SciencesPo Computational Economics
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1 Computational Economics: Constrained Optimization

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1.1 Constrained Optimisation

- Recall our generic definition of an optimization problem:

$$\min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } \begin{array}{ll} c_i(x) = 0, & i \in E \\ c_i(x) \geq 0, & i \in I \end{array}$$

- E is the set of *equality constraints* and I is the set of *inequality constraints*.
- **Defintion: The Feasible Set**: Let Ω be the set of points x that satisfy the constraints, i.e.

$$\Omega = x|c_i(x) = 0, i \in E; c_i(x) \geq 0, i \in I$$

- Then, a different way of writign our problem is

$$\min_{x\in\Omega} f(x)$$

- A vector x^* is a *local solution* to this problem if $x^* \in \Omega$ and there is a neighborhood $\mathcal{N}$ s.t.

$$f(x) \geq f(x^*), \forall x \in \mathcal{N} \cap \Omega$$

- **Definition: The Active Set**: Active set $\mathcal{A}(x)$ at any feasible x consists of the equality constraint indices from E together with the indices of the inequality constraints for i for which $c_i(x) = 0$; that is,

$$\mathcal{A}(x) = E \cup i \in I | c_i(x) = 0$$

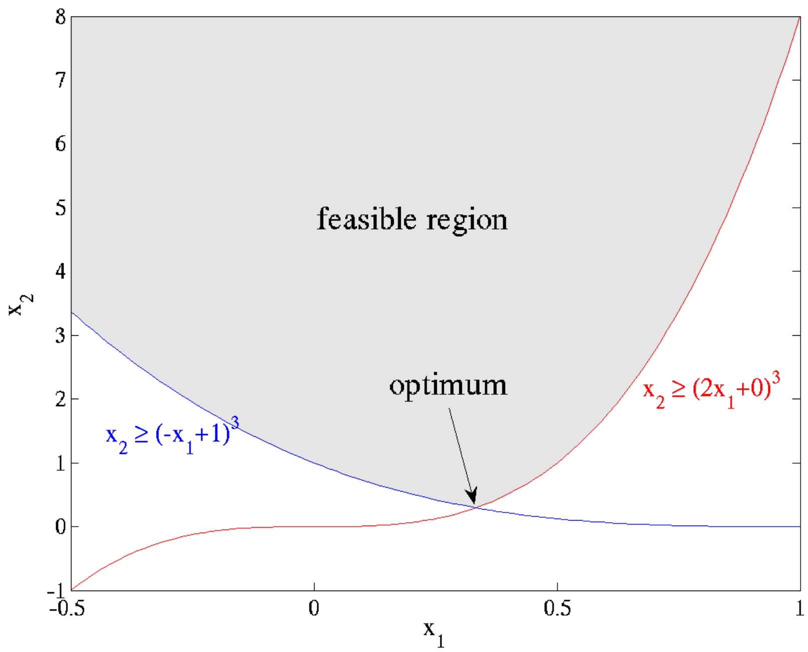
- At a feasible point x, the inequality constraint $i \in I$ is said to be *active* if $c_i(x) = 0$, and *inactive* if $c_i(x) > 0$

1.2 Nonlinear Constraints

- Consider the following problem

$$\min_{x \in \mathbb{R}^2} \sqrt{x_2} \text{ subject to } \begin{array}{r} x_2 \geq 0 \\ x_2 \geq (a_1 x_1 + b_1)^3 \\ x_2 \geq (a_2 x_1 + b_2)^3 \end{array}$$

- This configuration of constraints leads to the following feasible region for parameters $a_1 = 2, b_1 = 0, a_2 = -1, b_2 = 1$.



1.3 Example: 1 equality constraint

- consider

$$\min x_1 + x_2 \quad s.t. \quad x_1^2 + x_2^2 - 2 = 0$$

- constraint is a circle with radius $\sqrt{2}$ centered at 0. The solution must lie *on* that circle.
- Solution: $(-1, -1)$. Consider any other point on circle, like $(\sqrt{2}, 0)$.

