EE4C5 Digital Signal Processing

Lecture 13 – Using the DFT

This lecture

- Based on Chapter 10 of O&S
- All images from O&S book unless otherwise stated
- Some images from Ian Bruce of McMaster

DFT in practice

- Have looked at various properties of DFT
- Two important issues come up when using DFT
  - Choice of window
  - Use of zero-padding

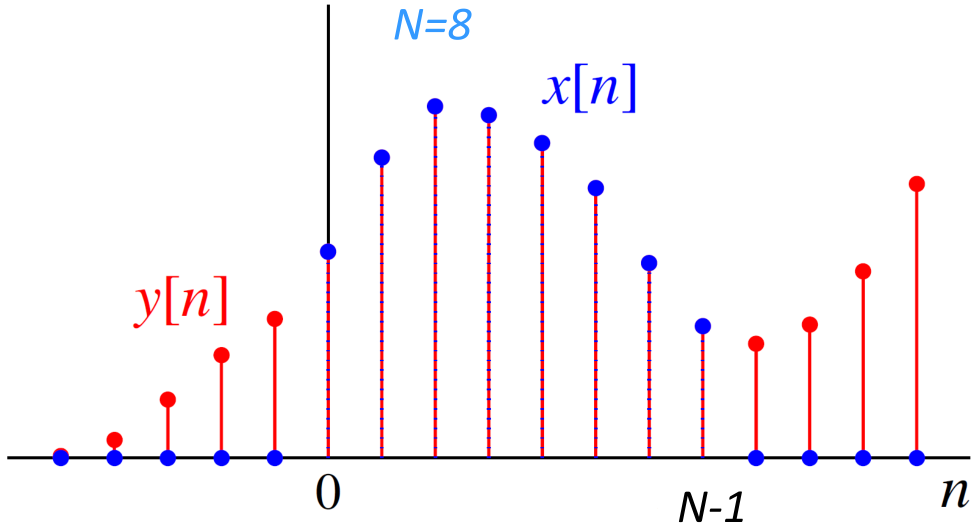
Window choice

Finite sequence

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{kn}{N}}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j2\pi\frac{kn}{N}}$$

- Examine DFT equations
  - Applied to a FINITE sequence
- Assumption that $x[n]$ is zero outside the range 0 to $N-1$
- Many signals you wish to analyse will not be finite or short
  - Windowing
  - Block-processing

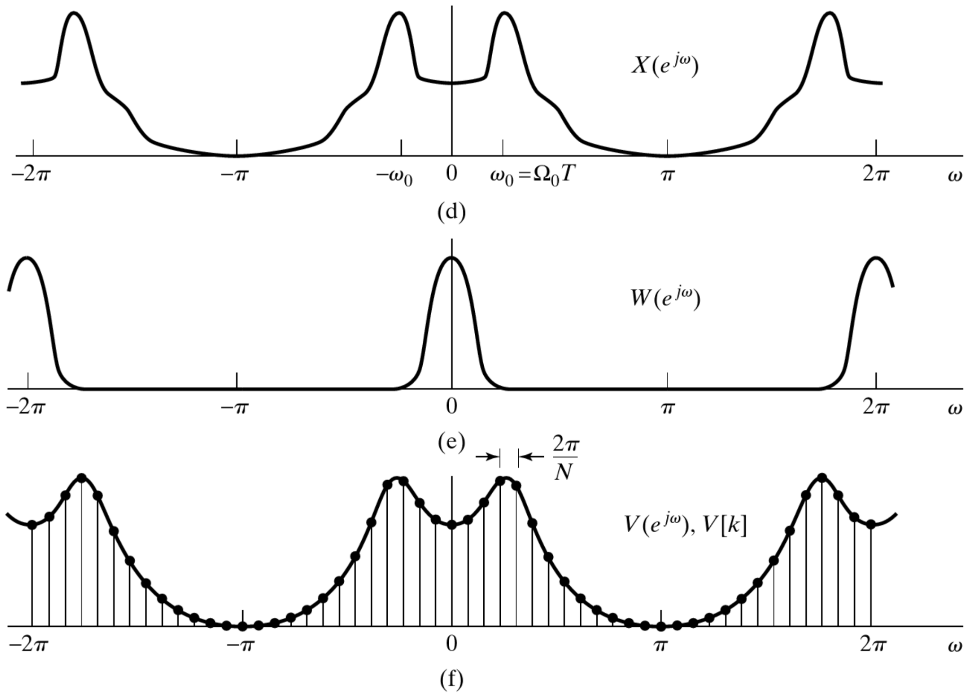


Longer sequence windowed to give x[n] with N samples

Impact of window

- Sequence $x[n]$ multiplied by finite duration window $w[n]$
- Gives finite-length sequence $v[n] = x[n]w[n]$
- Previously saw that windowing a discrete-time sequence (multiplication in the time-domain) is equivalent to (periodic) convolution in the frequency domain

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$



DFT…

- DFT of the windowed sequence

$$V[k] = \sum_{n=0}^{N-1} v[n] e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \ldots, N-1$$

- Equally spaced samples of the DTFT of v[n]: $V[k] = V(e^{j\omega})\big|_{\omega=2\pi k/N}$
- To explore this further:
  - Consider a sequence which is summation of two sinusoidal components

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1), \qquad -\infty < n < \infty$$

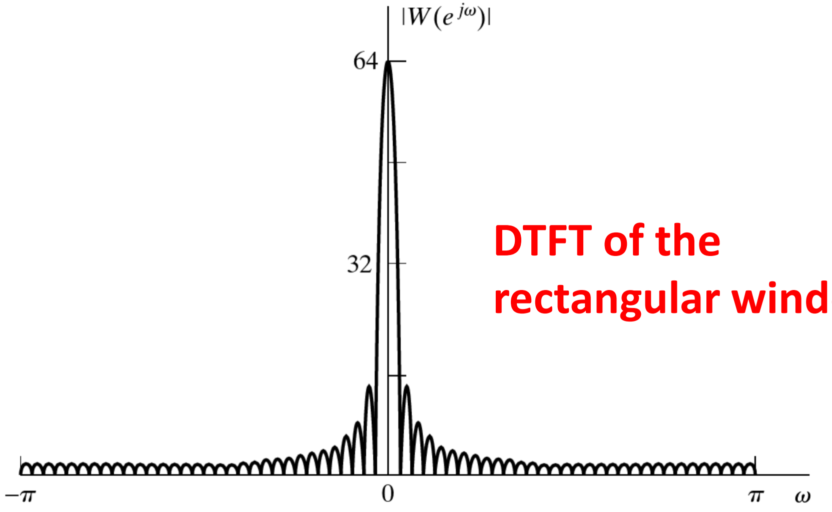
- Windowed sequence is now:

$$v[n] = A_0 w[n] \cos(\omega_0 n + \theta_0) + A_1 w[n] \cos(\omega_1 n + \theta_1)$$

