CNCM Online Round 1

CNCM Administration

Problems.

Problem 1. Pooki Sooki has 8 hoodies, and he may wear any of them throughout a 7 day week. He changes his hoodie exactly 2 times during the week, and will only do so at one of the 6 midnights. Once he changes out of a hoodie, he never wears it for the rest of the week. The number of ways he can wear his hoodies throughout the week can be expressed as $\frac{8!}{2^k}$. Find k.

Problem 2. Akshar is reading a 500 page book, with odd numbered pages on the left, and even numbered pages on the right. Multiple times in the book, the sum of the digits of the two opened pages are 18. Find the sum of the page numbers of the last time this occurs.

Problem 3. Define $S(N)$ to be the sum of the digits of N when it is written in base 10, and take $S^k(N) = S(S(\ldots(N)\ldots))$ with k applications of S. The *stability* of a number N is defined to be the smallest positive integer K where $S^K(N) = S^{K+1}(N) = S^{K+2}(N) = \ldots$. Let T_3 be the set of all natural numbers with stability 3. Compute the sum of the two least entries of T_3.

Problem 4. Consider all possible pairs of positive integers (a, b) such that $a \geq b$ and both $\dfrac{a^2+b}{a-1}$ and $\dfrac{b^2+a}{b-1}$ are integers. Find the sum of all possible values of the product ab.

Problem 5. Positive reals $a, b, c \leq 1$ satisfy $\frac{a+b+c-abc}{1-ab-bc-ca} = 1$. Find the minimum value of

$$\left(\frac{a+b}{1-ab} + \frac{b+c}{1-bc} + \frac{c+a}{1-ca}\right)^2$$

Problem 6. In triangle $\triangle ABC$ with $BC = 1$, the internal angle bisector of $\angle A$ intersects BC at D. M is taken to be the midpoint of BC. Point E is chosen on the boundary of $\triangle ABC$ such that ME bisects its perimeter. The circumcircle ω of $\triangle DEC$ is taken, and the second intersection of AD and ω is K, as well as the second intersection of ME and ω being L. If B lies on line KL and ED is parallel to AB, then the perimeter of $\triangle ABC$ can be written as a real number S. Compute $\lfloor 1000S \rfloor$.

Problem 7. Three cats–TheInnocentKitten, TheNeutralKitten, and TheGuiltyKitten labelled P_1, P_2, and P_3 respectively with $P_{n+3} = P_n$–are playing a game with three rounds as follows:

1. Each round has three turns. For round $r \in \{1, 2, 3\}$ and turn $t \in \{1, 2, 3\}$ in that round, player P_{t+1-r} picks a nonnegative integer. The turns in each round occur in increasing order of t, and the rounds occur in increasing order of r.
2. **Motivations:** Every player focuses primarily on maximizing the sum of their own choices and secondarily on minimizing the total of the other players' sums. TheNeutralKitten and TheGuiltyKitten have the additional tertiary priority of minimizing TheInnocentKitten's sum.
3. For round 2, player P_2 has no choice but to pick the number equal to what player P_1 chose in round 1. Likewise, for round 3, player P_3 must pick the number equal to what player P_2 chose in round 2.
4. If not all three players choose their numbers such that the values they chose in rounds 1,2,3 form an arithmetic progression in that order by the end of the game, all players' sums are set to -1 regardless of what they have chosen.
5. If the sum of the choices in any given round is greater than 100, all choices that round are set to 0 at the end of that round. That is, rules 2, 3, and 4 act as if each player chose 0 that round.
6. All players play optimally as per their motivations. Furthermore, all players know that all other players will play optimally (and so on.)

Let A and B be TheInnocentKitten's sum and TheGuiltyKitten's sum respectively. Compute $1000A + B$ when all players play optimally.