unit3

practice1

$$
\begin{aligned}
&1)\ \lim_{n\to\infty}\frac{100n^2+6}{3n^4+2n^2}=0\\
&2)\ \lim_{n\to\infty}\frac{100n}{6n^{1.5}+12}=\lim_{n\to\infty}\frac{100}{6n^{0.5}+12/n}=0\\
&3)\ \lim_{n\to\infty}\frac{10n^2}{7n^2logn}=0\\
&4)\ \lim_{n\to\infty}\frac{100n2^n+33n}{17n^2 2^n}=\lim_{n\to\infty}\frac{100+33/2^n}{17n}=0
\end{aligned}
$$

practice2

$$
\begin{aligned}
&1)O(n^3)\\
&2)O(n^{1.5})\\
&3)O(n^2logn)\\
&4)O(n^3)\\
&5)O(n2^n)
\end{aligned}
$$

practice3

$$
\begin{aligned}
&1)\ \lim_{n\to\infty}2n+7=\infty\\
&2)\ \lim_{n\to\infty}\frac{12n^2+8n+7}{n}=\infty\\
&3)\ \lim_{n\to\infty}\frac{5n^3+6n^2}{n^2}=\infty\\
&4)\ \lim_{n\to\infty}\frac{15n^3logn+16n^2}{n^3}=\infty
\end{aligned}
$$

practice4

$$
\begin{aligned}
&1)\Omega(n^3)\\
&2)\Omega(n^{1.5})\\
&3)\Omega(n^2logn)\\
&4)\Omega(n^3)\\
&5)\Omega(n2^n)
\end{aligned}
$$

practice5

$$
\begin{aligned}
&1)\ \lim_{n\to\infty}\frac{2n+7}{n^2}=0\\
&2)\ \lim_{n\to\infty}\frac{12n^2+8n+7}{n^3}=0\\
&3)\ \lim_{n\to\infty}\frac{5n^3+6n^2}{n^3logn}=0\\
&4)\ \lim_{n\to\infty}\frac{15n^3logn+16n^2}{n^4}=0
\end{aligned}
$$

practice6

$$
\begin{aligned}
&1)\Theta(n^3)\\
&2)\Theta(n^{1.5})\\
&3)\Theta(n^2logn)\\
&4)\Theta(n^3)\\
&5)\Theta(n2^n)
\end{aligned}
$$

---

practice7

思路：g(n) 是上限

$$
\begin{aligned}
&1)O(1)\\
&2)O(n)\\
&3)O(n^2)\\
&4)O(n^2logn)
\end{aligned}
$$

---

practice8

$$
\begin{aligned}
&1)O(m^2n^2+m^3n)\\
&2)O(m^2logn+m^2n^2)\\
&3)O(m^4+n^3+m^3n^2)\\
&4)O(mn^2+m^2n)
\end{aligned}
$$

---

practice9

1.

$$
\begin{aligned}
&\because \forall n \geq 1.5, \quad n^2 \leq 5n^2-6n \leq 6n^2\\
&\therefore 5n^2-6n=\Theta(n^2)
\end{aligned}
$$

2.

$$
\begin{aligned}
&\because \forall n \geq 1, \quad n! \leq n^n\\
&\therefore n! = O(n^n)
\end{aligned}
$$

3.

$$
\begin{aligned}
&\because \forall n > 0, \quad n^2 2^n < 2n^2 2^n + nlogn < 3n^2 2^n\\
&\therefore 2n^2 2^n + nlogn = \Theta(n^2 2^n)
\end{aligned}
$$

4.

$$
\begin{aligned}
&\because \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\\
&\therefore \sum_{i=0}^{n} i^2 = \Theta(n^3)
\end{aligned}
$$

5.

$$
\begin{aligned}
&\because \sum_{i=0}^{n} i^3 = (1+2+\cdots+n)^2 = (\frac{n(n+1)}{2})^2\\
&\therefore \sum_{i=0}^{n} i^3 = \Theta(n^4)
\end{aligned}
$$

6.

$$\because \forall n \geq 3, \quad n^{2n} < n^{2n} + 6*2^n < 2n^{2n}$$
$$\therefore n^{2n} + 6*2^n = \Theta(n^{2n})$$

7.

$$\because \forall n \geq 10^6, \quad n^3 < n^3 + 10^6 n^2 \leq 2n^3$$
$$\therefore n^3 + 10^6 n^2 = \Theta(n^3)$$

8.

$$\because \forall n \geq 1, \quad \frac{6n^3}{logn+1} \leq 6n^3$$
$$\therefore \frac{6n^3}{logn+1} = O(6n^3)$$

9.

$$\because \forall n \geq 1, \quad n^{1.001} \leq n^{1.001} + nlogn < 2n^{0.001}$$
$$\therefore n^{1.001} + nlogn = \Theta(n^{1.001})$$

10.