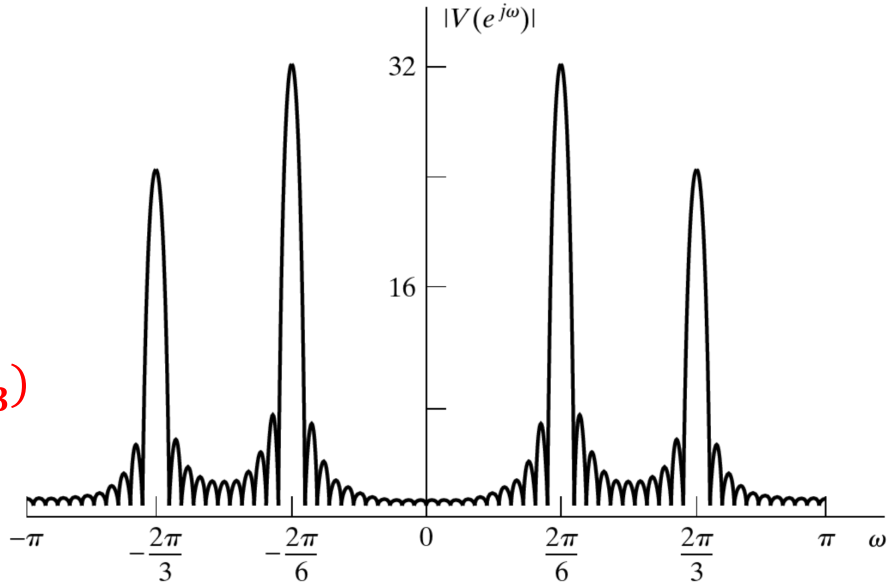
Specify the two sinusoids

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1), \qquad -\infty < n$$

- Sampling rate $1/T$ of 10 kHz
- $w[n]$ rectangular window of length 64
- Signal amplitude $A_0 = 1, A_1 = 0.75$
- Phase parameters $\theta_0 = \theta_1 = 0$

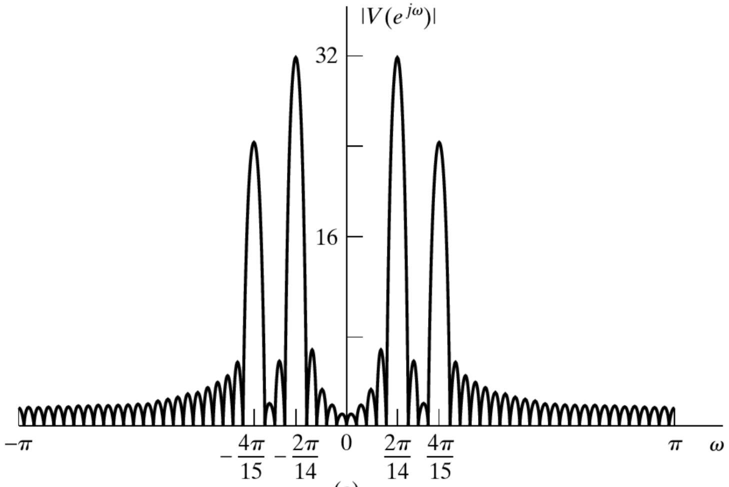


DTFT of the rectangular window



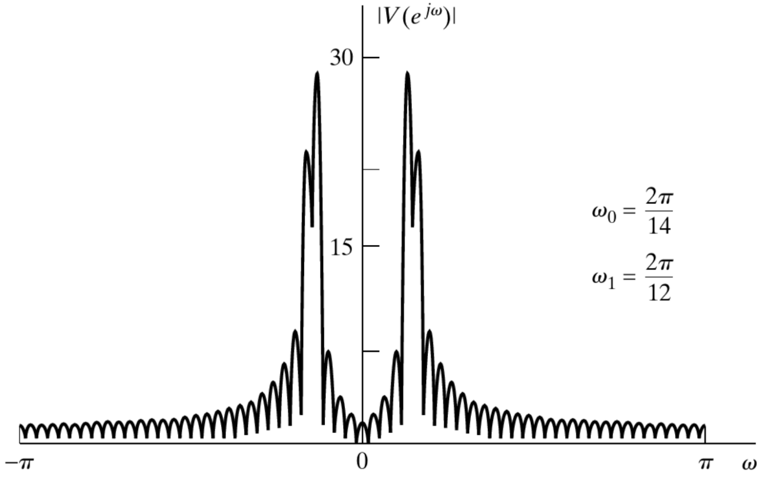
DTFT of the windowed sequence with $\omega_0 = (2\pi/6)$ and $\omega_1 = (2\pi/3)$

As the two frequencies get closer…



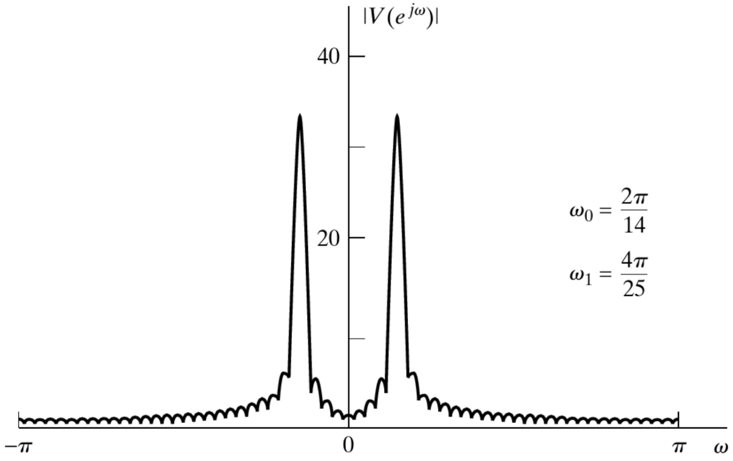
(c)

More overlap between the window replicas at ω_0 *and* ω_1 , and while two distinct peaks are present, the amplitude of the spectrum at $\omega = \omega_0$ is affected by the amplitude of the sinusoidal signal at frequency ω_1 and vice versa. This interaction is called **leakage**. The component at one frequency leaks into the vicinity of another component owing to the spectral smearing introduced by the window.



(d)

As the difference between the two frequencies gets smaller again, the leakage is even greater. Notice how side lobes adding out of phase can reduce the heights of the peaks.



(e)

The overlap between the spectrum windows at ω_0 *and* ω_1 is so significant that the two peaks that were previously visible have merged into one. In other words, with this window, the two frequencies will not be resolved in the spectrum.

