Problem Set #1, Measure Theory

OSM Lab: Math
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Exercise 1.3

1. $\mathcal{G}_1$ is not an algebra (and hence not a σ-algebra). $\mathcal{G}_1$ is not closed under complements. Fix $a \in \mathbb{R}$ and let $A = (-\infty, a)$ be an open set in $\mathbb{R}$. Then $A^c = [a, \infty)$ is not an open set in $\mathbb{R}$. Therefore, we have that $A \in \mathcal{G}_1$ but $A^c \notin \mathcal{G}_1$, which violates the definition of an algebra.

2. $\mathcal{G}_1$ is an algebra but not a σ-algebra. We first show that $\mathcal{G}_1$ is an algebra. Let $a = b \in \mathbb{R}$. Then $(a, b] = \emptyset \in \mathcal{G}_2$. $\mathcal{G}_2$ is also closed under complements. To see this, fix $a, b \in \mathbb{R}$. Then $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{G}_2$. Furthermore, $(-\infty, b]^c = (b, \infty) \in \mathcal{G}_2$, and $(a, \infty)^c = (-\infty, a] \in \mathcal{G}_2$. Finally, observe that $\mathcal{G}_2$ is closed under finite unions. A finite union of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) still results in a finite union of intervals of the same form. Therefore, $\mathcal{G}_2$ is an algebra. However, $\mathcal{G}_2$ is not a σ-algebra. An infinite (countable) union of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) does not belong to $\mathcal{G}_2$, by definition. Thus $\mathcal{G}_2$ is not a σ-algebra.

3. $\mathcal{G}_3$ is an algebra and a σ-algebra. The proof that $\emptyset \in \mathcal{G}_3$ and $\mathcal{G}_3$ is closed under complements is the same as the proof for $\mathcal{G}_2$. However, $\mathcal{G}_3$ contains countable unions of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) by definition; therefore $\mathcal{G}_3$ is closed under countable unions and is a σ-algebra.

Exercise 1.7

Let X be a nonempty set and $\mathcal{A}$ be a σ-algebra of X. The definition of a σ-algebra requires $\emptyset \in \mathcal{A}$. Furthermore, a σ-algebra is closed under complements, which implies that $\emptyset^c = X \in \mathcal{A}$. Therefore, to satisfy the definition, any σ-algebra must contain both $\emptyset$ and X, so that the set $\{\emptyset, X\}$ is the smallest σ-algebra. The power set of X, $\mathcal{P}(X)$, is the largest possible σ-algebra because it contains all possible subset of X. Recall that a σ-algebra is a family of subsets of X. Therefore, the largest possible family of subsets of X contains all subsets of X, which is precisely $\mathcal{P}(X)$.

Exercise 1.10

Let $\{S_\alpha\}$ be a family of σ-algebras on X. We show that $\cap_\alpha S_\alpha$ is also a σ-algebra.

Claim 1. $\emptyset \in \cap_\alpha S_\alpha$

Observe that $\emptyset \in S_\alpha$ for all α because $\{\mathcal{S}_\alpha\}$ is a family of σ-algebras. Therefore, $\emptyset \in \cap_\alpha S_\alpha$.

Claim 2. $\cap_\alpha S_\alpha$ is closed under complements.

Let $A \in \cap_\alpha S_\alpha$. Therefore, $A \in S_\alpha$ for all α. Each S_α is a σ-algebra, which implies that $A^c \in S_\alpha$ for all α. Therefore, $A^c \in \cap_\alpha S_\alpha$. Thus, $A \in \cap_\alpha S_\alpha$ implies $A^c \in \cap_\alpha S_\alpha$, so that $\cap_\alpha S_\alpha$ is closed under complements.

Claim 3. $\cap_\alpha S_\alpha$ is closed under countable unions.

Let $A_1, A_2, \ldots \in \cap_\alpha S_\alpha$. Therefore, $A_1, A_2, \ldots \in S_\alpha$ for all α. Each S_α is a σ-algebra, which implies that $\cup_{n=1}^\infty A_n \in S_\alpha$ for all α. Thus, $\cup_{n=1}^\infty A_n \in \cap_\alpha S_\alpha$, which shows that $\cap_\alpha S_\alpha$ is closed under countable unions.

Therefore, we have showed that $\cap_\alpha S_\alpha$ is indeed a σ-algebra.

Exercise 1.17

Let $(X, \mathcal{S}, \mu)$ be a measure space.

Claim 4. μ is montone: if $A, B \in \mathcal{S}$, $A \subset B$, then $\mu(A) \leq \mu(B)$.

