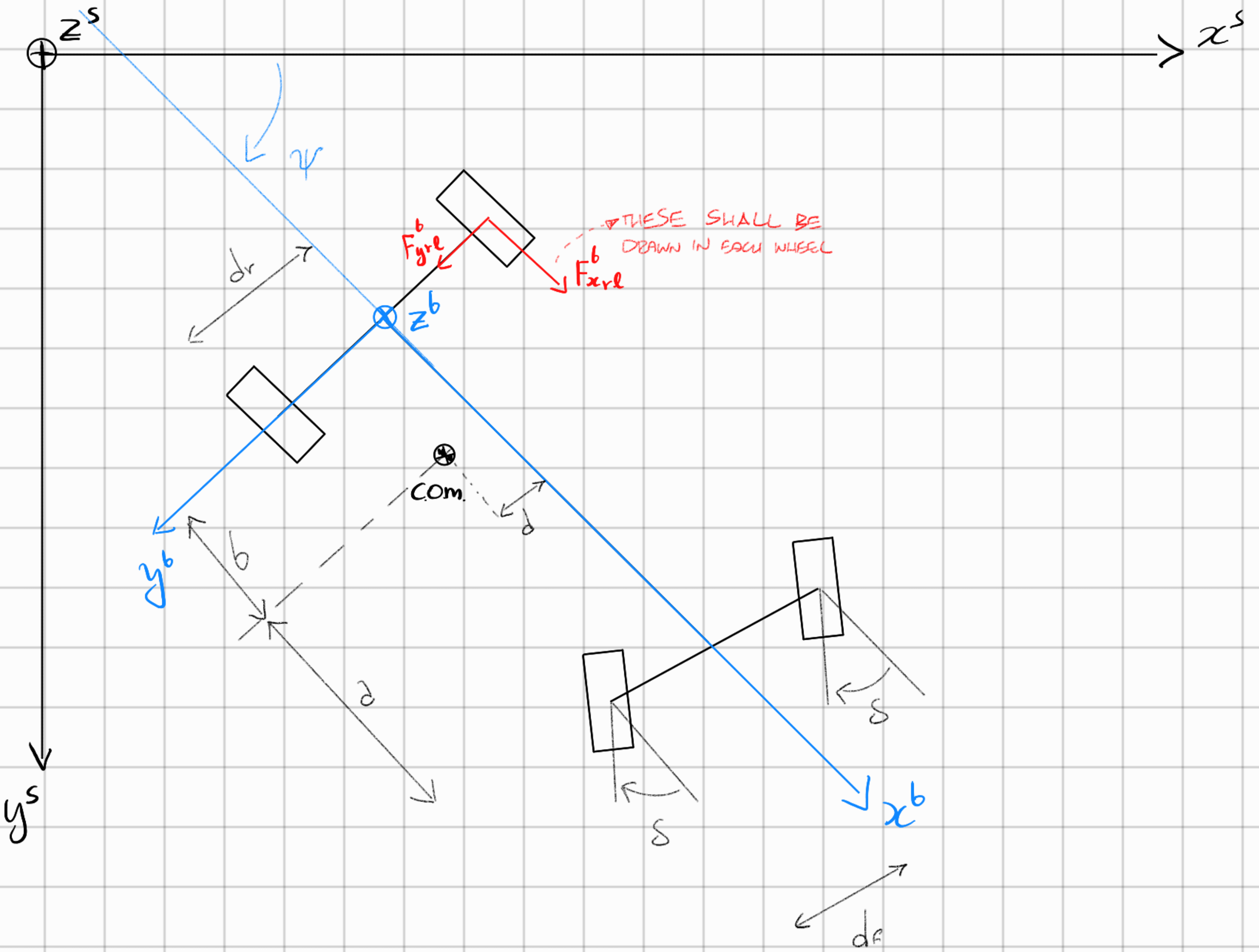


z^s
x^s
y^s
ψ
F^b_{yrl}
F^b_{xrl}
THESE SHALL BE DRAWN IN EACH WHEEL
z^b
d_r
COM
y^b
x^b
b
a
δ
d_f

PARAMETERS (TO BE PUT IN $\theta(:,1)$ IN THIS ORDER)

$m\,[kg]$: MASS OF THE VEHICLE

$d\,[m]$: DISTANCE OF THE COM TO THE LONG. AXLE ALONG y^b

$a\,[m]$: DISTANCE BETWEEN THE COG AND THE FRONT AXLE

$b\,[m]$: DISTANCE BETWEEN THE COG AND THE REAR AXLE

$I_{zz}\,[kg \cdot m^2]$: VEHICLE MOMENT OF INERTIA

$h\,[m]$: HEIGHT OF THE COM FROM THE GROUND

$I_{xz}\,[kg \cdot m^2]$: VEHICLE MOMENT OF INERTIA WITH RESPECT TO THE XZ PLANE (IT IS ONE OF THE CROSS COUPLING TERMS IN THE INERTIA TENSOR)

