State Estimation for Static Neural Networks With Time-Varying Delays Based on an Improved Reciprocally Convex Inequality

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***Abstract*— This brief is concerned with the problem of neural state estimation for static neural networks with time-varying delays. Notice that a Luenberger estimator can produce an estimation error irrespective of the neuron state trajectory. This brief provides a method for designing such an estimator for static neural networks with time-varying delays. First, in-depth analysis on a well-used reciprocally convex approach is made, leading to an *improved* reciprocally convex inequality. Second, the improved reciprocally convex inequality and some integral inequalities are employed to provide a tight upper bound on the time-derivative of some Lyapunov–Krasovskii functional. As a result, a novel bounded real lemma (BRL) for the resultant error system is derived. Third, the BRL is applied to present a method for designing suitable Luenberger estimators in terms of solutions of linear matrix inequalities with two tuning parameters. Finally, it is shown through a numerical example that the proposed method can derive less conservative results than some existing ones.**

***Index Terms*— Luenberger estimators, state estimation, static neural networks, time-varying delays.**

I. Introduction

Since more applications have been found in a wider range of areas, neural networks have gained increasing attention in the past decades, and a great number of results have been reported in the literature (see [1]–[5] and the references therein). Depending on the choice of either external or internal states of neurons as basic variables [4], a neural network is usually cast as either a static neural network model or a local field neural network model. Recalling existing results on neural networks, most of them focus on local field neural networks [5]–[7], while relatively few of them on static neural networks [8]–[10]. It is true that a static neural network can be equivalently transformed into a local field neural network under some assumption. However, this assumption may not be always satisfied for static neural networks [4].

Neuron state estimation is a fundamental issue when neural networks are applied in areas, such as static imagine processing, pattern recognition, and combinatorial optimization. In fact, a number of applications of neural networks are dependent closely on neuron states. Nevertheless, in a relatively large-scale neural network, it is often that only partial information on neuron states is available in network outputs [11], especially when external disturbance is imposed on the neural network involved. Thus, it is important to estimate the neuron states based on available measurements, such that some practical performance can be achieved for the successful applications of neural networks. Note that Luenberger estimators can produce an estimation error, which depends only on the neuron initial states rather than on the system trajectory [12]. Thus, Luenberger estimators are widely used in the practical estimation applications, e.g., to a class of artificial neural networks with stochastically corrupted measurements under Round-Robin protocol [13], and to discrete-time neural networks with randomly occurring quantization and missing measurements [14].

Due to a finite switching speed of amplifiers, time delays usually occur in the interaction between neurons when a neural network is implemented by large-scale integrated electronic circuits [3], [5]. The induced time delays may be constant, time-varying or stochastic, and they are commonly regarded as main factors responsible for performance degradation or instability of the neural network under consideration. For neural networks with time-varying delays, a large number of results focus on stability issues, while *relatively few* results focus on state estimation. State estimation for neural networks with time-varying delays is first investigated in [11], where some sufficient conditions on the existence of Luenberger estimators are presented. However, those conditions are irrespective of information of the delay size, and thus are of somewhat conservatism especially when the time delay is very small [15]. Since then, in order to derive less conservative criteria, delay-dependent state estimation issues come to the fore and a number of results are derived by employing a Lyapunov–Krasovskii functional method [16]–[18]. Among the results mentioned earlier, a reciprocally convex inequality [19] plays an important role in bounding the reciprocally convex combination appearing in the time-derivative of the Lyapunov–Krasovskii functional, leading to some less conservative criteria with lower complexity.

For state estimation on static neural networks with time-varying delays, a lot of effort has been made during the last decade. Based on a Lyapunov–Krasovskii functional method, together with some integral inequalities and the reciprocally convex inequality, delay-dependent sufficient conditions on the existence of suitable Luenberger estimators are derived, such that the resultant estimation error system achieves a prescribed disturbance attenuation level [16]. Recently, those sufficient conditions are improved [18] by introducing a specific Luenberger estimator by which the measurement error signal is not only used as an input of the estimator but also used to adjust the parameters in the activation function of the neural network under consideration. Although the results in [18] are of less conservatism than some existing ones, there still remains some room for improvement. On the one hand, the estimation of the time-derivative of the Lyapunov–Krasovskii functional is not tight enough since the Jensen inequality and the reciprocally convex approach are used; On the other hand, when designing a desired Luenberger estimator, an extra constraint is imposed on the activation functions. How to derive a less conservative delay-dependent condition to design desired Luenberger estimators is the main motivation of the study.

This brief deals with the problem of H_∞ state estimation for static neural networks with time-varying delay. First, a reciprocally convex inequality [19] is improved. The improved reciprocally convex inequality and some newly integral inequalities are used to provide a tight upper bound for the time derivative of the Lyapunov–Krasovskii functional, which leads to a less conservative delay-dependent bounded real lemma (BRL) for the resultant estimation error system. Second, a sufficient condition on the existence of

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suitable Lueberger estimators is established in terms of linear matrix inequalities (LMIs) with two tuning parameters. It is shown through a numerical example that the proposed method can deliver less conservative results than some existing ones.

Notation: The notations used in this brief are quite standard. For simplicity, He$\{A\}$ stands for $A + A^T$.