Impact of Window

- Reduced resolution
  - Depends on the width of the main lobe of $W\left(e^{j\omega}\right)$
- Leakage
  - Depends on the relative amplitude of the main lobe to the side lobes of $W\left(e^{j\omega}\right)$
- Recall from filter design that width of the main lobe and the relative side-lobe amplitude depend (mostly) on:
  - window length
  - shape (amount of tapering) of the window
- Take-home:
  - Effective spectral resolution of the DFT is not determined by the number of points N of the DFT but rather <u>*by the spectrum of the windowing function*</u>

Window choice matters

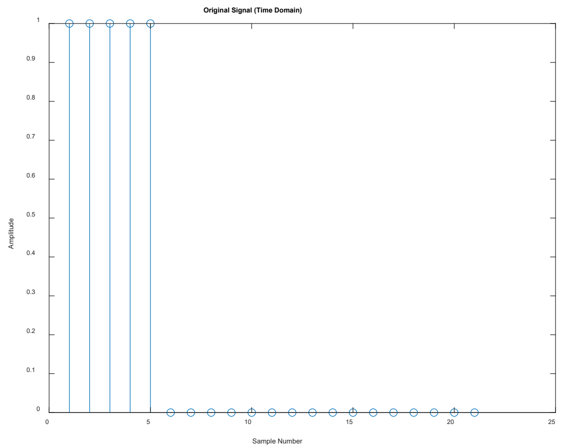
- Similar considerations as in our discussion re. filter design
- Rectangular window
  - Narrowest mainlobe but larger amplitude in sidelobes
- Other possible choices
  - E.g. Hann, Hamming, Bartlett
  - Can only change window length
- Kaiser window
  - More parameters – can control trade-off more

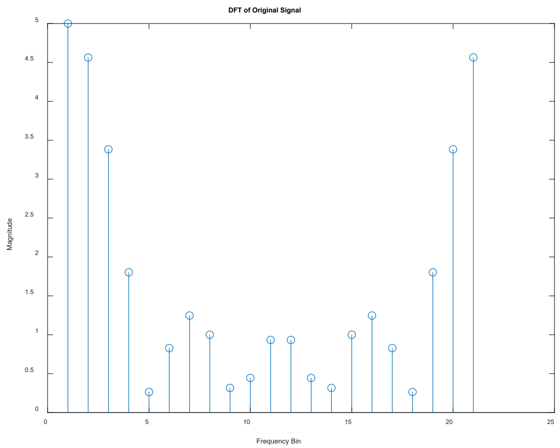
Zero-padding

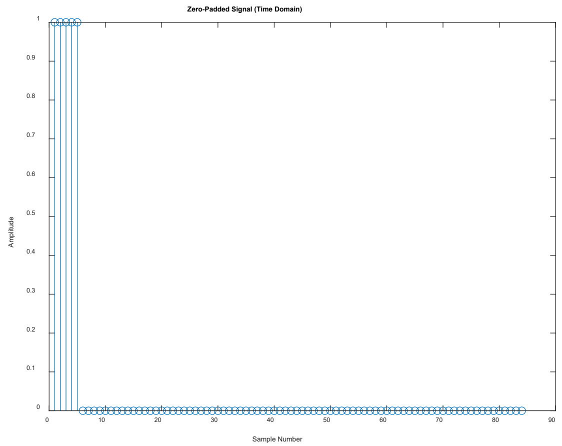
Zero-padding

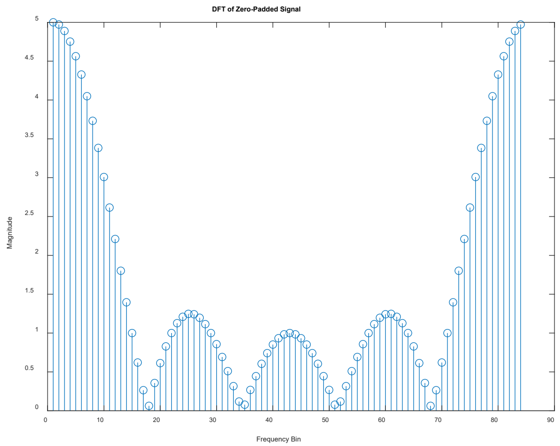
- Before taking the DFT, add zeroes at the end of the sequence to increase length
- Does not change information in the signal
- Increases the length of the DFT
  - And hence the number of points in the DFT spectrum

