Reduced-Order Observer Design for Servo System Using Duality to Discrete-Time Sliding-Surface Design

A. J. Mehta, *Member, IEEE*, B. Bandyopadhyay, *Senior Member, IEEE*, and A. Inoue

***Abstract*—This paper presents a design method of a discrete-time reduced-order observer using the duality to discrete-time sliding-surface design. First, the duality between the coefficients of the discrete-time reduced-order observer and the sliding-surface design is established, and then, the design method for the observer using the Riccati equation is explained. A discrete-time sliding-mode controller based on the proposed observer is designed and tested on a laboratory-type experimental servo system. The results show the efficacy of the reduced-order observer designed by the duality concept.**

***Index Terms*—Duality between reduced order observer and sliding surface design, reduced order observer, servo system, variable structure systems.**

I. INTRODUCTION

IN ORDER TO apply any state-feedback control law, it is necessary to have the state information. However, the states are seldom available for measurement. Therefore, one has to resort to an output-feedback control technique or has to use an observer-based design. Observers use the plant input and output signals to generate an estimate of the plant's state, which is then employed to close the control loop. The observer was first proposed and developed by Luenberger [1]–[3] in the early 1960s of the last century. Since then, several authors [4], [5] have been proposing the method for designing the observer. Among all the methods, the pole placement technique and the optimal observer design are widely used methods. Both methods require the observer system to be transformed into Gopinath's form [6]. However, when the observer system is transformed into Gopinath's form, the original controllable canonical form is lost. Inoue *et al.* [7], [8] proposed a method for an observer design in the continuous-time domain using the duality concept in which reduced-order observer coefficients are directly obtained from the sliding-surface design.

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In this paper, we establish the duality between the discrete-time sliding-surface design and the discrete-time reduced-order observer, and furthermore, a chatter-free discrete-time sliding-mode control (DSMC) law is designed for a laboratory servo system. The motivation behind this paper is that the SMC strategy is proven to be one of the most robust techniques and guarantee the state to converge to the surface in finite time. The main feature of the SMC is that the system states are forced to reach the sliding surface and slide along it toward the equilibrium state by an appropriate switching action. Once the states are in sliding mode, the closed-loop system dynamics are governed by the sliding surface and are independent of the disturbance and system uncertainties.

In recent years, considerable efforts have been put in the study of DSMC [9]–[13], [17]. To date, various reaching laws [9], [11], [12], [16] have been proposed for the design of DSMC. The sliding-surface design proposed in [14] is used here for establishing the duality with the discrete-time reduced-order observer. In this approach, the states are confined to the surface at each sampling instant.

The servo system is ideal for emulating control of modern industrial equipment such as spindle drives, turntables, conveyors, machine tools, and automated assembly machines [20]. For any servo system, position control is an essential issue. A good position-control servo system must have fast response without overshoot, good accuracy, and robustness against disturbance and parameter variation. The SMC is a right candidate for the application. Recently, in [18], DSMC algorithms based on the reaching-law principle is used for induction-motor position control. In this paper, the reduced-order-observer-based DSMC (where the observer is designed by the duality concept) is implemented on a laboratory experimental servo-system setup.

This paper is organized as follows. The discrete-time reduced-order observer review and problem statement is given in Section II. The duality to the discrete-time sliding-surface design is established in Section III. Sections IV and V deal with the design method of the discrete-time reduced-order observer using the discrete-time Riccati equation and the design of the reduced-order observer-based sliding-mode controller, respectively. The experimental servo-system setup is briefly explained in Section VI. The simulation and experimental results are discussed in Section VII, followed by the conclusions in Section VIII.

II. Discrete-Time Reduced-Order Observer

Consider the discrete-time linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ is full rank, $u \in \mathbb{R}^r$ is the control input, $C \in \mathbb{R}^{m\times n}$ such that CB is nonsingular, and $y \in \mathbb{R}^m$ is the output. We assume that (A, B) is completely controllable and that $r < n$.

Problem Statement

The problem is to design an $(n-m)$th-order observer to estimate state variable $x(k)$ from measurement $y(k)$ and input $u(k)$. The reduced-order state observer is given by

$$\begin{aligned} z(k+1) &= Dz(k) + Ey(k) + Fu(k) \\ \hat{x}(k) &= Pz(k) + Vy(k) \end{aligned} \tag{2}$$

where $z(k)$ is $\mathbb{R}^{(n-m)}$ and $\hat{x}(k)$ is an estimate of $x(k)$.

