

SciencesPo Computational Economics Spring 2017

Florian Oswald

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1 Numerical Dynamic Programming

Florian Oswald, Sciences Po, 2017

1.1 Intro

- Numerical Dynamic Programming (DP) is widely used to solve dynamic models.
- You are familiar with the technique from your core macro course.
- We will illustrate some ways to solve dynamic programs.
 1. Models with one discrete or continuous choice variable
 2. Models with several choice variables
 3. Models with a discrete-continuous choice combination
- We will go through:
 1. Value Function Iteration (VFI)
 2. Policy function iteration (PFI)
 3. Projection Methods
 4. Endogenous Grid Method (EGM)
 5. Discrete Choice Endogenous Grid Method (DCEGM)

1.2 Dynamic Programming Theory

- Payoffs over time are

$$U = \sum_{t=1}^{\infty} \beta^t u(s_t, c_t)$$

where $\beta < 1$ is a discount factor, s_t is the state, c_t is the control.

- The state (vector) evolves as $s_{t+1} = h(s_t, c_t)$.
- All past decisions are contained in s_t .

1.2.1 Assumptions

- Let $c_t \in C(s_t)$, $s_t \in S$ and assume u is bounded in $(c, s) \in C \times S$.
- Stationarity: neither payoff u nor transition h depend on time.
- Write the problem as

$$v(s) = \max_{s' \in \Gamma(s)} u(s, s') + \beta v(s')$$

- $\Gamma(s)$ is the constraint set (or feasible set) for s' when the current state is s

1.2.2 Existence

Theorem. Assume that $u(s, s')$ is real-valued, continuous, and bounded, that $\beta \in (0, 1)$, and that the constraint set $\Gamma(s)$ is nonempty, compact, and continuous. Then there exists a unique function $v(s)$ that solves the above functional equation.

Proof. [stokeylucas] [4] theorem 4.6.

2 Solution Methods

2.1 Value Function Iteration (VFI)

- Find the fix point of the functional equation by iterating on it until the distance between consecutive iterations becomes small.
- Motivated by the Bellman Operator, and it's characterization in the Continuous Mapping Theorem.

2.2 Discrete DP VFI

- Represents and solves the functional problem in \mathbb{R} on a finite set of grid points only.
- Widely used method.
 - Simple (+)
 - Robust (+)
 - Slow (-)
 - Imprecise (-)
- Precision depends on number of discretization points used.
- High-dimensional problems are difficult to tackle with this method because of the curse of dimensionality.

2.2.1 Deterministic growth model with Discrete VFI

- We have this theoretical model:

$$V(k) = \max_{0 < k' < f(k)} u(f(k) - k') + \beta V(k')$$

$$f(k) = k^\alpha$$

k_0 given

- and we employ the following numerical approximation:

$$V(k_i) = \max_{i'=1,2,\dots,n} u(f(k_i) - k_{i'}) + \beta V(i')$$

- The iteration is then on successive iterates of V : The LHS gets updated in each iteration!

$$\begin{aligned} V^{r+1}(k_i) &= \max_{i'=1,2,\dots,n} u(f(k_i) - k_{i'}) + \beta V^r(i') \\ V^{r+2}(k_i) &= \max_{i'=1,2,\dots,n} u(f(k_i) - k_{i'}) + \beta V^{r+1}(i') \\ &\dots \end{aligned}$$

- And it stops at iteration r if $d(V^r, V^{r-1}) < \text{tol}$
- You choose a measure of *distance*, $d(\cdot, \cdot)$, and a level of tolerance.
- V^r is usually an *array*. So d will be some kind of *norm*.
- maximal absolute distance
- mean squared distance

2.2.2 Exercise 1: Implement discrete VFI

2.3 Checklist

1. Set parameter values
2. define a grid for state variable $k \in [0, 2]$
3. initialize value function V
4. start iteration, repeatedly computing a new version of V .
5. stop if $d(V^r, V^{r-1}) < \text{tol}$.
6. plot value and policy function
7. report the maximum error of both wrt to analytic solution

```
In [1]: alpha      = 0.65
        beta       = 0.95
        grid_max   = 2 # upper bound of capital grid
        n          = 150 # number of grid points
        N_iter     = 3000 # number of iterations
        kgrid      = 1e-6:(grid_max-1e-6)/(n-1):grid_max # equispaced grid
        f(x) = x^alpha # defines the production function f(k)
        tol = 1e-9
```

```
Out[1]: 1.0e-9
```

2.4 Analytic Solution

- If we choose $u(x) = \ln(x)$, the problem has a closed form solution.
- We can use this to check accuracy of our solution.

```
In [2]: ab          = alpha * beta
        c1          = (log(1 - ab) + log(ab) * ab / (1 - ab)) / (1 - beta)
        c2          = alpha / (1 - ab)
```

```

# optimal analytical values
v_star(k) = c1 .+ c2 .* log(k)
k_star(k) = ab * k.^alpha
c_star(k) = (1-ab) * k.^alpha
ufun(x) = log(x)

```

Out[2]: ufun (generic function with 1 method)

In [4]: kgrid[4]

Out[4]: 0.04026943624161074

In [5]: # Bellman Operator

```

# inputs
# `grid`: grid of values of state variable
# `v0`: current guess of value function

# output
# `v1`: next guess of value function
# `pol`: corresponding policy function

#takes a grid of state variables and computes the next iterate of the value function.
function bellman_operator(grid,v0)

    v1 = zeros(n)      # next guess
    pol = zeros(Int,n)  # policy function
    w = zeros(n)       # temporary vector

    # loop over current states
    # current capital
    for (i,k) in enumerate(grid)

        # loop over all possible kprime choices
        for (iprime,kprime) in enumerate(grid)
            if f(k) - kprime < 0 #check for negative consumption
                w[iprime] = -Inf
            else
                w[iprime] = ufun(f(k) - kprime) + beta * v0[iprime]
            end
        end
        # find maximal choice
        v1[i], pol[i] = findmax(w) # stores Value und policy (index of optimal choice)
    end
    return (v1,pol) # return both value and policy function
end

# VFI iterator

