Lynx vs Hare Relationship

Linear Algebra Eigenvalue Project

To start, I simply chose a predator prey relationship that sounded interesting and in doing so I found that lynx and hare populations are very linked in a predator prey relationship. Since the two populations have a strong correlation I chose what I thought to be high numbers for my correlation coefficients. In this case, H_k represents number of hares in hundreds and L_k is simply the population of Lynx. The two equations I started with are as follows:

Lynx: $L_{k+1} = (0.45)L_k + (0.54)H_k$

Hare: $H_{k+1} = (-0.2)L_k + (1.3)H_k$

These equations can then be expressed in matrix form as such:

$$X_{k+1} = \begin{bmatrix} L_{k+1} \\ H_{k+1} \end{bmatrix} = \begin{bmatrix} 0.45 & 0.54 \\ -0.2 & 1.3 \end{bmatrix} \begin{bmatrix} L_k \\ H_k \end{bmatrix}$$

Using this matrix, I then calculated for it's eigenvalues, λ, using the determinant of $(A - \lambda I)$. This would be more challenging with a larger matrix, but it is quite simple for a $2x2$ matrix.

$$\det(A - \lambda I)$$
$$= \det \begin{bmatrix} 0.45-\lambda & 0.54 \\ -0.2 & 1.3-\lambda \end{bmatrix}$$
$$= (0.45-\lambda)(1.3-\lambda) - (-0.2)(0.54)$$
$$= \lambda^2 - 1.75\lambda + 0.585 + 0.108$$
$$= \lambda^2 - 1.75\lambda + 0.693$$

$$\lambda = \frac{1.75 \pm \sqrt{(1.75)^2 - 4(1)(0.693)}}{2(1)}$$

$$\lambda_1 = 1.1445 \quad \lambda_2 = 0.6055$$

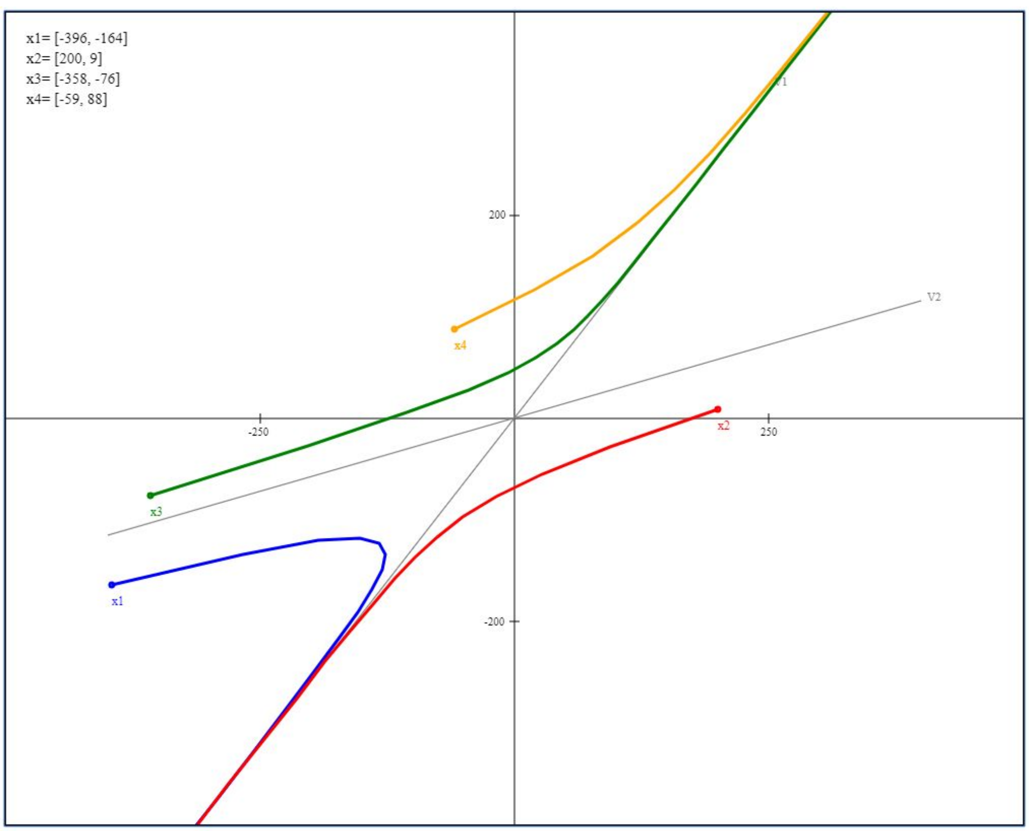
After having solved for the eigenvalues, I could solve for the eigenvectors by solving the homogeneous equation $(A-\lambda I)\widehat{V}=0$. V_1 *and* V_2 can be found simply by plugging in λ_1 *and* λ_2 to the homogeneous equation:

