Linear Programming

Which has absolutely nothing to do with programming

First, an aside

- It the paths through the residual graph G_f are chosen carefully, say using BFS and always the shortest path from s to t through G_f, then what is the worst case runtime of Ford-Fulkerson?

Optimization Problems

- Optimization problems are common in the real world
  - Ex. Maximize profit
  - Ex. Minimize error, minimize cost
- In general, such problems have two components:
  - A set of contraints that must be satisfied
  - An objective function to maximize or minimize, subject to the contraints

Example

- Web company want to buy new servers to replace outdated ones, and has two choices:
- Standard model
  - Costs $400
  - Uses 300W of power
  - Takes two shelves of server rack
  - Can handle 1000 hits/min
- Cutting-edge model
  - Costs $1600
  - Uses 500W of power
  - Takes one shelf of server rack
  - Can handle 2000 hits/min

Example (cont.)

- Budget is \$36,800
- Company has 44 shelves of server space
- Company has 12,200 Watts of power
- Question: How many units of each server should company purchase in order to maximize the number of hits it can serve every minute?
- Let x_1 = number of standard server
- Let x_2 = number of cutting-edge server
- Goal: maximize $1000x_1 + 2000x_2$ subject to contraints

Constraints:

- Cost: $400x_1 + 1600x_2 \leq 36,800$
- Shelves: $2x_1 + x_2 \leq 44$
- Power: $300x_1 + 500x_2 \leq 12200$
- $x_1 \geq 0, \; x_2 \geq 0$

Problem Summary

Therefore, this optimization problem can be summarized as follows:

$$
\begin{aligned}
\text{maximize:} \quad & z = 1000x_1 + 2000x_2 \\
\text{subject to:} \quad & 400x_1 + 1600x_2 \leq 36800 \\
& 2x_1 + x_2 \leq 44 \\
& 300x_1 + 500x_2 \leq 12200 \\
& x_1, x_2 \geq 0,
\end{aligned}
$$

Solving optimization problems that have
the above general form is known as *linear programming*.

Translating Problems into Linear Programs

- Determine the variables of the problem
- Find the quantity to optimize, and write it in terms of the variables
- Find all the constraints on the variables and write equations or inequalities to express these constraints
  - Be sure to include any implicit constraints on the range of values of variables
  - Make sure all equations are *linear*

Another Example

Objective function $\max x_1 + 6x_2$

Constraints

$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

- Chocolate company has two products
  - Pyramide and Pyramide Nuit
- produces x_1 boxes of Pyramide per day at $1 profit per box
- produces x_2 boxes of Pyramide per day at $6 profit per box
- Daily demand 200 boxes per day of Pyramide, 300 of Pyramide Nuit
- Current workforce can only produce 400 boxes of chocolate total per day

