OSM Bootcamp

Lecture 4

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2018

QuantEcon

Computational Aspects of Simulation

See

- Wk2_Dynamics/efficient_inventory_dynamics.ipynb

Stationary Distributions and Stationarity

Some marginal distributions have the special property of being fixed under updating

These are called **stationary**

More precisely, ψ^* is **stationary** for our model if

$$(X, \xi) \stackrel{\mathscr{D}}{=} \psi^* \times \phi \quad \implies \quad F(X, \xi) \stackrel{\mathscr{D}}{=} \psi^*$$

Example. Recall again the AR(1) model

$$X_{t+1} = \rho X_t + b + \sigma \xi_{t+1}, \qquad \{\xi_t\} \stackrel{\text{IID}}{\sim} N(0,1)$$

If $\psi_t = N(\mu_t, s_t^2)$, then

$$\psi_{t+1} = N(\mu_{t+1}, s_{t+1}^2)$$

where

$$\mu_{t+1} = \rho \mu_t + b \quad \text{and} \quad s_{t+1}^2 = \rho^2 s_t^2 + \sigma^2$$

(Why?)

Suppose now that $-1 < \rho < 1$ and

$$\mu_t = \frac{b}{1-\rho} \quad \text{and} \quad s_t = \frac{\sigma}{\sqrt{1-\rho^2}}$$

Then

$$\mu_{t+1} = \rho\mu_t + b = \rho\frac{b}{1-\rho} + b = \frac{b}{1-\rho} = \mu_t$$

Similarly, $s_{t+1} = s_t$ (check it)

Hence, $\psi_{t+1} = \psi_t$ and ψ_t is a stationary distribution

Some models have no stationary distribution

Example. Consider the AR(1) model

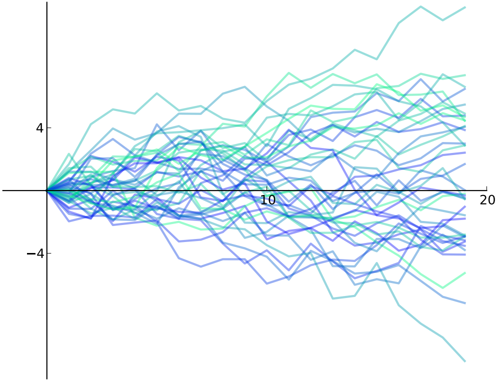
$$X_{t+1} = \rho X_t + b + \sigma \xi_{t+1}, \qquad \text{where } \{\xi_t\} \overset{\text{IID}}{\sim} N(0,1)$$

Suppose now that $\rho \geqslant 1$.

Then

$$\operatorname{var} X_{t+1} = \rho^2 \operatorname{var} X_t + \sigma^2 > \operatorname{var} X_t$$

Since the variance is always changing, the marginal distributions must be changing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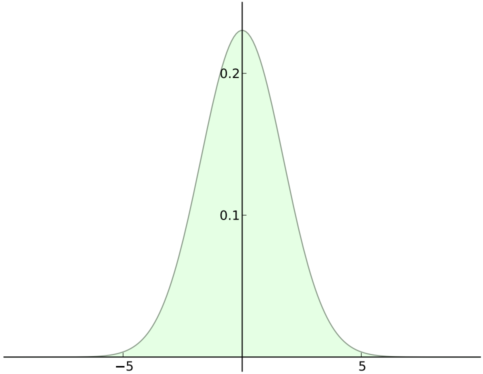