略，方式同9，用洛必达得到$\lim_{n\to\infty} \frac{logn}{n^{k+\epsilon}} = 0$即可

---

practice10

1.

$$\lim_{n\to\infty} \frac{5n^2-6n}{n^2} = 5$$
$$\lim_{n\to\infty} \frac{n^2}{5n^2-6n} = \frac{1}{5} < 5$$

2.

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

3.

$$\lim_{n\to\infty} \frac{2n^2 2^n + nlogn}{2n^2 2^n} = 1$$
$$\lim_{n\to\infty} \frac{2n^2 2^n}{2n^2 2^n + nlogn} = 1$$

4.

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n i^2}{n^3} = \frac{1}{3} < 3$$
$$\lim_{n\to\infty} \frac{n^3}{\sum_{i=0}^n i^2} = 3$$

5.

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n i^3}{n^4} = \frac{1}{4} < 4$$
$$\lim_{n\to\infty} \frac{n^4}{\sum_{i=0}^n i^3} = 4$$

6.

$$\lim_{n\to\infty}\frac{n^{2n}+6*2^n}{n^{2n}}=1$$
$$\lim_{n\to\infty}\frac{n^{2n}}{n^{2n}+6*2^n}=1$$

7.

$$\lim_{n\to\infty}\frac{n^3+10^6n^2}{n^3}=1$$
$$\lim_{n\to\infty}\frac{n^3}{n^3+10^6n^2}=1$$

8.

$$\lim_{n\to\infty}\frac{\frac{6n^3}{logn+1}}{n^3}=0$$

9.

$$\lim_{n\to\infty}\frac{n^{1.001}+nlogn}{n^{1.001}}=1$$
$$\lim_{n\to\infty}\frac{n^{1.001}+nlogn}{n^{1.001}}=1$$

10.

$$\lim_{n\to\infty}\frac{n^{k+\epsilon}+n^klogn}{n^{k+\epsilon}}=1,\quad k\geq 0\&\epsilon>0$$

---

practice11

1.

$$\lim_{n\to\infty}\frac{10n^2+9}{n}=\infty$$

2.

$$\lim_{n\to\infty}\frac{n^2logn}{n^2}=\infty$$

3.

$$\lim_{n\to\infty}\frac{n^2}{n^2/logn}=\infty$$

4.

$$\lim_{n\to\infty}\frac{n^32^n+6n^23^n}{n^32^n}=\infty$$

---

practice12

$$f(n)=\sum_{i=0}^{m}|a_i|n^i=n^m\sum_{i=0}^{m}|a_i|n^{i-m}\geq n^m,\quad n\geq 1(c=1)$$

---

practice13

$$f(n)=\sum_{i=0}^{m}|a_i|n^i=n^m\sum_{i=0}^{m}|a_i|n^{i-m}\geq n^m,\quad n\geq 1(c_1=1) \tag{1}$$

$$f(n)=\sum_{i=0}^{m}|a_i|n^i=n^m\sum_{i=0}^{m}|a_i|n^{i-m}\le n^m\sum_{i=0}^{m}|a_i|,\quad n\ge 1(c_2=\sum_{i=0}^{m}|a_i|) \tag{2}$$

---

practice14

$$\because \lim_{n\to\infty}\frac{f(n)}{g(n)}=0\Rightarrow f(n)=O(g(n))$$
$$\therefore \lim_{n\to\infty}\frac{g(n)}{f(n)}=\infty\Rightarrow f(n)\ne\Omega(g(n))$$
$$\therefore f(n)=o(g(n))$$

---

practice15

E5.

$$\text{证明略}$$

Ref:
https://en.wikipedia.org/wiki/Faulhaber's_formula

E6.

$$\sum_{i=0}^{n}r^i=\sum_{i=0}^{n}r^{i-n}r^n=r^n\sum_{i=0}^{n}r^{i-n},\quad r>1$$
$$\Rightarrow r^n<r^n\sum_{i=0}^{n}r^{i-n}\le r^n\sum_{i=0}^{n}r\Rightarrow\sum_{i=0}^{n}r^i=\Theta(r^n)=\oplus(r^n)$$

E7.

$$n!\approx\sqrt{2\pi n}(\frac{n}{e})^n=\Theta(\sqrt{n}(\frac{n}{e})^n)=\oplus(\sqrt{n}(\frac{n}{e})^n)$$

Ref:
https://en.wikipedia.org/wiki/Stirling's_approximation

E8.

$$\sum_{i=1}^{n}1/i\approx ln(n)+C=\Theta(logn)=\oplus(logn)$$

Ref:
https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)
https://en.wikipedia.org/wiki/Euler's_constant

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practice16

设 $\oplus=\Theta$
I1.

$$\lim_{n\to\infty}\frac{\sum_{n=a}^{b}f(n)}{\sum_{n=a}^{b}g(n)}=\lim_{n\to\infty}\frac{f(a)+\cdots+f(b)}{g(a)+\cdots+g(b)}=c_1>0$$
$$\lim_{n\to\infty}\frac{\sum_{n=a}^{b}g(n)}{\sum_{n=a}^{b}f(n)}=\lim_{n\to\infty}\frac{g(a)+\cdots+g(b)}{f(a)+\cdots+f(b)}=\frac{1}{c_1}>0$$
$$\because c1\text{为非零常量}$$
$$\therefore \text{令}\quad c=max\{c_1,1/c_1\},\quad \oplus=\Theta\text{时推论成立}$$
$$\therefore \oplus=O,\quad \oplus=\Omega\text{时，推论也成立}$$