```

```

#
## input
# `n`: number of grid points
# output
# `v_next`: tuple with value and policy functions after `n` iterations.
function VFI()
    v_init = zeros(n)      # initial guess
    for iter in 1:N_iter
        v_next = bellman_operator(kgrid,v_init) # returns a tuple: (v1,pol)
        # check convergence
        if maxabs(v_init.-v_next[1]) < tol
            errors = maxabs(v_next[1].-v_star(kgrid))
            perrors = maxabs(kgrid[v_next[2]].-k_star(kgrid))
            println("Found solution after $iter iterations")
            println("maximal value function error = $errors")
            println("maximal policy function error = $perrors")
            return v_next
        elseif iter==N_iter
            warn("No solution found after $iter iterations")
            return v_next
        end
        v_init = v_next[1] # update guess
    end
end

# plot
function plotVFI()
    v = VFI()
    figure("discrete VFI",figsize=(10,5))
    subplot(131)
    plot(kgrid,v[1],color="blue")
    plot(kgrid,v_star(kgrid),color="black")
    xlim(-0.1,grid_max)
    ylim(-50,-30)
    xlabel("k")
    ylabel("value")
    title("value function")

    subplot(132)
    plot(kgrid,kgrid[v[2]])
    plot(kgrid,k_star(kgrid),color="black")
    xlabel("k")
    title("policy function")

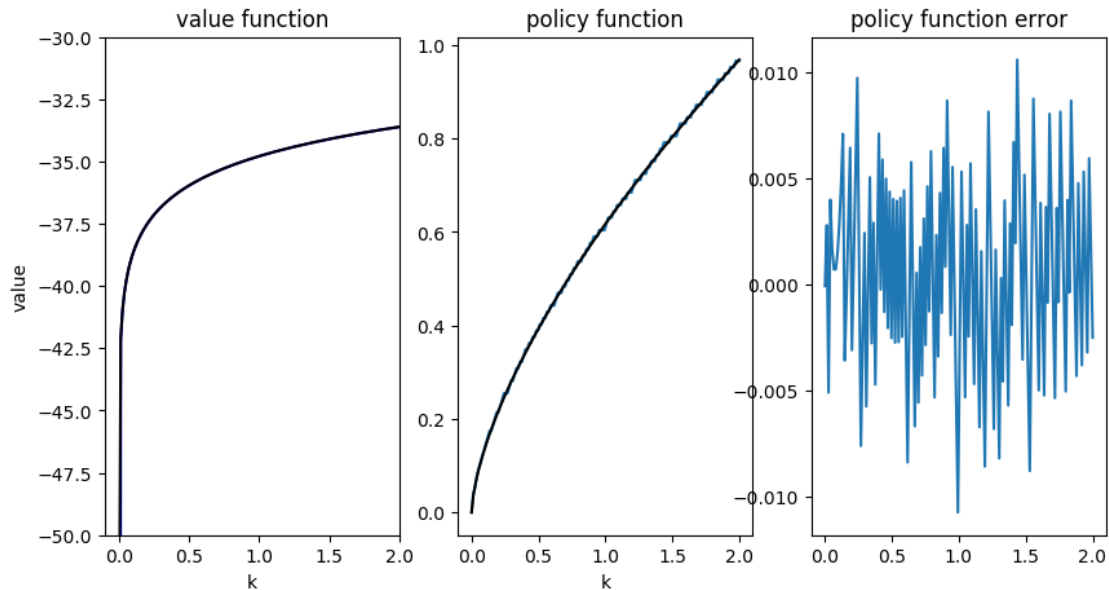
    subplot(133)
    plot(kgrid,kgrid[v[2]].-k_star(kgrid))
    title("policy function error")

```

```

end
using PyPlot
plotVFI()

```



```

Found solution after 448 iterations
maximal value function error = 121.49819145170268
maximal policy function error = 0.010775693497948935

```

Out [5]: PyObject <matplotlib.text.Text object at 0x328494ad0>

2.4.1 Exercise 2: Discretizing only the state space (not control space)

- Same exercise, but now use a continuous solver for choice of k' .
- in other words, employ the following numerical approximation:

$$V(k_i) = \max_{k' \in [0, \bar{k}]} \ln(f(k_i) - k') + \beta V(k')$$

- To do this, you need to be able to evaluate $V(k')$ where k' is potentially off the k grid.
- use `Interpolations.jl` to linearly interpolate V .
 - the relevant object is setup with function `interpolate((grid,), v, Gridded{Linear}())`
- use `Optim::optimize()` to perform the maximization.
 - you have to define an objective function for each k_i
 - do something like `optimize(objective, lb, ub)`

```

In [6]: using Interpolations
using Optim
function bellman_operator2(grid,v0)

    v1 = zeros(n)      # next guess
    pol = zeros(n)     # consumption policy function

    Interp = interpolate((collect(grid),), v0, Gridded{Linear}())

    # loop over current states
    # of current capital
    for (i,k) in enumerate(grid)

        objective(c) = - (log(c) + beta * Interp[f(k) - c])
        # find max of objective between [0,k^alpha]
        res = optimize(objective, 1e-6, f(k)) # Optim.jl
        pol[i] = f(k) - res.minimizer # k'
        v1[i] = -res.minimum
    end
    return (v1,pol) # return both value and policy function
end

function VFI2()
    v_init = zeros(n) # initial guess
    for iter in 1:N_iter
        v_next = bellman_operator2(kgrid,v_init) # returns a tuple: (v1,pol)
        # check convergence
        if maxabs(v_init.-v_next[1]) < tol
            errors = maxabs(v_next[1].-v_star(kgrid))
            perrors = maxabs(v_next[2].-k_star(kgrid))
            println("continuous VFI:")
            println("Found solution after $iter iterations")
            println("maximal value function error = $errors")
            println("maximal policy function error = $perrors")
            return v_next
        elseif iter==N_iter
            warn("No solution found after $iter iterations")
            return v_next
        end
        v_init = v_next[1] # update guess
    end
    return nothing
end

function plotVFI2()
    v = VFI2()
    figure("discrete VFI - continuous control",figsize=(10,5))
    subplot(131)

