Mathematical definitions for PyTelTools

Yishu Wang, Luc Duron

August 2017

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1 Introduction

We call informally a **2D mesh** $\mathcal{M}$, or simple **mesh**, a finite network where

— an **element** $\mathcal{T}$ is a triangle in $\mathbb{R}^2$; we write $\mathcal{T} \in \mathcal{M}$; its vertices $P_{1,2,3}$ are called **points** of the mesh;

— a **connection** is an common edge (two common vertices) between two elements;

such that

— every two elements can either intersect at one common vertex, intersect at one common edge (two common vertices), or have empty intersection;

— every element has at least one common edge with another element (the mesh is **connected**).

Furthermore, for some integer t between 0 and $T_{\max}$, a mesh can have a **valuation** $\mathcal{V}_t$ at time t, which assigns to each point of the mesh a real number, called the **value** of the point.

Other useful notations include

— $\mathrm{area}(\mathcal{T})$ is the area of the triangle $\mathcal{T}$;

— $\mathrm{area}(\mathcal{M}) = \sum_{\mathcal{T}\in\mathcal{M}} \mathrm{area}(\mathcal{T})$ is the total area of elements in the mesh $\mathcal{M}$;

— $\mathrm{size}(\mathcal{M})$ is the number of elements in the mesh $\mathcal{M}$.

2 Interpolation

Definition 1. Let $\mathcal{T}$ be a triangle in $\mathbb{R}^2$ of vertices $P_{1,2,3}$, for a point $Q \in \mathcal{T}$ (in the interior or on the boundary), the **barycentric coordinates** of Q is equal to

$$\lambda_1 = \frac{\mathrm{area}(\mathcal{T}_1)}{\mathrm{area}(\mathcal{T})}, \quad \lambda_2 = \frac{\mathrm{area}(\mathcal{T}_2)}{\mathrm{area}(\mathcal{T})}, \quad \lambda_3 = \frac{\mathrm{area}(\mathcal{T}_3)}{\mathrm{area}(\mathcal{T})}$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are triangles in $\mathbb{R}^2$ defined by the vertices $P_2\,P_3\,Q$, $P_1\,P_3\,Q$ and $P_1\,P_2\,Q$ (when Q is on the boundary of $\mathcal{T}$, one or two of the triangles $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are considered to be empty triangle).

Definition 2. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ and an element $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$, the **interpolated value of a point** $Q \in \mathcal{T}$ is equal to the dot product of the barycentric coordinates of Q and the values of $P_{1,2,3}$:

$$\lambda_1\,\mathcal{V}_t(P_1) + \lambda_2\,\mathcal{V}_t(P_2) + \lambda_3\,\mathcal{V}_t(P_3)$$

3 Sub-triangulation

3.1 Definition

Definition 3. A **sub-triangulation** $(\mathcal{M}^*,\, \mathcal{V}_t^*)$ of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the result of the operation that, for every element $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$,

— adds a point Q in the interior of $\mathcal{T}$ and splits $\mathcal{T}$ into three elements $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ of vertices $P_2\,P_3\,Q$, $P_1\,P_3\,Q$ and $P_1\,P_2\,Q$, called the splits of $\mathcal{T}$ at Q;

— extends $\mathcal{V}_t$ to Q and assigns to Q the interpolated value at Q.

We note $\mathrm{ST}(\mathcal{M},\, \mathcal{V}_t)$ the set of all sub-triangulations of $\mathcal{M}$ valuated at $\mathcal{V}_t$.

3.2 Invariance under sub-triangulation

Definition 4. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, a function f of $(\mathcal{M},\, \mathcal{V}_t)$ is **invariant under sub-triangulation** if f is constant in the set of all sub-triangulations of $\mathcal{M}$ valuated at $\mathcal{V}_t$:

$$\forall(\mathcal{M}^*,\, \mathcal{V}_t^*) \in \mathrm{ST}(\mathcal{M},\, \mathcal{V}_t), \quad f(\mathcal{M}^*,\, \mathcal{V}_t^*) = f(\mathcal{M},\, \mathcal{V}_t).$$

4 Flux

TODO: coming soon

Define mass flux

Define simplified fluxes :
- line flux
- area flux

5 Volume

5.1 Volume of an element

Definition 5. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **volume** $\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ (also called **net volume**) of vertices $P_{1,2,3}$ is the product of the area of $\mathcal{T}$ by one third the sum of the values of its three vertices:

$$\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t) = \frac{\mathrm{area}(\mathcal{T})}{3} \sum_{P_i} \mathcal{V}_t(P_i).$$

5.2 Explanation

The definition of the volume of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$ can be easily understood by picturing the **truncated right triangular prism** $\overline{\mathcal{T}}_t$ in $\mathbb{R}^3$ naturally induced by the valuation $\mathcal{V}_t$.