II. System Description

Consider the following static neural network with a time-varying delay and noise disturbance described by:

$$\begin{cases} \dot{u}(t) = -Au(t) + g(\pi(t)) + B_1 w(t) \\ \pi(t) = Wu(t-d(t)) + J \\ y(t) = C_0 u(t) + C_1 u(t-d(t)) + B_2 w(t) \\ z(t) = L_0 u(t) + L_1 u(t-d(t)) \\ u(\theta) = \phi(\theta), \quad \theta \in [-h,\ 0] \end{cases} \tag{1}$$

where $u = \mathrm{col}\{u_1, u_2, \dots, u_n\} \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the neuron state vector and the network measurement vector, respectively, $z \in \mathbb{R}^p$ is the vector to be estimated, $w(t) \in \mathbb{R}^q$ is the noise disturbance vector belonging to $\mathcal{L}_2[0,\infty)$, A, W, B_1, B_2, C_0, C_1, L_0, and L_1 are real matrices with compatible dimensions, and $g(u) = \mathrm{col}\{g_1(u_1), g_2(u_2), \dots, g_n(u_n)\}$ is the neuron activation function satisfying for $i = 1, 2, \dots, n$

$$0 \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le \ell_i \quad \forall s_1, s_2 \in \mathbb{R},\ s_1 \ne s_2 \tag{2}$$

where ℓ_i $(i = 1, 2, \dots, n)$ are known real scalars. $J = \mathrm{col}\{J_1, J_2, \dots, J_n\}$ is the external input vector and $\phi(\theta)$ is the initial condition function defined on $[-h, 0]$. The delay $d(t)$ is time varying and satisfies

$$\begin{cases} 0 \le d(t) \le h < \infty \\ -\infty \le \mu_1 \le \dot{d}(t) \le \mu_2 < \infty, \end{cases} \quad \forall t \in [0, \infty) \tag{3}$$

where h, μ_1 and μ_2 are real constants. In this brief, we are interested in seeking the following Luenberger estimator:

$$\begin{cases} \dot{\hat{u}}(t) = -A\hat{u}(t) + g(\hat{\pi}(t)) + K_1(y(t) - \hat{y}(t)) \\ \hat{\pi}(t) = W\hat{u}(t-d(t)) + J + K_2(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C_0\hat{u}(t) + C_1\hat{u}(t-d(t)) \\ \hat{z}(t) = L_0\hat{u}(t) + L_1\hat{u}(t-d(t)) \\ \hat{u}(\theta) = 0, \quad \theta \in [-h, 0] \end{cases} \tag{4}$$

where K_1 and K_2 are gain matrices to be determined.

Remark 1: One characteristic of the estimator (4) is that the measurement error signal $y(t) - \hat{y}(t)$ is not only used as an input of the estimator but also is used to adjust its activation function parameters. As a nonlinear observer, it is shown [20] that the Luenberger observer of the form (4) can effectively drive the observer error to zero without imposing a global Lipschitz restriction on nonlinearity. This kind of Luenberger estimator is also used to estimate system states for chaotic systems [21] and for delayed static neural networks [18].

Denote $x(t) := u(t) - \hat{u}(t)$ and $\tilde{e}(t) := z(t) - \hat{z}(t)$. Then

$$\begin{aligned} \tilde{\pi}(t) &:= \pi(t) - \hat{\pi}(t) \\ &= -K_2C_0x(t) + (W - K_2C_1)x(t-d(t)) - K_2B_2w(t) \end{aligned}$$

and the resultant estimation error system associated with (1) and (4) is given by

$$\begin{cases} \dot{x}(t) = -(A + K_1C_0)x(t) - K_1C_1x(t-d(t)) \\ \qquad\quad + f(\tilde{\pi}(t)) + (B_1 - K_1B_2)w(t) \\ \tilde{e}(t) = L_0x(t) + L_1x(t-d(t)) \\ x(\theta) = \phi(\theta), \quad \theta \in [-h, 0] \end{cases} \tag{5}$$

where $f(\tilde{\pi}(t)) := g(\pi(t)) - g(\hat{\pi}(t))$. It is easy to verify that the nonlinear functions $f_i(\tilde{\pi}_i(t))$ $(i = 1, 2, \dots, n)$ satisfy the following sector-bounded conditions:

$$f_i(\tilde{\pi}_i(t))[f_i(\tilde{\pi}_i(t)) - \ell_i\tilde{\pi}_i(t)] \le 0. \tag{6}$$

Thus, for any $T = \mathrm{diag}\{t_1, t_2, \dots, t_n\}$ with $t_i \ge 0$ $(i = 1, 2, \dots, n)$, the following is true:

$$0 \le -2f^T(\tilde{\pi}(t))T[f(\tilde{\pi}(t)) - \Lambda\tilde{\pi}(t)] \tag{7}$$

where $\Lambda := \mathrm{diag}\{\ell_1, \ell_2, \dots, \ell_n\}$. The problem to be addressed in this brief can be described as

Problem of an H_∞ State Estimation: For a prescribed $\gamma > 0$, design a suitable state estimator of the form (4) such that the resultant error system (5) with $w(t) \equiv 0$ is globally asymptotically stable, and the inequality $\|\tilde{e}(t)\|_2 < \gamma\|w(t)\|_2$ holds under zero initial conditions.

To end this section, we introduce the following result.

Lemma 1 [22]: Let R and Z be $n \times n$ constant real matrices satisfying $R = R^T > 0$ and $Z = Z^T > 0$, and $\omega : [a, b] \to \mathbb{R}^n$ a vector function, such that the integrations below are well defined, where a and b are two scalars with $b > a$. Then, the following two inequalities hold:

$$\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \ge \frac{1}{b-a}\zeta^T(\omega, a, b)\Gamma_1^T\mathcal{R}\Gamma_1\zeta(\omega, a, b) \tag{8}$$

$$\int_a^b (s-a)\dot{\omega}^T(s)Z\dot{\omega}(s)ds \ge \zeta^T(\omega, a, b)\Gamma_2^T\tilde{Z}\Gamma_2\zeta(\omega, a, b) \tag{9}$$

where $\mathcal{R} = \mathrm{diag}\{R, 3R, 5R\}$, $\tilde{Z} = \mathrm{diag}\{2Z, 4Z\}$ and

$$\begin{aligned} \zeta(\omega, a, b) := \mathrm{col}\Bigg\{&\omega(a), \omega(b), \frac{1}{b-a}\int_a^b \omega(s)ds, \\ &\frac{1}{(b-a)^2}\int_a^b (b-s)\omega(s)ds\Bigg\} \end{aligned} \tag{10}$$

$$\Gamma_1 := \begin{bmatrix} -I & I & 0 & 0 \\ I & I & -2I & 0 \\ -I & I & -6I & 12I \end{bmatrix}, \quad \Gamma_2 := \begin{bmatrix} 0 & I & -I & 0 \\ 0 & I & -4I & 6I \end{bmatrix}. \tag{11}$$

III. Improved Reciprocally Convex Inequality

First, we rewrite a reciprocally convex inequality [19] in the following lemma.

