Kochen-Specker Uncolorable Sets:

An Algorithmic Approach to Proving Contradictions

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Vector Spaces

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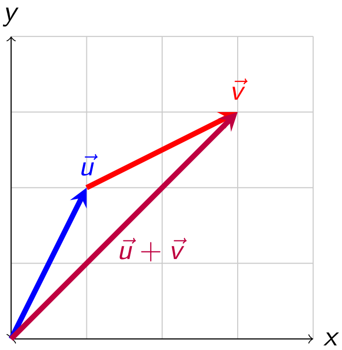


Figure 1: Vector addition

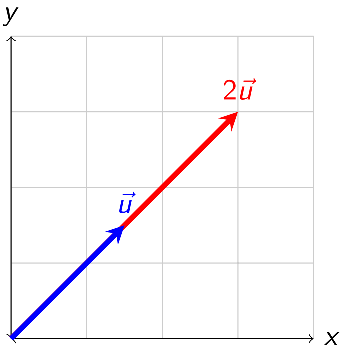


Figure 2: Scalar multiplication

Orthogonality

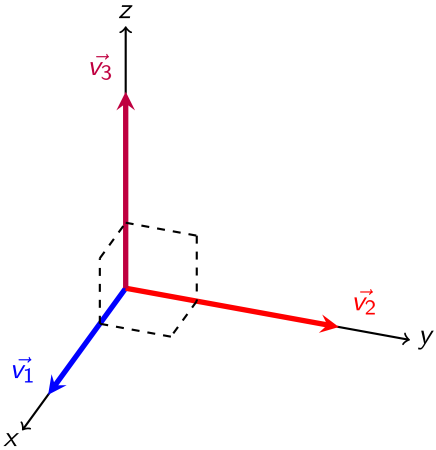
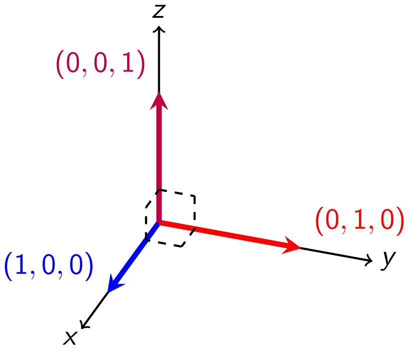
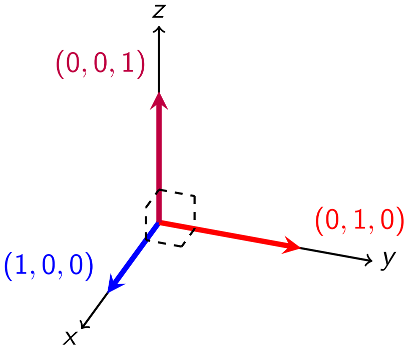


Figure 3: $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ are pairwise orthogonal.

Orthogonality



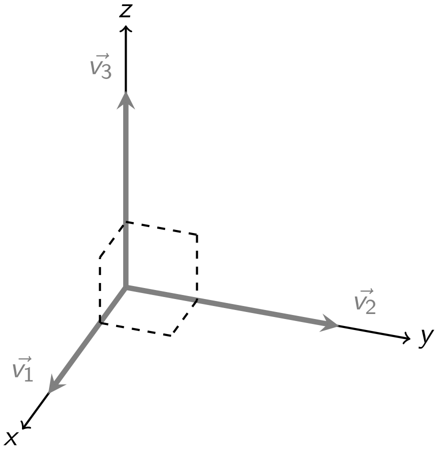
Orthogonality



Computing the dot product of each pair of vectors above will always yield 0:

$$(1, 0, 0) \cdot (0, 1, 0) = (1 \cdot 0) + (0 \cdot 1) + (0 \cdot 0) = 0 + 0 + 0 = 0.$$

Coloring and Contextuality



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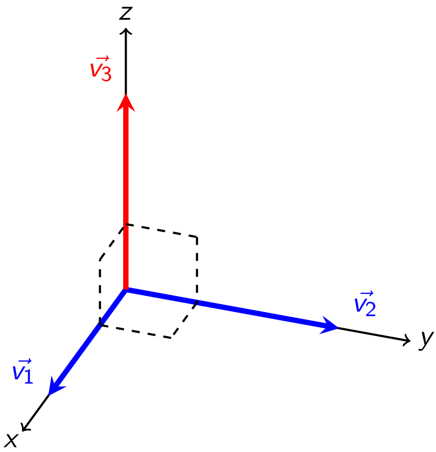

The Kochen-Specker (KS) Theorem

Let $\mathcal{S}$ be a set of vectors and consider the value function $f : \mathcal{S} \to \{0, 1\}$.

