Lecture Notes: A Dataflow Analysis Framework for WHILE3ADDR

17-355/17-665/17-819O: Program Analysis (Spring 2020)
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1 Defining a dataflow analysis

A dataflow analysis computes some dataflow information at each program point in the control flow graph.[1] We thus start by examining how this information is defined. We will use σ to denote this information. Typically σ tells us something about each variable in the program. For example, σ may map variables to abstract values taken from some set L:

$$\sigma \in \mathit{Var} \rightarrow L$$

L represents the set of abstract values we are interested in tracking in the analysis. This varies from one analysis to another. For example, consider a *zero analysis*, which tracks whether each variable is zero or not at each program point (Thought Question: Why would this be useful?). For this analysis, we define L to be the set $\{Z, N, \top\}$. The abstract value Z represents the value 0, N represents all nonzero values. $\top$ is pronounced "top", and we define it more concretely later it in these notes; we use it as a question mark, for the situations when we do not know whether a variable is zero or not, due to imprecision in the analysis.

Conceptually, each abstract value represents a set of one or more concrete values that may occur when a program executes. We define an abstraction function α that maps each possible concrete value of interest to an abstract value:

$$\alpha : \mathbb{Z} \rightarrow L$$

For zero analysis, we define α so that 0 maps to Z and all other integers map to N:

$$\alpha_Z(0) = Z$$
$$\alpha_Z(n) = N \text{where } n \neq 0$$

The core of any program analysis is how individual instructions in the program are analyzed and affect the analysis state σ at each program point. We define this using *flow functions* that map the dataflow information at the program point immediately *before* an instruction to the dataflow information *after* that instruction. A flow function should represent the semantics of the instruction, but abstractly, in terms of the abstract values tracked by the analysis. We will link semantics to the flow function precisely when we talk about correctness of dataflow analysis. For now, to approach the idea by example, we define the flow functions f_Z for zero analysis on WHILE3ADDR as follows:

[1] Refer to the first set of course notes for an overview of CFGs.

$$f_Z[\![x := 0]\!](\sigma) = [x \mapsto Z]\sigma \quad (1)$$
$$f_Z[\![x := n]\!](\sigma) = [x \mapsto N]\sigma \text{ where } n \neq 0 \quad (2)$$
$$f_Z[\![x := y]\!](\sigma) = [x \mapsto \sigma(y)]\sigma \quad (3)$$
$$f_Z[\![x := y \; op \; z]\!](\sigma) = [x \mapsto \top]\sigma \quad (4)$$
$$f_Z[\![\text{goto } n]\!](\sigma) = \sigma \quad (5)$$
$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!](\sigma) = \sigma \quad (6)$$

In the notation, the form of the instruction is an implicit argument to the function, which is followed by the explicit dataflow information argument, in the form $f_Z[\![I]\!](\sigma)$. (1) and (2) are for assignment to a constant. If we assign 0 to a variable x, then we should update the input dataflow information σ so that x maps to the abstract value Z. The notation $[x \mapsto Z]\sigma$ denotes dataflow information that is identical to σ except that the value in the mapping for x is updated to refer to Z. Flow function (3) is for copies from a variable y to another variable x: we look up y in σ, written $\sigma(y)$, and update σ so that x maps to the same abstract value as y.

We start with a generic flow function for arithmetic instructions (4). Arithmetic can produce either a zero or a nonzero value, so we use the abstract value $\top$ to represent our uncertainty. More precise flow functions are available based on certain instructions or operands. For example, if the instruction is subtraction and the operands are the same, the result will definitely be zero. Or, if the instruction is addition, and the analysis information tells us that one operand is zero, then the addition is really a copy and we can use a flow function similar to the copy instruction above. These examples could be written as follows (we would still need the generic case above for instructions that do not fit such special cases):

$$f_Z[\![x := y - y]\!](\sigma) = [x \mapsto Z]\sigma$$
$$f_Z[\![x := y + z]\!](\sigma) = [x \mapsto \sigma(y)]\sigma \quad \text{where } \sigma(z) = Z$$

Exercise 1. Define another flow function for some arithmetic instruction and certain conditions where you can also provide a more precise result than $\top$.

The flow function for branches ((5) and (6)) is trivial: branches do not change the state of the machine other than to change the program counter, and thus the analysis result is unaffected.

