Mathematics consideration for LENAPY.lnharmo

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Abstract

This documents contains details of the formula used in LENAPY for surface computation and various other geodetic problems used in lnharmo.

Contents

1 Notation

- R radius of the sphere
- λ longitude
- φ latitude for a sphere or geographic/geodetic latitude for an ellipsoid
- dS small surface element for integration
- a, b, f, e, e' ellipsoid variables
- a_E average Earth radius $= a(1-f)^{\frac{1}{3}}$
- ω geocentric latitude for an ellipsoid
- x, y, z cartesian coordinates
- dM small element of mass
- θ co-latitude
- l degree
- m order
- $C_{l,m}$ and $S_{l,m}$ Stokes coefficients
- $P_{l,m}$ associate Legendre polynomial
- M_{earth} mass of the Earth
- ρ_{water} water density $= 1000$ kg.m^{-3}
- ρ_{ave} average Earth density ≈ 4413 kg.m^{-3}
- k_l elastic loading Love number
- $H_w(\theta, \lambda)$ surface mass in Equivalent Water Height (EWH)
- $\eta(\theta, \lambda)$ surface gravity field without a specific unit.

2 Surface of a cell

In this section, we detail how to compute cell surface from a geometric object. The geometric object may be a sphere or an ellipsoid.

2.1 Sphere

The total surface of a sphere with a radius R is $4\pi R^2$. We characterize a cell on a sphere by its central longitude λ_0 and latitude φ_0 as well as its size in longitude $\Delta\lambda$ and in latitude $\Delta\varphi$.

The surface of the cell corresponds to

$$\begin{aligned} S_{cell} &= \int_{\lambda=\lambda_0-\frac{\Delta\lambda}{2}}^{\lambda_0+\frac{\Delta\lambda}{2}} \int_{\varphi=\varphi_0-\frac{\Delta\varphi}{2}}^{\varphi_0+\frac{\Delta\varphi}{2}} dS \\ &= \int_{\lambda=\lambda_0-\frac{\Delta\lambda}{2}}^{\lambda_0+\frac{\Delta\lambda}{2}} \int_{\varphi=\varphi_0-\frac{\Delta\varphi}{2}}^{\varphi_0+\frac{\Delta\varphi}{2}} R\cos(\varphi)d\varphi \times R d\lambda \\ &= R^2 \int_{\varphi=\varphi_0-\frac{\Delta\varphi}{2}}^{\varphi_0+\frac{\Delta\varphi}{2}} \cos(\varphi)d\varphi \int_{\lambda=\lambda_0-\frac{\Delta\lambda}{2}}^{\lambda_0+\frac{\Delta\lambda}{2}} d\lambda \\ &= R^2 \times (\sin(\varphi_0 + \frac{\Delta\varphi}{2}) - \sin(\varphi_0 - \frac{\Delta\varphi}{2})) \times \Delta\lambda \\ &= 2R^2\cos(\varphi_0)\sin(\frac{\Delta\varphi}{2})\Delta\lambda \end{aligned} \tag{1}$$

2.2 Ellipsoid

An ellipsoid is defined by two variables in a (semi major-axis), b (semi minor-axis), f (flattening), e (eccentricity), e' (eccentricity prime). Its surface is

$$S_{ellispoid} = 2\pi a^2 + \frac{\pi b^2}{e}\log\left(\frac{1+e}{1-e}\right) \tag{2}$$

We characterize a cell on an ellipsoid by its central longitude λ_0 and geodetic latitude φ_0 as well as its size in longitude $\Delta\lambda$ and in latitude $\Delta\varphi$. The minimum latitude is $\varphi_1 = \varphi_0 - \frac{\Delta\varphi}{2}$ and the maximal latitude is $\varphi_2 = \varphi_0 + \frac{\Delta\varphi}{2}$. The geodetic latitude φ is linked to the geocentric latitude ω by

$$\omega = \arctan\left((1-f)^2\tan(\varphi)\right) = \arctan\left(\frac{b^2}{a^2}\tan(\varphi)\right) \tag{3}$$

