transfer functions

Q2 (a)

Q3. is answered in the process

$$H_c(s) \xrightarrow[\text{transformation}]{\text{bilinear}} H(z)$$

continuous - time   discrete - time

$$\boxed{s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \quad (1)$$

$$\Leftrightarrow \left(1+\frac{1}{z}\right)\left(\frac{T_d}{2}\right)s = 1 - \frac{1}{z}$$

$$\Leftrightarrow \frac{1}{z}\left(\left(\frac{T_d}{2}\right)s + 1\right) = 1 - \left(\frac{T_d}{2}\right)s$$

$$\Leftrightarrow z = \frac{1 + \left(T_d/2\right)s}{1 - \left(\frac{T_d}{2}\right)s}$$

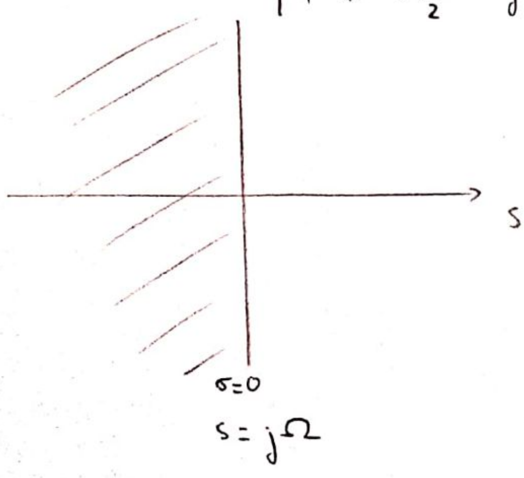
Now let's write $s = \sigma + j\Omega$

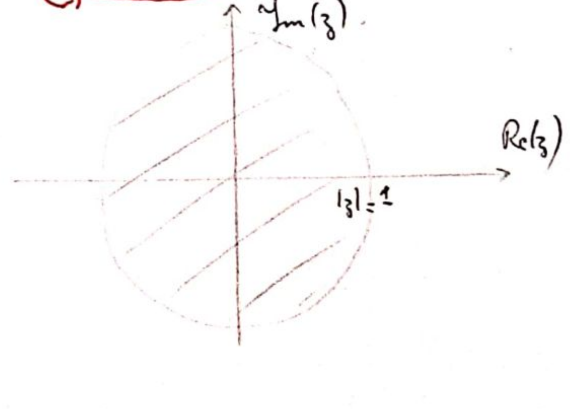
then $$z = \frac{1 + \sigma\frac{T_d}{2} + j\Omega\frac{T_d}{2}}{1 - \sigma\frac{T_d}{2} - j\Omega\frac{T_d}{2}}$$

Let's assume $\sigma < 0$, then $|z| < 1$

Proof: $$|z| = \frac{\left|1 + \sigma\frac{T_d}{2} + j\Omega\frac{T_d}{2}\right|}{\left|1 - \sigma\frac{T_d}{2} - j\Omega\frac{T_d}{2}\right|} = \frac{\sqrt{\left(1+\sigma\frac{T_d}{2}\right)^2 + \left(\Omega\frac{T_d}{2}\right)^2}}{\sqrt{\left(1-\sigma\frac{T_d}{2}\right)^2 + \left(\Omega\frac{T_d}{2}\right)^2}} < 1$$

is less than





Take $s = j\Omega$ / $\sigma = 0$

Then $z = \dfrac{1 + j\Omega \frac{T_d}{2}}{1 - j\Omega \frac{T_d}{2}}$. Obviously, $|z| = 1$

The $j\Omega$-axis maps to the unit circle.

We can rewrite: $e^{j\omega} = \dfrac{1 + j\Omega \frac{T_d}{2}}{1 - j\Omega \frac{T_d}{2}}$

$$\Leftrightarrow \quad s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right)$$

Separately, $\dfrac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \dfrac{\cancel{e^{-j\frac{\omega}{2}}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{\cancel{e^{-j\frac{\omega}{2}}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)} = \dfrac{2j \sin\left(\frac{\omega}{2}\right)}{2\cos\left(\frac{\omega}{2}\right)}$

So, $s = \dfrac{2j}{T_d}\tan\left(\dfrac{\omega}{2}\right)$

and $s = j\Omega$

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)$$

By choosing $T_d = 2$, one can find

$$H(z) = H(e^{j\omega}) = H_c(j\Omega)\Big|_{\Omega = \tan\left(\frac{\omega}{2}\right)}$$

Q2 (b) specifications

$$0.89 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.18 \qquad 0.6\pi \leq \omega \leq \pi$$

