EE4C5 Digital Signal Processing

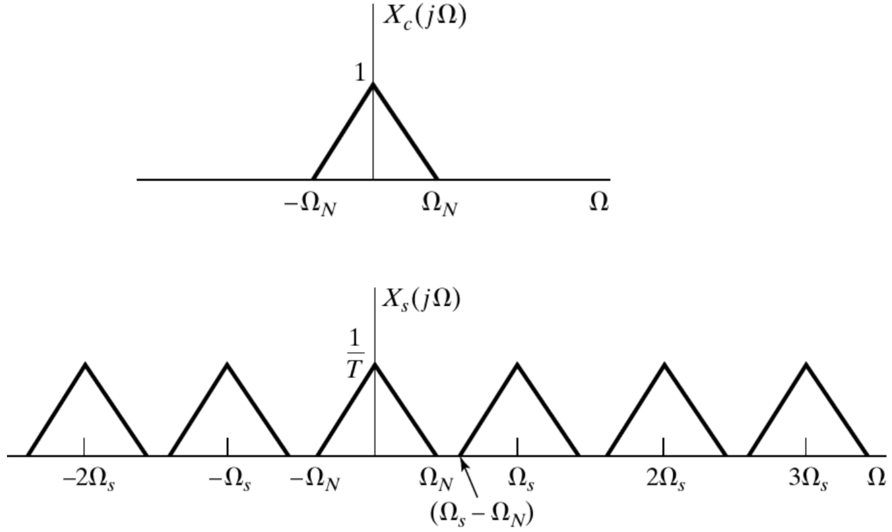
Lecture 5 – Decimation and Interpolation

This lecture

- Based on Chapter 4 of O&S
- All images from O&S book unless otherwise stated

Recall…

- The impulse train approximation $x_s(t)$ of the original continuous time signal $x_c(t)$ has (continuous) Fourier Transform:
- $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\big(j(\Omega - k\Omega_s)\big)$
- With T the sampling period, Ω is the continuous-time frequency in radians/s, and Ω_s $(= 2\pi f_s)$ is the angular sampling frequency in radians/s
- Saw that $X_s(j\Omega)$ comprises copies of $X_c(j\Omega)$ shifted by $k\Omega_s$ and scaled by $\frac{1}{T}$



Pay attention – discrete or continuous FT

- Can show that (how?)
  - $X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nt}$
- The DTFT gives
  - $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$
- So:
  - $X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T})$
- Thus the DTFT $X(e^{j\omega})$ is a frequency scaled version of $X_s(j\Omega)$, with the scaling specified by $\omega = \Omega T$
- The above relates the CTFT to the DTFT

Useful for later…

- Combining we get:
- $X\left(e^{j\Omega T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)$
- Or equivalently
- $X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$

Resampling a discrete-time sequence

- A continuous time signal $x_c(t)$ can be represented by the discrete sequence $x[n]$ where
  - $x[n] = x_c(nT), \ -\infty < n < \infty$
- Often wish to change the sampling rate of $x[n]$ such that
  - $x_1[n] = x_c(nT_1), \quad T_1 \neq T$
- This is resampling
- Could we reconstruct original $x_c(t)$ and use new period T_1?
  - Not practical
  - Restrict to discrete time operations

Reduce sampling rate by integer factor M

- $x_d[n] = x[n\mathrm{M}] = \; x_c(n\mathrm{M}T)$
- With $X_c(j\Omega)$ bandlimited to $\pm\Omega_N$ then $x_d[n]$ will be exact representation if $^{\pi}/_{T_d} \geq \Omega_N$ or equivalently $^{\pi}/_{MT} \geq \Omega_N$
- Can avoid aliasing if:
  - Original sampling rate is a least M times the Nyquist rate
  - Bandwidth of the sequence is first reduced by factor M by discrete-time filtering

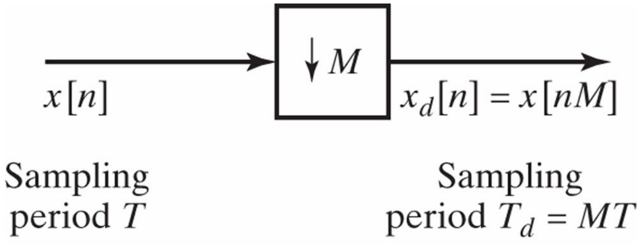


Figure 4.19 Representation of a compressor or discrete-time sampler.