The ellipsoid is a surface of revolution resulting from the rotation of an ellipse alongside the z-axis / semi minor-axis. $(x, y, z) \in$ ellipsoid are defined by the Cartesian equation $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$. For a certain z, $r(z) = \sqrt{x^2+y^2} = a\sqrt{1-\frac{z^2}{b^2}}$. We can note that $r'(z) = \frac{az}{b\sqrt{b^2-z^2}}$. For our cell, $z_1 = b\sin(\omega_1)$ and $z_2 = b\sin(\omega_2)$. The surface area of the cell can be found from the formula for a surface of revolution (Anton, 1998, sec. 8.5) that is

$$
\begin{aligned}
S_{cell} &= \int\limits_{\lambda=\lambda_0-\frac{\Delta\lambda}{2}}^{\lambda_0+\frac{\Delta\lambda}{2}} d\lambda \int\limits_{z=z_1}^{z_2} r(z)\sqrt{1+[r'(z)]^2}dz \\
&= \Delta\lambda a \int\limits_{z=z_1}^{z_2} \sqrt{1-\frac{z^2}{b^2}}\sqrt{1+\frac{a^2z^2}{b^2(b^2-z^2)}}dz \\
&= \Delta\lambda a \int\limits_{z=z_1}^{z_2} \frac{\sqrt{b^2-z^2}}{b}\frac{\sqrt{b^4-b^2z^2+a^2z^2}}{b\sqrt{(b^2-z^2)}}dz \\
&= \Delta\lambda a \int\limits_{z=z_1}^{z_2} \sqrt{1+\frac{(a^2-b^2)z^2}{b^4}}dz = \Delta\lambda a \int\limits_{z=z_1}^{z_2} \sqrt{1+\frac{e'^2}{b^2}z^2}dz \\
&= \Delta\lambda a \left[\frac{b}{2e'}\operatorname{arcsinh}\left(\frac{e'}{b}z\right) + \frac{z}{2}\sqrt{1+\frac{e'^2}{b^2}z^2}\right]_{z=b\sin(\omega_1)}^{b\sin(\omega_2)}
\end{aligned}
\tag{4}
$$

With the primitive of $\sqrt{1+u^2z^2}$ regarding z that is $\frac{1}{2u}\operatorname{arcsinh}(uz) + \frac{z}{2}\sqrt{1+u^2z^2}$.

Finally,

$$
\begin{aligned}
S_{cell} = \Delta\lambda ab[&\frac{1}{2e'}(\operatorname{arcsinh}(e'\sin(\omega_2)) - \operatorname{arcsinh}(e'\sin(\omega_1)))+ \\
&\frac{1}{2}\left(\sin(\omega_2)\sqrt{1+e'^2\sin^2(\omega_2)} - \sin(\omega_1)\sqrt{1+e'^2\sin^2(\omega_1)}\right)]
\end{aligned}
\tag{5}
$$

Note that the demonstration can also be made by using the first fundamental form of differential geometry to obtain the area of a portion of surface. And also by using developing the integral of ω instead of z.

3 Estimation of associated Legendre functions

For conversion operations between spatial and spherical harmonics domains, associated Legendre functions are computed using the recursive algorithm from Holmes and Featherstone, 2002. The implementation of this algorithm is based on the implementation from the Python library *pyshtools*, described in Wieczorek and Meschede, 2018 (FORTRAN functions called in the script), with an improvement introduced through the parallel computation of the associated Legendre functions along the 'latitude' dimension.

3.1 Discussion about the error in the manipulation on the higher degrees of the associated Legendre functions

The computation of the associated Legendre function values involves manipulation of very small numbers that can fall below the standard precision defined by the IEEE754-2008 norm. The higher the maximum degree used for the associated Legendre functions, the smaller these numbers can become (at high latitudes).

I (Hugo Lecomte) do not fully understand the implications of this potential issue. To simplify my understanding, the computer manipulates values close to zero that are so small it represents them either as zero or as the smallest representable number according to its precision (e.g., in Python *np.finfo(dtype).eps*). If the value is rounded to the smallest representable number, then an approximation occurs, and the computer manipulates a value that is too large compared to what it should be. This might result in an error that is not acceptable to the user. However, I do not have sufficient intuition or a sense of the order of magnitude to quantify this.

The associated Legendre functions are computed with a scaling factor (named *scalef* in the code) to mitigate this issue. Other implementations (e.g., Georges Balmino in GBSYNTMP) truncate the higher orders of the associated Legendre functions to avoid manipulating values considered sufficiently close to zero.

We choose to add a warning on the computation when the maximum degree is higher than 300 to inform the user that he should pay attention to these problem. A subsidiary validation regarding the associated Legendre functions computation up to degree 300 is shown in subsection 4.4.1.

Future validation can be made with SHTns from Nathanael Schaeffer.