Extend a sequence…









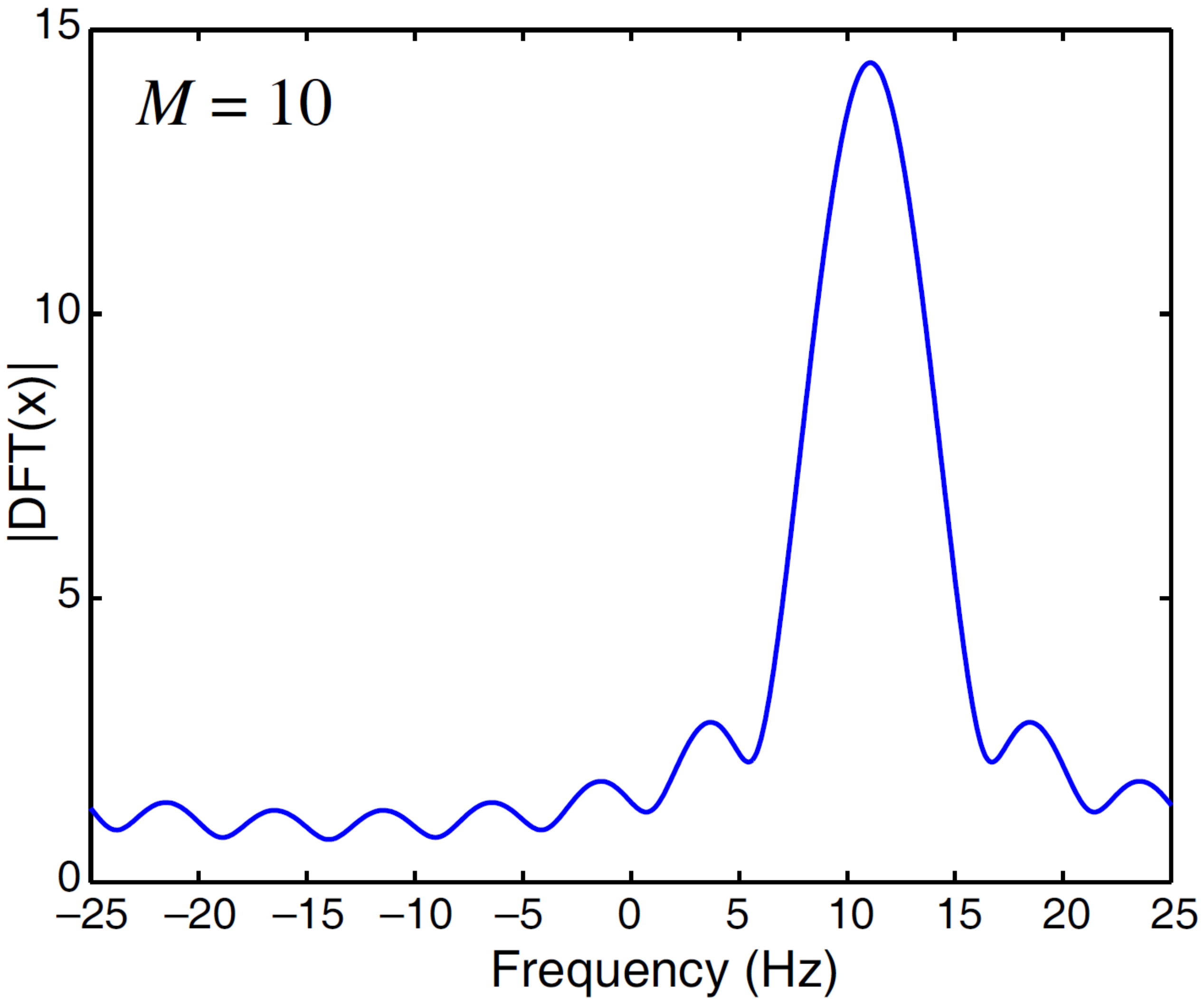
What zero-padding does

- By zero-padding on a windowed sequence, can *approximate* its DTFT more accurately
  - Increases the number of points in the DFT
- Zero-padding does NOT improve the resolution of spectral components
  - Simply interpolates
- The spectral resolution of the DTFT of the windowed sequence sets the effective spectral resolution of the DFT
  - Depends on original window length (before padding)

Illustration of this point

- Example (from Ian Bruce Lectures)
- Let two complex exponentials with the nearby frequencies $f_1 = 10\ Hz$ and $f_2 = 12\ Hz$ be sampled with the sampling interval $T = 0.02$ seconds and let us consider various rectangular window lengths $M = 10, 15, 30, 100, 300$ with zero-padding of each sequence to give $N = 512$ points

DFT with $M = 10$ and zero-padding to $N = 512$ points. The signals are <u>unresolved</u> because $f_2 - f_1 = 2\ \mathrm{Hz} < 1/(MT) = 5\ \mathrm{Hz}$.



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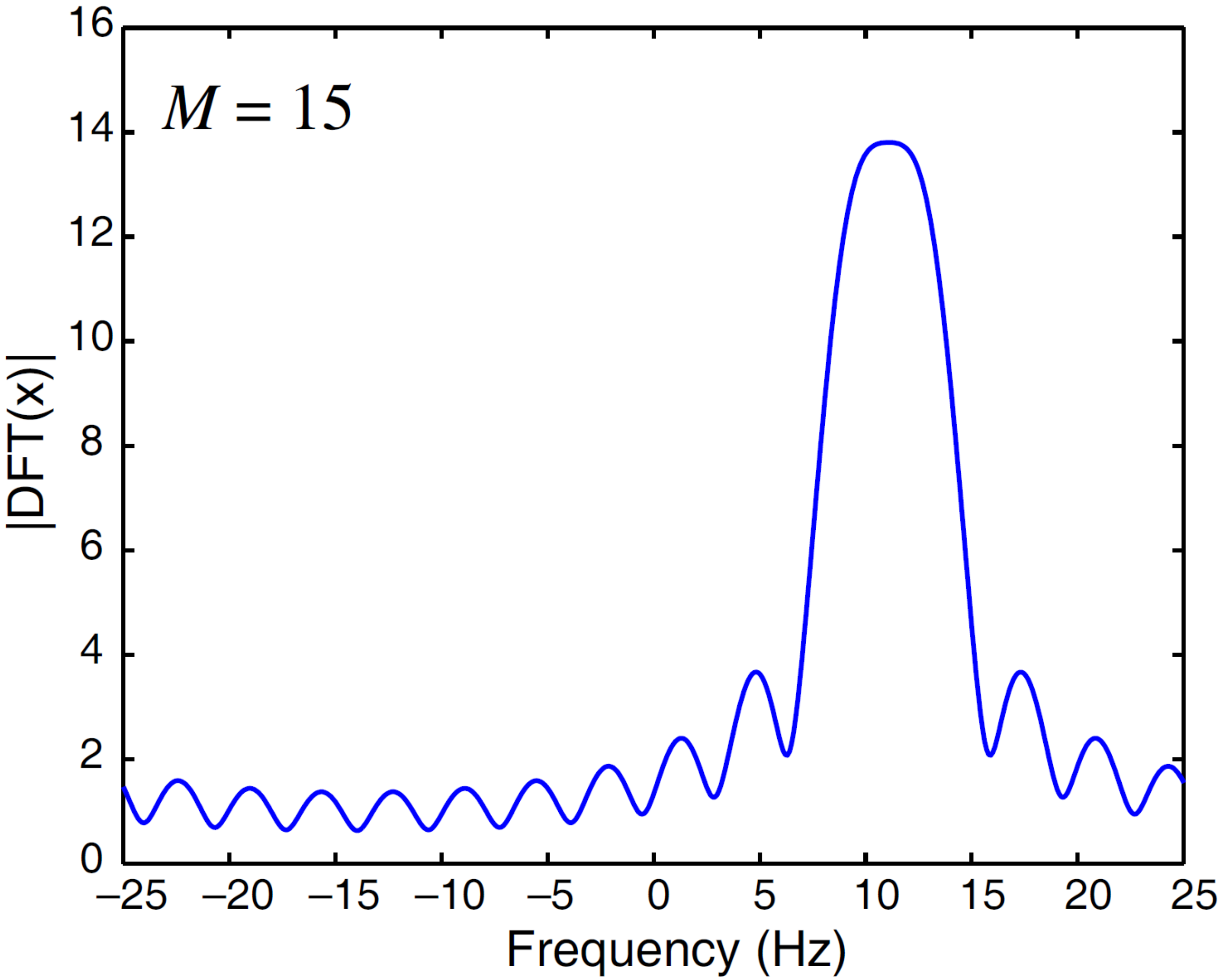
DFT with $M = 15$ and zero-padding to $N = 512$ points. The signals are still unresolved because $f_2 - f_1 = 2 \text{ Hz} < 1/(MT) \approx 3.3 \text{ Hz}$.