$d_r\,[m]$: HALF OF THE REAR TRUCK WIDTH

$d_f\,[m]$: HALF OF THE FRONT TRUCK WIDTH

$r_w\,[m]$: WHEEL RADIUS

$K_t\,(=0)$: TRACTION RATIO

$K_b\,(=0.5)$: BRAKE RATIO

C_f : PSEUDO CORNERING STIFFNESS OF THE FRONT WHEELS

C_r : PSEUDO CORNERING STIFFNESS OF THE REAR WHEELS

PT. 1 – ASSUMPTIONS AND LATERAL & LONGITUDINAL FORCES EXPRESSIONS

ASSUMPTIONS:

1) LONGITUDINAL FORCE(S) ARE DIRECTLY CONTROLLED

2) $F_{ijy} \propto \beta_{ij}$, WITH A LINEAR DEPENDANCE $\forall (i,j)$

NOTE: $i = \{f, r\}$ (f → FRONT, r → REAR), $j = \{r, l\}$ (r → RIGHT, l → LEFT)

3) F_{ijx} AND F_{ijy} ARE DECOUPLED $\forall (i,j)$

4) F_{ijx} AND F_{ijy} DEPEND LINEARLY ON F_{ijz} $\forall (i,j)$, SO:

$$F_{ijx} = -F_{ijz}\,\mu_{ijx}$$

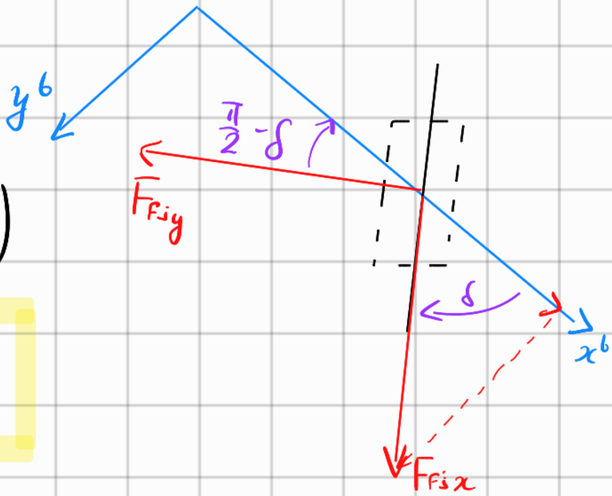
$$F_{ijy} = -F_{ijz}\,\mu_{ijy}(\beta_{ij})$$

WHILE REAR FORCES ARE ALREADY IN LOCAL REF. FRAME, FRONT ONES SHALL BE PROJECTED ALONG IT:

$$F^b_{Fjx} = F_{Fjx}\cos\delta - F_{Fjy}\cdot\sin\delta$$

$$= -F_{Fjz}\underbrace{\left(\mu_{Fjx}\cos\delta - \mu_{Fjy}(\beta_{Fj})\sin\delta\right)}_{\tilde{\mu}_{Fjx}}$$

$$\boxed{= -F_{Fjz}\,\tilde{\mu}^b_{Fjx}(\mu_{Fjx}, \beta_{Fj}, \delta)}$$



$$F^b_{Fjy} = F_{Fjx}\sin\delta + F_{Fjy}\cos\delta$$

$$= -F_{Fjz}\underbrace{\left(\mu_{Fjx}\sin\delta + \mu_{Fjy}(\beta_{Fj})\cos\delta\right)}_{\tilde{\mu}_{Fjy}}$$

$$\boxed{= -F_{Fjz}\,\tilde{\mu}^b_{Fjy}(\mu_{Fjy}, \beta_{Fj}, \delta)}$$

WHERE

IF $K_t = 0 \rightarrow$ REAR-WHEEL DRIVE

IF $K_b = 0.5 \rightarrow$ EQUAL BRAKE DISTRIBUTION F&R

$$\mu_{fix} = K_t u_t - K_b u_b$$

$$\mu_{rix} = (1-K_t) u_t - (1-K_b) u_b$$

AND

$$\mu_{fiy} = C_{fi} \cdot \beta_{fi}$$

$$\mu_{riy} = C_{ri} \, \beta_{ri}$$

WITH C_{ij} BEING KIND OF A CORNERING STIFFNESS

WITH "CONST" TO BE DEFINED, WHILE FROM KINEMATIC CONSIDERATIONS:

$$\beta_{ri} = \beta + \frac{r_w \cdot b}{v} \quad ; \quad \beta_{fi} = \beta + \delta - \frac{r_w \cdot a}{v}$$