If we note $(x_1,\, y_1)$, $(x_2,\, y_2)$, $(x_3,\, y_3)$ the coordinates of the three vertices $P_{1,2,3}$ of $\mathcal{T}$ in a Cartesian coordinate system, then the prism $\overline{\mathcal{T}}_t$ is defined by the following six vertices (figure 1):
- the **projected points** $\overline{P}_{1,2,3}$ of coordinates $(x_1,\, y_1,\, 0)$, $(x_2,\, y_2,\, 0)$, $(x_3,\, y_3,\, 0)$ which form the **base triangle** of the prism,
- the **valuated points** $\overline{P}^{\mathcal{V}_t}_{1,2,3}$ of coordinates $(x_1,\, y_1,\, \mathcal{V}_t(P_1))$, $(x_2,\, y_2,\, \mathcal{V}_t(P_2))$, $(x_3,\, y_3,\, \mathcal{V}_t(P_3))$ which form the **upper triangle** of the prism.

This prism is truncated because the upper triangle is not necessarily parallel to the base triangle ; it is also a right prism because the three lateral edges is always perpendicular to the base triangle.

It is known from the classical geometry that the volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges. The volume of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$ in definition 5 corresponds thus to the volume of its induced prism.

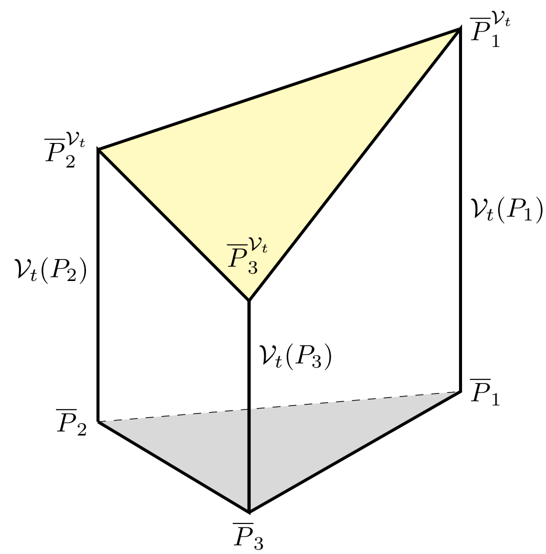

FIGURE 1 – The induced prism $\overline{\mathcal{T}}_t$ of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$

Since the valuation $\mathcal{V}_t$ can take positive and negative values, the upper triangle does not always lie on the half-space $z > 0$ or $z < 0$. When the upper triangle intersects the base triangle, the induced prism $\overline{\mathcal{T}}_t$ **is not a polyhedron**, but the volume defined above is still valid.

5.3 Volume of a mesh

Definition 6. The **volume** $\text{volume}_t^{\text{abs}}(\mathcal{M}, \mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the sum of the volumes of its elements:

$$\text{volume}_t(\mathcal{M}, \mathcal{V}_t) = \sum_{\mathcal{T} \in \mathcal{M}} \text{volume}_t(\mathcal{T}, \mathcal{V}_t).$$

Proposition 1. The volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 6 is invariant under sub-triangulation.

Proof. Let $\mathcal{M}$ be a mesh valuated at $\mathcal{V}_t$ and $(\mathcal{M}^*, \mathcal{V}_t^*) \in \text{ST}(\mathcal{M}, \mathcal{V}_t)$. Let $\mathcal{T}$ be any element of $\mathcal{M}$ and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ its splits in $\mathcal{M}^*$ at the point Q. First we write down the total volume of the three splits:

$$\begin{aligned}
&\sum_{k=1}^{3} \text{volume}_t(\mathcal{T}_k, \mathcal{V}_t^*) \\
&= \frac{\text{area}(\mathcal{T}_1)}{3} \left(\mathcal{V}_t^*(P_2) + \mathcal{V}_t^*(P_3) + \mathcal{V}_t^*(Q)\right) + \frac{\text{area}(\mathcal{T}_2)}{3} \left(\mathcal{V}_t^*(P_1) + \mathcal{V}_t^*(P_3) + \mathcal{V}_t^*(Q)\right) \\
&\quad + \frac{\text{area}(\mathcal{T}_3)}{3} \left(\mathcal{V}_t^*(P_1) + \mathcal{V}_t^*(P_2) + \mathcal{V}_t^*(Q)\right) \\
&= \frac{1}{3} \sum_{k=1}^{3} \left(\mathcal{V}_t^*(P_k) \sum_{\substack{j=1,2,3 \\ j \neq k}} \text{area}(\mathcal{T}_j) \right) + \frac{1}{3} \mathcal{V}_t^*(Q) \sum_{k=1}^{3} \text{area}(\mathcal{T}_k).
\end{aligned}$$

By the definition of sub-triangulation, the values $\mathcal{V}_t^*(P_k) = \mathcal{V}_t(P_k)$ and the value of Q is equal to

$$\mathcal{V}_t^*(Q) = \sum_{k=1}^{3} \lambda_k\, \mathcal{V}_t(P_k) = \frac{1}{\text{area}\,(\mathcal{T})} \sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)\, \mathcal{V}_t(P_k) = \frac{\sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)\, \mathcal{V}_t(P_k)}{\sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)},$$

thus,

$$\begin{aligned}
\sum_{k=1}^{3} \text{volume}_t(\mathcal{T}_k,\, \mathcal{V}_t^*) &= \frac{1}{3} \sum_{k=1}^{3} \left(\mathcal{V}_t(P_k) \sum_{\substack{j=1,2,3 \\ j \neq k}} \text{area}\,(\mathcal{T}_j) \right) + \frac{1}{3} \sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)\, \mathcal{V}_t(P_k) \\
&= \frac{1}{3} \sum_{k=1}^{3} \left(\mathcal{V}_t(P_k) \sum_{j=1}^{3} \text{area}\,(\mathcal{T}_j) \right) = \frac{\text{area}\,(\mathcal{T})}{3} \sum_{k=1}^{3} \mathcal{V}_t(P_i) \\
&= \text{volume}_t(\mathcal{T},\, \mathcal{V}_t).
\end{aligned}$$

Therefore,

$$\text{volume}_t(\mathcal{M}^*,\, \mathcal{V}_t^*) = \sum_{\mathcal{T} \in \mathcal{M}} \left(\sum_{\substack{\mathcal{T}_k \\ \text{splits of } \mathcal{T}}} \text{volume}_t(\mathcal{T}_k,\, \mathcal{V}_t^*) \right) = \sum_{\mathcal{T} \in \mathcal{M}} \text{volume}_t(\mathcal{T},\, \mathcal{V}_t) = \text{volume}_t(\mathcal{M},\, \mathcal{V}_t).$$

□

5.4 Positive and negative volumes of an element

Definition 7. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **positive volume** $\text{volume}_t^+(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$, is equal to

- 0, if $\mathcal{V}_t(P_i) \leqslant 0$ for $i = 1, 2, 3$ (the upper triangle lies in the half-space $z \leqslant 0$) ;
- its net volume $\text{volume}_t(\mathcal{T},\, \mathcal{V}_t)$, if $\mathcal{V}_t(P_i) \geqslant 0$ for $i = 1, 2, 3$ (the upper triangle lies in the half-space $z \geqslant 0$) ;
- otherwise, the upper triangle intersect the plane $z = 0$ at a segment $Q_1\, Q_2$ (figure 2). There are two cases:
  - if $\mathcal{T}$ has only one vertex, w.l.o.g. P_1, such that $\mathcal{V}_t(P_1) > 0$, the positive volume is equal to the volume of the tetrahedron defined by Q_1, Q_2, the projected point $\overline{P}_1$ and the valuated point $\overline{P}_1^{\mathcal{V}_t}$ of P_1 ;
  - otherwise, $\mathcal{T}$ has only one vertex, w.l.o.g. P_1, such that $\mathcal{V}_t(P_1) < 0$. The positive volume is equal to the difference between the net volume $\text{volume}_t(\mathcal{T},\, \mathcal{V}_t)$ and the volume of the tetrahedron defined by Q_1, Q_2, the projected point $\overline{P}_1$ and the valuated point $\overline{P}_1^{\mathcal{V}_t}$ of P_1.

