Classical Dynamics

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1 Newtonian mechanics and frames of reference

1.1 Newtonian mechanics

Mechanics encompasses statics, kinematics and dynamics.

Newtonian mechanics is non-relativistic ($v \ll c$) and classical ($Et \gg \hbar$). We assume that mass is independent of velocity, time or frame; that measurements of time and length are independent of the frame, and that all parameters can be known precisely.

1.2 Newton's laws of motion

Law (Newton's First Law of Motion)**.** A body remains at rest, or moves uniformly in a straight line, unless acted on by a force. (This is in fact Galileo's Law of Inertia)[1]

Law (Newton's Second Law of Motion)**.** The rate of change of momentum of a body is equal to the force acting on it (in both magnitude and direction).

Law (Newton's Third Law of Motion)**.** To every action there is an equal and opposite reaction: the forces of two bodies on each other are equal and in opposite directions.

Here "body" means either a particle or the centre of mass of an extended object.

The first law might seem redundant given the second if interpreted literally. According to the second law, if there is no force, then the momentum doesn't change. Hence the body remains at rest or moves uniformly in a straight line.

So why do we have the first law? Historically, it might be there to explicitly counter Aristotle's idea that objects naturally slow down to rest. However, some (modern) physicists give it an alternative interpretation:

Note that the first law isn't always true. Take yourself as a frame of reference. When you move around your room, things will seem like they are moving around (relative to you). When you sit down, they stop moving. However, in reality, they've always been sitting there still. On second thought, this is because you, the frame of reference, is accelerating, not the objects. The first law only holds in frames that are themselves not accelerating. We call these *inertial frames*.

Definition (Inertial frames)**.** *Inertial frames* are frames of reference which are not themselves accelerating. Newton's Laws only hold in inertial frames.

Then we can take the first law to assert that inertial frames exists. Even though the Earth itself is rotating and orbiting the sun, for most purposes, any fixed place on the Earth counts as an inertial frame.

1.3 The simple harmonic oscillator

Suppose we have a mass m moving in one dimension with coordinate x subject to restoring force $F = -kx$. We can write down the Newtonian equation of motion using the second law:

$$m\ddot{x} = -kx.$$

[1]Although actually Galileo believed in circular inertia.

Note that we can integrate the equation of motion to get a conserved quantity – the total energy, E.

$$\begin{aligned} m\ddot{x} + kx &= 0 \\ m\ddot{x}\dot{x} + kx\dot{x} &= 0 \\ \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 &= E \end{aligned} \tag{$*$}$$

From $(*)$ we can identify the kinetic and potential engergies, $T \equiv \frac{1}{2}m\dot{x}^2$ and $V \equiv \frac{1}{2}kx^2$ respectively. Note that the total energy $E = T + V$ is conserved. This conserved quantiy is also known as the Hamiltonian.

In common with many other dynamical systems, t does not appear explicitly in the equation of motion.

1.4 The energy method

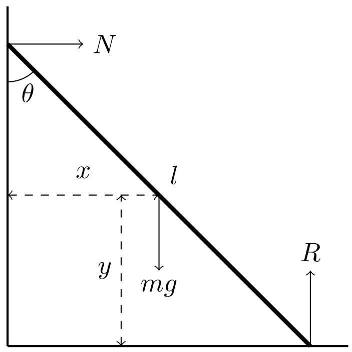
If, from physical grounds, we know that energy is conserved, then we can always derive the equations of motion of systems that only have one degree of freedom (such as the SHO) from the expressions for their kinetic and potential energies. We call this the *energy method*. (Note that in the case of the SHO, we use the fact that $\dot{x}$ is not always zero)

Sometimes we can derive the equations of motion of much more complicated systems with n degrees of freedom in a similar manner, however it is not rigorous. Despite this, it works for most of the systems studied in this course.

The more theoretically advanced methods of Lagrangian and Hamiltonian mechanics derive the equations of motion from a variational principle (see § 4). They are rigorous, but still use T and V (although in the combination $\mathcal{L} = T - V$).

In § 2 the energy method will be used to derive the equation of motion of a particle at radius r in a central force.

Example (leaning ladder)**.** A uniform ladder of length l leans against a smooth wall and rests on a smooth floor.