SEE ACEHV CH. 3 - SLIDE 15

F_{ijz} ARE GIVEN BY BALANCE OF FORCES

PT. 2 – STATE EQUATIONS IN TIME DOMAIN

STATE AND VERTICAL FORCES EQUATIONS

(TO BE DEMONSTRATED YET)

$$
\begin{bmatrix}
m & 0 & -md & 0 & \mu_{fr_x} & \mu_{fl_x} & \mu_{rr_x} & \mu_{rl_x} \\
0 & m & mb & 0 & \mu_{fr_y} & \mu_{fl_y} & \mu_{rr_y} & \mu_{rl_y} \\
-md & mb & \tilde{I}_{zz} & 0 & \mu_{fr_{xy}} & \mu_{fl_{xy}} & -d_r\mu_{rr_x} & d_r\mu_{rl_x} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & mh & \tilde{I}_{xz} & 0 & -d_f & d_f & -d_r & d_r \\
-mh & 0 & mhd & 0 & \ell & \ell & 0 & 0 \\
0 & 0 & 0 & 0 & -d_r & d_r & d_f & -d_f
\end{bmatrix}
\begin{bmatrix}
\dot{v}_x \\ \dot{v}_y \\ \dot{r} \\ \dot{\delta} \\ f_{fr_z} \\ f_{fl_z} \\ f_{rr_z} \\ f_{rl_z}
\end{bmatrix}
+
\begin{bmatrix}
-mr(v_y + br) \\ mr(v_x - dr) \\ mr(dv_y + bv_x) \\ 0 \\ 0 \\ mhr(v_x - dr) \\ mhv_y r + \tilde{I}_{xz} r^2 \\ 0
\end{bmatrix}
+
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ -mg \\ -mgd \\ mgb \\ 0
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ u_\delta \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\quad (1)
$$

WHERE

$$\tilde{I}_{zz} = I_{zz} + m(b^2 + d^2)$$

$$\tilde{I}_{xz} = I_{xz} + mbh$$ (I_{xz} IS THE CROSS-COUPLING TERM IN INERTIA TENSOR)

$$\mu_{frxy} = \ell \mu_{fry} - d_f \mu_{frx}$$

$$\mu_{flxy} = \ell \mu_{fly} + d_f \mu_{flx}$$

INPUTS ARE:

- $u_\delta = \dot{\delta}$
- u_b : BRAKING COEFFICIENT TORQUE
- u_t : TRACTION COEFFICIENT TORQUE