The estimate $\hat{x}(k)$ converges to $x(k)$ if coefficient matrices D, E, F, P, and V satisfy the conditions given by [2] and [4] as

$$\begin{aligned} &TA - DT = EC, \qquad F = TB \\ &PT + VC = I_n \qquad D \text{ is stable} \end{aligned} \tag{3}$$

where T is an $(n-m)\times n$ matrix.

The observer design is to find the coefficient matrices satisfying the aforementioned conditions (3).

III. Duality to Discrete-Time Sliding-Surface Design

Let the discrete-time sliding surface for the system in (1) be given by

$$H\left(x(k), k\right) = \{x(k) | \sigma(k) = Sx(k) = 0\} \tag{4}$$

where $\sigma(k)$ is of r-dimensional sliding function and S is $\mathbb{R}^{r\times n}$.

The following conditions have to be satisfied while selecting the matrix S of the sliding surface for DSMC.

1)

$$\det(SB) \neq 0. \tag{5}$$

2) On the surface

$$Sx(k) = 0. \tag{6}$$

Without loss of generality and using condition 1), we may change the switching surface σ of (4) to a new switching surface as

$$\sigma' = (SB)^{-1}\sigma \tag{7}$$

and then, the sliding-surface gain matrix becomes

$$S' = (SB)^{-1}S \tag{8}$$

and condition 1) becomes

$$1)' \quad S'B = I_r. \tag{9}$$

From this condition $1)'$, we may say that there exist an $n \times (n-r)$ matrix W and an $(n-r)\times n$ matrix U such that

$$rank[W \quad B] = n, S'W = 0 \text{ and } UB = 0. \tag{10}$$

Hence, condition $1)'$ is rewritten as

$$1)'' \quad \begin{bmatrix} U \\ S' \end{bmatrix} = [W \quad B]^{-1} \tag{11}$$

or, equivalently, from (11), we may write

$$WU + BS' = I_n. \tag{12}$$

This condition is equivalent to

$$\begin{aligned} \begin{bmatrix} U \\ S' \end{bmatrix} [W \quad B] &= \begin{bmatrix} UW & UB \\ S'W & S'B \end{bmatrix} \\ &= \begin{bmatrix} I_{n-r} & 0 \\ 0 & I_r \end{bmatrix}. \end{aligned} \tag{13}$$

Now, if we define state variables $x_1 \in \mathbb{R}^{n-r}$ and $x_2 \in \mathbb{R}^r$ as

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} U \\ S' \end{bmatrix} x(k) \tag{14}$$

then the state on the sliding surface $Sx(k) = 0$ and $S'x(k) = 0$ also. Thus, from (14), we may write $x_2(k) = 0$.

From (14), we may also define the state variables as

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = [W \quad B]^{-1} x(k) \tag{15}$$

or

$$[W \quad B] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = x(k). \tag{16}$$

From (14), the system dynamics on the sliding surface is given by

$$\begin{aligned} x_1(k) &= Ux(k) \\ x_1(k+1) &= Ux(k+1) \\ &= U\left(Ax(k) + Bu(k)\right) \\ &= UA\left(Wx_1(k) + Bx_2(k)\right) + UBu(k) \\ &= UAWx_1(k). \end{aligned} \tag{17}$$

Hence, condition 2) is given by

$$2)' \quad J = UAW \text{ is stable.} \tag{18}$$

Lemma 1: Condition $2)'$ is equivalent to the next condition

$$2)'' \quad AW - WJ = BL, \qquad J \text{ is stable} \tag{19}$$

where $L = S'AW$.

TABLE I
DUALITY TABLE

Observer Coefficients	Sliding Surface Design Coefficients
A	A^T
C	B^T
D	J^T
E	L^T
P	U^T
V	S'^T
T	W^T

Proof: If condition 2)$'$ holds, then substituting (12) and (18) into the left-hand side of (19) gives

$$\begin{aligned} AW - WJ &= AW - WUAW \\ &= AW - (I_n - BS')AW \\ &= BS'AW. \end{aligned} \tag{20}$$

Using L, we get the right-hand side of (19). ■

The design problem of sliding surface is to find matrices W, U, S', and J satisfying conditions 1)$''$ and 2)$''$.

