

Computing minimum lap-time trajectories for a single-track car with load transfer

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Abstract—In this paper we address the minimum lap-time problem for a single-track rigid car which includes tire models and load transfer. Given a planar track including lane boundaries, our goal is to find a trajectory of the car minimizing the lap time subject to tire and steering limits. By using a new set of coordinates, the time-dependent system is transformed into a “space-dependent” (and space-variant) system. The choice of a suitable set of coordinates and the partition of the dynamics into a “longitudinal” one and a “transverse” one, allows us to convert the minimum time problem into a fixed horizon constrained optimal control problem. Based on a projection operator nonlinear optimal control technique, we propose a minimum lap-time strategy to push the rigid car to the limit of its handling capabilities. Finally, we provide numerical computations that: (i) show the effectiveness of the proposed strategy, and (ii) allow us to highlight important features of minimum lap-time trajectories.

I. INTRODUCTION

A primary task in the automotive industry is to explore the dynamic capabilities of a vehicle and thereby improve its set-up so as to drive the fastest possible lap-time on a given track. Most companies use software tools helping their race car designers in gaining information on the vehicle performance before it is built. This set of *virtual prototyping* tools, if accurate and computationally efficient, may significantly reduce costs and time to find the best set-up.

Many computational strategies have been introduced in the literature to solve the minimum lap-time problem. Using a common classification, we can identify two main classes of strategies: quasi-steady state methods and transient optimal methods. The key idea of the quasi steady state methods is to approximate the vehicle trajectory as a concatenation of equilibrium conditions, see, for example, [1], [2]. The track is split into a sequence of segments of suitable length. For each segment it is assumed that the vehicle is in stationary conditions, that is the vehicle goes along the segment with constant speed and lateral acceleration. The lateral tire force needed to ensure the maximum lateral acceleration in the segment is computed. Then, using a combined tire model, the remaining tire forces are sought to compute the longitudinal acceleration. Even if this method provides a sufficiently realistic estimate of lap-time trajectories, it has two main limitations: (i) it computes the maximum speed profile for a given racing line, and (ii) important information

on transient dynamics are lost, especially when the car is at the limit of its capabilities. Conversely, transient optimal methods, based on optimal control techniques, generally aim at solving or approximating the minimum lap-time problem by maximizing or minimizing an objective function subject to the dynamics. Different objective functions are discussed in the literature: maximize the distance traveled along a race track in a given fixed time [3], maximize the vehicle speed in the final point of the maneuver [4], and minimize the lap-time [5], [6]. Different optimization methods are used to solve the minimum time optimal control problem. Based on Bellman’s Principle of Optimality, in [7] a semi-analytical method is proposed to generate minimum time velocity profiles for a point-mass model. An extension of that methodology is provided in [8] for an half-car model. In [4] the authors address the optimal control problem by using a commercial optimization software. In [5] and [6], the authors propose the use of the distance travelled along the road center line as independent variable. The authors solve the problem by applying the gradient descent method and the parallel shooting method, respectively. A comparison between pure steady state, quasi steady state, and transient optimal methods is discussed in [9].

The contributions of the paper are as follows. Based on the idea developed in [6], we provide a rigorous derivation of suitable transverse coordinates that allow to split the system dynamics into longitudinal and transverse ones. Using this transformation we set up an equivalent constrained optimal control problem solving the minimum lap-time problem. The main advantages with respect to the original problem are that: (i) the horizon is fixed and (ii) the time-varying state constraints are converted into space independent state constraints.

As main contribution of the paper, we develop an optimal control based strategy to effectively solve the optimal control problem and compute “realistic” lap-time trajectories. A constrained optimal control problem is in fact an infinite dimensional optimization problem and thus applying naively available numerical techniques could lead to local minima, i.e. to car trajectories that are significantly different from the actual minimum lap-time trajectory. The proposed strategy combines a barrier function relaxation, introduced in [10], to handle point-wise constraints with a continuation update rule. The main idea is the following. We start with a feasible car trajectory, i.e. a trajectory satisfying the state (lane) and input (steering and tire) constraints. Then we repeatedly apply the projection operator Newton method to solve the relaxed optimal control problem. At each iteration the re-

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laxed problem is modified by updating the barrier functional parameters (increasing the barrier climb) thus pushing the optimal trajectory toward the constraint boundaries. The interesting consequence of applying the proposed strategy is that the convergence to the optimal trajectory happens in an interior point fashion. This resembles, in our opinion, the learning process of a real driver in pushing the car toward its limit capabilities.