1.4 Example: 1 inequality constraint

- Let's modify this example to

$$\min x_1 + x_2 \text{ such that } 2 - x_1^2 - x_2^2 \geq 0$$

- constraint is the region inside a circle with radius $\sqrt{2}$ centered at 0. The solution must lie *on or inside* that circle.

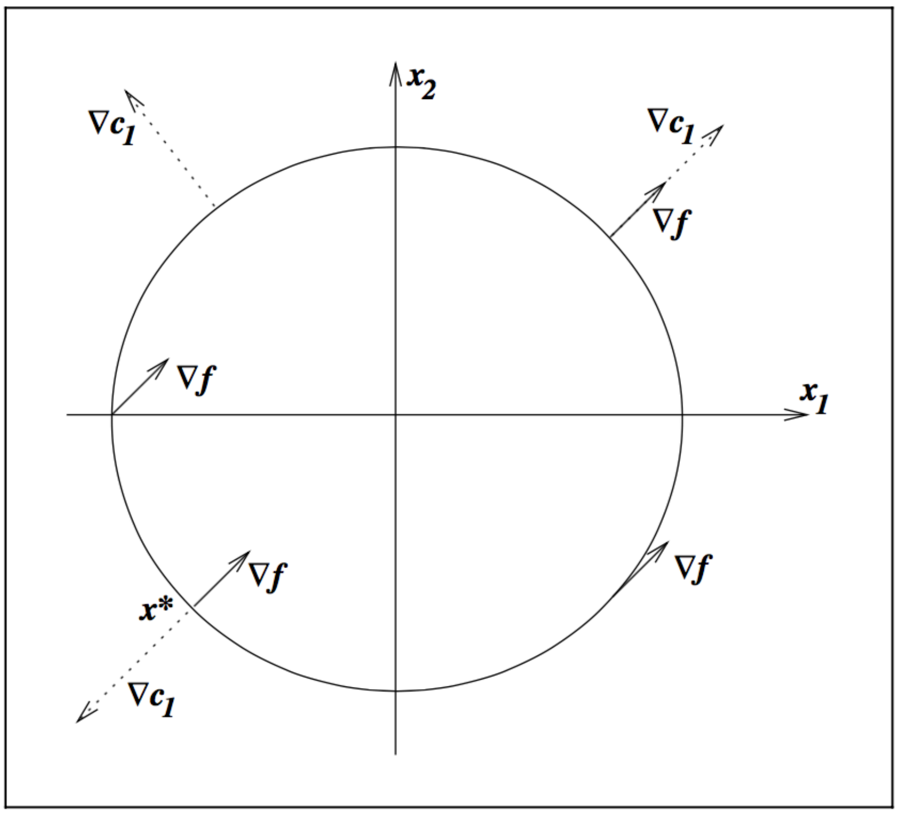


Figure 12.3 in [@nocedal-wright][2]

- Solution: $(-1,-1)$. Consider any other point on circle, like $(\sqrt{2},0)$.
- Two cases:

  1. x lies strictly inside the circle, and $c_1(x) > 0$
  2. x lies strictly on the circle, and $c_1(x) = 0$

- Complementarity condition.

1.5 Some Methods

- Penalty Function and Augmented Lagrangian Methods
- Sequential Quadratic Method
- Interior Point Method

1.6 Constrained Optimisation with `NLopt.jl`

- We need to specify one function for each objective and constraint.
- Both of those functions need to compute the function value (i.e. objective or constraint) *and* it's respective gradient.
- Notice that we can disregard $x_2 \geq 0$ here.
- `NLopt` expects contraints **always** to be formulated in the format

$$g(x) \leq 0$$

  where g is your constraint function
- The constraint function is formulated for each constraint at x. it returns a number (the value of the constraint at x), and it fills out the gradient vector, which is the partial derivative of the current constraint wrt x.
- There is also the option to have vector valued constraints, see the documentation.
- We set this up as follows:

