optimization

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1 Computational Economics: Unconstrained Optimization

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1.1 Some Taxonomy and Initial Examples

- In most of the examples to follow, we talk about *minimization* of a function f. Everything we do also applies to maximization, since $\min_x f(x) = \max_x -f(x)$.
- Here is a generic optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } \begin{array}{ll} c_i(x) = 0, & i \in E \\ c_i(x) \geq 0, & i \in I \end{array}$$

- This is a general way of writing an optimization problem. E are all indices as equality constraints, I are all inequality constraints.
- An example of such a problem might be

$$\min (x_1 - 2)^2 + (x_2 - 1)^2 \text{ s.t. } \begin{array}{l} x_1^2 - x_2 \leq 0 \\ x_1 + x_2 \leq 2 \end{array}$$

- Here is a picture of that problem taken from the textbook [@nocedal-wright] (for copyright reasons, I cannot show this in the online version of the slides.):

1.2 Kinds of problems considered

- Don't talk about stochastic optimization methods:
  - Simluated Annealing
  - MCMC
  - other Stochastic Search Methods
  - A gentle introduction is [@casella-R]

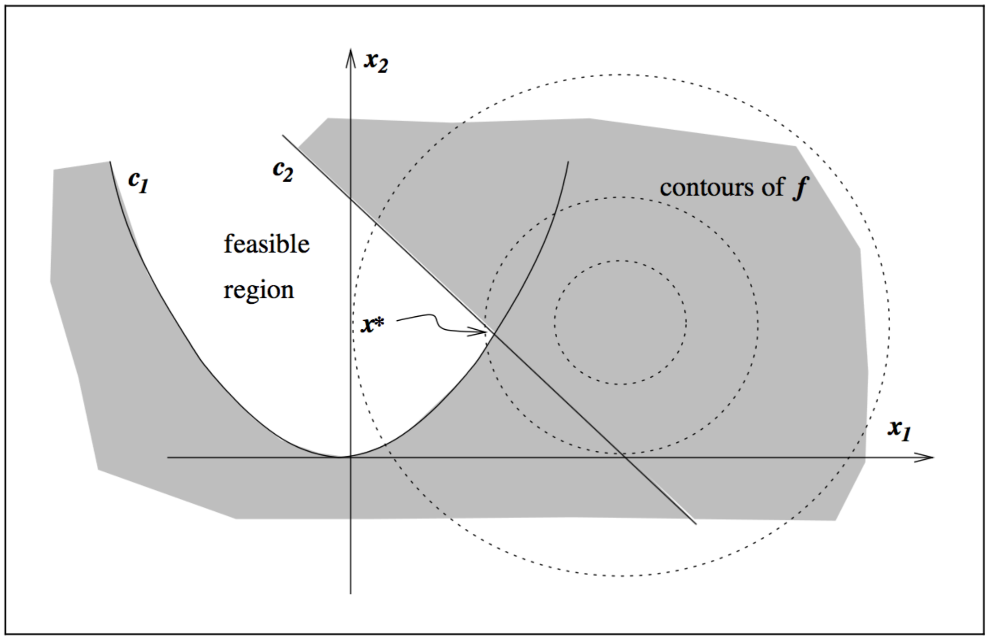


Figure 1.1 in [@nocedal-wright]

1.3 Transportation Problem

A chemical company has two factories F_1, F_2 and a dozen retail outlets $R_1, \ldots, R_{12}$. Each factory i can produce at most a_i tons of output each week. Each retail outlet j has a weekly demand of b_j tons per week. The cost of shipping from F_i to R_j is given by c_{ij}. How much of the product to ship from each factory to each outlet, minimize cost, and satisfy all constraints? let's call x_{ij} the number of tons shipped from i to j.

- A mathematical formulation of this problem is

$$
\begin{aligned}
\min \sum_{ij} c_{ij} x_{ij} \\
\text{subject to} \sum_{j=1}^{12} x_{ij} \leq a_i, \quad i = 1,2 \\
\sum_{i=1}^{2} x_{ij} \geq b_j, \quad j = 1, \ldots, 12 \\
x_{ij} \geq 0, \quad i = 1,2, j = 1, \ldots, 12
\end{aligned}
$$

- This is called a *linear programming* problem, because both objective function and all constrains are linear.
- With any of those being nonlinear, we would call this a non-linear problem.

1.4 Constrained vs Unconstrained

- There are many applications of both in economics.