Proof. Observe that we can write $B = (B \cap A^c) \cup (B \cap A) = (B \cap A^c) \cup A$, where the last inequality follows because $A \subset B$. By the definition of a measure, we have that $\mu(B) = \mu(B \cap A^c) + \mu(A)$, as $(B \cap A^c)$ and $(B \cap A) = A$ are clearly disjoint. Finally, note that by definition, a measure is nonnegative, so $\mu(B \cap A^c) \geq 0$. Therefore, $\mu(B) \geq \mu(A)$. □

Claim 5. μ is countably subadditive: if $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$, then $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$.

Proof. Let $A = \cup_{i=1}^\infty A_i$ and define $\{B_i\}_{i=1}^\infty$ as follows. Set $B_1 = A_1$, $B_2 = A_2 \cap A_1^c$, $B_3 = A_3 \cap (A_1 \cup A_2)^c$, and more generally, $B_i = A_i \cap (\cup_{n=1}^{i-1} A_i)^c$. Observe that $A = \cup_{i=1}^\infty B_i$. Also observe $B_i \subset A_i$, as we form B_i by intersecting A_i with other sets. Additionally, $B_i \cap B_j = \emptyset$ for all $i \neq j$, by construction. By monotonicity, proved above, we have that $\mu(B_i) \leq \mu(A_i)$. Therefore,

$$\mu(\cup_{i=1}^\infty A_i) = \mu(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \mu(B_i) \leq \sum_{i=1}^\infty \mu(A_i) \tag{1}$$

The second inequality follows by the disjointness of B_i and B_j for all $i \neq j$, and the fourth inequality follows by monotonicity. □

Exercise 1.18

Let $(X, \mathcal{S}, \mu)$ be a measure space and $B \in \mathcal{S}$. Define $\lambda : \mathcal{S} \to [0, \infty]$ by $\lambda(A) = \mu(A \cap B)$.

Claim 6. $\lambda(\emptyset) = 0$

By the definition of λ, $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$, because μ is a measure.

Claim 7. $\lambda(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \lambda(A_i)$ for any $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Fix $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$. By the definition of λ,

$$\begin{aligned}
\lambda(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap B) \\
&= \mu(\cup_{i=1}^{\infty}(A_i \cap B)) \\
&= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\
&= \sum_{i=1}^{\infty} \lambda(A_i \cap B)
\end{aligned}$$

The third equality follows because $A_i \cap A_j = \emptyset$ for all $i \neq j$ implies that $(A_i \cap B) \cap (A_j \cap B) = \emptyset$ for all $i \neq j$. Thus, as μ is a measure, we may break of the disjoint into a sum of the individual measures. Finally, the last inequality follows by the definition of λ.

Exercise 1.20

We use the following lemma in our proof:

Lemma 8. If $A_n \subset A_1$, then $\mu(A_1) - \mu(A_n) = \mu(A_1 \setminus A_n)$

Proof. We can write $A_1 = (A_1 \setminus A_n) \cup A_n$, which is a disjoint union, because $A_n \subset A_1$. Therefore, $\mu(A_1) = \mu(A_1 \setminus A_n) + \mu(A_n)$. The lemma follows. $\square$

Define $B_n = A_1 \setminus A_n$ for $n \in \mathbb{N}$ and let $B = \cup_{n=1}^{\infty} B_n$. Observe that $\{B_n\}_{n=1}^{\infty}$ forms an increasing sequence because $A_n \supset A_{n+1}$ for all n is a decreasing sequence of sets. Let $A = \cap_{n=1}^{\infty} A_n$. Then,

$$\begin{aligned}
\mu(A_1) - \mu(A) &= \mu(A_1 - A) && \text{(by the above lemma)} \\
&= \mu(B) && \text{(by the definition of } B\text{)} \\
&= \lim_{n\to\infty} \mu(B_n) && \text{(by part (i) of the theorem)} \\
&= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) && \text{(by the definition of } B_n\text{)}
\end{aligned}$$

Therefore, because $\mu(A_1) < \infty$, we may subtract $\mu(A_1)$ from both sides of the above sequence of equations to find that,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) \tag{2}$$

Exercise 2.10

Recall that μ^* is countably subadditive. Write $B \subset X$ as $B = (B \cap E) \cup (B \cap E^c)$. By countable subadditivity, we have that $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Therefore, if $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$, then it follows that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Therefore, satisfying the equality condition implies the condition (*) in Theorem 2.8.

