EE4C5 Digital Signal Processing

Lecture 12 – Introducing the DFT

This lecture

- Based on Chapter 8 of O&S
- Some material on CTFT and DTFT should have been in prior modules in e.g. mathematics, signals and systems
- All images from O&S book unless otherwise stated
- Some material based on Ian Bruce Lectures from McMaster University
  - Have reused some of his excellent images
- Some material from lectures based on https://dspfirst.gatech.edu/

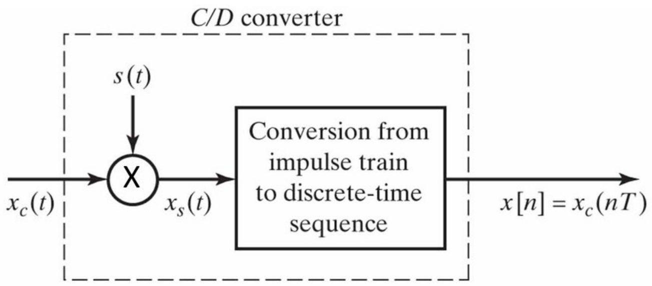
Family of Fourier Transforms

- Continuous-Time Fourier Transform (CTFT)
  - Continuous in time, frequency and amplitude
- Discrete-Time Fourier Transform (DTFT)
  - Discrete in time, continuous in frequency and amplitude
- Discrete Fourier Transform (DFT)
  - Discrete in time and frequency, continuous in amplitude
- Fast Fourier Transform (FFT)
  - Efficient algorithm to implement the DFT

DTFT – a recap

DTFT

- Back in Lecture 4, we considered a sampled continuous time signal $x_c(t)$ as being represented by a continuous-time modulated impulse-train signal $x_s(t)$ as:
- $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$
- $\qquad\quad = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$
- With $x[n] = x_c(nT)$ is the point-sampled discrete-time signal



Take CTFT

- Take the continuous-time Fourier transform of $x_s(t)$:
- $X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\Omega t}\, dt$
- $\qquad\qquad = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\, e^{-j\Omega t}\, dt$
- $\qquad\qquad = \sum_{n=-\infty}^{\infty} x[n]\ \int_{-\infty}^{\infty} \delta(t - nT)\, e^{-j\Omega t}\, dt$
- $\qquad\qquad = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega nT}\, dt$
- Where Ω is continuous-time frequency with units radians/sec

Yields the DTFT…

- Reformulate in terms of discrete-time frequency $\omega = \Omega T$ (with units of radians)
- $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$
- Above equation is the Discrete-Time Fourier Transform (DTFT) of the discrete-time sequence x[n]
- The frequency is represented by a continuous–time variable in ω
- The DTFT has infinite frequency resolution

Periodicity of DTFT

- The DTFT is periodic, with period 2π
- $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{=1}$
- $\qquad\qquad = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = X\left(e^{j(\omega+2\pi)}\right)$

Inverse DTFT

- Periodicity of DTFT means that only need consider one period of DTFT in taking inverse DTFT
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$
- Can show this is true by:
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$
- $= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega m} e^{j\omega n} d\omega$
- $= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m]\, e^{j\omega(n-m)} d\omega$
- $= \sum_{m=-\infty}^{\infty} x[m]\, \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega}_{\delta[n-m]} = x[n]$

Compare

- Continuous Time *<u>Fourier Series</u>*
- $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\frac{2\pi n}{T}t}$
- $X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$

- Discrete-Time FT
- $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$
- $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$

<u>Observe</u>: if you take the Fourier Series and replace:

$$x(t) \to X\left(e^{j\omega}\right); \quad X_n \to x[n]; \quad t \to -\omega; \quad T \to 2\pi;$$

You obtain the DTFT.

Significance?

- An important conclusion follows:
  - The DTFT is equivalent to Fourier series but applied to the “opposite” domain.
- In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients.
- In the DTFT, a periodic continuous spectrum is represented as a sum of exponentials weighted by discrete signal values.
- Important because:
  - DTFT can be derived directly from Fourier series.
  - All developments for Fourier series can be applied to DTFT.
  - Relationship between Fourier series and DTFT illustrates the duality between time and frequency domains.

