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1 Introduction

We call informally a **2D mesh** $\mathcal{M}$, or simple **mesh**, a finite network where

— an **element** $\mathcal{T}$ is a triangle in $\mathbb{R}^2$; we write $\mathcal{T} \in \mathcal{M}$; its vertices $P_{1,2,3}$ are called **points** of the mesh ;

— a **connection** is an common edge (two common vertices) between two elements ;

such that

— every two elements can either intersect at one common vertex, intersect at one common edge (two common vertices), or have empty intersection ;

— every element has at least one common edge with another element (the mesh is **connected**).

Furthermore, for some integer t between 0 and $T_{\max}$, a mesh can have a **valuation** $\mathcal{V}_t$ at time t, which assigns to each point of the mesh a real number, called the **value** of the point.

Other useful notations include

— $\text{area}(\mathcal{T})$ is the area of the triangle $\mathcal{T}$;

— $\text{area}(\mathcal{M}) = \sum_{\mathcal{T} \in \mathcal{M}} \text{area}(\mathcal{T})$ is the total area of elements in the mesh $\mathcal{M}$;

— $\text{size}(\mathcal{M})$ is the number of elements in the mesh $\mathcal{M}$.

2 Interpolation

Definition 1. Let $\mathcal{T}$ be a triangle in $\mathbb{R}^2$ of vertices $P_{1,2,3}$, for a point $Q \in \mathcal{T}$ (in the interior or on the boundary), the **barycentric coordinates** of Q is equal to

$$\lambda_1 = \frac{\text{area}(\mathcal{T}_1)}{\text{area}(\mathcal{T})}, \quad \lambda_2 = \frac{\text{area}(\mathcal{T}_2)}{\text{area}(\mathcal{T})}, \quad \lambda_3 = \frac{\text{area}(\mathcal{T}_3)}{\text{area}(\mathcal{T})}$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are triangles in $\mathbb{R}^2$ defined by the vertices $P_2\, P_3\, Q$, $P_1\, P_3\, Q$ and $P_1\, P_2\, Q$ (when Q is on the boundary of $\mathcal{T}$, one or two of the triangles $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are considered to be empty triangle).

Definition 2. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ and an element $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$, the **interpolated value of a point** $Q \in \mathcal{T}$ is equal to the dot product of the barycentric coordinates of Q and the values of $P_{1,2,3}$:

$$\lambda_1\, \mathcal{V}_t(P_1) + \lambda_2\, \mathcal{V}_t(P_2) + \lambda_3\, \mathcal{V}_t(P_3)$$

3 Sub-triangulation

3.1 Definition

Definition 3. A **sub-triangulation** $(\mathcal{M}^*,\, \mathcal{V}_t^*)$ of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the result of the operation that, for every element $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$,

— adds a point Q in the interior of $\mathcal{T}$ and splits $\mathcal{T}$ into three elements $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ of vertices $P_2\, P_3\, Q$, $P_1\, P_3\, Q$ and $P_1\, P_2\, Q$, called the splits of $\mathcal{T}$ at Q ;

— extends $\mathcal{V}_t$ to Q and assigns to Q the interpolated value at Q.

We note $\text{ST}(\mathcal{M},\, \mathcal{V}_t)$ the set of all sub-triangulations of $\mathcal{M}$ valuated at $\mathcal{V}_t$.

3.2 Constant, increasing or decreasing under sub-triangulation

Definition 4. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, a function f of $(\mathcal{M},\, \mathcal{V}_t)$ is **constant under sub-triangulation** if the values of f on the set of all sub-triangulations of $\mathcal{M}$ valuated at $\mathcal{V}_t$ are equal to its value on $(\mathcal{M},\, \mathcal{V}_t)$:

$$\forall(\mathcal{M}^*,\, \mathcal{V}_t^*) \in \mathrm{ST}(\mathcal{M},\, \mathcal{V}_t), \quad f(\mathcal{M}^*,\, \mathcal{V}_t^*) = f(\mathcal{M},\, \mathcal{V}_t).$$

