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Finite-time stability and finite-time weighted L_2-gain analysis for switched systems with time-varying delay

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Abstract: Finite-time stability and finite-time boundedness for a class of switched linear systems with time-varying delay are studied. Sufficient conditions which guarantee switched linear systems with time-varying delay finite-time stable or finite-time bounded are presented. These conditions are delay-dependent and are given in terms of linear matrix inequalities. Average dwell time of switching signals is also given such that switched linear systems are finite-time stable or finite-time bounded. Moreover, finite-time weighted L_2-gain of switched linear systems with time-varying delay are also given to measure its disturbance tolerance capability in the fixed time interval. Detail proofs are accomplished by using multiple Lyapunov-like functions. An example is employed to verify the efficiency of the proposed method.

1 Introduction

Switched systems consist of a family of subsystems described by differential or difference equations and a switching law that orchestrates switching between these subsystems. Lyapunov stability of switched systems is a basic topic in the research. Up to now, most of existing literatures related to stability of switched systems focused on Lyapunov asymptotic stability, which is defined over an infinite-time interval, such as [1–14]. However, in practice, one is interested in not only the qualitative behaviour of a dynamical system (usually Lyapunov stability), but also quantitative information (e.g. specific estimates of trajectory bounds over a fixed short time) [15], such as networked control systems [16, 17] and network congestion control [18].

In addition, a system could be Lyapunov stable but completely useless because it possesses undesirable transient performances. To study the transient performances of a system, the concept of short time stability, that is, finite-time stability, was introduced in [19]. Specifically, a system is said to be finite-time stable if, given a bound on the initial condition, its state remains within a prescribed bound in a fixed-time interval. Note that finite-time stability and Lyapunov asymptotic stability are independent concepts: a system could be finite-time stable but not Lyapunov asymptotically stable and vice versa [20]. Moreover, it should be emphasised that practical stability also studies the boundedness of system trajectory. Practical stability means that the system can run at an arbitrarily small neighbourhood around the equilibrium, which is also defined over an infinite-time interval.

Some early results on finite-time stability problems date back to the 1960s [19, 21, 22]. Recently, based on the linear matrix inequality (LMI) theory, many valuable results have been obtained [20, 23–34]. In addition, it should be pointed out that the authors of [35–39] have presented some results of finite-time stability for different systems, but finite-time stability in those papers that implies Lyapunov stability and finite-time convergence is different from that in this paper and [20, 23–34].

Switched systems with time-delay is ubiquitous in engineering control design, such as power systems [40, 41] and networked control systems [42, 43]. Owing to the interaction among continuous-time dynamics, discrete-time dynamics and time-delay, the dynamics of switched systems with time-delay are more complex than switched systems without time-delay and time-delay systems without switching. Therefore the study of switched systems with time-delay is very interesting and challenging. Moreover, there inevitably exist some external disturbances in practical systems. The disturbance tolerance capability can be measured by the largest bound on the energy of the disturbance, say L_2 gain. In order to measure the disturbance tolerance capability of switched time-delay systems in the fixed interval, finite-time weighted L_2 gain under zero initial condition of switched time-delay systems is discussed in this note.

LMI techniques is the most popular approach and have played an important role due to the fact that LMIs can be cast into a convex optimisation problem that can be handled efficiently [44–50]. Recently, based on LMI techniques, Lyapunov asymptotic stability of switched time-delay systems has been discussed, such as [51–54], but there are very few results available yet on finite-time stability of switched systems with time-delay, to the best of authors' knowledge.

Based on linear matrix inequalities, finite-time stability and stabilisation conditions for switched systems without time-delay or with constant delay were developed in [55, 56], respectively. In this paper, sufficient conditions for finite-time stability, finite-time boundedness and finite-time weighted L_2-gain of switched linear systems with time-varying delay are to be discussed. Although delay-independent stability conditions are simpler to apply, delay-dependent stability conditions are less conservative especially in the case when the time delay is small. So delay-dependent conditions will be presented in this paper. Our contributions are given as follows: (i) Definitions of finite-time stability, finite-time boundedness and finite-time weighted L_2-gain are extended to switched linear systems with time-varying delay. (ii) Sufficient conditions for finite-time boundedness, finite-time stability and finite-time weighted L_2-gain of switched linear systems with time-varying delay are given.

The paper is organised as follows. In Section 2, some notations and problem formulations are presented. In Section 3, based on linear matrix inequalities, sufficient conditions that guarantee finite-time stability, finite-time boundedness of switched linear systems with time-varying delay are given. Moreover, sufficient conditions that guarantee that switch linear systems have finite-time weighted L_2-gain are also presented. Finally, an example is presented to illustrate the efficiency of the proposed method in Section 4. Concluding remarks are given in Section 5.

2 Preliminaries and problem formulation

In this paper, $P > 0 (P \geq 0, P < 0, P \leq 0)$ denotes a symmetric positive-definite (positive-semidefinite, negative-definite, negative-semidefinite) matrix P. For any symmetric matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. The identity matrix of order n is denoted as I_n (or, simply, I if no confusion arises).

Consider a switched linear systems with time-varying delay as follows

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - \mathrm{d}(t)) + G_{\sigma(t)}\omega(t) \\ z(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}\omega(t), \quad t \geq 0 \\ x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0] \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state, $z(t) \in R^m$ is the output, $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$ and $D_{\sigma(t)}$ are constant real matrices, $\varphi(\theta)$ is a differentiable vector-valued initial function on $[-\tau, 0]$, $\omega(t)$ is time-varying exogenous disturbance, $\sigma(t) : [0, \infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal that is a piecewise constant function depending on time t or state $x(t)$, and m is the number of subsystems, $\mathrm{d}(t)$ denotes the time-varying delay satisfying either (a) or (b) below:

(a) $0 \leq \mathrm{d}(t) \leq \tau, \dot{d}(t) \leq d < 1$ for a known constant d, and
(b) $0 \leq \mathrm{d}(t) \leq \tau$.

The condition (b) means that the bound of the derivative of $\mathrm{d}(t)$ can be an unknown constant.

In this paper, we assume that the state of switched linear system does not jump at switching instants, that is, the trajectory $x(t)$ is everywhere continuous and switching signal $\sigma(t)$ has finite switching number in any finite interval time. As it was pointed out in [51], a continuous and piecewise differentiable initial condition guarantees the existence of the solutions of switched delay systems whose states does not jump at the switching instants. Corresponding to the switching signal $\sigma(t)$, we have the following switching sequence $\{x_0; (i_0, t_0), \ldots, (i_k, t_k), \ldots, | i_k \in M, k = 0, 1, \ldots\}$, in which t_0 is the initial time, x_0 is the initial state and the i_kth subsystem is activated when $t \in [t_k, t_{k+1})$.

Assumption 1: External disturbances $\omega(t)$ is time-varying and satisfies $\int_0^{T_f} \omega^T(t)\omega(t)\,\mathrm{d}t \leq d_\omega, \quad d_\omega \geq 0, [0, T_f]$ is the fixed finite-time interval.

It should be pointed out that the assumption about the external disturbances $\omega(t)$ in this paper, which is not constant, is different from that of [20, 26, 27].

Definition 1 [57]*:* For any $T \geq t \geq 0$, let $N_\sigma(t, T)$ denote the switching number of $\sigma(t)$ over (t, T). If $N_\sigma(t, T) \leq N_0 + \frac{T-t}{\tau_a}$ holds for $\tau_a > 0$ and an integer $N_0 \geq 0$, then τ_a is called an average dwell time and N_0 is called the chattering bound.

In the sequel, let us extend the definitions of finite-time stability and finite-time boundedness [20] to switched linear system with time-varying delay (1).

Definition 2: Given positive constants c_1, c_2, T_f with $c_1 < c_2$, a positive-definite matrix R, and a switching signal $\sigma(t)$. If

$$x(\overline{t_0})^T R x(\overline{t_0}) \leq c_1 \Rightarrow x(t)^T R x(t) < c_2, \forall t \in [0, T_f] \tag{2}$$

where $x(\overline{t_0})^T R x(\overline{t_0}) = \max\{\sup_{-\tau \leq \theta \leq 0}\{x(t_0 + \theta)\}, \sup_{-\tau \leq \theta \leq 0}\{\dot{x}(t_0 + \theta)\}\}^T R \max\{\sup_{-\tau \leq \theta \leq 0}\{x(t_0 + \theta)\}, \sup_{-\tau \leq \theta \leq 0}\{\dot{x}(t_0 + \theta)\}\}$, then switched systems (1) with $\omega(t) \equiv 0$ are said to be finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$.

Definition 3: Given positive constants c_1, c_2, T_f, d_ω with $c_1 < c_2$, $d_\omega \geq 0$, a positive definite matrix R, and a switching signal $\sigma(t)$. If $\forall \omega(t)$ satisfying Assumption 1

$$x(\overline{t_0})^T R x(\overline{t_0}) \leq c_1 \Rightarrow x(t)^T R x(t) < c_2, \forall t \in [0, T_f] \tag{3}$$

where $x(\overline{t_0})^T R x(\overline{t_0}) = \max\{\sup_{-\tau \leq \theta \leq 0}\{x(t_0 + \theta)\}, \sup_{-\tau \leq \theta \leq 0}\{\dot{x}(t_0 + \theta)\}\}^T R \max\{\sup_{-\tau \leq \theta \leq 0}\{x(t_0 + \theta)\}, \sup_{-\tau \leq \theta \leq 0}\{\dot{x}(t_0 + \theta)\}\}$, then switched systems (1) are said to be finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, R, \sigma)$.