Finally, we provide numerical computations to show the effectiveness of the proposed strategy both in terms of computation time and trajectory exploration. Indeed, the numerical computations highlight important features of the computed trajectories (also compared to the model without load transfer) that deserve further theoretical investigation.

The rest of the paper is organized as follows. In Section II, we briefly introduce the car model developed in previous works. In Section III we formulate the minimum lap-time problem (we define cost function, constraints and boundary conditions) and its equivalent formulation by using the transverse dynamics. Finally, in Section IV we describe the strategy to estimate the lap-time trajectory and provide numerical computations.

II. LT-CAR MODEL

In this section we briefly introduce the car model with load transfer (LT-CAR). The vehicle dynamics is based on a simplified car model developed in [11], [12]. In previous work, see [13], [14], we provide a rigorous derivation of the model based on the Lagrangian approach. The basic idea of this model is to introduce suitable constraints into the equations of motion which allow to compute the normal forces by means of the reaction forces. This allows to model load transfer without adding suspension models. The car model is a single planar rigid body with five degrees of freedom, the longitudinal, lateral, and vertical displacements, the yaw and pitch angles. It is constrained to move in a plane interacting with the road at two body-fixed contact points. The center of mass and the two contact points all lie within a plane with the center of mass located at distance b from the rear contact point and a from the front one, respectively. A planar view of the rigid car model is shown in Figure 1. The

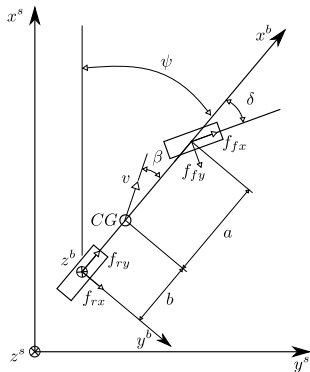


Fig. 1: LT-CAR model. The figure show the quantities used to describe the model.

body-frame of the car is attached at the rear contact point with x - y - z axes oriented in a forward-right-down (north-east-down) fashion, in accordance with the SAE J670e standard. The orientation R of the (unconstrained) rigid car model can be parameterized (using Roll-Pitch-Yaw parametrization) as follows

$$R = R(\psi, \theta) = R_z(\psi)R_y(\theta) = \begin{bmatrix} c_\psi c_\theta & -s_\psi & c_\psi s_\theta \\ s_\psi c_\theta & c_\psi & s_\psi s_\theta \\ -s_\theta & 0 & c_\theta \end{bmatrix},$$

where θ and ψ are respectively the pitch and yaw angles (we use the notation $c_\psi = \cos(\psi)$, etc.).

Each contact-point/road-plane interaction is modeled using a linear tire model. Before, we clarify our notation. We use a subscript “f” (“r”) for quantities of the front (rear) tire. When we want to give a generic expression that holds both for the front and the rear tire we just suppress the subscript. The linear tire model relies on the following assumptions: (i) the longitudinal force is directly controlled, (ii) the relationship between the lateral force f_y and the sideslip β is linear (w.r.t. the coefficient d_y), and (iii) the longitudinal and lateral forces, f_x and f_y , are decoupled. Notice that in Section III-B we add a suitable constraint such that the longitudinal and lateral forces become in fact coupled.

The rear and front forces tangent to the road plane, f_x and f_y , depend on the normal force, f_z , and on the longitudinal and lateral slips. In particular, we assume that the rear and front forces tangent to the road plane, depend linearly on the normal forces, that is $f_x = -f_z\mu_x$, $f_y = -f_z\mu_y$, where μ_x (μ_y) is the longitudinal (lateral) force coefficient.

We derive the dynamics of the system with respect to the following generalized coordinates¹ $q = [x, y, \psi, z, \theta]^T = [q_r, q_c]^T$. Then we constrain the contact points to the road plane in order to compute the normal tire forces as reaction forces. These forces are incorporated into the equations of motion allowing the explicit calculation of the front and rear contact point forces, f_{fz} and f_{rz} . Using Euler-Lagrange approach we are able to derive the car dynamics in the following form

$$\mathcal{M}(q_r, \mu) \begin{bmatrix} \ddot{q}_r \\ \lambda \end{bmatrix} + \mathcal{C}(q_r, \dot{q}_r) + \mathcal{G}(q_r) = 0, \quad (1)$$