In [1]:
```
function myfunc(x::Vector, grad::Vector)
    if length(grad) > 0
        grad[1] = 0
        grad[2] = 0.5/sqrt(x[2])
    end
    return sqrt(x[2])
end

function constraint(x::Vector, grad::Vector, a, b)
    if length(grad) > 0
            # modifies grad in place
        grad[1] = 3a * (a*x[1] + b)^2
        grad[2] = -1
    end
    return (a*x[1] + b)^3 - x[2]
end
using NLopt
# define an Opt object: which algorithm, how many dims of choice
opt = Opt(:LD_MMA, 2)
```

```
    # set bounds and tolerance
    lower_bounds!(opt, [-Inf, 0.])
    xtol_rel!(opt,1e-4)

    # define objective function
    min_objective!(opt, myfunc)
    # define constraints
    # notice the anonymous function
    inequality_constraint!(opt, (x,g) -> constraint(x,g,2,0), 1e-8)
    inequality_constraint!(opt, (x,g) -> constraint(x,g,-1,1), 1e-8)

    #ăcall optimize
    (minf,minx,ret) = optimize(opt, [1.234, 5.678])
```

```
Out[1]: (0.5443310477213124,[0.333333,0.296296],:XTOL_REACHED)
```

1.7 NLopt: Rosenbrock

- Let's tackle the rosenbrock example again.
- To make it more interesting, let's add an inequality constraint.

$$\min_{x\in\mathbb{R}^2}(1-x_1)^2+100(x_2-x_1^2)^2 \text{ subject to } 0.8-x_1^2-x_2^2 \geq 0$$

- in NLopt format, the constraint is $x_1+x_2-0.8\leq 0$

```
In [2]: function rosenbrock(x::Vector,grad::Vector)
            if length(grad) > 0
                    grad[1] = -2.0 * (1.0 - x[1]) - 400.0 * (x[2] - x[1]^2) * x[1]
                    grad[2] = 200.0 * (x[2] - x[1]^2)
            end
            return (1.0 - x[1])^2 + 100.0 * (x[2] - x[1]^2)^2
        end
        function r_constraint(x::Vector, grad::Vector)
            if length(grad) > 0
                grad[1] = 2*x[1]
                grad[2] = 2*x[2]
                end
                return x[1]^2 + x[2]^2 - 0.8
        end
        opt = Opt(:LD_MMA, 2)
        lower_bounds!(opt, [-5, -5.0])
        min_objective!(opt,(x,g) -> rosenbrock(x,g))
        inequality_constraint!(opt, (x,g) -> r_constraint(x,g))
        ftol_rel!(opt,1e-9)
        (minf,minx,ret) = optimize(opt, [-1.0,0.0])
```

```
Out[2]: (0.07588358473630112,[0.724702,0.524221],:FTOL_REACHED)
```

1.8 JuMP.jl

- Introduce `JuMP.jl`
- JuMP is a mathematical programming interface for Julia. It is like AMPL, but for free and with a decent programming language.
- The main highlights are:
  - It uses automatic differentiation to compute derivatives from your expression.
  - It supplies this information, as well as the sparsity structure of the Hessian to your preferred solver.
  - It decouples your problem completely from the type of solver you are using. This is great, since you don't have to worry about different solvers having different interfaces.
  - In order to achieve this, `JuMP` uses `MathProgBase.jl`, which converts your problem formulation into a standard representation of an optimization problem.
- Let's look at the readme
- The technical citation is Lubin et al [1]

1.9 JuMP: Quick start guide

- this is form the quick start guide
- please check the docs, they are excellent.

1.9.1 Step 1: create a model

- A model collects variables, objective function and constraints.
- it defines a solver to be used.

```
using Clp
m = Model(solver=ClpSolver())  # provide a solver

#ăDefine variables
@variable(m, x )              # No bounds
@variable(m, x >= lb )        # Lower bound only (note: 'lb <= x' is not valid)
@variable(m, x <= ub )        # Upper bound only
@variable(m, lb <= x <= ub )  # Lower and upper bounds


# we can create arrays of a variable
N = 2
@variable(m, x[1:M,1:N] >= 0 )

# or put them in a block
@variables m begin
    x
    y >= 0
    Z[1:10], Bin
    X[1:3,1:3], SDP
    q[i=1:2], (lowerbound = i, start = 2i, upperbound = 3i)
    t[j=1:3], (Int, start = j)
```

```
end

# Equivalent to:
@variable(m, x)
@variable(m, y >= 0)
@variable(m, Z[1:10], Bin)
@variable(m, X[1:3,1:3], SDP)
@variable(m, q[i=1:2], lowerbound = i, start = 2i, upperbound = 3i)
@variable(m, t[j=1:3], Int, start = j)

# bounds can depend on indices
@variable(m, x[i=1:10] >= i )
```

