A synthetic approach to
finite presentability of homotopy groups

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Abstract

This is an abridged version of an in-progress paper that presents a synthetic proof of the finite presentability of homotopy groups of spheres. We have included it in the `serre-finiteness` repository as an informal guide to the Agda formalization, which follows the argument described herein closely.

Contents

1 Stably almost finite spaces

We define the property of being "stably almost finite" in three steps, beginning with the notion of an "n-finite" space. This notion is based on the notion of a *finite CW complex*, which is familiar in ordinary homotopy theory. Section A.3 summarizes the required facts about finite CW complexes in homotopy type theory.

Definition 1.1. Let $n \geq 0$. A space X is *n-finite* if there exists a finite CW complex C together with an $(n-1)$-connected map $f : C \to X$.

We emphasize that the intended constructive (or type-theoretic) reading of this definition (as well as the later ones in this section) is *as a "mere" proposition*, and not as extra data or structure on X. This distinction is largely immaterial in classical mathematics, but required for the constructive validity of the fundamental Lemma 1.5. A consequence is that it makes no difference whether we interpret the phrase "a finite CW complex C" in this definition as referring to a a specific finite CW structure in the sense of Definition A.17, or only to a space that admits at least one such structure. We freely use whichever interpretation is more convenient in any given context.

An n-finite space is also n'-finite for each $n' < n$, because then an n-connected map is also n'-connected.

Example 1.2. A finite CW complex X is n-finite for all n, since we may take $C = X$, $f = \mathrm{id}$. More generally, a space admitting a CW structure with finitely many cells in each dimension (such as the infinite-dimensional real or complex projective space) is also n-finite for all n: take C to be the n-skeleton of the given CW structure and $f : C \to X$ the inclusion of the n-skeleton; f is $(n-1)$-connected by Lemma A.26. Any n-connected space X (for $n \geq 0$) is n-finite as well; since X is connected, we can in particular choose a point $x : 1 \to X$, and the map x is $(n-1)$-connected (Lemma A.9).

If X is 1-finite and $x : X$ then $\pi_1(X, x)$ is a finitely generated group, because if $f : C \to X$ is a 0-connected map from a finite CW complex to X, then we may choose $c : C$ with $f(c) = x$ and the map $\pi_1(C, c) \to \pi_1(X, x)$ is surjective. Therefore, the wedge $\bigvee_{i:\mathbb{N}} S^1$ of countably many circles is 0-finite but not 1-finite, because $\pi_1(\bigvee_{i:\mathbb{N}} S^1)$ is not finitely generated.

Lemma 1.3. *In the definition of "n-finite", we may take the CW complex C to be n-dimensional.*

Proof. Suppose X is n-finite, and choose a finite CW structure $C_\bullet$ of some dimension N together with an $(n-1)$-connected map $f : C_N \to X$. If $N < n$, we may extend $C_\bullet$ to an n-dimensional CW structure with $C_n = C_N$ (and with no k-cells for $k > n$). Otherwise, $N \geq n$. Then the inclusion $j : C_n \to C$ is also $(n-1)$-connected, by Lemma A.26, so C_n is an n-dimensional finite CW complex and $fj : C_n \to X$ is again $(n-1)$-connected. □

Remark 1.4. Using the lemma, it is not hard to see that in classical homotopy theory, a space X is n-finite if and only if it admits (up to homotopy equivalence) a CW structure with only finitely many cells in each dimension $0, \ldots, n$, but possibly infinitely many cells in each higher dimension. However, the latter condition is inappropriate or at best inconvenient in a constructive setting. We explain why in Remark 1.14.

Lemma 1.5. *Let $j : X \to Y$ be a map from an $(n-1)$-finite space X to an n-finite space Y, and let Z denote the cofiber of j. Then the space Z is n-finite.*

Proof. Choose an $(n-1)$-dimensional finite CW complex C with an $(n-2)$-connected map $f : C \to X$ (using Lemma 1.3), and a finite CW complex D with an $(n-1)$-connected map

$g : D \to Y$.

$$\begin{tikzcd} C \arrow[r, dashed, "l"] \arrow[d, "f"] & D \arrow[r, dashed] \arrow[d, "g"] & E \arrow[d, dashed, "h"] \\ X \arrow[r, "j"] & Y \arrow[r] & Z \end{tikzcd}$$

Since C is $(n-1)$-dimensional and g is $(n-1)$-connected, we may choose a map $l : C \to D$ completing the left square as shown above (Lemma A.27). The cofiber E of l is a finite CW complex by Theorem A.20, and the induced map $h : E \to Z$ between cofibers is $(n-1)$-connected, by Lemma A.11. $\square$

Remark 1.6. The "lift" $l : C \to D$ used in the previous proof is in general not unique, and cannot be constructed as a "continuous" function of the input data. This is the reason that the conclusion of the lemma must be propositionally truncated.

In fact, a version of the lemma statement which constructs a specific finite CW structure approximating Z from ones approximating X and Y is provably false in homotopy type theory. We sketch the proof. Consider the case where $X = S^{m-1}$, $Y = S^m$ $(m \geq 2)$, we restrict $k : X \to Y$ to be a pointed map, we choose fixed finite CW structures C and D on X and Y (such as the obvious ones with two cells), and we take n sufficiently large compared to m. Of course any such map k is homotopic in $\mathrm{Map}_*(S^{m-1}, S^m)$ to the constant map $z : S^{m-1} \to S^m$, but not uniquely: the element $z : \mathrm{Map}_*(S^{m-1}, S^m)$ has automorphism group $\mathbb{Z}$, with generator $\alpha : z = z$ corresponding under adjunction to the identification $\Sigma S^{m-1} \simeq S^m$.

Suppose that, as a function of $k : \mathrm{Map}_*(S^{m-1}, S^m)$, we could construct a finite CW structure $E(k)$ together with an $(n-1)$-connected map $h : E(k) \to Z(k)$, where $Z(k)$ denotes the cofiber of k. The cellular cohomology theory of [2], provides an isomorphism between $H^m(E(k))$ and the cohomology $H^m_{\text{cell}}(E(k))$ of a cochain complex of finitely generated free abelian groups built from the CW structure of $E(k)$. As $n \gg m$, h induces an isomorphism on H^n, so we would obtain a term

$$B : \prod (k : \mathrm{Map}_*(S^{m-1}, S^m)),\ (H^m(Z(k)) \cong_{\mathsf{Ab}} H^m_{\text{cell}}(E(k))).$$

Applying B to the automorphism $\alpha : z = z$, we obtain an automorphism of the isomorphism $B(z) : H^m(Z(z)) \cong_{\mathsf{Ab}} H^m_{\text{cell}}(E(z))$, which amounts to a commutative square as below.

$$\begin{tikzcd} H^m(Z(z)) \arrow[r, "B(z)", "\cong"'] \arrow[d, "\alpha_*"'] & H^m_{\text{cell}}(E(z)) \arrow[d, "\alpha_*"] \\ H^m(Z(z)) \arrow[r, "B(z)", "\cong"'] & H^m_{\text{cell}}(E(z)) \end{tikzcd}$$

Crucially, since the construction of $H^n_{\text{cell}}(E(k))$ from k factors through a *set* (consisting of the data of the length of the cellular cochain complex, the ranks of its free groups, and the matrices of integers encoding its differentials) the automorphism $\alpha : z = z$ must act trivially on $H^n_{\text{cell}}(E(z))$. On the other hand, one can compute that α acts nontrivially on $H^m(Z(z))$, a contradiction.

