EE4C5 Digital Signal Processing

Lecture 8 – Phase in filters

This lecture

- Partly based on Chapter 5 of O&S and general concepts in DTFT and signals & systems
- Chapter 4, Mitra book
- All other images from O&S book

Phase Response

- Considered the amplitude response of a filter $\left|H\left(e^{j\omega}\right)\right|$
- Also characterised by the phase response $\angle H\left(e^{j\omega}\right)$
- $H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\angle H\left(e^{j\omega}\right)}$
- Concepts:
  - Phase delay
  - Group delay

Distortionless transmission

- System gives distortionless transmission if the form of the signal passed through that system is unaffected:
  - output signal is a delayed and scaled replica of the input signal, i.e.
  - $y(t) = Gx(t - t_d)$ or $y[n] = Gx[n - n_d]$
- System must amplify or attenuate each frequency component uniformly
  - the magnitude response must be uniform within the signal frequency band.
- System must delay each frequency component by the same number of samples.

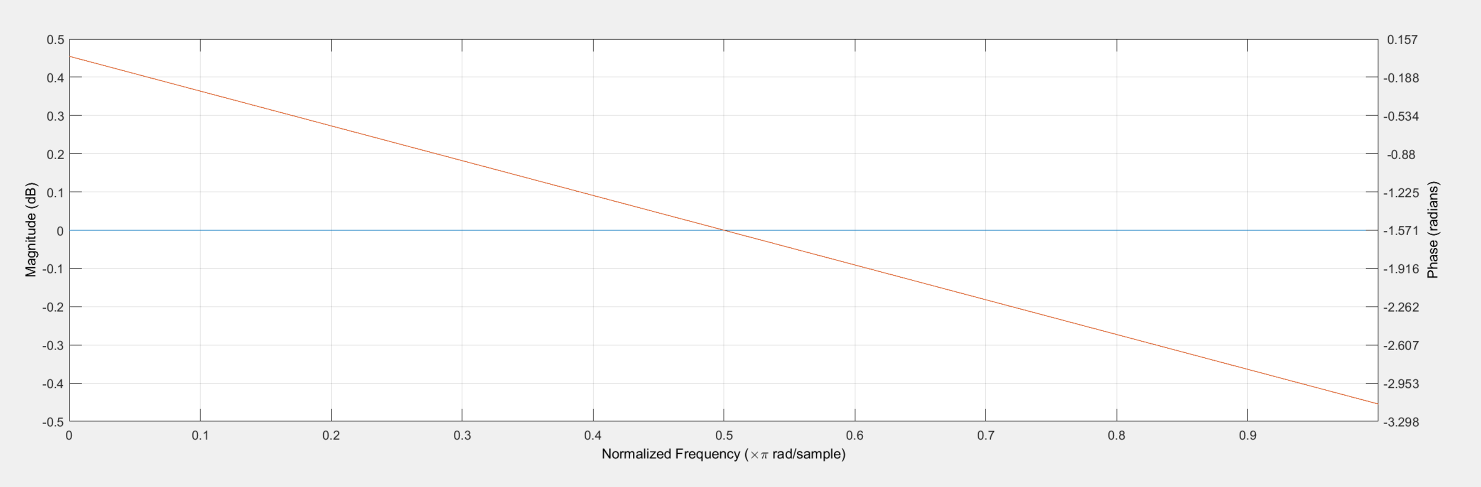
Ideal delay system

- $y[n] = x[n - n_d]$
- Can express h$[n] = \partial[n - n_d]$ $\qquad$ n_da positive fixed integer

- Take FT yields
  - $H(\omega) = \ e^{-j\omega n_d}$
- So:
  - $\left|H\left(e^{j\omega}\right)\right| = 1$
  - $\angle H\left(e^{j\omega}\right) = -\omega n_d$

Ideal delay system

- $y[n] = x[n - n_d]$
- Can express h$[n] = \partial[n - n_d]$ $\qquad$ n_d a positive fixed integer



- Take FT yields
  - $H(\omega) = \ e^{-j\omega n_d}$
- So:
  - $\left|H\left(e^{j\omega}\right)\right| = 1$ **CONSTANT**
  - $\angle H\left(e^{j\omega}\right) = -\omega n_d$ **LINEAR FUNCTION OF ω**