Definition 5. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, a function f of $(\mathcal{M},\, \mathcal{V}_t)$ is **increasing (resp. decreasing) under sub-triangulation** if the values of f on the set of all sub-triangulations of $\mathcal{M}$ valuated at $\mathcal{V}_t$ are greater than (resp. less than) or equal to its value on $(\mathcal{M},\, \mathcal{V}_t)$:

$$\forall(\mathcal{M}^*,\, \mathcal{V}_t^*) \in \mathrm{ST}(\mathcal{M},\, \mathcal{V}_t), \quad f(\mathcal{M}^*,\, \mathcal{V}_t^*) \geqslant f(\mathcal{M},\, \mathcal{V}_t)$$

$$(\text{resp. } \forall(\mathcal{M}^*,\, \mathcal{V}_t^*) \in \mathrm{ST}(\mathcal{M},\, \mathcal{V}_t), \quad f(\mathcal{M}^*,\, \mathcal{V}_t^*) \leqslant f(\mathcal{M},\, \mathcal{V}_t)\ ).$$

4 Volume

4.1 Volume of an element

Definition 6. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **volume** $\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ (also called **net volume**) of vertices $P_{1,2,3}$ is the product of the area of $\mathcal{T}$ by one third the sum of the values of its three vertices:

$$\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t) = \frac{\mathrm{area}(\mathcal{T})}{3} \sum_{P_i} \mathcal{V}_t(P_i).$$

4.2 Explanation

The definition of the volume of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$ can be easily understood by picturing the **truncated right triangular prism** $\overline{\mathcal{T}}_t$ in $\mathbb{R}^3$ naturally induced by the valuation $\mathcal{V}_t$.

If we note $(x_1,\, y_1)$, $(x_2,\, y_2)$, $(x_3,\, y_3)$ the coordinates of the three vertices $P_{1,2,3}$ of $\mathcal{T}$ in a Cartesian coordinate system, then the prism $\overline{\mathcal{T}}_t$ is defined by the following six vertices (figure 1):

- the **projected points** $\overline{P}_{1,2,3}$ of coordinates $(x_1,\, y_1,\, 0)$, $(x_2,\, y_2,\, 0)$, $(x_3,\, y_3,\, 0)$ which form the **base triangle** of the prism,
- the **valuated points** $\overline{P}^{\mathcal{V}_t}_{1,2,3}$ of coordinates $(x_1,\, y_1,\, \mathcal{V}_t(P_1))$, $(x_2,\, y_2,\, \mathcal{V}_t(P_2))$, $(x_3,\, y_3,\, \mathcal{V}_t(P_3))$ which form the **upper triangle** of the prism.

This prism is truncated because the upper triangle is not necessarily parallel to the base triangle; it is also a right prism because the three lateral edges is always perpendicular to the base triangle.

It is known from the classical geometry that the volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges. The volume of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$ in definition 6 corresponds thus to the volume of its induced prism.

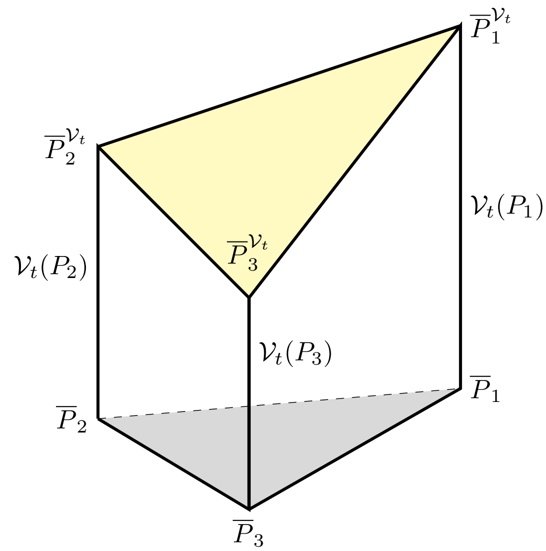

FIGURE 1 – The induced prism $\overline{\mathcal{T}}_t$ of an element $\mathcal{T}$ valuated at $\mathcal{V}_t$

Since the valuation $\mathcal{V}_t$ can take positive and negative values, the upper triangle does not always lie on the half-space $z > 0$ or $z < 0$. When the upper triangle intersects the base triangle, the induced prism $\overline{\mathcal{T}}_t$ **is not a polyhedron**, but the volume defined above is still valid.

