Problem Set #4, Optimization

OSM Lab: Math
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Exercise 6.6

Suppose

$$f(x, y) = 3x^2y + 4xy^2 + xy \tag{1}$$

Then,

$$\frac{\partial f}{\partial x} = 6xy + 4y^2 + y = y(6x + 4y + 1)$$
$$\frac{\partial f}{\partial y} = 3x^2 + 8xy + x = x(3x + 8y + 1)$$

We now find the critical points of f. Observe that $(0, 0)$ is a critical point. Similarly, if we set $x = 0$, then $(0, -\frac{1}{4})$ is a critical point. If we set $y = 0$, then $(-\frac{1}{3}, 0)$ is a critical point. Finally, observe that the values of x and y satisfying $6x + 4y = -1$ and $3x + 8y = -1$ will also be a critical point. This occurs at $(-\frac{1}{9}, -\frac{1}{12})$. Now, the Hessian is given by,

$$D^2 f(x, y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix} \tag{2}$$

Then,

$$D^2 f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad D^2 f(0, -\frac{1}{4}) = \begin{bmatrix} -\frac{3}{2} & -1 \\ -1 & 0 \end{bmatrix}$$
$$D^2 f(-\frac{1}{3}, 0) = \begin{bmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{bmatrix} \qquad D^2 f(0, -\frac{1}{4}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{8}{9} \end{bmatrix}$$

$D^2 f(0, -\frac{1}{4})$ is negative semidefinite, so $(0, -\frac{1}{4})$ is a local maximum. However, the remaining Hessian matrices are neither positive semidefinite nor negative semidefinite. Therefore, those corresponding points are saddle points.

Exercise 6.7

Define an unconstrained quadratic optimization problem as follows: the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is quadratic,

$$f(x) = x^T A x - b^T x + c \tag{3}$$

where $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

Claim 1. For any square matrix $A \in M_n(\mathbb{R})$, the matrix $Q = A^T + A$ is symmetric, and $x^TQx = x^TA^Tx + x^TAx = 2x^TAx$, so Equation 3 is equal to,

$$f(x) = \frac{1}{2}x^TQx - b^Tx + c \tag{4}$$

Proof. Observe that,

$$Q = A^T + A = A + A^T = (A^T + A)^T = Q^T \tag{5}$$

Therefore Q is symmetric. Additionally,

$$x^TQx = x^T(A^T + A)x = x^TA^Tx + x^TAx = 2x^TAx \tag{6}$$

Where the first equality follows from the definition of Q, the second from expanding the sum, and the third because x^TA^Tx and x^TAx are scalars. The transpose of a scalar is simply that scalar, so $(x^TA^Tx)^T = x^TAx$. Thus, $x^TA^Tx + x^TAx = 2x^TAx$. It follows that $x^TAx = \frac{1}{2}x^TQx$. Therefore, we may substitute this expression into Equation 3 to obtain Equation 4. □

Claim 2. Any minimizer x^* of f is a solution of the equation,

$$Q^Tx^* = b \tag{7}$$

Proof. First note that f is differentiable at x^* and x^* is an interior point. Therefore, the first order necessary condition requires that $f'(x^*) = 0$ if x^* is a local minimizer. Note that $f'(x) = Q^Tx - b$. Therefore, if x^* is a local minimizer of f, then $Q^Tx^* = b$. □

Claim 3. The quadratic minimization problem will have a unique solution if and only if Q is positive definite.

Proof. First suppose that Q is positive definite. Thus, it follows that $f''(x) > 0$ for all x. Then, Q is invertible, so that $x^* = Q^{-1}b$ for a local minimizer x^*. Thus, by the second-order sufficient condition, x^* is a unique minimizer of (4). Conversely, suppose that x^* is the unique minimizer of f. Then, by the second order necessary condition, we have that Q is positive semidefinite. Since x^* is the unique solution to $Q^Tx^* = b$, it must be that Q is invertible, so that it cannot have any zero eigenvalues. From this it follows that Q is positive definite. Thus, positive definiteness of Q guarentees a unique solution to $Q^Tx^* = b$, which in turn is the minimizer of Equation 4. □

Exercise 6.11

Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a > 0$. Let $x_0 \in \mathbb{R}$. Then, applying Newton's method, observe that,

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = -\frac{b}{2a} \tag{8}$$

Observe that $f'(x_1) = 2ax_1 + b = 2a(-\frac{b}{2a}) + b = 0$. Then, $f''(x_1) = 2a > 0$ since $a > 0$. Therefore, by the second-order sufficient condition, $x_1 = -\frac{b}{2a}$ is a unique local minimizer.