Properties of DTFT

- Linearity
- Shift property
- Modulation property
- Convolution property
- Multiplication property
- All covered in 3C1 Signals and Systems
- (Can be revised in Chapter 2 of O&S if needed)

TABLE 2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Shortcomings of DTFT

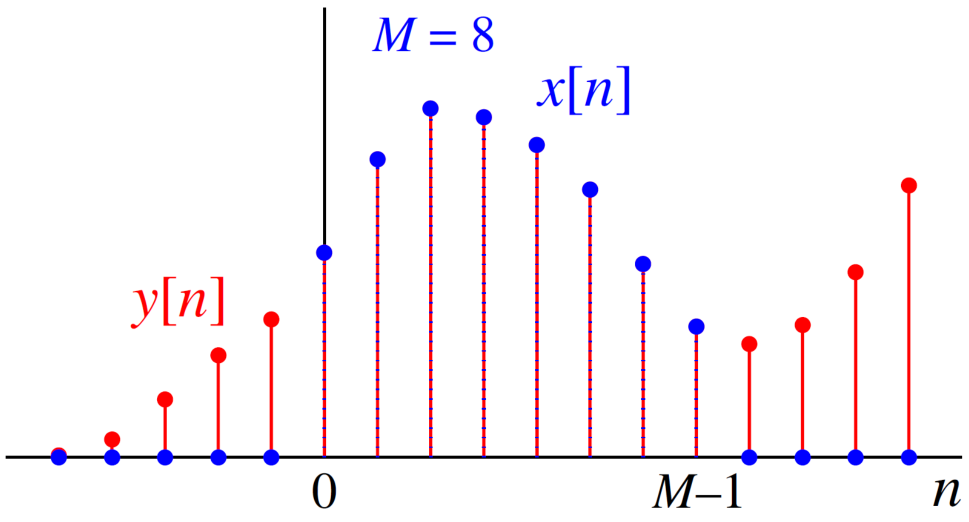
- The DTFT is not suitable for a real-time real-world DSP applications
  - Can only store only a finite number of samples
  - Can only compute the spectrum only at specific discrete values of ω.
- Many signals we encounter are finite in time
- Computationally complex
- Can be sensitive to sampling rate changes

The Discrete Fourier Transform (DFT)

Consider a finite sequence

- Consider x[n], a finite sequence of length M that can be obtained from a longer sequence y[n] by applying a rectangular window of length M

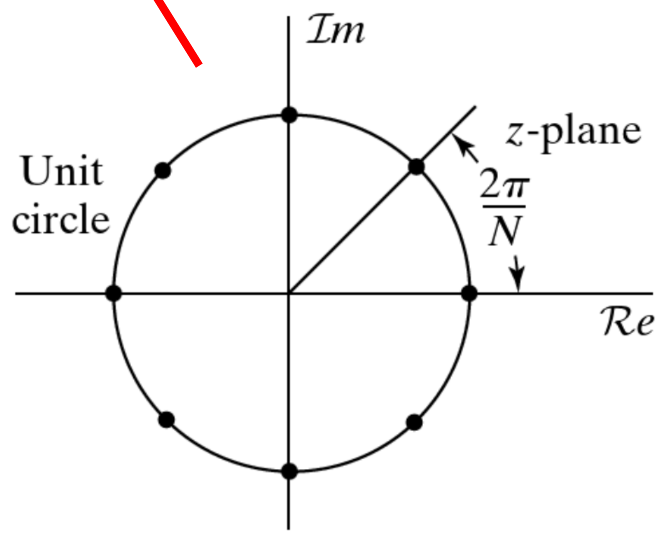
$$x[n] = \begin{cases} 0\,, & n < 0, \\ y[n]\,, & 0 \le n \le (M-1)\,, \\ 0\,, & n \ge M. \end{cases}$$



Move to frequency domain

- Now sample the DTFT spectrum $X(e^{j\omega})$ at N points to yield $X[k]$ as:
  - $X[k] = X(e^{jk\Delta\omega}), \quad \Delta\omega = \frac{2\pi}{N}$
- If N=M, then:
  - $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}}$ **DFT**
- The inverse:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\ e^{j2\pi\frac{kn}{N}}.$$