```

```

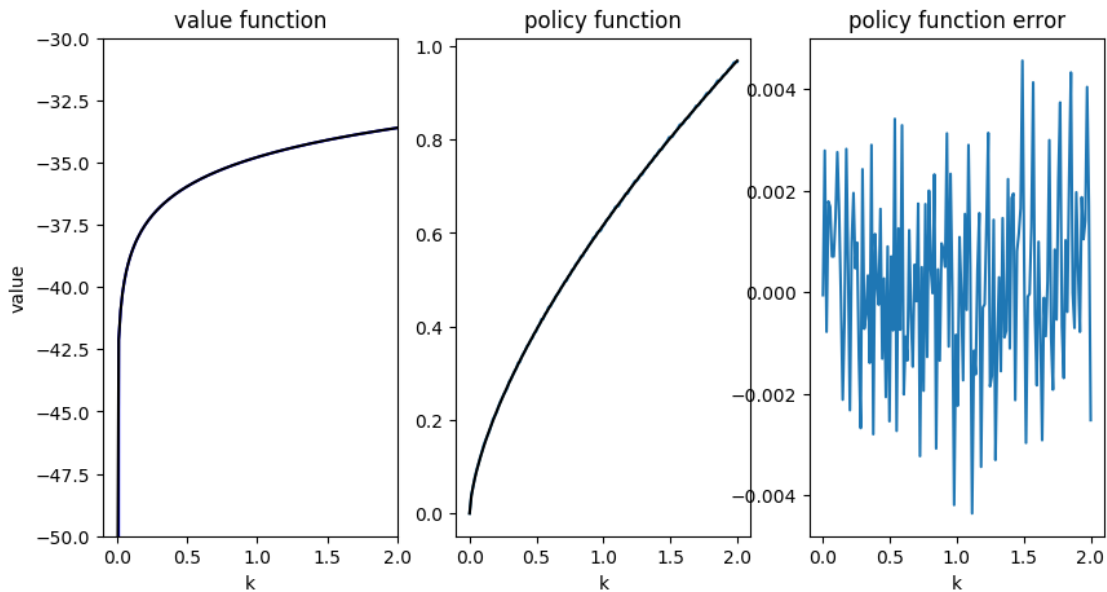
plot(kgrid,v[1],color="blue")
plot(kgrid,v_star(kgrid),color="black")
xlim(-0.1,grid_max)
ylim(-50,-30)
xlabel("k")
ylabel("value")
title("value function")

subplot(132)
plot(kgrid,v[2])
plot(kgrid,k_star(kgrid),color="black")
xlabel("k")
title("policy function")

subplot(133)
plot(kgrid,v[2].-k_star(kgrid))
xlabel("k")
title("policy function error")
end

```

```
plotVFI2()
```



continuous VFI:

Found solution after 439 iterations

maximal value function error = 121.10003370383764

maximal policy function error = 0.004564649814724597

Out[6]: PyObject <matplotlib.text.Text object at 0x32881c9d0>

2.5 Policy Function Iteration

- This is similar to VFI but we now guess successive *policy* functions
- The idea is to choose a new policy p^* in each iteration so as to satisfy an optimality condition. In our example, that would be the Euler Equation.
- We know that the solution to the above problem is a function $c^*(k)$ such that

$$c^*(k) = \arg \max_z u(z) + \beta V(f(k) - z) \quad \forall k \in [0, \infty]$$

- We **don't** directly solve the maximization problem outlined above, but it's first order condition:

$$\begin{aligned} u'(c^*(k_t)) &= \beta u'(c^*(k_{t+1})) f'(k_{t+1}) \\ &= \beta u'[c^*(f(k_t) - c^*(k_t))] f'(f(k_t) - c^*(k_t)) \end{aligned}$$

- In practice, we have to find the zeros of

$$g(k_t) = u'(c^*(k_t)) - \beta u'[c^*(f(k_t) - c^*(k_t))] f'(f(k_t) - c^*(k_t))$$

In [9]: *# Your turn!*

```
using Roots
function policy_iter(grid,c0,u_prime,f_prime)

    c1 = zeros(length(grid))      # next guess
    pol_fun = interpolate((collect(grid),), c0, Gridded{Linear}())

    # loop over current states
    # of current capital
    for (i,k) in enumerate(grid)
        objective(c) = u_prime(c) - beta * u_prime(pol_fun[f(k)-c]) * f_prime(f(k)-c)
        c1[i] = fzero(objective, 1e-10, f(k)-1e-10)
    end
    return c1
end

uprime(x) = 1.0 ./ x
fprime(x) = alpha * x.^(alpha-1)

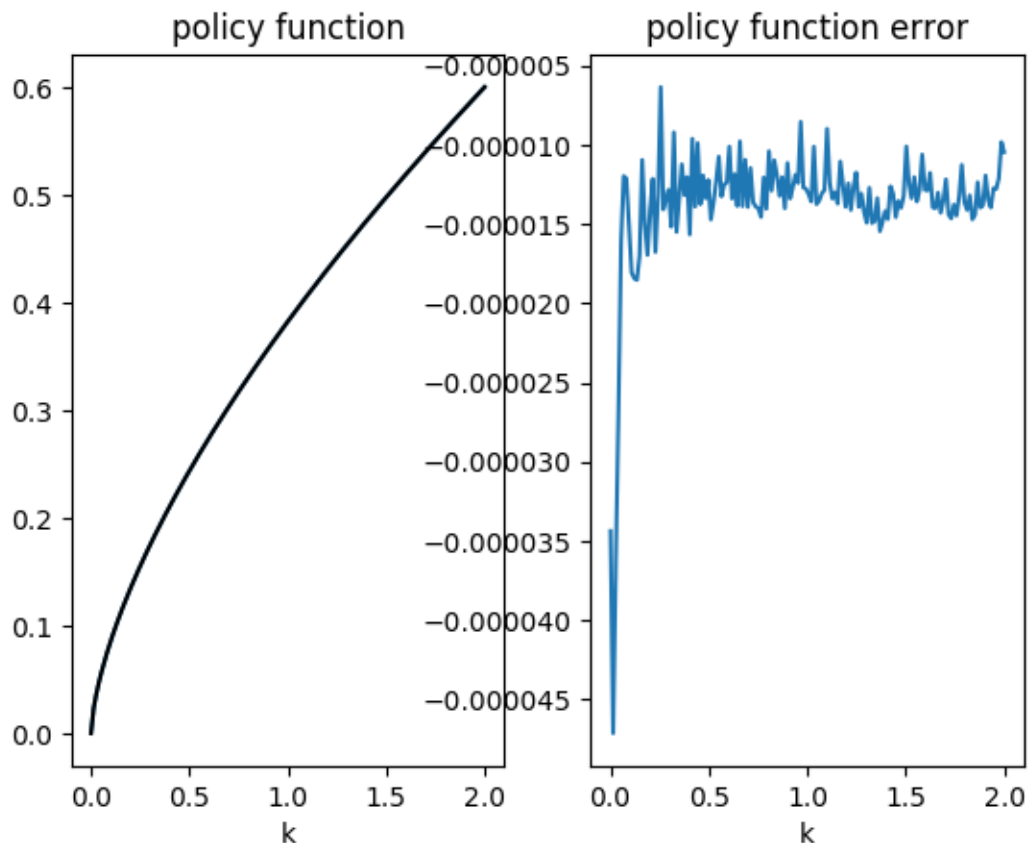
function PFI()
    c_init = kgrid
    for iter in 1:N_iter
        c_next = policy_iter(kgrid,c_init,uprime,fprime)
        # check convergence
        if maxabs(c_init.-c_next) < tol
```

```

        perrors = maxabs(c_next.-c_star(kgrid))
        println("PFI:")
        println("Found solution after $iter iterations")
        println("max policy function error = $perrors")
        return c_next
    elseif iter==N_iter
        warn("No solution found after $iter iterations")
        return c_next
    end
    c_init = c_next # update guess
end
end
function plotPFI()
    v = PFI()
    figure("PFI")
    subplot(121)
    plot(kgrid,v)
    plot(kgrid,c_star(kgrid),color="black")
    xlabel("k")
    title("policy function")

    subplot(122)
    plot(kgrid,v.-c_star(kgrid))
    xlabel("k")
    title("policy function error")
end
plotPFI()