Exercise 6.15

See Jupyter notebook.

Exercise 7.1

Claim 4. If S is a nonempty subset of V, then $conv(S)$ is convex.

Proof. Let $y_1, y_2 \in conv(S)$. Therefore, there exist $k \in \mathbb{N}$ and $(\lambda_i)_{i=1}^k$ such that $y = \sum_{i=1}^k \lambda_i y_i$, where each $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, and each $y_i \in S$. Similarly, there exist $l \in \mathbb{N}$ and $(\mu_i)_{i=1}^l$ such that $x = \sum_{i=1}^l \mu_i x_i$, where each $\mu_i \geq 0$, $\sum_{i=1}^l \mu_i = 1$, and each $x_i \in S$. Fix $\lambda \in [0, 1]$. Then,

$$\lambda x + (1-\lambda)y = \lambda \sum_{i=1}^{l} \mu_i x_i + (1-\lambda) \sum_{i=1}^{k} \lambda_i y_i \tag{9}$$

Consider the coefficients on the x_i's and y_i's. Note that,

$$\lambda(\sum_{i=1}^{l} \mu_i) + (1-\lambda) \sum_{i=1}^{k} \lambda_i = \lambda + (1-\lambda) = 1 \tag{10}$$

Observe that $\lambda x + (1-\lambda)y$ is indeed another convex combination of the elements of S. Therefore, $\lambda x + (1-\lambda)y \in conv(S)$ so $conv(S)$ is convex. $\square$

Exercise 7.2

Claim 5. A hyperplane is convex.

Proof. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $a \in V$ where $a \neq 0$ and $b \in \mathbb{R}$. Let $P = \{x \in V : \langle a, x \rangle = b\}$ be hyperplane in V. Let $x, y \in P$ and $\lambda \in [0, 1]$. By the definition of P, we have that $\langle a, x \rangle = b$ and $\langle a, y \rangle = b$. Therefore, by the linearity of the inner product, we have that,

$$\begin{aligned} b &= \lambda b + (1-\lambda)b \\ &= \lambda \langle a, x \rangle + (1-\lambda)\langle a, y \rangle \\ &= \langle a, \lambda x \rangle + \langle a, (1-\lambda)y \rangle \\ &= \langle a, \lambda x + (1-\lambda)y \rangle \end{aligned}$$

Therefore, $\langle a, \lambda x + (1-\lambda)y \rangle = b$, therefore $\lambda x + (1-\lambda)y \in P$, so that P is a convex set by definition. $\square$

Claim 6. A half space is convex.

Proof. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $a \in V$ where $a \neq 0$ and $b \in \mathbb{R}$. Let $H = \{x \in V : \langle a, x \rangle \leq b\}$ be hyperplane in V. Let $x, y \in H$ and $\lambda \in [0, 1]$. By the definition of H, we have that $\langle a, x\rangle \leq b$ and $\langle a, y\rangle \leq b$. Therefore, by the linearity of the inner product, we have that,

$$\begin{aligned} b &= \lambda b + (1-\lambda)b \\ &\geq \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \\ &= \langle a, \lambda x + (1-\lambda)y\rangle \end{aligned}$$

Therefore, $\langle a, \lambda x + (1-\lambda)y\rangle \leq b$, so that $\lambda x + (1-\lambda)y \in H$. Thus, H is convex. □

Exercise 7.4

Claim 7. Let $x, y, p \in \mathbb{R}^n$. Then,

$$\|x-y\|^2 = \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle \tag{11}$$

Proof. Observe that,

$$\begin{aligned} \|x-y\|^2 &= \|x-p+p-y\|^2 \\ &= \langle x-p+p-y, x-p+p-y\rangle \\ &= \langle x-p, x-p\rangle + \langle x-p, p-y\rangle + \langle p-y, x-p\rangle + \langle p-y, p-y\rangle \\ &= \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle \end{aligned}$$

□

Claim 8. Let $C \in \mathbb{R}^n$ be nonempty, closed, and convex. Let $p \in C$. Suppose that $\langle x-p, p-y\rangle \geq 0$ for all $y \in C$. Then, $\|x-y\| > \|x-p\|$ for all $y \in C$, $y \neq p$.

