第一章作业

1.5

(a)

$$P_{N_1,\dots,N_{r-1}}(n_1,\dots,n_{r-1}) = \frac{n!}{\prod_{i=1}^{r}}\prod_{i=1}^{r}P_{i^t}^{n},\text{其中}n_i=0,\dots,n,\text{且}\sum_{i=1}^{r}n_i=n.$$

(b)

$$\begin{gathered}
E[N_i]=nP_i, E[N_i^2]=nP_i-nP_1^2+n^2P_i^2,\\
E[N_j]=nP_j, E[N_j^2]=nP_1-nP_1^2+n^2P_j^2,\\
E[N_i,N_j]=E[E[N_i,N_j\mid N_j]],\\
E[N_i,N_j\mid N_j=m]=mE[N_i\mid N_j=m]=m(n-m)\frac{P_i}{1-P_j}\\
=\frac{nmP_j-m^2P_j}{1-P_j},\\
E[N_i,N_j]=\frac{nE[N_j]P_i-E[N_j^2]P_j}{1-P_j}=\frac{n^2P_iP_j-nP_iP_j+nP_j^2P_i-n^2P_j^2P_i}{1-P_j}\\
\frac{n^2P_jP_i(1-P_j)-nP_iP_j(1-P_j)}{1-P_j}=n^2P_iP_j-np_iP_j,\\
Cov(N_i,N_j)=E[N,N]-E[N_i][N_j]=-nP_iP_j, i\neq j.
\end{gathered}$$

(c)

$$\begin{gathered}
\text{令}I_j=\begin{cases}1 & \text{在时刻}j\text{有一个记录}\\ 0 & others\end{cases}\\
E[I_j]=(1-P_j)^n, Var[I_j]=(1-P_j)^n(1-(1-P_j)^n),\\
E[I_iI_j]=(1-P_i-P_j)^n, i\neq j,\\
\text{不出现的结果数}=\sum_{j=1}^{r}I_j,\\
E[\sum_{j=1}^{r}I_j]=\sum_{j=1}^{r}(1-P_j)^n, Var[\sum_{i=1}^{r}I_i]=\sum_{1}^{r}Var[I_j]+\sum\sum_{i\neq j}Cov[I_i,I_j],\\
Cov[I_i,I_j]=E[I_i,I_j]-E[I_i][I_j]=(1-P_i-P_j)^n-(1-P_i)^n-(1-P_j)^n,\\
Var[\sum_{i=1}^{r}I_i]=\sum_{i=1}^{r}(1-P_j)^n(1-(1-P_j)^n)+\sum\sum_{i\neq j}[(1-P_i-P_j)^n-(1-P_i)^n-(1-P_j)^n].
\end{gathered}$$

1.6

(a)

$$令I_j=\begin{cases}1 & 若结果j永不出现\\ 0 & others\end{cases}$$

$$N_n=\sum_{j=1}^{r} I_j, E[N_n]=\sum_{j=1}^{n} E[I_j]=\sum_{j=1}^{n}\frac{1}{j},$$

$$Var(N_n)=\sum_{j=1}^{n} Var(I_j)=\sum_{j=1}^{n}\frac{1}{j}(1-\frac{1}{j}),因为I_j都是独立的.$$

(b)

$$令T=minn:n>1且n出现在一个记录$$

$$T>n \Leftrightarrow X_1,X_2,\dots,X_n中的最大者$$

$$E[T]=\sum_{n=1}^{\infty}\frac{1}{n}=\infty,$$

$$P\{T=\infty\}\lim_{n\to\infty}P\{T>n\}=0.$$

(c)

$$以T_y记大于y的首次记录值的时刻，令XT_y是在时刻T_y的记录值.$$

$$\begin{aligned}P\{XT_y>x\mid T_y=n\}&=\{X_n>x\mid X_1<y,X_2<y,\dots,X_{n-1}<y,X_n>y\}\\&=P\{X_n>x\mid X_n>y\}\\&=\begin{cases}1 & x<y\\ \bar{F}(x)/\bar{F}(y) & x>y\end{cases}\end{aligned}$$

$$因为P\{X_n>x\mid T_y=n\}不依赖n,所以T_y和XT_y独立.$$

1.11

(a)

$$\frac{d^k}{dz^k}P(z)_{|z=0}=k!P\{X=k\}+\sum_{j=k+1}^{\infty}z^{j-k}P\{X=j\}=k!P\{X=k\}$$

(b)

$$\frac{P(-1)+P(1)}{2}=\frac{1}{2}\sum_{j=0,2,4,\dots}^{\infty}2P\{X=j\}=P\{X是偶数\}$$