Lemma 2: Let $\mathcal{R}_i \in \mathbb{R}^{m\times m}$ $(\mathcal{R}_i = \mathcal{R}_i^T > 0)$ and $\varpi_i \in \mathbb{R}^m$ $(i = 1, 2)$ and a scalar $\alpha \in (0, 1)$. If there exist real matrices $S \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} \mathcal{R}_1 & S \\ S^T & \mathcal{R}_2 \end{bmatrix} \ge 0 \tag{12}$$

the following inequality holds:

$$f(\alpha) := \frac{1}{\alpha}\varpi_1^T\mathcal{R}_1\varpi_1 + \frac{1}{1-\alpha}\varpi_2^T\mathcal{R}_2\varpi_2 \tag{13}$$

$$\ge\ \varpi_1^T\mathcal{R}_1\varpi_1 + 2\varpi_1^TS\varpi_2 + \varpi_2^T\mathcal{R}_2\varpi_2. \tag{14}$$

Then, we provide an improved inequality over Lemma 2.

Theorem 1: Let $\mathcal{R}_i \in \mathbb{R}^{m\times m}$ $(\mathcal{R}_i = \mathcal{R}_i^T > 0)$ and $\varpi_i \in \mathbb{R}^m$ $(i = 1, 2)$ and a scalar $\alpha \in (0, 1)$. If there exist real matrices M, N, $S \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} \mathcal{R}_1 - M & S \\ S^T & \mathcal{R}_2 \end{bmatrix} \ge 0, \quad \begin{bmatrix} \mathcal{R}_1 & S \\ S^T & \mathcal{R}_2 - N \end{bmatrix} \ge 0 \tag{15}$$

for $f(\alpha)$ defined in (13), then we have

$$f(\alpha) \geq g(M,N) \triangleq \varpi_1^T[\mathcal{R}_1 + (1-\alpha)M]\varpi_1 + 2\varpi_1^T S\varpi_2 + \varpi_2^T(\mathcal{R}_2 + \alpha N)\varpi_2. \tag{16}$$

Proof: Note that

$$f(\alpha) - g(M,N) = \begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix}^T \Phi \begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix}$$

where

$$\Phi := \begin{bmatrix} \frac{1-\alpha}{\alpha}\mathcal{R}_1 - (1-\alpha)M & -S \\ -S^T & \frac{\alpha}{1-\alpha}\mathcal{R}_2 - \alpha N\end{bmatrix}.$$

Thus, we only need to prove that $\Phi \geq 0$ holds if the matrix inequalities in (15) are satisfied. In fact, denoting $J := \mathrm{diag}\{-[\alpha/1-\alpha]^{1/2}I,\ [1-\alpha/\alpha]^{1/2}I\}$, we have

$$J^T\Phi J = \alpha\begin{bmatrix}\mathcal{R}_1 - M & S\\ S^T & \mathcal{R}_2\end{bmatrix} + (1-\alpha)\begin{bmatrix}\mathcal{R}_1 & S\\ S^T & \mathcal{R}_2 - N\end{bmatrix} \geq 0$$

which yields $\Phi \geq 0$. □

Remark 2: Compared with the reciprocally convex approach in Lemma 2, it is clear that the constraints in (15) are *weaker* than the one in (12). In fact, if (12) is satisfied, there exist sufficiently small matrices M and N (in the sense of the absolute value of its eigenvalues), such that (15) is satisfied. However, the reverse is not necessarily true, since M and N in (15) may be not semipositive definite.

We now consider an optimal problem described by

$$\text{OP1: } \max\{g(M,N) | M, N \in \mathbb{R}^{m\times m} \text{ satisfy (15)}\}.$$

After insight analysis into OP1, we obtain the following result.

Theorem 2: The solution to OP1 is given by $(M,N) = (M_{\mathrm{opt}}, N_{\mathrm{opt}})$, where

$$M_{\mathrm{opt}} = \mathcal{R}_1 - S\mathcal{R}_2^{-1}S^T, \quad N_{\mathrm{opt}} = \mathcal{R}_2 - S^T\mathcal{R}_1^{-1}S. \tag{17}$$

Proof: First, it is clear to see that $(M_{\mathrm{opt}}, N_{\mathrm{opt}})$ given in (17) satisfies (15). Then, due to $\mathcal{R}_1 > 0$ and $\mathcal{R}_2 > 0$, the matrix inequalities in (15) are equivalent to, respectively

$$\mathcal{R}_1 - M - S\mathcal{R}_2^{-1}S^T \geq 0, \quad \mathcal{R}_2 - N - S^T\mathcal{R}_1^{-1}S \geq 0$$

which follows that $M \leq M_{\mathrm{opt}}$ and $N \leq N_{\mathrm{opt}}$. Hence, one obtains that $g(M_{\mathrm{opt}}, N_{\mathrm{opt}}) \geq g(M,N)$. □

From Theorem 2, we immediately arrive at

Theorem 3: Let $\mathcal{R}_i \in \mathbb{R}^{m\times m}$ ($\mathcal{R}_i = \mathcal{R}_i^T > 0$) and $\varpi_i \in \mathbb{R}^m$ ($i = 1,2$) and a scalar $\alpha \in (0,1)$. Then, for any $S \in \mathbb{R}^{m\times m}$, the following inequality holds:

$$f(\alpha) \geq \varpi_1^T\big[\mathcal{R}_1 + (1-\alpha)\big(\mathcal{R}_1 - S\mathcal{R}_2^{-1}S^T\big)\big]\varpi_1 + 2\varpi_1^T S\varpi_2 + \varpi_2^T\big(\mathcal{R}_2 + \alpha\big(\mathcal{R}_2 - S^T\mathcal{R}_1^{-1}S\big)\big)\varpi_2. \tag{18}$$

Remark 3: The *significance* of Theorem 3 lies in two aspects: 1) the matrix constrain (12) on S is removed from (18) and 2) if the matrix S satisfies the constrain (12), the obtained lower bound $g(M_{\mathrm{opt}}, N_{\mathrm{opt}})$ for $f(\alpha)$ is *greater* than $g(0,0) = \varpi_1^T\mathcal{R}_1\varpi_1 + 2\varpi_1^T S\varpi_2 + \varpi_2^T\mathcal{R}_2\varpi_2$, which is derived by the reciprocally convex approach [19]. In this sense, the inequality (18) is referred to as *an improved reciprocally convex inequality*, which can be used to the dissipavity analysis and $l_2 - l_\infty$ filtering [23]–[25] for delayed systems.