Transposing (12) and (19), we may obtain

$$W^T A^T - J^T W^T = L^T B^T, \qquad J^T \text{ is stable}$$
$$U^T W^T + S'^T B^T = I_n \tag{21}$$

Comparing the conditions of (21) to the conditions of (3), the duality of coefficients are obtained, as given in Table I.

Using duality, a design scheme for a discrete-time reduced-order state observer is derived from the design of the discrete-time sliding surface and vice versa.

IV. DESIGN OF DISCRETE-TIME REDUCED-ORDER STATE OBSERVER

It has been proved by Mita and Chen [15] for a continuous-time system that the poles of the closed-loop system on the sliding surface are equal to the zeros of the transfer function. The same results for the discrete-time system are proved in Lemma 2 hereinafter.

Lemma 2: The poles of the closed-loop system on the sliding surface (the eigenvalues of J) are equal to the zeros of the pulse transfer function

$$S(zI_n - A)^{-1}B. \tag{22}$$

Proof: The proof is given in the Appendix. ■

The zeros of this pulse transfer function are assigned to be stable by matrix S. It can be obtained by solving the Riccati equation, as given in Theorem 1 hereinafter.

Theorem 1: For a positive definite matrix $Q > 0$ and $\rho(\varepsilon) < 1$, if matrix A_ε, the solution of P_s from the Riccati equation, and matrix S are defined, respectively, as

$$A_\varepsilon = A(\varepsilon) \tag{23}$$
$$P_s = Q + A_\varepsilon^T P_s A_\varepsilon + A_\varepsilon^T P_s B(R + B^T P_s B)^{-1} B^T P_s A_\varepsilon \tag{24}$$
$$S = B^T P_s \tag{25}$$

then all the zeros z_i $(i = 1, 2, \ldots, (n-r))$ of the pulse transfer function (22) are inside $\rho(\varepsilon)$ in the z plane, where $\rho(\cdot)$ denotes the spectral radius and $\varepsilon < 1$ is a number.

To formulate the design method using duality, first, we need to prove that the zeros of the pulse transfer function $S(zI_n - A)^{-1}B$ are equivalent to $S'(zI_n - A)^{-1}B$.

Lemma 3: The zeros of the pulse transfer function $S(zI_n - A)^{-1}B$ are equivalent to $S'(zI_n - A)^{-1}B$.

Proof: The zeros of the pulse transfer function $S(zI_n - A)^{-1}B$ are equal to the roots of (23)

$$P(z) = \det \begin{bmatrix} zI_n - A & B \\ S & 0 \end{bmatrix} = 0. \tag{26}$$

Then, we may write

$$\begin{bmatrix} zI_n - A & B \\ S & 0 \end{bmatrix} = \begin{bmatrix} zI_n - A & 0 \\ 0 & I_r \end{bmatrix} \times \begin{bmatrix} I_n & 0 \\ S & I_r \end{bmatrix} \times \begin{bmatrix} I_n & (zI_n - A)^{-1}B \\ 0 & -S(zI_n - A)^{-1}B \end{bmatrix}. \tag{27}$$

Thus, the roots of (27) are obtained by

$$\det \begin{bmatrix} zI_n - A & B \\ S & 0 \end{bmatrix} = \det(zI_n - A) \det\left(-S(zI_n - A)^{-1}B\right). \tag{28}$$

Substituting $S = SBS'$ from (8) in (28) gives

$$\begin{aligned} \det(zI_n - A) \det\left((SB)\left(-S'(zI_n - A)^{-1}B\right)\right) &= 0 \\ \det(zI_n - A) \det(SB) \det\left(-S'(zI_n - A)^{-1}B\right) &= 0 \\ \det(zI_n - A) \det\left(-S'(zI_n - A)^{-1}B\right) &= 0. \end{aligned} \tag{29}$$

Comparing (29) with (28), we may write

$$\det \begin{bmatrix} zI_n - A & B \\ S' & 0 \end{bmatrix} = 0.$$

Hence, the roots of (26) are equal to the roots of

$$\det \begin{bmatrix} zI_n - A & B \\ S' & 0 \end{bmatrix}. \tag{30}$$

■

Lemma 4: The poles of the observer (2) (eigenvalues of D) are equal to zeros of the pulse transfer function

$$C(zI_n - A)^{-1}V. \tag{31}$$

Proof: The proof is straightforward from Lemma 2 and Table I. ■

From Table I, we can obtain matrix V_R corresponding to S of the sliding-surface design and can get the zeros of the transfer function $C(zI_n - A)^{-1}V_R$ from Theorem 1. From (25), we may write