However, we can provide a better flow function for conditional branches if we distinguish the analysis information produced when the branch is taken or not taken. To do this, we extend our notation once more in defining flow functions for branches, using a subscript to the instruction to indicate whether we are specifying the dataflow information for the case where the condition is true (T) or when it is false (F). For example, to define the flow function for the true condition when testing a variable for equality with zero, we use the notation $f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_T(\sigma)$. In this case we know that x is zero so we can update σ with the Z lattice value. Conversely, in the false condition we know that x is nonzero:

$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_T(\sigma) = [x \mapsto Z]\sigma$$
$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_F(\sigma) = [x \mapsto N]\sigma$$

Exercise 2. Define a flow function for a conditional branch testing whether a variable $x < 0$.

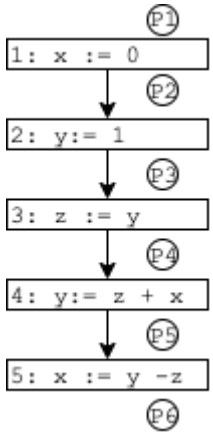
2 Running a dataflow analysis

The point of developing a dataflow analysis is to compute information about possible program states at each point in a program. For example, for of zero analysis, whenever we divide some expression by a variable x, we might like to know whether x must be zero (the abstract value Z) or may be zero (represented by $\top$) so that we can warn the developer.

2.1 Straightline code

One way to think of a simple dataflow analysis is that are statically simulating program execution, tracking only the information we care about. For each node in the CFG (each of which contains an instruction), we use the flow function to compute the dataflow analysis information at the program point immediately *after* that node from the information we had at the program point *before* that node. To demonstrate, consider the following simple program (left), with its control flow graph (middle):

$1: \quad x := 0$
$2: \quad y := 1$
$3: \quad z := y$
$4: \quad y := z + x$
$5: \quad x := y - z$



	x	y	z
P1	?	?	?
P2	Z	?	?
P3	Z	N	?
P4	Z	N	N
P5	Z	N	N
P6	$\top$	N	N

For such simple code, we can track analysis information using a table with a column for each program variable and a row for each program point (right, above).

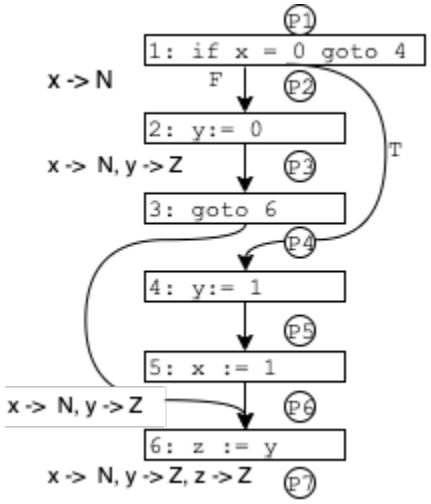
The first thing to notice is that, because flow functions operate on the abstract state for the program point immediately before a node, we need some kind of *initial* assumption (this confusion is illustrated by the ? in the cells of the table). We will return to this point in a moment, since those values don't influence the analysis for such simple, straight-line code.

Notice also that the analysis is imprecise at the end with respect to the value of x. We were able to keep track of which values are zero and nonzero quite well through instruction 4, using (in the last case) the flow function that knows that adding a variable known to be zero is equivalent to a copy. However, at instruction 5, the analysis does not know that y and z are equal, and so it cannot determine whether x will be zero. Because the analysis is not tracking the exact values of variables, but rather approximations, it will inevitably be imprecise in certain situations. However, in practice, well-designed approximations can often allow dataflow analysis to compute quite useful information.

2.2 Alternative Paths: Illustration

Things get more interesting in WHILE3ADDR code that contains `if` statements. An `if` statement introduces two possible paths through the program. Consider the following simple example (left), and its CFG (middle). We will begin by analyzing the first node as though the branch is not taken:

1 : if $x = 0$ goto 4
2 : $y := 0$
3 : goto 6
4 : $y := 1$
5 : $x := 1$
6 : $z := y$

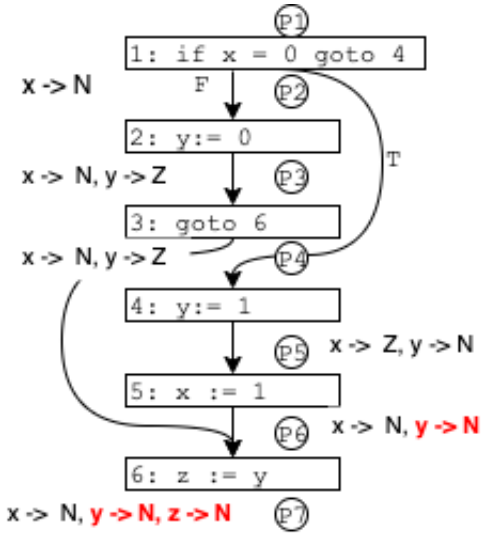


	x	y	z
P1	?	?	?
P2	N_F	?	?
P3	N	Z	?
P4	N	Z	?
P5			
P6	N?	Z?	??
P7	N??	Z??	Z??