see [14] for more details. From the equation (1) we have a dynamic model explicitly depending on the unconstrained coordinates q_r and an explicit expression for the normal forces that can be used to predict the load transfer (hence the name Load Transfer CAR). The control inputs of the car are the front wheel steer angle, δ , and the rear longitudinal force coefficient, μ_{rx} . Since the dynamics does not depend on the positions x and y , and the orientation ψ , we can work directly with the longitudinal velocity v_x , the lateral velocity v_y , and the yaw rate r . Expressing the dynamics in the body frame, the equations of motion are given by

$$\begin{aligned}\dot{q}_r &= f_{q_r}(q_r, q_v) \\ \dot{q}_v &= f_{q_v}(q_v, u)\end{aligned}\tag{2}$$

¹The coordinates $q_r = [x, y, \psi]^T$ are the *reduced* unconstrained car coordinates, while $q_c = [z, \theta]^T$ are the *constrained* coordinates.

$$\begin{bmatrix} m & 0 & 0 & \mu_{fx}(\delta, \beta_f) & \mu_{rx} \\ 0 & m & mb & \mu_{fy}(\delta, \beta_f) & \mu_{ry}(\beta_r) \\ 0 & mb & (I_{zz} + mb^2) & (a+b)\mu_{fy}(\delta, \beta_f) & 0 \\ 0 & 0 & 0 & -1 & -1 \\ -mh & 0 & 0 & a+b & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{r} \\ f_{fz} \\ f_{rz} \end{bmatrix} + \begin{bmatrix} -mbr^2 - mv_y r \\ mv_x r \\ mbv_x r \\ 0 \\ (I_{xz} + mhb)r^2 + mhv_y r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -mg \\ mgb \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

where $q_v = [v_x, v_y, r]^T$. In this way, we can decouple the dynamics of the vehicle, $\dot{q}_v = f_{q_v}(q_v, u)$, from its kinematics, $\dot{q}_r = f_{q_r}(q_r, q_v)$. The dynamics of the vehicle is described in (3): m is the vehicle's mass, h is the height of the mass center from the ground, I_{zz} and I_{xz} are the moments of inertia of the vehicle about the vertical axis and x - z plane, v_x and v_y are the longitudinal and lateral velocity, g is the gravity acceleration, δ is the steer angle, and β_f (β_r) is the front (rear) sideslip angle. Due to space limitations, we refer the reader to [14] for data and further details.

III. MINIMUM LAP-TIME PROBLEM

In this section we address the minimum lap-time problem. In order to formulate it, we need to define the cost function to be optimized and specify the state/input constraints and boundary conditions. In this work, we consider the problem of finding a trajectory of the LT-CAR system that minimizes the time T to complete a given track with: fixed initial point, track boundary constraints, and input control constraints:

$$\begin{aligned} \min \quad & \int_0^T 1 \, d\tau \\ \text{subj. to} \quad & \dot{q}_r = f_{q_r}(q_r, q_v) \quad q_r(0) = q_{r0} \\ & \dot{q}_v = f_{q_v}(q_v, u) \quad q_v(0) = q_{v0} \\ & c_w(x(t), y(t), t, w_{track}) \leq 0 \\ & c_\delta(\delta(t), \delta_{max}) \leq 0 \\ & c_t(q_v(t), d_{Xmax}, d_{Ymax}) \leq 0, \end{aligned} \quad (4)$$

where c_w , c_δ and c_t represent respectively the track boundary, steering and tire point-wise constraints. Specifically, the distance between the rear contact point and the center of the race track is limited by the track width w_{track} . We choose to constrain only the rear contact point to the race track for simplicity. The front wheel steer angle, δ , is bounded in module by δ_{max} . In order to take into account the limited grip of the tires, we constrain the longitudinal and lateral forces into an ellipse (friction ellipse), with maximum pure longitudinal (lateral) force defined by d_{Xmax} (d_{Ymax}).

