PD controller based on Second Order Sliding Mode Differentiation

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***Abstract*— The Proportional Derivative controller (PD) has been successfully implemented in many real-time applications. It is well-known, that the PD is composed by a proportional and a derivative terms of the signal error. However, the main problem in the implementation of this controller is related with the error signal derivative. Most of the existing results obtain the derivative based on first order filters. This approach is not useful if the signal is noisy and uncertain. In the present paper, the so-called Super-Twisting Algorithm (STA), that is a second order sliding mode approach, is implemented as a robust exact differentiator because it can reach the derivative of a signal in finite time. The closed loop stability of the proposed controller is analyzed in terms of a non-smooth Lyapunov function. Finite time convergence of the tracking error into a boundary layer is obtained. With a slightly modification in the control law with the addition of a discontinuous term, finite time convergence of the error to zero is obtained. Numerical results are given to show the difference between the classical PD and the proposed PD with the STA differentiator.**

I. Introduction

Linear time invariant systems can be regulated in terms of classical PD controllers if the mathematical model, the complete access to the state vector and a good derivative approximation are available. However, these assumptions can not be accomplished in real time applications if the system is uncertain and contains high frequency components. The principal problem arises when it is necessary the signal differentiation of a nosy signal[1]. If the equation that describes the signal error is completely known, a linear filter can be implemented in order to obtain the signal derivative. [2]. This method can bring numerical mistakes and the tracking performance only provides local stability. Also, it cannot be easily applied in nonlinear systems. Moreover, if the system presents some kind of perturbations and parametric uncertainties, a linear filter is not longer an option to obtain a numerical approximation of the signal derivative. The principal aspects to be considered in the implementation of a differentiator, is the presence of noise and high-frequency signal components. A differentiator could be called robust, if it is insensitive to high frequencies and the high frequency components can be addressed as noises. However, by the high frequencies, the differentiator would not be exact. High gain observers [3] and sliding mode based differentiators [1] have become an option to overcome the main problems in signal numerical differentiation.

The First Order Sliding Mode Technique (FSM) can provide finite time convergence and robustness against certain type of perturbations [4]. Also, inside the Sliding Mode Theory there exists the so-called Second Order Sliding Mode controllers (SSM), that maintains the main characteristics of classical sliding modes (FSM) and reduce the undesired high frequency oscillation called chattering [5]. If the structure of the signal to be differentiated is unknown, and some assumptions are accepted, the SSM can be used. A robust exact differentiator is presented in [1] based on a modification of the well known Super-Twisting Algorithm (STA). Recently in [6] and [7] new non-smooth Lyapunov functions were analyzed to proof the stability of the STA. In terms of these kind of functions, the conditions to ensure finite-time convergence for the STA are established.

In this paper, the STA is implemented as a robust exact differentiator to complete the structure given by a classical PD controller. By means of the non-smooth Lyapunov function presented in [6] the close loop stability of the controller is obtained. If the PD controller contains a discontinuous term finite time convergence to zero could be obtained. The rest of the paper is organized as follow, in Section II the class of nonlinear systems to deal with it is introduced, then the Super-Twisting Algorithm as a differentiator is described. In this section, the extended system that incorporates the STA to estimate the derivative of the signal error is given. In section III the main result is summarized in a theorem. Some modifications to the control law to get finite time convergence are stated in a brief corollary. Numerical results are presented in section IV, where the control of a simple pendulum is implemented. Finally in section V the conclusions are given.

II. Super-Twisting PD Controller

A. Class of Nonlinear Systems

Consider the nonlinear system described by the following nonlinear differential equations

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0 \tag{1}$$

where $x \in \Re^n$ is the state vector and x_0 is the initial condition, $f(\cdot) : \Re^n \to \Re^n$ is a nonlinear function that describes the system dynamics, $g(\cdot) : \Re^n \to \Re^n$ is a nonlinear function associated to the exogenous input $u \in \Re$. In this paper the control actions is designed such as a classical PD one given