Definition 4: Given positive constants c_1, c_2, T_f with $c_1 < c_2$, a positive-definite matrix R, and a switching signal $\sigma(t)$. If (2) holds for any switching signal $\sigma(t)$, then switched systems (1) with $\omega(t) \equiv 0$ are said to be uniformly finite-time stable with respect to (c_1, c_2, T_f, R).

Definition 5: Given positive constants c_1, c_2, T_f, d_ω with $c_1 < c_2$, $d_\omega \geq 0$, a positive-definite matrix R, and a switching signal $\sigma(t)$. If (3) holds for any switching signal $\sigma(t)$, then switched systems (1) are said to be uniformly finite-time bounded with respect to $(c_1, c_2, T_f, R, d_\omega)$.

Remark 1: It should be remarked that the concepts of finite-time stability and finite-time boundedness are different from the concept of reachable set. More detail discussions about the difference between two approaches can be found in Remark 4 of [20].

Remark 2: The meaning of 'uniformity' in Definitions 4 and 5 is with respect to the switching signal, rather than the time, which is similar to that of [1, 3].

Recently, disturbance attenuation properties for switched systems have been widely studied. In [5], weighted L_2-gain problem of switched systems in an infinite-time interval was introduced. Then, the case for a class switched linear systems with time-delay was discussed in [51]. In order to measure the disturbance tolerance capability of switched time-delay systems in the fixed interval, here, we investigate weighted L_2-gain in a fixed interval.

Definition 6: For $T_f > 0$, $k \geq 0$ and $\gamma > 0$, if under zero initial condition $\varphi(t) = 0, \forall t \in [-h, 0]$, it holds that

$$\int_0^{T_f} e^{-ks} z^T(s) z(s) \mathrm{d}s \leq \gamma^2 \int_0^{T_f} \omega^T(s)\omega(s)\mathrm{d}s \tag{4}$$

then switched systems (1) are said to have finite-time weighted L_2-gain.

3 Main results

3.1 Finite-time stability and finite-time boundedness

First, sufficient conditions for finite-time boundedness of switched systems with time-varying delay are given.

Theorem 1: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (a). For any $i \in M$, suppose that there exist matrices $P_i > 0, Q_i > 0$, $Z_i > 0$, $S_i > 0$, constants $\alpha_i \geq 0$, $\beta_i > 0$, $\mu \geq 1$

$$X_i = \begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i \\ * & X_{22}^i & X_{23}^i \\ * & * & X_{33}^i \end{pmatrix} \geq 0 \tag{5}$$

and any Y_i, T_i, L_i with appropriate dimensions such that

$$\begin{pmatrix} \varphi_{11}^i & \varphi_{12}^i & \varphi_{13}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \varphi_{23}^i & \tau B_i^T Z_i \\ * & * & \varphi_{33}^i & \tau G_i^T Z_i \\ * & * & * & -\tau Z_i \end{pmatrix} < 0 \tag{6}$$

$$\begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i & Y_i \\ * & X_{22}^i & X_{23}^i & T_i \\ * & * & X_{33}^i & L_i \\ * & * & * & Z_i \end{pmatrix} \geq 0 \tag{7}$$

$$P_i \leq \mu P_j, \quad Q_i \leq \mu Q_j, \quad Z_i \leq \mu Z_j, \quad \forall i, j \in M \tag{8}$$

$$\left(\lambda_2 + \frac{\tau^2}{2} e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4\right) c_1 + \beta\lambda_5 d_\omega < \lambda_1 c_2 \mu^{-N_0} e^{-\alpha T_f} \tag{9}$$

then for any switching signal σ with average dwell time satisfying $\tau_a > \tau_a^* = (T_f \ln\mu / F_m^*)$, switched systems (1) are finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, R, \sigma)$, where

$$\begin{aligned}
\varphi_{11}^i &= A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i \\
\varphi_{12}^i &= P_i B_i + \tau X_{12}^i - Y_i + T_i^T \\
\varphi_{13}^i &= P_i G_i + L_i^T + \tau X_{13}^i \\
\varphi_{22}^i &= \tau X_{22}^i - T_i^T - T_i - (1-d) Q_i \\
\varphi_{23}^i &= -L_i^T + \tau X_{23}^i; \varphi_{33}^i = \tau X_{33}^i - \beta_i S_i \\
\alpha &= \max_{\forall i\in M}(\alpha_i); \beta = \max_{\forall i\in M}(\beta_i) \\
\lambda_1 &= \min_{\forall i\in M}(\lambda_{\min}(\tilde{P}_i)) = \min_{\forall i\in M}(\lambda_{\min}(R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}})) \\
\lambda_2 &= \max_{\forall i\in M}(\lambda_{\max}(\tilde{P}_i)) = \max_{\forall i\in M}(\lambda_{\max}(R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}})) \\
\lambda_3 &= \max_{\forall i\in M}(\lambda_{\max}(\tilde{Z}_i)) = \max_{\forall i\in M}(\lambda_{\max}(R^{-\frac{1}{2}} Z_i R^{-\frac{1}{2}})) \\
\lambda_4 &= \max_{\forall i\in M}(\lambda_{\max}(\tilde{Q}_i)) = \max_{\forall i\in M}(\lambda_{\max}(R^{-\frac{1}{2}} Q_i R^{-\frac{1}{2}})) \\
\lambda_5 &= \max_{\forall i\in M}(\lambda_{\max}(S_i)); F_m^* = \ln(\lambda_1 c_2) - \alpha T_f - N_0 \ln\mu \\
&\quad - \ln\left[(\lambda_2 + \frac{\tau^2}{2} e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4) c_1 + \beta\lambda_5 d_\omega\right]
\end{aligned} \tag{10}$$

Proof: Choose a Lyapunov-like function $V(t) = V_{\sigma(t)}(t) = V_{1,\sigma(t)}(t) + V_{2,\sigma(t)}(t) + V_{3,\sigma(t)}(t)$, where

$$\begin{aligned}
V_{1,\sigma(t)}(t) &= x^T(t) P_{\sigma(t)} x(t) \\
V_{2,\sigma(t)}(t) &= \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) e^{\alpha_{\sigma(t)}(t-s)} Z_{\sigma(t)} \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
V_{3,\sigma(t)}(t) &= \int_{t-\mathrm{d}(t)}^t x^T(s) e^{\alpha_{\sigma(t)}(t-s)} Q_{\sigma(t)} x(s)\,\mathrm{d}s
\end{aligned} \tag{11}$$

Step 1: No losing the generality, when $t \in [t_k, t_{k+1})$, let $\sigma(t) = i$. Taking the derivative of $V(t)$ with respect to t along the trajectory of system (1) yields

$$\begin{aligned}
\dot{V}_{1,i}(t) &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) \\
&= x^T(t)(A_i^T P_i + P_i A_i) x(t) + x^T(t - \mathrm{d}(t)) B_i^T P_i x(t) \\
&\quad + x^T(t) P_i B_i x(t - \mathrm{d}(t)) + \omega^T(t) G_i^T P_i x(t) \\
&\quad + x^T(t) P_i G_i \omega(t)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_{2,i}(t) &= \alpha_i e^{\alpha_i t} \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) e^{-\alpha_i s} Z_i \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&\quad + \int_{-\tau}^0 \begin{bmatrix} \dot{x}^T(t) e^{\alpha_i(t-t)} Z_i \dot{x}(t) \\ -\dot{x}^T(t+\theta) e^{\alpha_i(t-t-\theta)} Z_i \dot{x}(t+\theta) \end{bmatrix} \mathrm{d}\theta \\
&= \alpha_i \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) e^{\alpha_i(t-s)} Z_i \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta + \tau\dot{x}^T(t) Z_i \dot{x}(t) \\
&\quad - \int_{t-\tau}^t \left[\dot{x}^T(s) e^{\alpha_i(t-s)} Z_i \dot{x}(s)\right] \mathrm{d}s \\
&\leq \alpha_i \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) e^{\alpha_i(t-s)} Z_i \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&\quad + x^T(t)(\tau A_i^T Z_i A_i) x(t) + x^T(t)(\tau A_i^T Z_i B_i) x(t - \mathrm{d}(t)) \\
&\quad + x^T(t - \mathrm{d}(t))(\tau B_i^T Z_i A_i) x(t) + x^T(t)(\tau A_i^T Z_i G_i)\omega(t) \\
&\quad + x^T(t - \mathrm{d}(t))(\tau B_i^T Z_i B_i) x(t - \mathrm{d}(t)) \\
&\quad + x^T(t - \mathrm{d}(t))(\tau B_i^T Z_i G_i)\omega(t) \\
&\quad + \omega^T(t)(\tau G_i^T Z_i A_i) x(t) \\
&\quad + \omega^T(t)(\tau G_i^T Z_i B_i) x(t - \mathrm{d}(t)) \\
&\quad + \omega^T(t)(\tau G_i^T Z_i G_i)\omega(t) \\
&\quad - \int_{t-\mathrm{d}(t)}^t \left[\dot{x}^T(s) e^{\alpha_i(t-s)} Z_i \dot{x}(s)\right] \mathrm{d}s
\end{aligned}$$

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$$
\begin{aligned}
\dot{V}_{3,i}(t) &= \alpha_i \int_{t-\mathrm{d}(t)}^{t} x^T(s) e^{\alpha_i(t-s)} Q_i x(s)\mathrm{d}s + x^T(t) Q_i x(t) \\
&\quad - (1-\dot{d}(t)) x^T(t-\mathrm{d}(t)) e^{\alpha_i \mathrm{d}(t)} Q_i x(t-\mathrm{d}(t)) \\
&\leq \alpha_i \int_{t-\mathrm{d}(t)}^{t} x^T(s) e^{\alpha_i(t-s)} Q_i x(s)\mathrm{d}s + x^T(t) Q_i x(t) \\
&\quad - (1-d) x^T(t-\mathrm{d}(t)) Q_i x(t-\mathrm{d}(t))
\end{aligned}
$$