This argument applies with equal force if "n-finite" is replaced by "stably n-finite" (to be introduced next), because cohomology is a stable invariant.

What *is* invariant under the action of α is the descending filtration of $H^m(Z(z))$ given by

$$F^0 = H^m(Z(z)) \supseteq F^1 = \ker\big(H^m(Z(z)) \to H^m(Y)\big) \supseteq F^2 = 0$$

together with the isomorphisms $F^0/F^1 \cong H^m(Y) \cong \mathbb{Z}$, $F^1/F^2 \cong H^{m-1}(X) \cong \mathbb{Z}$. This suggests that for the purpose of *untruncated* constructive arguments, it may be better to replace the notion of finite CW structure with another space-level notion that corresponds to this filtration in algebra. We have not pursued this direction.

Lemma 1.7. *If X is n-finite, then ΣX is $(n+1)$-finite.*

Proof. Apply Lemma 1.5 to the unique map $k : X \to 1$. □

Definition 1.8. We call a space X *stably n-finite* if there exists an integer $m \geq 0$ such that $\Sigma^m X$ is $(n+m)$-finite.

It follows immediately from the definition and Lemma 1.7 that:

- If X is stably n-finite, then $\Sigma^m X$ is $(n+m)$-finite for all sufficiently large m. In particular, given any finite collection $(X_i)_{i \in I}$ with X_i stably n_i-finite, we can find a single $m \geq 0$ such that $\Sigma^m X_i$ is $(n_i + m)$-finite for each i. We use this fact often.
- A space X is stably n-finite if and only if ΣX is stably $(n+1)$-finite. (Without "stably", we would have only the forward implication.)

Lemma 1.9. *If $X \to Y \to Z$ is a cofiber sequence and X and Z are stably n-finite, then so is Y.*

Proof. Choose m so that $\Sigma^m X$ and $\Sigma^m Z$ are stably $(n+m)$-finite. In the Puppe sequence beginning with $X \to Y \to Z$, we encounter at some point the cofiber sequence $\Sigma^m Z \to \Sigma^{m+1} X \to \Sigma^{m+1} Y$. Since $\Sigma^m Z$ is $(n+m)$-finite and $\Sigma^{m+1} X$ is $(n+m+1)$-finite, we deduce from Lemma 1.5 that $\Sigma^{m+1} Y$ is also $(n+m+1)$-finite, and so Y is stably n-finite. □

We now arrive at the main definition of this section.

Definition 1.10. A space X is *stably almost finite* if it is stably n-finite for every n.

Again we emphasize that the notion of "stably n-finite" is, by definition, propositionally truncated. That is, saying that X is stably almost finite does not mean that we can necessarily write down a *function* $m : \mathbb{N} \to \mathbb{N}$ such that $\Sigma^{m(n)} X$ is $(n + m(n))$-connected, much less any finite CW complex approximation depending functionally on n. We only ask that we can choose such an approximation for any given integer n, rather than for all integers n at once. This approach is sensible since our ultimate goal is to prove a theorem about a single abelian group $\pi_k X$, which depends on X only up to some "finite approximation".[1]

Observe that, for a pointed space X, X is stably almost finite if and only if ΣX is.

[1] But see Remark 1.14 regarding the choice of m.

Lemma 1.11. *If $X \to Y \to Z$ is a cofiber sequence and two of X, Y, Z are stably almost finite, then so is the third.*

Proof. If X and Z are stably almost finite then Y is as well by Lemma 1.9. For the other two cases we consider instead the "rotated" cofiber sequences $Y \to Z \to \Sigma X$ and $Z \to \Sigma X \to \Sigma Y$, applying the above observation. □

Example 1.12. Spaces admitting a CW structure with finitely many cells in each dimension are stably almost finite because they are already n-finite for each n (Example 1.2). On the other hand, there are also pointed spaces Y with $\pi_1(Y)$ nontrivial which are *acyclic*, meaning that ΣY is contractible. (See [3] for constructions of such spaces in homotopy type theory.) The wedge $X = \bigvee_{i:\mathbb{N}} Y$ of countably many copies of such a space is not 1-finite since $\pi_1(X)$ is not finitely generated (Example 1.2), but X is still stably almost finite, because $\Sigma X = \bigvee_{i:\mathbb{N}} \Sigma Y$ is contractible.

In another direction, any ∞-connected space X is also stably almost finite, since X is n-finite for each n (Example 1.2). In general, the ∞-connected spaces and ∞-connected maps are invisible to all the invariants and properties we consider in this paper.

The following remarks will not be used elsewhere in this paper, but may help clarify the notion of "stably almost finite".

Remark 1.13. In ordinary homotopy theory, a space X is said to be *homologically of finite type* if each homology group $H_n(X)$ is finitely presented (equivalently, finitely generated). Let us denote by $\mathcal{F}$ the class of such spaces. The class $\mathcal{F}$ has the following closure properties.

- $\mathcal{F}$ contains all finite CW complexes.
- If a space X admits arbitrarily highly-connected maps from spaces in $\mathcal{F}$, then $X \in \mathcal{F}$.
- $X \in \mathcal{F}$ if and only if $\Sigma X \in \mathcal{F}$.

These are elementary consequences of the Eilenberg–Steenrod axioms plus the fact that a highly-connected map induces an isomorphism on homology in low degrees.

The class of stably almost finite spaces is (almost by construction) the smallest class of spaces with the above closure properties. Thus, every stably almost finite space is homologically of finite type. The converse also holds: if X is homologically of finite type, we may assume X is simply connected (by suspending as needed) and then using the Hurewicz theorem, one can inductively construct arbitrarily good finite CW complex approximations to X. This argument also works in homotopy type theory for any homology theory satisfying the Eilenberg–Steenrod axioms and the Hurewicz theorem.

Thus, the condition "X is stably almost finite" precisely captures the condition "each homology group of X is finitely presentable" using space-level notions, rather than algebraic invariants.

Remark 1.14. Furthermore, if X is stably almost finite and simply connected, then X is actually already n-finite for each n. Hence for X a general stably almost finite space, $\Sigma^2 X$ is

n-finite for each n, i.e., we may take $m = 2$ in Definition 1.8. To prove this we will use our main Theorem 4.1.

Indeed, suppose X is stably almost finite and simply connected. We use the usual procedure (described for example in [8, Proposition 4.13]) to build a sequence of approximations $f_d : C_d \to X$ with C_d a d-dimensional finite CW complex and f_d an $(d-1)$-connected map for $d = 1, 2, \ldots, n$. At each stage we can choose a finite set of cells to continue the construction because all the homotopy groups that appear are finitely presentable, by Theorem 4.1.

Note however that if we try to continue this procedure forever to realize X as an *infinite-dimensional* CW complex of finite type, two problems might arise:

- At each step we need to choose a witness to a propositionally-truncated existence statement (the finite presentability of some abelian group). Hence, to make an infinite sequence of such choices, we seem to need some form of dependent choice.

- Even if we manage to make all these choices, we would only be able to deduce that the map from the resulting complex $C_\infty = \operatorname{colim}\big(C_1 \to C_2 \to C_3 \to \cdots\big)$ to X is ∞-connected, and not necessarily an equivalence.

It is for these reasons that the notion of "stably almost finite" is designed to require only finite CW complexes that approximate X (or its suspensions) arbitrarily well, rather than an entire infinite CW complex structure on X.