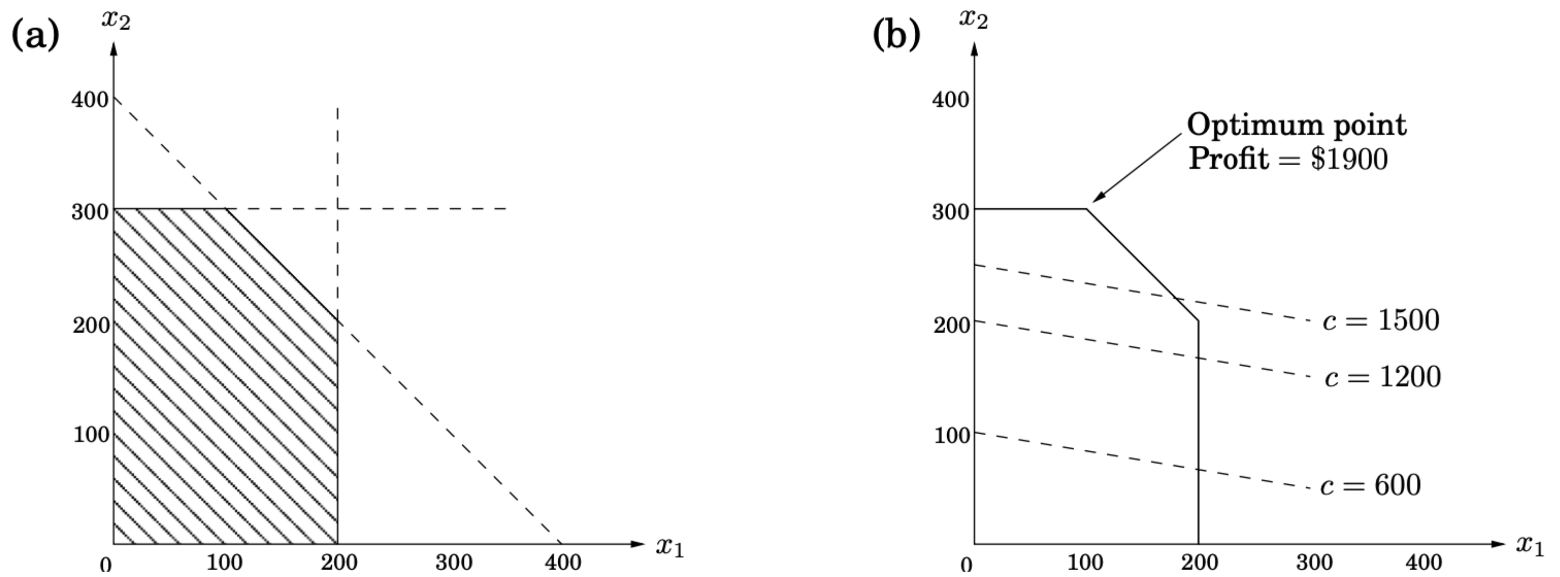
Another Example

Objective function $\max x_1 + 6x_2$

Constraints

$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

Figure 7.1 (a) The feasible region for a linear program. (b) Contour lines of the objective function: $x_1 + 6x_2 = c$ for different values of the profit c.

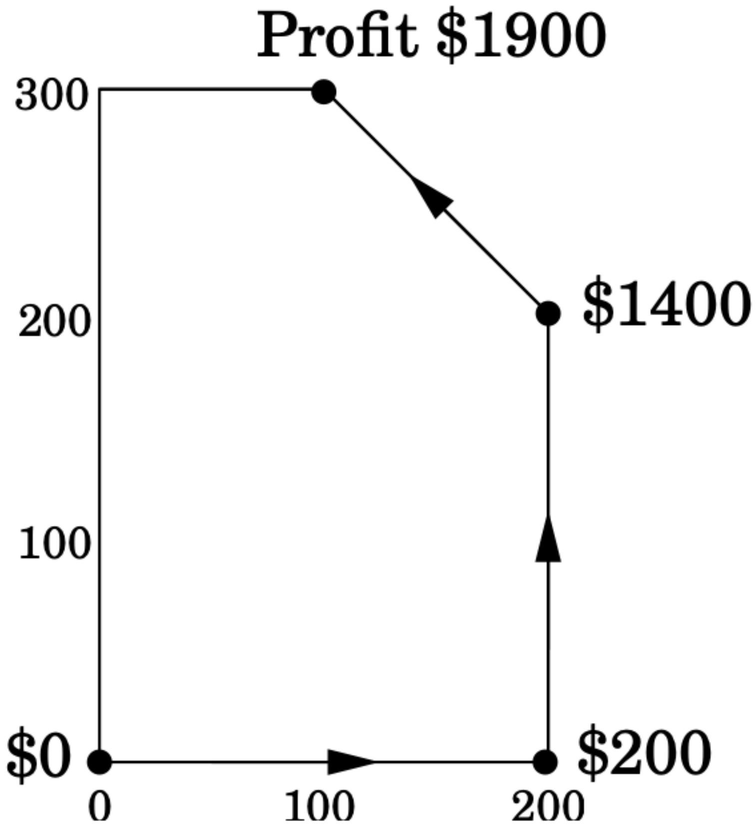


Generally:

- Optimal value occurs at a vertex of the feasible region
- Unless there is no optimal value
  - Constraints are too restrictive – no feasible region
  - Constraints so loose that the feasible region is unbounded
- Examples?

How to solve it: Simplex method

- Devised by George Dantzig in 1947
- Starts at a vertex, then moves from vertex to vertex in a hill-climbing manner
- Software implements this very efficiently
  - So we don't have to
  - And we don't: setting up problem is what it important!