$$S = B^T P_s. \tag{32}$$

Transposing it, we may write

$$S^T = P_s B. \tag{33}$$

From Table I, the corresponding matrix is obtained as

$$V_R = P_s C^T. \tag{34}$$

Similarly, from (8), we may write

$$S' = (SB)^{-1} S \tag{35}$$

Transposing it, we may write

$$S'^T = S^T (B^T S^T)^{-1}. \tag{36}$$

From Table I, the corresponding matrix is obtained as

$$V = V_R (CV_R)^{-1}. \tag{37}$$

Design Procedure for Discrete-Time Reduced-Order Observer

1) Obtain the solution for P_s from (23) and (24) for any positive $\varepsilon < 1$ and obtain matrix V as

$$V = V_R (CV_R)^{-1}, \quad \text{where } V_R = P_s C^T. \tag{38}$$

2) Find an $(n-m) \times n$ matrix T_0 satisfying

$$W_0 = \begin{bmatrix} T_0 \\ C \end{bmatrix} \quad \text{rank}[W_0] = n \tag{39}$$

and obtain the coefficients of observer T, D, E, F, P from Table I as

$$V_1 = T_0 V \quad T = T_0 - V_1 C \tag{40}$$

$$D = TAP \quad E = TAV \tag{41}$$

$$F = TB \quad P = W_0^{-1} \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix}. \tag{42}$$

Theorem 2: The coefficient matrices of the reduced-order observer obtained in previous steps 1) and 2) satisfy the observer conditions in (3).

Proof: As the poles of D are in the unit circle, D is stable. Using (40)

$$\begin{aligned} W_0(PT + VC) &= W_0 \left(W_0^{-1} \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} [T_0 - T_0 VC] + VC \right) \\ &= W_0. \end{aligned} \tag{43}$$

Substituting $D = TAP$ and $PT + VC = I_n$ into $TA - DT$

$$TA - DT = TA(I_n - PT) \tag{44}$$

$$= TAVC \tag{45}$$

$$= EC. \tag{46}$$

■

Remark: Matrix D depends on the selection of matrix T_0, which is not unique, but by Theorem 2, the poles of D are equal to the zeros of $C(zI_n - A)^{-1}V$, and the corresponding pulse transfer function is independent of the selection of T_0, so are the poles of D.

V. Design of DSMC Using the Reduced-Order Observer Designed by Duality

In order to design a DSMC, Bartoszewicz [12] has proposed a nonswitching-type DSMC strategy that guarantees finite-time convergence of the state trajectory to the sliding surface.

Consider a discrete-time system with matched uncertainty

$$x(k+1) = Ax(k) + Bu(k) + Bf(k) \tag{47}$$

$$y(k) = Cx(k) \tag{48}$$

and define the sliding function as

$$\sigma(k) = Sx(k). \tag{49}$$

with S such that $(SB)^{-1} \neq 0$.

The term $f \in \mathbb{R}^r$ in (47) denotes the matched external disturbances and is a bounded function. Let us assume that $\tilde{d}(k) = Bf(k)$. As $f(k)$ is bounded, $\tilde{d}(k)$ is also bounded, and so is $S\tilde{d}(k)$.

Let $d_l \leq d(k) = S\tilde{d}(k) \leq d_u$, with d_l being the lower bound and d_u being the upper bound and assumed to be known, and then, the mean and spread of the disturbance term are given as

$$d_0 = \frac{d_l + d_u}{2}$$

$$\delta_d = \frac{d_u - d_l}{2}.$$

Bartoszewicz's reaching law [12] is proposed as

$$\sigma(k+1) = d(k) - d_0 + \sigma_d(k+1) \tag{50}$$

where $\sigma_d(k)$ is an *a priori* known function such that the following applies.

1) If $\sigma(0) > 2\delta_d$, then

$$\sigma_d(0) = \sigma(0) \tag{51}$$

$$\sigma_d(k) \cdot \sigma_d(0) \geq 0, \text{ for any } k \geq 0 \tag{52}$$

$$\sigma_d(k) = 0, \text{ for any } k \geq k^* \tag{53}$$

$$|\sigma_d(k+1)| < |\sigma_d(k)| - 2\delta_d, \text{ for any } k < k^*. \tag{54}$$

The aforementioned relations state that the time-dependent hyperplane $\sigma_d(k)$ monotonically, and in a finite time, converges from its initial position $\sigma_d(0)$ to the sliding surface $\sigma(k) = 0$. Furthermore, in each control step, the hyperplane moves by a distance greater than $2\delta_d$. This implies that the reaching condition is satisfied, even in the case of the worst combination of disturbance.