Remark 1.15. In homological algebra, a module M over a ring R is called *pseudocoherent* [13, Definition 064Q] if there are arbitrarily long "partial" resolutions

$$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \cdots \to R^{\oplus a_0} \to M \to 0$$

of M by finitely generated free R-modules. (Note that the first map of this exact sequence need not be injective, so this is not a true resolution of M.) Equivalently, as an object of the derived category, M admits arbitrarily highly connected maps from finite complexes of finitely generated free R-modules.

In the setting of ∞-categories, Lurie uses the adjective "almost perfect" for the corresponding notion [11, Definition 7.2.4.10]. This is the origin of our terminology "stably almost finite": A space X is stably almost finite if its suspension spectrum $\Sigma^\infty X$ is almost perfect (as a module over the sphere spectrum). We avoid explicitly working with the stable homotopy category by using the fact that any finite spectrum with a map to $\Sigma^\infty X$ is realized by a map from a finite CW complex to X up to some desuspension (which is tracked by our m).

We end this section with some assorted lemmas that will be needed in the next section.

Lemma 1.16. *Let $g : X \to Y$ be an $(n-1)$-connected map.*

(i) If X is stably n-finite, then so is Y.

(ii) If Y is stably $(n-1)$-finite, then so is X.

Proof. For both parts it suffices to prove the statement without the word "stably", by choosing m sufficiently large and applying the statement to the map $\Sigma^m g : \Sigma^m X \to \Sigma^m Y$, which will be $(n+m-1)$-connected (Lemma A.12).

If X is n-finite then we choose a finite CW complex C together with an $(n-1)$-connected map $f : C \to X$; the pair (C, gf) then shows that Y is also n-finite.

Suppose then that Y is $(n-1)$-finite. By Lemma 1.3, we may choose an $(n-1)$-dimensional CW complex D together with an $(n-2)$-connected map $f : D \to Y$. Since $g : X \to Y$ is $(n-1)$-connected, there exists a lift $l : D \to X$, so that $gl = f$; see Lemma A.27. By the cancellation property for connected maps (Lemma A.13), we deduce that l is also $(n-2)$-connected, so that X is also $(n-1)$-finite. □

Lemma 1.17. *Let X_1 and X_2 be two spaces and suppose X_i is stably n_i-finite and k_i-connected ($k_i \geq -2$), with*

(*i*) *either $n_1 > k_1$,*

(*ii*) *or $n_1 = 0$.*

Then the join $X_1 * X_2$ is stably $(\min(n_1 + k_2, n_2 + k_1) + 2)$-finite.

In particular, the join of two stably almost finite pointed spaces is stably almost finite.

Proof. For $i = 1, 2$, choose integers m_i, finite CW complexes C_i and $(n_i + m_i - 1)$-connected maps $f_i : C_i \to \Sigma^{m_i} X_i$. If $n_1 = 0 \leq k_1$, then we specifically choose $m_1 = 0$, $C_1 = 1$, $f_1 : 1 \to X_1$ the inclusion of some point of X_1; this map f_1 is indeed (-1)-connected by Lemma A.13, because X is 0-connected. If $n_1 > k_1$, then C_1 is $(k_1 + m_1)$-connected because $\Sigma^{m_1} X_1$ is and f_1 is $(n_1 + m_1 - 1)$-connected, while if $n_1 = 0 \leq k_1$, then $C_1 = 1$ is also certainly $(k_1 + m_1)$-connected.

The join $C_1 * C_2$ admits a finite CW complex structure by Lemma A.25. The space $\Sigma^{m_2} X_2$ is $(k_2 + m_2)$-connected, so by Lemma A.15, the induced map

$$f_1 * f_2 : C_1 * C_2 \to (\Sigma^{m_1} X_1) * (\Sigma^{m_2} X_2) \simeq \Sigma^{m_1 + m_2}(X_1 * X_2)$$

has connectivity at least

$$\begin{aligned} &\min((n_1 + m_1 - 1) + (k_2 + m_2), (n_2 + m_2 - 1) + (k_1 + m_1)) + 2 \\ = &(\min(n_1 + k_2, n_2 + k_1) + 2) + (m_1 + m_2) - 1. \end{aligned}$$

Therefore, $X_1 * X_2$ is stably $(\min(n_1 + k_2, n_2 + k_1) + 2)$-finite. □

Lemma 1.18. *If X is a connected space that is stably n-finite, $n \geq 0$, then the j-fold join $X^{*j} = X * \cdots * X$ is $(2j-2)$-connected and stably $(n + 2j - 2)$-finite.*

Proof. Trivial for $j = 1$. Assuming it holds for X^{*j}, apply Lemma A.16 and Lemma 1.17 to deduce that $X^{*(j+1)} = X * X^{*j}$ is $2j$-connected and stably N-finite for

$$N = \min(n + (2j-2), (n + 2j - 2) + 0) + 2 = n + 2j.$$

Here, we use the first case of Lemma 1.17 if $n > 0$ and the second case if $n = 0$. □

Lemma 1.19. *The wedge, smash product and cartesian product of two stably almost finite* pointed *spaces X and Y are also stably almost finite.*

Proof. By Lemma 1.17, $X * Y$ is stably almost finite. We apply Lemma 1.11 to the following cofiber sequences in turn, deducing that the boxed space in each sequence is stably almost finite from the fact that the other two spaces are.

$$\begin{array}{ccccc} X & \longrightarrow & \boxed{X \vee Y} & \longrightarrow & Y \\ \boxed{X \wedge Y} & \longrightarrow & 1 & \longrightarrow & X * Y \\ X \vee Y & \longrightarrow & \boxed{X \times Y} & \longrightarrow & X \wedge Y \end{array}$$

□

2 The Ganea construction

In this section we show that under simple connectivity hypotheses, if two of the three spaces in a fiber sequence are stably almost finite, then so is the third. To do this, we use the Ganea construction, originally used by Ganea to obtain various connectivity results similar to the ones in the present article [7]. The Ganea construction is closely related to the join construction used by Rijke to construct image factorizations in homotopy type theory [12].

Theorem 2.1 (Ganea)**.** *Let*

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[d, "p"] \\ 1 \arrow[r, "b"] & B \end{tikzcd} \tag{1}$$

be a pullback square. Form the pushout square

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[d, "q"] \\ 1 \arrow[r] & E' \end{tikzcd} \tag{2}$$

along with the induced map $p' : E' \to B$. Then the fiber F' of p' over $b : 1 \to B$ is equivalent to the join $F * \Omega_b B$.

Proof. We can argue as follows in any locally cartesian closed ∞-category with pushouts. In the left diagram, the inner square is a pushout and the outer square is a pullback. These conditions are preserved by pulling back the diagram along $b : 1 \to B$, yielding the right diagram.

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[d] \arrow[ddr, bend left, "p"] & \\ 1 \arrow[r] \arrow[drr, bend right, "b"'] & E' \arrow[dr, "p'"] & \\ & & B \end{tikzcd} \qquad \begin{tikzcd} F \times_B 1 \arrow[r] \arrow[d] & E \times_B 1 \arrow[d] \arrow[ddr, bend left] & \\ 1 \times_B 1 \arrow[r] \arrow[drr, bend right] & F' \arrow[dr] & \\ & & 1 \end{tikzcd}$$

On the other hand, we have identifications $E \times_B 1 = F$ and $1 \times_B 1 = \Omega_b B$; since the outer square on the right is a pullback, $F \times_B 1$ can be identified with $F \times \Omega_b B$ and the maps out of it with the projections; and since the inner square is a pushout, F' can be identified with the join $F * \Omega_b B$. □

Ganea's theorem is also an instance of a theorem proved by Rijke in homotopy type theory [12, Theorem 2.2], namely that the fiber of the join of two maps (with the same target) is equivalent to the join of their fibers. (By the join of two maps with target B, we mean the join of these maps considered as objects in the slice over B.) Here, $p' : E' \to B$ is by construction the join of the maps $p : E \to B$ and $b : 1 \to B$. Consequently the fiber F' of p' over the basepoint $b : B$ is equivalent to the join of the fibers of p and b, which are respectively F and $\Omega_b B$ by definition.

