Problem Set #4, DSGE

OSM Lab: Econ
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DSGE Models

Exercise 1

We guess that the policy function in the Brock and Mirman model takes the following form: $K_{t+1} = Ae^{z_t}K_t^\alpha$. The Euler equation is given by,

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t\left[\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right] \tag{1}$$

To verify our guess, we substitute $K_{t+1} = Ae^{z_t}K_t^\alpha$ into each side of the Euler equation and see if we are able to find a value of A that will cause the LHS to equal the RHS. Starting with the LHS, observe that,

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} = \frac{1}{e^{z_t}K_t^\alpha(1-A)} \tag{2}$$

To simplify the RHS, observe that $E[z_{t+1}] = \rho z_t$. Then,

$$\begin{aligned}\beta E_t\left[\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right] &= \beta E_t\left[\frac{\alpha e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha - Ae^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha}\right] \\ &= \frac{\alpha\beta}{Ae^{z_t}K_t^\alpha(1-A)}\end{aligned}$$

Since we must have that the LHS equals the RHS, it must be that $\frac{\beta\alpha}{A} = 1$, so that $A = \alpha\beta$. Therefore, the policy function is given by $k_{t+1} = \Phi(k_t, z_t) = \alpha\beta e^{z_t}k_t^\alpha$.

Exercise 2

Consider the following functional forms:

$$\begin{aligned}u(c_t, l_t) &= \ln c_t + a\ln(1-l_t) \\ F(K_t, L_t, z_t) &= e^{z_t}K_t^\alpha L_t^{1-\alpha}\end{aligned}$$

Then, the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{3}$$
$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{4}$$
$$\frac{a}{1-l_t} = \frac{1}{c_t} w_t (1-\tau) \tag{5}$$
$$r_t = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} = \alpha e^{z_t} \left(\frac{l_t}{k_t}\right)^{1-\alpha} \tag{6}$$
$$w_t = (1-\alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} = (1-\alpha) e^{z_t} \left(\frac{k_t}{l_t}\right)^{\alpha} \tag{7}$$
$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{8}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{9}$$

We can't use the same tricks as Exercise 1 since households now optimize over both their leisure and consumption decisions.

Exercise 3

Consider the following functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1-l_t)$$
$$F_(K_t, L_t, z_t) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

Then, the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{10}$$
$$c_t^{-\gamma} = \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{11}$$
$$\frac{a}{1-l_t} = c_t^{-\gamma} w_t (1-\tau) \tag{12}$$
$$r_t = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} = \alpha e^{z_t} \left(\frac{l_t}{k_t}\right)^{1-\alpha} \tag{13}$$
$$w_t = (1-\alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} = (1-\alpha) e^{z_t} \left(\frac{k_t}{l_t}\right)^{\alpha} \tag{14}$$
$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{15}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(), \sigma_z^2) \tag{16}$$

Exercise 4

Consider the following function forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-l_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = e^{z_t}[\alpha K_t^\eta + (1-\alpha)L_t^\eta]^{\frac{1}{\eta}}$$

Then, the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{17}$$
$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{18}$$
$$\frac{a}{(1-l_t)^\xi} = c_t^{-\gamma} w_t (1-\tau) \tag{19}$$
$$r_t = \alpha e^{z_t} k_t^{\eta-1}[\alpha k_t^\eta + (1-\alpha)l_t^\eta]^{\frac{1-\eta}{\eta}} \tag{20}$$
$$w_t = (1-\alpha)e^{z_t} l_t^{\eta-1}[\alpha k_t^\eta + (1-\alpha)l_t^\eta]^{\frac{1-\eta}{\eta}} \tag{21}$$
$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{22}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(), \sigma_z^2) \tag{23}$$

Exercise 5

Consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^\alpha (L_t e_t^{z_t})^{1-\alpha}$$

and we assume $l_t = 1$. Then, by the labor market clearing condition, we know that $L_t = l_t = 1$. The following equations characterize the model:

$$c_t = (1-\tau)[w_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{24}$$
$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{25}$$
$$r_t = \alpha k_t^{\alpha-1}(l_t e_t^{z_t})^{1-\alpha} = \alpha\left(\frac{e^{z_t}}{k_t}\right)^{1-\alpha} \tag{26}$$
$$w_t = (1-\alpha)k_t^\alpha (e_t^{z_t})^{1-\alpha} \tag{27}$$
$$T_t = \tau[w_t + (r_t - \delta)k_t] \tag{28}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d(0, \sigma_z^2) \tag{29}$$