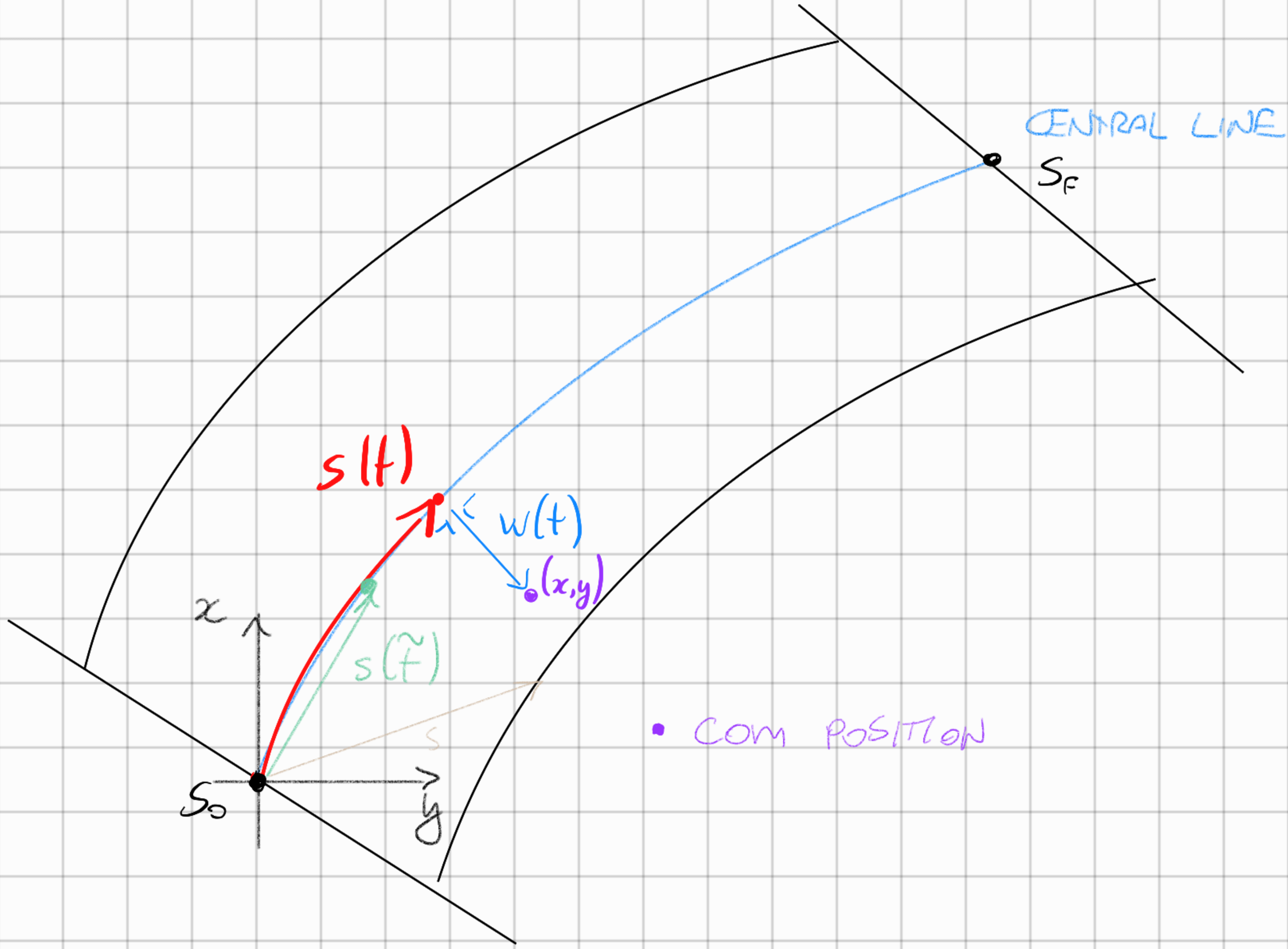
"INSIDE" μ_{fix}, μ_{rix}

PT. 3 – FROM TIME DOMAIN TO "S" DOMAIN

WE NEED TO GO FROM THE TIME COORDINATE t TO A LONGITUDINAL COORDINATE s. IN PARTICULAR:

① $s(t)$ REPRESENTS THE POSITION ALONG THE CENTER LINE OF THE TRACK

② $w(t)$ REPRESENTS THE DISPLACEMENT TRANSVERSE TO THE CENTRAL LINE.



$$\overset{*}{s}(t) = \int_0^t \sqrt{\dot{x}_{CB}^2(\tau) + \dot{y}_{CB}^2(\tau)}\, d\tau \Leftarrow$$

IF I CONSIDER TIME $\tilde{t}$: $s(\tilde{t}) = \sqrt{x_{cl}^2(\tilde{t}) + y_{cl}^2(\tilde{t})}$.

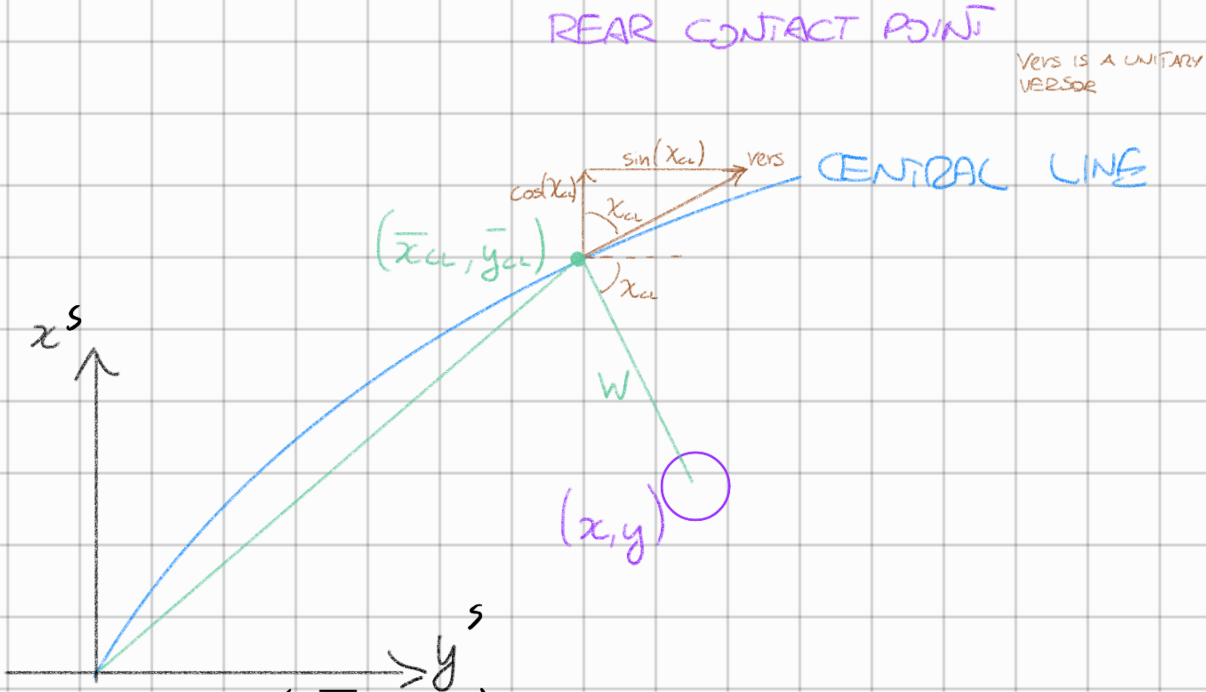
HOWEVER, SINCE A NATURAL CHOICE FOR THE NEW PARAMETRIZATION VARIABLE (IN PLACE OF TIME t) IS THE ARC-LENGTH CURVE ( ) $s(t)$:

(SINCE $\dot{s}(t) = \sqrt{\dot{x}_{cl}^2(t) + \dot{y}_{cl}^2(t)}$)

$$s(t) := \int_0^t \sqrt{\dot{x}_{cl}^2(\tau) + \dot{y}_{cl}^2(\tau)}\, d\tau$$

UNDER CERTAIN ASSUMPTIONS, IT CAN BE PROVED THAT $t \to s(t)$ IS STRICTLY INCREASING, SO THE MAPPING $t \to s(t)$ IS INVERTIBLE.