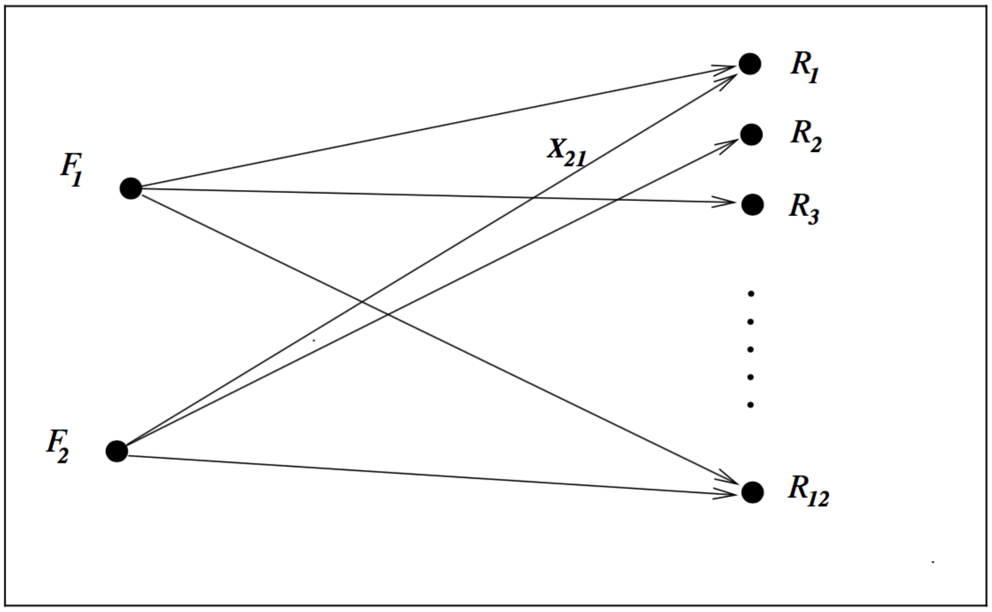


Figure 1.2 in [@nocedal-wright]

- Unconstrained: maximimum likelihood
- Constrained: MPEC
- It is sometimes possible to transform a constrained problem into an unconstrained one.

1.5 Convexity

- Convex problems are easier to solve.
- What is convex?

  A set $S \in \mathbb{R}^n$ is convex if the straight line segment connecting any two points in S lies entirely inside S. A function f is a convex function, if its domain S is a convex set, and for any two points $x, y \in S$, we have that

  $$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

  for all $\alpha \in [0, 1]$

- Simple instances of convex sets are the unit ball $\{y \in \mathbb{R}^n, \|y\|_2 \leq 1\}$, and any set defined by linear equalities and inequalities.
- *convex Programming* describes a special case of the introductory minimizatin problem where
  - the objective function is convex,
  - the equality constrains are linear, and
  - the inequality constraints are concave.

1.6 Optimization Algorithms

- All of the algorithms we are going to see employ some kind of *iterative* proceedure.
- They try to improve the value of the objective function over successive steps.

- The way the algorithm goes about generating the next step is what distinguishes algorithms from one another.
  - Some algos only use the objective function
  - Some use both objective and gradients
  - Some add the Hessian
  - and many variants more

1.7 Desirable Features of any Algorithm

- Robustness: We want good performance on a wide variety of problems in their class, and starting from *all* reasonable starting points.
- Efficiency: They should be fast and not use an excessive amount of memory.
- Accuracy: They should identify the solution with high precision.

1.8 A Word of Caution

- You should **not** normally attempt to write a numerical optimizer for yourself.
- Entire generations of Applied Mathematicians and other numerical pro's have worked on those topics before you, so you should use their work.
  - Any optimizer you could come up with is probably going to perform below par, and be highly likely to contain mistakes.
  - Don't reinvent the wheel.
- That said, it's very important that we understand some basics about the main algorithms, because your task is **to choose from the wide array of available ones**.

2 Unconstrained Optimization: What is a solution?

- A typical unconstrained optimization problem will look something like this:

$$\min_x f(x), \quad x \in \mathbb{R}^n$$

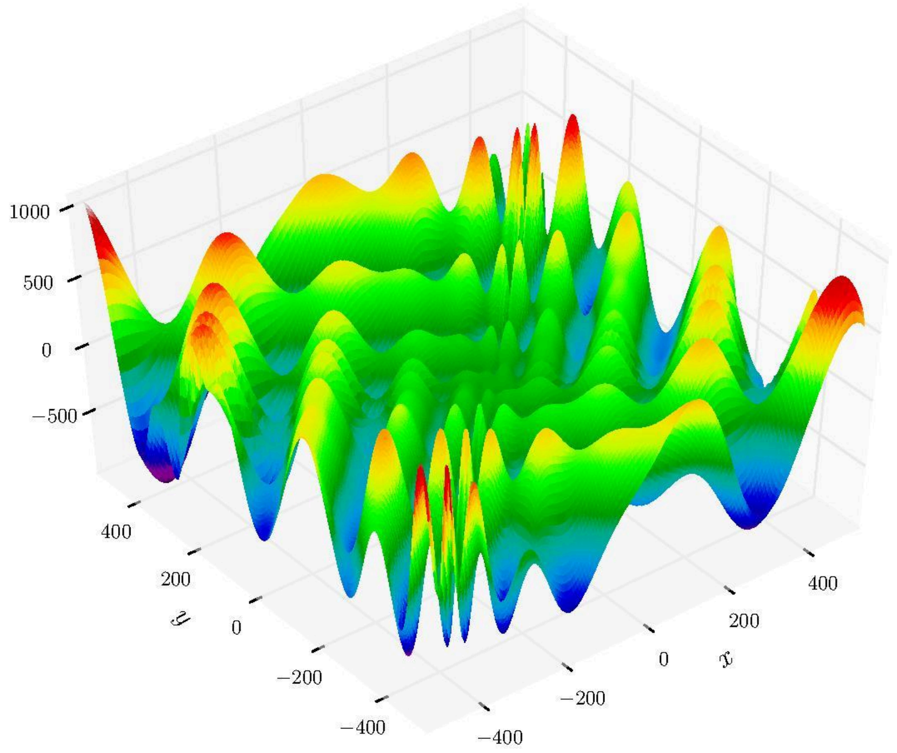
  and where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a smooth function.
- In general, we would always like to find a *global* minimizer, i.e. a point

$$x^* \text{ where } f(x^*) \leq f(x) \quad \forall x$$

- Since our algorithm is not going to visit many points in $\mathbb{R}^n$ (or so we hope), we can never be totally sure that we find a global optimizer.
- Most optimizers can only find a *local* minimizer. That is a point

$$x^* \text{ where } f(x^*) \leq f(x) \quad \forall x \in \mathcal{N}$$

  where $\mathcal{N}$ is a neighborhood around x^*.