4 Determine a spatial grid from SH

There are different units to represent the gravity field η onto the Earth's surface (sphere or ellipsoid). These units can be obtained from the Stokes coefficients with a certain factor ζ_l, which depends on the degree l :

$$\eta(\theta,\lambda) = \sum_{l=0}^{+\infty} \zeta_l \sum_{m=0}^{n} [C_{l,m}\cos(m\lambda) + S_{l,m}\sin(m\lambda)] P_{l,m}(\cos\theta) \tag{6}$$

Remark: in its current implementation, the conversion from spherical harmonics to a spatial representation only project the gravity field at the Earth's surface (sphere of radius R or an ellipsoid). The code might be improved to handle the height with the surface of reference. However, this can be overcome by changing the radius of the sphere of reference although it has some geodetic implications to be taken into account.

4.1 Direct units

The gravitational potential (*unit="potential"*) with unit in m^2.s^{-2}, uses $\zeta_l^V = \frac{GM}{r} \sim [L^2.T^{-2}]$.

The gravity field expressed as an acceleration in m.s^{-2} (*unit="microGal"*) uses $\zeta_l^g = \frac{GM(l+1)}{R^2} \sim [L.T^{-2}]$.

The geoid representation of the gravity field $N(\theta,\lambda)$ in meters compared to the ellipsoid of reference (Heiskanen and Moritz, 1967). In this case (*unit="mmgeoid"*), it uses $\zeta_l^N = \frac{GM}{R\,\gamma_0} \sim [L]$ for the ellipsoidal case or the simplification of this formula $\zeta_l^N = R \sim [L]$ for the spherical case. Here $\gamma_0(\varphi)$ is the normal acceleration obtained from the Somigliana formula Moritz, 1980.

4.2 Earth's elasticity units

The crustal elastic uplift (*unit="mecu"*) uses $\zeta_l^{ECU} = R\frac{h_l}{1+k_l} \sim [L]$, with k_l the potential Love numbers.

The crustal viscoelastic uplift (*unit="mecu"*) uses $\zeta_l^{VCU} = R\frac{2l+1}{2} \sim [L]$.

Equivalent Water Height (*unit="mewh"*), in meters, uses $\zeta_l^{EWH} = R\frac{\rho_{ave}}{3\rho_{water}}\frac{2l+1}{1+k_l} = \frac{GM_{earth}}{4\pi G\rho_{water}R^2}\frac{2l+1}{1+k_l} \sim [L]$ with ρ_{ave} the Earth's mean density (5517 kg.m^{-3}) and ρ_{water} the water density (≈ 1000 kg.m^{-3})

(Wahr et al., 1998).

The equivalent surface pressure in kg.m^{-1}.s^{-2} (*unit="pascal"*) uses $\zeta_l^{Pa} = R\frac{g\ \rho_{ave}(2l+1)}{3(1+k_l)} \sim [M.L^{-1}.T^{-2}]$ with g being the standard gravitational acceleration.

4.3 Ellipsoidal Earth

Before the projection of the Stokes coefficients on an ellipsoid, one might need to go from the Stokes coefficients that represent the full potential (W) to perturbed potential (T) by removing the normal potential (U) such as $T(M) = W(M) - U(M)$. It can be done with the function in *lenapy.utils.gravity* that is *apply_normal_zonal_correction()*. This function apply a correction on zonal $C_{2n,0}$ coefficients corresponding to the normal gravity field of an ellipsoid.

In the case of an ellipsoidal surface, the ζ_l values are multiplied by a fraction depending of the geocentric latitude and of the units (with the power k) $\zeta_l^{ellps} = \zeta_l \left(\frac{a}{r(\omega)}\right)^k$. The EWH formula is given by Ditmar, 2018.

4.4 Validation against other software

LENAPY's conversions have been compared with other software. It allows to test at the same time the associated Legendre functions estimations as well as the global conversion.

4.4.1 Validation with IGCEM calculation service

The ICGEM calculation service is a web-service that allows to compute grid from files stored in the website (https://icgem.gfz-potsdam.de/calcgrid).

Potential

Comparison between the conversion from SH to spatial grid with Lenapy has been compared to the ICGEM calculation service output considering the same Earth's parameters. The parameters used are

- $a = 6378137$ m
- $GM = 3.986004415e^{14}$ m^3.s^{-2}
- $f = \frac{1}{298.257650}$ for the ellipsoidal Earth.

To obtain the corresponding Earth's parameters for the field, it might need to be corrected with *change_reference()*.

The potential grid from Lenapy is the conversion of GO_CONS_GCF_2_DIR_R6.gfc coefficients to spatial grid for $l > 2$ and has an amplitude of ± 600 m^2.s^{-2} (Fig. 1).