Proof. By Claim 7, we know that $\|x-y\|^2 = \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle$. By the definition of a norm, and by squaring each norm, we know that $\|x-y\|^2, \|x-p\|^2, \|p-y\|^2$ are all weakly positive. Furthermore, by assumption, $\langle x-p, p-y\rangle$ is weakly positive. Then, since $y \neq p$, we know that $p-y \neq 0$ so that $\|p-y\|^2 > 0$. Therefore,

$$\begin{aligned} \|x-y\|^2 &= \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle \\ &\geq \|x-p\|^2 + \|p-y\|^2 \\ &> \|x-p\|^2 \end{aligned}$$

Therefore, $\|x-y\| > \|x-p\|$. □

Claim 9. If $z = \lambda y + (1-\lambda)p$, where $0 \leq \lambda \leq 1$, then,

$$\|x-z\|^2 = \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2 \tag{12}$$

Proof. By the linearity of the inner product, we have that,

$$\begin{aligned} \|x-z\|^2 &= \|x-\lambda y-(1-\lambda)p\|^2 \\ &= \|x-p+\lambda(p-y)\|^2 \\ &= \langle x-p, x-p\rangle + 2\lambda\langle x-p, p-y\rangle + \lambda^2\|p-y\|^2 \\ &= \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2 \end{aligned}$$

□

Claim 10. If p is a projection of x onto the convex set C, then $\langle x-p, p-y\rangle \geq 0$ for all $y \in C$.

Proof. By Claim 9, we know that, $\|x-z\|^2 = \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2 \|y-p\|^2$. Observe that $\|x-z\|^2, \|x-p\|^2, \lambda$ are all weakly positive. Therefore,

$$\begin{aligned} 0 &\leq \|x-z\|^2 = \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2 \|y-p\|^2 \\ &\leq 2\lambda\langle x-p, p-y\rangle + \lambda^2 \|y-p\|^2 \end{aligned}$$

Suppose $\lambda \in (0,1]$. It follows that $0 \leq 2\langle x-p, p-y\rangle + \lambda \|y-p\|^2$. □

Claim 11. A point $p \in C$ is the projection of x onto C if and only if

$$\langle x-p, p-y\rangle \geq 0 \tag{13}$$

for all $y \in C$.

Proof. First suppose $p \in C$ is the projection of x onto C. In Claim 10, set $\lambda = 0$. It follows that $\langle x-p, p-y\rangle \geq 0$ for all $y \in C$. Conversely, suppose $\langle x-p, p-y\rangle \geq 0$ for all $y \in C$. Then, by claim 8, we have that $\|x-y\| > \|x-p\|$ for all $y \in C$, $y \neq p$. Therefore, $p \in C$ is the projection of x onto C. □

Exercise 7.8

Claim 12. If $f : \mathbb{R}^m \to \mathbb{R}$ is convex, if $A \in M_{m\times n}(\mathbb{R})$, and if $b \in \mathbb{R}^m$, then the function $g : \mathbb{R}^m \to \mathbb{R}$ given by $g(x) = f(Ax+b)$ is convex.

Proof. Let $x_1, x_2 \in \mathbb{R}^m$ and let $\lambda \in [0,1]$. Then,

$$\begin{aligned} g(\lambda x_1 + (1-\lambda)x_2) &= f(A(\lambda x_1 + (1-\lambda)x_2) + b) \\ &= f(\lambda(Ax_1 + b) + (1-\lambda)(Ax_2 + b)) \\ &\leq \lambda f(Ax_1 + b) + (1-\lambda) f(Ax_2 + b) \qquad \text{(by the convexity of } f) \\ &= \lambda g(x_1) + (1-\lambda) g(x_2) \end{aligned}$$

Therefore, g is convex. □

Exercise 7.12

Claim 13. The set $PD_n(\mathbb{R})$ of positive-definite matrices in $M_n(\mathbb{R})$ is convex.