```



PFI:
 Found solution after 39 iterations
 max policy function error = 4.71194965185294e-5

WARNING: Method definition policy_iter(Any, Any, Any, Any) in module Main at In[8]:8 overwritten
 WARNING: Method definition uprime(Any) in module Main at In[8]:24 overwritten at In[9]:19.
 WARNING: Method definition fprime(Any) in module Main at In[8]:25 overwritten at In[9]:20.
 WARNING: Method definition PFI() in module Main at In[8]:29 overwritten at In[9]:23.
 WARNING: Method definition plotPFI() in module Main at In[8]:50 overwritten at In[9]:41.

Out[9]: PyObject <matplotlib.text.Text object at 0x32da2b690>

3 Projection Methods

- Many applications require us to solve for an *unknown function*
 - ODEs, PDEs
 - Pricing functions in asset pricing models

- Consumption/Investment policy functions
- Projection methods find approximations to those functions that set an error function close to zero.

3.1 Example: Growth, again

- We stick to our working example from above.
- We encountered the Euler Equation g for optimality.
- At the true consumption function c^* , $g(k) = 0$.
- We define the following function operator:

$$\begin{aligned} 0 &= u'(c^*(k)) - \beta u'[c^*(f(k) - c^*(k))]f'(f(k) - c^*(k)) \\ &\equiv (\mathcal{N}(\cdot^*))(k) \end{aligned}$$

- The Equilibrium solves the operator equation

$$0 = \mathcal{N}(c^*)$$

3.1.1 Projection Method example

1. create an approximation to c^* :

- find

$$\bar{c} \equiv \sum_{i=0}^n a_i k^i$$

which nearly solves

$$\mathcal{N}(\bar{c}) = 0$$

2. Compute Euler equation error function:

$$g(k; a) = u'(\bar{c}(k)) - \beta u'[\bar{c}(f(k) - \bar{c}(k))]f'(f(k) - \bar{c}(k))$$

3. Choose a to make $g(k; a)$ small in some sense

- *small in some sense*:
 - Least-squares: minimize sum of squared errors

$$\min_a \int g(k; a)^2 dk$$

- Galerkin: zero out weighted averages of Euler errors
- Collocation: zero out Euler equation errors at grid $k \in \{k_1, \dots, k_n\}$:

$$P_i(a) \equiv g(k_i; a) = 0, i = 1, \dots, n$$

3.1.2 General Projection Method

1. Express solution in terms of unknown function

$$\mathcal{N}(h) = 0$$

where $h(x)$ is the equilibrium function at state x

2. Choose a space for approximation
3. Find \bar{h} which nearly solves

$$\mathcal{N}(\bar{h}) = 0$$

3.1.3 Projection method exercise

- suppose we want to find effective supply of an oligopolistic firm in cournot competition.
- We want to know $q = S(p)$, how much is supplied at each price p .
- This function is characterized as

$$p + \frac{S(p)}{D'(p)} - MC(S(p)) = 0, \forall p > 0$$

- Take $D(p) = p^{-\eta}$ and $MC(q) = \alpha\sqrt{q} + q^2$.
- Our task is to solve for $S(p)$ in

$$p - \frac{S(p)p^{\eta+1}}{\eta} - \alpha\sqrt{S(p)} - S(p)^2 = 0, \forall p > 0$$

- No closed form solution. But collocation works!

TASK

1. solve for $S(p)$ by collocation
2. Plot residual function
3. Plot resulting $mS(p)$ together with market demand and $m = 1, 10, 20$ for market size.

```
In [ ]: using CompEcon
        using NLSolve
        function proj(n=25)
            alpha = 1.0
            eta    = 1.5
            a      = 0.1
            b      = 3.0
            basis = fundefn(:cheb,n,a,b) # cheby basis
            p     = funnode(basis)[1]   # collocation points
            c0 = ones(n)*0.3 # starting value is crucial!

            # your turn!
            function resid(c::Vector,result::Vector,p,basis,alpha,eta)
                q = funeval(c,basis,p)[1]
```

```

    q2 = zeros(q)
    for i in eachindex(q2)
        if q[i] < 0
            q2[i] = -20.0
        else
            q2[i] = sqrt(q[i])
        end
    end
    result[:] = p.+ q .*((-1/eta)*p.^(eta+1)) .- alpha*q2 .- q.^2
end

# need a closure around it
f_closure(x::Vector,r::Vector) = resid!(x,r,p,basis,alpha,eta)
res = nlsolve(f_closure,c0)

# plot residual function
x = collect(linspace(a,b,501))
y = similar(x)
resid(res.zero,y,x,basis,alpha,eta);
figure()
subplot(121)
plot(x,y)
xlabel("price")
ylabel("residual")
title("residual function")

subplot(122)
y = funeval(res.zero,basis,x)[1]
plot(y,x,label="supply 1")
plot(10*y,x,label="supply 10")
plot(20*y,x,label="supply 20")
d = x.^(-eta)
plot(d,x,label="demand")
legend()
ylabel("price")
xlabel("quantity")
title("Equilibrium")

# plot mS(p) and D(p) for m=1,10,20

return res
end

```

4 Endogenous Grid Method (EGM)

- Fast, elegant and precise method to solve consumption/savings problems

- One continuous state variable
- One continuous control variable

$$V(M_t) = \max_{0 < c < M_t} u(c) + \beta EV_{t+1}(R(M_t - c) + y_{t+1})$$

- Here, M_t is cash in hand, all available resources at the start of period t
 - For example, assets plus income.
- $A_t = M_t - c_t$ is end of period assets
- y_{t+1} is stochastic next period income.
- R is the gross return on savings, i.e. $R = 1 + r$.
- utility function can be of many forms, we only require twice differentiable and concave.