After the computation of the spherical Earth spatial grid with Lenapy, the difference with the ICGEM output are under the order of 10^{-9} compared to the original values (Fig. 2a).

After the computation of the ellipsoidal Earth spatial grid with Lenapy, the difference with the ICGEM output are under the order of 10^{-9} compared to the original values (Fig. 4b).

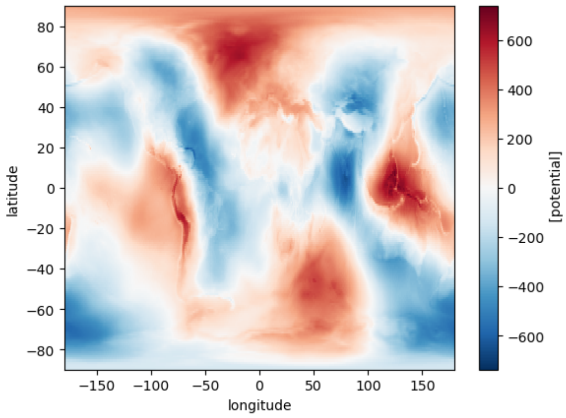
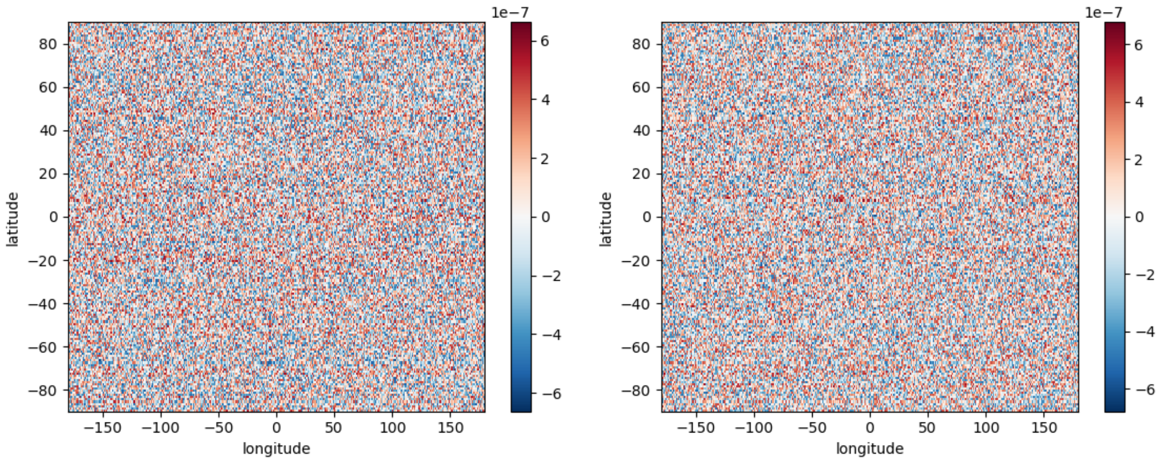


Figure 1: GO_CONS_GCF_2_DIR_R6.gfc in potential for $l > 2$ computed from Lenapy



(a) (b)

Figure 2: Comparison of the Lenapy output (potential) with ICGEM output (potential_ell) on a sphere (a) and on an ellipsoid (b)

Geoid height

Comparison between the conversion from SH to spatial grid with Lenapy has been compared to the ICGEM calculation service output considering the same Earth's parameters. This parameters are $a = 6378136.46$ m and $GM = 3.986004415e^{14}$ $\mathrm{m}^3.\mathrm{s}^{-2}$ for the spherical Earth and with also $f = \frac{1}{298.257650}$ for the ellipsoidal Earth.

The geoid grid from ICGEM is the conversion of GO_CONS_GCF_2_DIR_R6.gfc coefficients to spatial grid (Fig. 3).

After the computation of the spherical Earth spatial grid with Lenapy, the difference with the ICGEM output are under the order of 10^{-10} compared to the original values (Fig. 4a).

After the computation of the ellipsoidal Earth spatial grid with Lenapy, the difference with the ICGEM output are under the order of 10^{-7} compared to the original values (Fig. 4b). Note that the coefficients of the file read with Lenapy need to be corrected with *apply_normal_zonal_correction()* before the conversion.

