EE4C5 Digital Signal Processing

Lecture 2 – An Introduction

This lecture

- Based on Chapter 2 of O&S
- All images from O&S book unless otherwise stated

Discrete-time signals

- Sequence x of numbers with n^{th} number denoted as $x[n]$

$$x = [x[n]] \quad -\infty < n < \infty$$

- May be sampled from analog signal $x_a(t)$

$$x[n] = x_a(nT) \; -\infty < n < \infty$$

- T is sampling period, sampling frequency $\frac{1}{T}$
- Convenience – refer to $x[n]$ to denote the sequence and not just one sample

Discrete-time signal

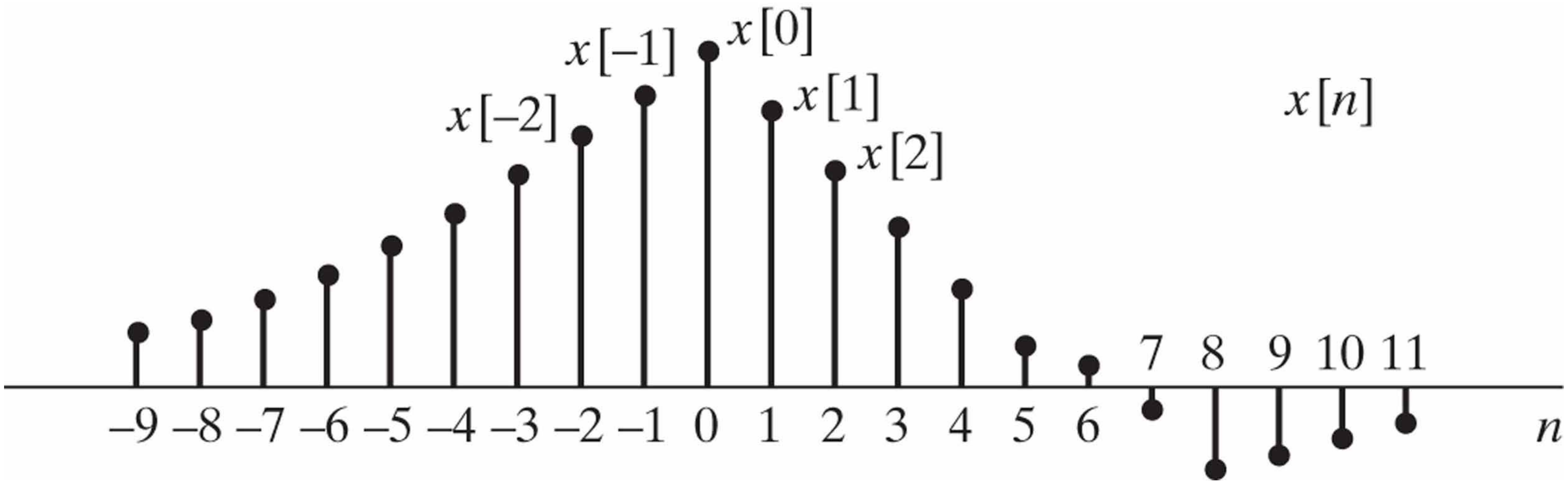


Figure 2.1 Graphic representation of a discrete-time signal.

