MGF-Localization: State Estimation and Propagation using Moment Generating Functions

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***Abstract*— The *de facto* method for nonlinear state estimation has long been the Extended Kalman Filter. However, one significant drawback of the method is that the underlying state distribution is represented using a Gaussian distribution. Such unimodal distributions fail to estimate the underlying state when there are symmetries in the map geometry. In this paper, we introduce state localization using a moment generating function to represent the underlying state. Such a state representation does not require the assumption of any specific underlying distribution function. As long as the distribution function is continuous over state space, the state evolves when new motion and sensor information is obtained.**

I. IMPACT

The potential impact of the proposed research is that the underlying distribution for the state no longer needs to be an unimodal Gaussian distribution. In fact, the distribution can be any arbitrary continuous function over state space. This allows state estimation when there are symmetries in the geometry of the map. Such an example is depicted in Figure 1. An EKF algorithm would not be able to simultaneously localize the state going left and going right. Instead, the algorithm would arbitrarily select a direction and reinforce this belief, even when there is a 50% probability of choosing the incorrect path.

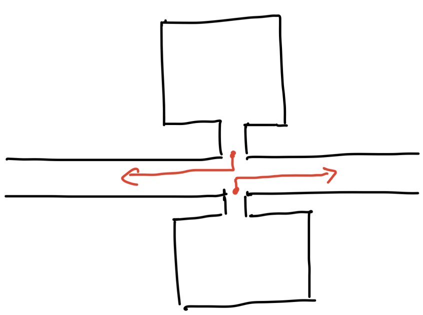

Fig. 1. Example of when EKF would not be able to localize

II. NOVELTY

This paper introduces a novel approach to represent the underlying state using a moment generating function (MGF) instead of the traditionally used probability density function (PDF). For a state variable $\mathbf{X}$, the moment generating function is defined as $M_{\mathbf{X}}(\mathbf{s}) = \mathbb{E}[e^{\mathbf{s}^T\mathbf{X}}]$. The equivalency between the two functions is guaranteed by the uniqueness property of moment generating functions [2]. I.e., the following information are equivalent and unique:

$$\{\textit{PDF of } X\} = \{\textit{MGF of } X\} = \{\textit{Central Moments of } X\}$$

Propagating the MGF has some unique advantages over the PDF, the most notable being linearity. Using the MGF also allows us to use an arbitrary continuous function to represent the state instead of assuming the Gaussian distribution.

III. METHODS

We assume a motion model of the form:

$$\tilde{\mathbf{X}}_t = A\mathbf{X}_{t-1} + \mathbf{b}_{t-1} \tag{1}$$

Where $\mathbf{b}_t = b\mathbf{u}_t$ is the control input.

The sensor model ($\mathbf{Z}_{t-1}$) is distribution agnostic and can be modeled by any continuous distribution. The motion model and the sensor model is fused using a linear combination to give the posterior state estimate:

$$\mathbf{X}_t = \alpha\mathbf{Z}_{t-1} + (1-\alpha)\tilde{\mathbf{X}}_t \tag{2}$$

Where $\alpha \in \mathbb{R}_{[0,1]}$.

Exploiting linearity and independence properties, the posterior MGF after propagating through Equations 1 and 2 is given by:

$$M_{\mathbf{X}_t}(\mathbf{s}) = M_{\mathbf{b}_{t-1}}(\mathbf{s})M_{\mathbf{Z}_{t-1}}(\alpha\mathbf{s})M_{\mathbf{X}_{t-1}}((1-\alpha)A\mathbf{s}) \tag{3}$$

Using elementary algebra and the evaluation property of MGFs:

$$\mathbb{E}[\mathbf{X}^k] = M_{\mathbf{X}}^{(k)}\Big|_{\mathbf{s}=0}$$

Equation 3 can be evaluated to

$$\begin{aligned}\mathbb{E}[\mathbf{X}_t^n] = \sum_{i=0}^{n}\sum_{j=0}^{n-j}\frac{n!}{i!j!(n-i-j)!} \times (1-\alpha)^i\mathbb{E}[\mathbf{b}_{t-1}^i] \\ \times \alpha^j\mathbb{E}[\mathbf{Z}_{t-1}^j] \times (A(1-\alpha))^{n-i-j}\mathbb{E}[\mathbf{X}_{t-1}^{n-i-j}]\end{aligned} \tag{4}$$

For multivariate state vector $\mathbf{X}$, the multiplications become tensor products and the evaluation property become n-dimensional derivative tensors.

