Q1 Answer is the same in all of a, b, c, d.

Refer to notes lecture 14, page 8.

Q2

(a) Gain in path shown. ($L \to R$)

$(+1)(+1)(W_N^0)(-1)(W_N^2)(1)$.

$-W_N^0 \cdot W_N^2$

but $W_N^0 = 1$ $\left(e^{-j0\frac{2\pi}{N}}\right)$

$\Rightarrow$ gain $-W_N^2$ overall on this path.

(b) from $x[7]$ to $X[2]$, there is only a single path.

In general, looking at the outputs $X[k]$, it may seem at first there are two paths arriving at any sample, but in fact if you look at the inputs, there is only <u>one</u> path between each input and each output.

Q2

Note this Q is Problem 6 in Chapter 9 of the O&S text

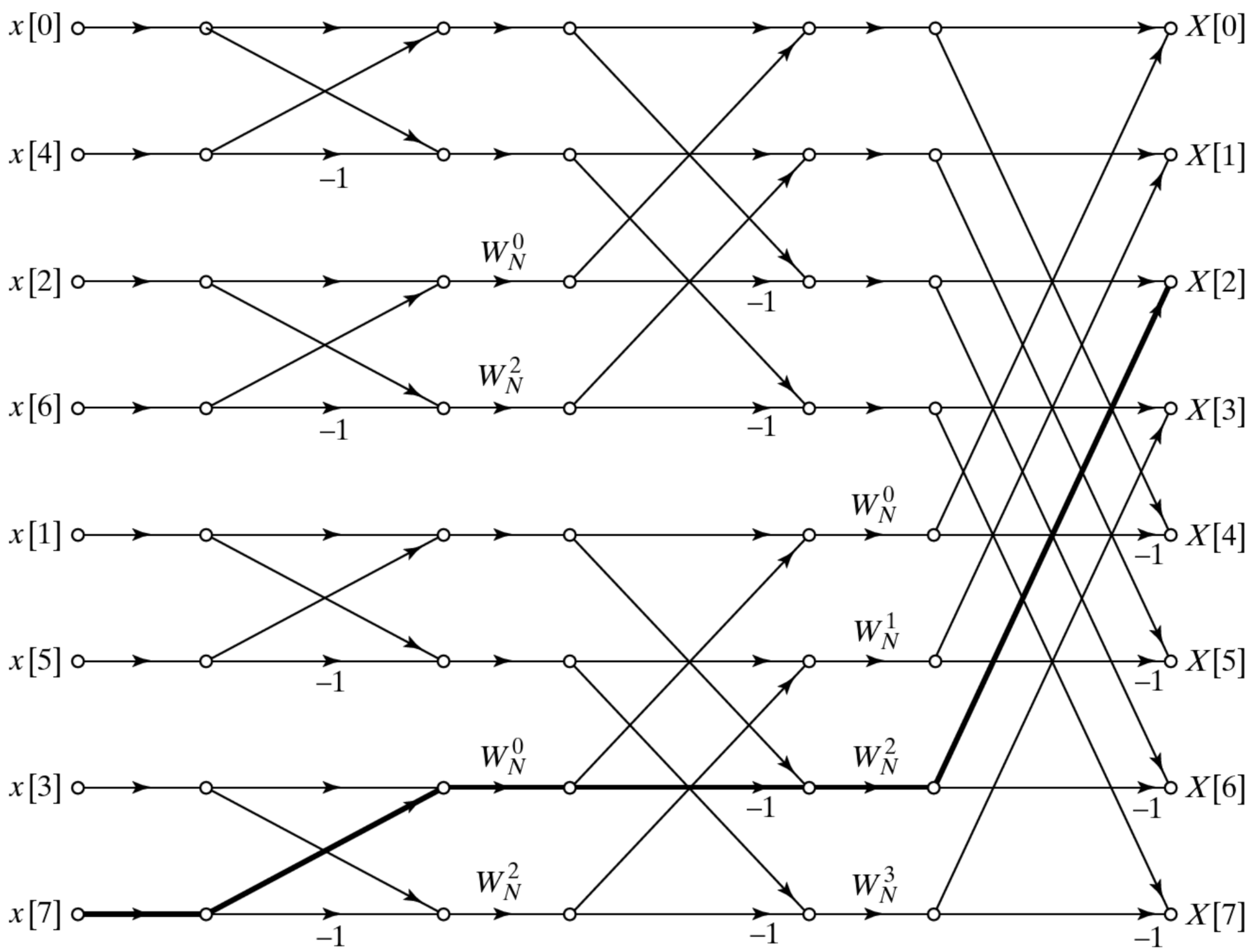


Figure P6 shows the graph representation of a decimation-in-time FFT algorithm for $N = 8$. The heavy line shows a path from sample $x[7]$ to DFT sample $X[2]$.