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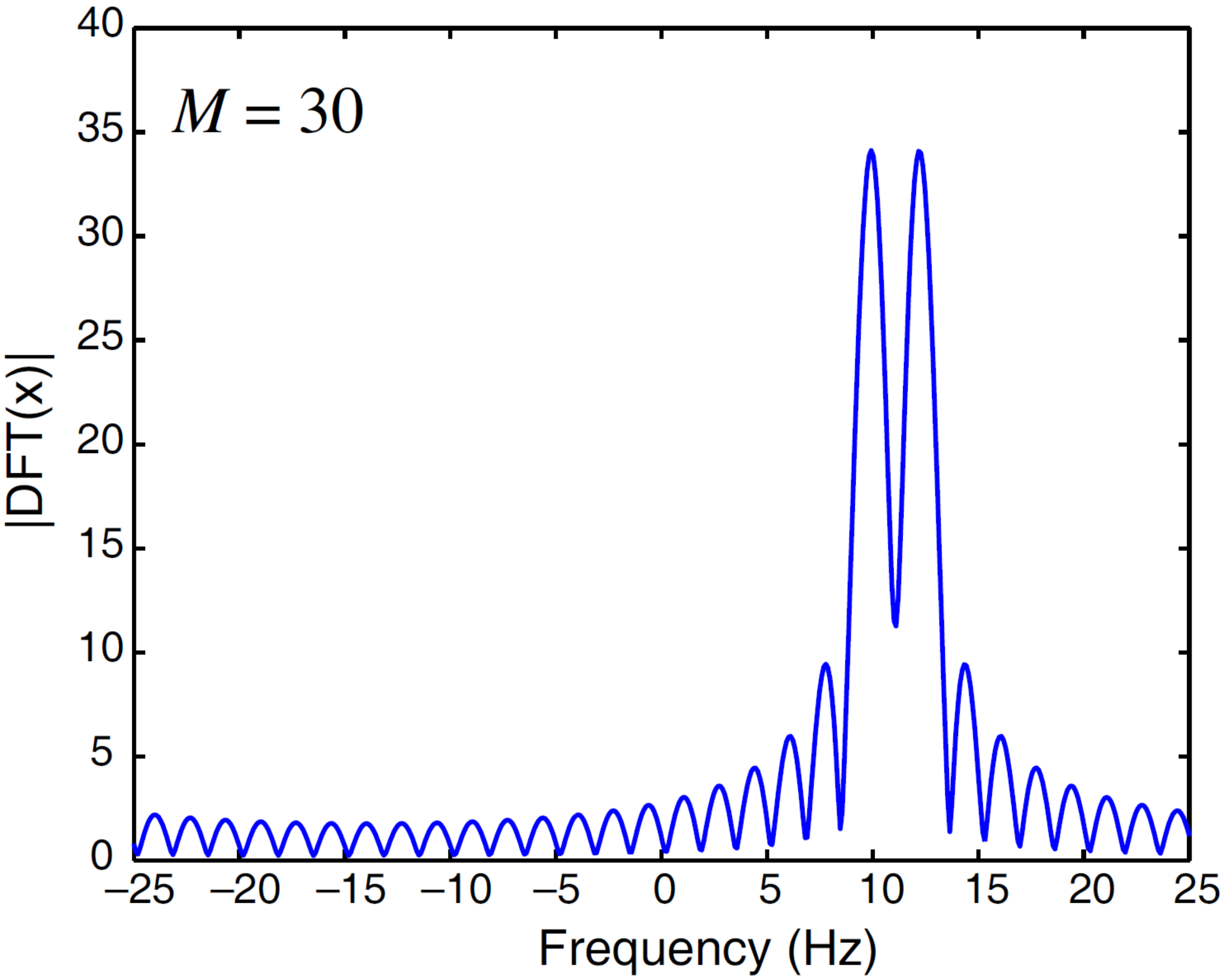
DFT with $M = 30$ and zero-padding to $N = 512$ points. The signals are now resolved because $f_2 - f_1 = 2\ \mathrm{Hz} > 1/(MT) \approx 1.7\ \mathrm{Hz}$.



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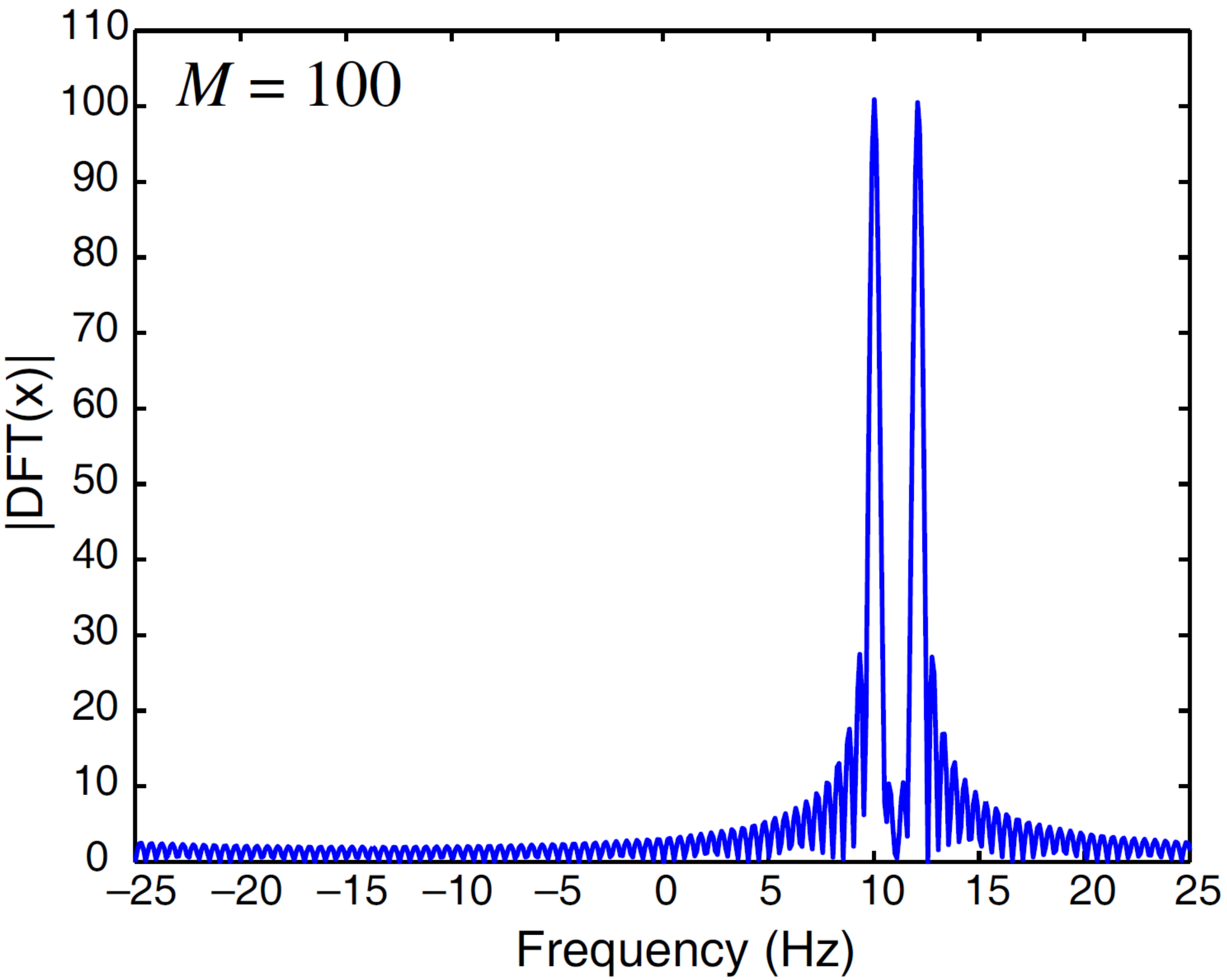
DFT with $M = 100$ and zero-padding to $N = 512$ points. The signals are now well resolved because $f_2 - f_1 = 2$ Hz $> 1/(MT) = 0.5$ Hz.