The statement of Ganea's theorem involves a construction transforming a fiber sequence $F \to E \to B$ into a new fiber sequence $F' \to E' \to B$ with the same base. By iterating this process, we obtain "Ganea's construction".

Construction 2.2. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fiber sequence of pointed spaces. We set $(F_0 \xrightarrow{j_0} E_0 \xrightarrow{p_0} B) = (F \xrightarrow{j} E \xrightarrow{p} B)$ and inductively construct a cofiber sequence $F_n \xrightarrow{j_n} E_n \xrightarrow{q_n} E_{n+1}$, with induced map $p_{n+1} : E_{n+1} \to B$, and then a fiber sequence $F_{n+1} \xrightarrow{j_{n+1}} E_{n+1} \xrightarrow{p_{n+1}} B$. These constructions assemble into a diagram

$$\begin{tikzcd}
F = F_0 \arrow[d, "j_0"] & F_1 \arrow[d, "j_1"] & F_2 \arrow[d, "j_2"] & \\
E = E_0 \arrow[r, "q_0"] \arrow[d, "p_0"] & E_1 \arrow[r, "q_1"] \arrow[d, "p_1"] & E_2 \arrow[r, "q_2"] \arrow[d, "p_2"] & \cdots \\
B & B & B &
\end{tikzcd}$$

in which the columns are fiber sequences, and each $F_n \xrightarrow{j_n} E_n \xrightarrow{q_n} E_{n+1}$ is a cofiber sequence.

By Ganea's theorem, $F_{n+1} \simeq F_n * \Omega B$ for each n, so we have the formula

$$F_n = F * \Omega B * \cdots * \Omega B \qquad \text{(with } n \text{ copies of } \Omega B). \tag{$*$}$$

In particular F_n is the join of $n+1$ pointed spaces, and so is at least $(n-1)$-connected.

For the rest of the section, we make the standing assumption that **the base B is connected**. In this case, *every* fiber of the map $p_n : E_n \to B$ is $(n-1)$-connected, not just the basepoint fiber F_n; so the map p_n itself is $(n-1)$-connected. Hence, as $n \to \infty$, we can regard the spaces E_n as "better and better approximations" to the space B (via the maps p_n). The "differences" between successive approximations are measured by the cofiber sequences $F_n \xrightarrow{j_n} E_n \xrightarrow{k_n} E_{n+1}$, and the formula $(*)$ ultimately lets us relate the spaces $E_0 = E$, F, ΩB and B.

We begin with the special situation in which we apply Construction 2.2 to the fiber sequence $\Omega B \to 1 \to B$.

Lemma 2.3. *If B is a connected pointed space and ΩB is stably almost finite, then B is also stably almost finite.*

Proof. Each fiber $F_n = \Omega B * \cdots * \Omega B$ (with $(n+1)$ copies) is stably almost finite by Lemma 1.17. Then $E_0 = 1$ and, by induction, each E_n is also stably almost finite (Lemma 1.11). Now, to prove that B is stably m-finite for any given m, we note that $p_m : E_m \to B$ is $(m-1)$-connected, and apply Lemma 1.16. □

Lemma 2.4. *If B is a simply connected pointed space which is stably almost finite, then ΩB is also stably almost finite.*

Proof. We prove by induction that ΩB is stably n-finite. Since ΩB is connected, the basepoint inclusion $1 \to \Omega B$ shows that ΩB is 0-finite.

Assume ΩB is stably n-finite. For large j, the map $p_j : E_j \to B$ is highly connected, so E_j is stably $(n+2)$-finite because B is (Lemma 1.16). For each $j \geq 1$, the fiber $F_j = (\Omega B)^{*(j+1)}$ is also stably $(n+2)$-finite, by Lemma 1.18. Hence, applying Lemma 1.9 repeatedly, we deduce that $\ldots$, E_2, E_1 are all stably $(n+2)$-finite. The cofiber sequence $\Omega B = F_0 \to 1 = E_0 \to E_1$ identifies E_1 with $\Sigma\Omega B$, so we deduce that ΩB is stably $(n+1)$-finite, completing the induction. □

For future use, we note the following consequence.

Lemma 2.5. *Let A be a finitely presentable abelian group and $n \geq 1$. Then the space $K(A,n)$ is stably almost finite.*

Proof. First suppose $n = 1$. By the structure theorem for finitely presented abelian groups (Theorem A.3), we may express A as the direct sum of a free abelian group $\mathbb{Z}^r$ and a finite abelian group G. Then $K(A,1) = K(\mathbb{Z}^r, 1) \times K(G,1) = (S^1)^r \times K(G,1)$. By Lemma 1.19, it suffices to prove that $K(G,1)$ is stably almost finite. But $\Omega K(G,1) = G$ is a finite set, so this follows from Lemma 2.3.

For $n \geq 2$, we use induction and Lemma 2.3 again, in view of the fact that $\Omega K(A,n) = K(A, n-1)$. □

Now we return to the case of a general fiber sequence $F \xrightarrow{j} E \xrightarrow{p} B$. Here B is a pointed space, but we do not assume that F and E are pointed.

Lemma 2.6. *If B is connected and F, B and ΩB are stably almost finite, then E is also stably almost finite.*

By Lemma 2.4, the assumption that ΩB is stably almost finite is redundant if B is simply connected.

Proof. The assumptions and the formula $(*)$ imply that B and each F_n is stably almost finite. We will prove that E is stably m-finite for each m. Since $p_{m+1} : E_{m+1} \to B$ is m-connected, Lemma 1.16 implies that E_{m+1} is stably m-finite. Now by backwards induction we deduce that E_m, $\ldots$, $E_0 = E$ are also stably m-finite using Lemma 1.9. □

Lemma 2.7. *If B and E are simply connected and stably almost finite, then F is also stably almost finite.*

Proof. Equip E with a basepoint which maps to the basepoint of B (which is possible as E is connected). By Lemma 2.4, the "rotated" fiber sequence $\Omega B \to F \to E$ satisfies the conditions of Lemma 2.6 (both ΩB and ΩE are stably almost finite). □

In the remainder of this section, we include some more closure properties of stably almost finite spaces. These facts are not needed for the proof of the main result, Theorem 4.1.

Lemma 2.8. *Lemma 2.7 is valid without the connectivity hypothesis on E.*

Proof. We reuse the argument of Lemma 2.4, with the information that ΩB is stably almost finite already in hand. Assume that F is stably n-finite for some n.

We know that E_j is stably $(n+2)$-finite for large j. Therefore, by the same argument as before, it will suffice to show that each fiber $F_j = F * (\Omega B)^{*j}$, $j \geq 1$, is also stably $(n+2)$-finite. We know

- F is stably n-finite and (-2)-connected;
- ΩB is stably almost finite and 0-connected. For $j \geq 1$, the same is true of $(\Omega B)^{*j}$.