Exercise 2.14

Recall that by Definition 1.11, $\sigma(\mathcal{O}) :=$ the smallest σ-algebra containing all open sets of X. We call $\sigma(\mathcal{O})$ the Borel σ-algebra of X written as $\mathcal{B}(X)$. Next, by the Carathéodory Construction, $\mathcal{M}$ is a σ-algebra. Additionally, $\mathcal{M}$ is the collection of all Lebesgue measurable sets. The collection of all Lebesgue measurable sets contains open sets. Therefore, because $\mathcal{B}(X)$ is defined as the intersection of all σ-algebra containing open sets, we must have that $\mathcal{B}(X) \subset \mathcal{M}$.

Exercise 3.1

Claim 9. Every countable subset of the real line has Lebesgue measure 0.

Proof. Let $A = \{a_1, a_2, \ldots\}$ be a countable subset of the real line. Fix $\epsilon > 0$. We construct a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ as follows. Let $I_1 = (a_1 - \frac{\epsilon}{2}, a_1 + \frac{\epsilon}{2})$, which has length ϵ. Similarly, let $I_2 = (a_2 - \frac{\epsilon}{4}, a_2 + \frac{\epsilon}{4})$, which has length $\frac{\epsilon}{2}$. For a general interval n, we write $I_n = (a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n})$, which has length $\frac{\epsilon}{2^{n-1}}$. Now, the sum of the lengths of the intervals is,

$$\sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon \tag{3}$$

By definition, the Lebesgue (outer) measure of A is,

$$\mu(A) = \inf\{\sum_{n=1}^{\infty}(d_n - c_n) : A \subset \bigcup_{n=1}^{\infty}(c_n, d_n]\} \tag{4}$$

Above, we demonstrated an open cover of A such that the sum of the intervals is arbitrarily small. Therefore, the measure of this open cover is arbitrarily small (yet weakly positive) and hence 0. Thus, $\mu(A) = 0$. □

Exercise 3.4

We show that the following conditions are equivalent:

1. $\{x \in X : f(x) < a\} \in \mathcal{M}$
2. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$
3. $\{x \in X : f(x) > a\} \in \mathcal{M}$
4. $\{x \in X : f(x) \leq a\} \in \mathcal{M}$

Proof. (1) $\implies$ (2): Suppose $\{x \in X : f(x) < a\} \in \mathcal{M}$. Observe that $f^{-1}([a, \infty)) = (f^{-1}(-\infty, a))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}([a, \infty)) \in \mathcal{M}$.

(2) $\implies$ (3): Suppose $\{x \in X : f(x) \geq a\} \in \mathcal{M}$. Observe that $f^{-1}((a, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}([a - \frac{1}{n}, \infty))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}(a, \infty) \in \mathcal{M}$.

(3) $\implies$ (4): Suppose $\{x \in X : f(x) > a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a]) = (f^{-1}(a, \infty))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}((-\infty, a]) \in \mathcal{M}$.

(4) $\implies$ (1): Suppose $\{x \in X : f(x) \leq a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a)) = \bigcap_{n=1}^{\infty} f^{-1}((-\infty, a + \frac{1}{n}))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}((a, \infty)) \in \mathcal{M}$. □

Exercise 3.7

Suppose f and g are measurable functions on $(X, \mathcal{M})$. Then the following are measurable:

1. $f + g$
2. $f \cdot g$
3. $\max(f, g)$
4. $\min(f, g)$
5. $|f|$

I prove (3), (4), and (5) directly from the definition of measurable functions and use results from Exercise 3.4 to rewrite the condition for measurability in equivalent forms to make the proofs easier.

1. Consider $F(f(x) + g(x)) = f(x) + g(x)$. Then F is continuous and by part 4 of Theorem 3.6, measurable. Therefore, $f + g$ is measurable.

2. Consdier $F(f(x) + g(x)) = f(x)g(x)$. Then F is continuous and by part 4 of Theorem 3.6, measurable. Therefore, $f \cdot g$ is measurable.

3. Because f and g are measurable functions on $(X, \mathcal{M})$, we have that for all $a \in \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$. Therefore, it follows that $\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$, so that $\max(f(x), g(x))$ is measurable.

4. The proof that $\min(f, g)$ is measurable is analogous to the proof of (3). The key observation here is that $\{x \in X : \min(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \min(f(x), g(x)) > a\} \in \mathcal{M}$, so that $\min(f(x), g(x))$ is measurable.

5. Observe that $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$. Both of these sets are in $\mathcal{M}$. $\mathcal{M}$ is closed under countable unions, therefore, $\{x \in X : |f(x)| > a\} \in \mathcal{M}$, so that $|f(x)|$ is measurable.