(a) What is the "gain" along the path that is emphasized in Figure P6?

(b) How many other paths in the flow graph begin at $x[7]$ and end at $X[2]$? Is this true in general? That is, how many paths are there between each input sample and each output sample?

(c) Now consider the DFT sample $X[2]$. By tracing paths in the flow graph of Figure P6, show that each input sample contributes the proper amount to the output DFT sample; i.e., verify that

$$X[2] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)2n}.$$

Q2 (c) DFT sample $X[2]$

Here need to trace path for all inputs $x[0]$ ---- $x[7]$, look at gains along each path, then sum.

Input	gain to $X[2]$	contributes.
$x[0]$	$(1)(1)(1)(1)(1)(1)$	$1\ x[0]$
$x[1]$	$(1)(1)(1)(1)\,W_N^2\,(1)$	$W_8^2\ x[1]$
$x[2]$	$(1)(1)(W_N^0)(-1)(1)(1)$	$-1\ x[2]$
$x[3]$	$(1)(1)(W_N^0)(-1)W_N^2(1)$	$-W_8^2\ x[3]$
$x[4]$	$(1)(1)(1)(1)(1)(1)$	$1\ x[4]$
$x[5]$	$(1)(1)(1)(1)\,W_N^2\,1$	$W_8^2\ x[5]$
$x[6]$	$(1)(1)\,W_N^0(-1)(1)(1)$	$-1\ x[6]$
$x[7]$	$(1)(1)\,W_N^0\cdot -1\,W_N^2\,1$	$-W_8^2\ x[7]$

From equation for DFT (as given in Q)

$$X[2] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)2n}$$

$$X[2] = x[0]e^{-j\frac{2\pi}{8}2(0)} + x[1]e^{-j\frac{2\pi}{8}2} + x[2]e^{-j\frac{2\pi}{8}2.2} + x[3]e^{-j\frac{2\pi}{8}2.3} + x[4]e^{-j\frac{2\pi}{8}2.4} + x[5]e^{-j\frac{2\pi}{8}2.5} + x[6]e^{-j\frac{2\pi}{8}2.6} + x[7]e^{-j\frac{2\pi}{8}2.7}$$

Q2

(c) (ctd)

Look at definition

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^2 = e^{-j\frac{2\pi}{N}\cdot 2}$$

$$W_8^2 = e^{-j\frac{2\pi}{8}\cdot 2} \quad \text{etc.}$$

$X[0]$ $e^{-j0} \rightarrow 1$ ✓

$X[1]$ e term is W_8^2 ✓

$X[2]$ $e^{-j\pi} \rightarrow -1$ ✓

$X[3]$ $e^{-j\frac{2\pi 2\cdot 3}{8}} \equiv e^{j\frac{2\pi}{8}\cdot 2}$ $\cancel{e^{j\frac{2\pi 4}{8}}} \rightarrow -1$

$= -W_8^2$

$X[4]$ $e^{j\frac{2\pi}{8}\cdot 8} = e^{j\pi} \rightarrow -1$

$X[5] = e^{-j\frac{2\pi}{8}2\cdot 5} = e^{j\frac{2\pi}{8}\cdot 2}$ $\cancel{e^{j\frac{2\pi 8}{8}}} \rightarrow 1$

$= W_8^2$

$X[6]$ $e^{-j\frac{2\pi}{8}2\cdot 6} = e^{-j3\pi} \rightarrow -1$

$X[7]$ $e^{-j\frac{2\pi}{8}2\cdot 7} = e^{-j\frac{2\pi}{8}\cdot 2}$ $\cancel{e^{-j\frac{2\pi 12}{8}}} \rightarrow -1$

$= -W_8^2$

Q3

Note this Q is Problem 16 from Chapter 9 of the O&S text

The butterfly in Figure P16 was taken from a decimation-in-time FFT with $N = 16$. Assume that the four stages of the signal flow graph are indexed by $m = 1, \ldots, 4$. What are the possible values of r for each of the four stages?

