Tutorial 2: Solutions

1. What is the purpose of a pulse-shaping filter in a digital communication system?

   Solution: Pulse shaping is a method for reducing the bandwidth of the square pulses coming out of the digital modulator by attenuating the side lobes energy. This filtering must be done considering the Nyquist criterion for distortion-less transmission, which establishes the conditions for avoiding ISI.

2. Consider a system with the input $x(t)$, the impulse response $h(t) = \frac{1}{\pi t}$ and the output $\hat{x}(t) = x(t) * h(t)$. Prove the following properties of this system.

   (a) If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$.

   (b) If $x(t) = \sin(\omega_0 t)$, then $\hat{x}(t) = -\cos(\omega_0 t)$.

   (c) $< x(t), \hat{x}(t) >= \int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.

   Solution:

   (a)

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda} d\lambda$$

$$\begin{aligned}
-\hat{x}(-t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{-t-\lambda} d\lambda \\
&= -\frac{1}{\pi}\int_{\infty}^{-\infty} \frac{x(-\gamma)}{-t+\gamma} (-d\gamma) \\
&= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\gamma)}{-t+\gamma} d\gamma \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\gamma)}{t-\gamma} d\gamma \\
&= \hat{x}(t)
\end{aligned}$$

   (b)

$$x(t) = \sin(\omega_0 t) \Rightarrow X(f) = \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)]$$

   The filter $h(t)$ is a Hilbert transformer. Hilbert transformer is a $\frac{\pi}{2}$ phase shifter for all the frequencies. In particular, Hilbert transform applies $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ phase shifts to the positive and negative frequencies, respectively. Hence, $H(f) = -j(2u(f)-1)$.

$$\Rightarrow \hat{X}(f) = X(f)H(f) = \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)] \times [-j(2u(f)-1)]$$
$$= \frac{1}{2}[-\delta(f-f_0) - \delta(f+f_0)] = -\frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$$
$$= -\mathcal{F}^{-1}\{\cos(2\pi f_0 t)\} \Rightarrow \hat{x}(t) = -\cos(\omega_0 t)$$

(c) From the result of part (a), one may realize that if $x(t)$ is an even function, $\hat{x}(t)$ is an odd function or vice-versa. Consequently, the product of the two is always an odd function and thus, their inner product is zero.

3. Given a digital stream generating 7 kbits/second and a system with a maximum transmission rate of 4 ksymbols/second.

   (a) Choose a suitable digital modulation scheme for transmission.

   (b) What is the bandwidth required using the modulation scheme chosen assuming a pulse shaping filter which is:

      i. An ideal LPF.

      ii. A raised cosine pulse with 75% excess bandwidth.

   (c) What bandwidth would be required if 64QAM was used with the raised cosine filter of the previous point?

   (d) Suggest a possible disadvantage of using 64QAM.

   Solution:

   (a) The digital stream generates 7 bits/second and thus there is no data for transmission at a higher rate. Given a maximum transmission rate of 4 ksymbols/second, we must choose a constellation which gives at least 7/4 kbits/symbol. We also wish to choose the modulation scheme with the lowest probability of error. The constellation with the lowest probability of error is BPSK, but this only allows transmission of 1 bit/symbol and thus cannot be used. 4-QAM allow transmission of 2 kbits/symbol and has the lowest probability of error of the remaining constellations (highest minimum distance). The modulation scheme we choose is 4-QAM for the above reasons.

   (b) i. Using 4-QAM, only 3.5 ksymbols/second are sent. Using a LPF as shaping filter the bandwidth of 4-QAM bandpass signal is equal to the symbol rate, i.e. 3.5 kHz. If the signal was transmitted in baseband the bandwidth would be half, 1.75 kHz.

      ii. Using a raised cosine pulse with 75% excess bandwidth, the bandwidth requirements are increased by 75% over the ideal LPF shaping filter. The bandwidth required in this case is 6.125 kHz for a bandpass signal, and 3.0625 kHz for baseband.