(c)

$$P(1)=\sum_{j=0}^{n}1^j\binom{n}{j}p^j(1-p)^{n-j}=1$$

$$P(-1)=\sum_{j=0}^{n}\binom{n}{j}(-p)^j(1-p)^{n-j}=(1-2p)^n$$

$$P\{X\ 是偶数\}=\frac{P(-1)+P(1)}{2}=\frac{1+(1-2p)^n}{2}$$

(d)

$$P(1)=\sum_{j=0}^{\infty}1^j\frac{\lambda^j e^{-\lambda}}{j!}=1$$
$$P(-1)=e^{-2\lambda}\sum_{j=0}^{\infty}\frac{(-\lambda)^j e^{\lambda}}{j!}=e^{-2\lambda}$$
$$P\{X\text{ 是偶数}\}=\frac{P(-1)+P(1)}{2}=\frac{1+e^{-2\lambda}}{2}$$

(e)

$$\begin{aligned}P(1)&=1\\ P(-1)&=\sum_{j=1}^{\infty}(-1)^j(1-p)^(j-1)p\\ &=-\frac{p}{2-p}\sum_{j=1}^{\infty}(p-1)^(j-1)(2-p)=-\frac{p}{2-p}\\ P\{X\text{ 是偶数}\}&=\frac{P(-1)+P(1)}{2}=\frac{1-p}{2-p}\end{aligned}$$

(f)

$$\begin{aligned}P(1)&=1\\ P(-1)&=\sum_{j=r}^{\infty}(-1)^j\binom{j-1}{r-1}p^r(1-p)^{j-r}\\ &=(-1)^r(\frac{p}{2-p})^r\sum_{j=r}^{\infty}(2-p)^r(p-1)^{j-r}=(-1)^r(\frac{p}{2-p})^r\\ P\{X\text{ 是偶数}\}&=\frac{P(-1)+P(1)}{2}=\frac{1}{2}[1+(-1)^r(\frac{p}{2-p})^r]\end{aligned}$$

1.17

(a)

令

$$\begin{aligned}F_{i,n}(x)&=P\{\text{第}i\text{个最小者}\leq x\mid X_n\leq x\}F(x)+P\{\text{第}i\text{个最小者}\leq x\mid X_n>x\}\bar{F}(x).\\ &=P\{X_{i-1,n-1}\leq x\}F(x)+P\{X_{i,n-1}\leq x\}\bar{F}(x).\end{aligned}$$

(b)

令

$$\begin{gathered}F_{i,n-1}(x)=P\{X_{i,n-1}\leq x\mid X_n\text{在第}i\text{个最小者中}\}i/n\\ +P\{X_{i,n-1}\leq x\mid X_n\text{不在第}i\text{个最小者中}\}(1-i)/n\\ =P\{X_{i+1,n}\leq x\}i/n+P\{X_{i,n}\leq x\}(1-i)/n,\\ X_n\text{是否在}X_1,\ldots,X_n\text{的第}i\text{个最小者中并不影响}X_{i,n}(i=1,\ldots,n)\text{的联合分布}\end{gathered}$$

1.20

$$x<1\text{时无法容纳任何随机区间, 因此}N(x)=0,M(x)=E[N(x)]=0$$

$x > 1$时, 令Y为第一个随机区间的左端点，$Y \sim U(0, x-1)$，
第一个随机区间把整个区间$(0, x)$分为长度分别为y和$x-y-1$的两部分，则有

$$\begin{aligned} M(x) &= E[N(x)] = E[E[N(x)|Y]] \\ &= E[N(y) + N(x-y-1) + 1] \\ &= \int_0^{x-1} (\frac{1}{x-1})[M(y) + M(x-y-1) + 1]dy \\ &= \frac{2}{x-1} \int_0^{x-1} M(y)dy + 1, \quad x > 1 \end{aligned}$$

1.22

$$\begin{aligned} Var(X \mid Y) &= E[(X - E(X \mid Y))^2 \mid Y] \\ &= E[X^2 - 2XE(X \mid Y) + (E(X \mid Y))^2 \mid Y] \\ &= E[X^2 \mid Y] - 2[E(X \mid Y)]^2 + E[(X \mid Y)^2] \\ &= E[X^2 \mid Y] + [E(X \mid Y)]^2, \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= E(E[X^2 \mid Y]) - (E[E[X \mid Y]])^2 \\ &= E[Var(X \mid Y) + (E[X \mid Y])^2] - (E[E[X \mid Y]])^2 \\ &= E[Var(X \mid Y)] + E[(E[X \mid Y])^2] - (E[E[X \mid Y]])^2 \\ &= E[Var(X \mid Y)] + Var[E(X \mid Y)] \end{aligned}$$

