Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

FACULTY OF ENGINEERING, MATHEMATICS & SCIENCE

SCHOOL OF ENGINEERING

Electronic and Electrical Engineering

Engineering
Senior Sophister
Annual Examinations

Hilary Term, 2018

Digital Signal Processing (4C5)

Date: 4th January 2018 **Venue: Goldsmith Hall** **Time: 09.30 – 11.30**

Dr. W. Dowling

Instructions to Candidates:

Answer THREE questions. All questions carry equal marks.

Materials permitted for this examination:

Mathematical Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.

Please indicate the make and model of your calculator on each answer book used.

Q.1 (a) A continuous-time signal, $x_a(t)$, has the Fourier transform, $X_a(j\omega)$, shown in Fig. Q1-1. The sequence $x[n]$ was derived from $x_a(t)$ by periodic sampling:

$$x[n] = x_a(nT)$$

where the sampling period $T = 1$ millisecond.

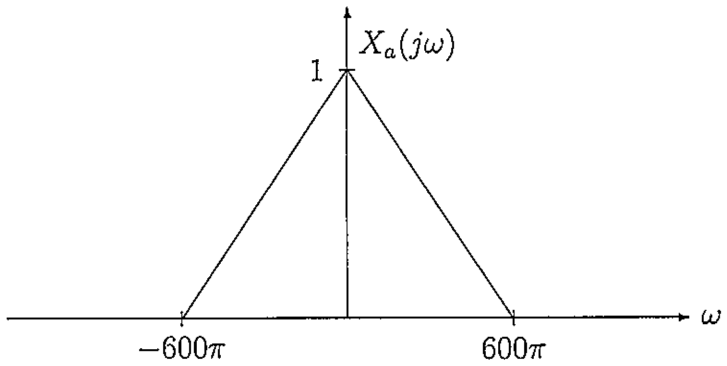

Fig. Q1-1

A system for sampling rate reduction by a factor of 1.5 is shown in Fig. Q1-2.

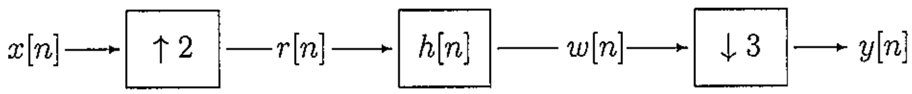

Fig. Q1-2

$$r[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$w[n] = \sum_{k=-\infty}^{\infty} r[k]h[n-k]$$

$$y[n] = w[3n]$$

The ideal discrete-time low-pass filter has a unit sample response, $h[n]$, and a frequency response, $H(e^{j\Omega})$, given by

$$H(e^{j\Omega}) = \begin{cases} 2, & |\Omega| < \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |\Omega| \leq \pi \end{cases}$$

continued ...

[Q.1 continued]

Let $R(e^{j\Omega})$ and $Y(e^{j\theta})$ denote the discrete-time Fourier transforms of the sequences $r[n]$ and $y[n]$ respectively.

(i) Sketch $R(e^{j\Omega})$ for $-\pi \le \Omega \le \pi$. **[5 marks]**

(ii) Sketch $Y(e^{j\theta})$ for $-\pi \le \theta \le \pi$. **[4 marks]**

(b) Using a rectangular window sequence, design a causal, 17-point, discrete-time generalised linear phase filter with a magnitude response which approximates the ideal band-pass magnitude response, $\left|H_{id}\left(e^{j\Omega}\right)\right|$, given by

$$\left|H_{id}\left(e^{j\Omega}\right)\right| = \begin{cases} 0, & |\Omega| < \frac{\pi}{4} \\ 1, & \frac{\pi}{4} < |\Omega| < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |\Omega| \le \pi \end{cases}$$

[11 marks]

Q.2 (a) The sequence $x[n]$ is zero for $n < 0$ and for $n > N-1$. Assume that $N = 2^M$, where M is a positive integer. Let $g[n] = x[2n]$, and $h[n] = x[2n+1]$.
Show that the N-point discrete Fourier transform (DFT) of the sequence $x[n]$ can be obtained by appropriately combining the $N/2$-point DFTs of the sequences $g[n]$ and $h[n]$.