978-0-7695-4882-1/12 $26.00 © 2012 IEEE
DOI 10.1109/Andescon.2012.38

by $u = -k_1 e - k_2 \dot{e}$ where e is the tracking error and it is defined as

$$e := x - x^* \tag{2}$$

with x^* being a smooth nonlinear trajectory to be tracked with bounded derivative

$$\begin{array}{ll} \dot{x}^* = h(x^*), & x^*(0) = x_0^* \\ |h(x^*)| \leq h^+, & \forall x^* \in \Re^n \end{array} \tag{3}$$

The following statements are assumed throughout this paper

- A1. The nonlinear function $f(\cdot)$ is continuous, that means $\|f(x) - f(x')\| \leq L_1 \|x - x'\| \ \forall x, x' \in \Re^n$, $L_1 \geq 0$;
- A2. The nonlinear function $g(\cdot)$ is continuous and satisfies $0 \leq g^- \leq \|g(\cdot)\| \leq g^+$.

B. Super-Twisting Algorithm

The STA is composed of two differential inclusions. The STA can be used as a controller [8], a state estimator [9] and as a robust exact differentiator (RED) [1]. To applied the STA as a RED, consider the following auxiliary equation $\dot{z} = w$, the second order sliding algorithm to keep $z - n = 0$, with $n(t)$ the function to be differentiated is

$$\begin{array}{c} w = w_1 - \lambda_1 |z - n| \, sign(z - n) \\ \dot{w}_1 = -\lambda_2 sign(z - n) \end{array} \tag{4}$$

where $\lambda_1, \lambda_2 > 0$. Here w is the output of the differentiator [1].

The derivative of the tracking error is given by $\dot{e} := \dot{x} - \dot{x}^*$, therefore the following equation is valid substituting (1) in (2)

$$\dot{e} = f(x) + g(x) k_1 e + g(x) k_2 \dot{e} - h(x^*) \tag{5}$$

the last equation is a kind of a implicit equation. If it is desired to proof the PD stability in close lop, equation (5) have to be written as

$$\dot{e} = (1 - g(x) k_2)^{-1} (f(x) + g(x) k_1 e - h(x^*)) \tag{6}$$

The stability of equation (6) can be obtained by means of a quadratic Lyapunov function. However this proof is outside the goal of this paper. Here it is assumed that $\dot{e}$ is obtained by means of the second order STA (4). Then, the following extended system describes the complete dynamics of the signal error with an adequate implementation of (4). If we define $\gamma := \hat{e} - e$ as the differentiator signal error with $\hat{e}$ being the estimate of the error,

$$\begin{array}{c} \dot{e} = f(x) + g(x) k_1 e_t + g(x) k_2 \dot{\gamma} - h(x^*) \\ \dot{\gamma} = \sigma - \lambda_1 |\gamma|^{1/2} sign(\gamma) \\ \dot{\sigma} = -\lambda_2 sign(\gamma) \end{array} \tag{7}$$

here $\gamma, \sigma \in \Re$ are the states of the Super-Twisting Algorithm, λ_1 and λ_2 are the gains to ensure the convergence of the algorithm.

III. Main Result

The problem statement to deal with, is to select, the adequate combinations of gains λ_1 and λ_2 for the STA and k_1 and k_2 for the PD controller, such that, the tracking error signal converges in finite time into a small boundary layer B_δ to be described according to the Lyapunov approach. The main result of this paper is addressed in the following theorem.

Theorem 1: Consider the nonlinear system given by (1), with the supplied control law (**??**) and the Robust Exact Differentiator given in (4), if the gains of the differentiator are selected as $\lambda_1 > 0$, $\lambda_2 > 0$ and the gains of the PD controller as $k_1 > 2l_1 + 4\Lambda (2g^-)^{-1}$ and $k_2 > 0$. Then, the error tracking trajectories (5) are confined in finite time into a boundary layer defined as $B_\delta := \frac{\eta}{\delta_1}$ with $\delta_1 = \frac{\lambda_{\min}^{1/2}(P) \lambda_{\min}(Q_1)}{\lambda_{\max}(P)}$ and $\eta = h^+ \Lambda^{-1} h^+$.

Proof: The proof of the previous theorem is given in the appendix. ■

The term η is related to the bound asked for $g(x)$ and it is responsible to drive the error (5) only into a boundary layer. If it is desired to drive (5) in finite time to zero, a second control law with a discontinuous term can be proposed. The next Corollary describes this modification.