I2.

$$\because max_{1\le i\le k}\{g_i(n)\}\le\sum_{i=1}^{k}g_i(n)\le k\cdot max_{1\le i\le k}\{g_i(n)\}$$
$$\therefore \sum_{i=1}^{k}g_i(n)=\oplus(max_{1\le i\le k}\{g_i(n)\})$$
$$\therefore \sum_{i=1}^{k}f_i(n)=\oplus(\sum_{i=1}^{k}g_i(n))=\oplus(max_{1\le i\le k}\{g_i(n)\})$$

I3.
略（参考 I1，求和变求积）

I4.

$$\because \lim_{n\to\infty}\frac{f_1(n)+f_2(n)}{g_1(n)+g_2(n)}=\lim_{n\to\infty}\frac{f_1(n)}{g_1(n)}+\lim_{n\to\infty}\frac{f_2(n)}{g_2(n)}\le c_1+c_2$$
$$\therefore f_1(n)+f_2(n)=O(g_1(n)+g_2(n))$$
$$\because f_1(n)=O(g_1(n))$$
$$\therefore \text{无法保证}\lim_{n\to\infty}\frac{g_1(n)}{f_1(n)}\text{存在极限}$$
$$\therefore \text{无法保证}\lim_{n\to\infty}\frac{g_1(n)+g_2(n)}{f_1(n)+f_2(n)}\text{存在极限}\Rightarrow f_1(n)+f_2(n)\ne\Omega(g_1(n)+g_2(n))\Rightarrow f_1(n)+f_2(n)\ne\Theta(g_1(n)+g_2(n))$$

I5.
略（参考 I4）

I6.
略（参考 I4）

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practice17

1. 错，反例:$f(n)=n^2=O(n^2),g(n)=n=O(n^2),\frac{f(n)}{g(n)}=n\ne O(1)=O(\frac{F(n)}{G(n)})$
2. 错，反例:$f(n)=n^2=O(n^3),g(n)=n=O(n),\frac{f(n)}{g(n)}=n\ne\Omega(n^2)=\Omega(\frac{F(n)}{G(n)})$
3. 错，反例:$f(n)=n^2=O(n^3),g(n)=n=O(n),\frac{f(n)}{g(n)}=n\ne\Theta(n^2)=\Theta(\frac{F(n)}{G(n)})$
4. 错，反例:$f(n)=n^2=\Omega(n^2),g(n)=n=\Omega(1),\frac{f(n)}{g(n)}=n\ne\Omega(n^2)=\Omega(\frac{F(n)}{G(n)})$
5. 错，反例:$f(n)=n^2=\Omega(n),g(n)=n=\Omega(1),\frac{f(n)}{g(n)}=n\ne O(1)=O(\frac{F(n)}{G(n)})$
6. 错，反例:$f(n)=n^2=\Omega(n^2),g(n)=n=\Omega(1),\frac{f(n)}{g(n)}=n\ne\Theta(n^2)=\Theta(\frac{F(n)}{G(n)})$
7. 对，证明:

$$
\begin{aligned}
&\text{有} f(n) = \Theta(F(n)), \quad g(n) = \Theta(G(n)), \quad \text{则设} \\
&\begin{cases}
\lim\limits_{n\to\infty} \dfrac{f(n)}{F(n)} = \lim\limits_{n\to\infty} \dfrac{F(n)}{f(n)} \le c1 \quad (c1 > 0) \\
\lim\limits_{n\to\infty} \dfrac{g(n)}{G(n)} = \lim\limits_{n\to\infty} \dfrac{g(n)}{G(n)} \le c2 \quad (c2 > 0)
\end{cases} \\
&\therefore \lim\limits_{n\to\infty} \frac{\frac{f(n)}{g(n)}}{\frac{F(n)}{G(n)}} = \lim\limits_{n\to\infty} \frac{f(n)G(n)}{F(n)g(n)} = c1 \cdot c2 > 0 \\
&\therefore \lim\limits_{n\to\infty} \frac{\frac{F(n)}{G(n)}}{\frac{f(n)}{g(n)}} = \frac{1}{\lim_{n\to\infty} \frac{f(n)G(n)}{F(n)g(n)}} = \frac{1}{c1 \cdot c2} < c1 \cdot c2 \\
&\therefore \frac{f(n)}{g(n)} = \Theta(\frac{F(n)}{G(n)})
\end{aligned}
$$

8. 对，因为 7 对
9. 对，因为 7 对

---

practice18

略

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practice19

1. $[100, +\infty)$
2. $[2, +\infty)$
3. $[1, 9]$
4. $[1] \cup [13747, +\infty)$

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practice20

略

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practice21

略