[8 marks]

(b) Draw the complete signal flow graph for an 8-point decimation-in-time fast Fourier transform (FFT) algorithm. **[12 marks]**

Q.3 (a) Let $H_c(s)$ denote the transfer function of a continuous-time filter. The transfer function of a discrete-time filter, $H(z)$, is obtained by applying the bilinear transformation to $H_c(s)$:

$$H(z) = H_c(s)\Big|_{s = (1 - z^{-1})/(1 + z^{-1})}$$

Show that the frequency responses of the discrete-time and continuous-time filters are related by

$$H\left(e^{j\Omega}\right) = H_c(j\omega)\Big|_{\omega = \tan(\Omega/2)}$$

[8 marks]

(b) A discrete-time high-pass filter with frequency response, $H\left(e^{j\Omega}\right)$, is to be designed to meet the following specifications:

$$\begin{array}{ll} 0.89 \leq \left|H\left(e^{j\Omega}\right)\right| \leq 1, & 0.6\pi \leq |\Omega| \leq \pi \\ \left|H\left(e^{j\Omega}\right)\right| \leq 0.18, & |\Omega| \leq 0.2\pi \end{array}$$

The filter is to be designed by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter.

Verify that a second order filter is sufficient to meet the specifications.

Determine the transfer function, $H(z)$, of the discrete-time filter.

Note that the transfer function of a second order Butterworth low-pass prototype filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

[12 marks]

Q.4 (a) Consider a stable, linear, shift-invariant filter with unit-sample response $h[n]$. The input to the filter, $x[n]$, is a real-valued sample sequence of a wide-sense stationary discrete-time random process. Let $y[n]$ denote the output sequence. Show that the input and output autocorrelation sequences, $\phi_{XX}[m]$ and $\phi_{YY}[m]$ respectively, are related by

$$\phi_{YY}[m] = \sum_{l=-\infty}^{\infty} v[l]\phi_{XX}[m-l],$$

where

$$v[l] = \sum_{k=-\infty}^{\infty} h[k]h[k-l].$$

[8 marks]

(b) Let $x[n]$ be a real white-noise sequence with zero mean and autocorrelation sequence $\phi_{XX}[m] = \sigma_X^2\,\delta[m]$, where $\delta[m]$ is the unit-sample sequence.
The sequence $x[n]$ is the input to a causal and stable linear shift-invariant filter. The output of the filter, $y[n]$, satisfies the following difference equation:

$$y[n] = a\,y[n-1] + b\,x[n]$$

where a and b are real constants, and $a > 0$.
Let $\Phi_{YY}(z)$ denote the z-transform of the output autocorrelation sequence, $\phi_{YY}[m]$.

(i) Show that

$$\Phi_{YY}(z) = \frac{\sigma_X^2\, b^2}{(1-a\,z^{-1})(1-a\,z)}, \quad a < |z| < \frac{1}{a}$$

[4 marks]

(ii) Obtain an expression for the output autocorrelation sequence, $\phi_{YY}[m]$. Note that the z-transform of the sequence $a^{|m|}$ is

$$\frac{1-a^2}{(1-a\,z^{-1})(1-a\,z)}, \quad a < |z| < \frac{1}{a}$$

[2 marks]

(iii) Express the power spectral density, $S_{YY}(\Omega)$, of the output process in terms of the magnitude of the frequency response of the system. **[6 marks]**

Q.5 (a) Let $x[n]$ denote an infinite-duration sequence with discrete-time Fourier transform $X(e^{j\Omega})$, and $x_1[n]$ a finite-duration sequence of length N whose N-point DFT is denoted by $X_1[k]$. Determine the relation between $x_1[n]$ and $x[n]$ if $X_1[k]$ and $X(e^{j\Omega})$ are related by

$$X_1[k] = X(e^{j\Omega})\,\big|_{\Omega=(2\pi/N)k}\,, \quad k = 0, 1, 2, \ldots, N-1.$$

[12 marks]

(b) Consider a finite-duration sequence $x[n]$ of length M such that $x[n] = 0$ for $n < 0$ and $n \geq M$. We want to compute samples of the discrete-time Fourier transform of $x[n]$ at the N equally spaced frequencies

$$\Omega_k = \frac{2\pi k}{N}, \quad k = 0, 1, \ldots, N-1.$$

Determine and justify procedures for computing the N samples of the discrete-time Fourier transform using only one N-point DFT for the following cases: (i) $N > M$; and (ii) $N < M$.

[8 marks]