Corollary 1: If the control input in (**??**) is modified to $u = -k_1 e_t - k_2 \dot{\gamma} - k_3 sign(e)$. Then, the error tracking trajectories (5) converge in finite time to zero and reaches that value at most after $T^* = \frac{2V^{1/2}(x_0)}{\delta_1}$ units of time.

Proof: The proof of this Corollary is given in the appendix. ■

IV. Numerical Results

As an illustration of the results presented in this paper, the PD control is applied in a nonlinear pendulum system. The performance of the PD controller and the modified PD controller are compared. Consider a pendulum which state model is given by

$$\begin{array}{c} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{J} u - \frac{MgL}{2J} \sin(x_1) - \frac{V_s}{J} x_2 + \xi_t \\ y = x_1 \end{array} \tag{8}$$

where $x_1 = \theta$ is the angle of oscillation, $x_2 = \dot{\theta}$ is the angular velocity, M is the pendulum mass, g is the gravitational force, L is the pendulum length, J is the arm inertia, V_s is the pendulum viscous friction coefficient and ξ_t is a bounded perturbation. the initial conditions were chosen as $x_1(0) = -1$ and $x_2(0) = 3$ for the model. The following numeric values were applied to simulate the pendulum model: $m_1 = 1.1kg$, $L = 1m$, $g = 9.81 \left(\frac{m}{s^2}\right)$ and $V_s = 0.18 \frac{kg \cdot m}{s^2}$. For simulation proposes the bounded perturbation is expressed as $\xi_t = 0.5 \sin(2t) + 0.5 \cos(5t)$ and the gains for the STA were chosen as $\lambda_1 = 10$ and $\lambda_" = 7$. For the PD controller $k_1 = 20$ and $k_2 = 23$. For the case when the discontinuous term is included $k_3 = 10$. In Fig. 1, the derivative of the signal error is depicted. It can be observer how the estimation process is faster when the STA is applied. In Fig. 2 the control signals are depicted. The chattering effect of the FSM control can be appreciated. To implement this modified controller, several actions can be used in order to reduce the high frequency oscillation. For instance, the concept of equivalent control have been studied in [10]. Also, the substitution of the signum function by a saturation one could be another option to reduce the chattering [11]. When the PD is computed for the system, the derivative obtained by the STA brings some advantages, as it can be seen in Fig. 3a and 3b. In this picture the states are driven to zero, the difference between the PD without SM is significant. It is important to mention, that the gains in the PD are equal for the three different controllers. While the simple PD takes more than five seconds to reach a boundary layer, the PD with STA achieved his task in less than one second. The advantage of the discontinuous injection is more appreciated if a zoom is performed into the figures. The plots in Fig. 3c and 3d are the enlarged version of 3a and 3b. In this plots, the controllers that not have discontinuous injections only can performance practical stability, that means, the error remains into a boundary layer; whereas the control injection with discontinuous part converges in finite-time to zero.