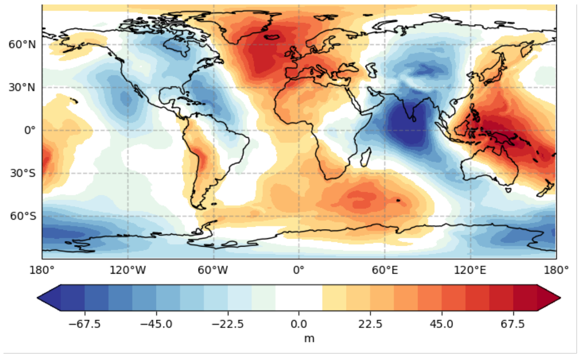
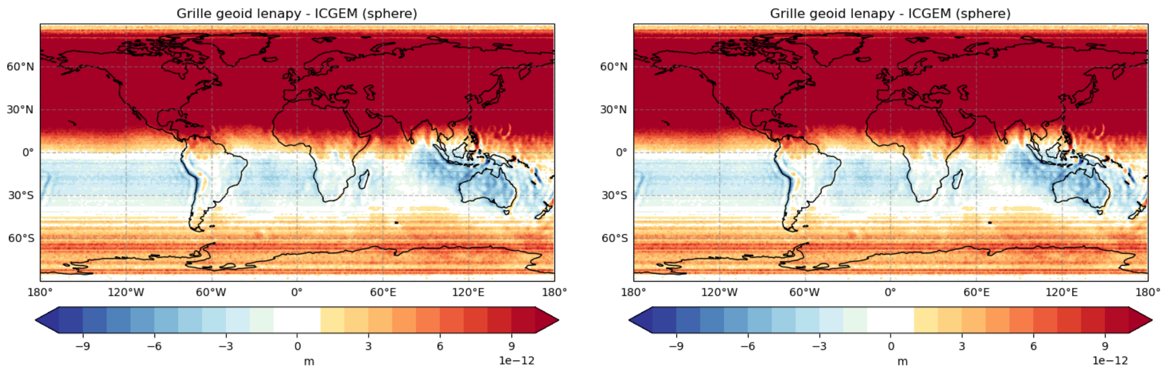


Figure 3: GO_CONS_GCF_2_DIR_R6.gfc in geoid height (labeled height anomaly on ICGEM calculation service)



(a) (b)

Figure 4: Comparison of the Lenapy output (mmgeoid) with ICGEM output (height anomaly) on a sphere (a) and on an ellipsoid (b)

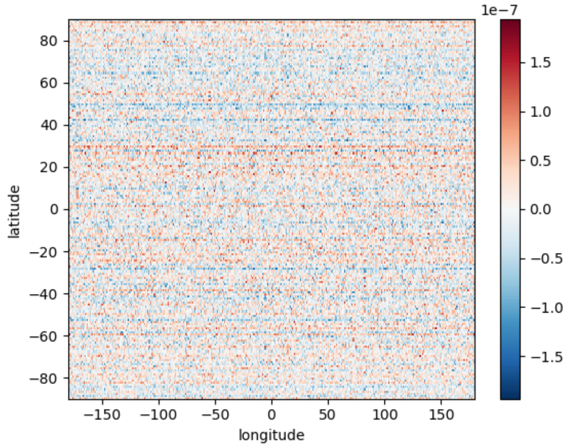
4.4.2 grates

The *grates* Python library allows to read and convert spherical harmonics based on object-oriented approach. *grates* have its own internal function to compute associated Legendre functions.

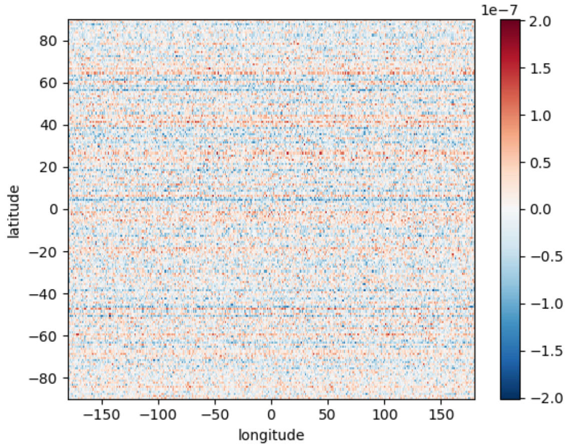
Potential : Comparison between the conversion from SH to spatial grid with Lenapy has been compared to the grates output considering the same Earth's parameters. Here input file is ITSG-Grace2018_n60_2004-01.gfc. After the computation of the spherical and ellipsoidal Earth spatial grid with Lenapy, the difference with the grates output are under the order of 10^{-9} compared to the original values (Fig. 5) with a linear pattern for equals latitudes.