4.3 Volume of a mesh

Definition 7. The **volume** $\text{volume}_t^{\text{abs}}(\mathcal{M}, \mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the sum of the volumes of its elements:

$$\text{volume}_t(\mathcal{M}, \mathcal{V}_t) = \sum_{\mathcal{T} \in \mathcal{M}} \text{volume}_t(\mathcal{T}, \mathcal{V}_t).$$

Proposition 1. The volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 7 is invariant under sub-triangulation.

Proof. Let $\mathcal{M}$ be a mesh valuated at $\mathcal{V}_t$ and $(\mathcal{M}^*, \mathcal{V}_t^*) \in \text{ST}(\mathcal{M}, \mathcal{V}_t)$. Let $\mathcal{T}$ be any element of $\mathcal{M}$ and

$\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ its splits in $\mathcal{M}^*$ at the point Q. First we write down the total volume of the three splits:

$$\begin{aligned}
&\sum_{k=1}^{3} \text{volume}_t(\mathcal{T}_k,\, \mathcal{V}_t^*) \\
&= \frac{\text{area}\,(\mathcal{T}_1)}{3}\,(\mathcal{V}_t^*(P_2) + \mathcal{V}_t^*(P_3) + \mathcal{V}_t^*(Q)) + \frac{\text{area}\,(\mathcal{T}_2)}{3}\,(\mathcal{V}_t^*(P_1) + \mathcal{V}_t^*(P_3) + \mathcal{V}_t^*(Q)) \\
&\quad + \frac{\text{area}\,(\mathcal{T}_3)}{3}\,(\mathcal{V}_t^*(P_1) + \mathcal{V}_t^*(P_2) + \mathcal{V}_t^*(Q)) \\
&= \frac{1}{3}\sum_{k=1}^{3}\left(\mathcal{V}_t^*(P_k)\sum_{\substack{j=1,2,3\\ j\neq k}} \text{area}\,(\mathcal{T}_j)\right) + \frac{1}{3}\,\mathcal{V}_t^*(Q)\,\text{area}\,(\mathcal{T}).
\end{aligned}$$

By the definition of sub-triangulation, the values $\mathcal{V}_t^*(P_k) = \mathcal{V}_t(P_k)$ and the value of Q is equal to

$$\mathcal{V}_t^*(Q) = \sum_{k=1}^{3} \lambda_k\, \mathcal{V}_t(P_k) = \frac{1}{\text{area}\,(\mathcal{T})}\sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)\, \mathcal{V}_t(P_k),$$

thus,

$$\begin{aligned}
\sum_{k=1}^{3} \text{volume}_t(\mathcal{T}_k,\, \mathcal{V}_t^*) &= \frac{1}{3}\sum_{k=1}^{3}\left(\mathcal{V}_t(P_k)\sum_{\substack{j=1,2,3\\ j\neq k}} \text{area}\,(\mathcal{T}_j)\right) + \frac{1}{3}\sum_{k=1}^{3} \text{area}\,(\mathcal{T}_k)\, \mathcal{V}_t(P_k) \\
&= \frac{1}{3}\sum_{k=1}^{3}\left(\mathcal{V}_t(P_k)\sum_{j=1}^{3} \text{area}\,(\mathcal{T}_j)\right) = \frac{\text{area}\,(\mathcal{T})}{3}\sum_{k=1}^{3} \mathcal{V}_t(P_i) \\
&= \text{volume}_t(\mathcal{T},\, \mathcal{V}_t).
\end{aligned}$$

Therefore,

$$\text{volume}_t(\mathcal{M}^*,\, \mathcal{V}_t^*) = \sum_{\mathcal{T}\in\mathcal{M}}\left(\sum_{\substack{\mathcal{T}_k\\ \text{splits of } \mathcal{T}}} \text{volume}_t(\mathcal{T}_k,\, \mathcal{V}_t^*)\right) = \sum_{\mathcal{T}\in\mathcal{M}} \text{volume}_t(\mathcal{T},\, \mathcal{V}_t) = \text{volume}_t(\mathcal{M},\, \mathcal{V}_t),$$

which proves the proposition. $\square$

4.4 Positive and negative volumes of an element

Definition 8. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **positive volume** $\text{volume}_t^+(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$, is equal to