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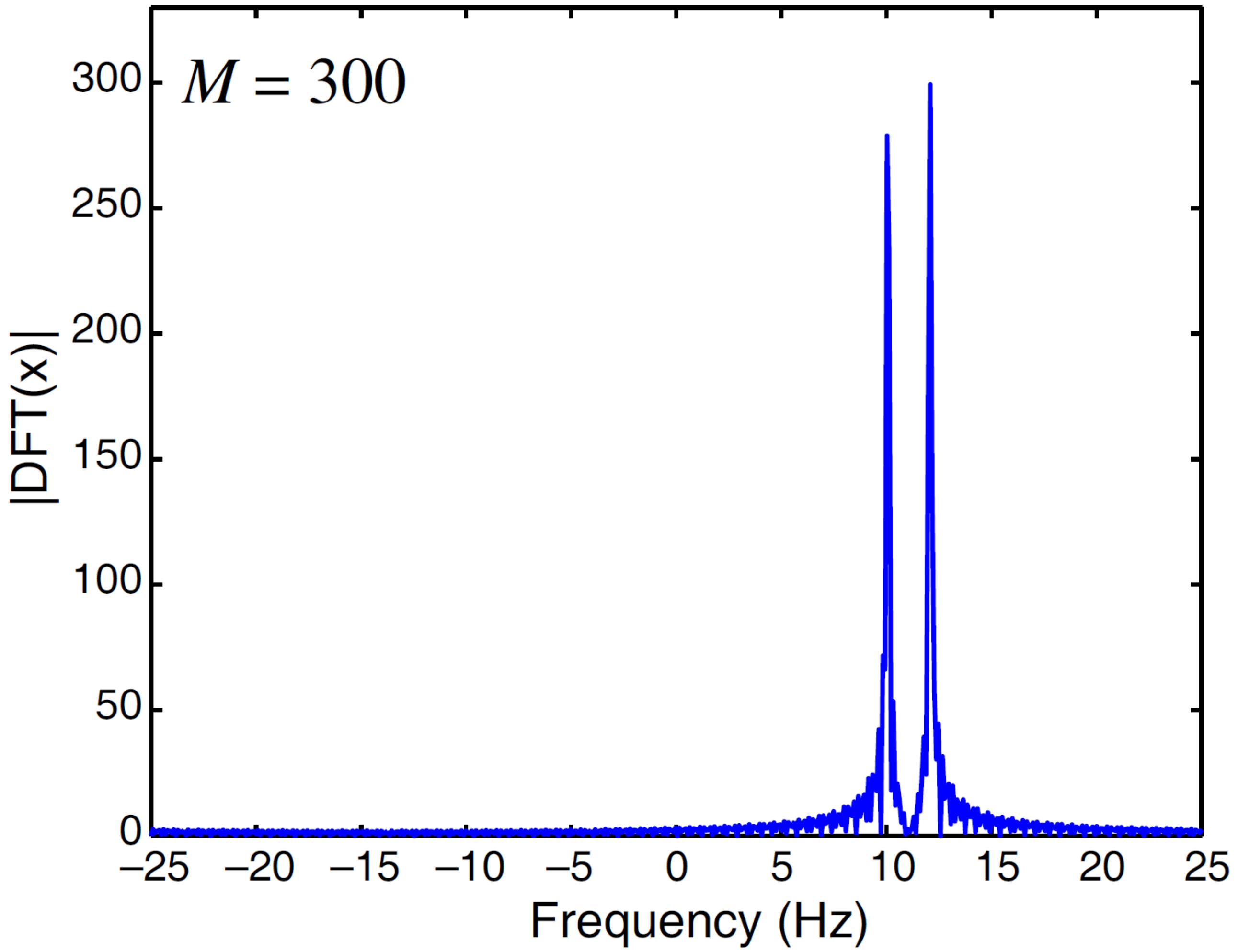
DFT with $M = 300$ and zero-padding to $N = 512$ points. The signals are now very well resolved because $f_2 - f_1 = 2$ Hz $> 1/(MT) \approx 0.17$ Hz.