Sampling a continuous-time signal

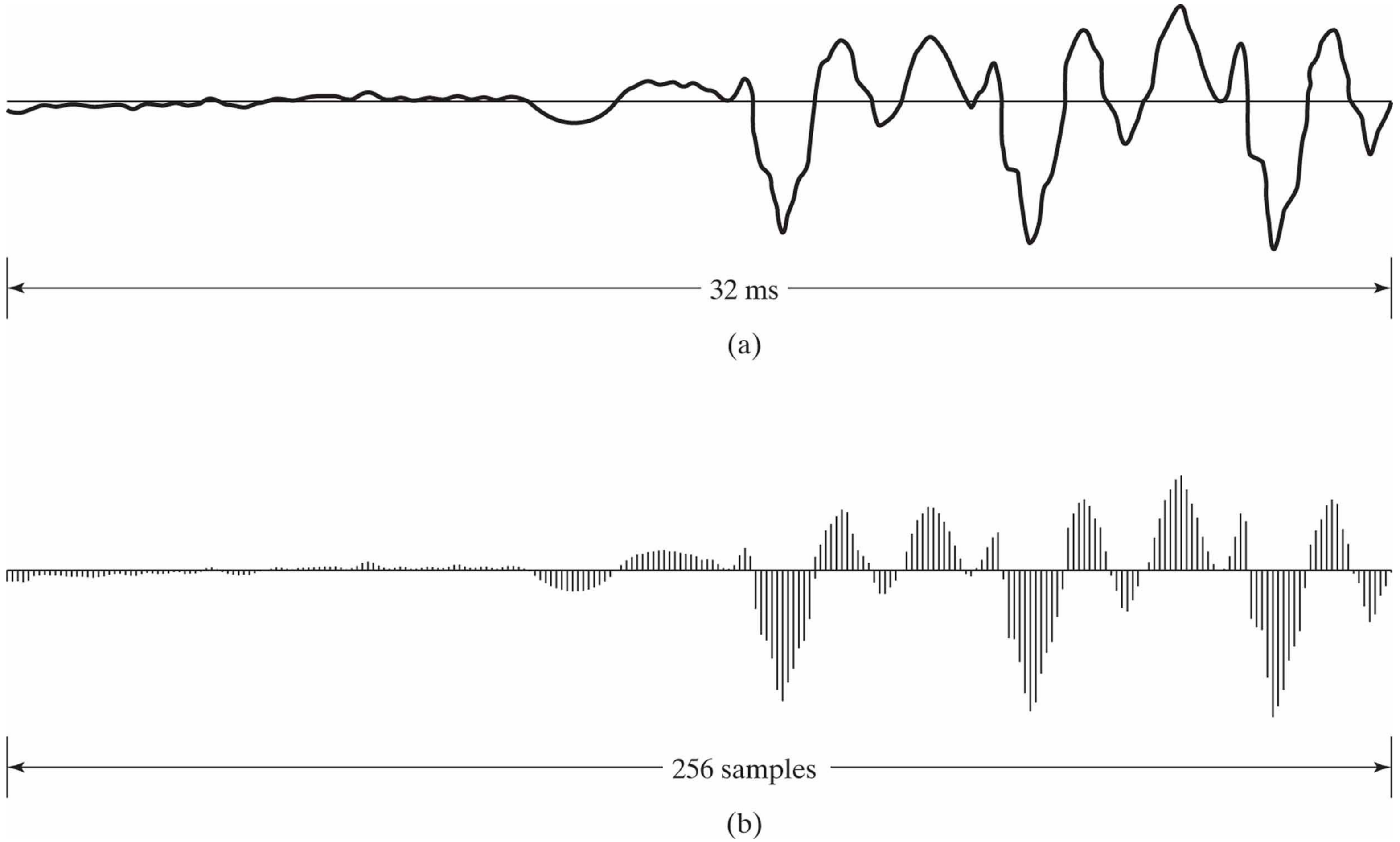
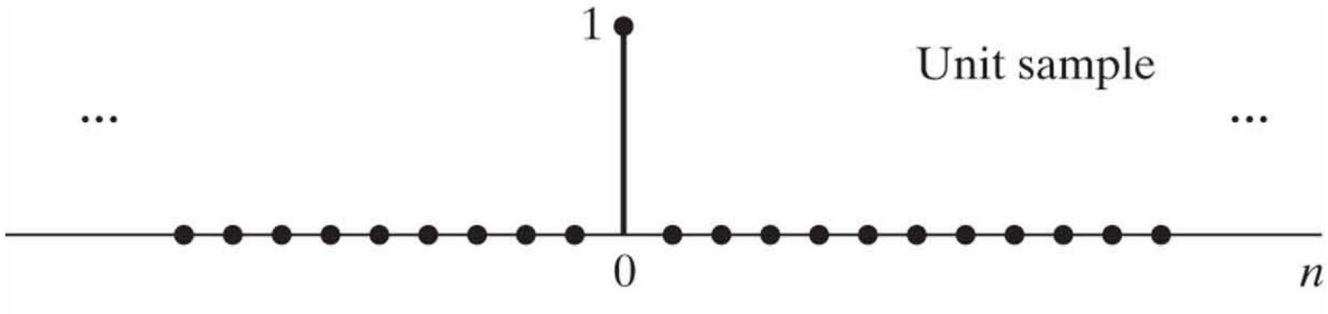