Figure: ψ_1

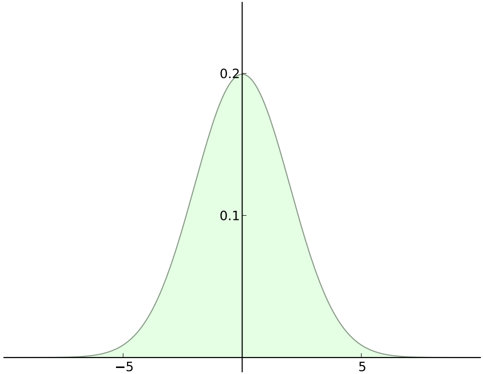


Figure: ψ_2

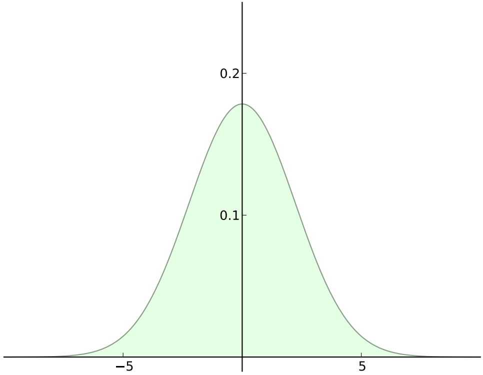


Figure: ψ_3

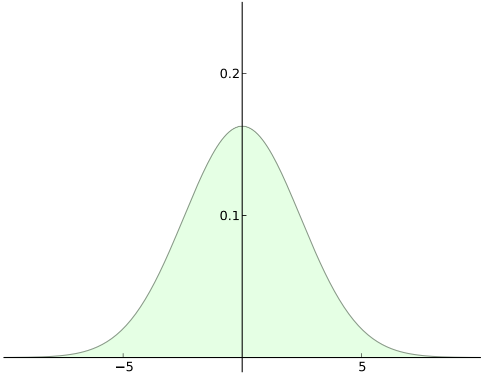


Figure: ψ_4

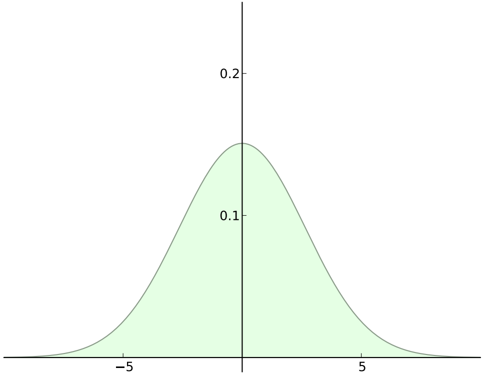


Figure: ψ_5

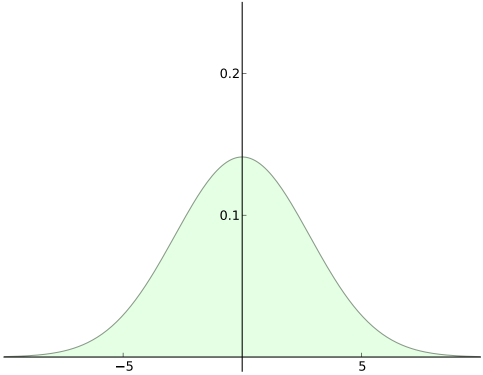


Figure: ψ_6

Asymptotic Stationarity

Some models have the property that

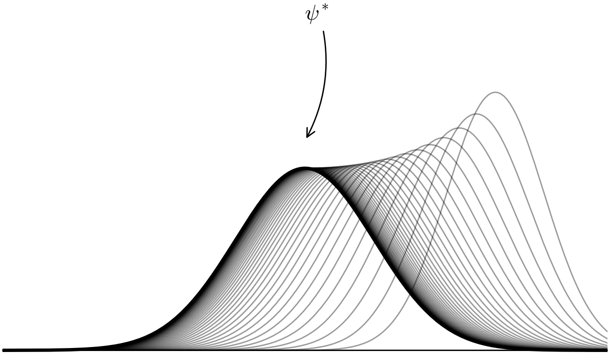
1. they have a unique stationary distribution ψ^*
2. $\psi_t \to \psi^*$ as $t \to \infty$ regardless of the initial condition

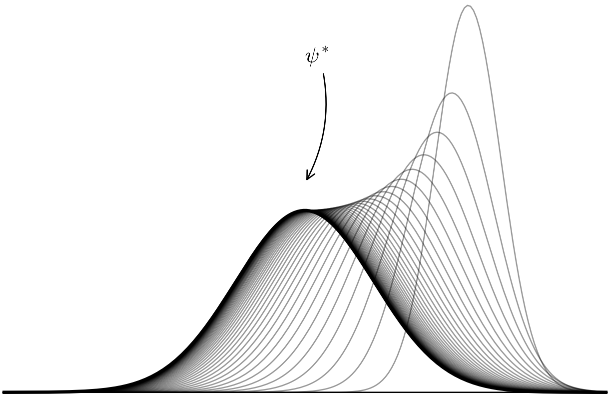
Such models are called **globally stable**

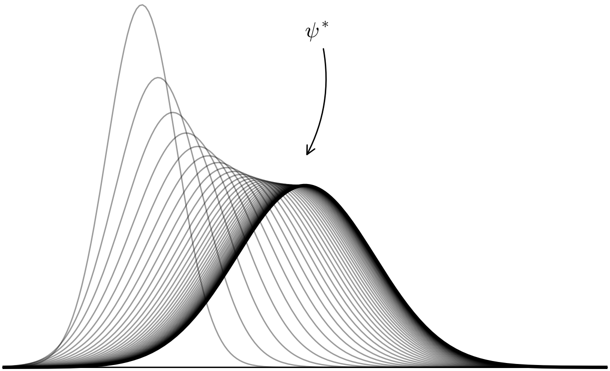
Example. For the linear AR(1) model

$$X_{t+1} = \rho X_t + b + \sigma \xi_{t+1}, \qquad \text{where } \{\xi_t\} \stackrel{\text{IID}}{\sim} N(0,1)$$