2) Otherwise, i.e., if $\sigma(0) < 2\delta_d$, then $\sigma_d(k) = 0$ for any $k \geq 0$.

The constant k^* in the aforesaid relations is a positive integer chosen by the designer in order to achieve good tradeoff between the fast convergence rate of the system and the magnitude of the control required to achieve this convergence rate.

The control law that satisfies the reaching law in (50) can be computed from (47) as

$$u(k) = -(SB)^{-1} \left(SAx(k) + d_0 - \sigma_d(k+1) \right). \tag{55}$$

Fig. 1. Industrial emulator servo system.

The states $x(k)$ in (55) are estimated by the reduced-order observer designed by the duality principle. Thus, the modified control law in terms of the estimated states $\hat{x}(k)$ can be written as

$$u(k) = -(SB)^{-1} \left(SA\hat{x}(k) + d_0 - \sigma_d(k+1)\right). \qquad (56)$$

The control law so designed guarantees that for any $k \geq k^*$, the system states satisfy inequality

$$|\sigma(k)| = |d(k-1) - d_0| \leq \delta_d. \qquad (57)$$

Hence, the states of the system settle within a quasi-sliding-mode band whose width is less than half of the width of the band achieved by the control law proposed in [9].

VI. Experimental Setup

A. Industrial Emulator Servo System

The laboratory experimental servo system called industrial emulator servo system shown in Fig. 1 is ideal for emulating control of modern industrial equipment such as spindle drives, turntables, conveyors, machine tools, and automated assembly machines [20].

The electromechanical plant shown in Fig. 1 consists of the emulator mechanism, its actuator, and sensors. The design features brushless dc servo motors for both drive and disturbance generation, high-resolution encoders, adjustable inertias, and changeable gear ratios. It consists of a drive motor (servo actuator), which is coupled via a timing belt to a drive disk with variable inertia. Another timing belt connects the drive disk to the speed reduction (SR) assembly, while a third belt completes the drive train to the load disk. The load and drive disks have variable inertia that may be adjusted by moving (or removing) the brass weights.

A disturbance motor connects to the load disk via a 4:1 SR and is used to emulate viscous friction and disturbances at the plant output. A brake below the load disk may be used to introduce Coulomb friction. The drive and disturbance motors are electrically driven by servo amplifiers and power supplies in the controller box. Gear ratios can be changed via selection of the sizes of the upper and lower pulleys in the SR assembly. The end-to-end gear ratio, i.e., the gear ratio between the load and drive disks, is given by [20]

$$n_g = (N/n_{pl}) * (n_{pd}/n) \qquad (58)$$

where

- n number of teeth on the drive disk pulley $= 12$;
- N number of teeth on the load disk pulley $= 72$;
- n_{pl} number of teeth on the bottom pulley of the SR assembly;
- n_{pd} number of teeth on the top pulley of the SR assembly.

High-resolution incremental encoders are coupled directly to the drive and load disks to measure the incremental displacement of the motor and the load shaft. Each encoder has a resolution of 4000 pulses/revolution. The encoders are routed through the controller box to interface directly with the digital signal processor (DSP) board via a gate array that converts their pulse signals to numerical values.

The system option provides two analog-output channels in the control box, which are connected to two 16-b digital-to-analog converters (DACs) that physically reside on the real-time controller. Each analog output has a range of ± 10 V ($-32\,768$ to $+32\,767$ counts) with respect to the analog ground. The outputs on these DACs are updated by the real-time controller as a low-priority task.

B. Controller Data Board and Firmware

The next subsystem is the real-time controller unit that contains the DSP-based real-time controller, servo actuator interfaces, servo amplifiers, and auxiliary power supplies. The controller also interprets trajectory commands and supports such functions as data acquisition, trajectory generation, and system health and safety checks. Two optional auxiliary DACs provide for real-time analog-signal measurement. The executive program is the user's interface to the system and supports controller specification, trajectory definition, data acquisition, plotting, system execution commands, and more.