Exercise 3.14

Proof. Let f be bounded, and fix $\epsilon > 0$. Then, there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$. Therefore, $x \in E_i^M$ for some i and all $x \in X$. Observe that there is an $N \in \mathbb{R}$ and $N \geq M$ such that $\frac{1}{2^N} < \epsilon$. Therefore, for all $x \in X$ and $n \geq N$, $||s_n(x) - f(x)|| < \epsilon$. Therefore, the convergence in part (1) of Theorem 3.13 is uniform. □

Exercise 4.13

To show that $f \in \mathscr{L}^1(\mu, E)$, we must show that both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite.

Recall that $||f|| = f^+ + f^-$. Also note that $0 \leq f^+$ and $0 \leq f^-$ by definition. Because $||f|| < M$ on E, then $0 \leq f^+ < M$ and $0 \leq f^- < M$ on E.

Then, by Proposition 4.5, because $\mu(E) < \infty$, we have that,

$$\int_E f^+ d\mu < M\mu(E) < \infty$$
$$\int_E f^- d\mu < M\mu(E) < \infty$$

Therefore, both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite. Then by definition, $f \in \mathscr{L}^1(\mu, E)$.

Exercise 4.14

We prove the contrapositive of this statement. To that end, suppose there exists a measurable set $\hat{E} \subset E$ such that f is infinite on $\hat{E}$. Here, we assume that f reaches positive infinity (without loss of generality, the proof for negative infinity or mixed between positive and negative infinity is analogous). It follows that,

$$\infty = \int_{\hat{E}} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu \tag{5}$$

The first inequality is proved in 4.16, below. However, this implies that $f \notin \mathscr{L}^1(\mu, E)$.

Exercise 4.15

Let $f, g \in \mathscr{L}^1(\mu, E)$. Define the set of simple functions $B(f) = \{s : 0 \leq s \leq f, s \text{ simple, measurable}\}$. Let $f \leq g$. If follows that $f^+ \leq g^+$ and $f^- \geq g^-$. Then following a similar proof to Proposition 4.7, we have that $B(f^+) \subset B(g^+)$ and $B(g^-) \subset B(f^-)$. These two relationships imply that $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E f^- d\mu \geq \int_E g^- d\mu$. Then by the definition of the Lebesgue integral, we observe that,

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu \tag{6}$$

Therefore, we have that,

$$\int_E f d\mu \leq \int_E g d\mu \tag{7}$$

Exercise 4.16

Following Definition 4.1, fix a simple function $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$, where $E_i \in \mathcal{M}$. Let $A \subset E \in \mathcal{M}$. Then, by the monotonicity of measures, we have that $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for all i. Therefore, combining this result with Definition 4.1, we have that,

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu \tag{8}$$

Now, by Definition 4.2, we have that,

$$\int_A f d\mu = \sup\{\int_A s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\}$$

and

$$\int_E f d\mu = \sup\{\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\}$$

Now because our choice of s was arbitrary, we have by Equation (8) that,

$$\int_A f d\mu \leq \int_E f d\mu \tag{9}$$

Because $f \in \mathscr{L}^1(\mu, E)$, by definition we have that $\int_E ||f|| d\mu < \infty$. Therefore, $\int_E f d\mu < \infty$. Finally, it follows that $\int_A f d\mu < \infty$, which in turn implies $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$, so that $f \in \mathscr{L}^1(\mu, A)$.

Exercise 4.21

Let $A, B \in \mathcal{M}$, $B \subset A$, $\mu(A - B) = 0$, and $f \in \mathscr{L}^1$. Then, by Proposition 4.6. we have that,

$$\int_{A-B} f d\mu = 0. \tag{10}$$

Recall that f^+ and f^- are non-negative $\mathcal{M}$-measurable functions because $f \in \mathscr{L}^1$. By Theorem 4.19, we have that $\mu_1(A) = \int_A f^+ d\mu$ and $\mu_2(A) = \int_A f^- d\mu$ are measures on $\mathcal{M}$. Therefore, by the definition of the Lesbesgue integral,

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \mu_1(A) - \mu_2(A) \tag{11}$$

Now, consider the disjoint union $A = (A - B) \cup B$. Because both $\mu_1(A)$ and $\mu_2(A)$ are measures, we have that $\mu_i(A) = \mu_i(A - B) + \mu_i(B)$ for $i = 1, 2$, because measures are additively separable on disjoint sets. Therefore, we have that $\mu_i(A) = \mu_i(B)$ for $i = 1, 2$ because $\mu(A - B) = 0$. Therefore,

$$\int_A f d\mu = \mu_1(B) - \mu_2(B) = \int_B f d\mu \tag{12}$$

This result clearly implies that

$$\int_A f d\mu \leq \int_B f d\mu \tag{13}$$