We can approach this in two ways:

1. Introduce forces N and R, take moments to get angular acceleration.

2. Use the energy method:

(a) Work out the potential energy of the centre of mass:

$$V = \frac{1}{2}mgl\cos\theta.$$

(b) Work out the kinetic energy of the centre of mass:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2,$$

where $I = \frac{1}{12}ml^2$ is the moment of inertia of a uniform rod about its centre.

(c) Conserve energy:

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta \\ &= \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta \\ \dot{E} &= \dot{\theta}\big(\frac{1}{3}\ddot{\theta}ml^2 - \frac{1}{2}mgl\sin\theta\big) = 0 \end{aligned}$$

(d) So the equation of motion is:

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta.$$

1.5 Dynamics of many-particle systems

For a system of n particles, the ith has mass m_i, position $\mathbf{r}_i$ and velocity $\mathbf{v}_i$. It experiences an external force $\mathbf{f}_i$ and internal forces $\mathbf{f}_{i,j}$ as a result of other particles j.

Definition (total mass)**.** The *total mass* of the system is

$$M = \sum_i m_i$$

Definition (centre of mass)**.** The *centre of mass* of the system, $\mathbf{R}$, is defined by

$$\mathbf{R} = \frac{1}{M}\sum_i m_i\mathbf{r}_i$$

The motion of the system as a result of an external force can be thought of as acting at the centre of mass.

Definition (total momentum)**.** The *total momentum* of the system is

$$\mathbf{P} = \sum_i m_i\dot{\mathbf{r}}_i = \sum_i \mathbf{p}_i$$

Definition (total angular momentum about the origin)**.** The *total angular momentum about the origin* is

$$\mathbf{J}_\mathrm{O} = \sum_i \big(\mathbf{r}_i \times (m_i\dot{\mathbf{r}}_i)\big) = \sum_i \big(\mathbf{r}_i \times \mathbf{p}_i\big)$$

Definition (total angular momentum about the origin)**.** The *total angular momentum about the origin* is

$$\mathbf{J}_{\text{CM}} = \sum_i \left(\left(\mathbf{r}_i - \mathbf{R}\right) \times \left(m_i \dot{\mathbf{r}}_i\right)\right) = \sum_i \left(\left(\mathbf{r}_i - \mathbf{R}\right) \times \mathbf{p}_i\right)$$

Definition (total force)**.** The *total force* acting on the system is

$$\mathbf{F} = \sum_i \mathbf{f}_i$$

Definition (total couple acting about the centre of mass)**.** The *total couple acting about the centre of mass* is

$$\mathbf{G}_{\text{CM}} \equiv \sum_i \left(\mathbf{r}_i \times \mathbf{f}_i\right)$$

Now we state and prove the rotational variant of Newton's second law.

Theorem (Newton's second law (rotational version))**.**

$$\dot{\mathbf{J}}_{\text{O}} = \mathbf{G}_{\text{O}}$$

Proof.

$$\begin{aligned}
\dot{\mathbf{J}}_{\text{O}} &= \sum_i \left(\dot{\mathbf{r}}_i \times \mathbf{p}_i + \mathbf{r}_i \times \dot{\mathbf{p}}_i\right) \\
&= \sum_i \left(\mathbf{r}_i \times \mathbf{f}_i\right) + \sum_i \sum_j \left(\mathbf{r}_i \times \mathbf{f}_{i,j}\right) \\
&= \mathbf{G}_{\text{O}} + \frac{1}{2} \sum_i \sum_j \left(\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{f}_{i,j}\right) \\
&= \mathbf{G}_{\text{O}}
\end{aligned}$$

□

We denote the kinetic, potential and total energies of the system T, U and E respectively. Note that T is composed of rotational and linear parts.

1.6 Galilean transformations

Inertial frames are not unique. If S is an inertial frame, then any other frame S' moving at constant velocity $\mathbf{v}$ relative to S is also an inertial frame. Provided the frames coincide at $t = 0$, we have

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r} - \mathbf{v}t \\
t' &= t
\end{aligned}$$

This a Galilean transformation or boost. In general a Galilean transformation combines a boost with translations of space and time, along with rotations (and reflections) of space.[2]

[2] These generate the Galilean group.

Law (Galilean relativity)**.** The *principle of relativity* asserts that physical laws are the same in all inertial frames. This implies that physical processes are identical:

- at every point in space
- at all times
- in whichever direction we face
- at whatever constant velocity we move

So the equations of Newtonian physics must have *Galilean invariance.*

Motion (velocity) is not absolute, it is only relative. However acceleration is absolute.