Global min at $f(512, 404.2319)$. By Gaortizg GFDL or CC BY-SA 3.0, via Wikimedia Commons

2.1 Global minization can be very hard sometimes.

2.2 (Unconstrained) Optimization in `Julia`

- Umbrella Organisation: `http://www.juliaopt.org`
  - We will make ample use of this when we talk about constrained optimsation.
  - The Julia Interface to the very well established C-Library NLopt is called `NLopt.jl`. One could use `NLopt` without constraints in an unconstrained problem.
- `Roots.jl`: Simple algorithms that find the zeros of a univariate function.
- Baseline Collection of unconstrained optimization algorithms: `Optim.jl`

2.3 Introducing `Optim.jl`

- Multipurpose unconstrained optimization package
  - provides 8 different algorithms with/without derivatives
  - univariate optimization without derivatives

2.4 The Golden Ratio or Bracketing Search for 1D problems

- A derivative-free method
- a Bracketing method
  - find the local minimum of f on $[a, b]$
  - select 2 interior points c, d such that $a < c < d < b$

    * $f(c) \leq f(d) \implies$ min must lie in $[a, d]$. replace b with d, start again with $[a, d]$
    * $f(c) > f(d) \implies$ min must lie in $[c, b]$. replace a with c, start again with $[c, b]$
  - how to choose b, d though?
  - we want the length of the interval to be independent of whether we replace upper or lower bound
  - we want to reuse the non-replaced point from the previous iteration.
  - these imply the golden rule:
  - new point $x_i = a + \alpha_i(b - a)$, where $\alpha_1 = \frac{3-\sqrt{5}}{2}, \alpha_2 = \frac{\sqrt{5}-1}{2}$
  - α_2 is known as the *golden ratio*, well known for it's role in renaissance art.

2.4.1 Bracketing Search in Julia

- The package `Optim.jl` provides an implementation of "Brent's Method" as well as the golden section search:

```
In [9]: using Plots
        using Optim
        f(x) = exp(x) - x^4
        minf(x) = -f(x)
        brent = optimize(minf,0,2,Brent())
        golden = optimize(minf,0,2,GoldenSection())
        println("brent = $brent")
        println("golden = $golden")
        plot(f,0,2)
```

```
brent = Results of Optimization Algorithm
 * Algorithm: Brent's Method
 * Search Interval: [0.000000, 2.000000]
 * Minimizer: 8.310315e-01
 * Minimum: -1.818739e+00
 * Iterations: 12
 * Convergence: max(|x - x_upper|, |x - x_lower|) <= 2*(1.5e-08*|x|+2.2e-16): true
 * Objective Function Calls: 13


WARNING: Method definition f(Any) in module Main at In[1]:2 overwritten at In[9]:3.
WARNING: Method definition minf(Any) in module Main at In[2]:3 overwritten at In[9]:4.


golden = Results of Optimization Algorithm
 * Algorithm: Golden Section Search
 * Search Interval: [0.000000, 2.000000]
 * Minimizer: 8.310315e-01
 * Minimum: -1.818739e+00
 * Iterations: 37
 * Convergence: max(|x - x_upper|, |x - x_lower|) <= 2*(1.5e-08*|x|+2.2e-16): true
 * Objective Function Calls: 38
```

```
In [11]: # how well does this do with many local minima?
         fun(x) = exp(x) - x^4 +sin(40*x)
         minf(x) = -fun(x)
         grid = collect(0:0.000001:2);
         # fun.(grid)
         # v,idx  = findmax(fun.(grid))
         v,idx  = findmax(Float64[fun.(x) for x in grid])
         println("grid maximizer is $(grid[idx])")
         golden = optimize(minf,0,2,GoldenSection())
         brent = optimize(minf,0,2,Brent())
         using Base.Test
         println("brent minimizer = $(brent.minimizer)")
         println("golden minimizer = $(golden.minimizer)")
         plot(fun,0,2)
```

```
grid maximizer is 0.824692
brent minimizer = 0.8310314519658002
golden minimizer = 0.8310314579502819


WARNING: Method definition fun(Any) in module Main at In[10]:2 overwritten at In[11]:2.
WARNING: Method definition minf(Any) in module Main at In[10]:3 overwritten at In[11]:3.
```