The steady-state versions of these equations are:

$$\bar{c} = (1-\tau)[\bar{w} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{30}$$
$$\bar{T} = \tau[\bar{w} + (\bar{r} - \delta)\bar{k}] \tag{31}$$
$$\tag{32}$$

and

$$\bar{c}^{-\gamma} = \beta E_t\left[\bar{c}^{-\gamma}[(\bar{r}-\delta)(1-\tau)+1]\right] \tag{33}$$

$$\bar{r} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha} = \alpha\left(\frac{e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \tag{34}$$

$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha} \tag{35}$$

$$\bar{z} = (1-\rho_z)\bar{z} + \rho_z\bar{z} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d(0, \sigma_z^2) \tag{36}$$

We can solve these equations analytically to find that,

$$\begin{aligned}
\bar{r} &= \frac{1-\beta}{\beta(1-\tau)} + \delta \\
\bar{k} &= \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\bar{w} &= (1-\alpha)\bar{k}^{\alpha} \\
\bar{c} &= (1-\tau)[\bar{w} + (\bar{r}-\delta)\bar{k}]\bar{T} \\
\bar{T} &= \tau[\bar{w} + (\bar{r}-\delta)\bar{k}]
\end{aligned}$$

See the Jupyter notebook for a numerical comparison of the algebraic and numerical solutions for the steady-state variables.

Exercise 6

Consider the following function forms:

$$\begin{aligned}
u(c_t, l_t) &= \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-l_t)^{1-\xi}-1}{1-\xi} \\
F(K_t, L_t, z_t) &= K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}
\end{aligned}$$

Then, the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$c_t = (1-\tau)[w_t l_t + (r_t-\delta)k_t] + k_t + T_t - k_{t+1} \tag{37}$$

$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1}-\delta)(1-\tau)+1]\right] \tag{38}$$

$$\frac{a}{(1-l_t)^{\xi}} = c_t^{-\gamma} w_t(1-\tau) \tag{39}$$

$$r_t = \alpha\left(\frac{l_t e^{z_t}}{k_t}\right)^{1-\alpha} \tag{40}$$

$$w_t = (1-\alpha)e^{z_t}\left(\frac{k_t}{l_t e^{z_t}}\right)^{\alpha} \tag{41}$$

$$T_t = \tau[w_t l_t + (r_t-\delta)k_t] \tag{42}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{43}$$

The steady state version of these equations are:

$$\bar{c} = (1-\tau)[\bar{w}\bar{l} + (\bar{r}-\delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{44}$$
$$\bar{c}^{-\gamma} = \beta E_t \left[\bar{c}^{-\gamma}[(\bar{r}-\delta)(1-\tau)+1]\right] \tag{45}$$
$$\frac{a}{(1-\bar{l})^{\xi}} = \bar{c}\bar{w}(1-\tau) \tag{46}$$
$$\bar{r} = \alpha\left(\frac{\bar{l}e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \tag{47}$$
$$\bar{w} = (1-\alpha)e^{\bar{z}}\left(\frac{\bar{k}}{\bar{l}e^{\bar{z}}}\right)^{\alpha} \tag{48}$$
$$\bar{T} = \tau[\bar{w}\bar{l} + (\bar{r}-\delta)\bar{k}] \tag{49}$$
$$\bar{z} = (1-\rho_z)\bar{z} + \rho_z\bar{z} + \epsilon^{\bar{z}}; \quad \epsilon_t^z \sim i.i.d.(0,\sigma_z^2) \tag{50}$$

Linearization Methods

Exercise 3

We have that,

$$E_t[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t] = 0 \tag{51}$$

and $\tilde{Z}_t = N\tilde{Z}_{t-1} + \epsilon_t$ and by hypothesis, $\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$. We use these relationships to put Equation 51 in terms of $\tilde{X}_{t-1}$ and $\tilde{Z}_t$. Then, by substitution and that $E[\epsilon_t] = 0$ for all t, we have that,

$$\begin{aligned}
0 &= E_t[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t] \\
&= E_t[F(P(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + Q(N\tilde{Z}_t + \epsilon_{t+1})) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + H\tilde{X}_{t-1} + L(N\tilde{Z}_{t-1} + \epsilon_t) + M\tilde{Z}_t] \\
&= [(FP+G)P + H]\tilde{X}_{t-1} + [(FQ+L)N + (FP+G)Q + M]\tilde{Z}_t
\end{aligned}$$

Perturbation Methods

Exercise 1

We compute the third derivative of $F(x(u),u)$ with respect to u and find that,

$$x_{uuu} = -\frac{F_{xxx}x_u^3 + 3(F_{xxu}x_u^2 + F_{uux}x_u + F_{xu}x_{uu} + F_{xx}x_u x_{uu}) + F_{uuu}}{F_x} \tag{52}$$

where all arguments are evaluated at u_0.