A. Longitudinal and transverse dynamics

In order to formulate an equivalent version of the minimum lap-time problem, we define a new set of coordinates and split the dynamics into a longitudinal one and a transverse one. We define the new set of coordinates by replacing the position coordinates with: a longitudinal coordinate s , representing the position along the center line of the track, and a lateral displacement w representing the displacement transverse to the central line. In our context, a natural choice

of the new parametrization is the arc-length of the curve, $s(t)$, that is defined as

$$s(t) = \int_0^t \sqrt{\dot{x}_{cl}^2(t) + \dot{y}_{cl}^2(t)} dt,$$

where $x_{cl}(\cdot)$ and $y_{cl}(\cdot)$ are the longitudinal and lateral coordinates of the center line. If the velocity is bounded away from zero, the mapping $t \mapsto s(t)$ is strictly increasing, so that we can compute its inverse. This allows us to define the state $q_r(t)$, $q_v(t)$, and input $u(t)$ trajectories as a function of the arc-length instead of time, that is, $\bar{q}_r(s)$, $\bar{q}_v(s)$, and $\bar{u}(s)$, respectively. From now on, the bar symbol indicates that a quantity is expressed as a function of the longitudinal parameters s rather than time t .

Given the arc-length parametrization of the track center line, $s \mapsto (\bar{x}_{cl}(s), \bar{y}_{cl}(s))$, we choose a set of local coordinates in a “tube” around it. The coordinates of the rear contact point (x, y) can be defined with respect to the arc-length of the point at minimum distance, that is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x}_{cl}(s) \\ \bar{y}_{cl}(s) \end{bmatrix} + \begin{bmatrix} -w \sin \bar{\chi}_{cl}(s) \\ w \cos \bar{\chi}_{cl}(s) \end{bmatrix}. \quad (5)$$

From now on, we use a prime symbol to denote the first derivative of the variable with respect to the arc-length s . For example, $\bar{x}'(s) = \frac{d\bar{x}(s)}{ds}$, while $\dot{x}(t) = \frac{dx(t)}{dt}$.

Remark 3.1: The transverse displacement w is orthogonal to the unit tangent vector $(\bar{x}'(s), \bar{y}'(s)) = (\cos \bar{\chi}_{cl}(s), \sin \bar{\chi}_{cl}(s))$. The inverse of the map $(s, w) \mapsto (x, y)$ is locally well defined around points (s, w) satisfying $w \bar{\sigma}_{cl}(s) < 1$, where $\bar{\sigma}_{cl} = \bar{\chi}'_{cl}$ is the curvature of the map at $(\bar{x}(s), \bar{y}(s))$. \square

By differentiating (5) with respect to the time t , we get the expression of \dot{s} and \dot{w} , that are

$$\begin{aligned} \dot{s} &= \frac{v_x \cos(\psi - \bar{\chi}_{cl}(s)) - v_y \sin(\psi - \bar{\chi}_{cl}(s))}{1 - \bar{\sigma}_{cl}(s)w}, \\ \dot{w} &= v_x \sin(\psi - \bar{\chi}_{cl}(s)) + v_y \cos(\psi - \bar{\chi}_{cl}(s)). \end{aligned}$$

Defining the local heading angle as $\mu = \psi - \bar{\chi}_{cl}(s)$, the nonlinear system (2) can be written with respect to the new set of coordinates $(s(\cdot), w(\cdot), \mu(\cdot), q_v(\cdot), u(\cdot))$:

$$\begin{aligned} \dot{s} &= \frac{v_x \cos \mu - v_y \sin \mu}{1 - \bar{\sigma}_{cl}(s)w} \\ \dot{w} &= v_x \sin \mu + v_y \cos \mu \\ \dot{\mu} &= \dot{\psi} - \bar{\sigma}_{cl}(s) \frac{v_x \cos \mu - v_y \sin \mu}{1 - \bar{\sigma}_{cl}(s)w} \\ \dot{q}_v &= f_{q_v}(q_v, u) \end{aligned} \quad (6)$$

The nonlinear differential equation (6) is the longitudinal and transverse dynamics form of the nonlinear system (2).

B. State-control constraints in transverse coordinates

The constraints defined in the minimum lap-time problem (4) can be re-written in the new set of coordinates. The constraint on the track boundary assumes a very simple form. Indeed, it is given by $|\bar{w}(s)| \leq w_{track}$. In order to have a smooth function defining the constraint, we rewrite it in the equivalent form

$$\left(\frac{\bar{w}(s)}{w_{track}}\right)^2 - 1 \leq 0, \forall s \in [0, S].$$

Remark 3.2: This constraint ensures also that the change of coordinates is well defined in a “tubular” neighborhood of the center line $(\bar{x}_{cl}(\cdot), \bar{y}_{cl}(\cdot))$. In fact, the condition $\bar{w}(s)\bar{\sigma}_{cl}(s) < 1$ in Remark 3.1 is true for all $s \in [0, S]$, provided that w_{track} is appropriately chosen. \square