Therefore, we can apply Lemma 1.17 to deduce that $F_j = F * (\Omega B)^{*j}$ is stably $(n+2)$-finite for $j \geq 1$. The same kind of backwards induction as before lets us deduce that E_1 is stably $(n+2)$-finite. Since $E = E_0$ is stably almost finite, the original fiber F must be stably $(n+1)$-finite, completing the induction. □

Lemma 2.9. *Suppose given a diagram $X \to Z \leftarrow Y$ of (unpointed) stably almost finite spaces, with X and Y connected and Z simply connected. Then the fiber product $X \times_Z Y$ is also stably almost finite (and connected).*

Proof. Equip X, Y, Z with basepoints to make the given diagram into a diagram of pointed spaces and pointed maps (which is possible as X and Z are connected). Then consider the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \times_Z Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ 1 \arrow[r] & Z \arrow[r, "\Delta"] & Z \times Z \end{tikzcd}$$

consisting of two pullback squares. By Lemma 1.19, $X \times Y$ and $Z \times Z$ are stably almost finite. Applying Lemma 2.7 to the outer rectangle, we learn that F is stably almost finite. Then by Lemma 2.6 applied to the left square, $X \times_Z Y$ is also stably almost finite. □

Likewise, we could obtain conditions under which the base of a fiber sequence $F \to E \to B$ is stably almost finite by applying Lemma 2.7 to $\Omega B \to F \to E$ and then using Lemma 2.3. As we will not need this, we omit the details.

3 The lowest homotopy group

In this section we prove:

Lemma 3.1. *Let $n \geq 2$. If X is an $(n-1)$-connected pointed space which is stably almost finite (or even just stably $(n+1)$-finite), then the abelian group $\pi_n X$ is finitely presentable.*

Ljungström and Pujet [9] have recently given a construction of the cellular homology of a finite CW complex and shown that it satisfies the Hurewicz theorem. Inspection of their proof shows that it produces as a byproduct the above statement for a finite CW complex X.

Proof. First, suppose that X is an $(n-1)$-connected finite CW complex. We use the proof of [9, Theorem 50], which shows that there is an n-connected map $\iota : C_f \to X$ from the cofiber of some map $f : \bigvee_A S^n \to \bigvee_B S^n$ between two finite wedges of n-spheres to X. The proof further shows that $\pi_n(C_f)$ is the cokernel of a group homomorphism between the free abelian groups on the finite sets A and B. In particular, $\pi_n(X) = \pi_n(C_f)$ is finitely presented.

Now, we treat the general case. Suppose X is an $(n-1)$-connected pointed space which is stably $(n+1)$-finite. Choose m and a finite CW complex C together with an $(n+m)$-connected map $f : C \to \Sigma^m X$. The space $\Sigma^m X$ is $(n+m-1)$-connected, so C is an $(n+m-1)$-connected finite CW complex. By the first part of the proof, $\pi_{n+m} C$ is finitely presentable. As f is $(n+m)$-connected, it induces an isomorphism on π_{n+m}, so $\pi_{n+m}(\Sigma^m X)$ is also finitely presentable. Finally, since X is $(n-1)$-connected and $n \geq 2$, the suspension map $\pi_n X \to \pi_{n+m}(\Sigma^m X)$ is an isomorphism by the Freudenthal suspension theorem. $\square$

Remark 3.2. While [9] was motivated in part by precisely this application, it is interesting to note that the actual cellular homology groups end up playing no role in the above proof. The key ingredient is [9, Theorem 46] which allows one to replace a CW structure on an $(n-1)$-connected space by a structure with a single 0-cell and no k-cells for $0 < k < n$. This theorem is proved entirely by homotopy-theoretic, connectivity-style arguments.

4 The main inductive argument

Theorem 4.1. *Let X be a simply connected pointed space which is stably almost finite. Then $\pi_n X$ is finitely presentable for each $n \geq 2$.*

Proof. Form the Postnikov tower $(||X||_n)_{n \geq 1}$ of X and for each n, let $X\langle n \rangle$ denote the n-connected cover of X, defined as the fiber of the truncation $\eta_n : X \to ||X||_n$. These constructions fit into a diagram

$$\begin{tikzcd}
 & X\langle 4\rangle \arrow[d, "\varphi_4"] & X\langle 3\rangle \arrow[d, "\varphi_3"] & X\langle 2\rangle \arrow[d, "\varphi_2"] & X\langle 1\rangle = X \arrow[d, "\varphi_1"] \\
 & X \arrow[d, "\eta_4"] & X \arrow[d, "\eta_3"] & X \arrow[d, "\eta_2"] & X \arrow[d, "\eta_1"] \\
\cdots \arrow[r] & ||X||_4 \arrow[r] & ||X||_3 \arrow[r] & ||X||_2 \arrow[r] & ||X||_1 = 1
\end{tikzcd}$$

and their homotopy groups satisfy

$$\pi_k(X\langle n\rangle) = \begin{cases} 0 & \text{if } k \le n, \\ \pi_k X & \text{if } k > n; \end{cases} \qquad \pi_k(||X||_n) = \begin{cases} \pi_k X & \text{if } k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Moreover, the fiber of each truncation map $||X||_n \to ||X||_{n-1}$ is a space with a single homotopy group $\pi_n X$ in degree n, hence is $K(\pi_n X, n)$. All the spaces appearing in the above diagram are simply connected.

We prove by induction on $n \geq 1$ that $X\langle n\rangle$ and $||X||_n$ are stably almost finite. This holds for $n = 1$ by assumption, so assume $n \geq 2$ and that the induction hypothesis holds for $n - 1$. Then:

- $\pi_n X\langle n-1\rangle = \pi_n X$ is finitely presentable, since $X\langle n-1\rangle$ is $(n-1)$-connected and stably almost finite (Lemma 3.1).
- $||X||_n$ is stably almost finite because (Lemma 2.6) it sits in a fiber sequence

  $$K(\pi_n X, n) \to ||X||_n \to ||X||_{n-1}$$

  and $K(\pi_n X, n)$ is stably almost finite by Lemma 2.5.
- $X\langle n\rangle$ is stably almost finite because it is the fiber of $\eta_n : X \to ||X||_n$, a map between simply connected stably almost finite spaces (Lemma 2.7).

In the course of the inductive argument we proved that the homotopy groups $\pi_n X$ for $n \geq 2$ are finitely presentable, as desired. □

Corollary 4.2. *For $n \geq 2$ and any k, the abelian group $\pi_n(S^k)$ is finitely presented.*

Proof. The pointed space S^k is stably almost finite and simply connected for $k \geq 2$, hence satisfies the hypotheses of the main theorem, while for $k \leq 1$ the groups in question are zero. □

A Constructive algebra and synthetic homotopy theory

In this appendix, we cover some material used in the main part of the paper which is standard in classical mathematics, but merits comment in our constructive and synthetic setting.

For the most part, we adopt the general setting and language of [14], except that we use the phrase "there exists" for what is there called *mere existence*. In other words, a statement like "there exists x such that $P(x)$" is always to be interpreted as the propositionally truncated type $\exists\,(x : X).\, P(x)$ (and never as $\Sigma\,(x : X).\, P(x)$, except when this type happens to already be a proposition). We caution the reader that many of the basic statements presented here (e.g., that the cofiber of a map between two finite CW complexes is a finite CW complex) would be false under the untruncated interpretation of "there exists". We make free use of quotients of sets by equivalence relations (including the aforementioned propositional truncations), function extensionality, the principle of unique choice, and so on, which distinguishes our setting from some other texts on constructive algebra, such as [10].