We require that

$$0.89 \leq |H_c(j\Omega)| \leq 1 \qquad 0 \leq \Omega \leq \tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_c(j\Omega)| \leq 0.18 \qquad \tan\left(\frac{0.6\pi}{2}\right) \leq \Omega \leq \infty$$

Since a continuous-time Butterworth filter has a monotonic magnitude response, this is equivalent to

$$|H_c(j\tan(0.1\pi))| \geqslant 0.89 \qquad (2.1)$$

$$|H_c(j\tan(0.3\pi))| \leqslant 0.18 \qquad (2.2)$$

The form of the magnitude-squared function for the Butterworth filter is :

$$\boxed{|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}} \qquad (3)$$

Solving for N and Ω_c with the equality sign in eqs (2.1) and (2.2) we obtain

$$1+\left(\frac{\tan(0.1\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89}\right)^2 \qquad (2.3)$$

$$1+\left(\frac{\tan(0.3\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.18}\right)^2$$

This yields:

$$\left(\frac{\tan(0.1\pi)}{\tan(0.3\pi)}\right)^{2N} = \frac{\left(\frac{1}{0.89}\right)^2-1}{\left(\frac{1}{0.18}\right)^2-1} = c^r$$

$$N = \frac{\log c^r}{2\log\left(\frac{\tan(0.1)\pi}{\tan(0.3)\pi}\right)}$$

$$N = 1.64$$

Thus the second order $(N=2)$ is sufficient to meet the specifications.

Substituting $N=2$ in (2.3), we obtain :

$$\Omega_c = \sqrt[4]{\left(\left(\frac{1}{0.89}\right)^2-1\right)/\tan(0.1\pi)^4} = 2.2$$

Scanné avec CamScanner

$$H(z) = H_c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$= \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\,\frac{1-z^{-1}}{1+z^{-1}} + 1}$$

$$= \frac{\left(1 + z^{-1}\right)^2}{\left(1+z^{-1}\right)^2 + \left(1-z^{-1}\right)^2 + \sqrt{2}\left(1-z^{-1}\right)^2}$$

$$= \frac{\left(\frac{z+1}{\cancel{z}}\right)^2}{\left(\frac{z+1}{\cancel{z}}\right)^2 + \left(\frac{z-1}{\cancel{z}}\right)^2 + \sqrt{2}\left(\frac{z-1}{\cancel{z}}\right)^2}$$

$$= \frac{(z+1)^2}{(z+1)^2 + (z-1)^2 + \sqrt{2}\,(z-1)^2}$$

$$= \ldots$$

Q4 $\quad 0.89 \leq |H(e^{j\omega})| \leq 1 \qquad 0.6\pi \leq \omega \leq \pi$

$\quad |H(e^{j\omega})| \leq 0.18 \qquad \omega \leq 0.2\pi$ } high-pass

Prewarp the critical discrete time frequencies to the corresponding analog frequencies.

For $\tan\left(\frac{0.3\pi}{1}\right) \leq \Omega \leq \tan\left(\frac{\pi}{2}\right)$, $\quad 0.89 \leq |H_c(j\Omega)| \leq 1$

$\Omega \leq \tan(0.1\pi)$, $\quad |H_c(j\Omega)| \leq 0.18$

Hence $|H_c(j\tan(0.1\pi))| \leq 0.18$

$|H_c(j\tan(0.3\pi))| \geq 0.89$

Butterworth filter magnitude response (cf. eq. (3)), this time high-pass

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega_c}{\Omega}\right)^{2N}}$$

where Ω_c is the 3-dB frequency and N is the order of the filter.

$$1 + \left(\frac{\Omega_c}{\tan(0.1\pi)}\right)^{2N} = \left(\frac{1}{0.18}\right)^2$$

$$1 + \left(\frac{\Omega_c}{\tan(0.3\pi)}\right)^{2N} = \left(\frac{1}{0.89}\right)^2$$

We find $N = 1.64$, the order of the filter must be an integer.