In the table above, the entry for x at P2 indicates the abstract value produced for the false condition on the branch, which is then used as input to analyze instruction 2 (and produce the state at P3). We can go right from P3 to P4 without any complexity. But, if we just continue "simulating" execution, we get to P6. It has *two* possible incoming edges, so two possible incoming states to use for the flow function for instruction 6. What to do? We have not yet analyzed a path through lines 4 and 5. The table shows the (questionable) values if we just use the state coming from P4 as "incoming" at instruction 6, and ignore what might have happened along that other path.

Perhaps turning to that alternative path, will give answers. Let's analyze instructions 4 and 5 as if we had taken the true branch at instruction 1:

	x	y	z	
P1	??	??	??	
P2	Z_T, N_F	??	??	
P3	N	Z	??	
P4	N	Z	??	
P5	Z	N	??	
P6	N?	N?	??	*note: different!*
P7	N??	Z??	Z??	*??????*



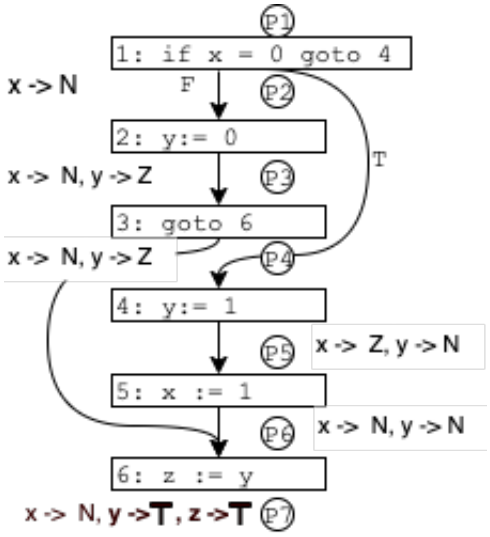
We have a dilemma. The first time we analyzed instruction 6, the incoming state had come

from instruction 3, where x was nonzero and y was zero. Now have, the incoming state coming from instruction 5 is different: x is still nonzero, but so is y!

We resolve this dilemma by *combining* the abstract values computed along the two paths for y. The incoming abstract values at P6 for y are N and Z. We represent this uncertainty with a new abstract value $\top$ (pronounced "top"). This value indicates that we do know know if y is zero or not, because we don't know how we reached this program location. We can apply similar logic to x, but because x is nonzero on both incoming paths, we can maintain our knowledge that x is nonzero. Thus, we should analyze instruction 6 with this combined incoming state: $\{x \mapsto N, y \mapsto \top\}$.

The corrected analysis, showing the *combined* state at P6, looks like:

	x	y	z	
P1	?	?	?	
P2	Z_T, N_F	?	?	
P3	N	Z	?	
P4	N	Z	?	
P5	Z	N	?	
P6	N	$\top$	?	*combined with P4*
P7	N	$\top$	$\top$	*corrected*



3 Join

The mechanism for combining analysis results along multiple paths is called a *join* operation, $\sqcup$. When taking two abstract values $l_1, l_2 \in L$, the result of $l_1 \sqcup l_2$ is an abstract value l_j that generalizes both l_1 and l_2.

To precisely define what "generalizes" means, we define a partial order $\sqsubseteq$ over abstract values, and say that l_1 and l_2 are at least as precise as l_j, written $l_1 \sqsubseteq l_j$. Recall that a partial order is any relation that is:

- reflexive: $\forall l : l \sqsubseteq l$
- transitive: $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$
- anti-symmetric: $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$

A set of values L that is equipped with a partial order $\sqsubseteq$, and for which the least upper bound of any two values in that ordering $l_1 \sqcup l_2$ is unique and is also in L, is called a *join-semilattice*. Any join-semilattice has a maximal element $\top$ (pronounced "top"). We require that the abstract values used in dataflow analyses form a join-semilattice. We will use the term lattice for short; as we will see below, this is the correct terminology for most dataflow analyses anyway. For zero analysis, we define the partial order with $Z \sqsubseteq \top$ and $N \sqsubseteq \top$, where $Z \sqcup N = \top$.