IV. H_∞ Performance Analysis and Design of State Estimators

In this section, we first present a novel BRL for the system (5) using the inequality (18). Then, this BRL is employed to design suitable neuron state estimators. In doing so, choose the following augmented Lyapunov–Krasovskii functional candidate as:

$$V(t,x_t) := V_1(t,x_t) + V_2(t,x_t) + V_3(t,x_t) \tag{19}$$

where $V_1(t,x_t) := \eta_1^T(t)P\eta_1(t)$ and

$$\begin{aligned}
\eta_1(t) &:= \mathrm{col}\{x(t), x(t-d(t)), x(t-h), (h-d(t))\rho_1(t), d(t)\rho_2(t), (h-d(t))\rho_3(t), d(t)\rho_4(t)\}\\
V_2(t,x_t) &:= \int_{t-d(t)}^{t}\eta_2^T(s)Q_1\eta_2(s)ds + \int_{t-h}^{t-d(t)}\eta_2^T(s)Q_2\eta_2(s)ds\\
V_3(t,x_t) &:= \int_{t-h}^{t}\dot{x}^T(s)[h\sigma(s)R + \sigma^2(s)Z]\dot{x}(s)ds
\end{aligned} \tag{20}$$

with $\eta_2(t) := \mathrm{col}\{\dot{x}(s), x(s)\}$ and $\sigma(s) := h - t + s$, and

$$\begin{aligned}
\rho_1(t) &:= \frac{1}{h-d(t)}\int_{t-h}^{t-d(t)} x(s)ds, \quad \rho_2(t) := \frac{1}{d(t)}\int_{t-d(t)}^{t} x(s)ds\\
\rho_3(t) &:= \frac{1}{(h-d(t))^2}\int_{t-h}^{t-d(t)}(t-d(t)-s)x(s)ds\\
\rho_4(t) &:= \frac{1}{d^2(t)}\int_{t-d(t)}^{t}(t-s)x(s)ds.
\end{aligned}$$

Now, we state and establish the following result.

Proposition 1: For given scalars $\gamma > 0$, $h > 0$, μ_1 and μ_2 satisfying (3), the system (5) subject to (7) is globally asymptotically stable with a prescribed level γ if there exist real matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, $Z > 0$, $T = \mathrm{diag}\{t_1, t_2, \ldots, t_n\} > 0$, X_1, X_2, X_3, and S with appropriate dimensions such that

$$\begin{bmatrix}
\Upsilon(h,d)|_{d\in\{\mu_1,\mu_2\}} & \Phi_1^T & \Phi_2 & \mathscr{C}_7^T & \mathscr{C}_5^T\Gamma_1^T S^T\\
\Phi_1 & -2T & \Phi_3 & 0 & 0\\
\Phi_2^T & \Phi_3^T & -\gamma^2 I & 0 & 0\\
\mathscr{C}_7 & 0 & 0 & -I & 0\\
S\Gamma_1\mathscr{C}_5 & 0 & 0 & 0 & -\mathcal{R} - 2\mathcal{Z}
\end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix}
\Upsilon(0,d)|_{d\in\{\mu_1,\mu_2\}} & \Phi_1^T & \Phi_2 & \mathscr{C}_7^T & \mathscr{C}_4^T\Gamma_1^T S\\
\Phi_1 & -2T & \Phi_3 & 0 & 0\\
\Phi_2^T & \Phi_3^T & -\gamma^2 I & 0 & 0\\
\mathscr{C}_7 & 0 & 0 & -I & 0\\
S^T\Gamma_1\mathscr{C}_4 & 0 & 0 & 0 & -\mathcal{R}
\end{bmatrix} < 0 \tag{22}$$

where $\mathcal{R} := \mathrm{diag}\{R, 3R, 5R\}$, $\mathcal{Z} := \mathrm{diag}\{Z, 3Z, 5Z\}$, e_i is the ith $n \times 10n$ row-block vector of the $10n \times 10n$ identity matrix ($i = 1, \ldots, 10$), and

$$\Phi_1 := \tilde{X} + T\Lambda\mathscr{C}_6, \quad \Phi_2 := \tilde{X}^T(B_1 - K_1B_2) \tag{23}$$

$$\Phi_3 := -T\Lambda K_2B_2, \quad \tilde{X} := X_1e_{10} + X_2e_1 + X_3e_2 \tag{24}$$

$$\Upsilon(d(t),\dot{d}(t)) := \Upsilon_1(d(t,\dot{d}(t)) - \Upsilon_2(d(t)) \tag{25}$$

$$\begin{aligned}
\Upsilon_1(d(t),\dot{d}(t)) &:= \mathrm{He}\{\mathscr{D}_1^T P\mathscr{D}_2 + \tilde{X}^T\mathscr{C}_0\} + h^2 e_{10}^T(R+Z)e_{10} + \mathscr{C}_1^T Q_1\mathscr{C}_1\\
&\quad + (1-\dot{d}(t))\mathscr{C}_3^T(Q_2 - Q_1)\mathscr{C}_3 - \mathscr{C}_2^T Q_2\mathscr{C}_2
\end{aligned} \tag{26}$$