In the same way, we define the constraint on the front wheel steer angle as

$$\left(\frac{\bar{\delta}(s)}{\delta_{max}}\right)^2 - 1 \leq 0, \forall s \in [0, S].$$

Finally, in order to prevent the tires to operate in a high saturation region, we consider the linear tire model and we constraint the total force acting on each wheel limited within the ellipse of maximum tire force, that is

$$\left(\frac{\bar{\mu}_x(s)}{d_{Xmax}}\right)^2 + \left(\frac{\bar{\mu}_y(s)}{d_{Ymax}}\right)^2 - 1 \leq 0, \forall s \in [0, S].$$

C. Equivalent formulation of the minimum lap-time problem

Now we address the minimum lap-time problem with respect to the new parametrization. We eliminate the time t and write the dynamics as a differential equation where the longitudinal coordinate s becomes the independent variable. The transverse coordinates, the generalized velocity coordinates and the input control at location s are $\bar{w}(s)$, $\bar{\mu}(s)$, $\bar{q}_v(s)$, $\bar{u}(s)$, and the dynamics (6) becomes

$$\begin{aligned} \bar{w}' &= (\bar{v}_x \sin \bar{\mu} + \bar{v}_y \cos \bar{\mu}) \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}}{\bar{v}_x \cos \bar{\mu} - \bar{v}_y \sin \bar{\mu}} \\ \bar{\mu}' &= \bar{r} \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}}{\bar{v}_x \cos \bar{\mu} - \bar{v}_y \sin \bar{\mu}} - \bar{\sigma}_{cl}(s) \\ \bar{v}_x' &= f_{q_v}^1(\bar{q}_v, \bar{u}) \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}}{\bar{v}_x \cos \bar{\mu} - \bar{v}_y \sin \bar{\mu}} \\ \bar{v}_y' &= f_{q_v}^2(\bar{q}_v, \bar{u}) \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}}{\bar{v}_x \cos \bar{\mu} - \bar{v}_y \sin \bar{\mu}} \\ \bar{r}' &= f_{q_v}^3(\bar{q}_v, \bar{u}) \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}}{\bar{v}_x \cos \bar{\mu} - \bar{v}_y \sin \bar{\mu}} \end{aligned}$$

where $f_{q_v}^1(\cdot)$, $f_{q_v}^2(\cdot)$, $f_{q_v}^3(\cdot)$, are the components of the function $f_{q_v}(\cdot)$. In compact form:

$$\bar{\mathbf{w}}'(s) = \bar{f}(\bar{\mathbf{w}}(s), \bar{u}(s), \bar{\sigma}_{cl}(s)), \quad (7)$$

where $\bar{\mathbf{w}}(\cdot) = [\bar{w}(\cdot), \bar{\mu}(\cdot), \bar{v}_x(\cdot), \bar{v}_y(\cdot), \bar{r}(\cdot)]^T$.

An equivalent formulation of the minimum lap-time problem, expressed in the new set of coordinates and in the new

variable s , is given by:

$$\begin{aligned} \min \quad & \int_0^S \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}(s)}{\bar{v}_x(s) \cos \bar{\mu}(s) - \bar{v}_y(s) \sin \bar{\mu}(s)} ds \\ \text{subj. to} \quad & \bar{\mathbf{w}}' = \bar{f}(\bar{\mathbf{w}}(s), \bar{u}(s), \bar{\sigma}_{cl}(s)) \\ & \bar{\mathbf{w}}(0) = \mathbf{w}_0 \\ & \left(\frac{\bar{w}(s)}{w_{track}}\right)^2 - 1 \leq 0 \\ & \left(\frac{\bar{\delta}(s)}{\delta_{max}}\right)^2 - 1 \leq 0 \\ & \left(\frac{\bar{\mu}_x(s)}{d_{Xmax}}\right)^2 + \left(\frac{\bar{\mu}_y(s)}{d_{Ymax}}\right)^2 - 1 \leq 0. \end{aligned} \quad (8)$$

The problems (4) and (8) are equivalent in the sense that trajectories that solve the problem (4) can be mapped to trajectories that solve (8) and viceversa. This last form has several benefits: the state dimension is reduced by one and the problem is now a fixed-horizon free-endpoint problem.

IV. NONLINEAR OPTIMAL CONTROL BASED MINIMUM LAP-TIME TRAJECTORY

In this section we describe the optimal control based strategies used to compute the minimum lap-time trajectory of the car vehicle and provide numerical computations showing their effectiveness. We address the constrained nonlinear optimal control problem by computing solutions of a suitable unconstrained nonlinear optimal control problem.