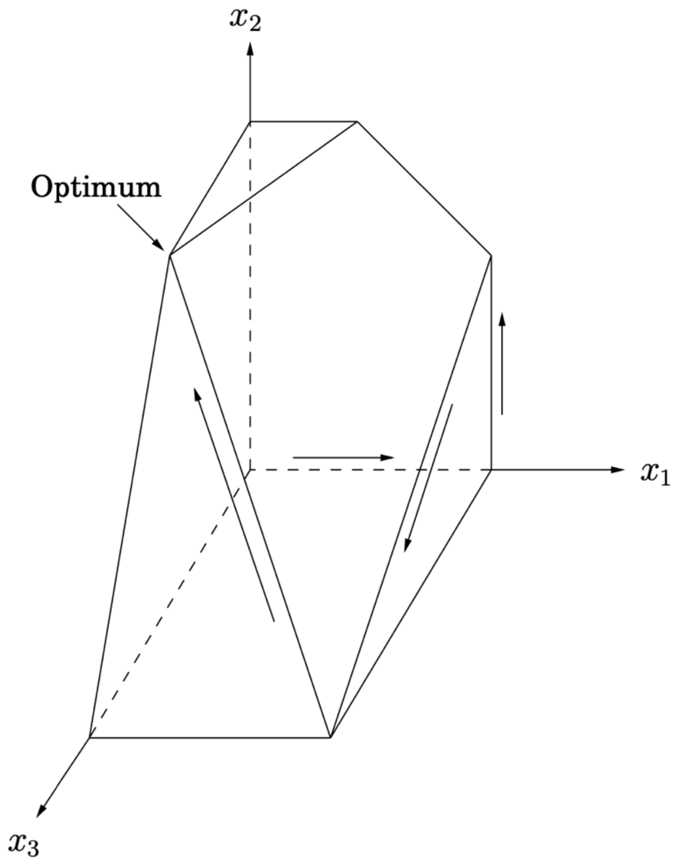


Another Example

$$
\begin{aligned}
\max \quad x_1 + 6x_2 + 13x_3 \\
x_1 \le 200 \\
x_2 \le 300 \\
x_1 + x_2 + x_3 \le 400 \\
x_2 + 3x_3 \le 600 \\
x_1, x_2, x_3 \ge 0
\end{aligned}
$$

- Chocolate company adds another product: Pyramide Luxe
- x_3 boxes per day at $13 profit per box
- Luxe and Nuit use the same packaging machine, but Luxe uses it 3 times as much, which introduces another constraint

Figure 7.2 The feasible polyhedron for a three-variable linear program.



$$\begin{array}{ccccccccc} (0,0,0) & & (200,0,0) & & (200,200,0) & & (200,0,200) & & (0,300,100) \\ \$0 & \longrightarrow & \$200 & \longrightarrow & \$1400 & \longrightarrow & \$2800 & \longrightarrow & \$3100 \end{array}$$

Some magic

$$
\begin{aligned}
\max \quad x_1 + 6x_2 + 13x_3 & \\
x_1 &\le 200 \\
x_2 &\le 300 \\
x_1 + x_2 + x_3 &\le 400 \\
x_2 + 3x_3 &\le 600 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

- Add second inequality to third.
- Add that sum to 4 times the fourth inequality.
- What do you get?
- What does that tell you?

Some magic

$$
\begin{aligned}
\max \quad x_1 + 6x_2 + 13x_3 & \\
x_1 &\le 200 \\
x_2 &\le 300 \\
x_1 + x_2 + x_3 &\le 400 \\
x_2 + 3x_3 &\le 600 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

- Add second inequality to third.
- Add that sum to 4 times the fourth inequality.
- What do you get?
- What does that tell you?
- Consider the coefficients you used: 0,1,1,4
  - Where did they come from?

As evidenced in our examples, a general linear program has many degrees of freedom.

1. It can be either a maximization or a minimization problem.
2. Its constraints can be equations and/or inequalities.
3. The variables are often restricted to be nonnegative, but they can also be unrestricted in sign.

- How can you turn a maximization problem into a minimization problem?
- How can you turn an inequality into an equation?
- How can you turn an equation into inequalities?
- How do you deal with a variable that is unrestricted in sign

As evidenced in our examples, a general linear program has many degrees of freedom.

1. It can be either a maximization or a minimization problem.
2. Its constraints can be equations and/or inequalities.
3. The variables are often restricted to be nonnegative, but they can also be unrestricted in sign.