$$\begin{aligned}
\Upsilon_2(d(t)) &:= \mathscr{C}_4^T\{\Gamma_1^T[\mathcal{R} + (1-\beta)(\mathcal{R} + 2\mathcal{Z})]\Gamma_1 + 2\Gamma_2^T\tilde{Z}\Gamma_2\}\mathscr{C}_4\\
&\quad + \mathscr{C}_5^T[\Gamma_2^T\tilde{Z}\Gamma_2 + (1+\beta)\Gamma_1^T\mathcal{R}\Gamma_1]\mathscr{C}_5 + \mathrm{He}\{\mathscr{C}_4^T\Gamma_1^T S\Gamma_1\mathscr{C}_5\}
\end{aligned} \tag{27}$$

$$\mathscr{C}_0 := -(A + K_1C_0)e_1 - K_1C_1e_2 - e_{10}$$

$$\mathscr{C}_1 := \mathrm{col}\{e_{10}, e_1\}, \quad \mathscr{C}_2 := \mathrm{col}\{e_9, e_3\}, \quad \mathscr{C}_3 := \mathrm{col}\{e_8, e_2\}$$

$$
\begin{aligned}
\mathscr{C}_4 &:= \mathrm{col}\{e_2, e_1, e_5, e_7\}, \quad \mathscr{C}_5 := \mathrm{col}\{e_3, e_2, e_4, e_6\}\\
\mathscr{C}_6 &:= -K_2C_0e_1 + (W - K_2C_1)e_2, \quad \mathscr{C}_7 := L_0e_1 + L_1e_2\\
\mathcal{D}_1 &:= \mathrm{col}\{e_1, e_2, e_3, (h-d(t))e_4, d(t)e_5, (h-d(t))e_6, d(t)e_7\}\\
\mathcal{D}_2 &:= \mathrm{col}\{e_{10}, (1-\dot d(t))e_8, e_9, (1-\dot d(t))e_2 - e_3,\\
&\qquad e_1 - (1-\dot d(t))e_2, (1-\dot d(t))e_4 + \dot d(t)e_6 - e_3,\\
&\qquad e_5 - (1-\dot d(t))e_2 - \dot d(t)e_7\} \qquad (28)
\end{aligned}
$$

with $\tilde Z = \mathrm{diag}\{2Z, 4Z\}$, Γ_1 and Γ_2 defined in (11), $\beta = d(t)/h$.

Proof: Taking the time-derivative of the Lyapunov–Krasovskii functional (19) along with the trajectory of the system (5) yields

$$\dot V_1(t, x_t) = 2\eta_1^T(t)P\dot\eta_1(t) \qquad (29)$$

$$
\begin{aligned}
\dot V_2(t, x_t) &= \eta_2^T(t)Q_1\eta_2(t) - \eta_2^T(t-h)Q_2\eta_2(t-h)\\
&\quad + (1-\dot d(t))\eta_2^T(t-d(t))(Q_2 - Q_1)\eta_2(t-d(t)) \qquad (30)
\end{aligned}
$$

$$\dot V_3(t, x_t) = h^2\dot x^T(t)(R+Z)\dot x(t) - \mathscr{I}(t) - 2\wp(t) \qquad (31)$$

where

$$\mathscr{I}(t) := h\int_{t-h}^{t}\dot x^T(s)R\dot x(s)ds \qquad (32)$$

$$\wp(t) := \int_{t-h}^{t}(h-t+s)\dot x^T(s)Z\dot x(s)ds. \qquad (33)$$

Denote $\xi(t) := \mathrm{col}\{x(t), x(t-d(t)), x(t-h), \rho_1(t), \rho_2(t), \rho_3(t), \rho_4(t), \dot x(t-d(t)), \dot x(t-h), \dot x(t)\}$. Then, the first equation in (5) can be rewritten as

$$0 = \mathscr{C}_0\xi(t) + f(u(t), \hat u(t)) + B_Kw(t) \qquad (34)$$

where $B_K := B_1 - K_1B_2$, and $\mathscr{C}_0$ is defined in (28). For any real matrices $X_j \in \mathbb{R}^{n\times n}$ $(j = 1, 2, 3)$, it is true that

$$2\xi^T(t)\tilde X^T[\mathscr{C}_0\xi(t) + f(u(t), \hat u(t)) + B_Kw(t)] = 0 \qquad (35)$$

where $\tilde X$ is defined in (24). From (29)–(31) and (35), one gets

$$
\begin{aligned}
\dot V(t, x_t) &= \xi^T(t)\Upsilon_1(d(t), \dot d(t))\xi(t) - \mathscr{I}(t) - 2\wp(t)\\
&\quad + 2\xi^T(t)\tilde X^T[f(u(t), \hat u(t)) + B_Kw(t)] \qquad (36)
\end{aligned}
$$

where $\Upsilon_1(d(t), \dot d(t))$ is defined in (26). Now, we estimate the integral term $\mathscr{I}(t) + 2\wp(t)$. Note that $\mathscr{I}(t) = \mathscr{I}_1(t) + \mathscr{I}_2(t)$ and $\wp(t) = \wp_1(t) + \wp_2(t) + \wp_3(t)$, where

$$
\begin{aligned}
\mathscr{I}_1(t) &:= h\int_{t-d(t)}^{t}\dot x^T(s)R\dot x(s)ds\\
\mathscr{I}_2(t) &:= h\int_{t-h}^{t-d(t)}\dot x^T(s)R\dot x(s)ds\\
\wp_1(t) &:= \int_{t-d(t)}^{t}(d(t)-t+s)\dot x^T(s)Z\dot x(s)ds\\
\wp_2(t) &:= (h-d(t))\int_{t-d(t)}^{t}\dot x^T(s)Z\dot x(s)ds\\
\wp_3(t) &:= \int_{t-h}^{t-d(t)}(h-t+s)\dot x^T(s)Z\dot x(s)ds.
\end{aligned}
$$