Proof. Let M and N be positive-definite matrices. Fix $\lambda \in [0,1]$ and $x \in \mathbb{R}^n$. Then, by the definition of positive-definite and λ being positive, we have that $x^T M x > 0$ and $x^T N x > 0$. Consider the convex combination $\lambda M + (1-\lambda)N$. Then,

$$x^T(\lambda M + (1-\lambda)N)x = \lambda x^T M x + (1-\lambda) x^T N x \tag{14}$$

If $\lambda \in (0,1)$, then Equation 14 is clearly strictly positive. Similarly, in the boundary cases ($\lambda = 0$ or $\lambda = 1$), we also have that one of the terms in Equation 14 will be strictly positive and the other 0, so that the entire term is strictly positive. Therefore, $\lambda x^T M x + (1-\lambda) x^T N x > 0$ so that $\lambda M + (1-\lambda)N$ is positive-definite, hence $PD_n(\mathbb{R})$ is convex. □

Claim 14. The function $f(X) = -\log(\det(X))$ is convex on $PD_n(\mathbb{R})$.

Proof. We first prove a series of lemmas.

Lemma 15. If for every $A, B \in PD_n(\mathbb{R})$ the function $g(t) : [0,1] \to \mathbb{R}$ given by $g(t) = f(tA + (1-t)B)$ is convex, then f is convex.

Proof. Observe that,

$$\begin{aligned} f(tA + (1-t)B) &= g(t) \\ &= g(1 \cdot t + (1-t) \cdot 0) \\ &\leq tg(1) + (1-t)g(0) \\ &= tf(A) + (1-t)f(B) \end{aligned}$$

Therefore, f is convex. □

Lemma 16. If A is positive definite then there is an S such that $S^H S = A$. Furthermore,

$$\begin{aligned} g(t) &= -\log(\det(S^H(tI + (1-t)(S^H)^{-1}BS^{-1})S)) \\ &= -\log(\det(A)) - \log(\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \end{aligned}$$

Proof. By Proposition 4.5.7 in Volume I, we know that if A is positive definite, then there exists a nonsingular matrix S such that $A = S^H S$. Now, recall that for matrices A and B, we have that $det(AB) = det(A)det(B)$. We can use this property and properties of logarithms to prove the rest of this lemma. To that end,

$$\begin{aligned} -\log(\det(S^H(tI + (1-t)(S^H)^{-1}BS^{-1})S)) &= -\log(\det(S^H)\det(tI + (1-t)(S^H)^{-1}BS^{-1}))\det(S)) \\ &= -\log(\det(S^H)\det(S)\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \\ &= -\log(\det(S^H S)\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \\ &= -\log(\det(S^H S)) - log(\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \\ &= -\log(\det(A)) - \log(\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \end{aligned}$$

□

Lemma 17.

$$g(t) = -\sum_{i=1}^{n} \log(t + (1-t)\lambda_i) + -\log(\det(A)) \tag{15}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $(S^H)^{-1}BS^{-1}$.

Proof. First note that the eigenvalues of $tI + (1-t)(S^H)^{-1}BS^{-1})$ are $\{t + (1-t)\lambda_i\}_{i=1}^n$. Indeed, suppose λ_i and x_i are an eigenvalue and eigenvector respectively of $(S^H)^{-1}BS^{-1}$. Then,

$$\begin{aligned} (tI + (1-t)(S^H)^{-1}BS^{-1}))x_i &= tx_i + (1-t)\lambda_i x_i \\ &= (t + (1-t)\lambda_i)x_i \end{aligned}$$

so that $(t + (1-t)\lambda_i)$ is an eigenvalue of $tI + (1-t)(S^H)^{-1}BS^{-1})$. Next, recall that the determinant of a matrix is the product of its eigenvalues. Therefore,

$$\begin{aligned} g(t) &= -\log(\det(A)) - \log(\det(tI + (1-t)(S^H)^{-1}BS^{-1})) \\ &= -\sum_{i=1}^{n} \log(t + (1-t)\lambda_i) + -\log(\det(A)) \end{aligned}$$

$\square$

Lemma 18. $g''(t) \geq 0$ for all $t \in [0, 1]$.