$N = 2$

We obtain then $\Omega_c = 0.76$

e second order Butterworth low pass prototype filter transfer function

given by :

$$H_p(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}$$

$$H_c(s) = H_p\left(\frac{\Omega_c}{s}\right)$$

$$= \frac{1}{\left(\frac{0.76}{s}\right)^2 + \sqrt{2}\left(\frac{0.76}{s}\right) + 1}$$

$$= \frac{s^2}{s^2 + 1.07\, s + 0.58}$$

bilinear transformation $s = \frac{1-z^{-1}}{1+z^{-1}}$

$$\boxed{H(z) = \frac{0.38 - 0.76\, z^{-1} + 0.38\, z^{-2}}{1 - 0.32\, z^{-1} + 0.13\, z^{-2}}}$$

Tutorial 3

Q1 Given the 3-dB cutoff frequency of the Butterworth filter, all that needed is to find the filter order N, that will give 40dB of attenuation at 3 kHz, or $\Omega_s = 2\pi \cdot 3000$. At the stopband cutoff frequency Ω_s, the magnitude of the frequency response squared is

$$|H_a(j\Omega)|^2_{\Omega=\Omega_s} = \left.\frac{1}{1 + \left(\frac{j\Omega}{j\underset{1.5\,kHz}{\Omega_c}}\right)^{2N}}\right|_{\Omega = \underset{3\,kHz}{\Omega_s}} = \frac{1}{1+2^{2N}}$$

Therefore, if we want the magnitude of the frequency response to be down 40dB at $\Omega_s = 2\pi \cdot 3000$, the magnitude squared must be no larger than 10^{-4}, or

$$\frac{1}{1+2^{2N}} \leq 10^{-4}$$

$$\Leftrightarrow \quad 1 \leq 10^{-4} + 10^{-4}\, 2^{2N}$$

$$\Leftrightarrow \quad 2^{2N} \geq \frac{1-10^{-4}}{10^{-4}}$$

$$\Leftrightarrow \quad 2N \log 2 \geq \log\left(\frac{10^{4} - 1}{\cancel{10^{4}}\ 1}\right)$$

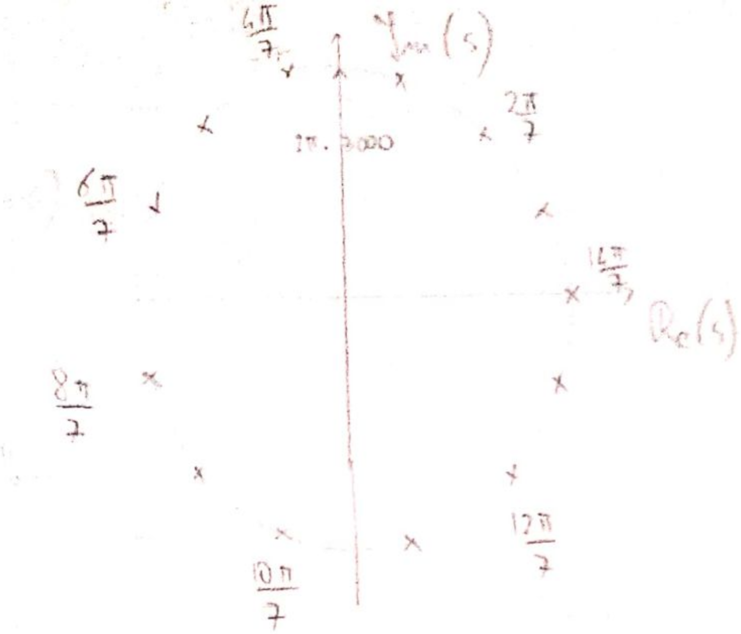
$$\Leftrightarrow \quad 2N \geq \frac{\log(10^4 - 1)}{\log 2} = 13.29 \Rightarrow \boxed{N = 7}$$

For a seventh-order Butterworth filter, the 14 poles of

$$H_a(s)\, H_a(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

lie on a circle of radius $2\pi \cdot 3000$, at angles of:

$$\theta_k = \frac{(N+1+2k)\pi}{2N} = \frac{2(4+k)\pi}{7} \quad k = 0, 1 \dots 13$$



The poles of $H_a(s)$ are the seven poles of $H_a(s)H_a(-s)$ that lie in the left-half s-plane, that is,

$$s_k = -\Omega_c e^{\pm jk\pi/7} \qquad k = 0,1,2,3$$

Except for the isolated pole at $s = -\Omega_c$, the remaining six poles occur in complex conjugate pairs. The conjugate pairs may be combined to form second-order factors with real coefficients to yield factors of the form

$$H_k(s) = \frac{1}{s^2 - 2\Omega_c \cos\left(\frac{k\pi}{7}\right)s + \Omega_c^2} \qquad k = 1,2,3$$

Thus, the system function of the seventh-order Butterworth filter is

$$\boxed{H_a(s) = \prod_{k=0}^{N-1} \frac{-s_k}{s - s_k} = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{3} \frac{\Omega_c^2}{s^2 - 2\Omega_c \cos\left(\frac{k\pi}{7}\right)s + \Omega_c^2}}$$