Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

FACULTY OF ENGINEERING, MATHEMATICS & SCIENCE

SCHOOL OF ENGINEERING

Electronic and Electrical Engineering

Engineering
Senior Sophister
Annual Examinations

Semester 1, 2018

Digital Signal Processing (4C5)

11th December 2018 **Venue: RDS - Simmons Court** **Time: 14.00 – 16.00**

Dr. W. Dowling

Instructions to Candidates:

Answer THREE questions. All questions carry equal marks.

Materials permitted for this examination:

Mathematical Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.

Please indicate the make and model of your calculator on each answer book used.

Q.1 (a) A discrete-time filter has a unit sample response, $h[n]$, that is zero for $n < 0$ and for $n > N-1$. Let $H(e^{j\Omega})$ denote the frequency response of the filter.
If $h[n] = h[N-1-n]$ and N is odd, show that

$$H(e^{j\Omega}) = e^{-j\Omega[(N-1)/2]}\left\{h\left[\frac{N-1}{2}\right] + \sum_{n=0}^{[(N-1)/2]-1} 2h[n]\cos\left[\Omega\left(n-\frac{N-1}{2}\right)\right]\right\}$$

[8 marks]

(b) An ideal discrete-time high-pass filter has a frequency response, $H_{id}(e^{j\Omega})$, given by

$$H_{id}\left(e^{j\Omega}\right) = \begin{cases} 0, & |\Omega| < \frac{\pi}{3}, \\ 1, & \frac{\pi}{3} < |\Omega| < \pi. \end{cases}$$

Obtain an expression for the unit-sample response of this filter. **[7 marks]**

(c) A 9-point Hamming window, $w_H[n]$, is given by

$$w_H[n] = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{4}\right), & -4 \le n \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Using the Hamming window, design a causal, 9-point, generalised linear phase filter that approximates the magnitude response of the ideal high-pass filter in part (b). **[5 marks]**

Q.2 (a) Show that the bilinear transformation, $s = (1 - z^{-1})/(1 + z^{-1})$, has the following properties:

(i) The imaginary axis in the s-plane maps to the unit circle in the z-plane.

[4 marks]

(ii) The left half of the s-plane maps to the inside of the unit circle in the z-plane.

[4 marks]

(b) A discrete-time low-pass filter with frequency response, $H(e^{j\Omega})$, is to be designed to meet the following specifications:

$$0.89 \le |H(e^{j\Omega})| \le 1, \qquad |\Omega| \le 0.2\pi,$$
$$|H(e^{j\Omega})| \le 0.18, \qquad 0.6\pi \le |\Omega| \le \pi$$

The filter is to be designed by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter. Verify that a second order filter is sufficient to meet the specifications. Determine the transfer function, $H(z)$, of the discrete-time filter.

Note that the transfer function of a second order Butterworth low-pass prototype filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$$

[12 marks]

Q.3 (a) A continuous-time filter has an impulse response $h_a(t)$. The unit-sample response of a discrete-time filter, $h[n]$, is given by

$$h[n] = T\, h_a(nT)$$

where T is a positive constant. Let $H_a(j\omega)$ and $H\left(e^{j\Omega}\right)$ denote the frequency response of the continuous-time filter and the frequency response of the discrete-time filter respectively. Starting from first principles, show that

$$H\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} H_a\left(j\left(\frac{\Omega}{T} - \frac{2\pi k}{T}\right)\right).$$

[12 marks]

(b) The sequence $x[n]$ is zero for $n < 0$ and for $n > N-1$. Assume that $N = 2^M$, where M is a positive integer. Let $g[n] = x[2n]$, and $h[n] = x[2n+1]$.
Show that the N-point discrete Fourier transform (DFT) of the sequence $x[n]$ can be obtained by appropriately combining the $N/2$-point DFTs of the sequences $g[n]$ and $h[n]$.

[8 marks]

Q.4 (a) Let $x[n]$ denote a finite-duration sequence of length M such that $x[n] = 0$ for $n < 0$ and $n \geq M$. Let $X(e^{j\Omega})$ denote the discrete-time Fourier transform transform of $x[n]$. If we sample $X(e^{j\Omega})$ at $\Omega = (2\pi/N)k, \ \ k = 0, 1, 2, \ldots, N-1$, we obtain

$$X_1[k] = X(e^{j2\pi k/N}), \qquad k = 0, 1, \ldots, N-1.$$

The number of samples, N, is *less than* the duration of the sequence, M; i.e. $N < M$. The sequence $x_1[n]$ is obtained as the inverse DFT of $X_1[k]$. Determine the relation between $x_1[n]$ and $x[n]$.

[12 marks]

(b) Consider two finite length sequences, $x[n]$ and $h[n]$, where both are zero for $n < 0$ and where

$$x[n] = 0, \quad n \geq 16$$

$$h[n] = 0, \quad n \geq 4.$$

The 16-point DFTs of the two sequences are multiplied and the inverse DFT computed. Let $r[n]$ denote this inverse DFT.
The sequence $y[n]$ is obtained by linearly convolving $x[n]$ and $h[n]$.
Specify the values of n for which $r[n]$ is guaranteed to be equal to $y[n]$.

[3 marks]

(c) A 10,000 point sequence is to be linearly convolved with a sequence that is 80 points long. The convolution is to be implemented using DFTs and inverse DFTs of length 512. If the overlap-add method is used, what is the minimum number of 512-point DFTs and the minimum number of 512-point inverse DFTs needed to implement the convolution for the entire 10,000 point sequence?

[5 marks]

Q.5 (a) Consider a stable, linear, shift-invariant system with unit-sample response $h[n]$. Let $x[n]$ be a real input sequence that is a sample sequence of a wide-sense stationary discrete-time random process. Let $y[n]$ denote the output sequence. Show that the input and output autocorrelation sequences, $\phi_{XX}[m]$ and $\phi_{YY}[m]$, respectively, are related by

$$\phi_{YY}[m] = \sum_{l=-\infty}^{\infty} v[l]\phi_{XX}[m-l],$$

where

$$v[l] = \sum_{k=-\infty}^{\infty} h[k]h[l+k].$$

[8 marks]

(b) Let $x[n]$ be a real white-noise sequence with zero mean and autocorrelation sequence $\phi_{XX}[m] = \sigma_X^2\,\delta[m]$, where $\delta[m]$ is the unit-sample sequence. The sequence $x[n]$ is the input to a linear shift-invariant system with unit-sample response $h[n] = a^n u[n]$, where $|a| < 1$ and $u[n]$ is the unit-step sequence.

(i) Find an expression for the output autocorrelation sequence, $\phi_{YY}[m]$.
[4 marks]

(ii) Express the power spectral density, $S_{YY}(\Omega)$, of the output process in terms of the magnitude of the frequency response of the system. **[6 marks]**

(iii) Determine the mean, m_Y, and the variance, σ_Y^2, of the output process.
[2 marks]