Geoid height : After the computation of the spherical and ellipsoidal Earth spatial grid with Lenapy, the difference with the grates output are under the order of 10^{-7} compared to the original values (Fig. 6) for the ellipsoidal Earth. The difference for the spherical Earth case are of the order of 10^{-4} seems to be related to the addition of the centrifugal effect from grates that is not added in Lenapy.

Geoid height : After the computation of the spherical and ellipsoidal Earth spatial grid with Lenapy (by carefully applying the same parameters and Love numbers), the difference with the grates output are under the order of 10^{-6} compared to the original values (Fig. 7) for the ellipsoidal Earth and spherical Earth



(a)

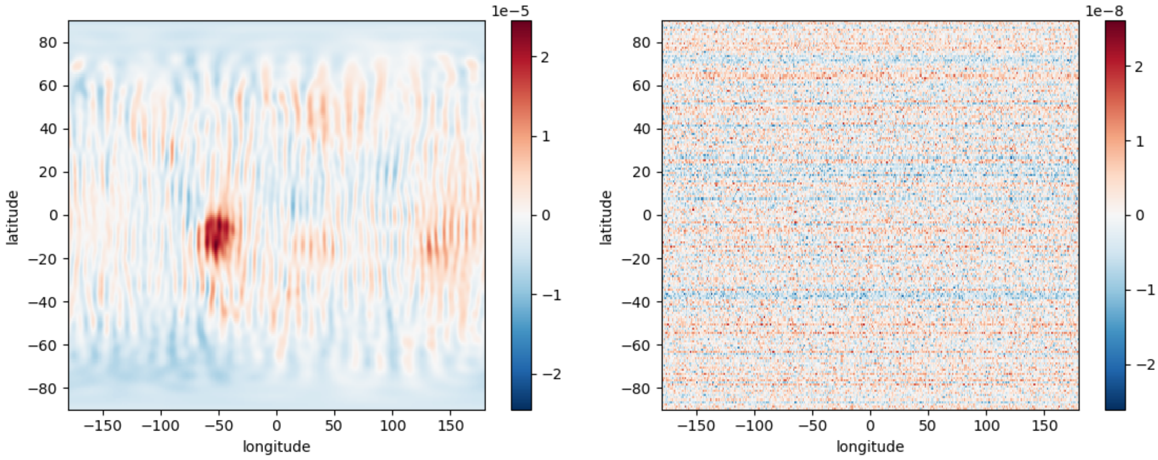


(b)

Figure 5: Comparison of the Lenapy output (potential) with grates output (potential) on a sphere (a) and on an ellipsoid (b).

4.4.3 Legacy internal validation

Internal LEGOS and Magellium validation has been made between Lenapy and an internal Python library called *Panis standalone* that uses associated Legendre functions from *Pyshtools*. The agreement between both library has been judged sufficient enough when the difference between both reach under 1% of errors with the Equivalent Water Height representation (differences mostly located at high latitudes).



(a) (b)

Figure 6: Comparison of the Lenapy output (mmgeoid) with grates output (geoid) on a sphere (a) and on an ellipsoid (b).

5 Determine SH from a spatial grid

From Chen et al., 2022 (eq. 3), Stokes coefficient can be deduced from the mass repartition

$$C_{l,m} = \frac{1}{M_{earth}(2l+1)} \iiint \left(\frac{r}{R}\right)^l P_{l,m}(\cos\theta)\cos(m\lambda)dM \tag{7}$$

We detail the computation for Equivalent Water Height (EWH) that is an unit to express gravity field anomalies in a form where the gravity field is generated by a thin layer of water located at the surface of a sphere / ellipsoid. Here H_w is the surface density in EWH and it can be retrieve at a certain longitude λ and co-latitude θ by

$$H_w(\theta,\lambda) = \sum_{l=0}^{\infty} \frac{R\rho_{ave}}{3\rho_{water}} \frac{2l+1}{1+k_l} \sum_{m=0}^{l} \left[C_{l,m}\cos(m\lambda) + S_{l,m}\sin(m\lambda)\right] P_{l,m}(\cos\theta) \tag{8}$$

With R the radius of the sphere over which the water layer is located, ρ_{ave} the average density of the Earth, ρ_{water} the water density, k_l the elastic loading Love number.