Apply Lemma 1 to obtain $\mathscr{I}_1(t) \ge \frac{1}{\beta}\zeta_1^T(t)\Gamma_1^T\mathcal{R}\Gamma_1\zeta_1(t)$ and

$$\mathscr{I}_2(t) \ge \frac{1}{1-\beta}\zeta_2^T(t)\Gamma_1^T\mathcal{R}\Gamma_1\zeta_2(t), \quad \wp_1(t) \ge \zeta_1^T(t)\Gamma_2^T\tilde Z\Gamma_2\zeta_1(t)$$

$$\wp_2(t) \ge \left(\frac{1}{\beta} - 1\right)\zeta_1^T(t)\Gamma_1^T\mathcal{Z}\Gamma_1\zeta_1(t), \quad \wp_3(t) \ge \zeta_2^T(t)\Gamma_2^T\tilde Z\Gamma_2\zeta_2(t)$$

where Γ_1 and Γ_2 are given in (11), and

$$\zeta_1(t) := \zeta(x, t-d(t), t), \quad \zeta_2(t) := \zeta(x, t-h, t-d(t))$$

with $\zeta(\cdot,\cdot,\cdot)$ being defined in (10). Thus, one has

$$
\begin{aligned}
\mathscr{I}(t) + 2\wp(t) &\ge \mathscr{F}(\beta) + 2\zeta_1^T(t)\big(\Gamma_2^T\tilde Z\Gamma_2 - \Gamma_1^T\mathcal{Z}\Gamma_1\big)\zeta_1(t)\\
&\quad + 2\zeta_2^T(t)\Gamma_2^T\tilde Z\Gamma_2\zeta_2(t) \qquad (37)
\end{aligned}
$$

where $\mathscr{F}(\beta) := (1/\beta)\zeta_1^T(t)\Gamma_1^T(\mathcal{R} + 2\mathcal{Z})\Gamma_1\zeta_1(t) + (1/(1-\beta))\zeta_2^T(t)\Gamma_1^T\mathcal{R}\Gamma_1\zeta_2(t)$. Applying (18) with $\varpi_1 = \Gamma_1\zeta_1(t)$, $\varpi_2 = \Gamma_1\zeta_2(t)$, $\mathcal{R}_1 = \mathcal{R} + 2\mathcal{Z}$ and $\mathcal{R}_2 = \mathcal{R}$, and $\alpha = \beta$ yields

$$
\begin{aligned}
\mathscr{F}(\beta) &\ge \zeta_1^T(t)\Gamma_1^T[\mathcal{R} + 2\mathcal{Z} + (1-\beta)(\mathcal{R} + 2\mathcal{Z} - S\mathcal{R}^{-1}S^T)]\Gamma_1\zeta_1(t)\\
&\quad + \zeta_2^T(t)\Gamma_1^T[\mathcal{R} + \beta(\mathcal{R} - S^T(\mathcal{R} + 2\mathcal{Z})^{-1}S)]\Gamma_1\zeta_2(t)\\
&\quad + 2\zeta_1^T(t)\Gamma_1^TS\Gamma_1\zeta_2(t) \qquad (38)
\end{aligned}
$$

which follows from (37) that:

$$\mathscr{I}(t) + 2\wp(t) \ge \xi^T(t)[\Upsilon_2(d(t)) - \Upsilon_{21}]\xi(t) \qquad (39)$$

where $\Upsilon_2(d(t))$ is defined in (27), and

$$
\begin{aligned}
\Upsilon_{21} &:= (1-\beta)\mathscr{C}_4^T\Gamma_1^TS\mathcal{R}^{-1}S^T\Gamma_1\mathscr{C}_4\\
&\quad + \beta\mathscr{C}_5^T\Gamma_1^TS^T(\mathcal{R} + 2\mathcal{Z})^{-1}S\Gamma_1\mathscr{C}_5.
\end{aligned}
$$

Substituting (39) into (36) yields

$$
\begin{aligned}
\dot V(t, x_t) &\le \xi^T(t)[\Upsilon(d(t), \dot d(t)) + \Upsilon_{21}]\xi(t)\\
&\quad + 2\xi^T(t)\tilde X^T[f(u(t), \hat u(t)) + B_Kw(t)] \qquad (40)
\end{aligned}
$$

where $\Upsilon(d(t), \dot d(t))$ is given in (25). Noticing (7), one has

$$\dot V(t, x_t) \le \tilde\xi^T(t)\begin{bmatrix}\Upsilon(d(t), \dot d(t)) + \Upsilon_{21} & \Phi_1^T & \Phi_2\\ \Phi_1 & -2T & \Phi_3\\ \Phi_2^T & \Phi_3^T & 0\end{bmatrix}\tilde\xi(t) \qquad (41)$$

where $\tilde\xi(t) := \mathrm{col}\{\xi(t), f(u(t), \hat u(t)), w(t)\}$ and Φ_1, Φ_2 and Φ_3 are given in (23) and (24), respectively. The rest of the proof can be completed similar to that in [18] or [26], and it is omitted due to page limitation. □

Remark 4: Proposition 1 presents a BRL for the resultant error system (5). The key point of the proof is how to deal with the integral terms $\mathscr{I}(t)$ in (32) and $\wp(t)$ in (33). The integral term $\wp(t)$ is rewritten as the sum of three terms, and each of them is bounded using newly integral inequalities in Lemma 1. Moreover, $\mathscr{I}(t) + 2\wp(t)$, rather than $\mathscr{I}(t)$ in some existing literature, is bounded by a reciprocally convex combination $\mathscr{F}(\beta)$ in (37). Finally, an improved reciprocally convex inequality (18) is applied to bound $\mathscr{F}(\beta)$ by a convex combination on the same variable β, which can be seen from (38).

In the sequel, based on Proposition 1, we present a sufficient condition to design suitable state estimators of the form (4).