As in [14], we use the term "set" for a type that is 0-truncated. By an *(abelian) group* we mean a set equipped with algebraic structure in the standard sense. We use the term *isomorphism* in the context of sets or set-based algebraic structures. A set A is *finite* if there exists a natural number n and an isomorphism between A and $\mathsf{Fin}\, n$, some fixed type with n elements (e.g., $\mathsf{Fin}\, n :\equiv \Sigma\,(a : \mathbb{N}).\, a < n$).

A.1 Finitely presented abelian groups

In classical mathematics, every finitely generated abelian group is finitely presented. Furthermore, the class of finitely generated abelian groups enjoys strong closure properties: it is closed under extensions and under passage to arbitrary subgroups or quotient groups. (In other words, it forms a "Serre class" or "Serre subcategory" [13, Definition 02MO] in the category of all abelian groups.) Finally, there is a structure theorem: each finitely generated abelian group is a direct sum of finitely many cyclic groups.

In a constructive setting, none of the above statements are provable.

Example A.1. Let P be any proposition and let $H \subseteq \mathbb{Z}$ be the subgroup defined by

$$H = \{\, x \in \mathbb{Z} \mid x = 0 \vee P \,\}.$$

The quotient map $q : \mathbb{Z} \to \mathbb{Z}/H$ is a homomorphism between two finitely generated abelian groups. But if its quotient H is finitely generated, then the proposition P must be decidable. (As $\mathbb{Z}$ has decidable equality, if $x_1, \ldots, x_k \in H \subseteq \mathbb{Z}$ generate H, then either $x_i \neq 0$ for some i, in which case P holds, or $x_i = 0$ for all i, in which case $H = 0$ so that $1 \notin H$ and $\neg P$ holds.)

By similar reasoning, if the finitely generated group $\mathbb{Z}/H$ is isomorphic to a finite direct sum of cyclic groups, then P holds if and only if this direct sum is trivial, again implying that P is decidable.

However, the structure theorem is provable constructively for finitely *presented* abelian groups; and in general, this condition is better behaved than the (classically equivalent) condition of being finitely generated.

Definition A.2. An abelian group A is *finitely presented* if there exists a homomorphism $f : \mathbb{Z}^m \to \mathbb{Z}^n$ between finitely generated free abelian groups and an isomorphism $A \cong \operatorname{coker} f$. We regard finitely presented abelian groups as a full subcategory of the category of all abelian groups.

In this definition, the isomorphism type of A is determined by the homomorphism $f : \mathbb{Z}^m \to \mathbb{Z}^n$, which can in turn be encoded as an n-by-m matrix of integers. We can then use the standard algorithm for computing the Smith normal form of an integer matrix to establish the structure theorem below. By a *cyclic group*, we mean an abelian group of the form $\mathbb{Z}/\ell$ for some $\ell \geq 0$.

Theorem A.3 (Structure theorem for finitely presented abelian groups)**.** *Let A be a finitely presented abelian group.*

(i) There exists an isomorphism

$$A \cong \mathbb{Z}/\ell_1 \oplus \cdots \oplus \mathbb{Z}/\ell_k$$

between A and a direct sum of finitely many cyclic groups.

(ii) Furthermore, we can choose the ℓ_i so that $1 \neq \ell_1 \mid \cdots \mid \ell_k$.

(iii) Furthermore, the ℓ_i as in part (ii) are uniquely determined by A.

Proof. The first two parts follow from the existence of Smith normal form for the matrix encoding the morphism f in a choice of finite presentation of A. These parts have been formalized in the Coq proof assistant [5], and the first part has also been formalized in Cubical Agda by Kang Rongji.

For the uniqueness, we include an elementary counting argument, as we could not find a reference with a suitably constructive proof. Note that for a cyclic group $G = \mathbb{Z}/\ell$ and a prime power p^a, $a \geq 1$, we have

$$p^{a-1}G/p^aG \cong \begin{cases} \mathbb{Z}/p, & \text{if } p^a \mid \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if $A \cong \mathbb{Z}/\ell_1 \oplus \cdots \oplus \mathbb{Z}/\ell_k$ with the ℓ_i as in part (ii), $p^{a-1}A/p^aA$ is a finite group of order p^m, where m is the number of ℓ_i divisible by p^a. This means that in the sequence $(\ell_k, \ldots, \ell_1, 1, 1, \ldots)$, the first m elements are divisible by p^a and the rest are not. If also $A \cong \mathbb{Z}/\ell'_1 \oplus \cdots \oplus \mathbb{Z}/\ell'_{k'}$, then this reasoning shows that

$$(\ell_k, \ldots, \ell_1, 1, 1, \ldots) = (\ell'_{k'}, \ldots, \ell'_1, 1, 1, \ldots)$$

because corresponding elements on each side are divisible by exactly the same prime powers. Therefore $(\ell_1, \ldots, \ell_k) = (\ell'_1, \ldots, \ell'_{k'})$. □

For the proof of our main Theorem 4.1, Theorem A.3 part (i) is the only property of finitely presented abelian groups that we need. (We use it in the proof of Lemma 2.5.)

We close this section with some comments related to the notion of Serre class from traditional homotopy theory. Example A.1 also demonstrates that the class of finitely presented abelian groups is still not (provably) closed under passage to arbitrary subgroups or quotient groups. In other words, we cannot prove constructively that finitely presented abelian groups form a Serre class. This means in particular that the standard proof of the Serre finiteness theorem doesn't go through directly in homotopy type theory, even if we take for granted the existence of the homology Serre spectral sequence (a construction of which is sketched in [6, Section 5.5]).

However, we can prove constructively that a homomorphism between *two* finitely presented abelian groups has finitely presented kernel and cokernel. This property can be abstracted as part of the following definition.

Definition A.4 ([13, Lemma 0754])**.** A class $\mathcal{C}$ of abelian groups is a *weak Serre class* if $0 \in \mathcal{C}$ and, for any short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0,$$

if two of A, B, C belong to $\mathcal{C}$, then so does the third.

Lemma A.5. *A homomorphism $f : A \to B$ between finitely presented abelian groups has finitely presented kernel and cokernel.*

Proof. This is proved in [5, Sections 3–4]. The setting of [5] is slightly different from ours in that they represent a finitely presented abelian group as the cokernel of a specified linear transformation (or matrix); essentially, as an "untruncated" version of our Definition A.2. Likewise, they represent homomorphisms by suitable equivalence class of matrices. This is sufficient because every homomorphism between two abelian groups with specified finite presentations admits some lift to a homomorphism between the presentations, by [5, Lemma 1]; this lemma is constructive because its proof involves choosing preimages under surjections of only finitely many elements (one for each generator and relation of the domain group). $\square$

Perhaps less well-known is that the finitely presented abelian groups, defined in the truncated sense as here, are also closed under extensions:

Lemma A.6. *In a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, if A and C are finitely presented, then so is B.*

Proof. Choose finite presentations of A and C, assembling them into the diagram below.

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\mathbb{Z}^{a_1} \arrow[r] & \mathbb{Z}^{a_0} \arrow[r] & A \arrow[r] \arrow[d] & 0 \\
 & & B \arrow[d] & \\
\mathbb{Z}^{c_1} \arrow[r] & \mathbb{Z}^{c_0} \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

We produce a finite presentation of B using the "horseshoe lemma". We refer the reader to [15, Lemma 2.2.8] for the proof. The only issue is to check that all the choices involved are justified constructively. Note that the abelian groups $\mathbb{Z}^{a_i}$, $\mathbb{Z}^{c_i}$ are *finitely generated* free abelian groups, and therefore projective. For instance, we can begin by lifting the surjection $\mathbb{Z}^{c_0} \to C$ along the surjection $g : B \to C$, by choosing preimages under g of the images of each of the finitely many generators of $\mathbb{Z}^{c_0}$ in C.