— 0, if $\mathcal{V}_t(P_i) \leqslant 0$ for $i = 1, 2, 3$ (the upper triangle lies in the half-space $z \leqslant 0$) ;

— its net volume $\text{volume}_t(\mathcal{T},\, \mathcal{V}_t)$, if $\mathcal{V}_t(P_i) \geqslant 0$ for $i = 1, 2, 3$ (the upper triangle lies in the half-space $z \geqslant 0$) ;

— otherwise, the upper triangle intersect the plane $z = 0$ at a segment $Q_1\, Q_2$ (figure 2). There are two cases:

— if $\mathcal{T}$ has only one vertex, w.l.o.g. P_1, such that $\mathcal{V}_t(P_1) > 0$, the positive volume is equal to the volume of the tetrahedron defined by Q_1, Q_2, the projected point $\overline{P}_1$ and the valuated point $\overline{P}_1^{\mathcal{V}_t}$ of P_1 ;

— otherwise, $\mathcal{T}$ has only one vertex, w.l.o.g. P_1, such that $\mathcal{V}_t(P_1) < 0$. The positive volume is equal to the difference between the net volume $\text{volume}_t(\mathcal{T},\, \mathcal{V}_t)$ and the volume of the tetrahedron defined by Q_1, Q_2, the projected point $\overline{P}_1$ and the valuated point $\overline{P}_1^{\mathcal{V}_t}$ of P_1.

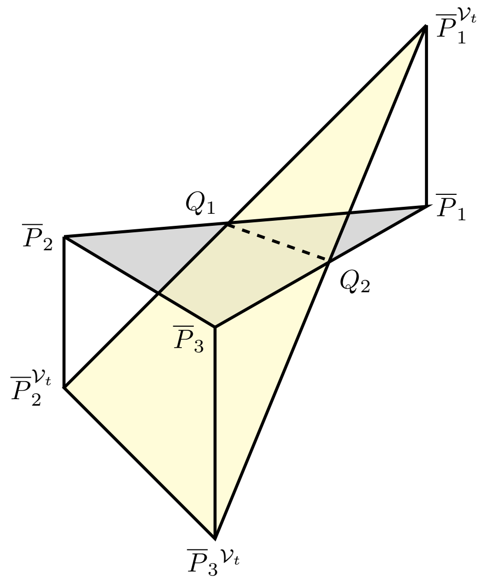

FIGURE 2 – When the upper triangle intersects the base triangle at a segment $Q_1\, Q_2$

Remark 1. In the third case, let $\mathcal{T}^+$ be the triangle defined by $\overline{P_1}$, Q_1 and Q_2, then

$$\text{area}\,(\mathcal{T}^+) = \frac{\mathcal{V}_t(P_1)^2}{|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_2)|\,|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_3)|}\,\text{area}\,(\mathcal{T}).$$

This can be easily showed by noting that

$$\overrightarrow{\overline{P_1}Q_1} = \frac{|\mathcal{V}_t(P_1)|}{|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_2)|}\,\overrightarrow{\overline{P_1P_2}}, \quad \overrightarrow{\overline{P_1}Q_2} = \frac{|\mathcal{V}_t(P_1)|}{|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_3)|}\,\overrightarrow{\overline{P_1P_3}}.$$

Since the volume of the tetrahedron is equal to the product of its base by one third its height, the positive volume of $\mathcal{T}$ in this case is equal to

$$\text{volume}^+(\mathcal{T},\, \mathcal{V}_t) = \frac{|\mathcal{V}_t(P_1)|^3}{3\,|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_2)|\,|\mathcal{V}_t(P_1) - \mathcal{V}_t(P_3)|}\,\text{area}\,(\mathcal{T}).$$