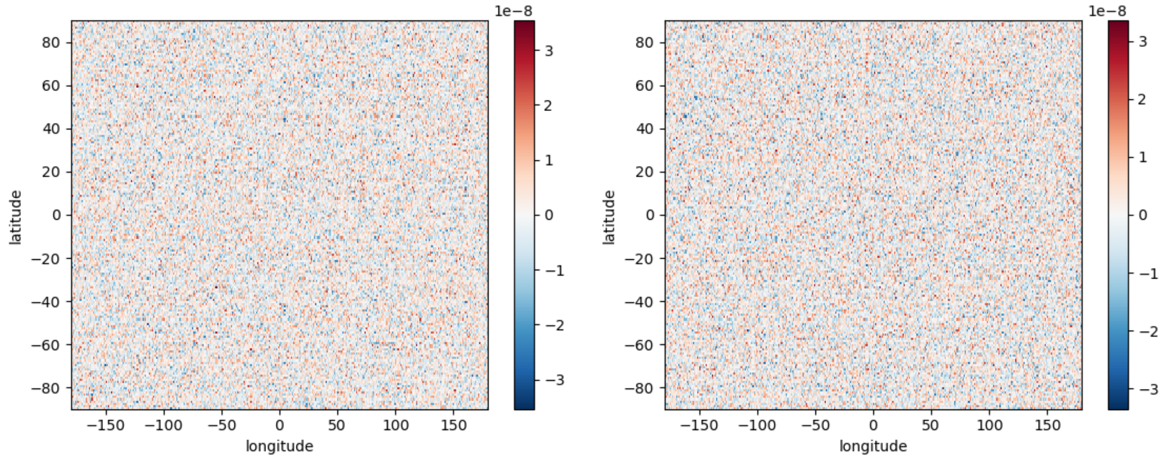
This expression can be found with the substitution of $\rho_{ave} = \frac{3M_{earth}}{4\pi R^3}$.

5.1 Over a sphere

The expression of $C_{l,m}$ can be found in Wahr et al., 1998

$$\begin{aligned} C_{l,m} &= \frac{1}{4\pi} \frac{3\rho_{water}}{R\rho_{ave}} \frac{1+k_l}{2l+1} \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{2\pi} H_w(\theta,\lambda) P_{l,m}(\cos\theta)\cos(m\lambda)\sin(\theta) d\theta d\lambda \\ &= \frac{1}{4\pi} \frac{3\rho_{water}}{R\rho_{ave}} \frac{1+k_l}{2l+1} \sum_{\lambda}\sum_{\theta} H_w(\theta,\lambda) P_{l,m}(\cos\theta)\cos(m\lambda) 2\sin(\theta)\sin(\frac{\Delta\theta}{2})\Delta\lambda \end{aligned} \tag{9}$$

With the discretization of the integral over sphere part.



(a) (b)

Figure 7: Comparison of the Lenapy output (mewh) with grates output (ewh) on a sphere (a) and on an ellipsoid (b).

5.2 Over an ellipsoid

With an ellipsoid, the equation contains a fraction that is not simplified corresponding to $\left(\frac{r(\theta)}{a}\right) = \left(\frac{1-f}{1-e^2\sin^2(\omega)}\right)$ (Ditmar, 2018)

$$\begin{aligned} C_{l,m} &= \frac{1}{S_{ellipsoid}} \frac{3\rho_{water}}{a\rho_{ave}} \frac{1+k_l}{2l+1} \int_{\lambda=0}^{2\pi} \int_{\omega=-\pi}^{\pi} \left(\frac{1-f}{1-e^2\sin^2(\omega)}\right)^{l+2} H_w(\omega,\lambda) P_{l,m}(\sin\omega)\cos(m\lambda) dS \\ &= \frac{1}{S_{ellipsoid}} \frac{3\rho_{water}}{a\rho_{ave}} \frac{1+k_l}{2l+1} \sum_{\lambda} \sum_{\omega} \left(\frac{1-f}{1-e^2\sin^2(\omega)}\right)^{l+2} H_w(\omega,\lambda) P_{l,m}(\sin\omega)\cos(m\lambda) \\ &\quad \Delta\lambda a \left[\frac{b}{2e'} \operatorname{arcsinh}\left(\frac{e'}{b}z\right) + \frac{z}{2}\sqrt{1+\frac{e'^2}{b^2}z^2}\right]_{z=b\sin(\omega_1)}^{b\sin(\omega_2)} \end{aligned} \tag{10}$$

As stated in Ditmar, 2018, this formula is the result of the simplification of the fraction $\left(\frac{a_E}{a}\right)^3$ (approximation of 0.3%) where a_E is the earth average radius so that the volume of the spherical earth with a radius $a_E = a(1-f)^{\frac{1}{3}}$ is equal to the volume of the ellipsoid. This simplification allow to be more coherent between the spherical case where $R = a$ and the ellipsoidal case. Taking this value into account would introduce a scale factor between both.