VII. Simulation and Experimental Results

The continuous-time linearized model of the rigid-body plant is given by [20]

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -8.4344 \end{bmatrix} x + \begin{bmatrix} 0 \\ 458.46 \end{bmatrix} (u+f)$$
$$y = [1 \quad 0]\, x \qquad (59)$$

and the states are defined as $x = [\bar{x}_1 \quad \bar{x}_2]^{\mathrm{T}}$. State $\bar{x}_1$ is the position of the load disc, and $\bar{x}_2$ is the speed of the same. The discrete-time model of the system with sampling time $T_s = 0.006$ s is obtained as

$$x(k+1) = Ax(k) + Bu(k) + Bf(k)$$
$$y(k) = Cx(k) \qquad (60)$$

where

$$A = \begin{bmatrix} 1 & 0.0059 \\ 0 & 0.9507 \end{bmatrix} \quad B = \begin{bmatrix} 0.0081 \\ 2.6823 \end{bmatrix} \quad C = [1 \quad 0]$$

It is assumed that the disturbance f in (59) does not change in the sampling interval, so the matched disturbance f in (59) remains matched in (60) also. Furthermore, the term $\tilde{d}(k) = Bf(k)$ in (60) is defined for the control signal analysis.

The displacement of the load disc is the output of the system, which is available for measurement. Input u is the voltage applied to the drive motor. The states are estimated by the reduced-order observer designed by the proposed method.

Reduced-Order Observer Design

1) Obtain sliding gain $S = [0.2845 \quad 0.0094]$ by solving the Riccati equation (24) for $\varepsilon = 0.9$.
2) Obtain the coefficients of the observer as $D = 0.9505$, $E = -9.1503 \times 10^{-4}$, $F = 2.6822$, $P = [0 \quad 1]^{\mathrm{T}}$, and $V = [1 \quad 0.0185]$ from step 2) of the design procedure.

The reduced-order observer (2) is written as

$$\begin{aligned} z(k+1) &= 0.9505z(k) - 9.1503y(k) + 2.6822u(k) \\ \hat{x}(k) &= [0 \quad 1]^{\mathrm{T}} z(k) + [1 \quad 0.0185]^{\mathrm{T}} y(k). \end{aligned}$$

Control Law Design

The DSMC law applied to the system is obtained as

$$u(k) = -36.2024\left([0.2845 \quad 0.0106]\hat{x}(k) + d_0 - \sigma_d(k+1)\right). \tag{61}$$

To verify the robustness and stability properties of the algorithm, first, a deflecting torque is applied at 24 s approximately as a disturbance. The simulation and experimental results in this case are shown in Figs. 2 and 3, respectively, for a reference trajectory. From the results, it is observed that the controller was able to move the states toward the surface in spite of the application of disturbance, and the output tracks once again the reference trajectory. Second, a disturbance $f(k) = 0.1\sin(0.7536k)$ is applied to the emulator system through the input channel, and in this case, the simulation and experimental results are shown in Figs. 4 and 5, respectively. The results show that the position state is tracking the reference trajectory without any overshoot and steady-state error. Moreover, the state remains within a band of the sliding surface that is $|\sigma(k)| \leq \delta_d$. This is also consistent with the theoretically calculated value from (57). This confirms the robustness and stability properties of the closed-loop system under the bounded matched disturbance.

VIII. Conclusion

In this paper, we have proposed the duality between a discrete-time reduced-order observer and the sliding-surface design. The design of the discrete-time observer based on the solution of the Riccati equation and the nonswitching-type SMC have been obtained from the same sliding-surface parameters. The reduced-order observer-based nonswitching-type sliding-mode controller has been designed for the industrial servo system to demonstrate the applicability and effectiveness of the proposed approach. The SMC law guarantees a stable system performance.