Definition 9. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **negative volume** $\text{volume}_t^-(\mathcal{T},\, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ is equal to the difference between its net volume and its positive volume:

$$\text{volume}_t^-(\mathcal{T},\, \mathcal{V}_t) = \text{volume}_t(\mathcal{T},\, \mathcal{V}_t) - \text{volume}_t^+(\mathcal{T},\, \mathcal{V}_t).$$

4.5 Absolute and quadratic volumes of an element, of a mesh

Definition 10. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **absolute volume** $\text{volume}_t^{\text{abs}}(\mathcal{T}, \mathcal{V}_t)$ **of an element** $\mathcal{T} \in \mathcal{M}$ of vertices $P_{1,2,3}$ is the product of the area of $\mathcal{T}$ by one third the sum of the absolute values of its three vertices:

$$\text{volume}_t^{\text{abs}}(\mathcal{T}, \mathcal{V}_t) = \frac{\text{area}(\mathcal{T})}{3} \sum_{k=1}^{3} |\mathcal{V}_t(P_i)| \,.$$

Definition 11. The **absolute volume** $\text{volume}_t^{\text{abs}}(\mathcal{M}, \mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the sum of the absolute volumes of its elements:

$$\text{volume}_t^{\text{abs}}(\mathcal{M}, \mathcal{V}_t) = \sum_{\mathcal{T} \in \mathcal{M}} \text{volume}_t^{\text{abs}}(\mathcal{T}, \mathcal{V}_t).$$

Proposition 2. The absolute volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 11 is decreasing under sub-triangulation. When $\mathcal{V}_t$ has the same sign for all points in $\mathcal{M}$, the absolute volume is constant under sub-triangulation.

Proof. Let $\mathcal{M}$ be a mesh valuated at $\mathcal{V}_t$ and $(\mathcal{M}^*, \mathcal{V}_t^*) \in \text{ST}(\mathcal{M}, \mathcal{V}_t)$. Let $\mathcal{T}$ be any element of $\mathcal{M}$ and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ its splits in $\mathcal{M}^*$ at the point Q. The total absolute volume of the three splits is equal to (see the proof of proposition 1):

$$\sum_{k=1}^{3} \text{volume}_t^{\text{abs}}(\mathcal{T}_k, \mathcal{V}_t^*) = \frac{1}{3} \sum_{k=1}^{3} \left(|\mathcal{V}_t^*(P_k)| \sum_{\substack{j=1,2,3 \\ j \neq k}} \text{area}(\mathcal{T}_j) \right) + \frac{1}{3} |\mathcal{V}_t^*(Q)| \, \text{area}(\mathcal{T}).$$

By the definition of sub-triangulation, the values $|\mathcal{V}_t^*(P_k)| = |\mathcal{V}_t(P_k)|$ and the value of Q is equal to

$$\mathcal{V}_t^*(Q) = \sum_{k=1}^{3} \lambda_k \, \mathcal{V}_t(P_k) = \frac{1}{\text{area}(\mathcal{T})} \sum_{k=1}^{3} \text{area}(\mathcal{T}_k) \, \mathcal{V}_t(P_k),$$

and by applying the inequality between the absolute value of sum and the sum of absolute values:

$$|\mathcal{V}_t^*(Q)| \leqslant \frac{\sum_{k=1}^{3} \text{area}(\mathcal{T}_k) \, |\mathcal{V}_t(P_k)|}{\text{area}(\mathcal{T})},$$

with equality when all three $\mathcal{V}_t(P_k)$ have the same sign. Thus,

$$\begin{aligned}
\sum_{k=1}^{3} \text{volume}_t^{\text{abs}}(\mathcal{T}_k, \mathcal{V}_t^*) &\leqslant \frac{1}{3} \sum_{k=1}^{3} \left(|\mathcal{V}_t(P_k)| \sum_{\substack{j=1,2,3 \\ j \neq k}} \text{area}(\mathcal{T}_j) \right) + \frac{1}{3} \sum_{k=1}^{3} \text{area}(\mathcal{T}_k) \, |\mathcal{V}_t(P_k)| \\
&\leqslant \frac{1}{3} \sum_{k=1}^{3} \left(|\mathcal{V}_t(P_k)| \sum_{j=1}^{3} \text{area}(\mathcal{T}_j) \right) \\
&\leqslant \frac{\text{area}(\mathcal{T})}{3} \sum_{k=1}^{3} |\mathcal{V}_t(P_i)| = \text{volume}_t^{\text{abs}}(\mathcal{T}, \mathcal{V}_t).
\end{aligned}$$