asymptotic stability holds if and only if $-1 < \rho < 1$


ψ^*


ψ^*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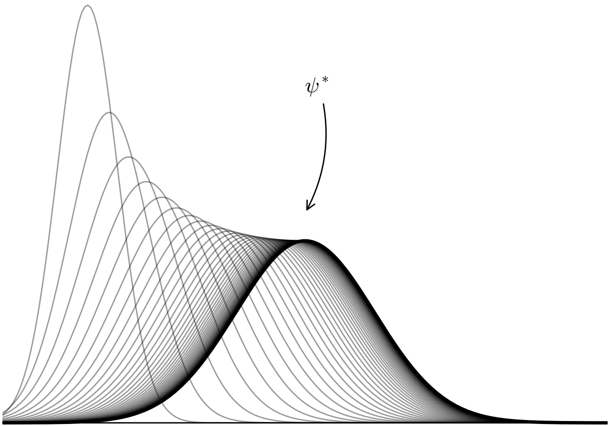

ψ^*

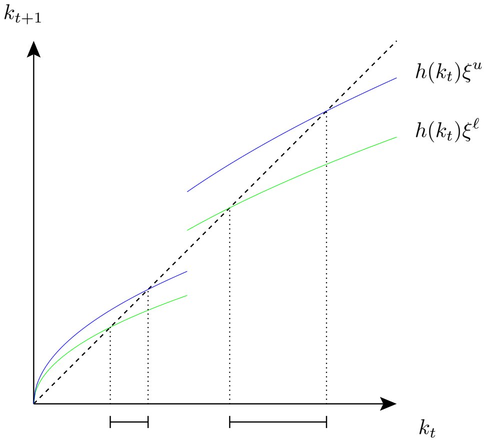
Global stability can fail because of insufficient **mixing**

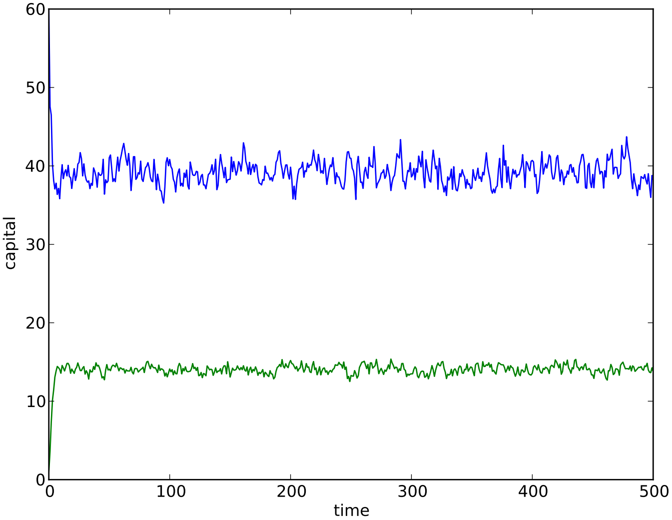
Example. The Azariadis–Drazen version of the Solow–Swan growth model

$$k_{t+1} = h(k_t)\xi_{t+1}$$

where

- $\xi_t \in [\xi^\ell, \xi^u]$
- h has a jump


k_{t+1}
$h(k_t)\xi^u$
$h(k_t)\xi^\ell$
k_t



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capital
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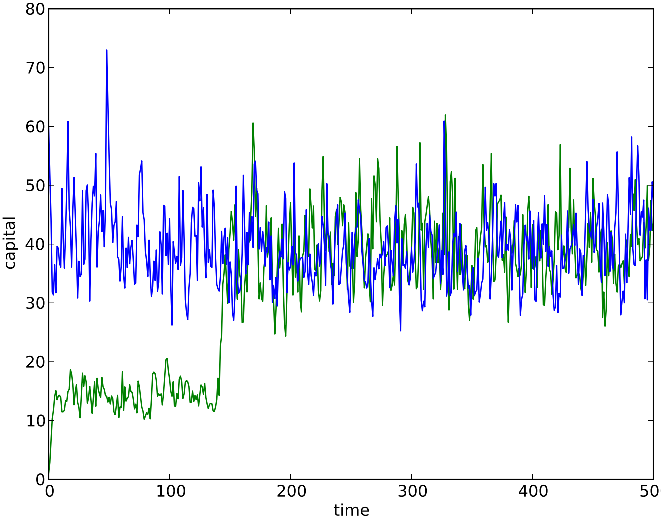
time