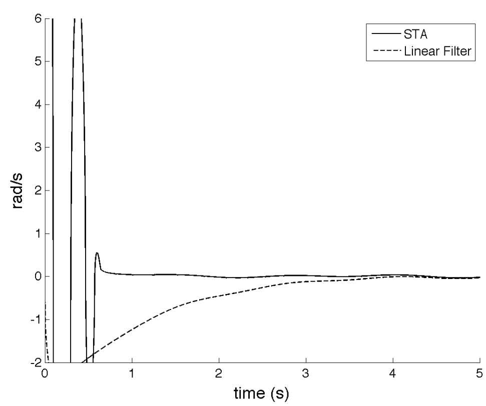

Figure 1. Error derivative estimation

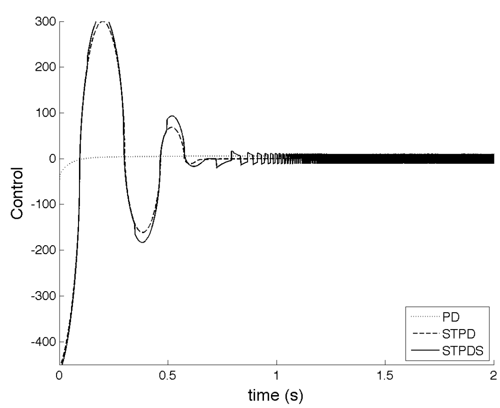

Figure 2. Control Signal

V. Conclusion

This paper explores the properties of a classical PD controller when the derivative is obtained by means of the STA. In the simulation results, the performance exhibited by this approach is better than the classical PD with the derivative obtained by a simple linear filter. Due to the presence of some kind of perturbations in the bounds asked for the nonlinear functions $f(x)$ and $g(x)$, the PD controller can only obtain finite time convergence into a boundary layer. The smaller the perturbations are, the smaller the zone of convergence is. To overcome this disadvantage, a second controller is applied with a discontinuous term. With this slightly modification finite time convergence to zero could be obtained.

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VI. Appendix

Proof: [Proof of Theorem 1] Consider the following candidate Lyapunov function [6] $V(\xi) = \xi^\top P \xi$ with $\xi := \begin{bmatrix} |\gamma|^{1/2} sign(\gamma) & \sigma & e_t \end{bmatrix}^\top$, and $P := \frac{1}{2}\begin{bmatrix} 4\lambda_2 + \lambda_1^2 & -\lambda_1 & 0 \\ -\lambda_1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Note that $V(x)$ is continuous but not differentiable in $\gamma = 0$. The ideas given in [6] are used here to avoid the point $\gamma = 0$. Therefore the proximal subgradient (PS), instead of the classical gradient is used [12]. Taking the derivative (by means of the PS) one gets