Therefore,

$$\mathrm{volume}_t^{\mathrm{abs}}(\mathcal{M}^*,\,\mathcal{V}_t^*) = \sum_{\mathcal{T}\in\mathcal{M}} \left(\sum_{\substack{\mathcal{T}_k \\ \text{splits of } \mathcal{T}}} \mathrm{volume}_t^{\mathrm{abs}}(\mathcal{T}_k,\,\mathcal{V}_t^*) \right) \leqslant \sum_{\mathcal{T}\in\mathcal{M}} \mathrm{volume}_t^{\mathrm{abs}}(\mathcal{T},\,\mathcal{V}_t) = \mathrm{volume}_t^{\mathrm{abs}}(\mathcal{M},\,\mathcal{V}_t),$$

which proves that the absolute volume is decreasing under sub-triangulation. When $\mathcal{V}_t$ has the same sign for all points in $\mathcal{M}$, the inequality becomes an equality and the absolute volume is constant under sub-triangulation. □

Definition 12. Given a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$, the **quadratic volume** $\mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T},\,\mathcal{V}_t)$ **of an element** $\mathcal{T}\in\mathcal{M}$ is the product of the area of $\mathcal{T}$ by one third the squared values of its three vertices:

$$\mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T},\,\mathcal{V}_t) = \frac{\mathrm{area}\,(\mathcal{T})}{3} \sum_{k=1}^{3} \mathcal{V}_t(P_i)^2.$$

Definition 13. The **quadratic volume** $\mathrm{volume}_t^{\mathrm{abs}}(\mathcal{M},\,\mathcal{V}_t)$ **of a mesh** $\mathcal{M}$ valuated at $\mathcal{V}_t$ is the sum of the quadratic volumes of its elements:

$$\mathrm{volume}_t^{\mathrm{quad}}(\mathcal{M},\,\mathcal{V}_t) = \sum_{\mathcal{T}\in\mathcal{M}} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T},\,\mathcal{V}_t).$$

Proposition 3. The quadratic volume of a mesh $\mathcal{M}$ valuated at $\mathcal{V}_t$ defined in definition 13 is decreasing under sub-triangulation. When $\mathcal{V}_t$ is constant on all points in $\mathcal{M}$, the quadratic volume is constant under sub-triangulation.

Proof. Let $\mathcal{M}$ be a mesh valuated at $\mathcal{V}_t$ and $(\mathcal{M}^*,\,\mathcal{V}_t^*)\in \mathrm{ST}(\mathcal{M},\,\mathcal{V}_t)$. Let $\mathcal{T}$ be any element of $\mathcal{M}$ and $\mathcal{T}_1,\,\mathcal{T}_2,\,\mathcal{T}_3$ its splits in $\mathcal{M}^*$ at the point Q. The total quadratic volume of the three splits is equal to (see the proof of proposition 1):

$$\sum_{k=1}^{3} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T}_k,\,\mathcal{V}_t^*) = \frac{1}{3}\sum_{k=1}^{3} \left(\mathcal{V}_t^*(P_k)^2 \sum_{\substack{j=1,2,3 \\ j\neq k}} \mathrm{area}\,(\mathcal{T}_j) \right) + \frac{1}{3}\,\mathcal{V}_t^*(Q)^2\,\mathrm{area}\,(\mathcal{T}).$$

By the definition of sub-triangulation, the values $\mathcal{V}_t^*(P_k)^2 = \mathcal{V}_t(P_k)^2$ and the value of Q is equal to

$$\mathcal{V}_t^*(Q) = \sum_{k=1}^{3} \lambda_k\,\mathcal{V}_t(P_k) = \frac{1}{\mathrm{area}\,(\mathcal{T})}\sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k),$$

so the squared value of Q is equal to

$$\mathcal{V}_t^*(Q)^2 = \frac{1}{\mathrm{area}\,(\mathcal{T})^2}\left(\sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)\right)^2.$$