Figure 2.2 (a) Segment of a continuous-time speech signal $x_a(t)$. (b) Sequence of samples $x[n] = x_a(nT)$ obtained from the signal in part (a) with $T = 125$ μs.

What's the sampling rate here?

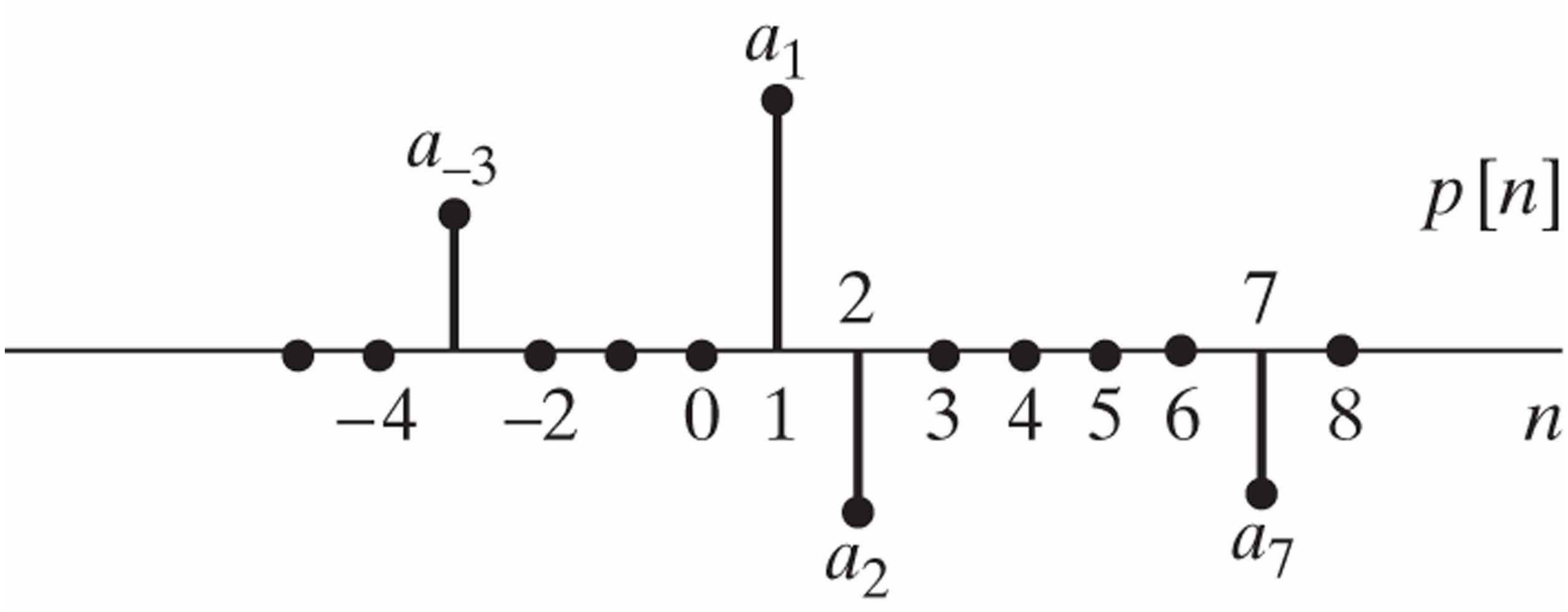
Unit sample sequence

- $\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$



- Analogous to unit impulse system
- Discrete-time impulse (or simply “an impulse”)
- Any arbitrary sequence can be represented as sum of scaled, delayed impulses

Expression for $p[n]$?