Definition 8. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **negative volume** $\text{volume}_t^-(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ is equal to the difference between its net volume and its positive volume:

$$\text{volume}_t^-(\mathcal{T},\, \mathcal{V}_t) = \text{volume}_t(\mathcal{T},\, \mathcal{V}_t) - \text{volume}_t^+(\mathcal{T},\, \mathcal{V}_t).$$

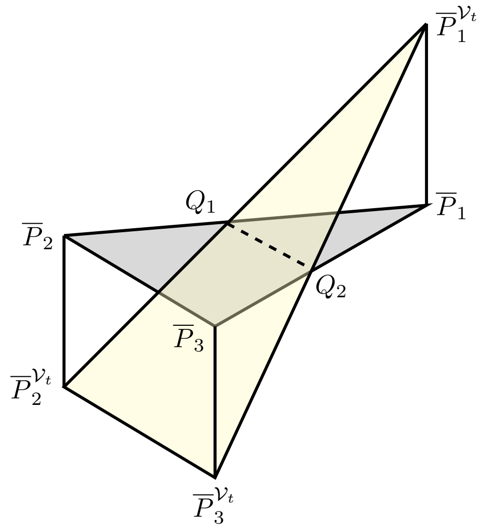

FIGURE 2 – When the upper triangle intersects the base triangle at a segment $Q_1\,Q_2$

5.5 Absolute and quadratic volumes of an element, of a mesh

Definition 9. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **absolute volume** $\text{volume}_t^{\text{abs}}(\mathcal{T},\,\mathcal{V}_t)$ **of an element** $\mathcal{T}\in\mathcal{M}$ of vertices $P_{1,2,3}$ is the product of the area of $\mathcal{T}$ by one third the sum of the absolute values of its three vertices:

$$\text{volume}_t^{\text{abs}}(\mathcal{T},\,\mathcal{V}_t) = \frac{\text{area}\,(\mathcal{T})}{3}\sum_{k=1}^{3}|\mathcal{V}_t(P_i)|\,.$$

Definition 10. The **absolute volume** $\text{volume}_t^{\text{abs}}(\mathcal{M},\,\mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the sum of the absolute volumes of its elements:

$$\text{volume}_t^{\text{abs}}(\mathcal{M},\,\mathcal{V}_t) = \sum_{\mathcal{T}\in\mathcal{M}}\text{volume}_t^{\text{abs}}(\mathcal{T},\,\mathcal{V}_t).$$

Proposition 2. The absolute volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 10 is invariant under sub-triangulation.

Proof. Analogous to the proof of proposition 1. □

Definition 11. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **quadratic volume** $\text{volume}_t^{\text{quad}}(\mathcal{T},\,\mathcal{V}_t)$ **of an element** $\mathcal{T}\in\mathcal{M}$ is equal to the volume of $\mathcal{T}$ valuated at $\mathcal{V}_t^2$:

$$\text{volume}_t^{\text{quad}}(\mathcal{T},\,\mathcal{V}_t) = \text{volume}_t(\mathcal{T},\,\mathcal{V}_t^2) = \frac{\text{area}\,(\mathcal{T})}{3}\sum_{k=1}^{3}\mathcal{V}_t^2(P_i).$$

Definition 12. The **quadratic volume** $\text{volume}_t^{\text{abs}}(\mathcal{M}, \mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the square root of the sum of the quadratic volumes of its elements:

$$\text{volume}_t^{\text{quad}}(\mathcal{M}, \mathcal{V}_t) = \sqrt{\sum_{\mathcal{T}\in\mathcal{M}} \text{volume}_t^{\text{quad}}(\mathcal{T}, \mathcal{V}_t)}.$$

Proposition 3. The quadratic volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 12 is invariant under sub-triangulation.

Proof. By definition,

$$\begin{aligned}
\text{volume}_t^{\text{quad}}(\mathcal{M}, \mathcal{V}_t) &= \sqrt{\sum_{\mathcal{T}\in\mathcal{M}} \text{volume}_t^{\text{quad}}(\mathcal{T}, \mathcal{V}_t)} \\
&= \sqrt{\sum_{\mathcal{T}\in\mathcal{M}} \text{volume}_t(\mathcal{T}, \mathcal{V}_t^2)} \\
&= \sqrt{\text{volume}_t(\mathcal{M}, \mathcal{V}_t^2)},
\end{aligned}$$

yet by proposition 1, the volume of $\mathcal{M}$ valuated at $\mathcal{V}_t^2$ is invariant under sub-triangulation, thus the quadratic volume of $\mathcal{M}$ valuated at $\mathcal{V}_t$ is also invariant under sub-triangulation. $\square$

6 Comparison between two meshes

6.1 Comparison between two identical meshes

Definition 13. Two meshes $\mathcal{M}_1$ and $\mathcal{M}_2$ are **identical** if the sets of their elements are equal:

$$\{\mathcal{T} \in \mathcal{M}_1\} = \{\mathcal{T} \in \mathcal{M}_2\}$$

.