Let $\xi(t) = (x^T(t), x^T(t-\mathrm{d}(t)), \omega^T(t))^T$. It is obviously $\tau\xi^T(t)X_i\xi(t) - \int_{t-\mathrm{d}(t)}^{t}[\xi^T(t)X_i\xi(t)]\mathrm{d}s \geq 0$, that is

$$
\begin{aligned}
&\tau\xi^T(t)X_i\xi(t) - \int_{t-\mathrm{d}(t)}^{t}[\xi^T(t)X_i\xi(t)]\mathrm{d}s \\
&\quad = x^T(t)(\tau X_{11}^i)x(t) + x^T(t-\mathrm{d}(t))(\tau X_{12}^i)^T x(t) \\
&\qquad + \omega^T(t)(\tau X_{13}^i)^T x(t) + x^T(t)(\tau X_{12}^i)x(t-\mathrm{d}(t)) \\
&\qquad + x^T(t-\mathrm{d}(t))(\tau X_{22}^i)x(t-\mathrm{d}(t)) \\
&\qquad + \omega^T(t)(\tau X_{23}^i)^T x(t-\mathrm{d}(t)) + \omega^T(t)(\tau X_{33}^i)x(t)\omega(t) \\
&\qquad + x^T(t)(\tau X_{13}^i)\omega(t) + x^T(t-\mathrm{d}(t))(\tau X_{23}^i)\omega(t) \\
&\qquad - \int_{t-\mathrm{d}(t)}^{t} \begin{bmatrix} x^T(t)(X_{11}^i)x(t) + \omega^T(t)(X_{13}^i)^T x(t) \\ +x^T(t-\mathrm{d}(t))(X_{12}^i)^T x(t) \\ +x^T(t)(X_{12}^i)x(t-\mathrm{d}(t)) \\ +x^T(t-\mathrm{d}(t))(X_{22}^i)x(t-\mathrm{d}(t)) \\ +\omega^T(t)(X_{23}^i)^T x(t-\mathrm{d}(t)) \\ +x^T(t-\mathrm{d}(t))(X_{23}^i)\omega(t) \\ +\omega^T(t)(X_{33}^i)x(t)\omega(t) \\ +x^T(t)(X_{13}^i)\omega(t) \end{bmatrix} \mathrm{d}s \geq 0
\end{aligned} \tag{12}
$$

From the Leibniz–Newton formula, we have

$$
\begin{aligned}
&2[x^T(t)Y_i + x^T(t-\mathrm{d}(t))T_i + \omega^T(t)L_i] \\
&\quad \times \left[x(t) - \int_{t-\mathrm{d}(t)}^{t}\dot{x}(s)\mathrm{d}s - x(t-\mathrm{d}(t))\right] = 0
\end{aligned} \tag{13}
$$

that is

$$
\begin{aligned}
&x^T(t)(Y_i^T + Y_i)x(t) - x^T(t)(Y_i + Y_i^T)x(t-\mathrm{d}(t)) \\
&\quad + x^T(t-\mathrm{d}(t))(T_i + T_i^T)x(t) + \omega^T(t)(L_i + L_i^T)x(t) \\
&\quad - \omega^T(t)(L_i + L_i^T)x(t-\mathrm{d}(t)) \\
&\quad - x^T(t-\mathrm{d}(t))(T_i + T_i^T)x(t-\mathrm{d}(t)) \\
&\quad - 2\int_{t-\mathrm{d}(t)}^{t} x^T(t)Y_i\dot{x}(s)\mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t} x^T(t-\mathrm{d}(t))T_i\dot{x}(s)\mathrm{d}s \\
&\quad - 2\int_{t-\mathrm{d}(t)}^{t} \omega^T(t)L_i\dot{x}(s)\mathrm{d}s = 0
\end{aligned} \tag{14}
$$

then one obtains (see equation at the bottom of the page)

where

$$
\xi(t) = (x^T(t), x^T(t-\mathrm{d}(t)), \omega^T(t))^T
$$

$$
\zeta(t) = (x^T(t), x^T(t-\mathrm{d}(t)), \omega^T(t), \dot{x}^T(s))^T
$$

$$
\Xi = \begin{pmatrix} \varphi_{11}^i + \tau A_i^T Z_i A_i & \varphi_{12}^i + \tau A_i^T Z_i B_i & \varphi_{13}^i + \tau A_i^T Z_i G_i \\ * & \varphi_{22}^i + \tau B_i^T Z_i B_i & \varphi_{23}^i + \tau B_i^T Z_i G_i \\ * & * & \varphi_{33}^i + \tau G_i^T Z_i G_i \end{pmatrix}
$$

$$
\begin{aligned}
\dot{V}_i(x(t)) - \alpha_i V_i(x(t)) &< x^T(t)(A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i + \tau A_i^T Z_i A_i)x(t) + x^T(t)(P_i B_i + \tau X_{12}^i - Y_i + T_i^T \\
&\quad + \tau A_i^T Z_i B_i)x(t-\mathrm{d}(t)) + x^T(t-\mathrm{d}(t))(B_i^T P_i + \tau(X_{12}^i)^T - Y_i^T + T_i + \tau B_i^T Z_i A_i)x(t) + x^T(t-\mathrm{d}(t)) \\
&\quad \times (\tau X_{22}^i - T_i^T - T_i - (1-d)Q_i + \tau B_i^T Z_i B_i)x(t-\mathrm{d}(t)) + x^T(t)(P_i G_i + \tau A_i^T Z_i G_i + L_i^T + \tau X_{13}^i)\omega(t) \\
&\quad + \omega^T(t)(G_i^T P_i + \tau G_i^T Z_i A_i + L_i + \tau(X_{13}^i)^T)x(t) + x^T(t-\mathrm{d}(t))(\tau B_i^T Z_i G_i - L_i^T + \tau X_{23}^i)\omega(t) \\
&\quad + \omega^T(t)(\tau G_i^T Z_i B_i - L_i + \tau(X_{23}^i)^T)x(t-\mathrm{d}(t)) + \omega^T(t)(\tau G_i^T Z_i G_i + \tau X_{33}^i)\omega(t) - \int_{t-\mathrm{d}(t)}^{t}[\dot{x}^T(s)e^{\alpha(t-s)}Z_i\dot{x}(s)]\mathrm{d}s \\
&\quad - \int_{t-\mathrm{d}(t)}^{t} \begin{bmatrix} x^T(t)(X_{11}^i)x(t) \\ +x^T(t-\mathrm{d}(t))(X_{12}^i)^T x(t) + \omega^T(t)(X_{13}^i)^T x(t) \\ +x^T(t)(X_{12}^i)x(t-\mathrm{d}(t)) \\ +x^T(t-\mathrm{d}(t))(X_{22}^i)x(t-\mathrm{d}(t)) \\ +\omega^T(t)(X_{23}^i)^T x(t-\mathrm{d}(t)) + x^T(t)(X_{13}^i)\omega(t) \\ +x^T(t-\mathrm{d}(t))(X_{23}^i)\omega(t) \\ +\omega^T(t)(X_{33}^i)x(t)\omega(t) \end{bmatrix} \mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t}[x^T(t)Y_i\dot{x}(s)]\mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t}[x^T(t-\mathrm{d}(t))T_i\dot{x}(s)]\mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t}[\omega^T(t)L_i\dot{x}(s)]\mathrm{d}s \\
&\leq x^T(t)(A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i + \tau A_i^T Z_i A_i)x(t) + x^T(t)(P_i B_i + \tau X_{12}^i - Y_i + T_i^T \\
&\quad + \tau A_i^T Z_i B_i)x(t-\mathrm{d}(t)) + x^T(t-\mathrm{d}(t))(B_i^T P_i + \tau(X_{12}^i)^T - Y_i^T + T_i + \tau B_i^T Z_i A_i)x(t) + x^T(t-\mathrm{d}(t)) \\
&\quad \times (\tau X_{22}^i - T_i^T - T_i - (1-d)Q_i + \tau B_i^T Z_i B_i)x(t-\mathrm{d}(t)) + x^T(t)(P_i G_i + \tau A_i^T Z_i G_i + L_i^T + \tau X_{13}^i)\omega(t) \\
&\quad + \omega^T(t)(G_i^T P_i + \tau G_i^T Z_i A_i + L_i + \tau(X_{13}^i)^T)x(t) + x^T(t-\mathrm{d}(t))(\tau B_i^T Z_i G_i - L_i^T + \tau X_{23}^i)\omega(t) + \omega^T(t) \\
&\quad \times (\tau G_i^T Z_i B_i - L_i + \tau(X_{23}^i)^T)x(t-\mathrm{d}(t)) + \omega^T(t)(\tau G_i^T Z_i G_i + \tau X_{33}^i - \beta_i S_i)\omega(t) \\
&\quad - \int_{t-\mathrm{d}(t)}^{t}[\dot{x}^T(s)Z_i\dot{x}(s)]ds + \beta_i\omega^T(t)S_i\omega(t) - \int_{t-\mathrm{d}(t)}^{t} \begin{bmatrix} x^T(t)(X_{11}^i)x(t) + x^T(t-\mathrm{d}(t))(X_{12}^i)^T x(t) \\ +\omega^T(t)(X_{13}^i)^T x(t) + x^T(t)(X_{12}^i)x(t-\mathrm{d}(t)) \\ +x^T(t-\mathrm{d}(t))(X_{22}^i)x(t-\mathrm{d}(t)) \\ +\omega^T(t)(X_{23}^i)^T x(t) + x^T(t)(X_{13}^i)\omega(t) \\ +x^T(t-\mathrm{d}(t))(X_{23}^i)\omega(t) + \omega^T(t)(X_{33}^i)x(t)\omega(t) \end{bmatrix} \mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t}[x^T(t)Y_i\dot{x}(s)]\mathrm{d}s \\
&\quad - 2\int_{t-\mathrm{d}(t)}^{t}[x^T(t-\mathrm{d}(t))T_i\dot{x}(s)]\mathrm{d}s - 2\int_{t-\mathrm{d}(t)}^{t}[\omega^T(t)L_i\dot{x}(s)]\mathrm{d}s = \xi^T(t)\Xi\xi(t) - \int_{t-\mathrm{d}(t)}^{t}[\zeta^T(t)\Theta\zeta(t)]ds + \beta_i\omega^T(t)S_i\omega(t)
\end{aligned}
$$