Reduce sampling rate by integer factor M

- Using $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$ (slide #5)
- DTFT of $x_d[n] = x[n\mathrm{M}] = \ x_c(n\mathrm{M}T)$
  - $X_d(e^{j\omega}) = \frac{1}{T_d}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T_d} - \frac{2\pi r}{T_d}\right)\right)$
- But with $T_d = MT$ can rewrite:
  - $X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$

Reduce sampling rate by integer factor M

- Using $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$ (slide #5)
- DTFT of $x_d[n] = x[n\mathrm{M}] = x_c(n\mathrm{M}T)$
  - $X_d(e^{j\omega}) = \frac{1}{T_d}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T_d} - \frac{2\pi r}{T_d}\right)\right)$
- But with $T_d = MT$ can rewrite:
  - $X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$
- The summation index r is related as
  - $r = i + kM$
- k and i integers, $-\infty < k < \infty$ and $0 \leq i \leq M - 1$

Want to relate back

Reduce sampling rate by integer factor M

- $X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{\mathrm{i}=0}^{\mathrm{M}-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)\right\}$
- Recognising that
  - $X\left(e^{j(\omega-2\pi i/M)}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega-2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)$
- Can arrive at:
  - $X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{\mathrm{i}=0}^{\mathrm{M}-1} X\left(e^{j(\omega-2\pi i/M)}\right)$
- Expresses FT of $x_d[n]$ in terms of FT of $x[n]$
- $X_d\left(e^{j\omega}\right)$ consists of copies of $X\left(e^{j\omega}\right)$ scaled by $\frac{1}{M}$, and frequency scaled by $\frac{1}{M}$ and shifted by $2\pi i$.

Avoid aliasing…

- $X_d\left(e^{j\omega}\right)$ periodic with period 2π (like all DTFT!!)
- Can avoid aliasing if $X\left(e^{j\omega}\right)$ is bandlimited, i.e.
- $X\left(e^{j\omega}\right) = 0 \quad \omega_N \leq |\omega| \leq \pi$
- And ${}^{2\pi}/_{M} \geq 2\omega_N$
- Example shown opposite has M=2, at the limit to avoid aliasing here