2.5 Beyond One Dimension

2.5.1 Introducing Rosenbrock's Banana function

The Banana function is defined by

$$f(x,y) = (a-x)^2 + b(y-x^2)^2$$

2.5.2 What is the minimum of that function?

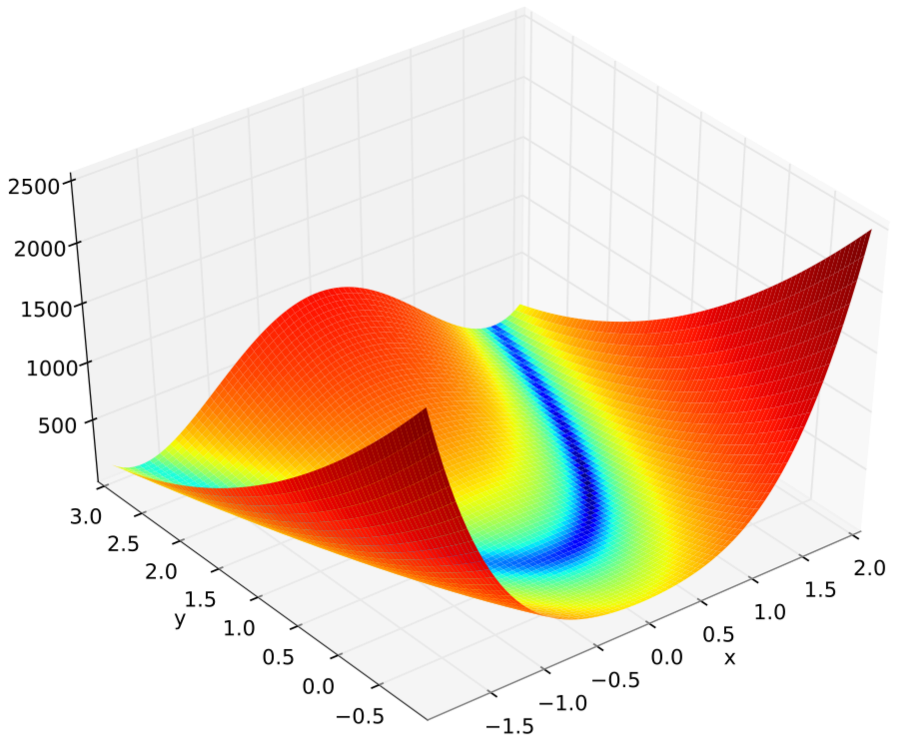
- For $a = 1, b = 100$, what is the global minimum of that function?
- What are the inputs one needs to supply to an algorithm in a more general example?

2.6 Rosenbrock Banana and Optim.jl

- We will use `Optim` for the rest of this lecture.
- We need to supply the objective function and - depending on the solution algorithm - the gradient and hessian as well.

```
In [12]: using Optim
         rosenbrock = Optim.UnconstrainedProblems.examples["Rosenbrock"]

         # contains:
         # function rosenbrock(x::Vector)
         #     return (1.0 - x[1])^2 + 100.0 * (x[2] - x[1]^2)^2
```



Banana for $a = 0$. By Gaortizg GFDL or CC BY-SA 3.0, via Wikimedia Commons

```
# end

# function rosenbrock_gradient!(x::Vector, storage::Vector)
#     storage[1] = -2.0 * (1.0 - x[1]) - 400.0 * (x[2] - x[1]^2) * x[1]
#     storage[2] = 200.0 * (x[2] - x[1]^2)
# end

# function rosenbrock_hessian!(x::Vector, storage::Matrix)
#     storage[1, 1] = 2.0 - 400.0 * x[2] + 1200.0 * x[1]^2
#     storage[1, 2] = -400.0 * x[1]
#     storage[2, 1] = -400.0 * x[1]
#     storage[2, 2] = 200.0
# end

# there are many other examples on Optim.UnconstrainedProblems
```

```
Out[12]: Optim.UnconstrainedProblems.OptimizationProblem("Rosenbrock",Optim.UnconstrainedProblem
```

2.7 Comparison Methods

- We will now look at a first class of algorithms, which are very simple, but sometimes a good starting point.
- They just *compare* function values.
- *Grid Search* : Compute the objective function at $G = \{x_1, \ldots, x_N\}$ and pick the highest value

of f.

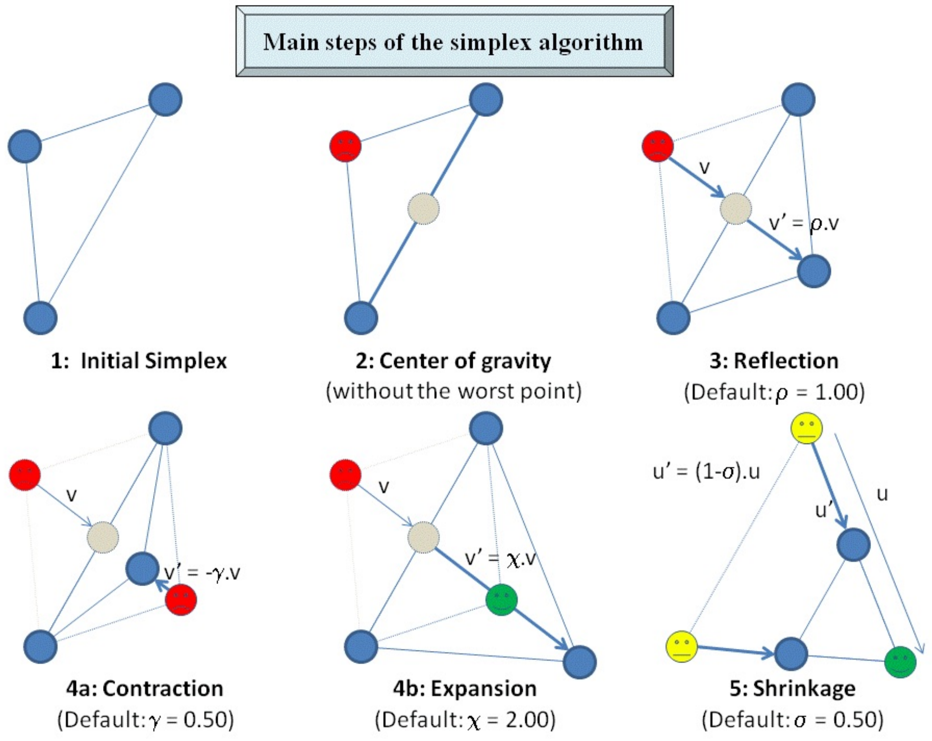
- This is very slow.
- It requires large N.
- But it's robust (will find global optimizer for large enough N)

```
In [13]: # grid search on rosenbrock
         grid = collect(-1.0:0.1:3);
         grid2D = [[i;j] for i in grid,j in grid];
         val2D = map(rosenbrock.f,grid2D);
         r = findmin(val2D);
         println("grid search results in minimizer = $(grid2D[r[2]])")
```

```
grid search results in minimizer = [1.0,1.0]
```