1.29

使用数学归纳法

此密度函数在$n=1$时成立，假设在$n=k$时也成立，则在$n=k+1$时,

$$\begin{aligned} f_{k+1}(t) &= \int_0^t f_k(x) f_1(t-x) dx \\ &= \int_0^t \lambda e^{-\lambda x} (\lambda x)^{k-1} \lambda e^{-\lambda(t-x)} / (k-1)! dx \\ &= \lambda e^{-\lambda t} (\lambda t)^n / (n)! \end{aligned}$$

1.34

$$\begin{aligned} P\{X_1 < X_2 \mid min(X_1, X_2) = t\} &= \frac{P\{X_1 < X_2, min(X_1, X_2) = t\}}{P\{min(X_1, X_2) = t\}} \\ &= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + \{X_2 = t, X_1 > t\}} \\ &= \frac{P\{X_1 = t\}P\{X_2 > t\}}{P\{X_1 = t\}P\{X_2 > t\} + \{X_2 = t\}P\{X_1 > t\}}, \end{aligned}$$

$$P\{X_2 = t\} = \lambda_2(t) P\{X_2 > t\}, P\{X_1 > t\} = \frac{P\{X_1 = t\}}{\lambda_1(t)},$$

$$P\{X_2 = t\}P\{X_1 > t\} = \frac{\lambda_2(t)}{\lambda_1(t)} P\{X_1 = t\}P\{X_2 > t\}$$

因此$P\{X_1 < X_2 \mid min(X_1, X_2) = t\} = \frac{1}{1 + \frac{\lambda_2(t)}{\lambda_1(t)}} = \frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}.$

1.35

(a)

$$\begin{aligned}M(t)E[exp\{-tX_t\}h(X_t)] &= M(t)\int_{-\infty}^{\infty} e^{-tx}h(x)f_t(x)dx \\ &= \int_{-\infty}^{\infty} h(x)f(x)dx \\ &= E[h(X)]\end{aligned}$$

(b)

$$\begin{aligned}M(t)e^{-ta}P\{X_t > a\} &= M(t)e^{-ta}\int_a^{\infty} \frac{e^{tx}f(x)}{M(t)}dx \\ &= \int_a^{\infty} e^{t(x-a)}f(x)dx \\ &\geq \int_a^{\infty} f(x)dx \\ &= P\{X > a\}\end{aligned}$$

(c)

$$\begin{aligned}f(x,t) &= M(t)e^{-ta} = e^{-ta}\int_{-\infty}^{\infty} e^{tx}f(x)dx \\ f_t'(x,t) &= e^{-2ta}(\int_{-\infty}^{\infty} e^{ta}xe^{tx}f(x)dx – a\int_{-\infty}^{\infty} e^{ta}e^{tx}f(x)dx) \\ &= e^{-ta}(\int_{-\infty}^{\infty} xe^{tx}f(x)dx – aM(t)) \\ &\text{当此导数等于0时}, E[X_{t*}] = a\end{aligned}$$

1.37

若峰值出现在时刻n则令$I_n = 1$, 否则令$I_n = 0$;

注意由于X_{n-1}, X_n或X_{n+1}中的每一个都等可能地是这3个中的最大者, 所以$E[I_n] = \dfrac{1}{3}$.

因为$\{I_2, I_5, I_8, \ldots\}, \{I_3, I_6, I_9, \ldots\}, \{I_4, I_7, I_10, \ldots\}$都是独立同分布序列，

由强大数定理推出，每个序列的前n项的平均以概率为1地收敛到$\dfrac{1}{3}$.

但是这蕴含了以概率为1地有$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} I_{n+1}/n = 1/3$.

1.39

$$E[T_1] = 1.$$

$$\text{对}i > 1,$$

$$E[T_i] = 1 + 1/2(E[\text{从}i-2\text{到}i\text{的时间}]) = 1 + 1/2(E[T_{i-1}] + E[T_i]),$$

故而

$$E[T_i] = 2 + E[T_i - 1], i < 1.$$

因此

$$E[T_2]=3, E[T_3]=5, E[T_i]=2i-1, i=1,\ldots,n.$$

若$T_{0,n}$是从0到n的步数，则

$$E[T_{0,n}]=E[\sum_{i=1}^{n} T_i]=2n(n+1)/2-n=n^2$$

1.40

$$\frac{1/n_2}{1/n_1+1/n_2+1/n_3}\frac{1/n_3}{1/n_1+1/n_3}+\frac{1/n_3}{1/n_1+1/n_2+1/n_3}\frac{1/n_2}{1/n_1+1/n_2}$$