EE4C5 Digital Signal Processing

Lecture 16 – An Introduction to Random Processes

This lecture

- Based on Chapter 2 of O&S, Appendix on Random Signals
  - Multiple other sources too though
  - Introduction to Random Processes, UDRC Summer School, 20th July 2015 James R. Hopgood
  - Advanced Digital Signal Processing and Noise Reduction, Saeed V. Vaseghi

Deterministic signals

- We've largely dealt with deterministic signals
  - Can express samples in a sequence with a mathematical expression, or give values, or rule etc
- Not always realistic or possible
  - Noise
  - Speech
- Random process is an important concept in signal processing

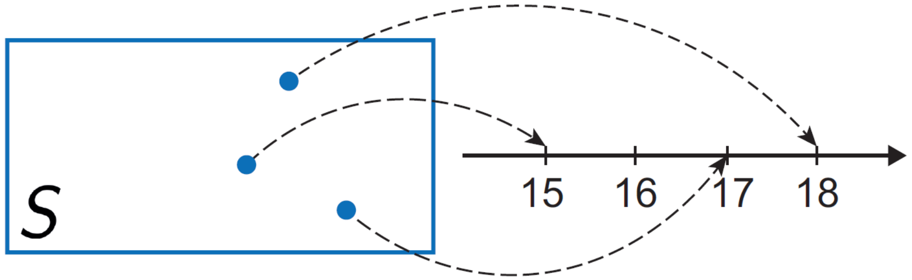
Background material

Random Variable

- Should have covered in some previous module
- Covered in 3E3 (for students who attended TCD Engineering)
  - 2022/23 Academic Year (Prof Arman Farhang)
  - Lecture 6 Discrete Random Variables
  - Lecture 7 Continuous Random Variables

Definition (based on 3E3 notes by Prof Farhang)

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.



A random variable is represented by an uppercase letter such as X and the particular value of the corresponding random variable is denoted by a lowercase letter.

The notation $X(s) = x$ means that x is the value associated with the outcome s by the random variable X.

More formally, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

Random variables (quick recap)

Discrete Random Variable		Continuous Random Variable	
A random variable with a finite (or countably infinite) range		A random variable with an interval (finite or infinite) of real numbers for its range while no possible value of the random variable has positive probability.	
e.g. #transmitted bits received in error in a communication system, #scratches on a surface, proportion of defective parts among 1000 tested.		e.g. electrical current, length, pressure, temperature, time, voltage, weight.	
probability mass function or the probability distribution – a probability for each value the variable can take		probability density function (PDF) – likelihood of the variable taking a specific value or falling in some range	
e.g. Binomial distribution, Poisson distribution		e.g. Normal distribution, Chi-Squared Distribution	
Mean	$\mu = E(X) = \sum_x x f(x)$	Mean	$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx.$
Variance	$\sigma^2 = E\{(X-\mu)^2\}$	Variance	$\sigma^2 = E(x^2) - [E(x)]^2$

Covariance

- Covariance of two random variables X and Y defined as:
- $\mathrm{cov}[X, Y] = \left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$
- $\quad\quad\quad\quad = [(X - \mu_X)(Y - \mu_Y)]$
- $\quad\quad\quad\quad = E(X\mathrm{Y}) - \mu_X \mu_Y$
- Covariance values:
  - positive when both variables tend to be above or below their respective means simultaneously
  - negative when one variable is above its mean while the other is below its mean.
- Measure of how much they change together but issues with scale
  - Correlation more useful

Correlation

- Two random variables X and Y have joint probability density function $p_{XY}(x, y)$
- The random variables are independent iff:
  - $p_{XY}(x, y) = p_X(x) p_Y(y) \quad \forall (x, y)$
- The correlation of X and Y measures the degree to which the pair of random variables are linearly related:
  - $\rho_{XY} = \frac{\mathrm{cov}[X, Y]}{\sigma_X \sigma_Y}$
  - Pearson's correlation coefficient
- The two RVs are uncorrelated if
  - $E(X\mathrm{Y}) = E(X)\, E(\mathrm{Y})$
  - Gives $\rho_{XY} = 0$