A. Optimization of trajectory functionals with constraints

From now on, we work only on the system defined with respect to the independent variable s . In order to not overburden the notation, we neglect the bar symbol on top of the variable. For example we write $\mathbf{w}(s)$ instead of $\bar{\mathbf{w}}(s)$.

We recall that a trajectory is a (state-input) curve $\xi = (\mathbf{w}(\cdot), u(\cdot))$ defined on $L_\infty[0, S]$ such that

$$\mathbf{w}'(s) = f(\mathbf{w}(s), u(s), \sigma_{cl}(s)).$$

for all $s \in [0, S]$, where $f : \mathbb{R}^5 \times \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{R}^5$ and $f \in \mathcal{C}^r$ with $0 \leq r \leq \infty$. To facilitate the local exploration of trajectories of this nonlinear system, we use the PRjection Operator based Newton method for Trajectory Optimization (PRONTO) developed in [15].

We are interested in developing unconstrained optimal control strategies that can be used to find approximate solutions to the constrained optimal control problem (8).

Based on the idea developed in [10], we use a barrier function relaxation to handle the constraints. Formally, let $c_j(\mathbf{w}(s), u(s))$, $j \in \{1, 2, 3\}$, denote the three constraints in problem (8). For a given (state-input) curve $\xi = (\mathbf{w}(\cdot), u(\cdot))$, a barrier functional can be defined as

$$b_\delta(\xi) = \int_0^S \sum_j \beta_\delta(-c_j(\mathbf{w}(\cdot), u(\cdot))) d\tau$$

where

$$\beta_\delta(x) = \begin{cases} -\log x, & x > \delta \\ \frac{k-1}{k} \left[\left(\frac{x-k\delta}{(k-1)\delta} \right)^k - 1 \right] - \log \delta, & x \leq \delta \end{cases}.$$

Using the barrier functional defined above, the relaxed version of problem (8) is given by

$$\begin{aligned} \min \quad & \int_0^S \frac{1 - \bar{\sigma}_{cl}(s)\bar{w}(s)}{\bar{v}_x(s) \cos \bar{\mu}(s) - \bar{v}_y(s) \sin \bar{\mu}(s)} \\ & + \epsilon \Sigma_j \beta_\delta (-c_j(\mathbf{w}(s), u(s))) ds \\ \text{subj. to} \quad & \bar{\mathbf{w}}'(s) = \bar{f}(\bar{\mathbf{w}}(s), \bar{u}(s), \bar{\sigma}_{cl}(s)) \quad \bar{\mathbf{w}}(0) = \mathbf{w}_0. \end{aligned} \quad (9)$$

Denoting \mathcal{T} the manifold of bounded trajectories $\xi = (\mathbf{w}(\cdot), u(\cdot))$ on $[0, S]$, and $h(\xi) = \int_0^S \frac{1 - \sigma_{cl}(s)w}{v_x \cos \mu - v_y \sin \mu} ds$ the cost functional of problem (8), we can write problem (9) in a more compact form as

$$\min_{\xi \in \mathcal{T}} h(\xi) + \epsilon b_\delta(\xi). \quad (10)$$

Using the projection operator defined in [15] to locally parametrize the trajectory manifold, we may convert the constrained optimization problem (10) into one of minimizing the unconstrained functional

$$g_{\epsilon, \delta}(\xi) = h(\mathcal{P}(\xi)) + \epsilon b_\delta(\mathcal{P}(\xi)). \quad (11)$$

The projection operator based Newton method is used to optimize the functional (11), as part of a continuation method to seek an approximate solution to (10). The strategy is to start with a reasonably large ϵ and δ . Then, for the current ϵ and δ , the problem $\min g_{\epsilon, \delta}(\xi)$ is solved using the Newton method starting from the current trajectory.

B. Exploration strategy

The projection operator Newton method, being a descent method, guarantees the convergence to a local minimum of the optimal control problem relaxation (9). A naive application of the method and, even more, a naive choice of the initial trajectory, ξ_0 , may let the algorithm converge to a (local minimum) trajectory that is too far from the optimal lap-time trajectory. In order to deal with this issue, we develop an exploration strategy based on a continuation idea to iteratively apply the projection operator Newton method.