Phase delay

- Phase delay defined by:
  - $\Theta(\omega) = -\frac{\angle H(e^{j\omega})}{\omega}$
- And measured in samples.
- The relative delay imposed on individual frequency components of the input signal .
- If this is constant => distortionless transmission.
- All frequency components delayed same amount.
- A filter with this property is a LINEAR-PHASE filter
  - Phase varies linearly with ω

Linear Phase

- In general, a linear phase frequency response has the form:
  - $H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha}$
- Phase $-\omega\alpha$
- Consider example of ideal low pass filter, linear phase and cut-off ω_c :
  - $H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$
- Impulse response:
  - $h_{lp}[n] = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}, \quad -\infty < n < \infty$

Ideal LPF with linear phase

- Has some interesting properties…
- If α is an integer, impulse response symmetric about $\alpha = n_d$
- $h_{lp}[2n_d - n] = \frac{\sin \omega_c(2n_d - n - n_d)}{\pi(2n_d - n - n_d)} = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} = h_{lp}[n]$
- A symmetric impulse response means linear phase.

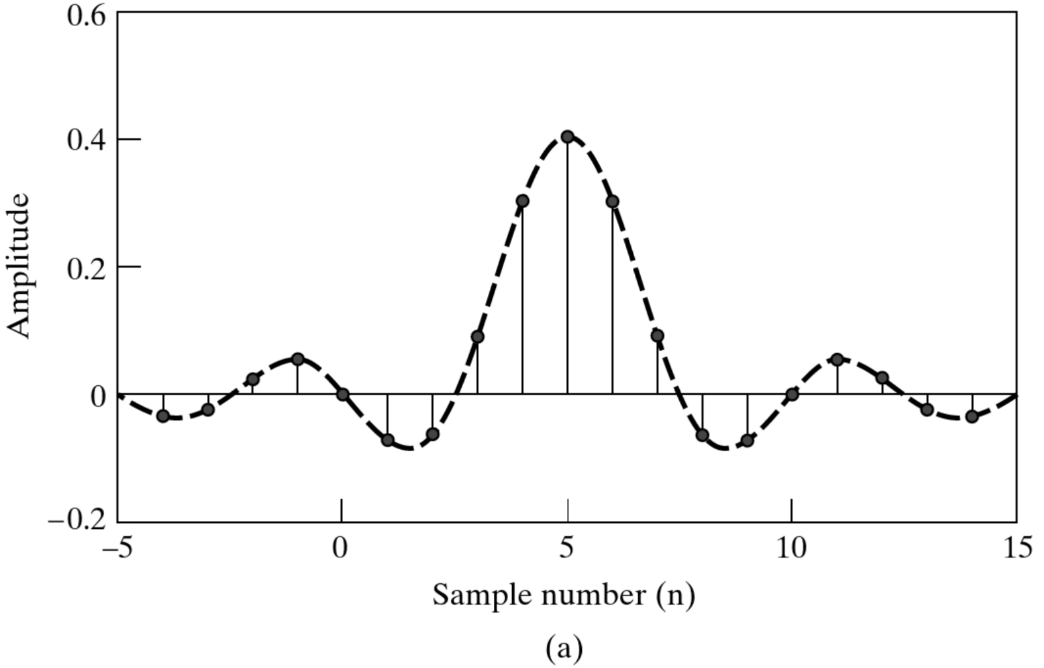


Figure 32 Ideal lowpass filter impulse responses, with $\omega_C = 0.4\pi$. (a) Delay $= \alpha = 5$.

Group Delay

- The group delay of a system is measures in samples as:
- $\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H\left(e^{j\omega}\right)\right\}$
- Linear phase results in constant group delay.
- Nonlinearity of the phase (or equivalently nonconstant group delay) results in time dispersion.

Generalised linear phase

- Useful properties even when not strict linear phase.
- A system is referred to as a generalised linear-phase system if its frequency response can be expressed in the form:
- $H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$
  - With $A(e^{j\omega})$ a real function of ω and
  - α and β are constants
- Notice that phase of such a system consists of constant terms added to the linear function $-\omega\alpha$ and also $-\omega\alpha + \beta$ is the equation of a straight line.