Applying the following form of the Cauchy-Schwarz inequality:

$$(ax+by+cz)^2 \leqslant (a^2+b^2+c^2)\,(x^2+y^2+z^2), \quad \forall a,b,c,x,y,z\in\mathbb{R}$$

on the real numbers

$$\begin{aligned}
&a = \sqrt{\mathrm{area}\,(\mathcal{T}_1)} && b = \sqrt{\mathrm{area}\,(\mathcal{T}_2)} && c = \sqrt{\mathrm{area}\,(\mathcal{T}_3)}\\
&x = \mathcal{V}_t(P_1)\,\sqrt{\mathrm{area}\,(\mathcal{T}_1)} && y = \mathcal{V}_t(P_2)\,\sqrt{\mathrm{area}\,(\mathcal{T}_2)} && z = \mathcal{V}_t(P_3)\,\sqrt{\mathrm{area}\,(\mathcal{T}_3)},
\end{aligned}$$

one obtains

$$\left(\sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)\right)^2 \leqslant \mathrm{area}\,(\mathcal{T}) \sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)^2$$

with equality when all three $\mathcal{V}_t(P_k)$ are equal.

So the second term in the total quadratic volume of the three splits is bounded by:

$$\frac{1}{3}\,\mathcal{V}_t^*(Q)^2\,\mathrm{area}\,(\mathcal{T}) = \frac{1}{3\,\mathrm{area}\,(\mathcal{T})}\left(\sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)\right)^2 \leqslant \sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)^2.$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{3} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T}_k,\,\mathcal{V}_t^*) &\leqslant \frac{1}{3}\sum_{k=1}^{3}\left(\mathcal{V}_t(P_k)^2 \sum_{\substack{j=1,2,3\\ j\neq k}} \mathrm{area}\,(\mathcal{T}_j)\right) + \frac{1}{3}\sum_{k=1}^{3} \mathrm{area}\,(\mathcal{T}_k)\,\mathcal{V}_t(P_k)^2\\
&\leqslant \frac{1}{3}\sum_{k=1}^{3}\left(\mathcal{V}_t(P_k)^2 \sum_{j=1}^{3} \mathrm{area}\,(\mathcal{T}_j)\right)\\
&\leqslant \frac{\mathrm{area}\,(\mathcal{T})}{3}\sum_{k=1}^{3} \mathcal{V}_t(P_i)^2 = \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T},\,\mathcal{V}_t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{volume}_t^{\mathrm{quad}}(\mathcal{M}^*,\,\mathcal{V}_t^*) &= \sum_{\mathcal{T}\in\mathcal{M}}\left(\sum_{\substack{\mathcal{T}_k\\ \text{splits of } \mathcal{T}}} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T}_k,\,\mathcal{V}_t^*)\right)\\
&\leqslant \sum_{\mathcal{T}\in\mathcal{M}} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T},\,\mathcal{V}_t) = \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{M},\,\mathcal{V}_t),
\end{aligned}$$

which proves that the quadratic volume is decreasing under sub-triangulation. When $\mathcal{V}_t$ is constant on all points in $\mathcal{M}$, the inequality becomes an equality and the quadratic volume is constant under sub-triangulation. □

5 Comparison between two meshes

5.1 Comparison between two identical meshes

Definition 14. Two meshes $\mathcal{M}_1$ and $\mathcal{M}_2$ are **identical** if the sets of their elements are equal:

$$\{\mathcal{T}\in\mathcal{M}_1\} = \{\mathcal{T}\in\mathcal{M}_2\}$$

.

Definition 15. A **reference mesh** is a mesh $\mathcal{M}_0$ with a constant valuation $\mathcal{V}_t = \mathcal{V}_0$ for all time t.