2.8 Bracketing for Multidimensional Problems: Nelder-Mead

- The Goal here is to find the simplex containing the local minimizer x^*
- In the case where f is n-D, this simplex has $n + 1$ vertices
- In the case where f is 2-D, this simplex has $2 + 1$ vertices, i.e. it's a triangle.
- The method proceeds by evaluating the function at all $n + 1$ vertices, and by replacing the worst function value with a new guess.
- this can be achieved by a sequence of moves:
  - reflect
  - expand
  - contract
  - shrink movements.



- this is a very popular method. The matlab functions `fmincon` and `fminsearch` implements it.
- When it works, it works quite fast.
- No derivatives required.

```
In [14]: optimize(rosenbrock, [0.0, 0.0], NelderMead())
```

```
Out[14]: Results of Optimization Algorithm
          * Algorithm: Nelder-Mead
          * Starting Point: [0.0,0.0]
          * Minimizer: [0.9999710322210338,0.9999438685860869]
          * Minimum: 1.164323e-09
          * Iterations: 74
          * Convergence: true
            *  ((y-)š)/n < 1.0e-08: true
            * Reached Maximum Number of Iterations: false
          * Objective Function Calls: 108
```

- But.

2.9 Bracketing for Multidimensional Problems: Comment on Nelder-Mead

Lagarias et al. (SIOPT, 1999): At present there is no function in any dimension greater than one, for which the original Nelder-Mead algorithm has been proved to converge to a minimizer.

Given all the known inefficiencies and failures of the Nelder-Mead algorithm [...], one might wonder why it is used at all, let alone why it is so extraordinarily popular.

2.10 Reminder: Optimality Conditions

2.10.1 Notation

- Unless otherwise noted, we have $x \in \mathbb{R}^n$ as an n element vector.
- The **gradient** of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is denoted $\nabla f : \mathbb{R}^n \mapsto \mathbb{R}^n$ a it returns a vector

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

- It's **hessian** is a function denoted $\nabla^2 f(x)$ or $H_f : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ and returns an (n,n) matrix given by

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{pmatrix}$$

2.10.2 Optimality Conditions

- **First Order Necessary Conditions**: If f is continously differentiable and x^* is a local minimizer of f, then $\nabla f(x^*) = 0$.
- **Second Order Necessary Conditions**: If f is twice continuously differentiable and x^* is a local minimizer of f, then $\nabla f(x^*) = 0$ *and* $H_f(x^*)$ is positive semi-definite, i.e. we have $s^T H_f(x^*)s \geq 0$ for all $s \in \mathbb{R}^n$.
- **Second Order Sufficient Conditions**: If f is twice continuously differentiable and at x^* we have that $\nabla f(x^*) = 0$ *and* $H_f(x^*)$ is positive definite, i.e. $s^T H_f(x^*)s > 0, s \neq 0$ then x^* is a local minimizer of f.
- More sophisticated solvers make ample use of those.

2.11 Reminder: Sufficient Conditions for Global Optimality

- If f is convex, then any local minimizer x^* is a global minimizer.
- If f is convex and differentiable on $\mathbb{R}^n$, then any point x^* is a global minimizer if and only if it is a stationary point, i.e. if f$\nabla f(x^*) = 0$.

2.12 Reminder: Taylor's Theorem

- Many of the ensuing methods are based on Taylor's theorem, so let's remind ourselves of it:

- Suppose that $f \in C^{n+1}[a,b]$, and $x, x^0 \in [a,b]$. Then

$$\begin{aligned} f(x) =& f(x^0) + f'(x^0)(x-x^0) + f''(x^0)\frac{(x-x^0)^2}{2} \\ &+ \cdots + f^{(n)}(x^0)\frac{(x-x^0)^n}{n!} + R_{n+1}(x) \end{aligned}$$

where $R_{n+1}(x) = \wr\left(\|(x-x^0)\|^{n+1}\right)$ is reminder term that converges at a rate $n+1$ to zero, i.e. * we say a function f is $\wr(\|x\|^n)$ if $\lim_{x\to 0} \|f(x)\|/\|x\|^n = 0$. * we say a function f is $\mathcal{O}(\|x\|^n)$ if $\lim_{x\to 0} \|f(x)\|/\|x\|^n < \infty$

2.13 A quick Note on Computing Derivatives

- Finite Differences
- Automatic Differentiation
- We have talked about this in a separate session.
- For now just remember that if we don't supply analytic gradients, and the algorithm requires them, this often triggers a numerical approximation of the gradient known as finite differences. This is most of the times a slow proceedure.