First, we define the center line of the track. The center line is described by the following parameters (related to each other): the longitudinal and lateral coordinates with respect to a fixed spatial-frame, $x_{cl}(\cdot)$ and $y_{cl}(\cdot)$, the radius of curvature, $\sigma_{cl}(\cdot)$, and the tangent angle, $\chi_{cl}(\cdot)$. For the numerical computations, we consider a chicane track. Then, we have to design the initial trajectory to initialize the projection operator Newton method. The initial trajectory is based on the center line of the track. In particular, given the curvature of the center line, $\sigma_{cl}(\cdot)$, we choose a suitable constant velocity profile, v_0 , and, for $s \in [0, S]$, we define the following (state-input) curve $\xi_0 = [0, 0, v_0, 0, v_0 \sigma_{cl}(\cdot), 0, 0]^T$. This curve is not a trajectory of the LT-CAR system (i.e. it does not satisfy the dynamics (7)). By using the trajectory tracking projection operator defined in [15], the curve is projected onto the set of system trajectories to get a suitable initial trajectory, i.e. $\xi_0^1 = \mathcal{P}(\xi_0)$.

Now, with the center line track and the initial trajectory in hand, we repeatedly perform the following steps.

We compute the optimal lap-time trajectory by using the Projection operator Newton method, $\xi_{opt}^1 = \text{PRONTO}(\xi_0^1)$. Then, at the i -th step, we update the initial trajectory, ξ_0^i , with the previous optimal trajectory, ξ_{opt}^{i-1} , and the constraints parameters, $\epsilon \leftarrow \epsilon/6$ and $\delta \leftarrow \delta/8$, to solve problem (9) with an updated barrier functional. Next, we give a pseudo code description of the exploration strategy.

Algorithm 1 Minimum lap-time strategy

Given: center line track $(x_{cl}(\cdot), y_{cl}(\cdot))$, $\chi_{cl}(\cdot)$, and $\sigma_{cl}(\cdot)$;

set a “suitable” initial velocity and compute $\xi_0(\sigma_{cl}(\cdot))$;

compute transverse dynamics $\mathbf{w}' = f(\mathbf{w}, u, \sigma_{cl})$;

compute initial trajectory $\xi_0^1 = \mathcal{P}(\xi_0)$;

set $\epsilon = 1$, $\delta = 1$;

for $i = 1, 2, \dots$ **do**

set the functional

$$g_{\epsilon, \delta}(\xi) = h(\mathcal{P}(\xi)) + \epsilon b_\delta(\mathcal{P}(\xi))$$

compute: $\xi_{opt}^i = \text{PRONTO}(\xi_0^i)$;

update: $\epsilon = \epsilon/6$, $\delta = \delta/8$, and $\xi_0^{i+1} = \xi_{opt}^i$;

end for

Output: $\xi_{opt} = \xi_{opt}^{end}$.

C. Lap-time for the chicane track

We compute the minimum lap-time trajectory for a chicane track. The minimum lap-time (optimal) trajectory is shown in Figure 2. We stress that the initial trajectory is feasible. By setting as initial longitudinal velocity $v_{x0} = 20[\text{m/s}]$, the constraints are not active, see Figures 2d, 2e, and 2f. At the beginning of the track, the vehicle accelerates to gain the best velocity, Figure 2b, and moves on the left of the track to gain the most favorable position to enter the right turn. Then, the steer angle increases, Figure 2d, to steer the vehicle up to the apex point. In this point the velocity reaches a local minimum value in order to maintain cornering grip. After the first apex point, the vehicle proceeds straight for a short interval, where it accelerates and, immediately after that, it decelerates in sharply fashion. The steer angle changes sign to allow the vehicle to enter the left turn. We can observe an overshoot for the steer angle. We believe that this behavior is actually the one appearing in race cars and, thus, needs further investigation. Then the vehicle moves on the left of the track (second apex point) to gain the most favorable position to exit the turn. Finally, the vehicle starts to accelerate with maximum traction force. We emphasize the significant and rapid load transfer in the two turns: 30% of the total vehicle mass is shifted to the rear axle in the acceleration phase and to the front axle in the braking phase. In Figure 2f we observe that the vehicle performance is limited by the tire performance (dash red line): the rear tire works on the ellipse constraint for almost all the track (green asterisk). The minimum maneuver time is 15.3768[s].