1.10 Objective and Constraints

- We can easily add objective and constraint functions:

```
@constraint(m, x[i] - s[i] <= 0)  # Other options: == and >=
@constraint(m, sum(x[i] for i=1:numLocation) == 1)
@objective(m, Max, 5x + 22y + (x+y)/2) # or Min
```

- This is fully integrated with Julia. you can use the `generator` syntax for sums:

```
@objective(sum(x[i] + y[i]/pi for i = I1, j = I2 if i+j < some_val))
```

```
In [3]: ##ăSimple example
        using JuMP
        using Clp

        m = Model(solver = ClpSolver())
        @variable(m, 0 <= x <= 2 )
        @variable(m, 0 <= y <= 30 )

        @objective(m, Max, 5x + 3*y )
        @constraint(m, 1x + 5y <= 3.0 )

        print(m)

        status = solve(m)

        println("Objective value: ", getobjectivevalue(m))
        println("x = ", getvalue(x))
        println("y = ", getvalue(y))
```

```
Max 5 x + 3 y
Subject to
 x + 5 y  3
 0  x  2
 0  y  30
```

```
Objective value: 10.6
x = 2.0
y = 0.2
```

In [4]:
```
# JuMP: Rosenbrock Example
# Instead of hand-coding first and second derivatives, you only have to give `JuMP` expr
# Here is an example.


using JuMP
using Ipopt

let

    m = Model(solver=IpoptSolver())

    @variable(m, x)
    @variable(m, y)

    @NLobjective(m, Min, (1-x)^2 + 100(y-x^2)^2)

    solve(m)

    println("x = ", getvalue(x), " y = ", getvalue(y))

end
```

```
******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:        3

Total number of variables...:        2
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:        0
Total number of inequality constraints...:        0
```

```
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  1.0000000e+00 0.00e+00 2.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  9.5312500e-01 0.00e+00 1.25e+01  -1.0 1.00e+00    -  1.00e+00 2.50e-01f  3
   2  4.8320569e-01 0.00e+00 1.01e+00  -1.0 9.03e-02    -  1.00e+00 1.00e+00f  1
   3  4.5708829e-01 0.00e+00 9.53e+00  -1.0 4.29e-01    -  1.00e+00 5.00e-01f  2
   4  1.8894205e-01 0.00e+00 4.15e-01  -1.0 9.51e-02    -  1.00e+00 1.00e+00f  1
   5  1.3918726e-01 0.00e+00 6.51e+00  -1.7 3.49e-01    -  1.00e+00 5.00e-01f  2
   6  5.4940990e-02 0.00e+00 4.51e-01  -1.7 9.29e-02    -  1.00e+00 1.00e+00f  1
   7  2.9144630e-02 0.00e+00 2.27e+00  -1.7 2.49e-01    -  1.00e+00 5.00e-01f  2
   8  9.8586451e-03 0.00e+00 1.15e+00  -1.7 1.10e-01    -  1.00e+00 1.00e+00f  1
   9  2.3237475e-03 0.00e+00 1.00e+00  -1.7 1.00e-01    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  2.3797236e-04 0.00e+00 2.19e-01  -1.7 5.09e-02    -  1.00e+00 1.00e+00f  1
  11  4.9267371e-06 0.00e+00 5.95e-02  -1.7 2.53e-02    -  1.00e+00 1.00e+00f  1
  12  2.8189505e-09 0.00e+00 8.31e-04  -2.5 3.20e-03    -  1.00e+00 1.00e+00f  1
  13  1.0095040e-15 0.00e+00 8.68e-07  -5.7 9.78e-05    -  1.00e+00 1.00e+00f  1
  14  1.3288608e-28 0.00e+00 2.02e-13  -8.6 4.65e-08    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 14

                                   (scaled)                 (unscaled)
Objective...............:   1.3288608467480825e-28    1.3288608467480825e-28
Dual infeasibility......:   2.0183854587685121e-13    2.0183854587685121e-13
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   2.0183854587685121e-13    2.0183854587685121e-13


Number of objective function evaluations             = 36
Number of objective gradient evaluations             = 15
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 14
Total CPU secs in IPOPT (w/o function evaluations)   =      0.118
Total CPU secs in NLP function evaluations           =      0.029

EXIT: Optimal Solution Found.
x = 0.9999999999999899 y = 0.9999999999999792
```