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$$\Theta = \begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i & Y_i \\ * & X_{22}^i & X_{23}^i & T_i \\ * & * & X_{33}^i & L_i \\ * & * & * & Z_i \end{pmatrix}$$

Thus, we have

$$\dot{V}_i(x(t)) - \alpha_i V_i(x(t)) < \beta_i \omega^T(t) S_i \omega(t) \tag{15}$$

and then

$$V_i(x(t)) < e^{\alpha_i(t-t_k)} V_i(x(t_k)) + \int_{t_k}^{t} e^{\alpha_i(t-s)} \beta_i \omega^T(s) S_i \omega(s)\,\mathrm{d}s \tag{16}$$

Step 2: Without loss of generality, assume that $\sigma(t_k) = i, \sigma(t_k^-) = j$ at switching instant t_k. Since $\mu \geq 1, P_i \leq \mu P_j, Q_i \leq \mu Q_j, Z_i \leq \mu Z_j, \forall i, j \in M$, one obtains

$$V_{\sigma(t_k)}(x(t_k)) \leq \mu V_{\sigma(t_k^-)}(x(t_k^-)) \tag{17}$$

where $x(t_k^-) = \lim_{v \to 0-} x(t_k + v)$. Since $\alpha = \max_{\forall i \in M}(\alpha_i)$, $\beta = \max_{\forall i \in M}(\beta_i)$, for any $t \in (0, T_f)$, it follows from (16) and (17) that

$$\begin{aligned} V_{\sigma(t)}(x(t)) &< e^{\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(x(t_k^-)) \\ &\quad + \int_{t_k}^{t} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_k)} \omega(s)\,\mathrm{d}s \end{aligned} \tag{18}$$

Using the iterative method, we have (see equation at the bottom of the page)

On the other hand

$$\begin{aligned} V(x(t)) &\geq x^T(t) P_{\sigma(t)} x(t) \geq \lambda_{\min}(\tilde{P}_{\sigma(t)}) x^T(t) R x(t) \\ &= \lambda_1 x^T(t) R x(t) \\ V_{\sigma(0)}(x(0)) &\leq \lambda_{\max}(\tilde{P}_{\sigma(0)}) x^T(0) R x(0) \\ &\quad + \lambda_{\max}(\tilde{Z}_{\sigma(0)}) \int_{-\tau}^{0} \int_{\theta}^{0} \dot{x}^T(s)\, e^{\alpha_{\sigma(0)}(-s)} R \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad + \tau e^{\alpha\tau} \lambda_{\max}(\tilde{Q}_{\sigma(0)}) \sup_{-\tau \leq \theta \leq 0} \{x(\theta)^T R x(\theta)\} \\ &\leq \lambda_{\max}(\tilde{P}_{\sigma(0)}) \sup_{-\tau \leq \theta \leq 0} \{x(\theta)^T R x(\theta)\} \\ &\quad + \frac{\tau^2}{2} e^{\alpha\tau} \lambda_{\max}(\tilde{Z}_{\sigma(0)}) \sup_{-\tau \leq \theta \leq 0} \{\dot{x}(\theta)^T R \dot{x}(\theta)\} \\ &\quad + \tau e^{\alpha\tau} \lambda_{\max}(\tilde{Q}_{\sigma(0)}) \sup_{-\tau \leq \theta \leq 0} \{x(\theta)^T R x(\theta)\} \\ &\leq \left(\lambda_2 + \frac{\tau^2}{2} e^{\alpha\tau} \lambda_3 + \tau e^{\alpha\tau} \lambda_4\right) x(\overline{t_0})^T R x(\overline{t_0}) \\ &\leq \left(\lambda_2 + \frac{\tau^2}{2} e^{\alpha\tau} \lambda_3 + \tau e^{\alpha\tau} \lambda_4\right) c_1 \end{aligned}$$

Thus, one obtains

$$\begin{aligned} x^T(t) R x(t) &\leq \frac{V(x(t))}{\lambda_1} \\ &< \frac{e^{\alpha T_f} \mu^{N_0} \mu^{\frac{T_f}{\tau_a}} (V_{\sigma(0)}(x(0)) + \beta \lambda_5 d_\omega)}{\lambda_1} \end{aligned}$$

$$\begin{aligned} V_{\sigma(t)}(x(t)) &< e^{\alpha(t-t_k)} \mu \quad V_{\sigma(t_k^-)}(x(t_k^-)) + \int_{t_k}^{t} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t)} \omega(s)\,\mathrm{d}s \\ &< e^{\alpha(t-t_{k-2})} \mu^2 V_{\sigma(t_{k-2})}(x(t_{k-2})) + \mu^2 \int_{t_{k-2}}^{t_{k-1}} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-2})} \omega(s)\,\mathrm{d}s \\ &\quad + \mu \int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-1})} \omega(s)\,\mathrm{d}s + \int_{t_k}^{t} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t)} \omega(s)\,\mathrm{d}s \\ &< \cdots < e^{\alpha t} \mu^{N(0,t)} V_{\sigma(0)}(x(0)) + \mu^{N(0,t)} \int_0^{t_1} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(0)} \omega(s)\,\mathrm{d}s + \mu^{N(t_1,t)} \int_{t_1}^{t_2} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-2})} \omega(s)\,\mathrm{d}s \\ &\quad + \cdots + \mu \int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-1})} \omega(s)\,\mathrm{d}s + \int_{t_k}^{t} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t)} \omega(s)\,\mathrm{d}s \\ &< e^{\alpha T_f} \mu^{N(0,T_f)} V_{\sigma(0)}(x(0)) + \mu^{N(0,T_f)} \int_0^{t_1} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(0)} \omega(s)\,\mathrm{d}s + \mu^{N(t_1,T_f)} \int_{t_1}^{t_2} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-2})} \omega(s)\,\mathrm{d}s \\ &\quad + \cdots + \mu \int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t_{k-1})} \omega(s)\,\mathrm{d}s + \int_{t_k}^{t} e^{\alpha(t-s)} \beta \omega^T(s) S_{\sigma(t)} \omega(s)\,\mathrm{d}s \\ &\leq e^{\alpha T_f} \mu^{N_0 + \frac{T_f}{\tau_a}} V_{\sigma(0)}(x(0)) + \int_0^{T_f} e^{\alpha(t-s)} \beta \mu^{N(s,T_f)} \omega^T(s) S_{\sigma(t)} \omega(s)\,\mathrm{d}s \\ &\leq e^{\alpha T_f} \mu^{N_0} \mu^{\frac{T_f}{\tau_a}} V_{\sigma(0)}(x(0)) + e^{\alpha T_f} \mu^{N_0} \mu^{\frac{T_f}{\tau_a}} \beta \max_{\forall \sigma(s) \in M} (\lambda_{\max}(S_{\sigma(s)})) \int_0^{T_f} \omega^T(s) \omega(s)\,\mathrm{d}s \\ &\leq e^{\alpha T_f} \mu^{N_0} \mu^{\frac{T_f}{\tau_a}} (V_{\sigma(0)}(x(0)) + \beta \lambda_5 d_\omega) \end{aligned}$$

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$$\leq \frac{(\lambda_2 + \frac{\tau^2}{2}e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4)c_1 + \beta\lambda_5 d_\omega}{\lambda_1} \times \mu^{N_0} e^{\alpha T_f} \mu^{\frac{T_f}{\tau_a}} \tag{19}$$

The following proof can be divided into two cases.

Case 1: $\mu = 1$, which is a trivial case, from (9), $x^T(t)Rx(t) < c_2 e^{-\alpha T_f} e^{\alpha T_f} = c_2$.
Case 2: $\mu > 1$, from (9), $F_M^* > 0$. Then, we have

$$\frac{T_f}{\tau_a} < \frac{\ln\left(\frac{\lambda_1 c_2 \mu^{-N_0} e^{-\alpha T_f}}{(\lambda_2 + \frac{\tau^2}{2}e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4)c_1 + \beta\lambda_5 d_\omega)}\right)}{\ln \mu} \tag{20}$$

Substituting (20) into (19) yields

$$\begin{aligned} x^T(t)Rx(t) &< \frac{e^{\alpha T_f}\mu^{N_0}\mu^{\frac{T_f}{\tau_a}}(V_{\sigma(0)}(x(0)) + \beta\lambda_5 d_\omega))}{\lambda_1} \\ &\leq \frac{(\lambda_2 + \frac{\tau^2}{2}e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4)c_1 + \beta\lambda_5 d_\omega)}{\lambda_1} \\ &\quad \times \mu^{N_0} e^{\alpha T_f} \mu^{\frac{T_f}{\tau_a}} \\ &< c_2 \end{aligned}$$

□

Corollary 1: Suppose that the time-varying delay d(t) satisfying (b). If (6) with $Q_i = 0$, (7), (8) and (9) hold, then for any switching signal σ with average dwell time satisfying $\tau_a > \tau_a^* = (T_f \ln \mu / F_m^*)$, switched systems (1) are finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, R, \sigma)$, where the parameters are same to those in Theorem 1.