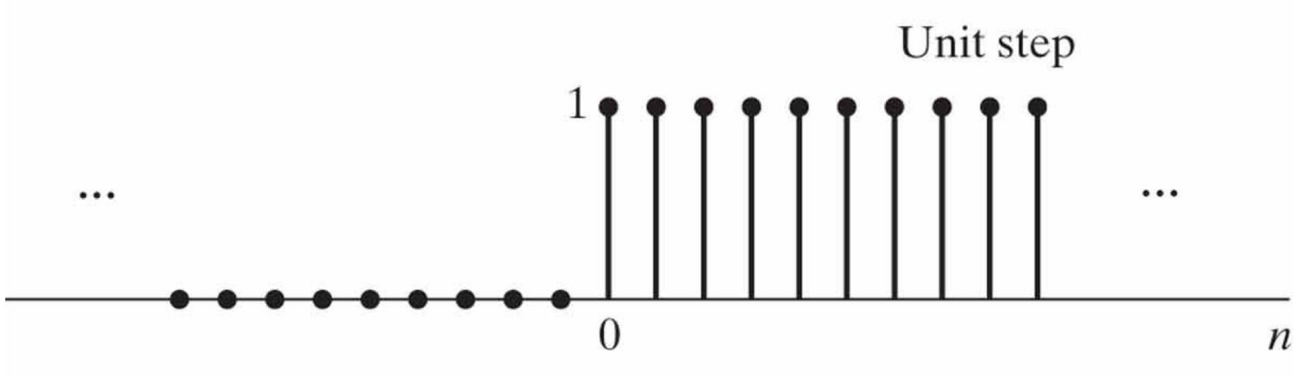
- General expression: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