In [5]:
```
# not bad, right?
# adding the constraint from before:
```

```julia
let

    m = Model(solver=IpoptSolver())

    @variable(m, x)
    @variable(m, y)

    @NLobjective(m, Min, (1-x)^2 + 100(y-x^2)^2)
    @NLconstraint(m,x^2 + y^2 <= 0.8)

    solve(m)

    println("x = ", getvalue(x), " y = ", getvalue(y))

end
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        2
Number of nonzeros in Lagrangian Hessian...:        5

Total number of variables...:        2
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:        0
Total number of inequality constraints...:        1
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        1

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  1.0000000e+00 0.00e+00 2.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  9.5312500e-01 0.00e+00 1.25e+01  -1.0 5.00e-01    -  1.00e+00 5.00e-01f  2
   2  4.9204994e-01 0.00e+00 9.72e-01  -1.0 8.71e-02    -  1.00e+00 1.00e+00f  1
   3  2.0451702e+00 0.00e+00 3.69e+01  -1.7 3.80e-01    -  1.00e+00 1.00e+00H  1
   4  1.0409466e-01 0.00e+00 3.10e-01  -1.7 1.46e-01    -  1.00e+00 1.00e+00f  1
   5  8.5804626e-02 0.00e+00 2.71e-01  -1.7 9.98e-02    -  1.00e+00 1.00e+00h  1
   6  9.4244879e-02 0.00e+00 6.24e-02  -1.7 3.74e-02    -  1.00e+00 1.00e+00h  1
   7  8.0582034e-02 0.00e+00 1.51e-01  -2.5 6.41e-02    -  1.00e+00 1.00e+00h  1
   8  7.8681242e-02 0.00e+00 2.12e-03  -2.5 1.12e-02    -  1.00e+00 1.00e+00h  1
   9  7.6095770e-02 0.00e+00 6.16e-03  -3.8 1.37e-02    -  1.00e+00 1.00e+00h  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  7.6033892e-02 0.00e+00 2.23e-06  -3.8 3.99e-04    -  1.00e+00 1.00e+00h  1
  11  7.5885642e-02 0.00e+00 2.07e-05  -5.7 7.99e-04    -  1.00e+00 1.00e+00h  1
```

```
  12  7.5885428e-02 0.00e+00 2.74e-11  -5.7 1.38e-06    -  1.00e+00 1.00e+00h  1
  13  7.5883585e-02 0.00e+00 3.19e-09  -8.6 9.93e-06    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 13

                                   (scaled)                 (unscaled)
Objective...:   7.5883585442440671e-02    7.5883585442440671e-02
Dual infeasibility...:   3.1949178858070582e-09    3.1949178858070582e-09
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   2.5454985882932001e-09    2.5454985882932001e-09
Overall NLP error...:   3.1949178858070582e-09    3.1949178858070582e-09


Number of objective function evaluations             = 20
Number of objective gradient evaluations             = 14
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 20
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 14
Number of Lagrangian Hessian evaluations             = 13
Total CPU secs in IPOPT (w/o function evaluations)   =      0.004
Total CPU secs in NLP function evaluations           =      0.002

EXIT: Optimal Solution Found.
x = 0.7247018392092258 y = 0.5242206029480763
```

1.11 JuMP: Maximium Likelihood

- Let's redo the maximum likelihood example in JuMP.
- Let μ, σ^2 be the unknown mean and variance of a random sample generated from the normal distribution.
- Find the maximum likelihood estimator for those parameters!
- density:

$$f(x_i|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

- Likelihood Function

$$\begin{aligned} L(\mu, \sigma^2) = \Pi_{i=1}^N f(x_i|\mu, \sigma^2) &= \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (x_i-\mu)^2\right) \\ &= \left(\sigma^2 2\pi\right)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^N (x_i-\mu)^2\right) \end{aligned}$$

- Constraints: $\mu \in \mathbb{R}, \sigma > 0$

- log-likelihood:

$$\log L = l = -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \mu)^2$$

- Let's do this in JuMP.