Proof: The proof is similar to that of Theorem 1; therefore it is omitted here. □

Sufficient conditions for finite-time stability of switched systems with time-varying delays are given as follows.

Theorem 2: Suppose that the time-varying delay d(t) satisfying (a). For any $i \in M$, suppose that there exist matrices $P_i > 0, Q_i > 0$, $Z_i > 0$, constants $\alpha_i \geq 0$, $\beta_i > 0$, $\mu \geq 1$

$$X_i = \begin{pmatrix} X_{11}^i & X_{12}^i \\ * & X_{22}^i \end{pmatrix} \geq 0 \tag{21}$$

and any Y_i, T_i with appropriate dimensions such that

$$\begin{pmatrix} \varphi_{11}^i & \varphi_{12}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \tau B_i^T Z_i \\ * & * & -\tau Z_i \end{pmatrix} < 0 \tag{22}$$

$$\begin{pmatrix} X_{11}^i & X_{12}^i & Y_i \\ * & X_{22}^i & T_i \\ * & * & Z_i \end{pmatrix} \geq 0 \tag{23}$$

$$P_i \leq \mu P_j, \quad Q_i \leq \mu Q_j, \quad Z_i \leq \mu Z_j, \quad \forall i, j \in M \tag{24}$$

$$(\lambda_2 + \tau e^{\alpha\tau}\lambda_3)c_1 + \lambda_4\beta d < c_2 \mu^{-N_0} e^{-\alpha T_f}\lambda_1 \tag{25}$$

then for any switching signal σ with average dwell time satisfying $\tau_a > \tau_a^* = (T_f \ln \mu / F_m^*)$ switched systems (1) are finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where

$$\begin{aligned} \varphi_{11}^i &= A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i \\ \varphi_{12}^i &= P_i B_i + \tau X_{12}^i - Y_i - T_i^T \\ \varphi_{22}^i &= \tau X_{22}^i - T_i^T - T_i - (1-d)e^{-\alpha\tau} Q_i \\ \alpha &= \max_{\forall i \in M}(\alpha_i) \\ \lambda_1 &= \min_{\forall i \in M}(\lambda_{\min}(\tilde{P}_i)) = \min_{\forall i \in M}(\lambda_{\min}(R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}})) \\ \lambda_2 &= \max_{\forall i \in M}(\lambda_{\max}(\tilde{P}_i)) = \max_{\forall i \in M}(\lambda_{\max}(R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}})) \\ \lambda_3 &= \max_{\forall i \in M}(\lambda_{\max}(\tilde{Z}_i)) = \max_{\forall i \in M}(\lambda_{\max}(R^{-\frac{1}{2}} Z_i R^{-\frac{1}{2}})) \\ \lambda_4 &= \max_{\forall i \in M}(\lambda_{\max}(\tilde{Q}_i)) = \max_{\forall i \in M}(\lambda_{\max}(R^{-\frac{1}{2}} Q_i R^{-\frac{1}{2}})) \\ F_m^* &= \ln(\lambda_1 c_2) - \ln[(\lambda_2 + \tau e^{\alpha\tau}\lambda_3)c_1 + \lambda_4 \beta d] \\ &\quad - \alpha T_f - N_0 \ln \mu \end{aligned}$$

Proof: Let $\omega(t) \equiv 0$, the proof procedure is similar to that of Theorem 1, and it is easy to obtain the conclusion. □

Corollary 2: Suppose that the time-varying delay d(t) satisfying (b). If (22) with $Q_i = 0$, (23), (24) and (25) hold, then for any switching signal σ with average dwell time satisfying $\tau_a > \tau_a^* = \frac{T_f \ln \mu}{F_m^*}$, switched systems (1) are finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where the parameters are same to those in Theorem 2.

Proof: The proof is similar to that of Theorem 2, therefore it is omitted here. □

Remark 3: The results in this note are based on free weighting matrix method, which is a well-known method for the stability analysis of time-delay systems. Moreover, the proofs of the main results are really resorted to the related reference such as [47]. However, unlike the case of asymptotical stability for switched systems, the function $\dot{V}(t)$ in the proof procedure of Theorems 1 and 2 is no need to be negative definite or negative semidefinite. Actually, if the exogenous disturbance $\omega(t) = 0$ and the constants $\alpha_i < 0$ $(\forall i \in M)$, then $\dot{V}(t)$ will be a negative-definite function. In this case, switched system (1) is asymptotically stable on the infinite interval $[0, +\infty)$ if the average dwell time $\tau_a > (-(\ln \mu)/\alpha)$ as that was proved in [51]. In view of this, the results in this note are more general than those in [47] to some extent.

Now, sufficient conditions which guarantee uniform finite-time boundedness and uniform finite-time stability of switched systems (1) are given.

Theorem 3: Suppose that the time-varying delay d(t) satisfying (a). If there exist matrices $P > 0, Q > 0$, $Z > 0$, $S > 0$, constants $\alpha \geq 0$, $\beta > 0$

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ * & X_{22} & X_{23} \\ * & * & X_{33} \end{pmatrix} \geq 0 \tag{26}$$

and any Y, T, L with appropriate dimensions such that

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \tau A^T Z \\ * & \varphi_{22} & \varphi_{23} & \tau B^T Z \\ * & * & \varphi_{33} & \tau G^T Z \\ * & * & * & -\tau Z \end{pmatrix} < 0 \quad (27)$$

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & Y \\ * & X_{22} & X_{23} & T \\ * & * & X_{33} & L \\ * & * & * & Z \end{pmatrix} \geq 0 \quad (28)$$

then switched systems (1) are uniformly finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, R)$ where

$$\varphi_{11} = A_i^T P + PA_i + Y + Y^T + Q - \alpha P + \tau X_{11}$$
$$\varphi_{12} = PB_i + \tau X_{12} - Y - T^T$$
$$\varphi_{13} = PG_i + L^T + \tau X_{13}$$
$$\varphi_{22} = \tau X_{22} - T^T - T\_(1-d)e^{-\alpha\tau} Q$$
$$\varphi_{23} = -L^T + \tau X_{23}; \ \varphi_{33} = \tau X_{33} - \beta S$$

Proof: Choose a common Lyapunov-like function $V(t) = V_1(t) + V_2(t) + V_3(t)$, where

$$\begin{aligned} V_1(t) &= x^T(t)Px(t) \\ V_2(t) &= \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\alpha(t-s)} Z\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ V_3(t) &= \int_{t-\mathrm{d}(t)}^{t} x^T(s)e^{\alpha(t-s)}Qx(s)\,\mathrm{d}s \end{aligned} \quad (29)$$

The proof procedure is similar to that of Theorem 1, and it is not difficult to obtain the conclusion. □

Corollary 3: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (b). If (27) with $Q = 0$ and (28) hold, then switched systems (1) are uniformly finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, R)$, where the parameters are same to those in Theorem 3.

Proof: The proof is similar to that of Theorem 3; therefore it is omitted here. □

Theorem 4: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (a). If there exist matrices $P > 0, Q > 0$, $Z > 0$, constants $\alpha \geq 0$

$$\begin{pmatrix} X_{11} & X_{12} \\ * & X_{22} \end{pmatrix} \geq 0 \quad (30)$$

and any Y, T with appropriate dimensions such that

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \tau A^T Z \\ * & \varphi_{22} & \tau B^T Z \\ * & * & -\tau Z \end{pmatrix} < 0 \quad (31)$$

$$\begin{pmatrix} X_{11} & X_{12} & Y \\ * & X_{22} & T \\ * & * & Z \end{pmatrix} \geq 0 \quad (32)$$

then switched systems (1) are uniformly finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where

$$\begin{aligned} \varphi_{11} &= A_i^T P + PA_i + Y + Y^T + Q - \alpha P + \tau X_{11} \\ \varphi_{12} &= PB_i + \tau X_{12} - Y - T^T \\ \varphi_{22} &= \tau X_{22} - T^T - T - (1-d)e^{-\alpha\tau} Q \end{aligned} \quad (33)$$

Proof: Choose a common Lyapunov-like function as Theorem 3. The proof procedure is similar to that of Theorem 2. □

Corollary 4: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (b). If (31) with $Q = 0$ and (32) hold, then switched systems (1) are uniformly finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where the parameters are same to those in Theorem 4.

Proof: The proof is similar to that of Theorem 4; therefore it is omitted here. □

3.2 *Finite-time weighted L_2-gain analysis*

In this subsection, a restricted L_2 gain analysis of switched linear systems with time-varying delays (1), that is, finite-time weighted L_2 gain with zero initial conditions analysis, is discussed in the following theorem.