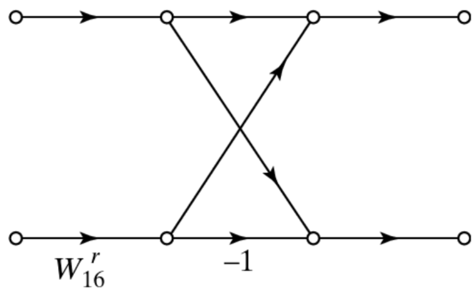
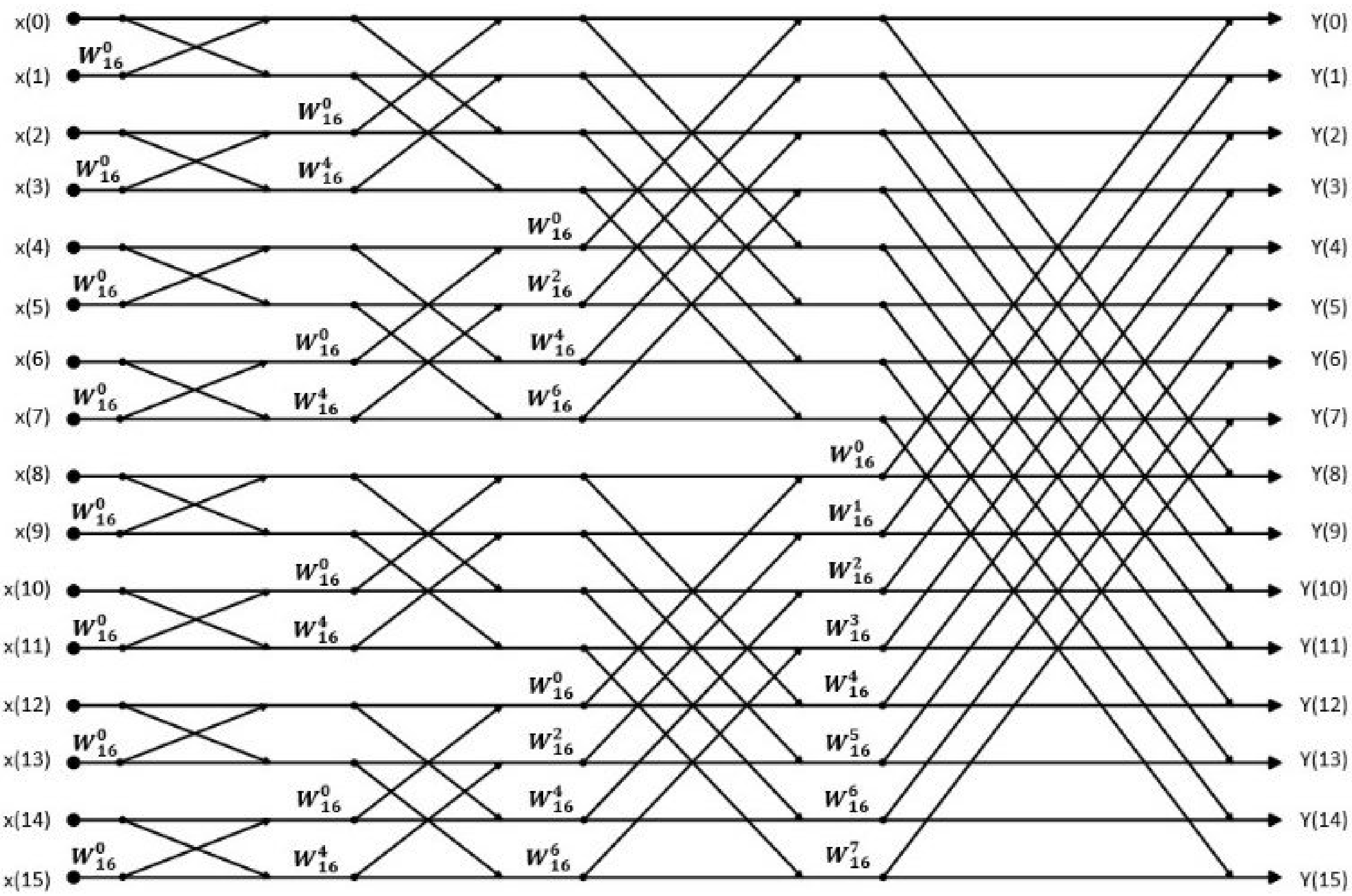