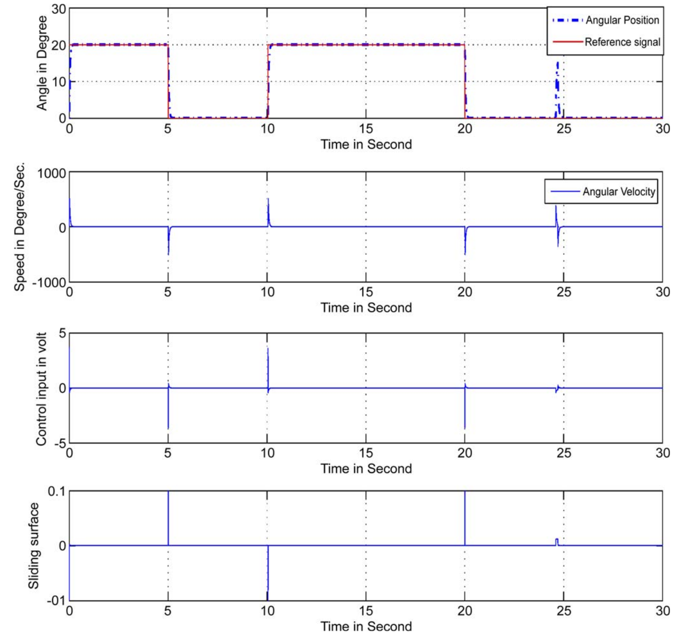

Fig. 2. Simulation results with the proposed reduced-order observer design method for tracking a reference trajectory without disturbance.

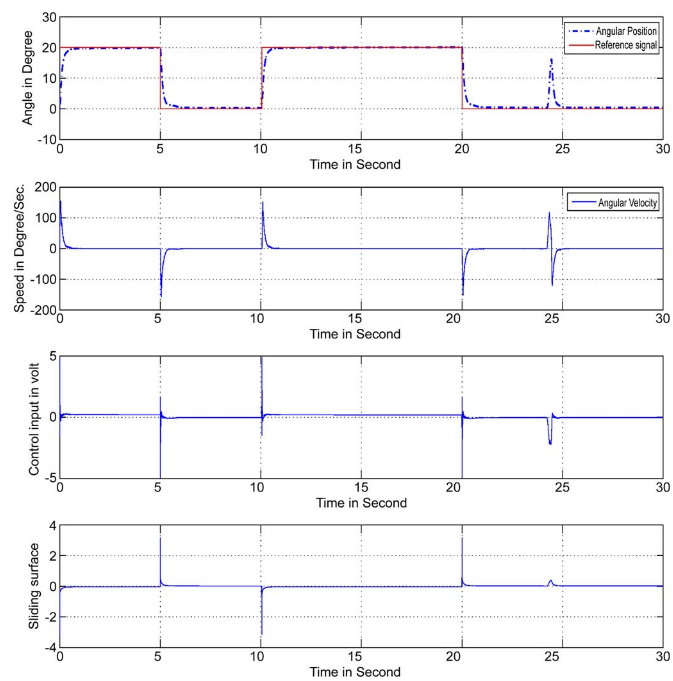

Fig. 3. Experimental results with the proposed reduced-order observer design method for tracking a reference trajectory without disturbance.

Appendix

Proof of Lemma 2: Consider the discrete-time system (1)

$$x(k+1) = Ax(k) + Bu(k).$$

Using relation (14), The system can be written in the form of

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) \tag{62}$$

$$x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_{21}u(k) \tag{63}$$

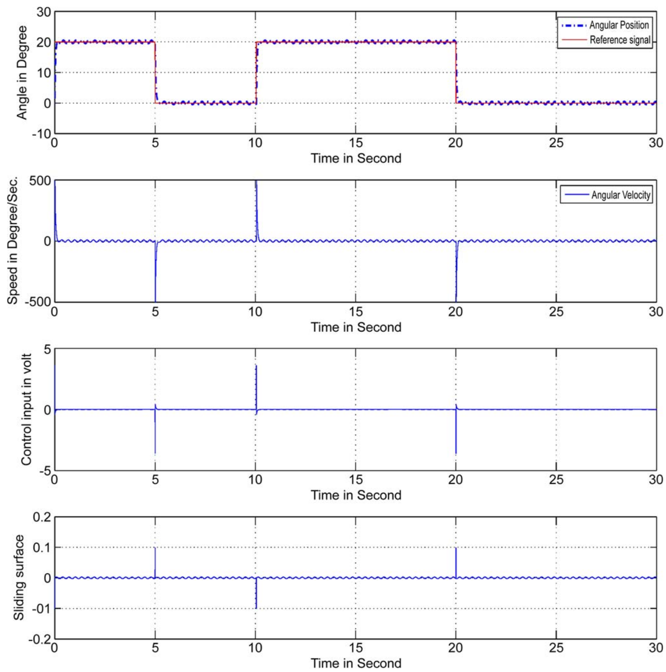

Fig. 4. Simulation results with the proposed reduced-order observer design method for tracking a reference trajectory with matched disturbance.