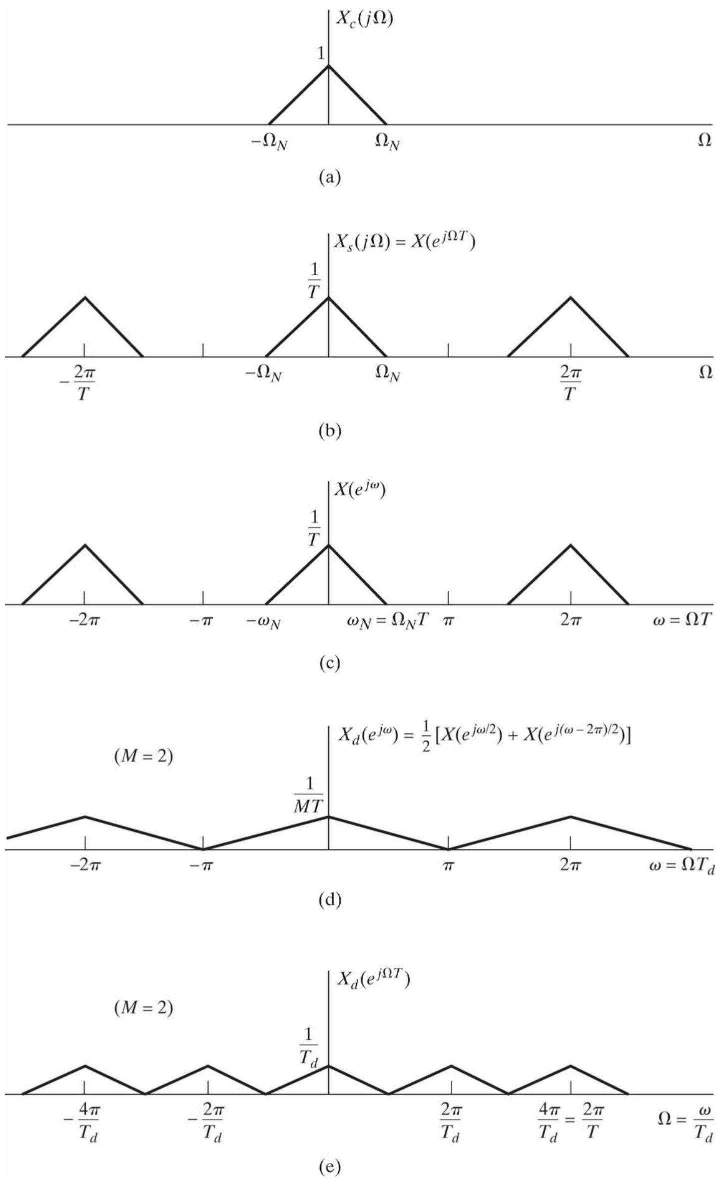


Figure 4.20 Frequency-domain illustration of downsampling.

Aliasing with M=3

- Will get aliasing with M=3
- Prefilter to avoid this and bandlimit the signal

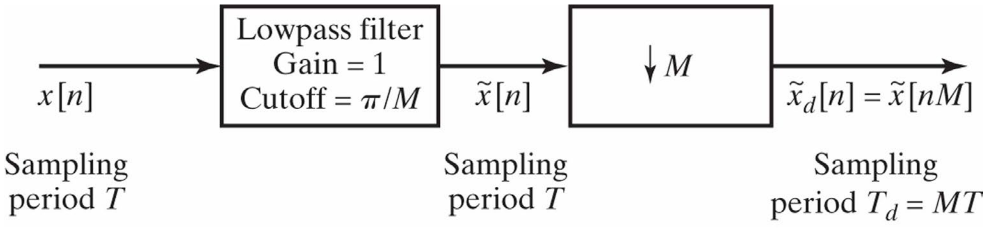


Figure 4.22 General system for sampling rate reduction by *M*.