Figure P16

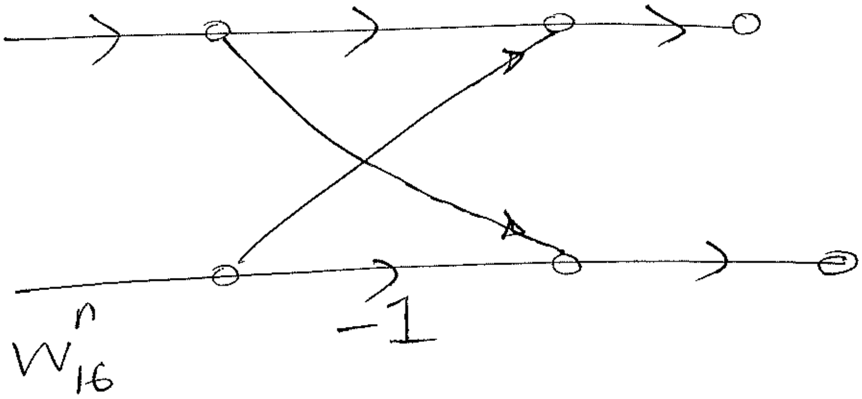


Source:

https://www.researchgate.net/publication/304297645_FPGA_design_and_implementation_of_radix-2_Fast_Fourier_Transform_algorithm_with_16_and_32_points/figures?lo=1

The diagram above is useful for reference. With extra samples, each stage will have more samples to process. Note changes in twiddle factors.

Q3.



Decimation in time with $N = 16$

Get 4 stages in our signal flow graph (recall 3 for $N = 8 = 2^3$, hence now 4 stages, always ν stages with 2^ν)

Stage M	values of r for W_{16}^{r}
1	0
2	0, 4 i.e. W_{16}^{0} W_{16}^{4}
3	0, 2, 4, 6
4	0, 1, 2, 3, 4, 5, 6, 7

Q4.

In lectures, we showed that for a sequence $x[n]$ which is zero for $n < 0$ and for $n > N - 1$ and with $N = 2^v$, where v is a positive integer that if we et $g[n] = x[2n]$ and $\mathrm{h}[n] = x[2n + 1]$ then the N-point discrete Fourier transform (DFT) of the sequence $x[n]$ can be obtained by appropriately combining the $N/2$-point DFTs of the sequences $g[n]$ and $\mathrm{h}[n]$.

Extend this, using a similar method, to show that the sequences $g[n]$ and $\mathrm{h}[n]$ can be again decimated.

Using your findings, draw the flow graph for the case N=8, showing how the 4-point (N/2) DFTs can be performed using instead the 2-point (N/4) point DFTs.

Q4.

In the lectures we showed that we can write N-point DFT of x[n] in terms of N/2 point DFTs of g[n] and h[n] where they are the even & odd samples in x[n] as defined in the question.

Proof was done in lectures and we showed that:

$$X[k] = \sum_{n=0}^{N/2-1} g[n] W_{N/2}^{kn} + W_N^{k} \sum_{n=0}^{N/2-1} h[n] W_{N/2}^{kn}$$

Decimate g[n] and h[n] further ⇒ divide by factor of 2 again.

Lets look at g[n] first.