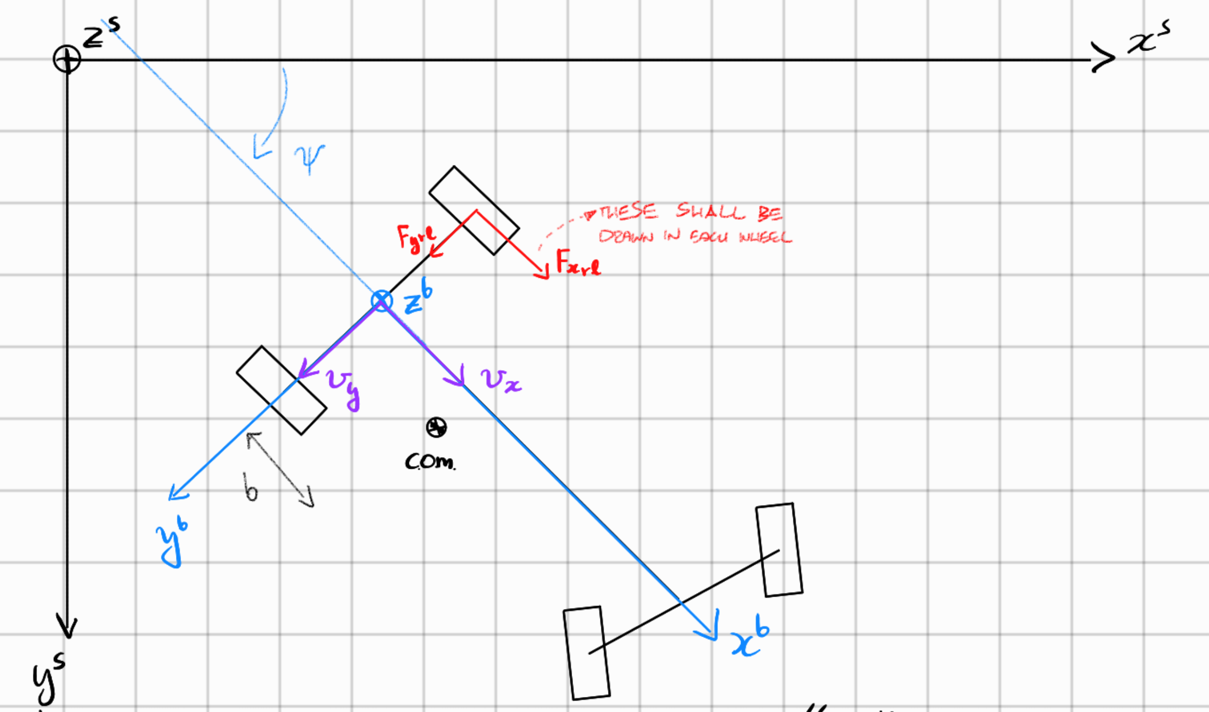
THIS ALLOWS TO DEFINE EVERYTHING AS FCN OF s INSTEAD OF t.



$$\begin{cases} x = \bar{x}_{cl}(s) - w \sin(\bar{\chi}_{cl}(s)) \\ y = \bar{y}_{cl}(s) + w \cos(\bar{\chi}_{cl}(s)) \end{cases}$$

STARTING FROM THIS SYSTEM OF EQUATIONS,

WE NOW WANT TO FIND EXPRESSIONS FOR THE TWO STATE DERIVATIVES $\dot{s}$ AND $\dot{w}$ TO PUT IN THE MODEL STARTING



NEGLECTING THE CONTRIBUTION FOR "$b\dot{\psi}$", THE VELOCITY OF THE REAR CONTACT POINT $(\dot{x}, \dot{y})$ IN THE GLOBAL REF. FRAME CAN BE EXPRESSED AS FCN OF THE COM VELOCITIES IN THE LOCAL REF. FRAME (v_x, v_y) AS:

$$\begin{cases} \dot{x} = v_x \cos\psi - v_y \sin\psi & (1a) \\ \dot{y} = v_x \sin\psi + v_y \cos\psi & (1b) \end{cases}$$