2.14 Two Strategies: Line Search and Trust Region

- We only provide an overview of methods here. If you want to *really* know the details about those algorithms, I invite you to consult [@nocedal-wright].

2.14.1 The Line Search Strategy

- An algorithm from the line search class chooses a direction $p_k \in \mathbb{R}^n$ and searches along that direction starting from the current iterate $x_k \in \mathbb{R}^n$ for a new iterate $x_{k+1} \in \mathbb{R}^n$ with a lower function value.
- After deciding on a direction p_k, one needs to decide the *step length* α to travel by solving

$$\min_{\alpha > 0} f(x_k + \alpha p_k)$$

- In practice, solving this exactly is too costly, so algos usually generate a sequence of trial values α and pick the one with the lowest f.

2.14.2 The Trust Region Strategy

- Here we construct a *model function* m_k that is similar to f around x_k.
- We acknowledge that m_k is decent approximation of f only in some *region*.
- The problem is then to find a candidate step length p by solving

$$\min_{p} m_k(x_k + p)$$

  where $x_k + p$ lies inside the trust region.
- If candidate p does not produce a value lower than $f(x_k)$, we must have had a too large trust region, shrink it, and do it again.
- Usually the trust region is a ball $\|p\|_2 \leq \Delta$, where Δ is called the *trust region radius*, but elliptical and box regions are possible.
- A common definition of the model function is a quadratic of the form

$$m_k(x_k + p) = f(x_k) + p^T \nabla f(x_k) + \frac{1}{2} p^T B(x_k) p$$

  where gradient and matrix $B(x_k)$ are evaluated at the current iterate, so the model function is in agreement to first order with f at the current guess.
- The matrix $B(x_k)$ is either the Hessian H_f or some approximation to it.

2.15 Trust Region Example

- Suppose we have $f(x) = 10(x_2 - x_1^2)^2 + (1 - x_1)^2$. At point $x_k = (0, 1)$ gradient and hessian are:
- ?

$$\nabla f(x_k) = \begin{bmatrix} -2 \\ 20 \end{bmatrix}, \qquad \nabla^2 f(x_k) = \begin{bmatrix} -38 & 0 \\ 0 & 20 \end{bmatrix}$$

- In this figure, we use $B(x_k) = \nabla^2 f(x_k)$.
- After each unsuccesful step, the new candidate step will be shorter, and will go in a different direction
- This is the main difference to line search methods.
- Main difference between the two methods: order in which they change *direction* and *step length*
  - line search fixes direction p_k and finds right distance α_k
  - trust region fixes an appropriate radius Δ_k

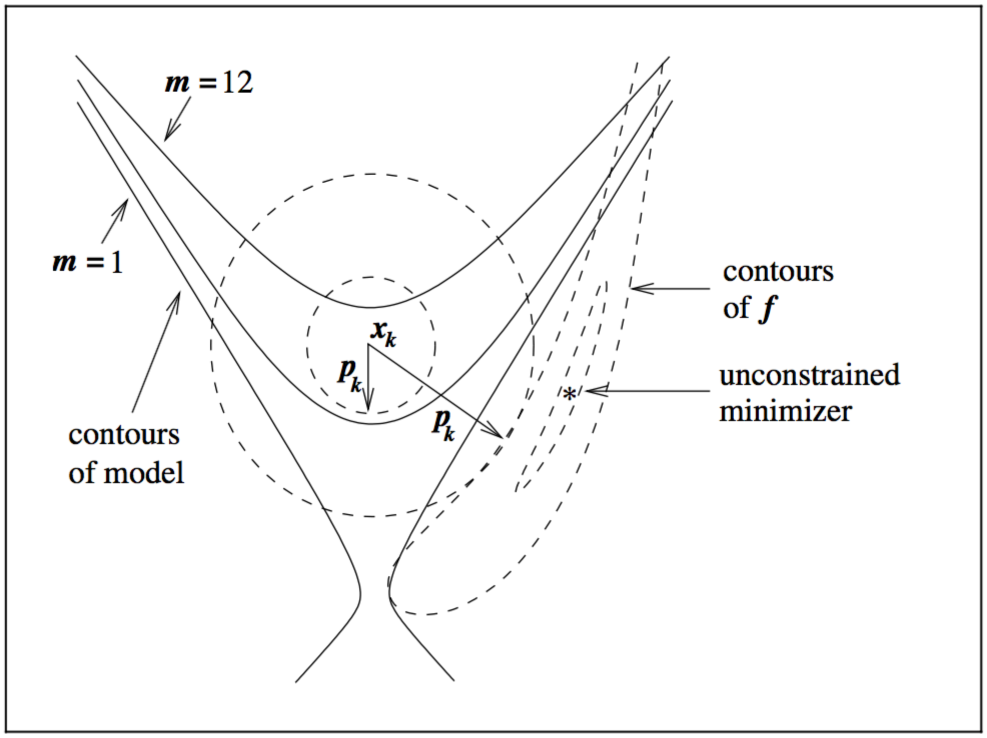


Figure 2.4 of [@nocedal-wright]