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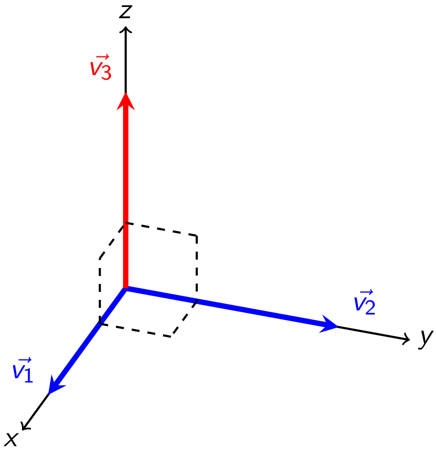

$$f(\vec{v_1}) = 0 \quad f(\vec{v_2}) = 0 \quad f(\vec{v_3}) = 1$$

The Kochen-Specker (KS) Theorem

Kochen-Specker, 1967

There is a finite set $\mathcal{S} \subset \mathbb{R}^3$ such that there is no function $f : \mathcal{S} \to \{0, 1\}$ satisfying

$$f(\vec{u}) + f(\vec{v}) + f(\vec{w}) = 1$$

for all triples $(\vec{u}, \vec{v}, \vec{w})$ of mutually orthogonal vectors in $\mathcal{S}$.

$\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$

The Kochen-Specker (KS) Theorem

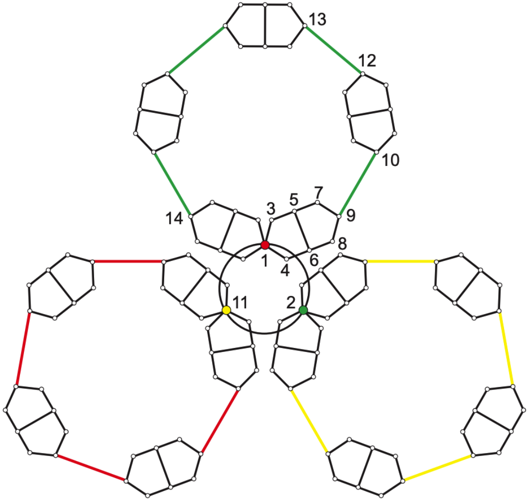

Figure 4: A graphical representation of the set of 117 vectors in the original proof of the KS Theorem (Budroni et al, 2022).

A Kochen-Specker (KS) System

$$\mathcal{S}(N) = \{\vec{v} = (v_1, v_2, v_3) \in \mathbb{Z}^3 : v_1^2 + v_2^2 + v_3^2 \text{ divides a power of } N\}$$

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Example

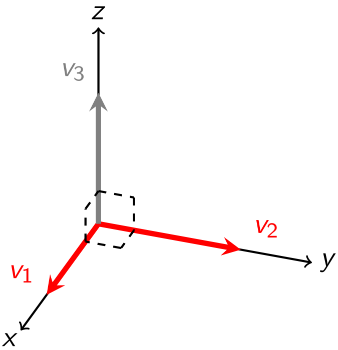
Let $N = 30$ and consider $\vec{v} = (5, 2, 1)$. We determine that

$$5^2 + 2^2 + 1^1 = 25 + 4 + 1 = 30.$$

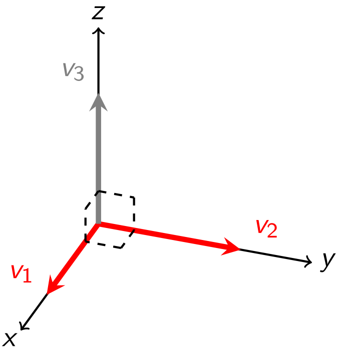
Thus, $\vec{v} \in \mathcal{S}(30)$.

What does our algorithm tell us?

Contradictions



Contradictions

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In [3]: vector_coloring(30)
A contradiction has been identified from coloring (2, 1, 0), the cross product of (0, 0, 1) and (1, -2, 0).
Out[3]: False

In [4]: vector_coloring(462)
A contradiction has been identified from coloring (-3, 2, 8), the cross product of (2, -5, 2) and (4, 2, 1).
Out[4]: False