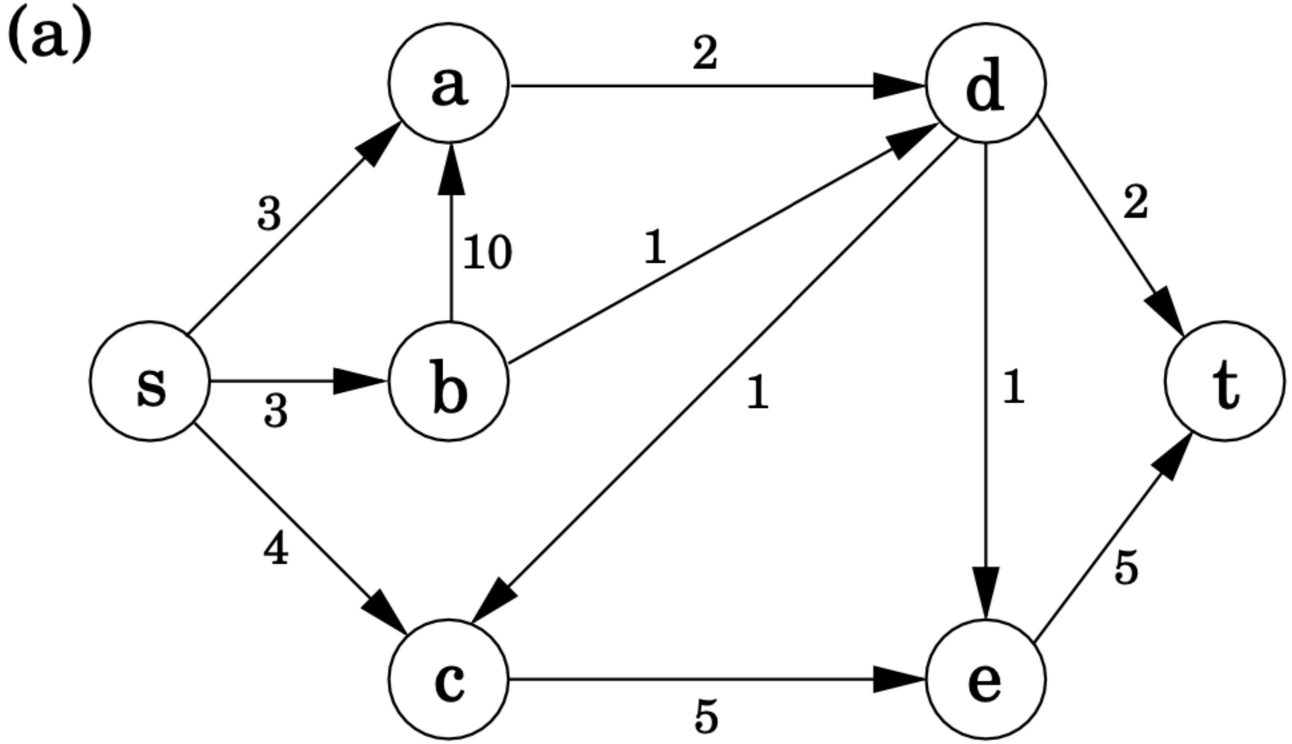
- Bottom line: any form can be converted into any other form
- Standard form:
  - Variables all nonnegative
  - Constraints are all equations
  - Objective function is to be minimized

- Bottom line: any form can be converted into any other form
- Standard form:
  - Variables all nonnegative
  - Constraints are all equations
  - Objective function is to be minimized

$$
\begin{aligned}
\max \ & x_1 + 6x_2 \\
x_1 &\le 200 \\
x_2 &\le 300 \\
x_1 + x_2 &\le 400 \\
x_1, x_2 &\ge 0
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\min \ & -x_1 - 6x_2 \\
x_1 + s_1 &= 200 \\
x_2 + s_2 &= 300 \\
x_1 + x_2 + s_3 &= 400 \\
x_1, x_2, s_1, s_2, s_3 &\ge 0
\end{aligned}
$$