In order to illustrate the importance of the load transfer, we compute the minimum lap-time trajectory for the rigid car model without load transfer, called bicycle model. The comparison shows some interesting behaviors. Even if the

speed profile may seem similar at a first look, they present a significant difference in the braking points and in the braking and acceleration rates, see Figure 2b. The anticipated braking of the LT-CAR model can be seen also in Figure 2e. The different rate of acceleration can be explained in the following way. When the rear-wheel drive transmission vehicle accelerates, the μ_{rx} saturates. This happens for both the bicycle model and the LT-CAR. However, due to the effect of the load transfer, the weight of the LT-CAR shifts to the rear axle increasing the longitudinal traction force. Clearly, this phenomenon is not captured at all by the bicycle model. This explains why the minimum lap-time of the bicycle model, 16.02[s], is greater than the LT-CAR one. These aspects reveal that the bicycle model is missing important dynamic effects (mainly the load transfer) that will appear on the real vehicle.

V. CONCLUSIONS

In this paper we addressed the minimum lap-time problem for a single track rigid car that takes into account tire models and load transfer. We provided an optimization strategy to estimate the lap-time on a given path. This strategy is based on an equivalent formulation of the optimal control problem and combines optimal control techniques with a continuation method to find a (candidate) minimum lap-time trajectory. The strategy allows us to capture important dynamic aspects of the vehicle with a low computational effort. A detailed theoretical study of the computed minimum lap-time trajectories deserves further investigation. In fact, this work can be seen as a preliminary step to find the best set up to improve the performance of the vehicle. The effectiveness of the strategy has been shown for a chicane track and a comparison with the vehicle without load transfer is discussed.

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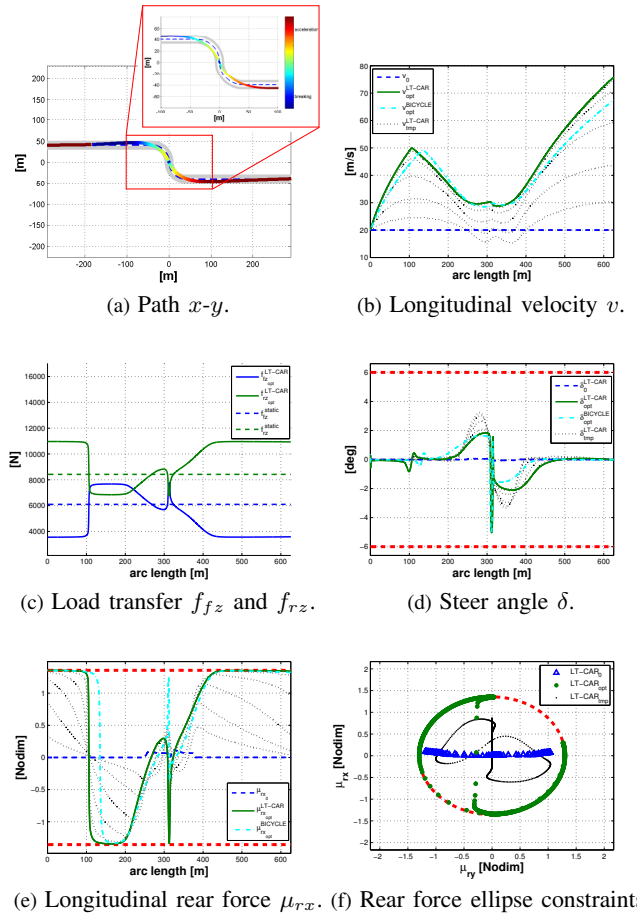


Fig. 2: Minimum lap-time trajectory. In subplot (a), the dash and solid lines represent the center line of the track and the LT-CAR minimum time race line. In (b) the LT-CAR initial velocity (dashed line), the LT-CAR minimum time velocity (solid line), and the minimum time velocity (dash-dot line) of the bicycle model are shown. In (c) the optimal LT-CAR load transfer (solid line) is compared with the static load (dashed line). The optimal control inputs are plotted in (d)-(e). The optimal friction ellipse is shown in (f). Temporary optimal trajectories are in light dot lines. Constraints: $w_{track} = 6[m]$, $\delta = 6[deg]$, $d_{x,max} = 1.48[Nodim]$, $d_{y,max} = 1.22[Nodim]$. The intermediate optimal trajectory approach the corresponding minimum lap-time trajectory for the constrained problem as $\epsilon \rightarrow 0$. The sequence of approximate transit time is: 28.05[s], 22.19[s], 18.28[s], 16.20[s], 15.52[s].

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