$$G[k] = \sum_{n=0}^{N/2-1} g[n] W_{N/2}^{nk}$$

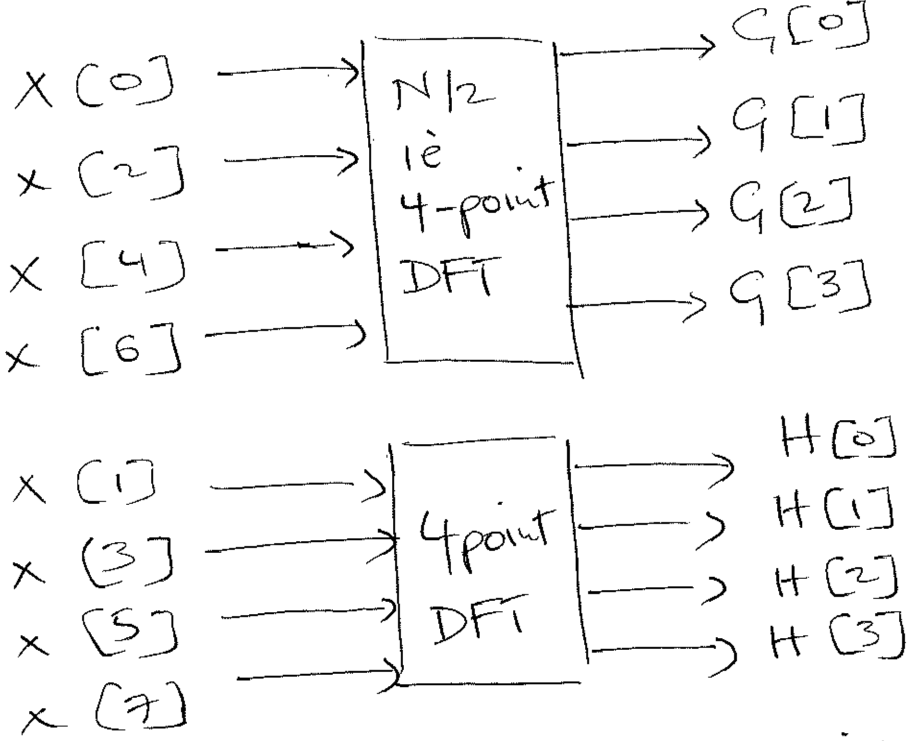
Divide into even and odd like before

$$= \sum_{n \text{ even}}^{N/2-1} g[n] W_{N/2}^{nk} + \sum_{n \text{ odd}}^{N/2-1} g[n] W_{N/2}^{nk}$$

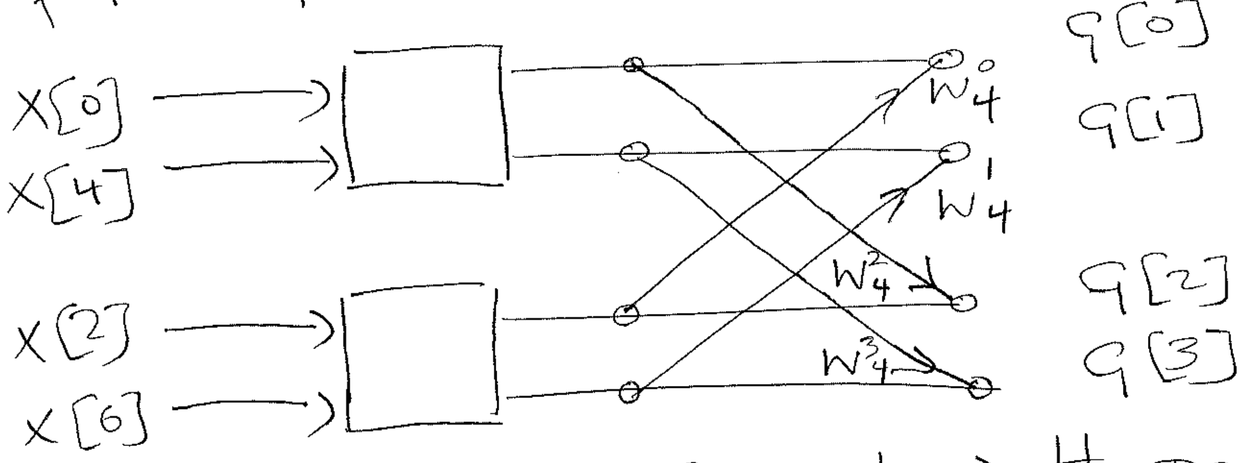
Which just like in original proof gives

$$G(k) = \sum_{n=0}^{N/4-1} g[2n] W_{N/4}^{nk} + W_{N/2}^{k} \sum_{n=0}^{N/4-1} g[2n+1] W_{N/4}^{nk}$$

For case $N=8$, had ~~N/4~~ $N/2$ point DFT replacing



If we go to $N/4$ point, ie 2point DFT for G elements this becomes.



and similar for $h \to H$ part.

Note twiddle factors W_4^0, W_4^1 etc are in form $W_{N/2}^k$

Can map $W_{N/2} \to W_N^2$

So $W_4^0 \to W_8^0$, $W_4^1 \to W_8^2$ etc. (See slide 10 in lecture 14)

Q5

You need to analyse the frequency content of a signal with 1024 samples. How many complex multiplications are required for a straightforward DFT calculation? Compare this with the number of complex multiplications needed for a 1024-point FFT.

Q5. 1024 samples.

Direct DFT $\Rightarrow$ N^2 complex mults.

FFT $o(N)$ complex mults.

$\Rightarrow$ 1,048,576 Vs 1024