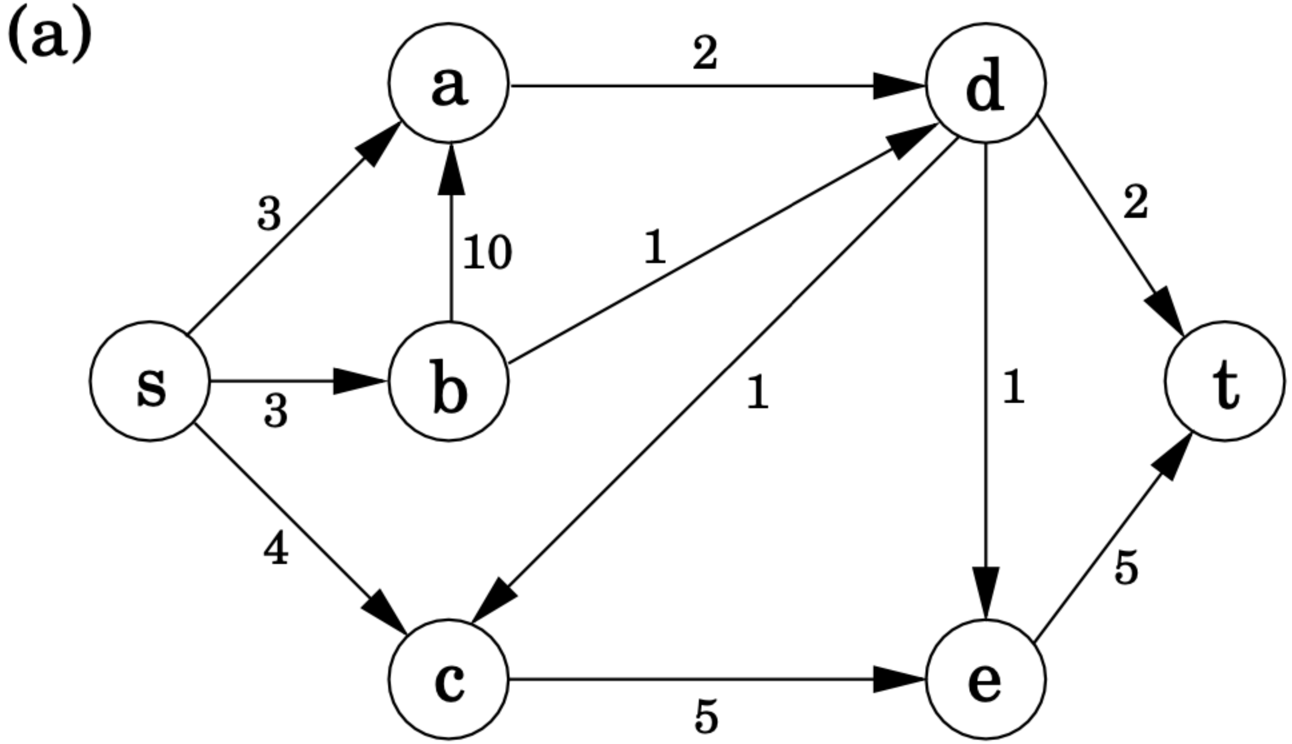
Another Example: Max flow

- You set it up



Another Example: Max flow

- This problem has a built in proof of the validity of your solution.



- So does this

$$
\begin{aligned}
\max \quad & x_1 + 6x_2 + 13x_3 \\
x_1 & \leq 200 \\
x_2 & \leq 300 \\
x_1 + x_2 + x_3 & \leq 400 \\
x_2 + 3x_3 & \leq 600 \\
x_1, x_2, x_3 & \geq 0
\end{aligned}
$$

- So does this, though we haven't seen it yet.
  - But let's look into it.
- Fact: simplex method yields x_1=100, x_2 = 300, for total profit of 1900

$$
\begin{array}{ll}
\textbf{Objective function} & \max x_1 + 6x_2 \\
\textbf{Constraints} & x_1 \le 200 \\
 & x_2 \le 300 \\
 & x_1 + x_2 \le 400 \\
 & x_1, x_2 \ge 0
\end{array}
$$

- Fact: simplex method yields x_1=100, x_2 = 300, for total profit of 1900
- Add one times first inequality to 6 times second. What does this tell you?

Objective function	$\max x_1 + 6x_2$
Constraints	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

- Fact: simplex method yields x_1=100, x_2 = 300, for total profit of 1900
- Now use the multipliers (0,5,1). What does this tell you?

Objective function	$\max x_1 + 6x_2$
Constraints	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

- So, where do these multipliers come from?
- Let's call them y_1, y_2, and y_3
- Note first: all three must be nonnegative. Why?

Multiplier	**Inequality**
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- Any feasible solution to this *dual* LP must be an upper bound on the original *primal* LP

$$x_1 + 6x_2 \quad \leq \quad 200y_1 + 300y_2 + 400y_3 \qquad \textbf{if} \qquad \left\{ \begin{array}{c} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{array} \right\}.$$

- It looks like we have this

Therefore, finding the set of multipliers that gives the best upper bound on our original LP is tantamount to solving a new LP:

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

- Any feasible solution to this *dual* LP is an upper found on the *primal* LP

Therefore, finding the set of multipliers that gives the best upper bound on our original LP is tantamount to solving a new LP:

$$
\begin{aligned}
\min\ \ 200y_1 + 300y_2 + 400y_3 \\
y_1 + y_3 \geq 1 \\
y_2 + y_3 \geq 6 \\
y_1, y_2, y_3 \geq 0
\end{aligned}
$$

- Any feasible solution to this *dual* LP is an upper found on the *primal* LP
- So if we find a solution to the primal and dual that are equal, we have a proof of optimality!