2.16 Line Search Methods: Which Direction to go?

2.16.1 Steepest Descent

- The direction $-\nabla f(x_k)$ is an obvious choice: Among all possible directions, along this one f decreases most rapidly.
- This claim can be verified using Taylor's theorem.
- The **steepest descent method** is a line search method that moves along

$$p_k = -\nabla f(x_k)$$

- The step length α_k can be chosen in many different ways.
- There are many other *descent* directions, steepest descent is but one of them.

```
In [15]: # there is a dedicated LineSearch package: https://github.com/JuliaNLSolvers/LineSearch
         GD = optimize(rosenbrock.f, rosenbrock.g!,[0.0, 0.0],GradientDescent())
         GD1 = optimize(rosenbrock.f, rosenbrock.g!,[0.0, 0.0],GradientDescent(),Optim.Options(i
         GD2 = optimize(rosenbrock.f, rosenbrock.g!,[0.0, 0.0],GradientDescent(),Optim.Options(i

         println("gradient descent = $GD")
         println("\n")
         println("gradient descent 2 = $GD1")
         println("\n")
         println("gradient descent 3 = $GD2")
```

```
gradient descent = Results of Optimization Algorithm
 * Algorithm: Gradient Descent
 * Starting Point: [0.0,0.0]
 * Minimizer: [0.9356732500354086,0.875073922357589]
 * Minimum: 4.154782e-03
 * Iterations: 1000
 * Convergence: false
   * |x - x'| < 1.0e-32: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
   * |g(x)| < 1.0e-08: false
   * f(x) > f(x'): false
   * Reached Maximum Number of Iterations: true
 * Objective Function Calls: 3532
 * Gradient Calls: 3532


gradient descent 2 = Results of Optimization Algorithm
 * Algorithm: Gradient Descent
 * Starting Point: [0.0,0.0]
 * Minimizer: [0.9978398797724763,0.9956717950747302]
 * Minimum: 4.682073e-06
 * Iterations: 5000
 * Convergence: false
   * |x - x'| < 1.0e-32: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
   * |g(x)| < 1.0e-08: false
   * f(x) > f(x'): false
   * Reached Maximum Number of Iterations: true
 * Objective Function Calls: 17532
 * Gradient Calls: 17532


gradient descent 3 = Results of Optimization Algorithm
 * Algorithm: Gradient Descent
 * Starting Point: [0.0,0.0]
 * Minimizer: [0.9999999914304203,0.9999999828109042]
 * Minimum: 7.368706e-17
 * Iterations: 20458
 * Convergence: true
   * |x - x'| < 1.0e-32: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
   * |g(x)| < 1.0e-08: true
   * f(x) > f(x'): false
   * Reached Maximum Number of Iterations: false
 * Objective Function Calls: 71635
 * Gradient Calls: 71635
```

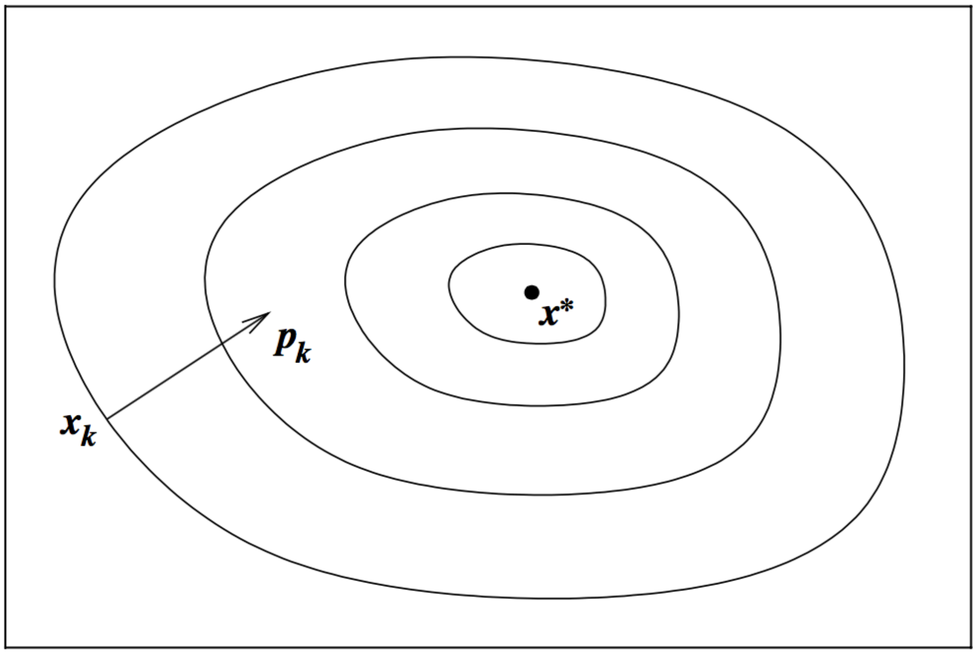


Figure 2.5 of [@nocedal-wright]