We have now considered all the elements necessary to define a dataflow analysis:

- a lattice $(L, \sqsubseteq)$
- an abstraction function α
- a flow function f
- initial dataflow analysis assumptions, σ_0

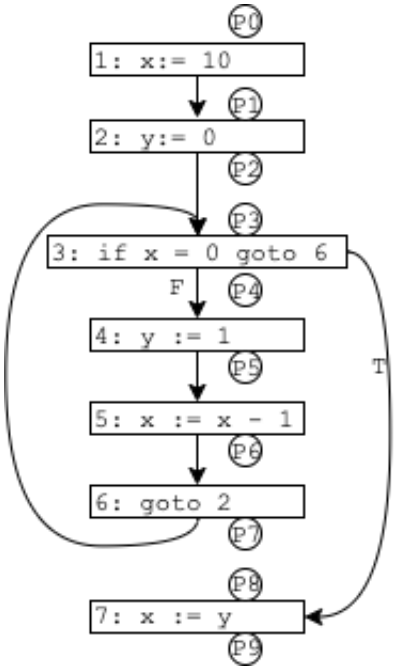
Note that the theory of lattices answers that side question that came up in the very first example: what should we assume about the value of input variables (the question marks in our example tables)? If we do not know anything about the value x can be, one good choice is to assume it can be anything. That is, in the initial environment σ_0, variables' initial state is mapped to $\top$.

Finally, as a small diagram point: in the interest of clarity, we sometimes elide program points between nodes when we can. That is, in the example above, the state going into instruction 3 is exactly the state coming out of instruction 2, so we label a single program point P3. However, when we need to join multiple paths to determine the incoming state at a node, we often differentiate the two program points in our CFG diagrams.

3.1 Dataflow analysis of loops

We now consider WHILE3ADDR programs with loops. Our intuition above, which simply analyzed the two paths induced by the `if` statement separately, no longer works so well. A loop produces a potentially unbounded number of program paths, and we want our analysis to take only bounded time. Consider the following simple looping example:[2]

1 : $x := 10$
2 : $y := 0$
3 : if $x = 0$ goto 7
4 : $y := 1$
5 : $x := x - 1$
6 : goto 3
7 : $x := y$



	x	y	
P0	$\top$	$\top$	
P1	N	$\top$	
P2	N	Z	
P3	N	Z	*first time through...*
P4	N_F	Z	
P5	N	N	
P6	$\top$	N	
P7	$\top$	N	
P8	Z_t	N	*first time through...*
P9	N	N	*first time through...*

The right-hand side above shows the straightforward straight-line analysis of the path that runs the loop exactly once. Thinking back to our handling of `if` above, we might now reconsider instruction 3, joining the states at P2 and P7 to create a new P3. For x, $N \sqcup \top = \top$. For y, $Z \sqcup N = \top$. This changes the incoming values at instruction 3. We can now choose between two paths once

[2] I provide the CFG for reference but omit the annotations in the interest of a cleaner diagram. Notice that I differentiate P2 and P3 because of the join, as well as P7 and P8, since they don't both come from instruction 6.

again. We will choose (arbitrarily, for now) to stay within the loop, and reconsider instruction 4. We have new incoming information (at P4, where both x and y are now $\top$). But, since instruction 4 assigns 1 to y, we still know that y is nonzero at P5. The updated input data does not change the analysis results at P5.

A quick check shows that going through the remaining instructions in the loop, even back to instruction 3, the analysis information will no longer change. That is because the flow functions are deterministic: given the same input analysis information and the same instruction, they will produce the same output analysis information.

We say that the dataflow analysis has reached a *fixed point* (or fixpoint). In mathematics, a fixed point of a function is a data value v that is mapped to itself by the function, i.e., $f(v) = v$. In analysis, the mathematical function is the flow function, and the fixed point is a tuple of the dataflow analysis values at each program point. If we invoke the flow function on the fixed point, the analysis results do not change (we get the same fixed point back).