Therefore, finding the set of multipliers that gives the best upper bound on our original LP is tantamount to solving a new LP:

$$
\begin{aligned}
\min\ 200y_1 + 300y_2 + 400y_3 \\
y_1 + y_3 \geq 1 \\
y_2 + y_3 \geq 6 \\
y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Duality theorem *If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.*

It turns out that the dual of max flow as an LP problem is the min-cut problem.

- Given the primal, can you read off the dual?

Objective function	$\max x_1 + 6x_2$	$\min\ 200y_1 + 300y_2 + 400y_3$
Constraints	$x_1 \leq 200$	$y_1 + y_3 \geq 1$
	$x_2 \leq 300$	$y_2 + y_3 \geq 6$
	$x_1 + x_2 \leq 400$	$y_1, y_2, y_3 \geq 0$
	$x_1, x_2 \geq 0$	

- Find the dual

$$
\begin{aligned}
\max \quad x_1 + 6x_2 + 13x_3 & \\
x_1 &\leq 200 \\
x_2 &\leq 300 \\
x_1 + x_2 + x_3 &\leq 400 \\
x_2 + 3x_3 &\leq 600 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

- Find the dual

$$\begin{aligned} \max \ 12x_1 + 3x_2 \ + 7x_3 & \\ 2x_1 + 4x_3 &\leq 100 \\ 5x_2 + 6x_3 &\leq 200 \\ 4x_1 + 8x_2 + 5x_3 &\leq 300 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

- Find the dual

$$\begin{aligned} \max\ \ 12x_1 + 3x_2\ \ + 7x_3 \\ 2x_1 + 4x_3 \le 100 \\ 5x_2 + 6x_3 \le 200 \\ 4x_1 + 8x_2 + 5x_3 \le 300 \\ x_1, x_2, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ \ 100y_1 + 200y_2\ \ + 300y_3 \\ 2y_1 + 4y_3 \ge 12 \\ 5y_2 + 8y_3 \ge 3 \\ 4y_1 + 6y_2 + 5y_3 \ge 7 \\ y_1, y_2, y_3 \ge 0 \end{aligned}$$

Somebody has already written a massive LP with thousands of variables and millions of constraints, describing a factory's yearly operation, with variables for things like steel, iron, copper, worker-hours, electricity, widgets produced, gizmos produced, numbers of intermediate parts, worker accidents, and many more things, as well as many existing constraints between them. The LP is currently trying to maximize the single profit variable p. Your company runs this LP to decide all sorts of things, like how many gizmos to manufacture, how many workers to hire, how to allocate workers to projects, etc.

You do not need to think about this massive pre-existing LP, but need only describe how to add/change this LP to achieve your goals, which are not currently a part of the LP.

You can modify the LP by saying things like "Add a variable ...", "Add the contraint...", "Modify/change the objective by doing ...". For each edit to the LP that you make, explain what that edit is for. Let's answer a few questions related to this. Remember that you do not know what the rest of the LP is - it could be anything.

We will assume that if there is *no* solution to the constraints provided, then the (possibly modified) LP algorithm will return "infeasible".

(EXAMPLE) Your company is going to buy special storage units for its products. Each storage unit has a form fitted space for one gizmo g, one widget w, one doohickey d and one thingamajig t, and costs u dollars. You need to buy enough storage units to store all your manufactured goods. Keep in minds that a gimzo can't fit in a spot designed for a widget, etc., so any excess storage space can't be reused.

How would you modify the LP to achieve this? Explain. Existing variables in the LP:

g - number of gizmos

w - number of widgets

d - number of doohickeys

t - number of thingamajigs

(EXAMPLE) Let W be the set of all workers. Your company's workers all work h_w hours per week. These hours are allocated to tasks by other constraints in the LP. Due to new rules, workers that work more than 40 hours per week are given an overtime bonus b_w per hour worked above 40. This bonus counts against the profit p of the company.

Modify the LP to take this into account.

Existing variables in the LP:

For each worker w: h_w - number of hours worker w works.