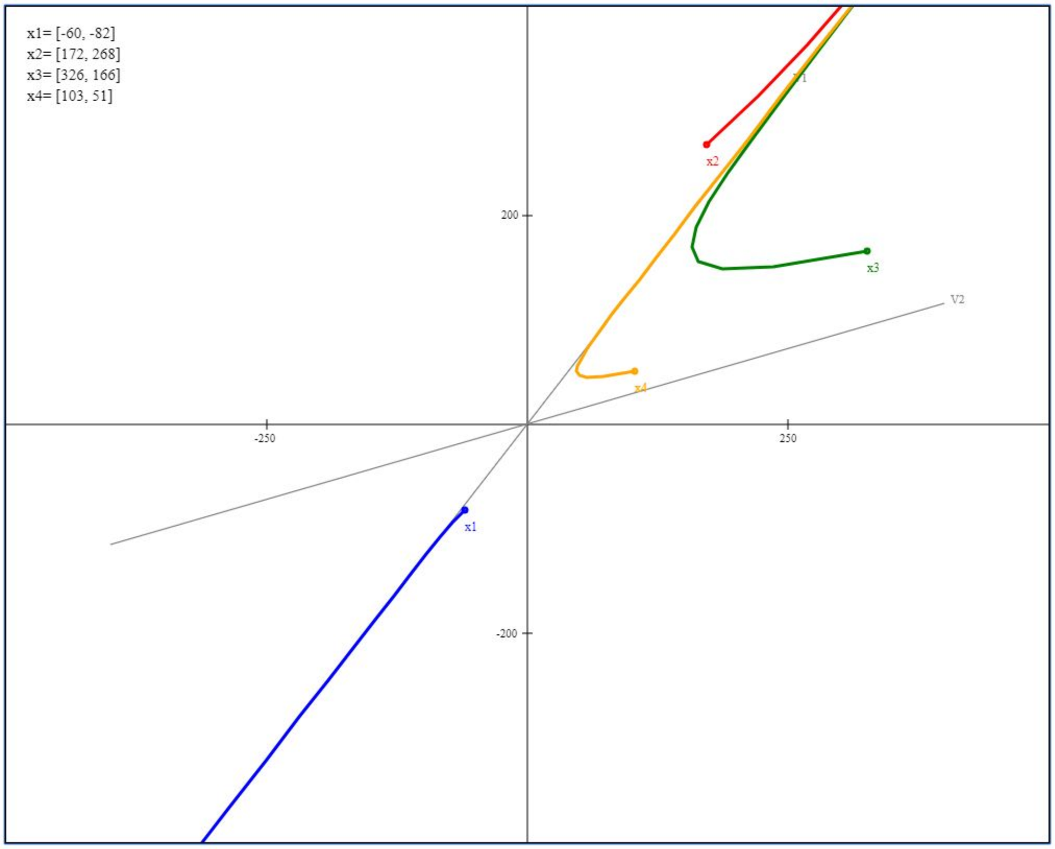
$$(A-\lambda_1 I)\vec{V_1}=0$$
$$(A-1.1445I)\vec{V_1}=0$$
$$\begin{bmatrix} -0.6945 & 0.54 \\ -0.2 & 0.1555 \end{bmatrix}\vec{V_1}=0$$
$$\begin{bmatrix} -0.6945 & 0.54 \\ -0.6945 & 0.54 \end{bmatrix}\vec{V_1}=0$$
$$\text{let } V_1=\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
$$\begin{bmatrix} -0.6945 & 0.54 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=0$$
$$-0.6945a_1+0.54a_2=0$$
$$\text{let } a_1=1$$
$$0.54a_2=0.6945$$
$$a_2=1.2861$$
$$V_1=\begin{bmatrix} 1 \\ 1.2861 \end{bmatrix}$$

$$(A-\lambda I)\vec{V}=0$$
$$(A-0.6055I)\vec{V}=0$$
$$\begin{bmatrix} -0.1555 & 0.54 \\ -0.2 & 0.6945 \end{bmatrix}\vec{V_2}=0$$
$$\begin{bmatrix} -0.1555 & 0.54 \\ -0.1555 & 0.54 \end{bmatrix}\vec{V_2}=0$$
$$\text{let } V_2=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$\begin{bmatrix} -0.1555 & 0.54 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}=0$$
$$-0.1555b_1+0.54b_2=0$$
$$\text{let } b_1=1$$
$$0.54b_2=0.1555$$
$$b_2=0.2880$$
$$V_2=\begin{bmatrix} 1 \\ 0.2880 \end{bmatrix}$$

The question then becomes, what do these two vectors even mean. Well in the case of my starting data, where H and L represent Hare and Lynx populations, V_1 *and* V_2 represent what ratio the populations will trend toward or trend away from depending on the starting conditions of each population. In the first case, there would be roughly 130 (128.61) hares for every lynx and in the second case, approximately 30 (28.8) hares for every lynx. While there is a short term trend towards V_2 in some cases, the long term trend is always towards the ratio represented by V_1. This trend is illustrated by the graphs on the following pages.


x1= [-307, -31]
x2= [-177, -147]
x3= [-303, -237]
x4= [-79, 6]
V1
V2
200
-200
-250
250
x1
x2
x3
x4


x1= [-396, -164]
x2= [200, 9]
x3= [-358, -76]
x4= [-59, 88]
V1
V2
200
-200
-250
250
x1
x2
x3
x4



These graphs were obtained from a program I built using CodeSkulptor. CodeSkulptor is an online environment that runs python in a browser. (The link to the project can be found at: https://goo.gl/sa1Mvt). The program chooses random starting locations for points on the canvas, and then uses the equations I started with to add each subsequent point to the line that is drawn. The canvas can be resized by changing the c_height and c_width variables, and the graph will adapt accordingly.

One important trend to expand on is the fact that points chosen close to eigenvector 2 will tend to follow its general trajectory towards the origin and then slowly move away from it and instead start to follow eigenvector 1. Because of this, we can conclude that a 1:130 lynx to hare population is the most natural balance for the two populations to be in. However, we can also say that when the two populations initial count is close to 1:30 ratio, it will take a longer amount of time for the populations to reach what is their more natural ratio. This makes intuitive sense, as the less hares there are relative to number of lynx, the slower their population will be to grow.