Proof. By direct calculation from the above lemma,

$$g'(t) = -\sum_{i=1}^{n} \frac{(1-\lambda_i)}{\log(t + (1-t)\lambda_i)} \tag{16}$$

and then,

$$g''(t) = \sum_{i=1}^{n} \frac{(1-\lambda_i)^2}{\log(t + (1-t)\lambda_i)^2} \tag{17}$$

Observe that $g''(t)$ is composed of a sum of squares and is therefore weakly positive. $\square$

Together these lemmas show that $f(X) = -\log(\det(X))$ is convex on $PD_n(\mathbb{R})$. $\square$

Exercise 7.13

Claim 19. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above, then f is constant.

Proof. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above but f is not constant. Therefore, there exist points $x_1, x_2 \in \mathbb{R}^n$ such that $f(x_1) \neq f(x_2)$. Fix $\lambda \in [0, 1]$. Finally, suppose $f(x) \leq M$ for all x. Now, observe that,

$$\begin{aligned} f(x_1) &= f\left(\lambda \frac{x_1 - (1-\lambda)x_2}{\lambda} + (1-\lambda)x_2\right) \\ &\leq \lambda f\left(\frac{x_1 - (1-\lambda)x_2}{\lambda}\right) + (1-\lambda)f(x_2) && \text{(by the convexity of } f\text{)} \end{aligned}$$

Next, observe that by this inequality, we have that,

$$\frac{f(x_1) - (1-\lambda)f(x_2)}{\lambda} \leq f\left(\frac{x_1 - (1-\lambda)x_2}{\lambda}\right) \leq M \tag{18}$$

by the boundedness of f from above. Now observe that,

$$\frac{f(x_1) - f(x_2)}{\lambda} \leq \frac{f(x_1) - (1-\lambda)f(x_2)}{\lambda} \tag{19}$$

Therefore, we have that,

$$\frac{f(x_1) - f(x_2)}{\lambda} \leq M \tag{20}$$

However, as let λ go to 0, the term on the LHS grows without bound, which contradicts the derived inequality. Therefore, we have reached a contradiction, and it must be that f is constant. □

Exercise 7.20

Claim 20. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $-f$ is also convex, then f is affine.

Proof. First note that if f is affine, then there exist a linear transformation L and $c \in \mathbb{R}$ such that $f(x) = L(x) + c$. An implication of f being affine is that for $\lambda \in [0,1]$ and $x_1, x_2 \in \mathbb{R}^n$, we have that

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= L(\lambda x_1 + (1-\lambda)x_2) + c \\ &= \lambda L x_1 + (1-\lambda) L x_2 + c \quad \text{(since } L \text{ is a linear transformation)} \\ &= \lambda(f(x_1) - c) + (1-\lambda)(f(x_2) - c) + c \qquad \text{(since } f \text{ is affine)} \\ &= \lambda f(x_1) + (1-\lambda) f(x_2) \end{aligned}$$

Now, for the sake of contradiction, suppose f and $-f$ are convex but f is not affine. Then, since f is convex, for all $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \tag{21}$$

Similarly, since $-f$ is convex, for all $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$-f(\lambda x_1 + (1-\lambda)x_2) \leq -\lambda f(x_1) - (1-\lambda) f(x_2) \tag{22}$$

which implies $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$. Therefore, putting (21) and (22) together, we have that $f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda) f(x_2)$. However, this equation contradicts the implication of being affine shown above. Therefore, f must be affine. □

Exercise 7.21

Proof. First suppose that x^* is a local minimizer of $f(x)$. Therefore, there exists a neighborhood Ω such that $f(x^*) \leq f(x)$ for all $x \in \Omega$. Now, since ϕ is strictly increasing, applying ϕ will preserve inequalities, so we have that $\phi(f(x^*)) \leq \phi(f(x))$ for all $x \in \Omega$. Therefore, x^* is a local minimizer of $\phi(f(x))$. Next, suppose x^* is a local minimizer of $\phi(f(x))$. Therefore, there exists a neighborhood Ω' such that $\phi(f(x^*)) \leq \phi(f(x))$ for all $x \in \Omega'$. Now, since ϕ is strictly increasing, it is one-to-one, and hence has an inverse. Similarly, ϕ^{-1} is also strictly increasing. Therefore, we may apply ϕ^{-1} to both sides of inequalities and have them preserved. If follows that $f(x^*) \leq f(x)$ for all $x \in \Omega'$, so that x^* is a local minimizer of f. □