5.3 Note on the LENAPY implementation

This estimation of Stokes coefficients is a direct approach with a formula application. One can use an indirect approach that consist of an inversion of the matrix between Stokes coefficients to grid (approach used in previous PANIS standalone). It corresponds to : $A_{Stokes}X = B_{grid}$ so in the other way, $B_{grid}X^{-1} = A_{Stokes}$

Normally both approaches are equivalent (although a numerical inversion might unstable numerically) but no mathematical verification has been made here.

6 Mass conservation

6.1 Over a sphere

To get a mass that is null trough time (and constant), one can add a negative mass over the ocean cell corresponding to the global mass at each time. By this way, $C_{0,0}$ coefficient computed for the resulting grid is equal to 0.

In EWH, the height h that is added to the ocean corresponds to

$$h = -\frac{\sum_{\lambda}\sum_{\theta} H_w(\theta, \lambda) \times \text{surface}}{\text{ocean surface}} = -\frac{\text{global mass}}{\text{ocean surface}} \tag{11}$$

6.2 Implementation of mass conservation over a sphere in LENAPY

The projection of a zonal stokes coefficient $C_{2l,0}$ over a sphere (or an ellipsoid) can have a global mass that is not null due to the discretisation of the projection. This is not the case for $C_{2l+1,0}$ coefficients due to the symmetry in latitude and for all other coefficients due to the symmetries in longitude that ensure a global mass null (at the computer precision) if the grid is regular.

One can artificially resolve this problem in LENAPY by using the argument "force_mass_conservation=True" in the function *lenapy.utils.harmo.sh_to_grid()*. To resolve this problem, the function project all coefficients except $C_{0,0}$ on the grid and then remove the artifact global mass uniformly over the sphere and then add the grid resulting from $C_{0,0}$.

6.3 Over an ellipsoid

For an ellipsoid, the problem is more complex. A global mass that is null does not means that $C_{0,0} = 0$ because

$$C_{0,0} = \frac{3\rho_{water}}{a\rho_{ave}} \sum_{\lambda}\sum_{\theta} \left(\frac{1-f}{\sqrt{1-e^2 \sin^2\theta}} \right)^2 H_w(\omega, \lambda) d\Omega \tag{12}$$

That means that removing a height h given by an adaptation of equation (11) does not implies a null $C_{0,0}$ because of the dependency in latitude with θ.

To get a null $C_{0,0}$, the height h_{C_0} need to be added on the whole ellipsoid

$$h_{C_0} = -\frac{\sum_{\lambda}\sum_{\theta} \left(\frac{1-f}{\sqrt{1-e^2 \sin^2\theta}} \right)^2 \times H_w(\theta, \lambda) \times \text{surface}}{\sum_{\lambda}\sum_{\theta} \left(\frac{1-f}{\sqrt{1-e^2 \sin^2\theta}} \right)^2 \times \text{surface}} \tag{13}$$

But with this, the resulting mass of the grid is not equal to 0 and other coefficients (like $C_{2,0}$) are also modified.

In the ellipsoidal case, the user has to be aware of the fact that a global mass that is null does not correspond to $C_0 = 0$.

7 Error due to discretisation

To assess and acknowledge the imperfection of determining Stokes coefficient from a punctual grid, here is the result of a test of conservation.

Starting from a grid corresponding to GRACE gravity field from CNES unfiltered on the first month of the product, we consider only the gravity field content over the oceans with a mask. The resulting grid has a global mass of 4.19×10^{21} kg (larger than the true ocean mass value due to the leakage to the continent mass over the ocean).

All the values are given for a grid on 1^o by 1^o.

7.1 For a sphere

The mass loss during the back and forth between grid and spherical harmonics is linear. It corresponds to a fraction of $3{\times}10^{-6}$ of the global mass at each back and forth between grid and spherical harmonics.

With the argument "force_mass_conservation=True", the mass gain is also linear correspond to $1{\times}10^{-7}$ of the global mass.

7.2 For an ellipsoid

The mass loss during the back and forth between ellipsoidal grid and spherical harmonics is linear. It corresponds to $3{\times}10^{-4}$ of the global mass at each back and forth.

Note that on an ellipsoidal surface, the discretisation error is larger than on a spherical surface.

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