In this way, we can run the proof of the horseshoe lemma for two steps, extending the original diagram to one of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathbb{Z}^{a_1} \arrow[r] \arrow[d] & \mathbb{Z}^{a_0} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}^{a_1} \oplus \mathbb{Z}^{c_1} \arrow[r] \arrow[d] & \mathbb{Z}^{a_0} \oplus \mathbb{Z}^{c_0} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}^{c_1} \arrow[r] \arrow[d] & \mathbb{Z}^{c_0} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact middle row. In particular, B is finitely presented. □

Remark A.7. The untruncated version of the previous result is provably false in homotopy type theory, much like the situation for Lemma 1.5. To see this, consider the type (with notation following [4]) $\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$, of which an element is an abelian group E together with a short exact sequence $0 \to \mathbb{Z} \xrightarrow{f} E \xrightarrow{g} \mathbb{Z} \to 0$, with basepoint the split exact sequence $0 \to \mathbb{Z} \xrightarrow{i_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{p_2} \mathbb{Z} \to 0$. By [4, Corollary 2.1.4] the automorphisms of the split exact sequence form the group $\pi_1(\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$. In particular, $\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$ is not a 0-type.

Suppose now that an untruncated version of Lemma A.6 held, in the sense that we had a function assigning to each $(0 \to \mathbb{Z} \xrightarrow{f} E \xrightarrow{g} \mathbb{Z} \to 0) : \mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$ a specific finite presentation $\mathbb{Z}^m \to \mathbb{Z}^n \to E \to 0$ of E. By the untruncated version of Theorem A.3, we could obtain a specific isomorphism between E and a finite sum of cyclic groups. Because $\mathbb{Z}$ is a projective abelian group, we know that this isomorphism must take the form $s(E) : E \cong \mathbb{Z} \oplus \mathbb{Z}$. This function s is then a section of the first component projection

$$T :\equiv \Sigma\,(0 \to \mathbb{Z} \xrightarrow{f} E \xrightarrow{g} \mathbb{Z} \to 0 : \mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})).\,(E \cong \mathbb{Z} \oplus \mathbb{Z}) \longrightarrow \mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}).$$

However, by rearranging Σ-types, T is just the type of all short exact sequences of the form $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0$, and therefore certainly a 0-type. So $\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$ cannot be a retract of T or it too would be a 0-type.

Corollary A.8. *Finitely presented abelian groups form a weak Serre class.*

Proof. Combine Lemmas A.5 and A.6. □

Weak Serre classes have some useful properties for homological algebra. If $\mathcal{C}$ is a weak Serre class, then:

(i) The homology groups of a complex of abelian groups in $\mathcal{C}$ are also in $\mathcal{C}$.

(ii) If $A_0 \to A_1 \to A_2 \to A_3 \to A_4$ is exact, and A_0, A_1, A_3, $A_4 \in \mathcal{C}$, then $A_2 \in \mathcal{C}$.

In particular, the integral (co)homology groups of a finite CW complex are finitely presented abelian groups. This can be shown by using either (i) or (ii) plus the Mayer–Vietoris sequence and induction, depending on the definition of the (co)homology theory in question.

In fact, it is also possible to carry out the standard proof of Theorem 4.1 via the homology Serre spectral sequence using only the fact that finitely presented abelian groups form a *weak* Serre class—at the cost of considerable extra complexity. This has been shown in unpublished work by the authors and (independently) by David Wärn.

A.2 Connectivity

We collect some basic results about connectivity that cannot be found in [14]. The indicated results have been formalized in the Cubical Agda library.

We remind the reader that, following the convention of [14], a map $f : X \to Y$ is said to be n-connected when all fibers of f are n-connected spaces. (This corresponds to what is called an $(n+1)$-connected (or $(n+1)$-connective) map or an $(n+1)$-equivalence in the classical homotopy theory literature.) A (-1)-connected map is the same as a surjection (or effective epimorphism) and every map (and every space) is (-2)-connected. We also caution the reader that in the Cubical Agda library, the `isConnected` and `isConnectedFun` predicates shift these connectivity levels by 2 (so that connectivity levels are indexed by natural numbers).

The n-connected maps are closed under composition [14, Lemma 7.5.6]. We often use the following fact:

Lemma A.9 ([14, Lemma 7.5.11]). *Let X be a space and $x : 1 \to X$ a point of X. Then X is n-connected if and only if the map x is $(n-1)$-connected*

A map is called n-truncated if all its fibers are n-truncated spaces. The n-connected and n-truncated maps form an orthogonal factorization system [14, Theorems 7.6.6, 7.6.7]. In particular, the natural map $\eta : X \to ||X||_n$ from a space X to its n-truncation is n-connected.

Lemma A.10 (`isConnectedPushout→`). *Consider a map of spans, as below.*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "f"'] \arrow[dd] & & B \arrow[dr, "g"] & \\
& A' \arrow[rr] \arrow[dd] & & B' \\
C \arrow[dr, "h"'] & & & \\
& C' & &
\end{tikzcd}$$

If f is $(n-1)$-connected and g and h are n-connected, then the induced map between the pushouts $B \amalg_A C \to B' \amalg_{A'} C'$ is n-connected.

Lemma A.11. *Given a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r] & B' \end{tikzcd}$$

in which f is $(n-1)$-connected and B is n-connected, the induced map between the cofibers $B/A \to B'/A'$ is n-connected.

Proof. Apply the previous lemma with $C = C' = 1$. $\square$

Lemma A.12 (`isConnectedSuspFun`)**.** *Let $g : X \to Y$ be an n-connected map. Then $\Sigma^m g : \Sigma^m X \to \Sigma^m Y$ is $(n+m)$-connected.*

Lemma A.13 (`isConnectedFunCancel'`)**.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be maps, and suppose that g is n-connected and gf is $(n-1)$-connected. Then f is $(n-1)$-connected.*

The analogues for the smash product of the following statements about the join are proved in Brunerie's PhD thesis [1, §4.3]. Those statements imply the ones here because there is a natural equivalence $X * Y \simeq \Sigma(X \wedge Y)$, and suspension increases connectivity by at least 1.

Lemma A.14 (`joinConnected`)**.** *If $f : X_1 \to X_2$ is n-connected and Y is k-connected, then $f * Y : X_1 * Y \to X_2 * Y$ is $(n+k+2)$-connected.*

Lemma A.15. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be maps which are respectively n_1-, n_2-connected, and assume that X_1 (not Y_1!), Y_2 are respectively k_1-, k_2-connected. Then $f_1 * f_2 : X_1 * X_2 \to Y_1 * Y_2$ is $(\min(n_1 + k_2, n_2 + k_1) + 2)$-connected.*

Proof. Factor $f_1 * f_2 : X_1 * X_2 \to Y_1 * Y_2$ as the composition

$$X_1 * X_2 \xrightarrow{X_1 * f_2} X_1 * Y_2 \xrightarrow{f_1 * Y_2} Y_1 * Y_2$$

and use Lemma A.14 and its symmetric version. $\square$

Lemma A.16. *If X and Y are respectively n_1-, n_2-connected then $X * Y$ is $(n_1 + n_2 + 2)$-connected.*

Proof. Apply the previous lemma with $k_1 = n_1$, $k_2 = n_2$. $\square$

A.3 Finite CW complexes

The following definition is lightly adapted from [9].