1.7 Frames in relative motion

Suppose we have frame S in which $m\ddot{\mathbf{r}} = \mathbf{F}$, with $\mathbf{F}$ known due to physical causes. What is the apparent equation of motion in a moving frame S'?

In the case that $\mathbf{r}' = \mathbf{r} - \mathbf{R}(t)$, if we suppose that the axes of S and S' remain parallel and that $t' = t$ (which is always the case classically):

$$\ddot{\mathbf{r}'} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}}$$

So the equation of motion becomes:

$$m\ddot{\mathbf{r}'} = F - m\ddot{\mathbf{R}}$$

Note the appearance of a fictitious force, $-m\ddot{\mathbf{R}}$, which is due to the use of an accelerated frame and is proportional to the mass, as is the case for all fictitious forces.

1.8 Rotating frames

Suppose S is an inertial frame, and S' is a frame rotating about the z-axis with angular velocity $\omega = \dot{\theta}$ with respect to S. In order to understand motion in S', we need to relate the basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ of S and S' respectively.

Consider a particle at rest in S'. From the perspective of S, its velocity is

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

where $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ is the angular velocity vector (aligned with the rotation axis). This formula also applies to the basis vectors of S'.

$$\dot{\mathbf{e}_i'} = \boldsymbol{\omega} \times \mathbf{e}_i'$$

Given a time dependent vector $\mathbf{a}$, we can express it in the $\mathbf{e}_i'$ basis as follows:

$$\mathbf{a} = a_i'(t)\mathbf{e}_i'$$

From the perspective of S', $\mathbf{e}_i'$ is constant and so

$$(\dot{\mathbf{a}})_{S'} = \dot{a_i'}\mathbf{e}_i'$$

However, in S', $\mathbf{e}'_i$ is not constant. We apply the product rule to obtain the time derivative of $\mathbf{a}$:

$$(\dot{\mathbf{a}})_S = \dot{a}_i \mathbf{e}'_i + a'_i \boldsymbol{\omega} \times \mathbf{e}'_i = (\dot{\mathbf{a}})_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

This identity applies to all vectors and can be written as an operator equation:

Proposition. If S is an inertial frame, and S' is rotating relative to S with angular velocity $\boldsymbol{\omega}$, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times$$

We can apply this to the position vector twice to get:

2 Orbits

2.1 Orbits and force power laws

2.2 Kepler's Laws

3 Rigid body dynamics

4 Introduction to Lagrangian mechanics

So far, we have methods that work provided we only have one degree of freedom. In the case that we have more than one, we need a new method.

Example (free particle)**.**

$$\begin{aligned}
V &= 0 \\
T &= \frac{m(\dot{\mathbf{r}}_i \dot{\mathbf{r}}_i)}{2} \\
E &= T + V \\
\dot{E} &= m\dot{\mathbf{r}}_i \ddot{\mathbf{r}}_i = 0
\end{aligned}$$

This gives us 3 equations.

Consider Newton II: in both the rotational and linear cases, it equates a "force" to the rate of change of a "momentum". It makes sense[3] that we can invent generalised "forces", similarly related to generalised "momenta". This is exploited by Lagrangian mechanics.

Definition (Lagrangian)**.** A *Lagrangian*, $\mathcal{L}$, is a function of a generalised coordinate q, evolving with t such that

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V$$

where T is the kinetic energy measured in an inertial frame, and V is the potential energy.

[3] We do not derive this here. (yet...)

We then can then define a conjugate momentum $\mathbf{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i}$. As per our earlier discussion, we have an equivalent to Newton II:

$$\dot{\mathbf{p}}_i = \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i}$$

We call this the *Euler-Lagrange* equation.

Example (free particle (again))**.**

$$\begin{aligned}
T &= \frac{m(\dot{\mathbf{r}}_i \dot{\mathbf{r}}_i)}{2} \\
V &= 0 \\
\mathcal{L} &= T - V \\
\mathbf{p}_i &= \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} = m\dot{\mathbf{r}}_i \\
\dot{\mathbf{p}}_i &= \frac{\partial \mathcal{L}}{\partial \mathbf{r}_i} = 0 \\
\implies m\dot{\mathbf{r}}_i &= \text{const.}
\end{aligned}$$

5 Normal modes

6 Elasticity

7 Fluid dynamics