Definition 14. A **reference mesh** is a mesh $\mathcal{M}_0$ with a constant valuation $\mathcal{V}_t = \mathcal{V}_0$ for all time t.

6.1.1 Mean signed deviation

Definition 15. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **mean signed deviation (MSD)** with respect to $\mathcal{M}_0$ is the function $\text{MSD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0, \mathcal{V}_t)$ to the volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$, divided by the total area of $\mathcal{M}_0$:

$$\text{MSD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0, \mathcal{V}_t) = \frac{\text{volume}_t(\mathcal{M}_0, \mathcal{V}_t - \mathcal{V}_0)}{\text{area}\,(\mathcal{M}_0)}.$$

Proposition 4. The mean signed distance is invariant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. By proposition 1, the volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$ is invariant under sub-triangulation. Therefore the mean signed deviation is also invariant under sub-triangulation. □

6.1.2 Mean absolute deviation

Definition 16. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **mean absolute error (MAD)** with respect to $\mathcal{M}_0$ is the function $\mathrm{MAD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0,\, \mathcal{V}_t)$ to the absolute volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$, divided by the total area of $\mathcal{M}_0$:

$$\mathrm{MAD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t) = \frac{\mathrm{volume}_t^{\mathrm{abs}}(\mathcal{M}_0, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}.$$

Proposition 5. The mean absolute distance is invariant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. By proposition 2, the absolute volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$ is invariant under sub-triangulation. Therefore the mean absolute deviation is also invariant under sub-triangulation. □

6.1.3 Root mean square deviation

Definition 17. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **root mean square deviation (RMSD)** with respect to $\mathcal{M}_0$ is the function $\mathrm{RMSD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0,\, \mathcal{V}_t)$ to the quadratic volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$, divided by the total area of $\mathcal{M}_0$:

$$\mathrm{RMSD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t) = \frac{\mathrm{volume}_t^{\mathrm{quad}}(\mathcal{M}_0, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}.$$

Proposition 6. The root mean square deviation is invariant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. By proposition 3, the quadratic volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$ is invariant under sub-triangulation. Therefore the root mean square deviation is also invariant under sub-triangulation. □

6.1.4 Element-wise signed deviation, deviation distribution

Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **deviation distribution function** F_X is the empirical distribution function of the variable

$$X = \frac{\mathrm{size}\,(\mathcal{M}_0)\,\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)},$$

called the **element-wise signed deviation**, measured from the set of all elements $\mathcal{T} \in \mathcal{M}_0$:

$$F_X(x) = \frac{1}{\mathrm{size}\,(\mathcal{M})}\,\mathrm{card}\left\{\mathcal{T} \in \mathcal{M}_0 \;\middle|\; \frac{\mathrm{size}\,(\mathcal{M})\,\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M})} \leqslant x\right\}, \quad x \in \mathbb{R}.$$

Remark 1. The empirical mean of the element-wise signed deviation is equal to the mean signed deviation defined in 15:

$$
\begin{aligned}
\mathbb{E}(X) &= \frac{1}{\operatorname{size}(\mathcal{M}_0)} \sum_{\mathcal{T} \in \mathcal{M}_0} \left(\frac{\operatorname{size}(\mathcal{M}_0)\, \operatorname{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\operatorname{area}(\mathcal{M}_0)} \right) \\
&= \sum_{\mathcal{T} \in \mathcal{M}_0} \left(\frac{\operatorname{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\operatorname{area}(\mathcal{M}_0)} \right) \\
&= \frac{\operatorname{volume}_t(\mathcal{M}_0,\, \mathcal{V}_t - \mathcal{V}_0)}{\operatorname{area}(\mathcal{M}_0)} \\
&= \operatorname{MSD}_{\mathcal{M}_0, \mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t),
\end{aligned}
$$

and is therefore invariant under sub-triangulation.