

# SciencesPo Computational Economics Spring 2017

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## 0.0.1 Numerical Integration

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## 0.1 Numerical Approximation of Integrals

- We will focus on methods that represent integrals as weighted sums.
- The typical representation will look like:

$$E[G(\epsilon)] = \int_{\mathbb{R}^N} G(\epsilon)w(\epsilon)d\epsilon \approx \sum_{j=1}^J \omega_j G(\epsilon_j)$$

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- $N$  is the dimensionality of the integration problem.
- $G : \mathbb{R}^N \mapsto \mathbb{R}$  is the function we want to integrate wrt  $\epsilon \in \mathbb{R}^N$ .
- $w$  is a density function s.t.  $\int_{\mathbb{R}^n} w(\epsilon)d\epsilon = 1$ .
- $\omega$  are weights such that (most of the time)  $\sum_{j=1}^J \omega_j = 1$ .
- We will look at normal shocks  $\epsilon \sim N(0_N, I_N)$
- in that case,  $w(\epsilon) = (2\pi)^{-N/2} \exp(-\frac{1}{2}\epsilon^T \epsilon)$
- $I_N$  is the  $n$  by  $n$  identity matrix, i.e. there is no correlation among the shocks for now.
- Other random processes will require different weighting functions, but the principle is identical.
- For now, let's say that  $N = 1$

## 0.2 Quadrature Rules

- We focus exclusively on those and leave Simpson and Newton Cowtes formulas out.
  - This is because Quadrature is the method that in many situations gives high accuracy with lowest computational cost.
- Quadrature provides a rule to compute weights  $w_j$  and nodes  $\epsilon_j$ .
- There are many different quadrature rules.
- They differ in their domain and weighting function.

- [https://en.wikipedia.org/wiki/Gaussian\\_quadrature](https://en.wikipedia.org/wiki/Gaussian_quadrature)
- In general, we can convert our function domain to a rule-specific domain with change of variables.

### 0.3 Gauss-Hermite: Expectation of a Normally Distributed Variable

- There are many different rules, all specific to a certain random process.
- Gauss-Hermite is designed for an integral of the form

$$\int_{-\infty}^{+\infty} e^{-x^2} G(x) dx$$

and where we would approximate

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x) dx \approx \sum_{i=1}^n \omega_i G(x_i)$$

- Now, let's say we want to approximate the expected value of function  $f$  when it's argument  $z \sim N(\mu, \sigma^2)$ :

$$E[f(z)] = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right) f(z) dz$$

### 0.4 Gauss-Hermite: Expectation of a Normally Distributed Variable

- The rule is defined for  $x$  however. We need to transform  $z$ :

$$x = \frac{(z-\mu)^2}{2\sigma^2} \Rightarrow z = \sqrt{2}\sigma x + \mu$$

- This gives us now (just plug in for  $z$ )

$$E[f(z)] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} \exp(-x^2) f(\sqrt{2}\sigma x + \mu) dx$$

- And thus, our approximation to this, using weights  $\omega_i$  and nodes  $x_i$  is

$$E[f(z)] \approx \sum_{j=1}^J \frac{1}{\sqrt{\pi}} \omega_j f(\sqrt{2}\sigma x_j + \mu)$$

## Using Quadrature in Julia

- <https://github.com/ajt60gaibb/FastGaussQuadrature.jl>

In [1]: `#Pkg.add("FastGaussQuadrature")`

```
using FastGaussQuadrature
```

```
np = 3
```

```
rules = Dict{"hermite" => gausshermite(np),  
            "chebyshev" => gausschebyshev(np),
```

```

        "legendre" => gausslegendre(np),
        "lobatto" => gausslobatto(np))

using DataFrames

integ = DataFrame(Rule=Symbol[Symbol(x) for x in keys(rules)],nodes=[x[1] for x in val
INFO: Precompiling module FastGaussQuadrature.WARNING: Method definition midpoints(Base.Range{
WARNING: Method definition midpoints(AbstractArray{T, 1} where T) in module Base at deprecated

```

Out[1]: 4E3 DataFrames.DataFrame. Omitted printing of 1 columns

Row	Rule	nodes
1	lobatto	[-1.0, 0.0, 1.0]
2	hermite	[-1.22474, -8.88178e-16, 1.22474]
3	legendre	[-0.774597, 0.0, 0.774597]
4	chebyshev	[-0.866025, 6.12323e-17, 0.866025]