Once we have reached a fixed point for the loop, further analysis of the loop will not be useful. Therefore, we will proceed to analyze statement 7. The final analysis results are as follows:

	x	y	
P0	$\top$	$\top$	
P1	N	$\top$	
P2	N	Z	
P3	$\top$	$\top$	*join*
P4	N_F	$\top$	*updated*
P5	N	N	*already at fixed point*
P6	$\top$	N	*already at fixed point*
P7	$\top$	N	*already at fixed point*
P8	Z_T	$\top$	*updated*
P9	$\top$	$\top$	*updated*

Quickly simulating a run of the program program shows that these results correctly approximate actual execution. The uncertainty in the value of x at P6 and P7 is real: x *is* nonzero after these instructions, except the last time through the loop, when it is zero. The uncertainty in the value of y at the end shows analysis imprecision: this loop always executes at least once, so y will be nonzero at these points. However, the analysis (as currently formulated) cannot tell this for certain, so it reports that it cannot tell if y is zero or not. This is safe—it is always correct to say the analysis is uncertain—but not as precise as would be ideal.

The benefit of analysis, however, is that we can gain correct information about all possible executions of the program with only a finite amount of work. In our example, we only had to analyze the loop statements at most twice each before reaching a fixed point. This is a significant improvement over the actual program execution, which runs the loop 10 times. We sacrificed precision in exchange for coverage of all possible executions, a classic tradeoff.

How can we be confident that the results of the analysis are correct, besides simulating every possible run of a (possibly very complex) program? The intuition behind correctness is the invariant that at each program point, the analysis results approximate all the possible program values that could exist at that point. If the analysis information at the beginning of the program correctly approximates the program arguments, then the invariant is true at the beginning of program execution. One can then make an inductive argument that the invariant is preserved. In particular, when the program executes an instruction, the instruction modifies the program's state. As long

as the flow functions account for every possible way that instruction can modify state, then at the analysis fixed point they will have correctly approximated actual program execution. We will make this argument more precise in a future lecture.

3.2 A convenience: the $\bot$ abstract value and complete lattices

To define an algorithm for dataflow anlaysis more precisely, we need to be more concrete about how to compute incoming states for CFG nodes with multiple incoming edges (like instruction 3, above). We've been ignoring these in our "one path at a time" approach so far, but this is a handwave for didactic purposes.

Instead, it is more precise and consistent to say that analyzing an instruction *always* uses the incoming dataflow analysis information from *all* instructions that could precede it. However, for instruction 3, this requires a dataflow value from instruction 6, even if instruction 6 has not yet been analyzed. We could do this if we had a dataflow value that is always ignored when it is joined with any other dataflow value. In other words, we need a abstract dataflow value $\bot$ (pronounced "bottom") such that $\bot \sqcup l = l$.

$\bot$ plays a dual role to the value $\top$: it sits at the bottom of the dataflow value lattice. For all l, we have the identity $l \sqsubseteq \top$ and correspondingly $\bot \sqsubseteq l$. There is an greatest lower bound operator *meet*, $\sqcap$, which is dual to $\sqcup$. The meet of all dataflow values is $\bot$.

A set of values L that is equipped with a partial order $\sqsubseteq$, and for which both least upper bounds $\sqcup$ and greatest lower bounds $\sqcap$ exist in L and are unique, is called a *complete lattice*.

This provides an elegant solution to the problem mentioned above. We initialize σ at every program point in the program, except at entry, to $\bot$, indicating that the instruction there has not yet been analyzed. We can then *always* merge all input values to a node, whether or not the sources of those inputs have been analysed, because we know that any $\bot$ values from unanalyzed sources will simply be ignored by the join operator $\sqcup$, and that if the dataflow value for that variable will change, we will get to it before the analysis is completed.

4 Analysis execution strategy

Our informal strategy above, which considers all paths until the dataflow analysis information reaches a fixed point, can be simplified. The argument for correctness outlined above implies that for correct flow functions, it doesn't matter how we get to the analysis fixed point (it would be surprising if analysis correctness depended on which branch of an `if` statement we explored first!). It is in fact possible to run the analysis on program instructions in any order we choose. As long as we continue doing so until a reaching a fixed point, the final result will be correct. The simplest correct algorithm for executing dataflow analysis can therefore be stated as follows:

```
for Node n in cfg
    results[n] = ⊥
results[0] = initialDataflowInformation

while not at fixed point
    pick a node n in program
    input = join { results[j] | j in predecessors(n) }
    output = flow(n, input)
    results[n] = output
```

Or, equivalently:

```
for Node n in cfg
    input[n] = ⊥
input[0] = initialDataflowInformation

while not at fixed point
    pick a node n in program
    output = flow(n, input[n])
    for Node j in sucessors(n)
        input[j] = input[j] ⊔ output
```

In the code above, the termination condition is expressed abstractly ("not at fixed point"). It can easily be checked by keeping track, when we process each node, whether the new results have changed compared to what we previously had stored for that node. If the results do not change for any node, the analysis has reached a fix point.