KNOWING THE EXPRESSION FOR $\dot{x}$ AND $\dot{y}$, WE WANT TO DIFFERENTIATE WRT TIME t THE FOUND SYSTEM:

$$\begin{cases} x = \bar{x}_{cl}(s(t)) - w\sin(\bar{\chi}_{cl}(s(t))) \\ y = \bar{y}_{cl}(s(t)) + w\cos(\chi_{cl}(s(t))) \end{cases}$$

$$\begin{cases} \dot{x} = \frac{d}{dt}\left(x_{cl}(s(t))\right) - \frac{d}{dt}\left(w\sin(\bar{\chi}_{cl})\right) & (2a) \\ \dot{y} = \frac{d}{dt}\left(y_{cl}(s(t))\right) + \frac{d}{dt}\left(w\cos(\bar{\chi}_{cl})\right) & (2b) \end{cases}$$

$\circledast$ $\frac{d}{dt}\left(\bar{x}_{cl}(s)\right)$

$$\frac{d}{dt}\left(\bar{x}_{cl}(s(t))\right) = \frac{\partial \bar{x}_{cl}}{\partial s}\cdot\frac{\partial s}{\partial t} = \cos\left(\bar{\chi}_{cl}\right)\cdot\dot{s} \qquad (3a)$$

$\circledast$ $\frac{d}{dt}\left(w\cdot\sin\left(\bar{\chi}_{cl}(s)\right)\right)$

$$\frac{d}{dt}\left(w\cdot\sin\left(\bar{\chi}_{cl}(s)\right)\right) =$$

$$\dot{w}\cdot\sin\left(\bar{\chi}_{cl}(s)\right) + w\,\frac{\partial\left[\sin\left(\bar{\chi}_{cl}(s)\right)\right]}{\partial s}\cdot\frac{\partial s}{\partial t} =$$

$$\dot{w}\sin\left(\bar{\chi}_{cl}(s)\right) + w\cos\left(\bar{\chi}_{cl}\right)\frac{\partial\bar{\chi}_{cl}}{\partial s}\,\dot{s} =$$

$$\dot{w}\sin\left(\bar{\chi}_{cl}\right) + w\bar{\sigma}_{cl}\,\dot{s}\cos\left(\bar{\chi}_{cl}\right) \qquad (4a)$$

PUTTING (1a), (3a), (4a) TOGETHER IN (2a)

$$v_x\cos\psi - v_y\sin\psi = \cos(\bar{\chi}_{cl})\dot{s} - \dot{w}\sin\left(\bar{\chi}_{cl}\right) - w\bar{\sigma}\dot{s}\cos\left(\bar{\chi}_{cl}\right)$$

$$v_x\cos\psi - v_y\sin\psi = \dot{s}\left(1 - w\bar{\sigma}\right)\cos\left(\bar{\chi}_{cl}\right) - \dot{w}\sin\left(\bar{\chi}_{cl}\right)$$

NOW, LOOKING AT y:

$$\dot{y} = \frac{d}{dt}\left(\bar{y}_{cl}(s)\right) + \frac{d}{dt}\left(w\cos\left(\bar{\chi}_{cl}(s)\right)\right)$$

ALREADY FOUND

② $\frac{d}{dt}\left(\bar{y}_{cl}(s)\right)$

SAME AS FOR $\bar{x}_{cl}(s)$:

$$\frac{d}{dt}\left(\bar{y}_{cl}(s)\right) = \dots = \sin\left(\bar{\chi}_{cl}\right)\dot{s} \qquad (3b)$$

② $\frac{d}{dt}\left(w\cos\left(\bar{\chi}_{cl}(s)\right)\right)$ ... SAME AS FOR x ...

$$\frac{d}{dt}\left(w\cos\left(\bar{\chi}_{cl}(s)\right)\right) = \dot{w}\cos\left(\bar{\chi}_{cl}\right) - w\bar{\sigma}_{cl}\dot{s}\sin\left(\bar{\chi}_{cl}\right) \qquad (4b)$$

PUTTING (1b), (3b), (4b) TOGETHER IN (2b)

$$v_x\sin\psi + v_y\cos\psi = \sin\left(\bar{\chi}_{cl}\right)\dot{s} + \dot{w}\cos\left(\bar{\chi}_{cl}\right) - w\bar{\sigma}_{cl}\dot{s}\sin\left(\bar{\chi}_{cl}\right)$$

$$v_x\sin\psi + v_y\cos\psi = \dot{s}\left(1 - w\bar{\sigma}_{cl}\right)\sin\left(\bar{\chi}_{cl}\right) + \dot{w}\cos\left(\bar{\chi}_{cl}\right)$$

THEN THE SYSTEM OF EQUATIONS TO BE SOLVED WRT $\dot{s}$ AND $\dot{w}$ IS:

$$\begin{cases} v_x \cos\psi - v_y \sin\psi = \dot{s}\left(1 - w\bar{\sigma}\right)\cos\left(\bar{\chi}_{cl}\right) - \dot{w}\sin\left(\bar{\chi}_{cl}\right) \\ v_x \sin\psi + v_y \cos\psi = \dot{s}\left(1 - w\bar{\sigma}_{cl}\right)\sin\left(\bar{\chi}_{cl}\right) + \dot{w}\cos\left(\bar{\chi}_{cl}\right) \end{cases}$$