In [6]:
```
#  Copyright 2015, Iain Dunning, Joey Huchette, Miles Lubin, and contributors
#  example modified
using JuMP
using Distributions

distrib = Normal(4.5,3.5)
n = 10000

data = rand(distrib,n);

m = Model(solver=IpoptSolver())

@variable(m, mu, start = 0.0)
@variable(m, sigma >= 0.0, start = 1.0)

@NLobjective(m, Max, -(n/2)*log(2*sigma^2)-sum((data[i] - mu) ^ 2 for i = 1:n)/(2*sigma^

solve(m)
println(" = ", getvalue(mu),", mean(data) = ", mean(data))
println("^2 = ", getvalue(sigma)^2, ", var(data) = ", var(data))
```

```
This is Ipopt version 3.12.4, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:        3

Total number of variables...:        2
                     variables with only lower bounds:        1
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:        0
Total number of inequality constraints...:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  1.7211927e+05 0.00e+00 1.01e+02  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.2147584e+05 0.00e+00 9.58e+01  -1.0 9.27e+00    -  1.00e+00 5.00e-01f  2
```

```
   2  7.4758242e+04 0.00e+00 3.97e+01  -1.0 2.37e-01    -  8.94e-01 1.00e+00f  1
   3  4.9624139e+04 0.00e+00 1.64e+01  -1.0 3.08e-01    -  1.00e+00 1.00e+00f  1
   4  3.6644573e+04 0.00e+00 6.64e+00  -1.0 3.88e-01    -  1.00e+00 1.00e+00f  1
   5  3.0384758e+04 0.00e+00 2.60e+00  -1.0 4.69e-01    -  1.00e+00 1.00e+00f  1
   6  2.7756981e+04 0.00e+00 9.40e-01  -1.0 5.20e-01    -  1.00e+00 1.00e+00f  1
   7  2.6938155e+04 0.00e+00 2.73e-01  -1.7 4.72e-01    -  1.00e+00 1.00e+00f  1
   8  2.6791555e+04 0.00e+00 5.45e-02  -1.7 3.06e-01    -  1.00e+00 1.00e+00f  1
   9  2.6784961e+04 0.00e+00 2.83e-03  -2.5 8.44e-02    -  1.00e+00 1.00e+00f  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  2.6784947e+04 0.00e+00 6.72e-06  -3.8 4.30e-03    -  1.00e+00 1.00e+00f  1
  11  2.6784947e+04 0.00e+00 1.69e-09  -5.7 6.80e-05    -  1.00e+00 1.00e+00f  1
  12  2.6784947e+04 0.00e+00 4.02e-13  -8.6 1.02e-06    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 12

                                   (scaled)                 (unscaled)
Objective...:   8.4800120957055221e+00    2.6784947040164145e+04
Dual infeasibility...:   4.0182611534353866e-13    1.2692070597731996e-09
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   2.5064395506978949e-09    7.9168343001320250e-06
Overall NLP error...:   2.5064395506978949e-09    7.9168343001320250e-06


Number of objective function evaluations             = 18
Number of objective gradient evaluations             = 13
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 12
Total CPU secs in IPOPT (w/o function evaluations)   =      0.006
Total CPU secs in NLP function evaluations           =      0.023

EXIT: Optimal Solution Found.
 = 4.490925092754868, mean(data) = 4.490925092754868
^2 = 12.417569220495565, var(data) = 12.418811091779508
```

References

[1] Iain Dunning, Joey Huchette, and Miles Lubin. JuMP: A modeling language for mathematical optimization. *arXiv:1508.01982 [math.OC]*, 2015.

[2] Jorge Nocedal and Stephen Wright. *Numerical optimization*. Springer Science & Business Media, 2006.