Unit step sequence

- $u[n] = \begin{cases} 1, n \geq 1 \\ 0, n < 0 \end{cases}$

- $u[n] = \sum_{k=-\infty}^{n} \delta[n]$

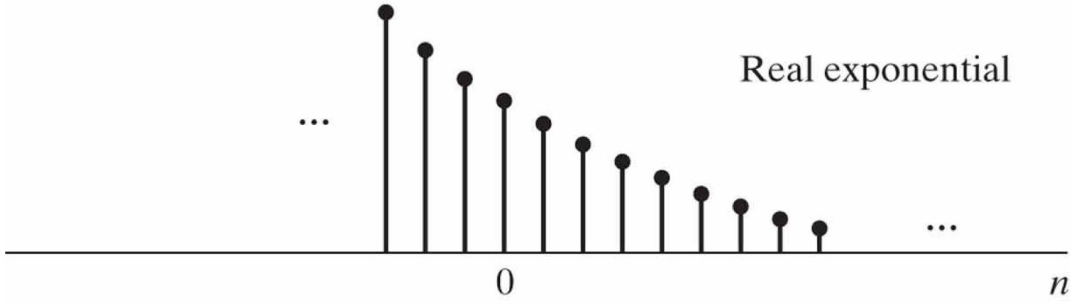
- $u[n] = \sum_{k=0}^{n} \delta[n-k]$



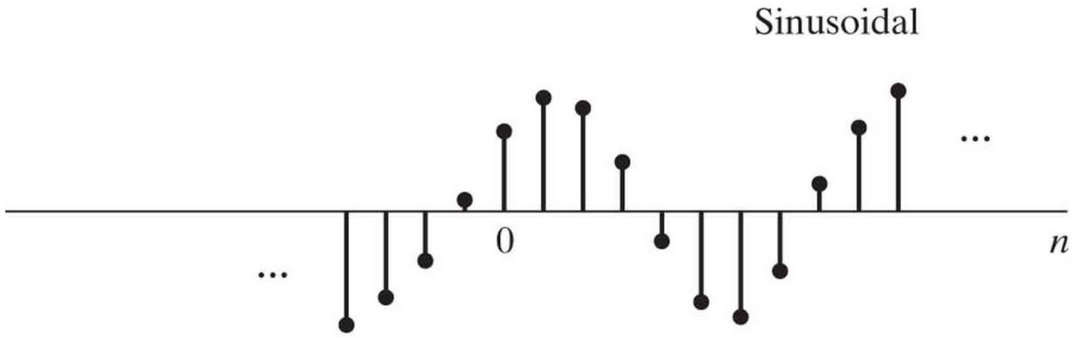
Other sequences

- Exponential sequence
- $x[n] = A\alpha^n$
- Real if A and α real

- $x[n] = A\cos(\omega_0 n + \phi)$where A is the amplitude, ω_0 is the angular frequency, and ϕ is the phase of $x[n]$
- Pay attention to periodicity!



(c)



Useful properties

- Right-sided $x[n] = 0, n < N_{min}$
- Left-sided $x[n] = 0, n > N_{max}$
- Finite length $x[n] = 0, n \notin [N_{min}, N_{max}]$
- Causal $x[n] = 0, n < 0$
- Finite Energy $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

Discrete-time systems

$$y[n] = T\{[x[n]]\}$$

Figure 2.6 Representation of a discrete-time system, i.e., a transformation that maps an input sequence $x[n]$ into a unique output sequence $y[n]$.

Ideal delay system

- $y[n] = x[n - n_d]$
- n_d positive integer => delay of n_d samples

Moving average

- $y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$
- Average around a window of length $(M_1 + M_2 + 1)$