AFTER SOME "TRIVIAL AND CUMBERSOME" COMPUTATIONS ONE CAN OBTAIN (I USED MATLAB):

$$\dot{s} = \frac{v_x \cos(\bar{\chi}_{cl})\cos(\psi) + v_x \sin(\bar{\chi}_{cl})\sin(\psi) + v_y \cos(\psi)\sin(\bar{\chi}_{cl}) - v_y \cos(\bar{\chi}_{cl})\sin(\psi)}{1 - \bar{\sigma}_{cl} W}$$

$$\dot{W} = v_y \cos(\bar{\chi}_{cl})\cos(\psi) + v_y \sin(\bar{\chi}_{cl})\sin(\psi) - \left[v_x \cos(\psi)\sin(\bar{\chi}_{cl}) - v_x \cos(\bar{\chi}_{cl})\sin(\psi)\right]$$

WE HAVE THAT:

...PROSTAPHERESIS FORMULAE...

$$\cos(\bar{\chi}_{cl})\cos(\psi)+\sin(\bar{\chi}_{cl})\sin(\psi) = \cos(\psi-\bar{\chi}_{cl})$$

...PROSTAPHERESIS FORMULAE...

$$\cos(\psi)\sin(\bar{\chi}_{cl})-\cos(\bar{\chi}_{cl})\sin(\psi) = \sin(\bar{\chi}_{cl}-\psi) =$$

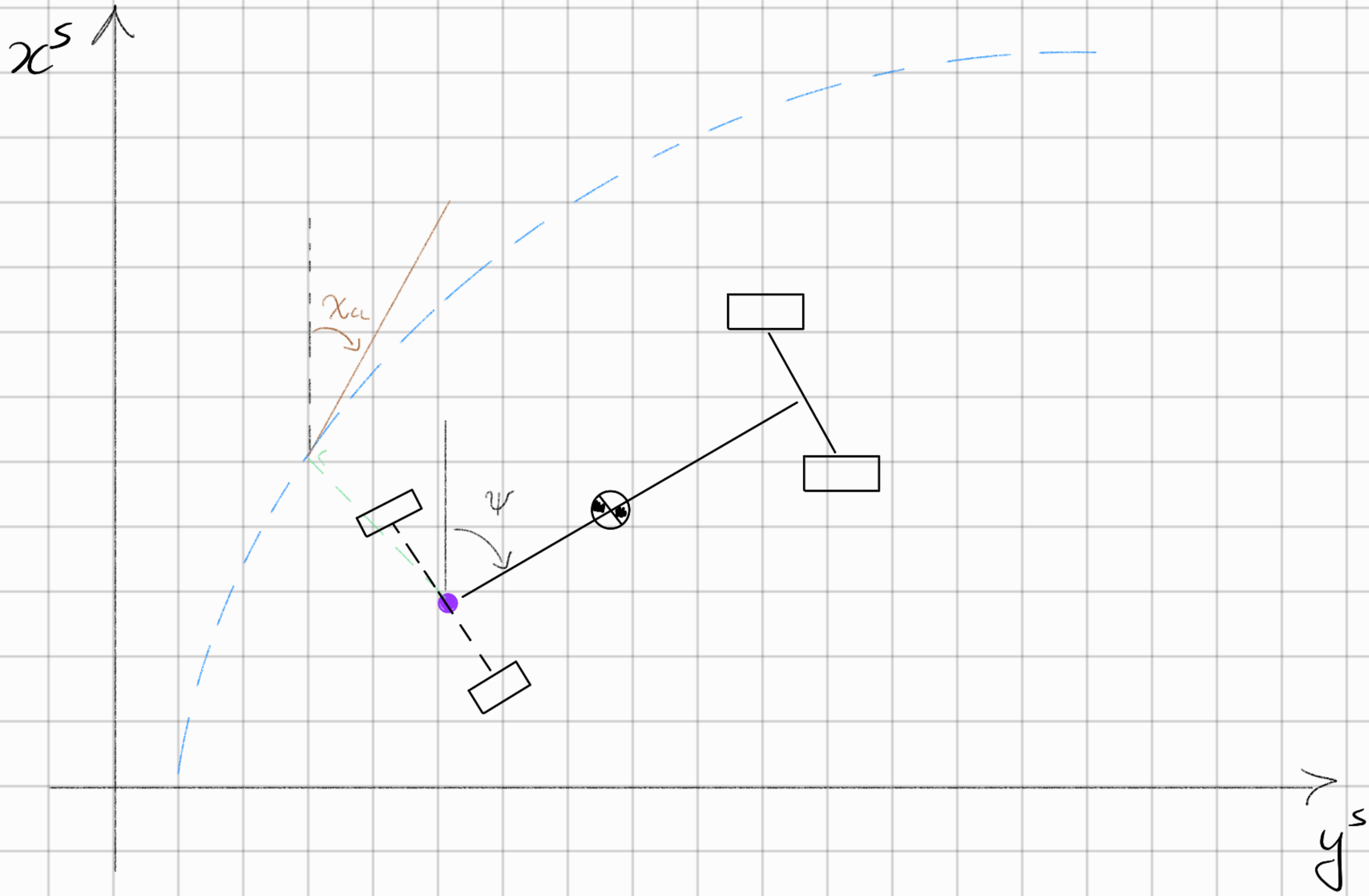
$$-\sin(\psi-\bar{\chi}_{cl})$$

EXPLOITING THAT WE GET:

$$\boxed{\begin{aligned}\dot{s} &= \frac{v_x\cos(\psi-\bar{\chi}_{cl})-v_y\sin(\psi-\bar{\chi}_{cl})}{1-\bar{\rho}_{cl}W} \\ \dot{W} &= v_x\sin(\psi-\bar{\chi}_{cl})+v_y\cos(\psi-\bar{\chi}_{cl})\end{aligned}}$$

... ALSO THE "LOCAL HEADING ANGLE" IS NEEDED... (see next page...

WE DEFINE THE LOCAL HEADING ANGLE AS:



WE DEFINE THE HEADING ANGLE $\mu = \psi - \bar{\chi}_{cl}$; ITS MEANING CAN BE INTERPRETED AS FOLLOWS:

- IF $\mu = 0$, THEN THE LONGITUDINAL AXIS OF THE VEHICLE IS PARALLEL TO THE LOCAL TANGENT OF THE CURVE, WHICH IS GOOD

WE HAVE THAT:

$$\dot{\mu} = \dot{\psi} - \dot{\bar{\chi}}_{cl} = \dot{\psi} - \frac{\partial \bar{\chi}_{cl}}{\partial s} \cdot \dot{s}$$

So:

$$\boxed{\dot{\mu} = \dot{\psi} - \bar{\sigma}_{cl} \cdot \frac{v_x \cos(\psi - \bar{\chi}_{cl}) - v_y \sin(\psi - \bar{\chi}_{cl})}{1 - \bar{\sigma}_{cl} W}}$$

THE EXPRESSION FOR THE OTHER STATES IN TIME DOMAIN CAN BE FOUND AT PAGE 4.

$$\dot{v}_x = f_{qv}^1(q_v, u)$$

$$\dot{v}_y = f_{qv}^2(q_v, u)$$

$$\dot{r} = f_{qv}^3(q_v, u)$$

$$\dot{\delta} = f_{qv}^4(q_v, u) \quad (\text{IN THIS CASE, IT'S JUST } \dot{\delta} = u_\delta)$$

$$\dot{\psi} = r$$

EVENTUALLY, ONE CAN OBTAIN THE STATE EQUATIONS IN THE INDEPENDENT COORDINATE $s(t)$ SINCE FOR THE GENERIC STATE VARIABLE $\alpha(t)$ ONE CAN WRITE THAT:

$$\dot{\alpha}(t) = f_\alpha(x,u) \longrightarrow \frac{d\alpha}{ds} \cdot \frac{ds}{dt} = f_\alpha(x,u) \longrightarrow$$

$$\frac{d\alpha}{ds} = \alpha'(s) = f_\alpha(x,u) \cdot \dot{s}^{-1}$$

WITH $\dot{s}^{-1} = \dfrac{1 - \rho_{cl}(s)\, w(s)}{v_x(s)\cos\mu(s) - v_y(s)\sin(\mu(s))}$

SO EVENTUALLY:

$$W'(s) = \frac{dw}{dt} \cdot \frac{dt}{ds} = \left(v_x \sin(\psi - \bar{\chi}_{cl}) + v_y \cos(\psi - \bar{\chi}_{cl}) \right) \cdot \overbrace{(\dots)}^{\dot{s}^{-1}}$$

$$\mu'(s) = \dot{\mu} \cdot \dot{s}^{-1} = \left(r - \bar{\sigma}_{cl} \cdot \frac{v_x \cos(\psi - \bar{\chi}_{cl}) - v_y \sin(\psi - \bar{\chi}_{cl})}{1 - \bar{\sigma}_{cl} W} \right) (\dots)$$

$$v_x'(s) = \dot{v}_x \cdot \dot{s}^{-1} = f^1_{qv}(q_v, u) \cdot (\dots)$$

$$v_y'(s) = \dot{v}_y \cdot \dot{s}^{-1} = f^2_{qv}(q_v, u) \cdot (\dots)$$

$$r'(s) = \dot{r} \cdot \dot{s}^{-1} = f^3_{qv}(q_v, u) \ (\dots)$$

$$\delta'(s) = \dot{\delta} \cdot \dot{s}^{-1} = f^4_{qv}(q_v, u) \ (\dots)$$

$$\psi'(s) = \dot{\psi} \cdot \dot{s}^{-1} = r \cdot (\dots)$$

$$u = [u_\delta; u_t; u_b]$$