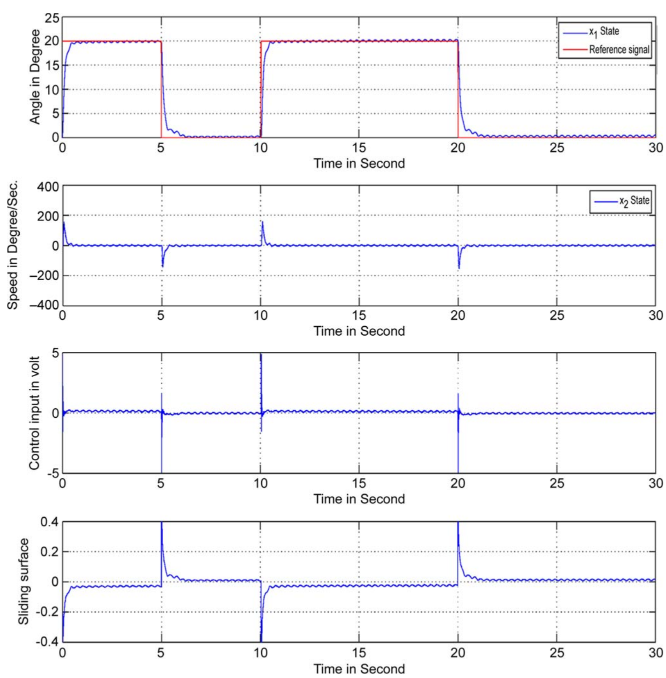

Fig. 5. Experimental results with the proposed reduced-order observer design method for tracking a reference trajectory with matched disturbance.

where $x_1 \in \mathbb{R}^{n-r}$ and $x_2 \in \mathbb{R}^r$. B_{21} is nonsingular, and A_{11}, A_{12}, A_{21}, and A_{22} are of appropriate dimensions. The sliding function is

$$\sigma(k) = Sx(k) \tag{64}$$

where

$$S = [S_1 \quad S_2] \tag{65}$$

in which S_2 is nonsingular.

During ideal sliding, $Sx(k) = 0$. Thus, $S_1x_1 + S_2x_2 = 0$. Substituting for $x_2(k)$ in (62), the ideal sliding motion is obtained as

$$x_1(k+1) = (A_{11} - A_{12}M)x_1(k) \tag{66}$$

where $M = S_2^{-1}S_1$.

Comparing with (17), we may write

$$J = A_{11} - A_{12}M. \tag{67}$$

Now, the zeros of the pulse transfer function $S(zI - A)^{-1}B$ are equal to the roots of equation

$$P(z) = \det \begin{bmatrix} zI - A & B \\ S & 0 \end{bmatrix} = 0 \tag{68}$$

called Rosenbrock's system matrix [22], [23]. Equivalently, it can be written as

$$\begin{aligned} \det(P(z)) &= \det \begin{bmatrix} zI - A & B \\ S & 0 \end{bmatrix} = 0 \\ &= \det \begin{bmatrix} zI - A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & B_{21} \\ S_1 & S_2 & 0 \end{bmatrix} = 0. \end{aligned} \tag{69}$$

As B_{21} is assumed to be nonsingular

$$\det(P(z)) = 0 \Leftrightarrow \det \begin{bmatrix} zI - A_{11} & -A_{12} \\ S_1 & S_2 \end{bmatrix} = 0.$$

This is equivalent to

$$\det\left(\begin{bmatrix} I & A_{12}S_2^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} zI-(A_{11}-A_{12}M) & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ M & I \end{bmatrix}\right) = 0. \tag{70}$$

This is equivalent to

$$\det \begin{bmatrix} zI - (A_{11} - A_{12}M) & 0 \\ 0 & S_2 \end{bmatrix} = 0. \tag{71}$$

As

$$\det \begin{bmatrix} I & A_{12}S_2^{-1} \\ 0 & I \end{bmatrix} = I \quad \det \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} = I \tag{72}$$

Equation (70) is equivalent to

$$\det \begin{bmatrix} zI - (A_{11} - A_{12}M) & 0 \\ 0 & S_2 \end{bmatrix} = 0 \tag{73}$$

This means

$$\det(P(z)) = 0 \Leftrightarrow \det(zI - (A_{11} - A_{12}M)) = 0. \tag{74}$$

Therefore, the invariant zeros of (A, B, S) are the eigenvalues of $(A_{11} - A_{12}M)$ that is equivalent to J in (18) i.e., the eigenvalues of the reduced-order sliding motion.

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