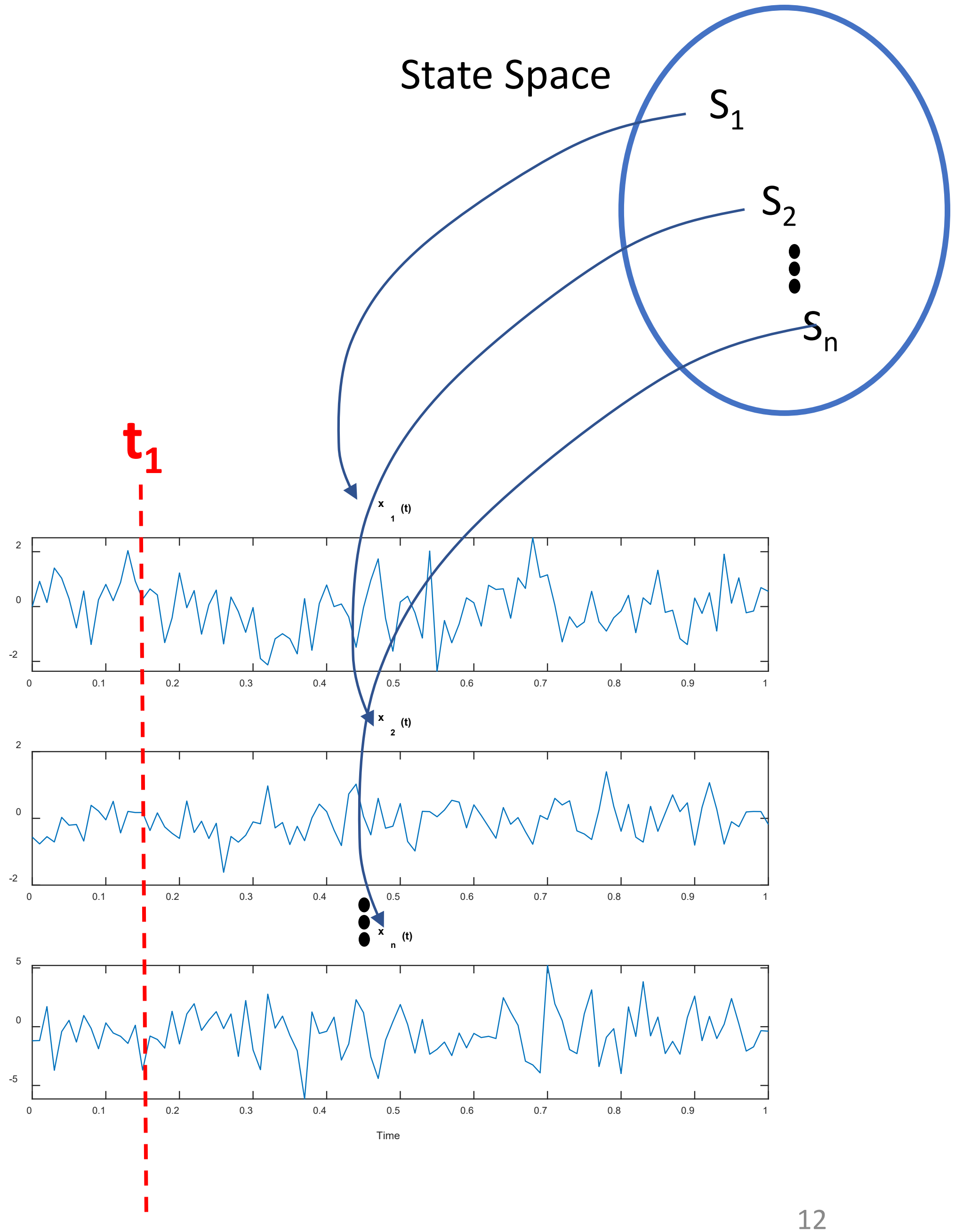
Independence and correlation

- If two random variables X and Y are independent
  - => they are uncorrelated
- If two random variables are uncorrelated
  - they may or may not be independent
- Not the same thing!!
- Independence implies that information about one RV provides no information about the other RV.
- Uncorrelated means there is no linear relationship

Random Process

What’s a random process?

- A collection of random variables, characterised by a set of probability distribution functions that are a function of time
- Widely used as mathematical models of systems and signals that appear to vary randomly
- Often referred to as stochastic process, random function
- Each random variable in the collection takes values from the same mathematical space known as the state space (or sample space or ensemble) of the process
- A random process can be well described by a suitable probabilistic model



Sampling

- Conside a random process $X(t)$ represented by the set of sample functions $\{x_j(t)\}$ for $j = 1,2 \ldots n$
- Each sample function $x_j(t)$ has a probability of occurrence, $p(s_j)$ for $j = 1,2 \ldots, n$
- Consider we observe the set of $x_j(t)$ at some time instance $\mathrm{t} = t_1$ (diagram previous slide)
- Then each sample point s_j of the sample space S has an associated $x_j(t_1)$ and probability $p(s_j)$
- The resulting collection $\{x_j(t)\}$ is a random variable $X(t_1)$
- Would get different random variable $X(t_2)$ at another time instance $\mathrm{t} = t_2$

Note the difference

- For a random variable the output of the experiment is mapped to a number
- For a random process the output of the experiment is mapped to a waveform which is a function of time