5.1.1 Mean signed deviation

Definition 16. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **mean signed deviation (MSD)** with respect to $\mathcal{M}_0$ is the function $\mathrm{MSD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0,\, \mathcal{V}_t)$ to the volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$, divided by the total area of $\mathcal{M}_0$:

$$\mathrm{MSD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t) = \frac{\mathrm{volume}_t(\mathcal{M}_0, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}.$$

Proposition 4. The mean signed deviation is invariant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. By proposition 1, the volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$ is invariant under sub-triangulation. Therefore the mean signed deviation is also invariant under sub-triangulation. □

5.1.2 Mean absolute deviation

Definition 17. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **mean absolute error (MAD)** with respect to $\mathcal{M}_0$ is the function $\mathrm{MAD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0,\, \mathcal{V}_t)$ to the absolute volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$, divided by the total area of $\mathcal{M}_0$:

$$\mathrm{MAD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t) = \frac{\mathrm{volume}_t^{\mathrm{abs}}(\mathcal{M}_0, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}.$$

Proposition 5. The mean absolute deviation is decreasing under sub-triangulation. When $\mathcal{V}_t - \mathcal{V}_0$ has the same sign for all points in $\mathcal{M}_0$, the absolute volume is constant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. The result is directly deduced from proposition 2. □

5.1.3 Root mean square deviation

Definition 18. Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **root mean square deviation (RMSD)** with respect to $\mathcal{M}_0$ is the function $\mathrm{RMSD}_{\mathcal{M}_0,\mathcal{V}_0}$ in $\mathbb{R}_+$ that maps $(\mathcal{M}_0,\, \mathcal{V}_t)$ to the square root of the quadratic volume of $\mathcal{M}_0$ valuated at $\mathcal{V}_t - \mathcal{V}_0$ divided by the total area of $\mathcal{M}_0$:

$$\mathrm{RMSD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t) = \sqrt{\frac{\sum_{\mathcal{T}\in\mathcal{M}_0} \mathrm{volume}_t^{\mathrm{quad}}(\mathcal{T}, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}}.$$

Proposition 6. The root mean absolute deviation is decreasing under sub-triangulation. When $\mathcal{V}_t - \mathcal{V}_0$ is constant on all points in $\mathcal{M}_0$, the quadratic volume is constant under sub-triangulation.

Proof. It is clear that a sub-triangulation does not change the total area of the mesh. The result is directly deduced from proposition 3. □

5.1.4 Element-wise signed deviation, deviation distribution

Let $\mathcal{M}_0$ be a reference mesh with valuation $\mathcal{V}_0$. For any mesh valuated at $\mathcal{V}_t$ and identical to $\mathcal{M}_0$, the **deviation distribution function** F_X is the empirical distribution function of the variable

$$X = \frac{\mathrm{size}\,(\mathcal{M}_0)\,\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)},$$

called the **element-wise signed deviation (EWSD)**, measured from the set of all elements $\mathcal{T} \in \mathcal{M}_0$:

$$F_X(x) = \frac{1}{\mathrm{size}\,(\mathcal{M}_0)}\,\mathrm{card}\left\{\mathcal{T} \in \mathcal{M}_0 \;\middle|\; \frac{\mathrm{size}\,(\mathcal{M}_0)\,\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)} \leqslant x\right\}, \quad x \in \mathbb{R}.$$

Remark 2. The empirical mean of the element-wise signed deviation is equal to the mean signed deviation defined in 16:

$$\begin{aligned}
\mathbb{E}(X) &= \frac{1}{\mathrm{size}\,(\mathcal{M}_0)} \sum_{\mathcal{T}\in\mathcal{M}_0} \left(\frac{\mathrm{size}\,(\mathcal{M}_0)\,\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}\right) \\
&= \sum_{\mathcal{T}\in\mathcal{M}_0} \left(\frac{\mathrm{volume}_t(\mathcal{T},\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)}\right) \\
&= \frac{\mathrm{volume}_t(\mathcal{M}_0,\, \mathcal{V}_t - \mathcal{V}_0)}{\mathrm{area}\,(\mathcal{M}_0)} \\
&= \mathrm{MSD}_{\mathcal{M}_0,\mathcal{V}_0}(\mathcal{M}_0,\, \mathcal{V}_t),
\end{aligned}$$

and is invariant under sub-triangulation by proposition 4.