(c) Using 64-QAM, we may send 6 bits/symbol which corresponds to 1.166 ksymbols/second. The bandwidth used is then 2.041 kHz for bandpass transmission, which is a smaller bandwidth than using 4-QAM. For baseband transmission the bandwidth would be 1.0205 kHz.

(d) The disadvantage is the higher probability of error for the same transmit power, due to the smaller distance between the constellation points. This is an example of the tradeoff between bandwidth and probability of error.

4. A signal is modulated using 8-PSK with a raised cosine pulse shape onto a carrier at 300 MHz. This signal is then transmitted over a noiseless, nondistorting channel and demodulated using a carrier at 300 MHz with a carrier timing offset of 5 ns and sampled at the ideal sampling instant. Draw an illustration of the signal space representation of the demodulated signal

(a) Sketch the signal space diagram of the demodulated received data with ideal receive carrier timing synchronization.

(b) Sketch the signal space diagram of the demodulated received data with the timing offset specified above.

(c) What effect would be observed if the signal were demodulated by a carrier at 295 MHz?

Solution:

(a) The signal space diagram of the ideal timing synchronisation matches the one on the left in the figure below.

(b) The in-phase part of the modulated signal is of the form:

$$m(t) = A(t)\cos(2\pi f_c t)$$

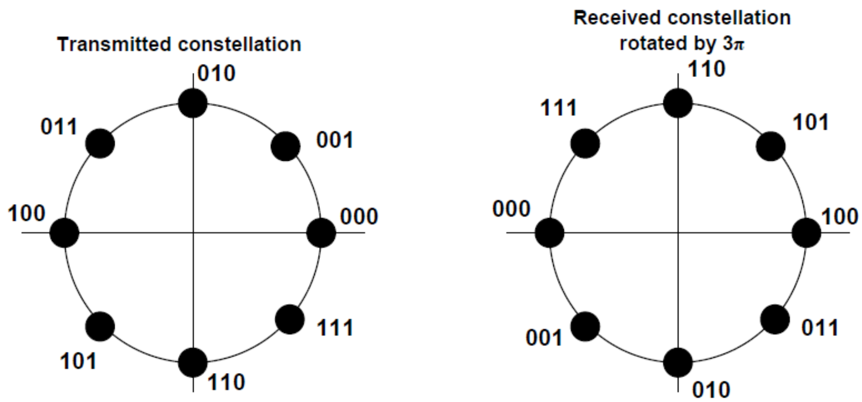
demodulation is performed by multiplication by $\cos(2\pi f_c t)$ which in this case has a timing offset of 5 ns. Thus, the signal becomes:

$$\begin{aligned} d(t) &= m(t)\cos(2\pi f_c(t+t_0)) \\ &= A(t)\cos(2\pi f_c t)\cos(2\pi f_c(t+t_0)) \\ &= \frac{1}{2}A(t)\left[\cos(2\pi f_c t_0) + \cos(4\pi f_c t + 2\pi f_c t_0)\right] \end{aligned}$$

The second term is removed by the low-pass filter, leaving (since the 1/2 is irrelevant to our signal space diagram)

$$d(t) = A(t)\cos(2\pi f_c t_0)$$

which corresponds to a rotation of: $\phi = 2\pi f_c t_0 = 2\pi \times 300 \times 10^6 \times 5 \times 10^{-9} = 3\pi$ rad. With a similar line of derivations, the quadrature part will also be rotated by the same amount. Thus, the whole signal space diagram is rotated by 3π radian (approx. 540 degrees).



(c) As discussed in the class, frequency offset between the transmitter and receiver corresponds to an anti-clockwise rotation of the signal space by $\psi(t) = 2\pi f_0 t = 2\pi \times (300 - 295) \times 10^6 \times t$ which is a function of time. This means that the signal space diagram is further rotated at each sample.