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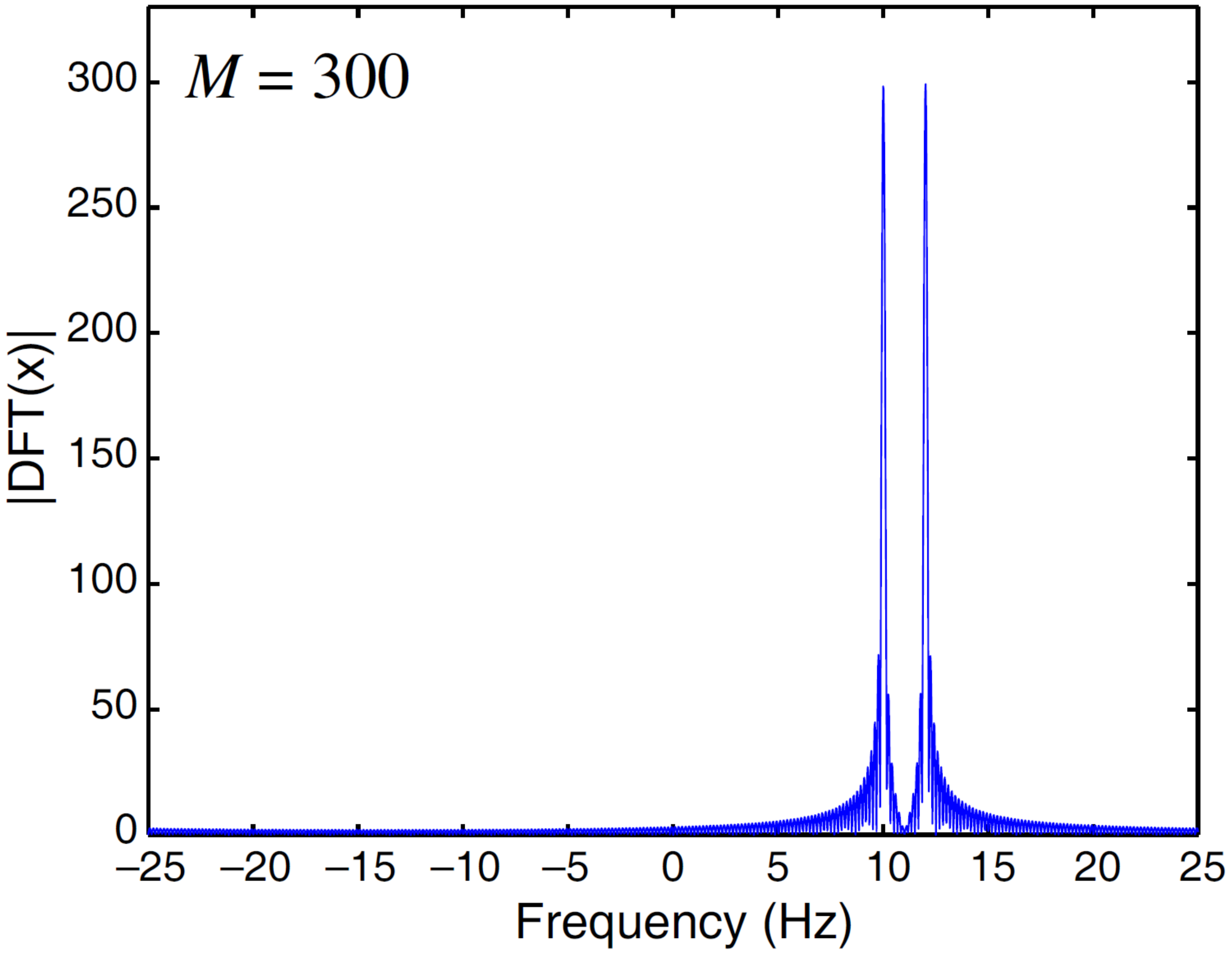
A better representation of the previous DFT plot can be obtained by zero-padding to $N = 2048$ points.



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Circular Convolution

Circular Convolution (lecture 12)

- With
  - $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- and
  - $\mathrm{y}[n] \overset{DFT}{\longleftrightarrow} \mathrm{Y}[k]$
- Then
  - $X[k]\mathrm{Y}[k] \overset{DFT}{\longleftrightarrow} x[n] \circledast \mathrm{y}[n]$
- $\circledast$ denotes circular convolution

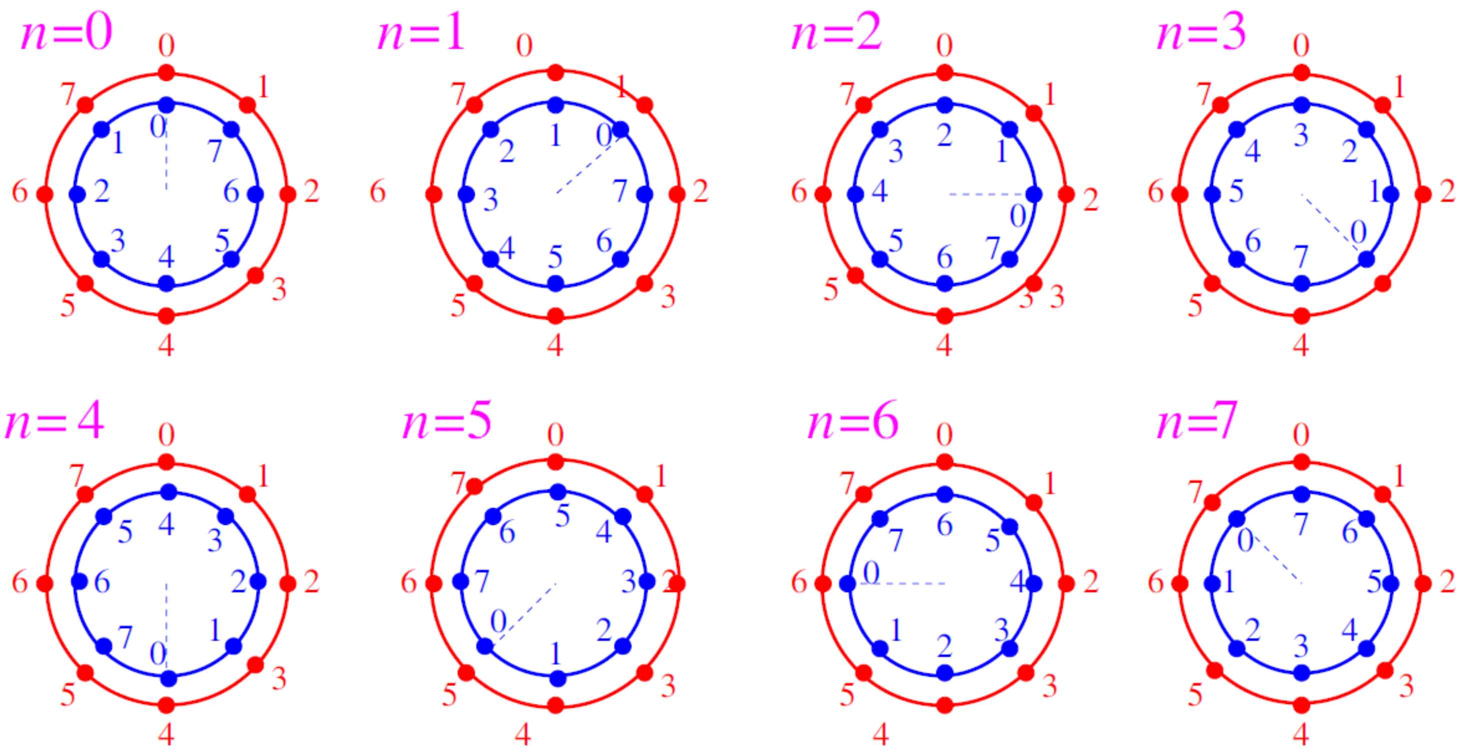
Implementing

- Matrix multiplication method
- Concentric circle method

Concentric Circles

1. Draw two concentric circles
2. Graph the N samples of x[n] at equally spaced points around the outer circle in a clockwise direction
3. Starting at the same point as for $x[n]$, now graph the N samples of $y[n]$ at equally spaced points around the inner circle, going in an anti-clockwise direction
4. For the first sample, multiply all corresponding samples on the two circles and sum the products to generate the output.
5. Rotate the inner circle by one sample in the clockwise direction. Keep the outer circle fixed. Repeat step 4 to generate the next output sample.
6. Repeat Step 5 until the inner circle first sample coincides with the first sample of the exterior circle once more. That completes one rotation.
7. The length of the resulting output sequence is N