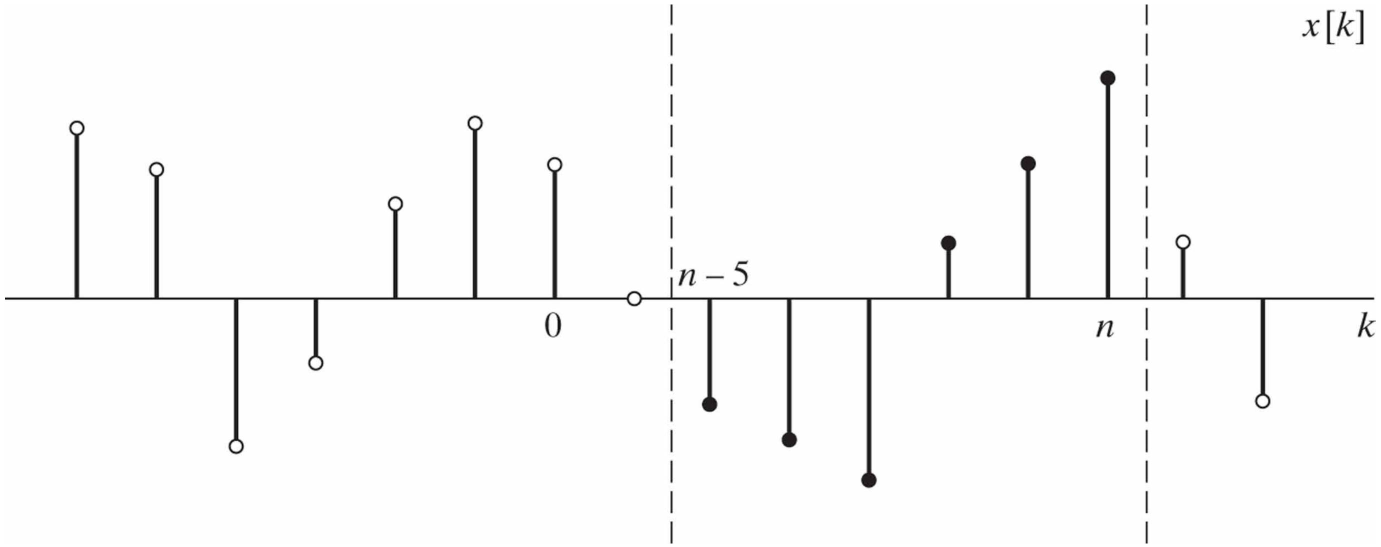


Figure 2.7 Sequence values involved in computing a moving average with M_1 = 0 and M_2 = 5.

Memoryless Systems

- Output $y[n]$ only depends on input $x[n]$ for any n
- E.g.
- $y[n] = (x[n])^2$

- Is the moving average system memoryless?

Linear Systems

- Principle of superposition
- Say $y_1[n]$ is output when input to a system is $x_1[n]$
- And $y_2[n]$ is output when input to that SAME system is $x_2[n]$
- That system is linear IF and ONLY IF:
  - $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$
- And
  - $T\{ax_1[n]\} = aT\{x_1[n]\}$
- With a an arbitrary constant
- Draw as system?

Linear Systems

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    - $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$ **Additive property**
- And
    - $T\{ax_1[n]\} = aT\{x_1[n]\}$ **Scaling (or homogeneity) property**
- With a an arbitrary constant
- Draw as system?

For you…

- Show the accumulator system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is a linear system

Time-invariance

- Shift-invariance
- Shift in input => corresponding shift in output
- Input $x[n]$ gives output $y[n]$ from TI system
- Then $x_1[n] = x[n - n_0]$ yields $y_1[n] = y[n - n_0]$
- See examples in O&S

Causality

- For all n_0 the output at index $n = n_0$ depends only on input values $n \leq n_0$
- Nonanticipative

- $y[n] = x[n+1] - x[n]$ **Non-causal**
- $y[n] = x[n] - x[n-1]$ **Causal**

Stability

- Bounded-input bounded-output
- BIBO stable
- $|x[n]| \leq B_x < \infty, \forall n$
- $|y[n]| \leq B_y < \infty, \forall n$

LTI systems

- Both linear and time-invariant
- Consider any general sequence can be represented as linear combination of delayed impulses (see slide #7 earlier!)
- Derive (done in lecture) convolution relationship
- $y[n] = x[n] * h[n]$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \mathrm{h}[n-k]$