Generalised linear phase #2

- Group delay is constant:
  - $\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H\left(e^{j\omega}\right)\right\} = \alpha$

- What about the phase delay?
  - $\Theta(\omega) = -\frac{\angle H\left(e^{j\omega}\right)}{\omega} = \alpha - \frac{\beta}{\omega}$
  - i.e. NOT constant
  - => Generalised linear phase can introduce phase distortions
- In practice, can design filters with generalised linear phase to have near distortionless transmission of the envelope of bandlimited signal

Linear-Phase FIR Responses

- It is nearly impossible to design a linear-phase IIR transfer function
- It is always possible to design an FIR transfer function with an exact linear-phase response
- Consider a causal FIR response $H(e^{j\omega})$, with $h[n]$ with $M+1$ coefficients
  - Note it is said to have order M
  - $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$

Linear-Phase FIR Responses

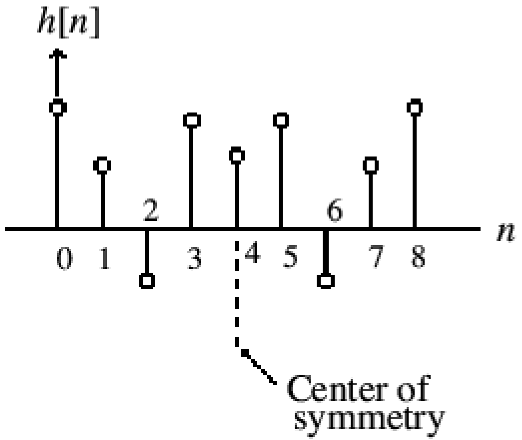
- The response has a linear phase, if its impulse response $h[n]$ is either symmetric:
  - $h[n] = h[M-n] \quad 0 \le n \le M$

- or is antisymmetric:
  - $h[n] = -h[M-n] \quad 0 \le n \le M$

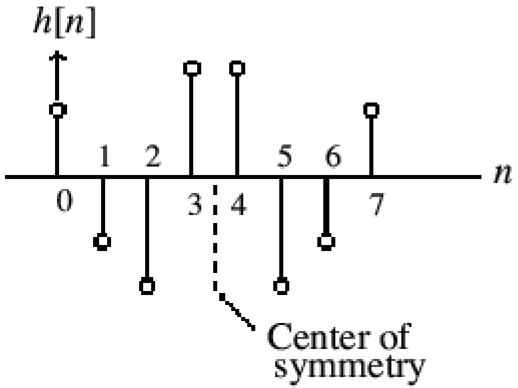
Linear-Phase FIR Responses

- Length of the impulse response can be either even or odd
- => can define 4 types of linear-phase FIR responses
  - Symmetric odd/even
  - Antisymmetric odd/even

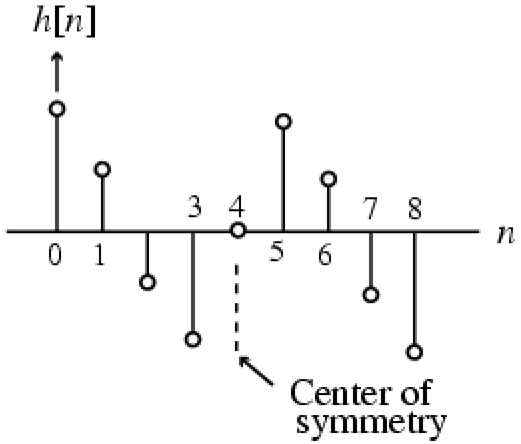
Linear-Phase FIR Responses



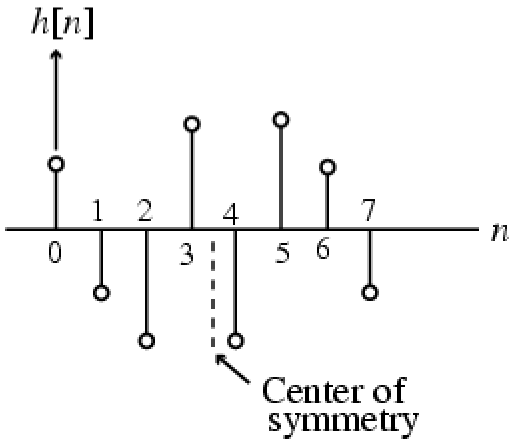
Type 1: $M = 8$



Type 2: $M = 7$



Type 3: $M = 8$



Type 4: $M = 7$

(Mitra book)