Theorem 5: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (a). For any $i \in M$, suppose that there exist matrices $P_i > 0, Q_i > 0$, $Z_i > 0$, constants $\alpha_i \geq 0$, $\gamma_i > 0$, $\mu \geq 1$, such that

$$X_i = \begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i \\ * & X_{22}^i & X_{23}^i \\ * & * & X_{33}^i \end{pmatrix} \geq 0 \quad (34)$$

$$\begin{pmatrix} \varphi_{11}^i & \varphi_{12}^i & \varphi_{13}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \varphi_{23}^i & \tau B_i^T Z_i \\ * & * & \varphi_{33}^i & \tau G_i^T Z_i \\ * & * & * & -\tau Z_i \end{pmatrix} < 0 \quad (35)$$

$$\begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i & Y_i \\ * & X_{22}^i & X_{23}^i & T_i \\ * & * & X_{33}^i & L_i \\ * & * & * & Z_i \end{pmatrix} \geq 0 \quad (36)$$

$$P_i \leq \mu P_j, \quad Q_i \leq \mu Q_j, \quad Z_i \leq \mu Z_j, \quad \forall i, j \in M \quad (37)$$

$$\gamma^2 d_\omega < \lambda_1 c_2 \mu^{-N_0} e^{-\alpha T_f} \quad (38)$$

and average dwell time of switching signal σ satisfies

$$\tau_a > \tau_a^* = \max\left\{\frac{T_f \ln\mu}{\ln(\lambda_1 c_2) - \ln[\gamma^2 d_\omega] - \alpha T_f - N_0 \ln\mu}, \frac{\ln\mu}{\alpha}\right\}$$

then switched systems (1) are finite-time bounded with respect to $(0, c_2, T_f, d_\omega, R, \sigma)$ and has weighted L_2 gain γ, where

$$\varphi_{11}^i = A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i + C_i^T C_i$$
$$\varphi_{12}^i = P_i B_i + \tau X_{12}^i - Y_i + T_i^T$$
$$\varphi_{13}^i = P_i G_i + L_i^T + \tau X_{13}^i + C_i^T D_i$$
$$\varphi_{22}^i = \tau X_{22}^i - T_i^T - T_i - (1-d)Q_i$$
$$\varphi_{23}^i = -L_i^T + \tau X_{23}^i; \ \varphi_{33}^i = \tau X_{33}^i - \gamma_i^2 I + D_i^T D_i$$
$$\alpha = \max_{\forall i \in M}(\alpha_i); \gamma = \max_{\forall i \in M}(\gamma_i)$$
$$\lambda_1 = \min_{\forall i \in M}(\lambda_{\min}(\tilde{P}_i)) = \min_{\forall i \in M}(\lambda_{\min}(R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}}))$$

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Proof: Let $\tilde{\varphi}_{11}^i = A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i + \tau X_{11}^i, \tilde{\varphi}_{13}^i = P_i G_i + L_i^T + \tau X_{13}^i, \tilde{\varphi}_{33}^i = \tau X_{33}^i - \gamma_i^2 I$. Assuming condition (35) is satisfied, we have

$$\begin{pmatrix} \varphi_{11}^i & \varphi_{12}^i & \varphi_{13}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \varphi_{23}^i & \tau B_i^T Z_i \\ * & * & \varphi_{33}^i & \tau G_i^T Z_i \\ * & * & * & -\tau Z_i \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_{11}^i & \varphi_{12}^i & \tilde{\varphi}_{13}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \varphi_{23}^i & \tau B_i^T Z_i \\ * & * & \tilde{\varphi}_{33}^i & \tau G_i^T Z_i \\ * & * & * & -\tau Z_i \end{pmatrix} + \begin{pmatrix} C_i^T C_i & 0 & C_i^T D_i & 0 \\ 0 & 0 & 0 & 0 \\ D_i^T C_i & 0 & D_i^T D_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} < 0$$

Note that

$$\begin{pmatrix} C_i^T C_i & 0 & C_i^T D_i & 0 \\ 0 & 0 & 0 & 0 \\ D_i^T C_i & 0 & D_i^T D_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_i^T \\ 0 \\ D_i^T \\ 0 \end{pmatrix} \begin{pmatrix} C_i & 0 & D_i & 0 \end{pmatrix} \geq 0 \tag{39}$$

which is implied

$$\begin{pmatrix} \tilde{\varphi}_{11}^i & \varphi_{12}^i & \tilde{\varphi}_{13}^i & \tau A_i^T Z_i \\ * & \varphi_{22}^i & \varphi_{23}^i & \tau B_i^T Z_i \\ * & * & \tilde{\varphi}_{33}^i & \tau G_i^T Z_i \\ * & * & * & -\tau Z_i \end{pmatrix} < 0 \tag{40}$$

From Theorem 1, conditions (37), (38) and (40) guarantee the switched system (1) is finite-time bounded with respect to $(0, c_2, T_f, d_\omega, R, \sigma)$.

Step 1: Choose a Lyapunov-like function $V(t) = V_{\sigma(t)}(t) = V_{1,\sigma(t)}(t) + V_{2,\sigma(t)}(t) + V_{3,\sigma(t)}(t)$ as in the proof of Theorem 1. No losing the generality, when $t \in [t_k, t_{k+1})$, let $\sigma(t) = i$. Similar to the proof procedure of Step 1 of Theorem 1, one obtains

$$\begin{aligned} &\dot{V}_i(x(t)) - \alpha_i V_i(x(t)) - \gamma_i^2 \omega^T(t)\omega(t) + z(t)^T z(t) \\ &\quad \leq x^T(t)(A_i^T P_i + P_i A_i + Y_i + Y_i^T + Q_i - \alpha_i P_i \\ &\qquad + \tau X_{11}^i + \tau A_i^T Z_i A_i + C_i^T C_i)x(t) + x^T(t) \\ &\qquad \times (P_i B_i + \tau X_{12}^i - Y_i + T_i^T + \tau A_i^T Z_i B_i)x(t - \mathrm{d}(t)) \\ &\qquad + x^T(t - \mathrm{d}(t))(B_i^T P_i + \tau (X_{12}^i)^T - Y_i^T + T_i + \tau B_i^T Z_i A_i) \\ &\qquad \times x(t) + x^T(t - \mathrm{d}(t))(\tau X_{22}^i - T_i^T - T_i - (1-d)Q_i \\ &\qquad + \tau B_i^T Z_i B_i)x(t - \mathrm{d}(t)) + x^T(t)(P_i G_i + \tau A_i^T Z_i G_i + L_i^T \\ &\qquad + \tau X_{13}^i + C_i^T D_i)\omega(t) + \omega^T(t)(G_i^T P_i + \tau G_i^T Z_i A_i + L_i \\ &\qquad + \tau (X_{13}^i)^T + D_i^T C_i)x(t) + x^T(t - \mathrm{d}(t))(\tau B_i^T Z_i G_i - L_i^T \\ &\qquad + \tau X_{23}^i)\omega(t) + \omega^T(t)(\tau G_i^T Z_i B_i - L_i + \tau (X_{23}^i)^T) \\ &\qquad \times x(t - \mathrm{d}(t)) + \omega^T(t)(\tau G_i^T Z_i G_i + \tau X_{33}^i - \gamma_i^2 I \\ &\qquad + D_i^T D_i)\omega(t) - \int_{t-\mathrm{d}(t)}^t [\dot{x}^T(s) Z_i \dot{x}(s)]\,\mathrm{d}s \\ &\qquad - \int_{t-\mathrm{d}(t)}^t \begin{bmatrix} x^T(t)(X_{11}^i)x(t) + x^T(t-\mathrm{d}(t))(X_{12}^i)^T x(t) \\ +\omega^T(t)(X_{13}^i)^T x(t) \\ +x^T(t)(X_{12}^i)x(t-\mathrm{d}(t)) \\ +x^T(t-\mathrm{d}(t))(X_{22}^i)x(t-\mathrm{d}(t)) \\ +\omega^T(t)(X_{23}^i)^T x(t) + x^T(t)(X_{13}^i)\omega(t) \\ +x^T(t-\mathrm{d}(t))(X_{23}^i)\omega(t) \\ +\omega^T(t)(X_{33}^i)x(t)\omega(t) \end{bmatrix} \mathrm{d}s \\ &\qquad - 2\int_{t-\mathrm{d}(t)}^t [x^T(t) Y_i \dot{x}(s)]\mathrm{d}s \\ &\qquad - 2\int_{t-\mathrm{d}(t)}^t [x^T(t-\mathrm{d}(t)) T_i \dot{x}(s)]\,\mathrm{d}s \\ &\qquad - 2\int_{t-\mathrm{d}(t)}^t [\omega^T(t) L_i \dot{x}(s)]\,\mathrm{d}s \\ &\quad = \xi^T(t)\Xi\xi(t) - \int_{t-\mathrm{d}(t)}^t [\zeta^T(t)\Theta\zeta(t)]\,\mathrm{d}s \end{aligned}$$

where

$$\xi(t) = (x^T(t), x^T(t-\mathrm{d}(t)), \omega^T(t))^T$$
$$\zeta(t) = (x^T(t), x^T(t-\mathrm{d}(t)), \omega^T(t), \dot{x}^T(s))^T$$
$$\Xi = \begin{pmatrix} \varphi_{11}^i + \tau A_i^T Z_i A_i & \varphi_{12}^i + \tau B_i^T Z_i A_i & \varphi_{13}^i + \tau A_i^T Z_i G_i \\ * & \varphi_{22}^i + \tau B_i^T Z_i B_i & \varphi_{23}^i + \tau G_i^T Z_i G_i \\ * & * & \varphi_{33}^i + \tau B_i^T Z_i G_i \end{pmatrix}$$
$$\Theta = \begin{pmatrix} X_{11}^i & X_{12}^i & X_{13}^i & Y_i \\ * & X_{22}^i & X_{23}^i & T_i \\ * & * & X_{33}^i & L_i \\ * & * & * & Z_i \end{pmatrix}$$