4.1 EGM after [carroll2006method]

- [carroll2006method] [1] introduced this method.
- The idea is as follows:
 - Instead of using non-linear root finding for optimal c (see above)
 - fix a grid of possible end-of-period asset levels A_t
 - use structure of model to find implied beginning of period cash in hand.
 - We use euler equation and envelope condition to connect M_{t+1} with c_t

4.1.1 Recall Traditional Methods: VFI and Euler Equation

- Just to be clear, let us repeat what we did in the beginning of this lecture, using the M_t notation.

$$V(M_t) = \max_{0 < c < M_t} u(c) + \beta EV_{t+1}(R(M_t - c) + y_{t+1})$$

$$M_{t+1} = R(M_t - c) + y_{t+1}$$

4.1.2 VFI

1. Define a grid over M_t .
2. In the final period, compute

$$V_T(M_T) = \max_{0 < c < M_t} u(c)$$

3. In all preceding periods t , do

$$V_t(M_t) = \max_{0 < c_t < M_t} u(c_t) + \beta EV_{t+1}(R(M_t - c_t) + y_{t+1})$$

4. where optimal consumption is

$$c_t^*(M_t) = \arg \max_{0 < c_t < M_t} u(c_t) + \beta EV_{t+1}(R(M_t - c_t) + y_{t+1})$$

4.1.3 Euler Equation

- The first order condition of the Bellman Equation is

$$\frac{\partial V_t}{\partial c_t} = 0$$

$$u'(c_t) = \beta E \left[\frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \right] \quad (FOC)$$

- By the Envelope Theorem, we have that

$$\frac{\partial V_t}{\partial M_t} = \beta E \left[\frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \right]$$

by FOC

$$\frac{\partial V_t}{\partial M_t} = u'(c_t)$$

true in every period:

$$\frac{\partial V_{t+1}}{\partial M_{t+1}} = u'(c_{t+1})$$

- Summing up, we get the Euler Equation:

$$u'(c_t) = \beta E [u'(c_{t+1})R]$$

4.1.4 Euler Equation Algorithm

1. Fix grid over M_t
2. In the final period, compute

$$c_T^*(M_T) = \arg \max_{0 < c_T < M_T} u(c_T)$$

3. With optimal $c_{t+1}^*(M_{t+1})$ in hand, backward recurse to find c_t from

$$u'(c_t) = \beta E [u'(c_{t+1}^*(R(M_t - c_t) + y_{t+1}))R]$$

4. Notice that if M_t is small, the euler equation does not hold.
 - In fact, the euler equation would prescribe to *borrow*, i.e. set $M_t < 0$. This is ruled out.
 - So, one needs to tweak this algorithm to check for this possibility
5. Homework.

4.2 The EGM Algorithm

Starts in period T with $c_T^* = M_T$. For all preceding periods:

1. Fix a grid of *end-of-period* assets A_t
2. Compute all possible next period cash-in-hand holdings M_{t+1}

$$M_{t+1} = R * A_t + y_{t+1}$$

- for example, if there are n values in A_t and m values for y_{t+1} , we have $\dim(M_{t+1}) = (n, m)$
3. Given that we know optimal policy in $t + 1$, use it to get consumption at each M_{t+1}

$$c_{t+1}^*(M_{t+1})$$

4. Invert the Euler Equation to get current consumption compliant with an expected level of cash-on-hand, given A_t

$$c_t = (u')^{-1} (\beta E [u'(c_{t+1}^*(M_{t+1}))R|A_t])$$

5. Current period *endogenous* cash on hand just obeys the accounting relation

$$M_t = c_t + A_t$$

#Core of a simple implementation

```
type iidModel <: Model
```

```
  # computation grids
```

```
  avec::Vector{Float64}
```

```
  yvec::Vector{Float64}    # income support
```

```
  ywgt::Vector{Float64}    # income weights
```

```
  # intermediate objects (na,ny)
```

```
  m1::Array{Float64,2}     # next period cash on hand (na,ny)
```

```
  c1::Array{Float64,2}     # next period consumption
```

```
  ev::Array{Float64,2}
```

```
  # result objects
```

```
  C::Array{Float64,2}      # consumption function on (na,nT)
```

```
  S::Array{Float64,2}      # savings function on (na,nT)
```

```
  M::Array{Float64,2}      # endogenous cash on hand on (na,nT)
```

```
  V::Array{Float64,2}      # value function on (na,nT). Optional.
```

```
  Vzero::Array{Float64,1}  # value of saving zero
```

```
end
```

```
function EGM!(m::iidModel,p::Param)
```

```
  # final period: consume everything.
```

```
  m.M[:,p.nT] = m.avec
```

```
  m.C[:,p.nT] = m.avec
```

```
  m.C[m.C[:,p.nT].<p.cfloor,p.nT] = p.cfloor
```

```
  # preceding periods
```

```
  for it in (p.nT-1):-1:1
```

```
    # interpolate optimal consumption from next period on all cash-on-hand states
```

```

# using C[:,it+1] and M[:,it+1], find c(m,it)

tmpx = [0.0; m.M[:,it+1] ]
tmpy = [0.0; m.C[:,it+1] ]
for ia in 1:p.na
    for iy in 1:p.ny
        m.c1[ia+p.na*(iy-1)] = linearapprox(tmpx,tmpy,m.m1[ia+p.na*(iy-1)],1,p.na)
        # m.c1[ia,iy] = linearapprox(tmpx,tmpy,m.m1[ia,iy],1,p.na) # equivalent
    end
end

# get expected marginal value of saving: RHS of euler equation
# beta * R * E[ u'(c_{t+1}) ]
Eu = p.R * p.beta .* up(m.c1,p) * m.ywgt

# get optimal consumption today from euler equation: invert marginal utility
m.C[:,it] = iup(Eu,p)

# floor consumption
m.C[m.C[:,it].<p.cfloor,it] = p.cfloor

# get endogenous grid today
m.M[:,it] = m.C[:,it] .+ m.avec
end
end