How do we know the algorithm will terminate? The intuition is as follows. We rely on the choice of a node to be fair, so that each node is eventually considered. As long as the analysis is not at a fixed point, some node can be analyzed to produce new results. If our flow functions are well-behaved (technically, if they are monotone, as we will discuss in a future lecture) then each time the flow function runs on a given node, either the results do not change, or they get become more approximate (i.e., they are higher in the lattice). Later runs of the flow function consider more possible paths through the program and therefore produce a more approximate result which considers all these possibilities. If the lattice is of finite height—meaning there are at most a finite number of steps from any place in the lattice going up towards the $\top$ value—then this process must terminate eventually. More concretely: once an abstract value is computed to be $\top$, it will stay $\top$ no matter how many times the analysis is run. The abstraction only flows in one direction.

Although the simple algorithm above always terminates and results in the correct answer, it is still not always the most efficient. Typically, for example, it is beneficial to analyze the program instructions in order, so that results from earlier instructions can be used to update the results of later instructions. It is also useful to keep track of a list of instructions for which there has been a change since the instruction was last analyzed in the result dataflow information of some predecessor. Only those instructions need be analyzed; reanalyzing other instructions is useless since their input has not changed. Kildall captured this intuition with his worklist algorithm, described in pseudocode as:

```
for Node n in program
    input[n] = ⊥
input[0] = initialDataflowInformation
worklist = { firstNode }

while worklist is not empty
    take a Node n off the worklist
    output = flow(n, input[n])
    for Node j in succs(n)
        if output ⋢ input[n]
            input[j] = input[j] ⊔ output
            add j to worklist
```

The algorithm above is very close to the generic algorithm declared previously, except the worklist that chooses the next instruction to analyze and determines when a fixed point is reached.

We can reason about the performance of this algorithm as follows. We only add a node to the worklist when the input data to it changes. The input for a given node can only change h times, where h is the height of the lattice. Thus we add at most $n * h$ nodes to the worklist, where n is the number of nodes/instructions in the program. After running the flow function for a node, however, we must test all its successors to find out if their input has changed. This test is done once for each edge, for each time that the source node of the edge is added to the worklist: thus at most $e * h$ times, where e is the number of control flow edges in the successor graph between instructions. If each operation (such as a flow function, $\sqcup$, or $\sqsubseteq$ test) has cost $O(c)$, then the overall cost is $O(c * (n + e) * h)$, or $O(c * e * h)$ because n is bounded by e.

The algorithm above is still abstract: We have not defined the operations to add and remove instructions from the worklist. We would like adding to the work list to be a set addition operation, so that no instruction appears in it multiple times. If we have just analysed the program with respect to an instruction, analyzing it again will not produce different results.

That leaves a choice of which instruction to remove from the worklist. We could choose among several policies, including last-in-first-out (LIFO) order or first-in-first-out (FIFO) order. In practice, the most efficient approach is to identify the strongly-connected components (i.e. loops) in the control flow graph of components and process them in topological order, so that loops that are nested, or appear in program order first, are solved before later loops. This works well because we do not want to do a lot of work bringing a loop late in the program to a fixed point, then have to redo that work when dataflow information from an earlier loop changes.

Within each loop, the instructions should be processed in reverse postorder, the reverse of the order in which each node is last visited when traversing a tree. Consider the example from Section 2.2 above, in which instruction 1 is an `if` test, instructions 2–3 are the then branch, instructions 4–5 are the else branch, and instruction 6 comes after the `if` statement. A tree traversal might go as follows: 1, 2, 3, 6, 3 (again), 2 (again), 1 (again), 4, 5, 4 (again), 1 (again). Some instructions in the tree are visited multiple times: once going down, once between visiting the children, and once coming up. The postorder, or order of the last visits to each node, is 6, 3, 2, 5, 4, 1. The reverse postorder is the reverse of this: 1, 4, 5, 2, 3, 6. Now we can see why reverse postorder works well: we explore both branches of the if statement (4–5 and 2–3) before we explore node 6. This ensures that we do not have to reanalyze node 6 after one of its inputs changes.

Although analyzing code using the strongly-connected component and reverse postorder heuristics improves performance substantially in practice, it does not change the worst-case performance results described above.