economic Dynamics (with E. Prescott)*. Harvard University Press, 1989.

Figure 2: Optimal Consumption Rules for Agent Working Today ($d_{t-1} = 1$).

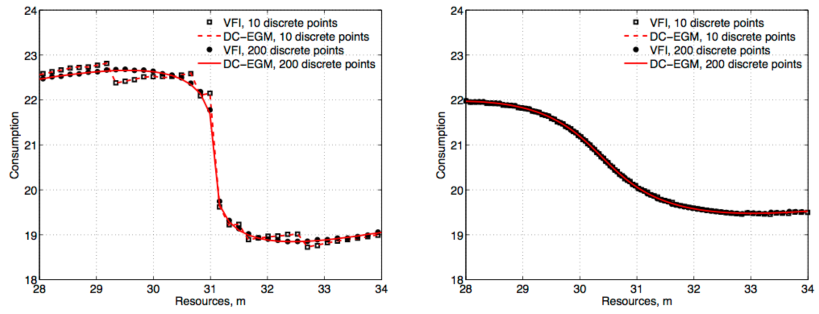
(a) Without income uncertainty, $\sigma_\eta = 0$ (b) With income uncertainty, $\sigma_\eta = \sqrt{.005}$



Notes: The plots show optimal consumption rules of the worker who decides to continue working in the consumption-savings model with retirement in period $t = T - 5$ for a set of taste shock scales σ_ε in the absence of income uncertainty, $\sigma_\eta = 0$, (left panel) and in presence of income uncertainty, $\sigma_\eta = \sqrt{.005}$, (right panel). The rest of the model parameters are $R = 1$, $\beta = 0.98$, $y = 20$.

Figure 3: Artificial Discontinuities in Consumption Functions, $\sigma_\eta^2 = 0.01$, $t = T - 3$.

(a) $\sigma_\varepsilon = 0$ (b) $\sigma_\varepsilon = 0.05$



Notes: Figure 3 illustrates how the number of discrete points used to approximate expectations regarding future income affects the consumption functions from value function iteration (VFI) and the DC-EGM. Panel (a) illustrates how using few (10) discrete equiprobable points to approximate expectations produce severe approximation error when there is *no* taste shocks. Panel (b) illustrates how moderate smoothing ($\sigma_\varepsilon = .05$) significantly reduces this approximation error.

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[@iskhakovRust2014] figure 2

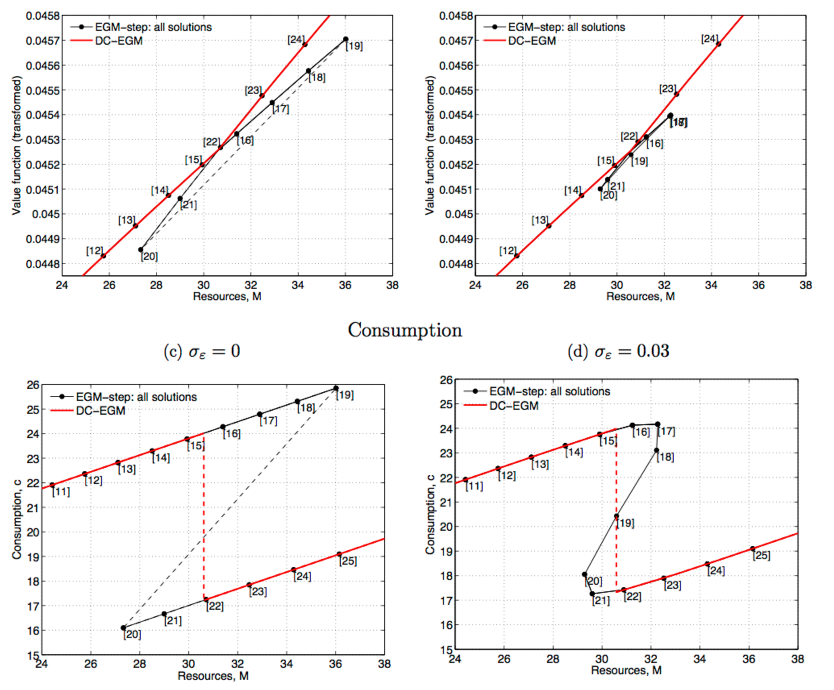
Figure 4: Non-concave regions and the elimination of the secondary kinks in DC-EGM.

Value function (transformed)

(a) $\sigma_\varepsilon = 0$ (b) $\sigma_\varepsilon = 0.03$

Consumption

(c) $\sigma_\varepsilon = 0$ (d) $\sigma_\varepsilon = 0.03$



Notes: The plots illustrate the output from the EGM-step of the DC-EGM algorithm (Algorithm 1) in a non-concave region. The dots are indexed with the index j of the ascending grid over the end-of-period wealth $\vec{A} = \{A^1, \ldots, A^G\}$ where $A^j > A^{j-1}, \forall j \in \{2, \ldots, G\}$. The connecting lines show the d_t-specific value functions $v_t(\vec{M}_t|d_t)$ and the consumption function $c_t(\vec{M}_t|d_t)$ linearly interpolated on the endogenous grid $\vec{M}_t$. computed on this grid are the outputs. The left panels illustrate the deterministic case without taste shocks, while in the right panels $\sigma_\varepsilon = 0.03$. The "true" solution, after applying the DC-EGM algorithm is illustrated with a solid red line. Dashed lines illustrate discontinuities. The solution is based on $G = 70$ grid points in $\vec{A}$, $R = 1$, $\beta = 0.98$, $y = 20$, $\sigma_\eta = 0$.

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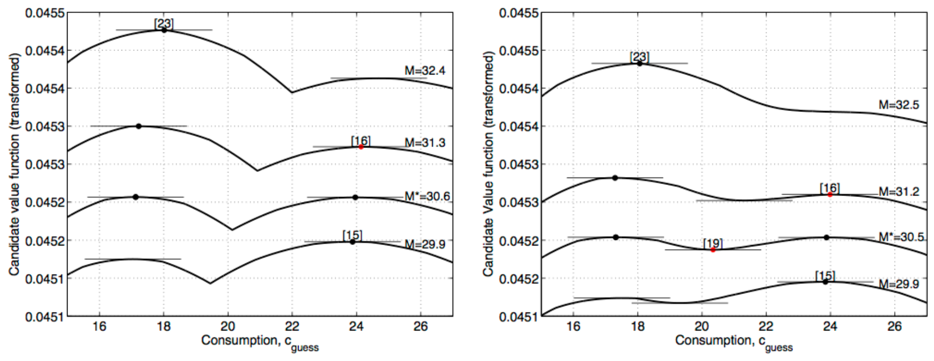
[@iskhakovRust2014] figure 4

Figure 5: Local maxima and multiple solutions of the Euler equation.

(a) $\sigma_\varepsilon = 0$ (b) $\sigma_\varepsilon = 0.03$



Notes: The figure plots the maximand of the equation (10), which defines the discrete choice specific value function $v_t(M_t|d_t = 1)$, for the case of $\sigma_\varepsilon = 0$ (panel a) and $\sigma_\varepsilon = 0.03$ (panel b). Horizontal lines indicate the critical points found or approximated by the EGM step of DC-EGM algorithm. The points are indexed with the same indexes as in Figure 4 and the black dots represent global maxima. Model parameters are identical to those of Figure 4.

[@iskhakovRust2014] figure 4