```

4.3 Discrete Choice EGM

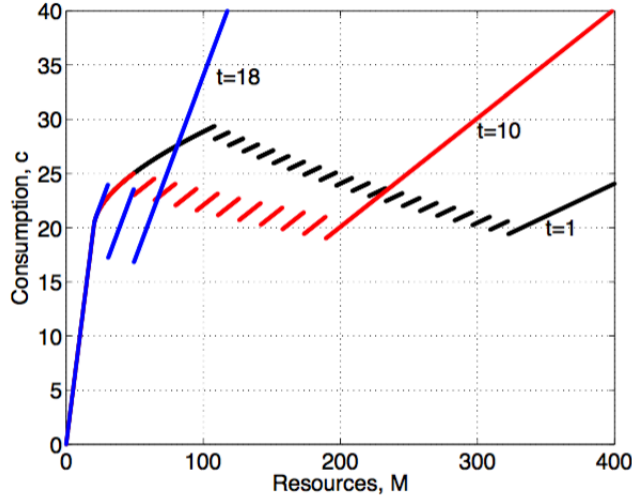
- This is a method developed by Fedor Iskhakov, Thomas Jorgensen, John Rust and Bertel Schjerning.
- Reference: [IskhakovRust2014] [3]
- Suppose we have several discrete choices (like "work/retire"), combined with a continuous choice in each case (like "how much to consume given work/retire").
- Let $d = 0$ mean to retire.
- Write the problem of a worker as

$$\begin{aligned}
 V_t(M_t) &= \max [v_t(M_t|d_t = 0), v_t(M_t|d_t = 1)] \\
 &\text{with} \\
 v_t(M_t|d_t = 0) &= \max_{0 < c_t < M_t} u(c_t) + \beta EW_{t+1}(R(M_t - c_t)) \\
 v_t(M_t|d_t = 1) &= \max_{0 < c_t < M_t} u(c_t) - 1 + \beta EV_{t+1}(R(M_t - c_t) + y_{t+1})
 \end{aligned}$$

- The problem of a retiree is

$$W_t(M_t) = \max_{0 < c_t < M_t} u(c_t) + \beta EW_{t+1}(R(M_t - c_t))$$

- Our task is to compute the optimal consumption functions $c_t^*(M_t|d_t = 0)$, $c_t^*(M_t|d_t = 1)$



[@iskhakovRust2014] figure 1

4.3.1 Problems with Discrete-Continuous Choice

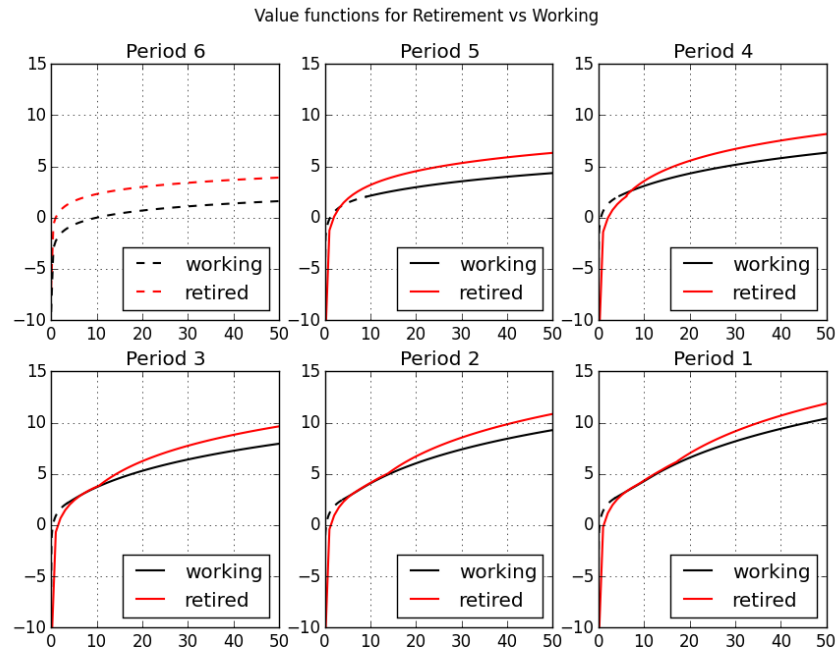
- Even if all conditional value functions v are concave, the *envelope* over them, V , is in general not.
- [@clausenenvelope] [2] show that there will be a kink point \bar{M} such that

$$v_t(\bar{M}|d_t = 0) = v_t(\bar{M}|d_t = 1)$$

- We call any such point a **primary kink** (because it refers to a discrete choice in the **current period**)
- V is not differentiable at \bar{M} .
- However, it can be shown that both left and right derivatives exist, with

$$V^-(\bar{M}) < V^+(\bar{M})$$

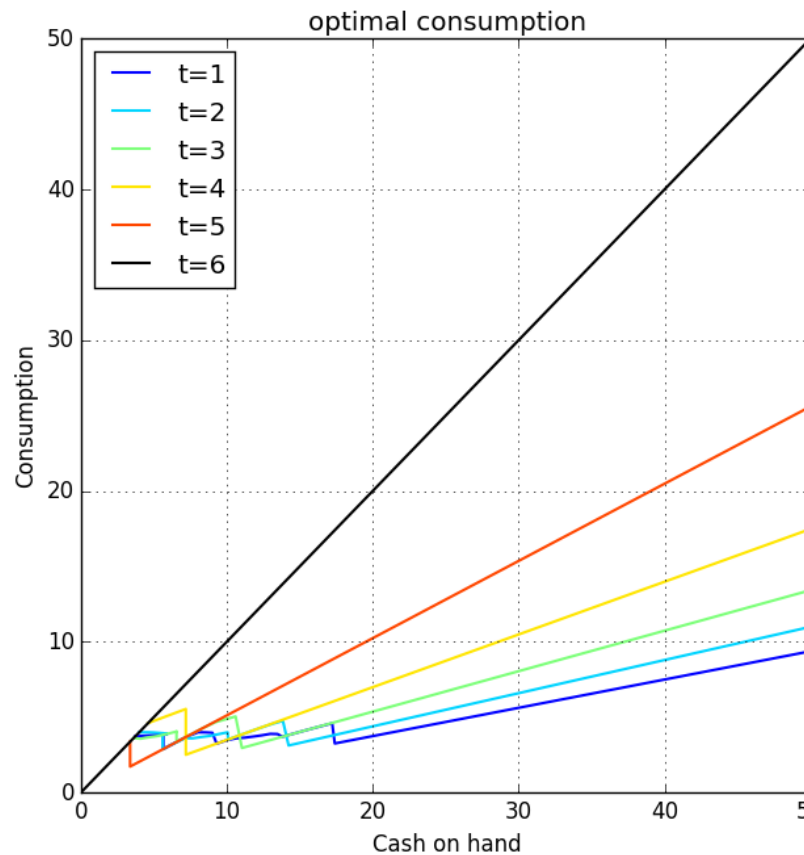
- Given that the value of the derivative changes discretely at \bar{M}_t , the value function in $t - 1$ will exhibit a discontinuity as well:
 - v_{t-1} depends on V_t .
 - Tracing out the optimal choice of c_{t-1} implies next period cash on hand M_t , and as that hits \bar{M}_t , the derivative jumps.
 - The derivative of the value function determines optimal behaviour via the Euler Equation.
 - We call a discontinuity in v_{t-1} arising from a kink in V_t a **secondary kink**.
- The kinks propagate backwards.
- [@iskhakovRust2014] [3] provide an analytic example where one can compute the actual number of kinks in period 1 of T .
- Figure 1 in [@clausenenvelope]:



github/floswald

4.3.2 Kinks

- Refer back to the work/retirement model from before.
- 6 period implementation of the DC-EGM method:
- [Iskhakov @ cemmap 2015: Value functions in T-1](#)
- [Iskhakov @ cemmap 2015: Value functions in T-2](#)
- [Iskhakov @ cemmap 2015: Consumption function in T-2](#)



- Optimal consumption in 6 period model:

4.3.3 The Problem with Kinks

- Relying on fast methods that rely on first order conditions (like euler equation) will fail.
- There are multiple zeros in the Euler Equation, and a standard Euler Equation approach is not guaranteed to find the right one.
- picture from Fedor Iskhakov's master class at [cemmap 2015](#):

4.3.4 DC-EGM Algorithm

1. Do the EGM step for each discrete choice d
2. Compute d -specific consumption and value functions
3. compare d -specific value functions to find optimal switch points
4. Build envelope over d -specific consumption functions with knowledge of which optimal d applies where.

4.3.5 But EGM relies on the Euler Equation?!

- Yes.
- An important result in [[@clausenenvelope](#)] is that the Euler Equation is still the necessary condition for optimal consumption
 - Intuition: marginal utility differs greatly at $\epsilon + \bar{M}$.

- No economic agent would ever locate at \bar{M} .
- This is different from saying that a procedure that tries to find the zeros of the Euler Equation would still work.
 - this will pick the wrong solution some times.
- EGM finds **all** solutions.
 - There is a procedure to discard the "wrong ones". Proof in [iskhakovRust2014]

4.3.6 Adding Shocks

- This problem is hard to solve with standard methods.
- It is hard, because the only reliable method is VFI, and this is not feasible in large problems.
- Adding shocks to non-smooth problems is a widely used remedy.
 - think of "convexifying" in game theoretic models
 - (Add a lottery)
 - Also used a lot in macro
- Adding shocks does indeed help in the current model.
 - We add idiosyncratic taste shocks: Type 1 EV.
 - Income uncertainty:
 - In general, the more shocks, the more smoothing.
- The problem becomes

$$V_t(M_t) = \max [v_t(M_t|d_t = 0) + \sigma_\epsilon \epsilon_t(0), v_t(M_t|d_t = 1) + \sigma_\epsilon \epsilon_t(1)]$$

$$v_t(M_t|d_t = 1) = \max_{0 < c_t < M_t} \log(c_t) - 1 + \beta \int EV_{t+1}(R(M_t - c_t) + y\eta_{t+1})f(d\eta_{t+1})$$

where the value for retirees stays the same.

4.3.7 Adding Shocks

4.3.8 Full DC-EGM

- Needs to discard *false* solutions.
- Criterion:
 - grid in A_t is **increasing**
 - Assuming concave utility function, the function

$$A(M|d) = M - c(M|d)$$

is **monotone non-decreasing**

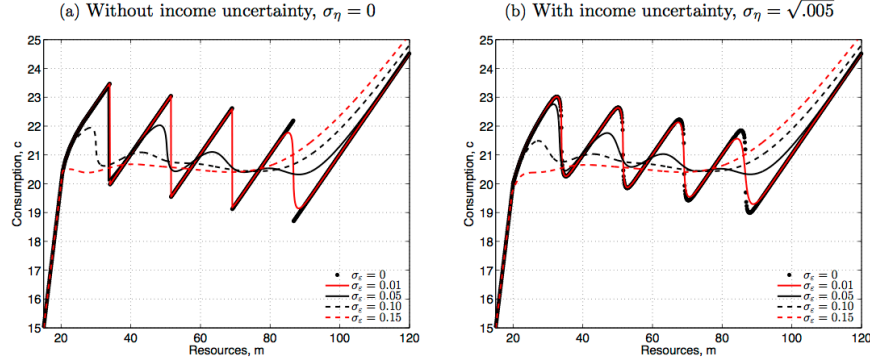
- This means that, if you go through A_i , and find that

$$M_t(A^j) < M_t(A^{j-1})$$

you know you entered a non-concave region

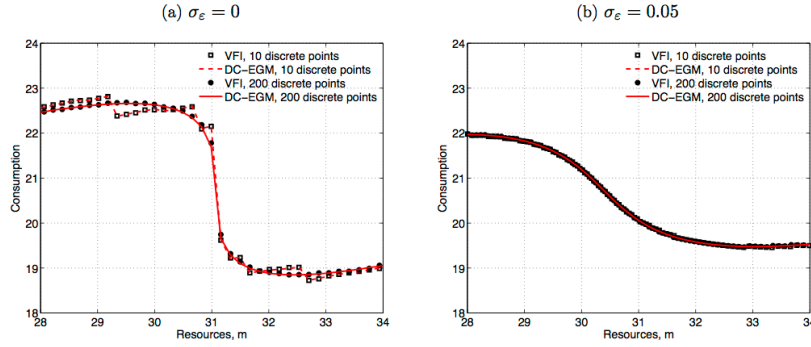
- The Algorithm goes through the upper envelope and *prunes* the *inferior* points M from the endogenous grids.
- Precise details of Algorithm in paper.
- Julia implementation on [floswald/ConsProb.jl](#)

Figure 2: Optimal Consumption Rules for Agent Working Today ($d_{t-1} = 1$).