2.17 Line Search Methods: The Newton Direction

- Probably the most important descent direction.
- In vector notation, the 2nd order taylor series approximation to $f(x_k + p)$ is

$$f(x_k + p) \approx f(x_k) + p^T \nabla f(x_k) + \frac{1}{2} p^T \nabla^2 f(x_k) p \equiv m_k(p)$$

- the Newton direction is obtained by finding the vector p that minimizes $m_k(p)$, i.e. by setting the derivative of $m_k(p)$ to zero.
- We obtain

$$p_k^N = -(\nabla^2 f(x_k))^{-1} \nabla f(x_k)$$

- The newton direction is reliable if the discrepancy between truth and model m is not too large at x_k.
- The biggest drawback is the need to compute the Hessian. This can be difficult analytically at times, and overly expensive numerically.

```
In [7]: optimize(rosenbrock.f, rosenbrock.g!, rosenbrock.h!, [0.0, 0.0], Newton(),Optim.Options(
```

```
Iter     Function value   Gradient norm
     0     1.000000e+00     2.000000e+00
     1     8.431140e-01     1.588830e+00
     2     6.776980e-01     3.453340e+00
     3     4.954645e-01     4.862093e+00
     4     3.041921e-01     2.590086e+00
     5     1.991512e-01     3.780900e+00
     6     9.531907e-02     1.299090e+00
     7     5.657827e-02     2.445401e+00
     8     2.257807e-02     1.839332e+00
     9     6.626125e-03     1.314236e+00
    10     8.689753e-04     5.438279e-01
    11     4.951399e-06     7.814556e-02
    12     9.065070e-10     6.017046e-04
    13     9.337686e-18     1.059738e-07
    14     3.081488e-31     1.110223e-15
```

```
Out[7]: Results of Optimization Algorithm
         * Algorithm: Newton's Method
         * Starting Point: [0.0,0.0]
         * Minimizer: [0.9999999999999994,0.9999999999999989]
         * Minimum: 3.081488e-31
         * Iterations: 14
         * Convergence: true
           * |x - x'| < 1.0e-32: false
           * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
           * |g(x)| < 1.0e-08: true
           * f(x) > f(x'): false
           * Reached Maximum Number of Iterations: false
         * Objective Function Calls: 58
         * Gradient Calls: 58
```

2.18 Quasi-Newton Methods

- In response to the difficulties of getting the Hessian, quasi-newton methods propose to approximate $B(x_k)$ with something *similar* to the hessian.
- Taylors Theorem implies that

$$\nabla^2 f(x_{k+1} - x_k) \approx \nabla f(x_{k+1}) - \nabla f(x_k)$$

  and so we choose a B matrix that mimics this property.
- This leads to the *secant condition*

$$B_{k+1}(x_{k+1} - x_k) = f(x_{k+1}) - \nabla f(x_k)$$

- There are different ways to update the hessian in this way.
- One of the best known is the BFGS method (after Broydon, Fletcher, Goldfarb and Shanno).
- Those methods get the search direction by using B_k instead of the exact Hessian, i.e.

$$p_k = -B_k^{-1} \nabla f(x_k)$$

```
In [17]: @show optimize(rosenbrock.f, rosenbrock.g!, rosenbrock.h!,  [-1.0, 3.0], BFGS());
         # low memory BFGS
         @show optimize(rosenbrock.f, rosenbrock.g!, rosenbrock.h!,  [0.0, 0.0], LBFGS());
```

```
optimize(rosenbrock.f,rosenbrock.g!,rosenbrock.h!,[-1.0,3.0],BFGS()) = Results of Optimization A
 * Algorithm: BFGS
 * Starting Point: [-1.0,3.0]
 * Minimizer: [0.9999999999999956,0.999999999999987]
 * Minimum: 1.707144e-27
 * Iterations: 39
 * Convergence: true
   * |x - x'| < 1.0e-32: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
   * |g(x)| < 1.0e-08: true
   * f(x) > f(x'): false
   * Reached Maximum Number of Iterations: false
 * Objective Function Calls: 176
 * Gradient Calls: 176
optimize(rosenbrock.f,rosenbrock.g!,rosenbrock.h!,[0.0,0.0],LBFGS()) = Results of Optimization A
 * Algorithm: L-BFGS
 * Starting Point: [0.0,0.0]
 * Minimizer: [1.0000000000000007,1.000000000000001]
 * Minimum: 5.374115e-30
 * Iterations: 21
 * Convergence: true
   * |x - x'| < 1.0e-32: false
   * |f(x) - f(x')| / |f(x)| < 1.0e-32: false
   * |g(x)| < 1.0e-08: true
   * f(x) > f(x'): false
   * Reached Maximum Number of Iterations: false
 * Objective Function Calls: 90
 * Gradient Calls: 90
```

2.19 Practical Considerations

2.19.1 Stopping criteria

- In all of the above examples, we did not alter the default values for stopping criteria.
- There are different things you could focus on as a stopping criterion with `Optim`, and similarly in most solver packages.

  - `xtol`: changes in x from one iterate to the next
  - `f_tol`: perentage changes in f from one iterate to the next
  - `g_tol`: absolute value of the gradient beign smaller than this number.

2.20 Some Applications in Economics

- Maximum Likelihood Estimation
- The Nested Fixed Point Algorithm

  - [@rust-bus] is a maximum likelihood estimation with an inner loop that solves a dynamic programming problem.
  - [@BLP] is a GMM estimation with an inner fixed point problem.