- Let $X(t_1)$, $X(t_2), \ldots, X(t_k)$ be the random variables obtained at times $t_1, t_2, \ldots, t_k$
- Those k random variables are completely characterised by their joint probability distribution
- $p_{X(t_1)\,,\,X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k)$

Stationarity

- The random process $X(t)$ is strictly stationary if the joint probability distribution is time-invariant
- $p_{X(t_1)\,,\,X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k) = p_{X(t_{1+\tau})\,,\,X(t_{2+\tau}),\ldots,X(t_{k+\tau})}(x_1, x_2, \ldots, x_k)$
- All parameters such as mean, variance, and higher order moments of the process are time-invariant.
- In practice there are degrees of stationarity since one set of statistics of a process could be stationary, but another set might be time-varying
    - Strict sense stationarity
    - Wide sense stationarity

Statistics of interest

- Mean of the random process $X(t)$ is given by
  - $\mu_X(t_k) = E[X(t_k)]$
- Where $X(t_k)$ is the random variable through the observation of the random process at time t_k
- The autocorrelation of random process $X(t)$ is a function of two time-variables, t_k and t_i
  - $R_X(t_k, t_i) = E[X(t_k)X(t_i)]$

Strict Sense Stationary Process

- $\mu_X(t_k) = \mu_X \quad \forall\, t_k$ i.e. constant
- $R_X(t_k, t_i)$ is time independent, so instead can write:
  - $R_X(\tau) = E[X(t)X(t+\tau)]$
- Note correlation plays central role in signal processing and communications
- Measure of similarity of the outcomes of the random process at different time instances
- For zero-mean signal, $R_X(0)$ is the signal power

Wide Sense Stationary Process (WSS)

- A "relaxed" form of stationarity
- Mean value of the random variable $X(t_j)$ is (again) independent of choice of t_j
  - $\mu_X(t_j) = \mu_X \quad \forall\, t_j$ i.e. constant

- Autocorrelation $R_X(t_j, t_a)$depends on the time difference $t_a - t_j$ (lag) and the starting point
  - $R_X(t_j, t_a) = E[X(t_j)X(t_a)]$

Note

- Strict sense stationary process is always wide sense stationary
- Converse is NOT true
    - Except if the process is Gaussian

Other forms of stationarity

- Cyclostationary process
  - Signal whose statistical properties vary cyclically with time
  - Several interleaved stationary processes
  - E.g. Pulse-Amplitude Modulation
- Locally stationary or quasi-stationary
  - Statistical properties change slowly over short periods of time.
  - Globally nonstationary, but approximately locally stationary, and are
  - Modelled as if the statistics actually are stationary over a short segments
  - E.g. speech

Nonstationary Process

- Statistics of the process all vary with time
- Video, audio, financial data, meteorological data, biomedical signals
    - all nonstationary

Ergodic Processes, time-averaged statistics

- May have just one realisation of a random process
  - Need to estimate statistical parameters e.g. mean, correlation from just this example
- This gives time-averaged statistics
  - From a single realisation
- “True” or ensemble statistics require calculation over entire space of realisations
- Ergodic processes are those for which time-averaged and ensemble averages statistics are equal
- Definition: *A stationary random process is said to be ergodic if it exhibits the same statistical characteristics along the time dimension of a single realisation as across the space (or ensemble) of the different realisations of the process*

Importance of Ergodicity

- Simplifies treatment of a process
  - Can make predictions based on observation of a suitably long sample of the process
- Practical use examples:
  - wireless communications :channel estimation (estimating the impulse response) made easier as you can rely on statistics obtained from one realisation
  - Image processing: simplifies image compression algorithms

Discrete sequences

- Changes the maths, but not the concepts presented so far
- Autocorrelation sequence of a real-valued random process
  - $r_{XX}[n+m, m] = E\big[X[n+m]X[n]\big]$
- If the random process is WSS
  - $r_{XX}[n+m, m] = r_{XX}[m]$
  - i.e. a function of the time difference m
- $E[x[n]^2] = r_{XX}[0]$

Classes of Random Processes

- Markov Process
- Markov Chain Process
- Gaussian Process
- Multivariate Gaussian Process
- Mixture Gaussian Process
- Poisson Process
- Autoregressive Process
- (not an exhaustive list)

Recommended Reading & other material

- Series of short videos on random processes (Iain explains) (accessible)
  - https://www.youtube.com/watch?v=W28-96AhF2s&list=PLx7-Q20A1VYKRLHUMSt2YOORrVz8iH-Kq
- Preprint of the textbook “Random Processes for Engineers” (more advanced)
  - https://hajek.ece.illinois.edu/Papers/randomprocJuly14.pdf