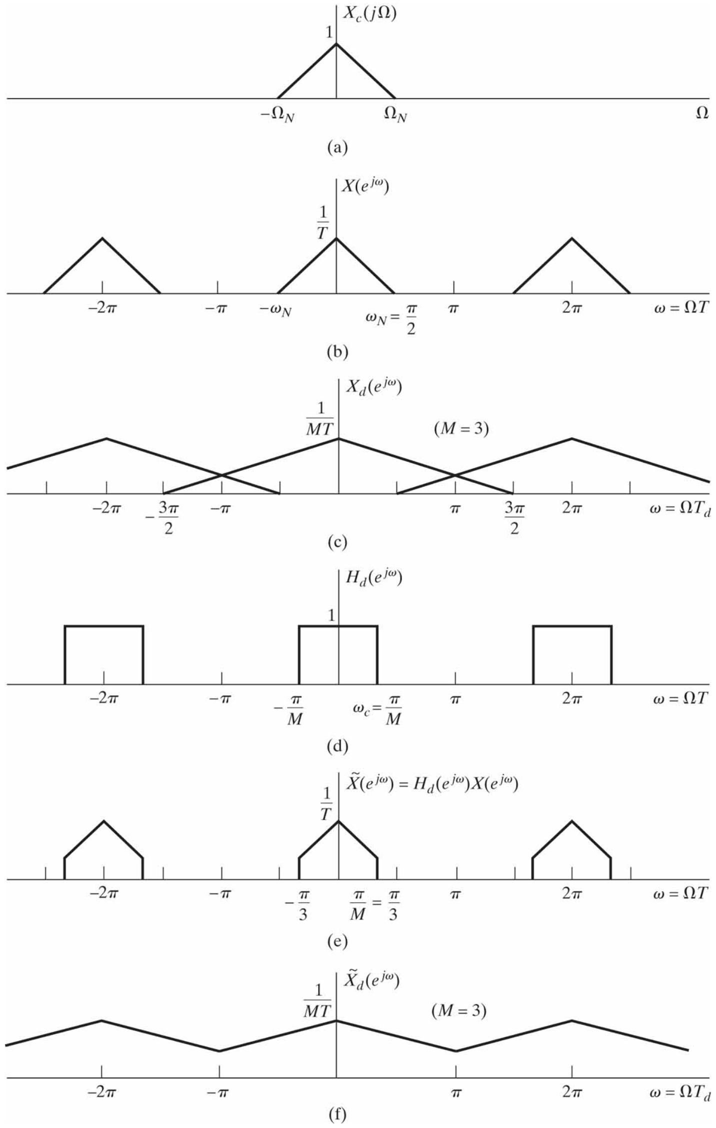


Figure 4.21 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

Increase sampling rate by integer factor

- Increase sampling rate of $x[n]$ by integer factor L
- Want
  - $x_i[n] = x_c(nT_i), \quad T_i = {}^{T}/_{L}$
- Where you can take the samples from
  - $x[n] = x_c(nT)$
- This is upsampling
  - $x_i[n] = \mathrm{x}[{}^{n}/_{L}] = x_c(n\,{}^{T}/_{L}), \quad n = 0, \pm L, \pm 2L, \ldots$
-

System

- Note distinction between $x_i[n]$ and $x_e[n]$
- $x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$
- $x_e[n] = \begin{cases} \mathrm{x}[{}^{n}/_{L}], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & otherwise \end{cases}$
- Expander
- LPF the reconstructs the sequence

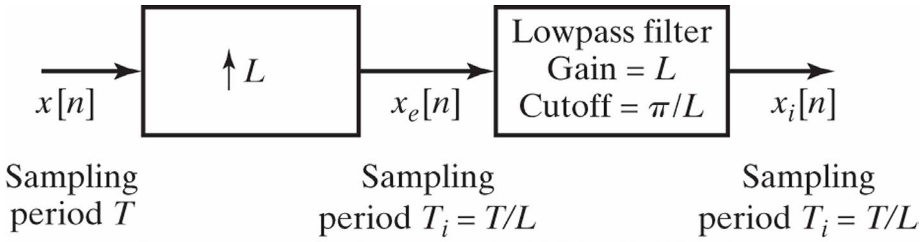
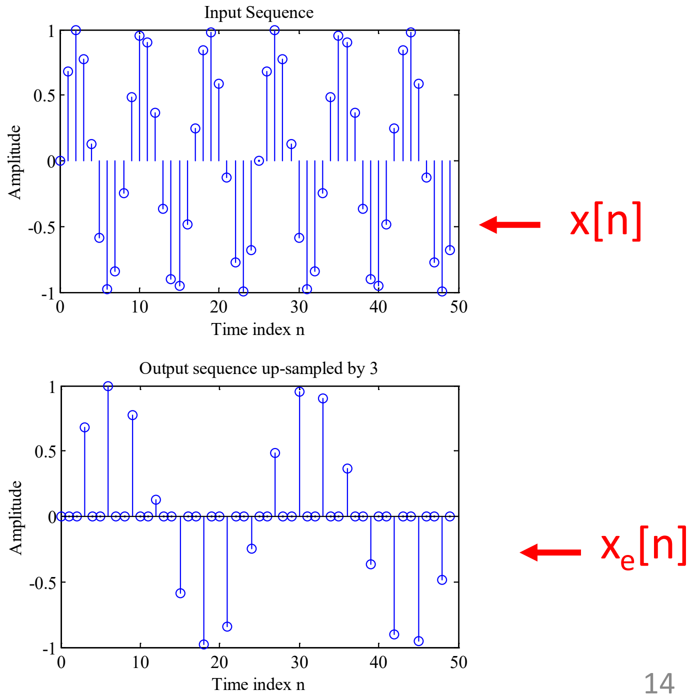


Figure 4.23 General system for sampling rate increase by L.