Proposition 2: For given real scalars $\gamma > 0$, $h > 0$, μ_1, and μ_2 satisfying (3), ε_1 and ε_2, the problem of an H_∞ state estimation is solvable if there exist real matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, $Z > 0$, $T = \mathrm{diag}\{t_1, t_2, \ldots, t_n\} > 0$, X_1, Y_1, Y_2, and S with appropriate dimensions such that

$$\begin{bmatrix}\tilde\Upsilon(h, d)|_{d\in\{\mu_1,\mu_2\}} & \tilde\Phi_1^T & \tilde\Phi_2 & \mathscr{C}_7^T & \mathscr{C}_5^T\Gamma_1^TS^T\\ \tilde\Phi_1 & -2T & \tilde\Phi_3 & 0 & 0\\ \tilde\Phi_2^T & \tilde\Phi_3^T & -\gamma^2I & 0 & 0\\ \mathscr{C}_7 & 0 & 0 & -I & 0\\ S\Gamma_1\mathscr{C}_5 & 0 & 0 & 0 & -\mathcal{R} - 2\mathcal{Z}\end{bmatrix} < 0 \qquad (42)$$

$$\begin{bmatrix}\tilde\Upsilon(0, d)|_{d\in\{\mu_1,\mu_2\}} & \tilde\Phi_1^T & \tilde\Phi_2 & \mathscr{C}_7^T & \mathscr{C}_4^T\Gamma_1^TS\\ \tilde\Phi_1 & -2T & \tilde\Phi_3 & 0 & 0\\ \tilde\Phi_2^T & \tilde\Phi_3^T & -\gamma^2I & 0 & 0\\ \mathscr{C}_7 & 0 & 0 & -I & 0\\ S^T\Gamma_1\mathscr{C}_4 & 0 & 0 & 0 & -\mathcal{R}\end{bmatrix} < 0 \qquad (43)$$

where $\tilde\Upsilon(d(t), \dot d(t)) = \tilde\Upsilon_1(d(t), \dot d(t)) - \Upsilon_2(d(t))$ with

$$
\begin{aligned}
\tilde\Upsilon_1(d(t), \dot d(t)) &:= \mathrm{He}\{\mathscr{D}_1^TP\mathscr{D}_2 - \Pi_0^T\tilde{\mathscr{C}}_0\} + h^2e_{10}^T(R+Z)e_{10}\\
&\quad + \mathscr{C}_1^TQ_1\mathscr{C}_1 + (1-\dot d(t))\mathscr{C}_3^T(Q_2 - Q_1)\mathscr{C}_3\\
&\quad - \mathscr{C}_2^TQ_2\mathscr{C}_2
\end{aligned}
$$

TABLE I
ACHIEVED MINIMUM H_∞ PERFORMANCE LEVEL $\gamma_{\min}$ FOR DIFFERENT VALUES OF $\Lambda = \lambda I$

λ	1.2	1.3	1.4	1.49	1.6	2.1
Th3.1 [16]	1.0919	2.2097	11.901	–	–	–
Th 1 [18]	0.6536	0.9805	2.2387	55.988	–	–
Prop 2	0.3973	0.4746	0.5549	0.6501	0.8289	27.489

$$\begin{aligned}\tilde{\Phi}_1 &= X_1\Pi_0 + \Lambda[-Y_1C_0e_1 + (TW - Y_1C_1)e_2]\\ \tilde{\Phi}_2 &= \Pi_0^T(X_1^TB_1 - Y_2B_2), \quad \tilde{\Phi}_3 = -\Lambda Y_1B_2\\ \tilde{\mathscr{C}}_0 &= (X_1A + Y_2C_0)e_1 + Y_2C_1e_2 + X_1e_{10}\end{aligned}$$

and $\Pi_0 := e_{10} + \varepsilon_1 e_1 + \varepsilon_2 e_2$. The other notations are the same as those in Proposition 1. Moreover, the gain matrices K_1 and K_2 of the H_∞ state estimator are given by $K_1 = X_1^{-T}Y_2$ and $K_2 = T^{-1}Y_1$.

Proof: Note that $T < 0$ and $-X_1^T - X_1 + h^2(R + Z) < 0$ hold if the matrix inequalities in (22) are satisfied. Thus, T and X_1 are nonsingular. After some mathematical manipulation, it is not difficult to complete the proof, which is omitted due to page limitation. □

Remark 5: Proposition 2 provides a method for designing suitable H_∞ state estimators of the form (4) for the static neural network (1). As long as a set of LMIs are satisfied, an H_∞ state estimator is readily obtained. It is worth pointing out that, although this kind of H_∞ state estimator is also designed in [18], the gain K_2 is given by $(T\Lambda)^{-1}Y_1$, which means that the matrix $\Lambda = \text{diag}\{\ell_1, \ell_2, \ldots, \ell_n\}$ needs to be invertible. If a specific ℓ_i is equal to zero, the gain K_2 cannot be designed due to that Λ is singular in this situation. However, Proposition 2 is of higher complexity than [18]. In fact, the number of decision variables required in Proposition 2 is $39.5n^2 + (7.5 + 2m)n$, which is more than $10n^2 + (4 + 2m)n$ of [18, Th. 1].

V. NUMERICAL EXAMPLE

In this section, we take the example from [18] to show the effectiveness of the proposed method. Consider the system (1) with $A = \text{diag}\{1.06, 1.42, 0.88\}$, $L_0 = 0$ and

$$L_1 = \begin{bmatrix}1 & 0 & 0.5\\ 1 & 0 & 1\\ 0 & -1 & 1\end{bmatrix}, \quad W = \begin{bmatrix}-0.32 & 0.85 & -1.36\\ 1.1 & 0.41 & -0.5\\ 0.42 & 0.82 & -0.95\end{bmatrix}$$
$$C_0 = \begin{bmatrix}1 & 0.5 & 0\\ 0 & -0.5 & 0.6\end{bmatrix}, \quad C_1 = \begin{bmatrix}0 & 1 & 0.2\\ 0 & 0 & 0.5\end{bmatrix}$$
$$B_1 = [0.2 \quad 0.2 \quad 0.2]^T, \quad B_2 = [0.4 \quad -0.3]^T.$$

The activation function $g(u)$ satisfies (2) and the time-varying delay $d(t)$ satisfies (3). Denote $\Lambda = \text{diag}\{\ell_1, \ell_2, \ell_3\}$.