## Quadrature in more dimensions: Product Rule

- If we have  $N > 1$ , we can use the product rule: this just takes the *kronecker product* of all univariate rules.
- The what?

```

In [2]: A = [1 2;3 4]
        B = [1;10]
        kron(A,B)
        kron(B,A)

```

Out[2]: 4E2 Array{Int64,2}:

1	2
3	4
10	20
30	40

- This works well as long as  $N$  is not too large. The number of required function evaluations grows exponentially.

$$E[G(\epsilon)] = \int_{\mathbb{R}^N} G(\epsilon) w(\epsilon) d\epsilon \approx \sum_{j_1=1}^{J_1} \cdots \sum_{j_N=1}^{J_N} \omega_{j_1}^1 \cdots \omega_{j_N}^N G(\epsilon_{j_1}^1, \dots, \epsilon_{j_N}^N)$$

where  $\omega_{j_1}^1$  stands for weight index  $j_1$  in dimension 1, same for  $\epsilon$ .

- Total number of nodes:  $J = J_1 J_2 \cdots J_N$ , and  $J_i$  can differ from  $J_k$ .

#### 0.4.1 Example for $N = 3$

- Suppose we have  $\epsilon^i \sim N(0, 1), i = 1, 2, 3$  as three uncorrelated random variables.
- Let's take  $J = 3$  points in all dimensions, so that in total we have  $J^N = 27$  points.

- We have the nodes and weights from before in `rules["hermite"]`.

```
In [3]: rules["hermite"][1]
        repeat(rules["hermite"][1], inner=[1], outer=[9])
```

```
Out[3]: 27-element Array{Float64,1}:
```

```
-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474

-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474
-8.88178e-16
 1.22474
-1.22474
```

```
In [4]: nodes = Any[]
        push!(nodes, repeat(rules["hermite"][1], inner=[1], outer=[9])) # dim1
        push!(nodes, repeat(rules["hermite"][1], inner=[3], outer=[3])) # dim2
        push!(nodes, repeat(rules["hermite"][1], inner=[9], outer=[1])) # dim3
        weights = kron(rules["hermite"][2], kron(rules["hermite"][2], rules["hermite"][2]))
        df = hcat(DataFrame(weights=weights), DataFrame(nodes, [:dim1, :dim2, :dim3]))
```

```
Out[4]: 27x4 DataFrames.DataFrame
```

Row	weights	dim1	dim2	dim3
1	0.0257793	-1.22474	-1.22474	-1.22474
2	0.103117	-8.88178e-16	-1.22474	-1.22474
3	0.0257793	1.22474	-1.22474	-1.22474
4	0.103117	-1.22474	-8.88178e-16	-1.22474
5	0.412469	-8.88178e-16	-8.88178e-16	-1.22474
6	0.103117	1.22474	-8.88178e-16	-1.22474

7	0.0257793	-1.22474	1.22474	-1.22474
8	0.103117	-8.88178e-16	1.22474	-1.22474
9	0.0257793	1.22474	1.22474	-1.22474
10	0.103117	-1.22474	-1.22474	-8.88178e-16
11	0.412469	-8.88178e-16	-1.22474	-8.88178e-16
16	0.103117	-1.22474	1.22474	-8.88178e-16
17	0.412469	-8.88178e-16	1.22474	-8.88178e-16
18	0.103117	1.22474	1.22474	-8.88178e-16
19	0.0257793	-1.22474	-1.22474	1.22474
20	0.103117	-8.88178e-16	-1.22474	1.22474
21	0.0257793	1.22474	-1.22474	1.22474
22	0.103117	-1.22474	-8.88178e-16	1.22474
23	0.412469	-8.88178e-16	-8.88178e-16	1.22474
24	0.103117	1.22474	-8.88178e-16	1.22474
25	0.0257793	-1.22474	1.22474	1.22474
26	0.103117	-8.88178e-16	1.22474	1.22474
27	0.0257793	1.22474	1.22474	1.22474

- Imagine you had a function  $g$  defined on those 3 dims: in order to approximate the integral, you would have to evaluate  $g$  at all combinations of  $\text{dimx}$ , multiply with the corresponding weight, and sum.