Increase sampling rate by integer factor

- Frequency domain interpretation, can epxpress the FT as:
- $X_e\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty}(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL])\, e^{-j\omega n}$
- $\quad = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X_e\left(e^{j\omega L}\right)$
- So FT of output of expander is a frequency scaled version of FT of the input, $\omega \rightarrow \omega L$ with ω now normalised by
- $\omega = \Omega T_i$

Get the desired signal

- $X_i(e^{j\omega})$ can be obtained from $X_e(e^{j\omega})$ by correcting the amplitude scaling from $^1/_T$ to $^1/_{T_i}$ and removing all frequency-scaled images of $X_c(j\Omega)$ except at integer multiples of 2π
- This requires a lowpass filter, gain of L and cut-off frequency $^\pi/_L$

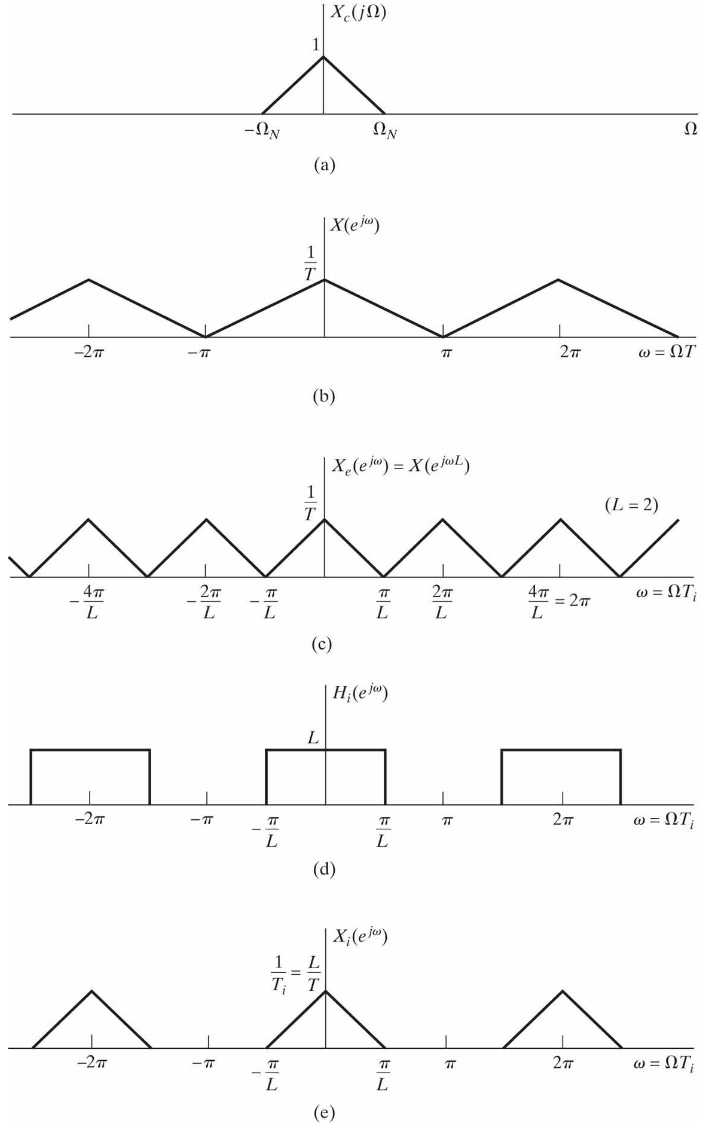


Figure 4.24 Frequency-domain illustration of interpolation.