$$\dot{V} = 2\xi^\top P_1 \dot{\xi} \tag{9}$$

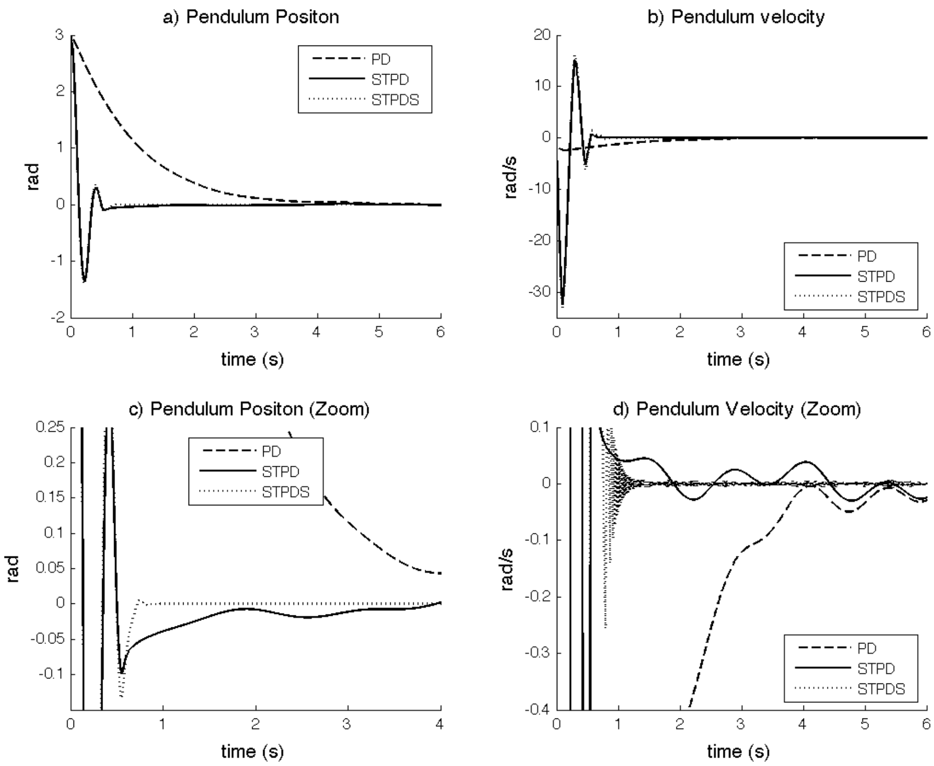

Figure 3. System Trajectories

with $\dot{\xi} := \begin{bmatrix} \frac{1}{2}|\sigma|^{-1/2}\dot{\gamma} & \dot{\sigma} & \dot{e}_t \end{bmatrix}^\top$. Taking the values of (7) in (9) and applied the so-called lambda inequality as $e_t h^+ p_3 \leq e_t p_3 \Lambda p_3 e_t + h^+ \Lambda^{-1} h^+$ it can be shown that $\dot{V}$ becomes $\dot{V}(x) \leq -|\gamma|^{-1/2} \xi_1^\top Q_1 \xi_1 + \xi^\top Q_2 \xi + \eta$ with $\eta := h^+ \Lambda^{-1} h^+$ and $\xi_1 := \begin{bmatrix} |\gamma|^{1/2} sign(\gamma) & \sigma \end{bmatrix}^\top$ and

$$Q_1 := \frac{\lambda_1}{2}\begin{bmatrix} 2\lambda_2 + \lambda_1^2 & -\lambda_1 \\ -\lambda_1 & 1 \end{bmatrix}$$

$$Q_2 := \begin{bmatrix} 0 & 0 & \frac{1}{2}p_3 g^+ k_2 \lambda_1 \\ 0 & 0 & -\frac{1}{2}p_3 g^+ k_2 \\ \frac{1}{2}p_3 g^+ k_2 \lambda_1 & \frac{1}{2}p_3 g^+ k_2 \lambda_1 & -2l_1 - 4\Lambda + g^- k_1 2 \end{bmatrix} \tag{10}$$

then Q_1 is positive definite if $\lambda_1, \lambda_2 > 0$ whereas Q_2 is positive semidefinite if $k_1 > 2l_1 + 4\Lambda(2g^-)^{-1}$. Note that $V(x)$ is positive definite and radially unbounded if $\lambda_1, \lambda_2 > 0$, that means

$$\lambda_{\min}(P)\|\xi\|^2 \leq V(x) \leq \lambda_{\max}(P)\|\xi\|^2 \tag{11}$$

Since $V(x) \leq -|\gamma|^{-1/2}\lambda_{\min}(Q_1)\|\xi\|^2 - \lambda_{\min}(Q_2)\|\xi\| + \eta$. Using (11) and the fact $|\gamma|^{1/2} \leq \|\xi\|^2 \leq \frac{V^{1/2}(x)}{\lambda_{\min}^{1/2}(P)}$it follows that

$$V(x) \leq \delta_1 V^{1/2}(x) - \delta_2 V(x) + \eta \tag{12}$$

with

$$\delta_1 = \frac{\lambda_{\min}^{1/2}(P)\lambda_{\min}(Q_1)}{\lambda_{\max}(P)} \qquad \delta_2 = \frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)} \tag{13}$$

By the comparison lemma [11] and the theorem given in [13] the error trajectories converge in finite time to a boundary layer delimited by $B_\delta := \frac{\eta}{\delta_1}$ and this concludes the proof. ■

Proof: [Proof of Corollary 1] Following the same methodology, Q_1 has the same values as in equation (10) and

$$Q_2 := \begin{bmatrix} 0 & 0 & \frac{1}{2}p_3 g^+ k_2 \lambda_1 \\ 0 & 0 & -\frac{1}{2}p_3 g^+ k_2 \\ \frac{1}{2}p_3 g^+ k_2 \lambda_1 & \frac{1}{2}p_3 g^+ k_2 \lambda_1 & -2l_1 + g^- k_3 2 \end{bmatrix}$$

then if $k_1 = l_1 (g^-)^{-1}$ and $k_3 = h^+ (g^+)^{-1}$ then equation (12) turns into $V(x) \leq \delta_1 V^{1/2}(x) - \delta_2 V(x)$ with δ_1 and δ_2 as in (13). By the comparison Lemma it follows easily that $V(x(t))$, and therefore $x(t)$ converges to zero in finite time and reaches that value at most after $T^* = \dfrac{2V^{1/2}(x_0)}{\delta_1}$ units of time. ■

VII. Acknowledgments

The authors would like to thank the Instituto Politécnico Nacional (Secretaría Académica, COFAA, SIP, and CIC), the CONACyT, the SNI, and the ICyTDF (grants PIUTE10-77 and PICSO10-85) for their economical support to develop this work.