By virtue of (35) (36), we have

$$\dot{V}_i(x(t)) - \alpha_i V_i(x(t)) < \gamma_i^2 \omega^T(t)\omega(t) - z^T(t)z(t) \tag{41}$$

therefore

$$V_i(x(t)) < e^{\alpha_i(t-t_k)} V_i(x(t_k)) + \int_{t_k}^t e^{\alpha_i(t-s)} [\gamma_i^2 \omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s \tag{42}$$

Step 2: Without loss of generality, assume that $\sigma(t_k) = i, \sigma(t_k^-) = j$ at switching instant t_k. Noticing that $x(t_k) = x(t_k^-)$. Since $\mu \geq 1, P_i \leq \mu P_j, Q_i \leq \mu Q_j, Z_i \leq \mu Z_j, \forall i, j \in M$, one obtains

$$V_{\sigma(t_k)}(x(t_k)) \leq \mu V_{\sigma(t_k^-)}(x(t_k^-)) \tag{43}$$

where $x(t_k^-) = \lim_{v \to 0^-} x(t_k + v)$. Since $\alpha = \max_{\forall i \in M}(\alpha_i)$, $\beta = \max_{\forall i \in M}(\beta_i)$, for any $t \in (0, T_f)$, it follows from (42) and (43) that

$$V_{\sigma(t)}(x(t)) < e^{\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(x(t_k^-)) + \int_{t_k}^t e^{\alpha(t-s)} [\gamma_{\sigma(t_k)}^2 \omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s \tag{44}$$

Using the iterative method, we have

$$\begin{aligned}
V_{\sigma(t)}(x(t)) &< e^{\alpha t}\mu^{N(0,t)}V_{\sigma(0)}(x(0)) + \mu^{N(t_1,t)}\int_0^{t_1} e^{\alpha(t-s)} \\
&\quad \times [\gamma^2_{\sigma(0)}\omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s + \mu^{N(t_2,t)} \\
&\quad \times \int_{t_1}^{t_2} e^{\alpha(t-s)}[\gamma^2_{\sigma(t_1)}\omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s \\
&\quad + \cdots + \mu\int_{t_{k-1}}^{t_k} e^{\alpha(t-s)} \\
&\quad \times [\gamma^2_{\sigma(t_{k-1})}\omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s \\
&\quad + \int_{t_k}^{t} e^{\alpha(t-s)}[\gamma^2_{\sigma(t_k)}\omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s \\
&\le e^{\alpha t}\mu^{N(0,t)}V_{\sigma(0)}(x(0)) + \int_0^t e^{\alpha(t-s)}\mu^{N(s,T_f)} \\
&\quad \times [\gamma^2_{\sigma(s)}\omega^T(s)\omega(s) - z^T(s)z(s)]\,\mathrm{d}s
\end{aligned}$$

Under the zero initial condition, one obtains

$$0 < \int_0^t e^{\alpha(t-s)}\mu^{N(s,t)}[\gamma^2\omega^T(s)\omega(s) - z^T(s)z(s)]\mathrm{d}s \tag{45}$$

which implies that

$$\begin{aligned}
&\int_0^t e^{\alpha(t-s)+N(s,t)\ln\mu}[z^T(s)z(s)]\,\mathrm{d}s \\
&\quad < \int_0^t e^{\alpha(t-s)+N(s,t)\ln\mu}[\gamma^2\omega^T(s)\omega(s)]\,\mathrm{d}s
\end{aligned} \tag{46}$$

Multiplying both sides of (46) by $e^{-N(0,t)\ln\mu}$ yields

$$\begin{aligned}
&\int_0^t e^{\alpha(t-s)-N(0,s)\ln\mu}[z^T(s)z(s)]\mathrm{d}s \\
&\quad < \int_0^t e^{\alpha(t-s)-N(0,s)\ln\mu}\left[\gamma^2\omega^T(s)\omega(s)\right]\mathrm{d}s
\end{aligned} \tag{47}$$

Since $\tau_a \ge \frac{\ln\mu}{\alpha}$, then $0 \le N_\sigma(0,s) \le N_0 + \frac{s}{\tau_a} \le N_0 + \frac{\alpha s}{\ln\mu}$. Substituting the inequality into (47) yields

$$\begin{aligned}
&\mu^{-N_0}\int_0^t e^{\alpha t}e^{-2\alpha s}[z^T(s)z(s)]\mathrm{d}s \\
&\quad < \mu^{-N_0}\int_0^t e^{\alpha t}e^{-2\alpha s}[\gamma^2\omega^T(s)\omega(s)]\,\mathrm{d}s
\end{aligned} \tag{48}$$

Setting $t = T_f$, we have

$$\begin{aligned}
&\mu^{-N_0}\int_0^{T_f} e^{\alpha T_f}e^{-2\alpha s}[z^T(s)z(s)]\mathrm{d}s \\
&\quad < \mu^{-N_0}\int_0^{T_f} e^{\alpha T_f}e^{-2\alpha s}[\gamma^2\omega^T(s)\omega(s)]\,\mathrm{d}s
\end{aligned} \tag{49}$$

therefore let $k = 2\alpha$, one obtains

$$\begin{aligned}
\int_0^{T_f} e^{-ks}[z^T(s)z(s)]\,\mathrm{d}s &= \int_0^{T_f} e^{-2\alpha s}[z^T(s)z(s)]\mathrm{d}s \\
&< \int_0^{T_f} e^{-2\alpha s}[\gamma^2\omega^T(s)\omega(s)]\,\mathrm{d}s \\
&< \gamma^2\int_0^{T_f}[\omega^T(s)\omega(s)]\mathrm{d}s
\end{aligned} \tag{50}$$

□

Remark 4: It should be noted that the L_2 gain γ is related to the integral bound d_ω of the disturbance in Theorem 5. We can calculate the L_2 gain for the energy bounded disturbance simultaneously based on the precondition that switched systems are finite-time bounded.

Corollary 5: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (b). If (35) with $Q_i = 0$, (36), (37) and (38) hold, and average dwell time of switching signal σ satisfies

$$\begin{aligned}
\tau_a &> \tau_a^* \\
&= \max\left\{\frac{T_f\ln\mu}{\ln(\lambda_1 c_2) - \ln[\gamma^2 d_\omega] - \alpha T_f - N_0\ln\mu}, \frac{\ln\mu}{\alpha}\right\}
\end{aligned}$$

then switched systems (1) are finite-time bounded with respect to $(0, c_2, T_f, d_\omega, R, \sigma)$ and has weighted L_2 gain γ, where the parameters are same to those in Theorem 5.

Proof: The proof is similar to that of Theorem 5, therefore it is omitted here. □

Based on Theorems 3 and 5, sufficient conditions which guarantee uniform finite-time boundedness and uniform finite-time stability of switched systems (1) are given.

Theorem 6: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (a). If there exist matrices $P > 0, Q > 0, Z > 0, S > 0$ and constants $\alpha \ge 0,\ \beta > 0,\ \mu \ge 1$, such that

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ * & X_{22} & X_{23} \\ * & * & X_{33} \end{pmatrix} \ge 0 \tag{51}$$

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \tau A^T Z \\ * & \varphi_{22} & \varphi_{23} & \tau B^T Z \\ * & * & \varphi_{33} & \tau G^T Z \\ * & * & * & -\tau Z \end{pmatrix} < 0 \tag{52}$$

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & Y \\ * & X_{22} & X_{23} & T \\ * & * & X_{33} & L \\ * & * & * & Z \end{pmatrix} < 0 \tag{53}$$

then switched systems (1) are uniformly finite-time bounded with respect to $(0, c_2, T_f, d_\omega, R, \sigma)$ and has weighted L_2 gain γ, where

$$\begin{aligned}
\varphi_{11} &= A_i^T P + PA_i + Y + Y^T + Q - \alpha P + \tau X_{11} + C_i^T C_i \\
\varphi_{12} &= PB_i + \tau X_{12} - Y + T^T \\
\varphi_{13} &= PG_i + L^T + \tau X_{13} + C_i^T D_i \\
\varphi_{22} &= \tau X_{22} - T^T - T - (1-d)Q \\
\varphi_{23} &= -L^T + \tau X_{23}; \varphi_{33} = \tau X_{33} - \gamma^2 I + D_i^T D_i
\end{aligned}$$

Proof: Choose a common Lyapunov-like function as Theorem 3. The proof procedure is similar to that of Theorem 5. □

Corollary 6: Suppose that the time-varying delay $\mathrm{d}(t)$ satisfying (b). If (52) with $Q = 0$ and (53) hold, then switched systems (1) are uniformly finite-time bounded with respect to $(0, c_2, T_f, d_\omega, R, \sigma)$ and has weighted L_2 gain γ, where the parameters are same to those in Theorem 6.

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Proof: The proof is similar to that of Theorem 5, therefore it is omitted here. □

Remark 5: It should be pointed out that the potential assumption of the previous results in this note is that each subsystem should be finite-time bounded or finite-time stable. If one subsystem of the switched systems is not finite-time bounded or finite-time stable, the results in this paper may fail. However, the results may be extended to such cases by virtue of the method in the related references, such as [58–61], which is our future research topic.

Remark 6: In this note, finite-time stability and finite-time weighted L_2 gain of switched systems are related to the bound of the time-varying delay. Some novel methods such as delay partitioning method [62–64] and input–output approach [65] can approximate the time-varying delay and reduce the complexity and conservatism for some time-delay systems to some extent. It is an interesting topic for our future research to use these methods to improve the method proposed in this note.