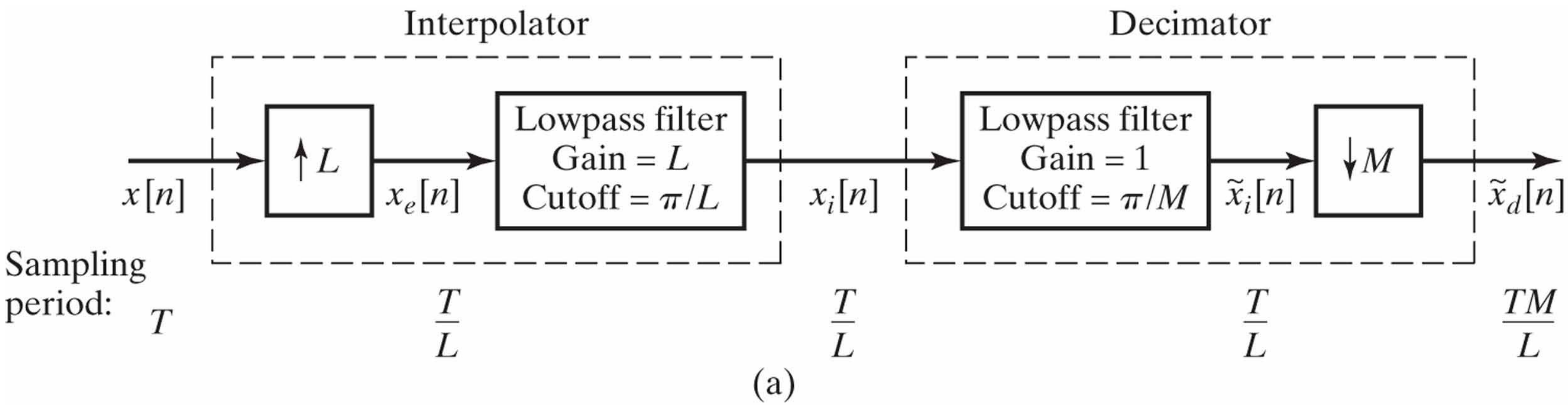
Interpolation

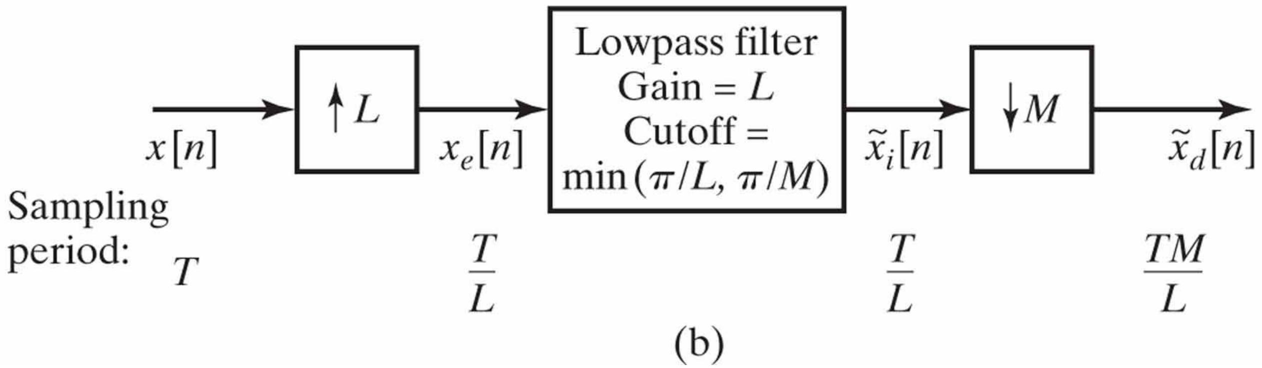
- Can view this as interpolation in the time domain
- Impulse response of the LPF is
  - $h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}$
- Can see that:
  - $x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin(\pi(n-kL)/L)}{\pi(n-kL)/L}$
- Ideal case $h_i[0] = 1$ and $h_i[n] = 0 \; n = \pm L, \pm 2L, \ldots$
- Can show (you should try!) this gives:
  - $x_i[n] = x[n/L] = x_c[nT/L] = x_c(nT_i)$

Practical Interpolation

- We looked at the ideal LPF
- Can only be approximated in reality
- Linear interpolation can be useful
  - Simplicity/accuracy tradeoff

Change sample rate by non-integer factor





Combine decimation and interpolation

Figure 4.29 (a) System for changing the sampling rate by a noninteger factor. (b) Simplified system in which the decimation and interpolation filters are combined.

Required Reading & other material

- Oppenheim & Schafer, Chapter 4, particularly 4.6