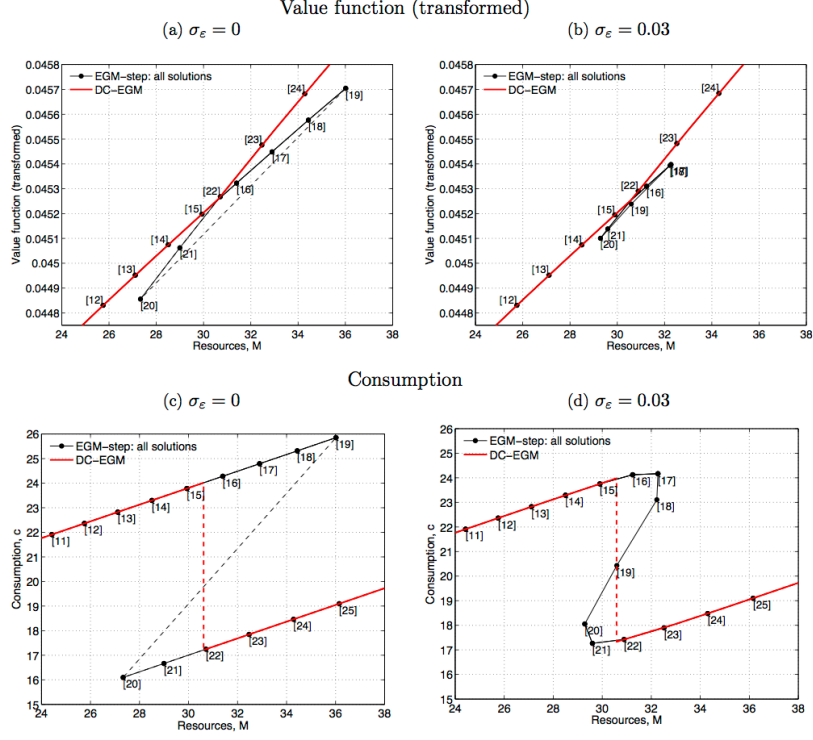
Notes: The plots show optimal consumption rules of the worker who decides to continue working in the consumption-savings model with retirement in period $t = T - 5$ for a set of taste shock scales σ_ϵ in the absence of income uncertainty, $\sigma_\eta = 0$, (left panel) and in presence of income uncertainty, $\sigma_\eta = \sqrt{.005}$, (right panel). The rest of the model parameters are $R = 1$, $\beta = 0.98$, $y = 20$.

Figure 3: Artificial Discontinuities in Consumption Functions, $\sigma_\eta^2 = 0.01$, $t = T - 3$.



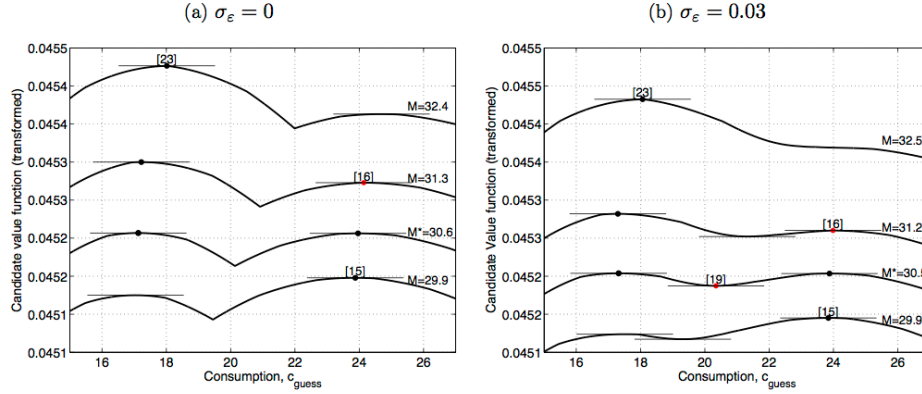
Notes: Figure 3 illustrates how the number of discrete points used to approximate expectations regarding future income affects the consumption functions from value function iteration (VFI) and the DC-EGM. Panel (a) illustrates how using few (10) discrete equiprobable points to approximate expectations produce severe approximation error when there is no taste shocks. Panel (b) illustrates how moderate smoothing ($\sigma_\epsilon = .05$) significantly reduces this approximation error.

Figure 4: Non-concave regions and the elimination of the secondary kinks in DC-EGM.



Notes: The plots illustrate the output from the EGM-step of the DC-EGM algorithm (Algorithm 1) in a non-concave region. The dots are indexed by the index j of the ascending grid over the end-of-period wealth $\vec{A} = \{A^1, \dots, A^G\}$ where $A^j > A^{j-1}, \forall j \in \{2, \dots, G\}$. The connecting lines show the d_t -specific value functions $v_t(\vec{M}_t|d_t)$ and the consumption function $c_t(\vec{M}_t|d_t)$ linearly interpolated on the endogenous grid \vec{M}_t . computed on this grid are the outputs. The left panels illustrate the deterministic case without taste shocks, while in the right panels $\sigma_\varepsilon = 0.03$. The “true” solution, after applying the DC-EGM algorithm is illustrated with a solid red line. Dashed lines illustrate discontinuities. The solution is based on $G = 70$ grid points in \vec{A} , $R = 1$, $\beta = 0.98$, $y = 20$, $\sigma_\eta = 0$.

Figure 5: Local maxima and multiple solutions of the Euler equation.



Notes: The figure plots the maximand of the equation (10), which defines the discrete choice specific value function $v_t(M_t|d_t = 1)$, for the case of $\sigma_\varepsilon = 0$ (panel a) and $\sigma_\varepsilon = 0.03$ (panel b). Horizontal lines indicate the critical points found or approximated by the EGM step of DC-EGM algorithm. The points are indexed with the same indexes as in Figure 4 and the black dots represent global maxima. Model parameters are identical to those of Figure 4.

[@iskhakovRust2014] figure 4

References

- [1] Christopher D Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics letters*, 91(3):312–320, 2006.
- [2] A. Clausen and C. Strub. Envelope theorems for non-smooth and non-concave optimization. <https://andrewclausen.net/research.html>, 2013.
- [3] Fedor Iskhakov, John Rust, Bertel Schjerning, and Thomas Jorgensen. Estimating Discrete-Continuous Choice Models: Endogenous Grid Method with Taste Shocks. *SSRN working paper*, 2014.
- [4] Nancy Stokey and R Lucas. *Recursive Methods in Economic Dynamics (with E. Prescott)*. Harvard University Press, 1989.