FIR filters with linear phase

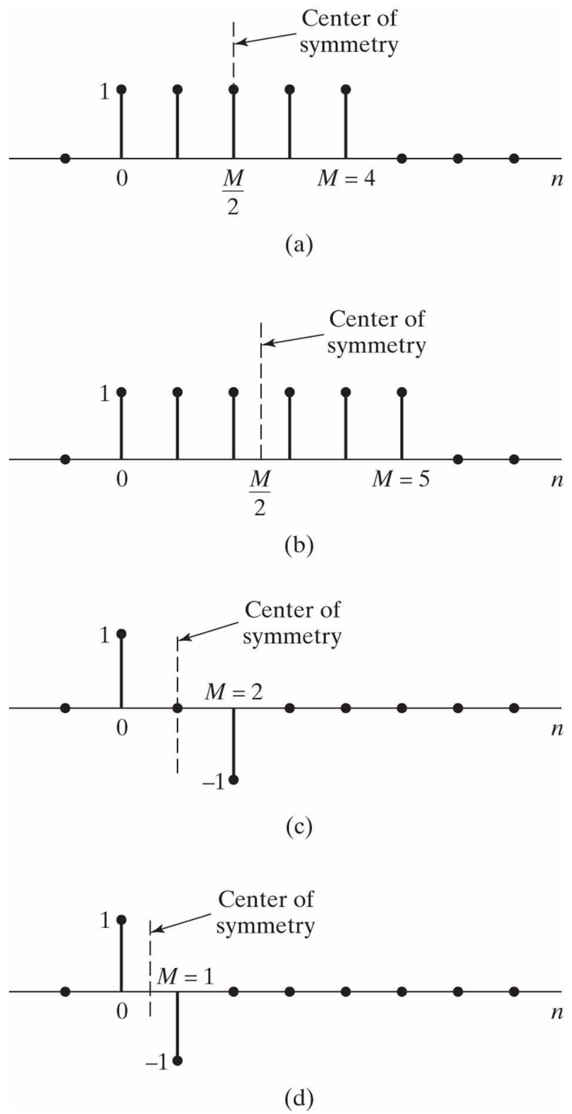


Figure 5.33 Examples of FIR linear-phase systems.

(a)Type I, M even, $h[n] = h[M - n]$.

(b)Type II, M odd, $h[n] = h[M - n]$.

(c)Type III, M even, $h[n] = -h[M - n]$.

(d)Type IV, M odd, $h[n] = -h[M - n]$.

Linear-Phase FIR Transfer Functions

- **Type 1**: Symmetric Impulse Response with Odd Length
- In this case, the degree M is even
- Assume M = 8 for simplicity
- The transfer function H(z) is given by

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$
$$+ h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

Linear-Phase FIR Transfer Functions

- Because of symmetry, we have h[0] = h[8], h[1] = h[7], h[2] = h[6], and h[3] = h[5]
- Thus, we can write

$$
\begin{aligned}
H(z) &= h[0](1+z^{-8}) + h[1](z^{-1}+z^{-7}) \\
&\quad + h[2](z^{-2}+z^{-6}) + h[3](z^{-3}+z^{-5}) + h[4]z^{-4} \\
&= z^{-4}\{h[0](z^{4}+z^{-4}) + h[1](z^{3}+z^{-3}) \\
&\quad + h[2](z^{2}+z^{-2}) + h[3](z+z^{-1}) + h[4]\}
\end{aligned}
$$

Required Reading & other material

- Oppenheim & Schafer, Chapter 2, section 2.1
- Group delay vs phase delay: https://www.youtube.com/watch?v=ox-CyJVpJEM&t=571s
  - Gets into more detail later in video, first part useful for now