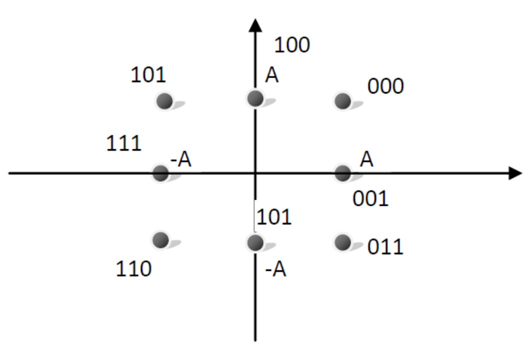
5. A telecommunications company wishes to transmit a signal with a bit rate equal to 750 kbit/s through a radio channel centred at 250 MHz and bandwidth 300 kHz. To do so, a modulator which transmits the following waveforms is used, where $\psi_1(t)$ and $\psi_2(t)$ being orthonormal signals.

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\cos(\frac{2\pi n t}{T}) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and

$$\psi_2(t) = \begin{cases} -\sqrt{\frac{2}{T}}\sin(\frac{2\pi n t}{T}) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The constellation and coding employed by the digital modulator is the following figure



(a) Show that $\psi_1(t)$ and $\psi_2(t)$ are orthonormal.
(b) Determine the symbol period T of the system.
(c) Calculate the value of n so the carrier frequency is equal to 250 MHz.
(d) If a raised cosine filter is used for pulse shape, what is the maximum roll-off factor that can be used so the transmitted signal fits in the bandwidth available?
(e) If $A = 5.164 \times 10^{-3}$, calculate the average symbol energy, average bit energy and average transmitted power.

Solution:

(a) $< \psi_1(t), \psi_2(t) >= \int_0^T \left(\sqrt{\frac{2}{T}}\cos(\frac{2\pi nt}{T})\right)\left(-\sqrt{\frac{2}{T}}\sin(\frac{2\pi nt}{T})\right) dt = \frac{2}{T}\int_0^T \frac{1}{2}\sin(\frac{4\pi nt}{T})dt = 0$. Therefore they are orthogonal.

$$E_1 = \int_0^T \psi_1^2(t)dt = \int_0^T \frac{2}{T}\cos^2(\frac{2\pi nt}{T})dt = \frac{1}{T}\int_0^T \left(1+\cos(\frac{4\pi nt}{T})\right)dt \quad (3)$$
$$= \frac{1}{T}\int_0^T (1+\cos(4\pi f_c t))\, dt = 1 \quad (4)$$

Similarly we have $E_2 = 1$. That is $\psi_1(t)$ and $\psi_2(t)$ are normalised in energy. Overall, they are orthonormal.

(b) The modulator has a set of 8 possible symbols, which require 3 bits-per-symbol to represent all values. The symbol rate will be 3 times smaller than the bit rate, this is $f_{sym} = 250$ ksymbol/s. So the symbol period is $T = \frac{1}{250\times10^3} = 4\mu$s.

(c)
$$f_c = \frac{n}{T} = nf_{sym} \Rightarrow n = \frac{f_c}{f_{sym}} = \frac{250\times10^6}{250\times10^3} = 1000 \quad (5)$$

(d) When a raised cosine pulse shape filter is used, the bandpass signal bandwidth varies between f_{sym} (ideal brick-wall filter case or Nyquist case) and $2f_{sym}$. In this case the bandwidth would vary between 250 kHz and 500 kHz. A bandwidth of 300 kHz corresponds to $300/250 = 1.2$ or 20% extra bandwidth more than in the Nyquist filter case. This means that the maximum value for the roll-off factor is $\beta = 0.2$.

(e) Symbol energy:

$$E_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2 = \frac{1}{8}\sum_{i=1}^{M}\left(4A^2 + 4\left(\sqrt{2}A\right)^2\right) = 40\mu\text{J} \quad (6)$$

Bit energy:

$$E_b = \frac{E_s}{\text{\#bits per symbol}} = \frac{E_s}{3} = 13.3\mu\text{J} \quad (7)$$

Average transmitted power

$$P = \frac{E_s}{T} = \frac{40\times10^{-6}}{4\times10^{-6}} = 10\text{ W} \quad (8)$$