Circular convolution with N=8



• - $x[n]$ spread clockwise

• - $y[n]$ spread counterclockwise

In class

$$
\begin{aligned}
x[n] &= \{1, -1, -1, -1, 1, 0, 1, 2\}, \\
y[n] &= \{5, -4, 3, 2, -1, 1, 0, -1\},
\end{aligned}
$$

Details of solution

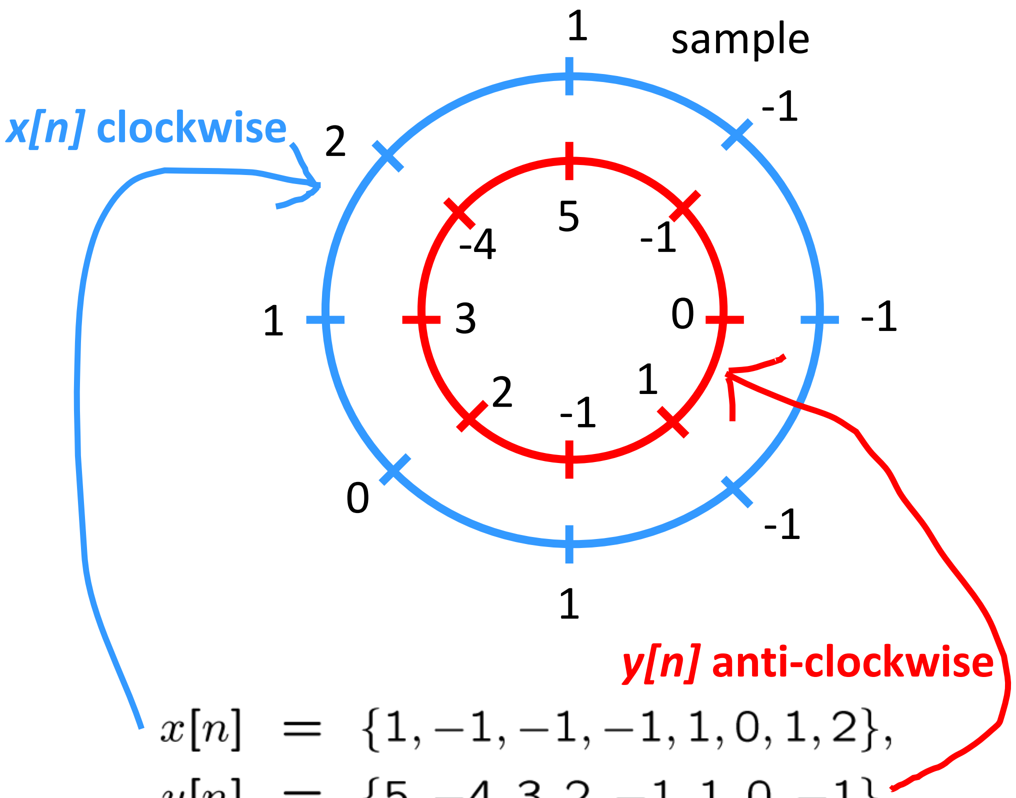
$$\begin{aligned}
z[0] &= x[0]\,y[0]+x[1]\,y[7]+x[2]\,y[6]+x[3]\,y[5]+x[4]\,y[4] \\
&+ x[5]\,y[3]+x[6]\,y[2]+x[7]\,y[1]=-1
\end{aligned}$$

$$\begin{aligned}
z[1] &= x[0]\,y[1]+x[1]\,y[0]+x[2]\,y[7]+x[3]\,y[6]+x[4]\,y[5] \\
&+ x[5]\,y[4]+x[6]\,y[3]+x[7]\,y[2]=1
\end{aligned}$$

$$\begin{aligned}
z[2] &= x[0]\,y[2]+x[1]\,y[1]+x[2]\,y[0]+x[3]\,y[7]+x[4]\,y[6] \\
&+ x[5]\,y[5]+x[6]\,y[4]+x[7]\,y[3]=6
\end{aligned}$$

$$\begin{aligned}
z[3] &= x[0]\,y[3]+x[1]\,y[2]+x[2]\,y[1]+x[3]\,y[0]+x[4]\,y[7] \\
&+ x[5]\,y[6]+x[6]\,y[5]+x[7]\,y[4] = -4 \\
z[4] &= x[0]\,y[4]+x[1]\,y[3]+x[2]\,y[2]+x[3]\,y[1]+x[4]\,y[0] \\
&+ x[5]\,y[7]+x[6]\,y[6]+x[7]\,y[5] = 5 \\
z[5] &= x[0]\,y[5]+x[1]\,y[4]+x[2]\,y[3]+x[3]\,y[2]+x[4]\,y[1] \\
&+ x[5]\,y[0]+x[6]\,y[7]+x[7]\,y[6] = -8 \\
z[6] &= x[0]\,y[6]+x[1]\,y[5]+x[2]\,y[4]+x[3]\,y[3]+x[4]\,y[2] \\
&+ x[5]\,y[1]+x[6]\,y[0]+x[7]\,y[7] = 4 \\
z[7] &= x[0]\,y[7]+x[1]\,y[6]+x[2]\,y[5]+x[3]\,y[4]+x[4]\,y[3] \\
&+ x[5]\,y[2]+x[6]\,y[1]+x[7]\,y[0] = 7
\end{aligned}$$

Move *y[n]* in centre circle one place clockwise each sample



$$\begin{aligned}
x[n] &= \{1,-1,-1,-1,1,0,1,2\}, \\
y[n] &= \{5,-4,3,2,-1,1,0,-1\},
\end{aligned}$$

Required Reading & other material

- Oppenheim & Schafer, Chapter 10
- Video on frequency resolution and zero-padding
  - https://www.youtube.com/watch?v=oh7WvhlkxnU&t=39s
- Circular convolution:
  - https://thewolfsound.com/circular-vs-linear-convolution-whats-the-difference/
- General video on convolution for time analysis: Convolution in the time domain
  - https://www.youtube.com/watch?v=9Hk-RAIzOaw
- Matrix method for convolution (only if interested)
  - https://ccrma.stanford.edu/~jos/fp/Cyclic_Convolution_Matrix.html