Here there is path dependence rather than global stability

To regain stability, we need more mixing



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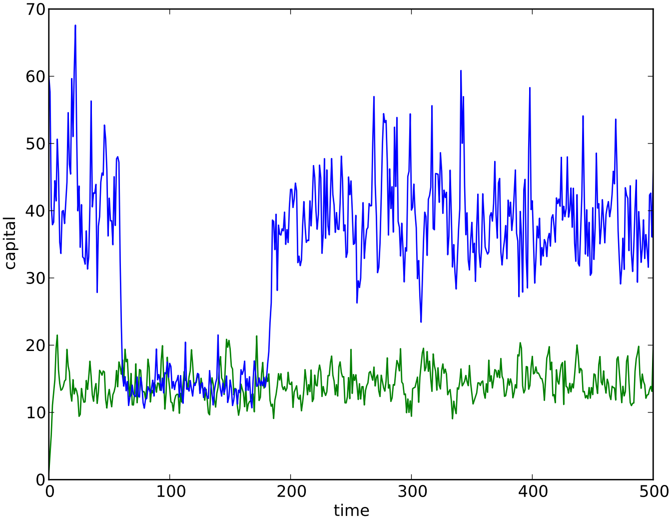

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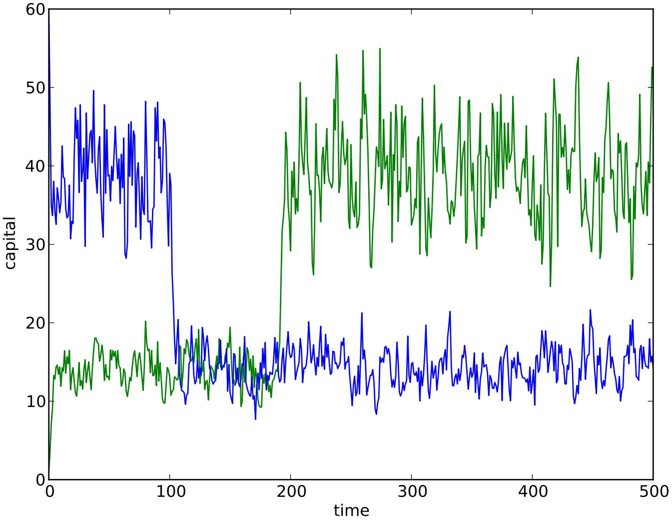

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Ergodicity

Globally stable Markov models have a special property: **Ergodicity**

Consider again the model

$$X_{t+1} = F(X_t, \xi_{t+1}), \qquad \text{where } \{\xi_t\} \overset{\text{IID}}{\sim} \phi$$

Suppose globally stable with stationary distribution ψ^*

Then, for any "nice" function $h \colon \mathbb{X} \to \mathbb{R}$ and any initial condition x_0,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} h(X_t) = \int h(x) \psi^*(x) \, \mathrm{d}x$$

with probability one

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How can we use

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} h(X_t) = \int h(x) \psi^*(x) \, \mathrm{d}x \, ?$$

Example. With $h(x) = x$ we get

$$\frac{1}{n} \sum_{t=1}^{n} X_t \to \int x \, \psi^*(x) \, \mathrm{d}x = \text{mean of stationary dist}$$

Example. With $B \subset \mathbb{X}$ and $h(x) = \mathbb{1}\{x \in B\}$ we get

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{1}\{X_t \in B\} \to \int \mathbb{1}\{x \in B\} \, \psi^*(x) \, \mathrm{d}x = \psi^*(B)$$

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Example. Consider the consumption model of **Schorfheide, Song and Yaron, Econometrica, 2018**

$$g_t := \ln(C_{t+1}/C_t) = \mu_c + z_t + \sigma_{c,t}\,\eta_{c,t+1},$$

where

$$z_{t+1} = \rho\, z_t + (1-\rho^2)^{1/2}\,\sigma_{z,t}\, v_{t+1},$$

$$\sigma_{i,t} = \varphi_i\,\bar\sigma \exp(h_{i,t}),$$

$$h_{i,t+1} = \rho_{h_i} h_i + \sigma_{h_i}\xi_{i,t+1}, \quad i \in \{z, c\}$$

- shocks are IID standard normal

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This model is complicated — how can we understand it?

Fact. If ρ, ρ_{h_c} and ρ_{h_z} are all in $(0,1)$, then this model is globally stable

Therefore it has a unique stationary distribution and is ergodic

We can learn about the stationary distribution by simulation

Example. For the mean of stationary consumption, simulate and compute

$$\frac{1}{n} \sum_{t=1}^{n} g_t$$

- see `Wk2_Dynamics/sim_ssy_consumption.ipynb`

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Extra reading

Review **linear state space models** by reading

- https://lectures.quantecon.org/py/linear_models.html

Read the discussions of

- stationarity distributions
- ergodicity – probabilities for cross sectional are equal to time series averages sample paths