First, we investigate effects of the activation function $g(u)$ on the achievable minimum H_∞ performance level $\gamma_{\min}$.

Let $h = 0.8$ and $\mu_2 = -\mu_1 = 0.6$. Suppose that $\Lambda = \lambda I$ ($\lambda > 0$). For different values of λ, we calculate the minimum H_∞ performance level $\gamma_{\min}$ using Proposition 2 with $\varepsilon_1 = 1.5$ and $\varepsilon_2 = 0.25$ [16, Th. 3.1], [18, Th. 1]. The obtained results are listed in Table I, from which one can see that the larger λ ($\in (0, 1.6]$) is given, the larger $\gamma_{\min}$ is obtained. However, if increasing λ to 1.6 or above, the methods in [16] and [18] fail to make any conclusion on the feasibility of the H_∞ state estimation problem, while Proposition 2 can derive a feasible solution to the problem with a minimum H_∞ performance level $\gamma_{\min} = 0.8289$ for $\lambda = 1.6$ and $\gamma_{\min} = 27.4882$ for $\lambda = 2.1$.

Second, we investigate effects of the delay upper bound h on the achievable minimum H_∞ performance level $\gamma_{\min}$.

Let $\Lambda = \text{diag}\{1.2, 1.3, 1.4\}$ and $\mu_2 = -\mu_1 = 0.5$. For different delay upper bounds of h, the achieved minimum H_∞ performance

TABLE II
ACHIEVED MINIMUM H_∞ PERFORMANCE LEVEL $\gamma_{\min}$ FOR DIFFERENT VALUES OF h

h	0.6	0.7	0.8	0.9	1
Th3.1 [16]	0.9383	1.3676	2.4425	7.1801	–
Th 1 [18]	0.4826	0.5953	1.0052	32.083	–
Prop 2 ($K_2 = 0$)	0.4283	0.5245	0.6598	0.8749	1.2665
Prop 2	0.3750	0.4218	0.4760	0.5676	0.6683

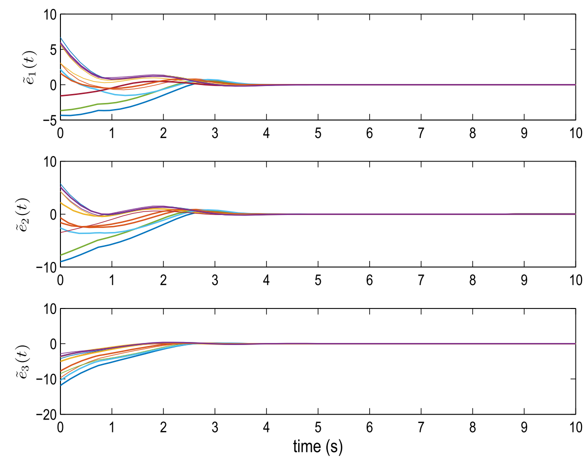

Fig. 1. Estimation error $\tilde{e}_i(t) = z_i(t) - \hat{z}_i(t)$ ($i = 1, 2, 3$) derived by (4) with (44) under ten different initial conditions generated randomly.

$\gamma_{\min}$ is listed in Table II by employing Proposition 2 with $\varepsilon_1 = 1.5$ and $\varepsilon_2 = 0.25$ [16, Th. 3.1], [18, Th.1]. This table shows that, a larger delay upper bound h (≤ 1.6) produces a larger minimum H_∞ level $\gamma_{\min}$, leading to the worse H_∞ performance of the resultant estimation error system.

From Table II, for some delay upper bound h, Proposition 2 can derive a smaller H_∞ level $\gamma_{\min}$ than [16, Th. 3.1 and [18, Th. 1]. Specifically, for $h = 1$, [16, Th. 3.1] and [18, Th. 1] cannot draw any conclusion on the existence of H_∞ state estimators for this example. However, by applying Proposition 2, it is found that the problem of H_∞ state estimation is solvable with the H_∞ performance $\gamma_{\min} = 0.6683$ for the estimation error system, and the corresponding estimator parameters can be obtained as

$$K_1 = \begin{bmatrix}-0.0735 & -0.6992\\ 0.3157 & -0.0526\\ 0.1762 & -0.3806\end{bmatrix}, \quad K_2 = \begin{bmatrix}0.7181 & 0.7627\\ 0.3073 & 0.0750\\ 0.6314 & 0.9427\end{bmatrix}. \tag{44}$$

Associated with this state estimator, the estimation errors $\tilde{e}_i(t)$ ($i = 1, 2, 3$) are illustrated in Fig. 1, where the initial values $\phi(\theta)$ are generated randomly and the time-varying delay is given by $d(t) = 0.4 + 0.6\sin(5t/6)$. Clearly, this estimator can produce good estimation on the neuron states.

Moreover, if setting $K_2 = 0$ in (4), the measurement error signal is not used to adjust the activation function parameters. In this situation, we also calculate the achieved H_∞ lever $\gamma_{\min}$ using Proposition 2 with $K_2 = 0$. The obtained results are listed in Table II, which shows that, by adjusting the parameters in the activation functions, the Luenberger estimator (4) with $K_2 \neq 0$ indeed can produce much better performance than (4) with $K_2 = 0$.

For further comparison, we calculate the admissible upper bound $h_{\max}$ of h, such that the error system is globally asymptotically stable. Applying Proposition 2 yields $h_{\max} = 1.64$, which is larger than 0.95 by [16] and 0.98 by [18].

VI. Conclusion

The problem of H_∞ state estimation for static neural networks with time-varying delays has been studied. An improved reciprocally convex inequality has been introduced to formulate a BRL incorporating with some integral inequalities and the free-weighting matrix technique. This BRL has been used to derive an LMI-based method for designing suitable Luenberger estimators. A numerical example has been given to demonstrate the effectiveness of the proposed method.

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