#### 0.4.2 Alternatives to the Product Rule

- Monomial Rules: They grow only linearly.
- Please refer to [juddbook] [1] for more details.

### 0.5 Monte Carlo Integration

- A widely used method is to just draw  $N$  points randomly from the space of the shock  $\epsilon$ , and to assign equal weights  $\omega_j = \frac{1}{N}$  to all of them.
- The expectation is then

$$E[G(\epsilon)] \approx \frac{1}{N} \sum_{j=1}^N G(\epsilon_j)$$

- This in general a very inefficient method.
- Particularly in more than 1 dimensions, the number of points needed for good accuracy is very large.
- Monte Carlo has a rate of convergence of  $\mathcal{O}(n^{-0.5})$

### 0.6 Quasi Monte Carlo Integration

- Uses non-product techniques to construct a grid of uniformly spaced points.
- The researcher controls the number of points.
- We need to construct equidistributed points.
- Typically one uses a low-discrepancy sequence of points, e.g. the Weyl sequence:

- $x_n = nv$  where  $v$  is an irrational number and  $\{\cdot\}$  stands for the fractional part of a number. for  $v = \sqrt{2}$ ,

$$x_1 = \{1\sqrt{2}\} = \{1.4142\} = 0.4142, x_2 = \{2\sqrt{2}\} = \{2.8284\} = 0.8284, \dots$$

- Other low-discrepancy sequences are Niederreiter, Haber, Baker or Sobol.
- Quasi Monte Carlo has a rate of convergence of close to  $\mathcal{O}(n^{-1})$
- [The wikipedia entry is good.](#)

```
In [5]: # Pkg.add("Sobol")
        using Sobol
        using Plots
        s = SobolSeq(2)
        p = hcat([next(s) for i = 1:1024]...)
        scatter(p[:,1], p[:,2], m=(:red,:dot,1.0), legend=false)
```

ArgumentError: Module Sobol not found in current path.  
Run `Pkg.add("Sobol")` to install the Sobol package.

Stacktrace:

```
[1] _require(::Symbol) at ./loading.jl:428

[2] require(::Symbol) at ./loading.jl:398

[3] include_string(::String, ::String) at ./loading.jl:515
```

## ## Correlated Shocks

- We often face situations where the shocks are in fact correlated.
  - One very typical case is an AR1 process:

$$z_{t+1} = \rho z_t + \varepsilon_t, \varepsilon \sim N(0, \sigma^2)$$

- The general case is again:

$$E[G(\epsilon)] = \int_{\mathbb{R}^N} G(\epsilon) w(\epsilon) d\epsilon \approx \sum_{j_1=1}^{J_1} \cdots \sum_{j_N=1}^{J_N} \omega_{j_1}^1 \cdots \omega_{j_N}^N G(\epsilon_{j_1}^1, \dots, \epsilon_{j_N}^N)$$

- Now  $\epsilon \sim N(\mu, \Sigma)$  where  $\Sigma$  is an N by N variance-covariance matrix.
- The multivariate density is

$$w(\epsilon) = (2\pi)^{-N/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(\epsilon - \mu)^T(\epsilon - \mu)\right)$$

- We need to perform a change of variables before we can integrate this.

- Given  $\Sigma$  is symmetric and positive semi-definite, it has a Cholesky decomposition,

$$\Sigma = \Omega\Omega^T$$

where  $\Omega$  is a lower-triangular with strictly positive entries.

- The linear change of variables is then

$$v = \Omega^{-1}(\epsilon - \mu)$$

- Plugging this in gives

$$\sum_{j=1}^J \omega_j G(\Omega v_j + \mu) \equiv \sum_{j=1}^J \omega_j G(\epsilon_j)$$

where  $v \sim N(0, I_N)$ .

- So, we can follow the exact same steps as with the uncorrelated shocks, but need to adapt the nodes.

## 0.7 References

- The Integration part of these slides are based on [maliar-maliar] [2] chapter 5

## References

- [1] Kenneth L. Judd. *Numerical methods in economics*. The MIT Press, 1998.
- [2] Lilia Maliar and Serguei Maliar. Numerical methods for large scale dynamic economic models. *Handbook of Computational Economics*, 3:325, 2013.