In [5]:
```

Finite KS Colorable Sets

Coloring	Vectors
0	(-1, 1, 1), (0, 0, 1), (0, 1, -1), (0, 1, 0), (0, 1, 1), (1, -2, 1), (1, -1, 1), (1, 0, 1), (1, 1, -2), (1, 1, -1), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 1)
1	(-2, 1, 1), (-1, 1, 2), (-1, 2, 1), (1, -1, 0), (1, -1, 2), (1, 0, -1), (1, 0, 0), (1, 2, -1), (2, -1, 1), (2, 1, -1), (2, 1, 1)

Finite KS Colorable Sets

Coloring	Vectors
0	(-1, 1, 1), (0, 0, 1), (0, 1, -1), (0, 1, 0), (0, 1, 1), (1, -2, 1), (1, -1, 1), (1, 0, 1), (1, 1, -2), (1, 1, -1), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 1)
1	(-2, 1, 1), (-1, 1, 2), (-1, 2, 1), (1, -1, 0), (1, -1, 2), (1, 0, -1), (1, 0, 0), (1, 2, -1), (2, -1, 1), (2, 1, -1), (2, 1, 1)

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**********************************
We are currently at depth 1.
**********************************
We'll assume that (-1, 1, 1) is colored 0.
We'll assume that (-1, 1, 1) is colored 1.
Every vector in our original finite set of vectors has been colored by assuming (-1, 1, 1) was colored 0: {(1, 0, 0): 1, (0, 0, 1): 0, (0, 1, 0): 0, (1, 0, -1): 1, (1, 0, 1): 0,
(1, 1, 0): 0, (1, -1, 0): 1, (0, 1, 1): 0, (0, 1, -1): 0, (1, 2, 1): 0, (1, -1, 1): 0, (1, -2, 1): 0, (1, 1, 2): 0, (1, 1, -2): 0, (1, 1, -1): 0, (1, 1, 1): 0, (-1, 2, 1): 1, (2,
 1, -1): 1, (-1, 1, 2): 1, (2, -1, 1): 1, (-2, 1, 1): 1, (-1, 1, 1): 0, (1, -1, 2): 1, (2, 1, 1): 1, (1, 2, -1): 1}
Every vector in our original finite set of vectors has been colored by assuming (-1, 1, 1) was colored 1: {(1, 0, 0): 1, (0, 0, 1): 0, (0, 1, 0): 0, (1, 0, -1): 1, (1, 0, 1): 0,
(1, 1, 0): 0, (1, -1, 0): 1, (0, 1, 1): 0, (0, 1, -1): 0, (1, 2, 1): 0, (1, -1, 1): 0, (1, -2, 1): 0, (1, 1, 2): 0, (1, 1, -2): 0, (1, 1, -1): 0, (1, 1, 1): 0, (-1, 2, 1): 1, (2,
 1, -1): 1, (-1, 1, 2): 1, (2, -1, 1): 1, (-2, 1, 1): 1, (-1, 1, 1): 1, (2, 1, 1): 0, (1, -1, 2): 0, (1, 2, -1): 0}
The uncolored vector sets following each assumption share no common elements.
Out[7]: False

In [8]:
```
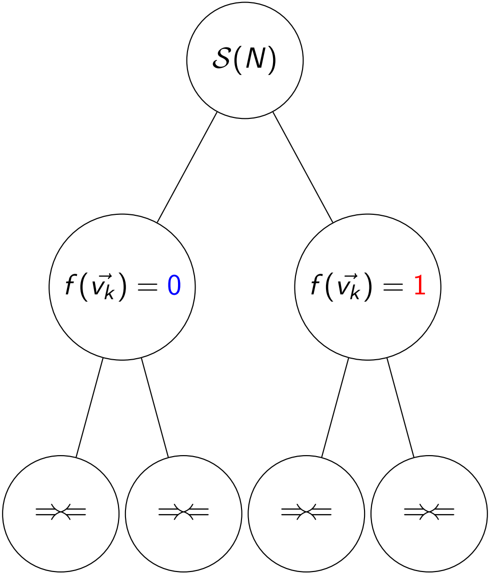

Further Assumptions and Future Directions



Acknowledgements

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References

- Ben-Zvi, Michael, et al. "A Kochen–Specker theorem for integer matrices and noncommutative spectrum functors." Journal of Algebra 491 (2017): 280-313.
- Budroni, Costantino, et al. "Kochen-specker contextuality." Reviews of Modern Physics 94.4 (2022): 045007.
- Cortez, Ida, and Manuel L. Reyes. "A set of integer vectors with no Kochen-Specker coloring." arXiv preprint arXiv:2211.13216 (2022).

Uncolorable Sets of Integer Vectors

Let N be a positive squarefree integer. We define

$$\begin{aligned} \mathcal{S}_n(N) &= \{\vec{v} \in \mathbb{Z}^n : ||\vec{v}||^2 \text{ is a unit in } \mathbb{Z}[1/N]\} \\ &= \{\vec{v} \in \mathbb{Z}^n : ||\vec{v}||^2 \text{ divides a power of } N\}. \end{aligned}$$

Question

For which positive squarefree integers N is the set of vectors $\mathcal{S}(N) := \mathcal{S}_3(N)$ Kochen-Specker uncolorable?

(Un)colorable Sets of Integer Vectors

Let M be an integer such that M divides a squarefree integer N. It follows that $\mathcal{S}(M) \subseteq \mathcal{S}(N)$. Furthermore,

$$\mathcal{S}(N) \text{ colorable} \implies \mathcal{S}(M) \text{ colorable},$$
$$\mathcal{S}(M) \text{ uncolorable} \implies \mathcal{S}(N) \text{ uncolorable}.$$

Ben-Zvi et al., 2017

If $\mathcal{S}(N)$ is KS uncolorable, then 6 divides N.

Bub, 1996

If 30 divides N, then the set $\mathcal{S}(N)$ is KS uncolorable.