Definition A.17. An *n-dimensional finite CW structure $C_\bullet$* consists of

- a sequence of spaces $C_{-1} = \varnothing$, C_0, …, C_n;

- a sequence of natural numbers $c_0, \ldots, c_n$;
- for $0 \le k \le n$, specified pushout squares of the form

$$\begin{tikzcd} \mathsf{Fin}\, c_k \times S^{k-1} \arrow[r, "p_1"] \arrow[d] & \mathsf{Fin}\, c_k \arrow[d] \\ C_{k-1} \arrow[r] & C_k \end{tikzcd} \tag{1}$$

  where $\mathsf{Fin}\, c_k$ denotes the discrete space with c_k points.

The *realization* of such a CW structure is defined to be the last space, C_n. We call a space X an *n-dimensional finite CW complex* if there exists an n-dimensional finite CW structure $C_\bullet$ with $X = C_n$; we also call such a $C_\bullet$ an "(n-dimensional) finite CW structure on X". We call X a *finite CW complex* if it is an n-dimensional finite CW complex for some n.

Outside of certain examples and remarks, we only ever consider finite CW complexes. We sometimes omit the word "finite" when the finiteness is not relevant. In the definition above, we are careful to distinguish between a *CW structure*—the data of the C_k, c_k and pushout squares—and a *CW complex*, a space with the mere property that it is the realization of some CW structure. Otherwise, we use the standard terminology relevant to CW complexes: c_k is the number of k-cells of $C_\bullet$, the left vertical map of (1) is the attaching map for the k-cells, and so on. By the "k-skeleton" of a CW structure $C_\bullet$, depending on the context, we mean either the k-dimensional CW structure obtained by truncating $C_\bullet$ at dimension k, or the space C_k (which is a k-dimensional CW complex).

Example A.18. The empty space $\varnothing$ is the unique (-1)-dimensional CW complex.

Example A.19. The point $X = 1$ is a 0-dimensional finite CW complex, since we can take $C_0 = 1$, $c_0 = 1$ and the unique pushout square

$$\begin{tikzcd} 1 \times S^{-1} \arrow[r] \arrow[d] & 1 \arrow[d] \\ C_{-1} \arrow[r] & C_0 \end{tikzcd}$$

recalling that $S^{-1} = \varnothing$. In general, the 0-dimensional finite CW complexes are the finite sets.

We next establish an "induction principle" for finite CW complexes (Corollary A.22) saying that the class of finite CW complexes is the smallest class of spaces that contains $\varnothing$ and 1 and is closed under pushouts. The only hard part is to prove that finite CW complexes actually are closed under pushouts; this is proved in [9] using a version of the cellular approximation theorem (which relies crucially on the fact that we work with *finite* CW complexes).

Theorem A.20 ([9, Corollary 15], `isFinCWPushout`). *The class of finite CW complexes is closed under pushouts.*

In particular, each sphere S^n is a finite CW complex (also easily checked directly) and if X is a finite CW complex, then so is its suspension ΣX.

Lemma A.21 (`elimPropFinCW`)**.** *Let $\mathcal{C}$ be a class of spaces satisfying the following conditions:*

(i) $\varnothing \in \mathcal{C}$*;*

(ii) $1 \in \mathcal{C}$*;*

(iii) for any pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

if X, Y, $Z \in \mathcal{C}$, then $W \in \mathcal{C}$.

Then every finite CW complex belongs to $\mathcal{C}$.

Proof. First, $\mathcal{C}$ contains each sphere S^{n-1}, by induction on n: $S^{-1} = \varnothing$ and $S^n = 1 \amalg_{S^{n-1}} 1$. Next, $\mathcal{C}$ is closed under finite coproducts: the empty coproduct by (i), and binary coproducts by (iii). Therefore, $\mathcal{C}$ contains all spaces of the form $\mathsf{Fin}\, c_k \times S^{k-1} = \coprod_{i:\mathsf{Fin}\, c_k} S^{k-1}$ and $\mathsf{Fin}\, c_k = \coprod_{i:\mathsf{Fin}\, c_k} 1$. Then for any n-dimensional finite CW structure $C_\bullet$, each C_k belongs to $\mathcal{C}$ for each k by induction on k, using (iii). Now if X is a finite CW complex, then "$X \in \mathcal{C}$" is a proposition, so we may choose $n : \mathbb{N}$ and an n-dimensional finite CW structure $C_\bullet$ with $X = C_n$, showing that $X \in \mathcal{C}$. □

Corollary A.22. *The class of CW complexes is the smallest class of spaces that contains $\varnothing$ and 1 and is closed under pushouts.*

Using this induction principle, we establish further closure properties of the class of finite CW complexes.

Lemma A.23 (`isFinCW×`)**.** *If X and Y are finite CW complexes, then so is $X \times Y$.*

Proof. Let $\mathcal{C}$ denote the class of all spaces Z such that $X \times Z$ is a finite CW complex. It is enough to check that $\mathcal{C}$ satisfies the conditions of Lemma A.21, as then in particular $Y \in \mathcal{C}$. We have $X \times \varnothing = \varnothing \in \mathcal{C}$ and $X \times 1 = X \in \mathcal{C}$ (as X is a finite CW complex by assumption). Given a pushout square

$$\begin{array}{ccc} Z_0 & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ Z_2 & \longrightarrow & Z_3 \end{array}$$

if $X \times Z_i$ is a finite CW complex for $i = 0$, 1, 2, then so is

$$X \times Z_3 = (X \times Z_1) \amalg_{X \times Z_0} (X \times Z_2)$$

by Theorem A.20. This shows that $\mathcal{C}$ is closed under pushouts. □

Remark A.24. A similar argument shows that the class of finite CW complexes is closed under Σ-types (`isFinCWΣ`). This generalizes the above result, because $X \times Y = \sum_{x:X} Y$.

Lemma A.25 (`isFinCWJoin`). *If X and Y are finite CW complexes, then so is $X * Y$.*

Proof. By definition, the join $X * Y$ fits in a pushout square

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X * Y \end{tikzcd}$$

so this follows from Theorem A.20 and Lemma A.23. □

We end this section with two basic facts relating the dimension of a finite CW complex and the connectivity of maps.

Lemma A.26 (`mapFromNSkel`). *Let $C_\bullet$ be an n-dimensional finite CW structure. The inclusion of the k-skeleton $C_k \to C_n$ is a $(k-1)$-connected map.*

Proof. The inclusion $C_j \to C_{j+1}$ is $(j-1)$-connected by [9, Lemma 12], so the composition

$$C_k \to C_{k+1} \to \cdots \to C_n$$

of inclusions is $(k-1)$-connected. □

Lemma A.27 (`liftFromNDimFinCW`). *In any diagram*

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "k"] & Z \end{tikzcd}$$

with X an n-dimensional finite CW complex and $g : Y \to Z$ an $(n-1)$-connected map, there exists a lift $l : X \to Y$ making the triangle commute.

Proof. Choose an n-dimensional finite CW structure $C_\bullet$ with $X = C_n$. The map $f : \varnothing = C_{-1} \to C_n$ is a composition of finitely many pushouts of maps of the form $S^m \to 1$ with $m \leq n-1$. Because g is $(n-1)$-connected, we can successively extend the map $\varnothing \to Y$ over each of these cell attachments to a lift $l : X \to Y$ of $k : X \to Z$. □

Note that this proof depends on the fact that X is a *finite* CW complex, so that we encounter only finitely many lifting problems, and therefore can choose a solution for each of them.

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