Practical convolution

- Revise – we will revisit in next lecture

Properties of LTI System

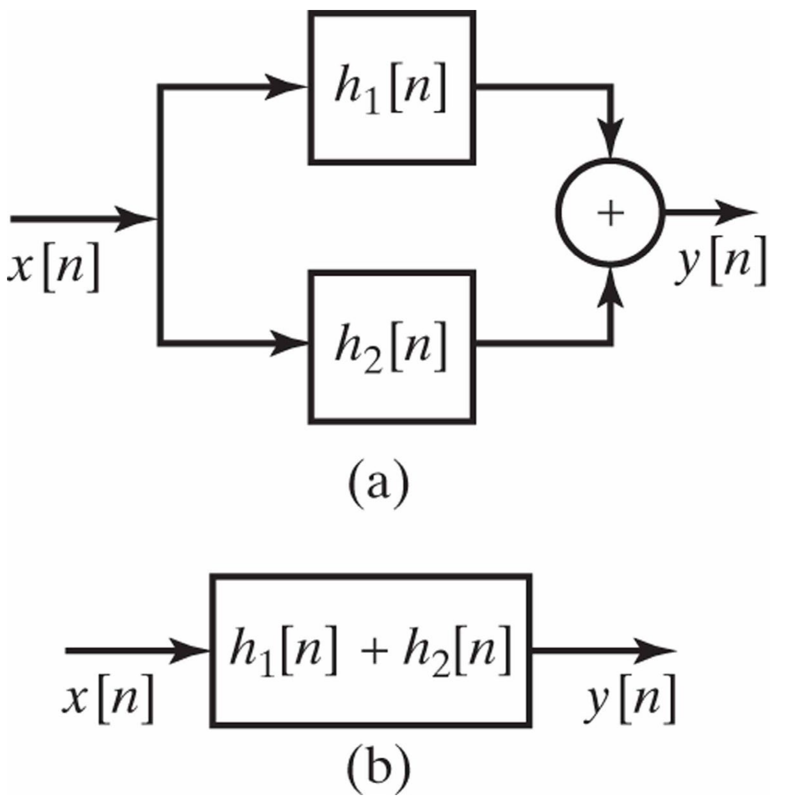

Figure 2.11 (a) Parallel combination of LTI systems. (b) An equivalent system.

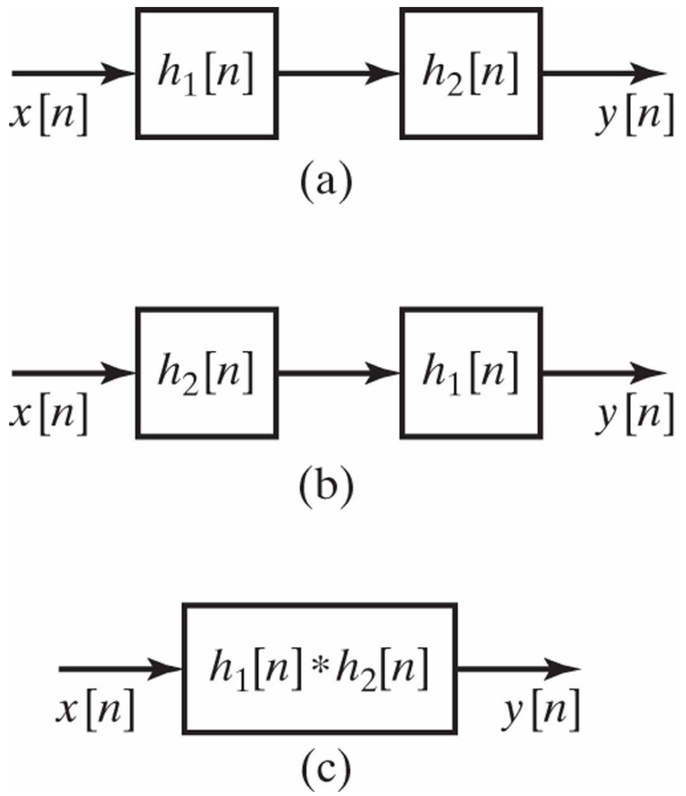

Figure 2.12 (a) Cascade combination of two LTI systems. (b) Equivalent cascade. (c) Single equivalent system.

Other properties of LTI system

- Is the system causal?
  - $h[n] = 0 \qquad n < 0$
- Is the system stable?
  - Impulse response absolutely summable
  - $B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Fourier Transform representation of signals

- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
- Where:
- $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega n}$
- This pair of equations form the Fourier representation for the sequence
- Discrete-time Fourier Transform (DTFT), forms:
- $X(e^{j\omega})= X_R(e^{j\omega}) + jX_{Im}(e^{j\omega})$
- $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

Frequency response

- $\mathrm{h}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega$

Convolution Theorem

- $x[n] \overset{\mathcal{F}}{\leftrightarrow} X\left(e^{j\omega}\right)$
- $h[n] \overset{\mathcal{F}}{\leftrightarrow} H\left(e^{j\omega}\right)$
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{h}[n-k] = x[n] * h[n]$
- $\mathrm{Y}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$

Required Reading & other material

- Oppenheim & Schafer, Chapter 2
- TCD's matlab portal: https://uk.mathworks.com/academia/tah-portal/trinity-college-dublin-729365.html
- Convolution – practical realisation: https://www.youtube.com/watch?v=W56uw9GUvxU