4 Numerical examples

From a viewpoint of computation, it should be noted that the conditions in Theorem 1, are not standard LMIs. However, once some values are fixed for α_i, β_i, these conditions, that is, (5)–(7), can be translated into LMIs conditions and thus solved involving Matlab's LMI control toolbox. In addition, as in [20], condition (9) can also be guaranteed by LMIs conditions once the values of α_i, β_i are fixed.

Specifically, condition (9) can be guaranteed by the following LMIs conditions: for any $i \in M$, there exist some positive numbers λ_1, λ_2, λ_3, λ_4 and λ_5 such that

$$\lambda_1 I < R^{-\frac{1}{2}} P_i R^{-\frac{1}{2}} < \lambda_2 I; \quad R^{-\frac{1}{2}} Z_i R^{-\frac{1}{2}} < \lambda_3 I$$
$$R^{-\frac{1}{2}} Q_i R^{-\frac{1}{2}} < \lambda_4 I; \quad S_i < \lambda_5 I$$
$$\left(\lambda_2 + \frac{\tau^2}{2} e^{\alpha\tau}\lambda_3 + \tau e^{\alpha\tau}\lambda_4\right) c_1 + \beta\lambda_5 d_\omega < \lambda_1 c_2 \mu^{N_0} e^{-\alpha T_f}$$

Remark 7: The conditions proposed in this note are not LMIs conditions from a viewpoint of computation. It should be noted that conservatism does exist because that they are translated into LMIs conditions by fixing some parameters in order to be easily calculated.

Now, an example is employed to verify the proposed method in this paper.

Example 1: Consider a switched linear system with time-varying delay as follows

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - \mathrm{d}(t)) + G_{\sigma(t)}\omega \tag{54}$$

with

$$A_1 = \begin{pmatrix} 0 & -0.7 \\ 1.46 & -0.45 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5 & -1.5 \\ 0.7 & 0 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 0 & -0.3 \\ 0.54 & 0.45 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.5 & -0.5 \\ 0.3 & 0 \end{pmatrix}$$
$$G_1 = G_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{d}(t) = 0.2 - 0.2\sin(t)$$
$$\omega(t) = [0.01\sin(t), 0.009\cos(2t+1)]^T, x(t) = \begin{pmatrix} 0.44 \\ 0 \end{pmatrix}$$
$$t \in [-0.4, 0]$$

A straightforward calculation gives $\tau = 0.4$, $d = 0.2$. The values of c_1, c_2, T_f, d_ω and matrix R are given as follows

$$c_1 = 0.2, \quad c_2 = 400, \quad T_f = 12, \quad R = I, d_\omega = 0.0011$$

Solving (5)–(7) and (9) for $\alpha_i = 0.1$, $\beta_i = 0.5$, $(\forall i \in M)$ leads to feasible solutions

$$P_1 = 1.0 \times 10^3 \times \begin{pmatrix} 1.0318 & 0.0002 \\ 0.0002 & 1.0293 \end{pmatrix}$$
$$P_2 = 1.0 \times 10^3 \times \begin{pmatrix} 1.0293 & -0.0002 \\ -0.0002 & 1.0318 \end{pmatrix}$$
$$Q_1 = 1.0 \times 10^3 \times \begin{pmatrix} 2.4462 & -0.0031 \\ -0.0031 & 2.4462 \end{pmatrix}$$
$$Q_2 = 1.0 \times 10^3 \times \begin{pmatrix} 2.4462 & 0.0031 \\ 0.0031 & 2.4462 \end{pmatrix}$$
$$Z_1 = 1.0 \times 10^{-14} \times \begin{pmatrix} 0.6119 & -0.0060 \\ -0.0060 & 0.6138 \end{pmatrix}$$
$$Z_2 = 1.0 \times 10^{-14} \times \begin{pmatrix} 0.6138 & 0.0060 \\ 0.0060 & 0.6119 \end{pmatrix}$$
$$S_1 = 1.0 \times 10^5 \times \begin{pmatrix} 2.3523 & 0.0820 \\ 0.0820 & 2.6861 \end{pmatrix}$$
$$S_2 = 1.0 \times 10^5 \times \begin{pmatrix} 2.6861 & -0.0820 \\ -0.0820 & 2.3523 \end{pmatrix}$$
$$Y_1 = 1.0 \times 10^{-11} \times \begin{pmatrix} 0.6088 & 0.5591 \\ 0.5591 & -0.2996 \end{pmatrix}$$
$$Y_2 = 1.0 \times 10^{-11} \times \begin{pmatrix} -0.2996 & -0.5591 \\ -0.5591 & 0.6088 \end{pmatrix}$$
$$T_1 = 1.0 \times 10^{-11} \times \begin{pmatrix} -0.9258 & -0.1869 \\ -0.1869 & 0.5876 \end{pmatrix}$$
$$T_2 = 1.0 \times 10^{-11} \times \begin{pmatrix} 0.5876 & 0.1869 \\ 0.1869 & -0.9258 \end{pmatrix}$$
$$L_1 = 1.0 \times 10^{-11} \times \begin{pmatrix} -0.4987 & -0.7871 \\ -0.7871 & -0.5459 \end{pmatrix}$$
$$L_2 = 1.0 \times 10^{-11} \times \begin{pmatrix} -0.5459 & 0.7871 \\ 0.7871 & -0.4987 \end{pmatrix}$$
$$\mu = 1.0201$$

Calculating average dwell time of switching signal, one obtains $\tau_a^* = 3.3663$. Then, by virtue of Theorem 1, for any switching signal $\sigma(t)$ with average dwell time $\tau_a > \tau_a^*$, switched system (1) is finite-time bounded with respect to $(c_1, c_2, T_f, d_\omega, I, \sigma)$. Fig. 1*a* shows the state trajectory over $0 \sim 12$ under a periodic switching signal $\sigma_1(t)$ with interval time $\triangle T = 3.4$. From Fig. 1*b*, it is easy to see that system (1) is finite-time bounded. If the switching is too frequent, it is possible that the system is not finite-time bounded any more. For instance, simulation curves of system (1) under a periodic switching signal σ_2 with interval time $\triangle T = 1.2$ are shown Fig. 2.

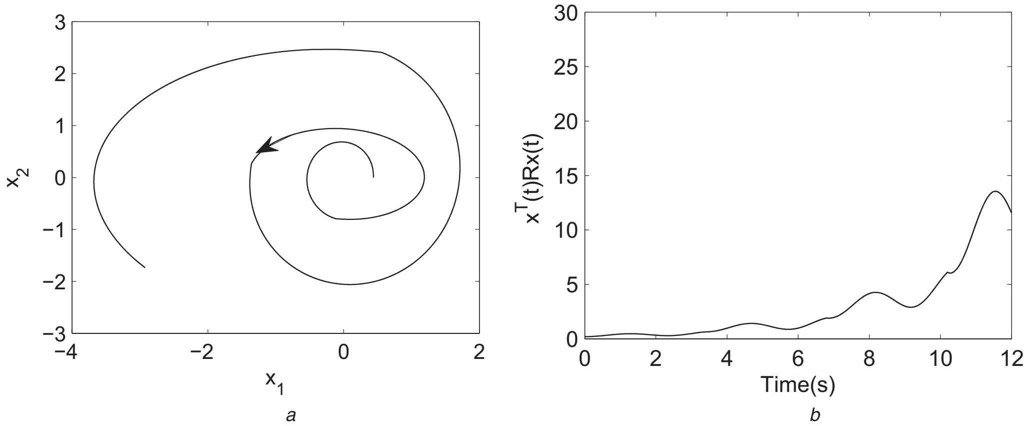

Fig. 1 *System response under switching signal σ_1*

a Phase plot of state $x(t)$
b Time history of $x(t)^T Rx(t)$

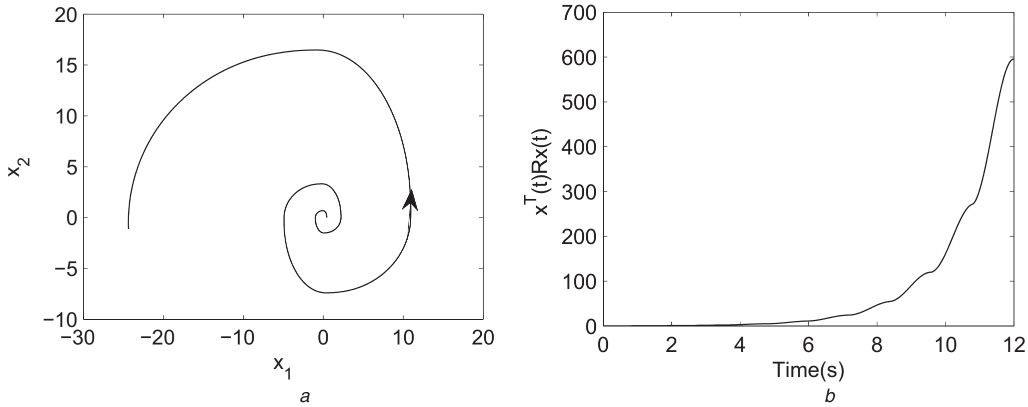

Fig. 2 *System response under switching signal σ_2*

a Phase plot of state $x(t)$
b Time history of $x(t)^T Rx(t)$

5 Conclusion

In this paper, finite-time stability, finite-time boundedness and finite-time weighted L_2-gain for a class of switched linear systems with time-varying delay have been investigated. Based on linear matrix techniques and multiple Lyapunov-like function method, delay-dependent conditions which guarantee that the switched linear systems with time-varying delay is finite-time stable or finite-time bounded and has finite-time weighted L_2-gain have been provided, respectively.

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