Given the first n central moments of the prior state ($\mathbf{X}_{t-1}$), control input ($\mathbf{u}_{t-1}$), and the sensor measurement ($\mathbf{Z}_{t-1}$), we can calculate the posterior n central moments and theoretically recover the state distribution.

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IV. METRICS FOR SUCCESS

Our proposed MGF-Localization method will be evaluated based on the following key metrics:

Localization Accuracy: We will assess the accuracy of the estimated state by comparing it with the ground truth pose using metrics such as Mean Square Error (MSE) and Mean Absolute Error (MAE). Additionally, we will evaluate the method's ability to localize accurately in environments with symmetrical features, where traditional approaches may struggle.

Efficiency and Convergence: We will evaluate the algorithm's efficiency by measuring both its computational runtime and the number of iterations required to converge to an accurate estimate. Comparisons will be made against the Extended Kalman Filter (EKF) or other baseline methods to assess improvements in speed, memory requirements, and scalability for real-time applications.

Robustness to Noise and Uncertainty: We will analyze the algorithm's performance under varying levels of sensor noise and motion uncertainty. Additionally, we will examine failure cases where EKF struggles due to incorrect unimodal Gaussian assumptions, assessing whether MGF-Localization provides a more reliable alternative in such scenarios.

V. CHALLENGES

Our proposed MGF-Localization method has following challenges:

Computational Complexity of Moment-Based Representations: The proposed method relies on moment generating functions (MGFs) and their higher-order derivatives for estimating the state. For high-dimensional state spaces, computing and propagating higher-order moments could become computationally expensive.

Sensitivity to Sensor Noise and Motion Uncertainty: Since the method combines motion and sensor models in a weighted manner (α), choosing the optimal weight for different environments and sensor characteristics could be challenging. If sensor noise is high or motion uncertainty is large, the estimated moments may diverge from the true state distribution.

Real-World Applicability and Robustness: While the method is distribution-agnostic, its performance in real-world scenarios is unclear. Ensuring robustness across different environments would requeir extensive testing.

VI. RESOURCES

Data from HW1 & HW2: We will leverage the datasets and frameworks provided in HW1 and HW2 as the foundation for our evaluation. By integrating our MGF-Localization method into the existing assignments, we can directly compare its performance against particle filter localization and the Extended Kalman Filter (EKF). These resources provide a controlled environment for assessing accuracy, robustness, and computational efficiency, allowing for a thorough analysis of MGF-Localization's advantages and limitations.

KITTI Odometry Dataset [1]: The KITTI odometry dataset offers stereo images, LiDAR, and GPS-IMU data with accurate ground-truth trajectories for real-world autonomous driving scenarios. Its diverse road conditions and dynamic elements introduce challenges like sensor noise and varying motion speeds. By benchmarking against KITTI, we can validate our method's ability to handle non-Gaussian uncertainties in large-scale outdoor localization.

TUM RGB-D SLAM Dataset [3]: The TUM RGB-D dataset provides RGB, depth, and IMU data with precise ground-truth trajectories, making it ideal for evaluating localization methods in indoor environments. Its controlled settings allow for reliable error analysis, while challenging sequences with motion blur and textureless surfaces test robustness. This dataset helps assess how well our MGF-based localization handles perceptual ambiguities and map symmetries in structured indoor scenes.

VII. TIMELINE

The project will be executed in four phases.

Literature Review and Theoretical Development (Week 9): We will conduct a comprehensive review of existing localization methods, focusing on the limitations of EKF and the potential advantages of moment-generating functions (MGF). We will also develop a more detailed theoretical formulation, and validate the mathematical properties and assumptions of MGF-Localization.

Implementation and Preliminary Testing (Week 11): We will implement the MGF-based localization algorithm in a controlled simulation environment. We will first replace the localization algorithm in the first homework, and then compare MGF-Localization with particle filter localization to assess its advantages and limitations. Additionally, we will survey classical SLAM implementations and benchmark MGF-Localization against EKF in simple scenarios to understand its relative performance.

Experimental Evaluation (Week 12): We will test the algorithm on real-world, indoor datasets with symmetric poses such as the TUM RGB-D Dataset to assess its effectiveness in practical environments. Performance will be evaluated based on key metrics, including accuracy, efficiency, and robustness under varying conditions.

Documentation and Finalization (Week 13): We will document the experiments and results, and organize them into a structured report. Additionally, we will analysis MGF-Localization's limitations, weaknesses, and potentials.

VIII. REFERENCE

[1] Andreas Geiger, Philip Lenz, and Raquel Urtasun. "Are we ready for Autonomous Driving? The KITTI Vision Benchmark Suite". In: *Conference on Computer Vision and Pattern Recognition (CVPR)*. 2012.

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