Sampling of the DTFT spectrum

DFT Analysis and Synthesis Equations

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j2\pi\frac{kn}{N}}$$

Analysis

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \, e^{j2\pi\frac{kn}{N}}$$

Synthesis

DFT – alternative formulation

$$X[k] = \sum_{n=0}^{N-1} x[n]\ e^{-j2\pi\frac{kn}{N}}$$

Analysis

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\ e^{j2\pi\frac{kn}{N}}$$

Synthesis

Rewrite

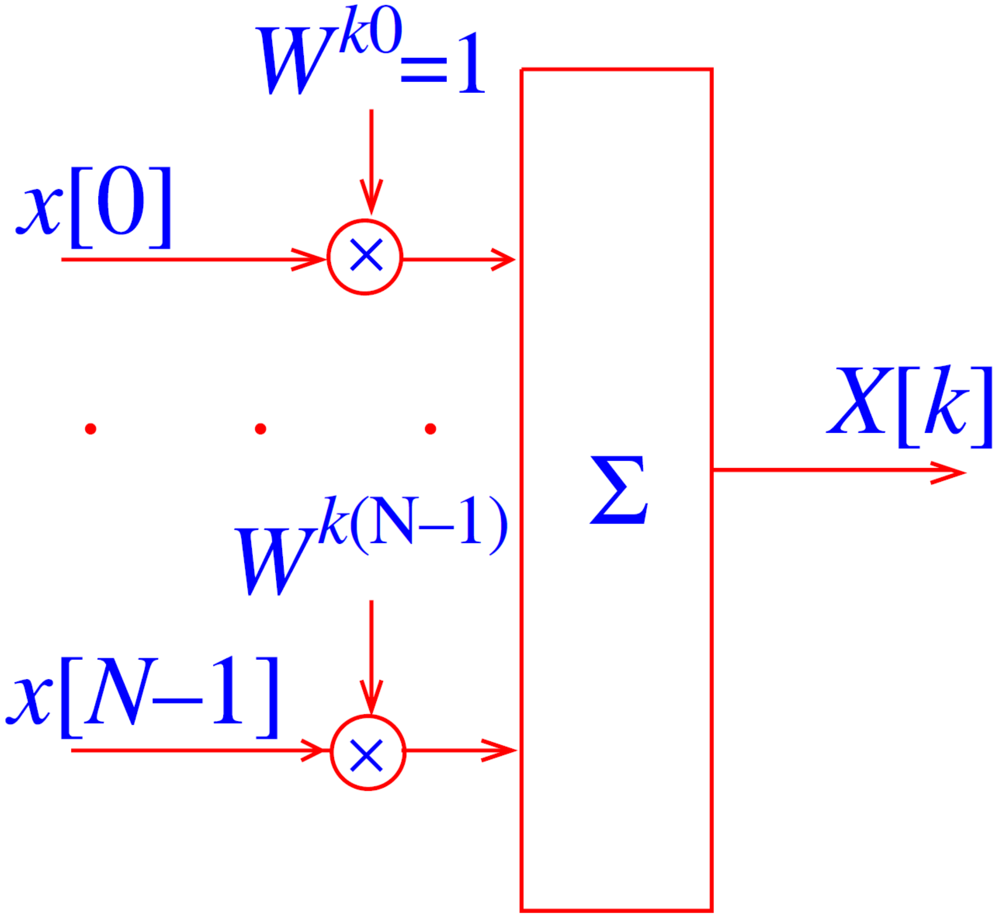
$$X[k] = \sum_{n=0}^{N-1} x[n]\ W^{kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\ W^{-kn}$$

With $W = e^{-j\frac{2\pi}{N}}$

Schematic

- With $W = e^{-j\frac{2\pi}{N}}$



Properties of the DFT

Periodicity

- DFT spectrum $X[k]$ is periodic, with period N
- Recall DTFT periodic with period 2π
- Can show that:
  - $X[k+N] = X[k]e^{-j2\pi n} = X[k]$

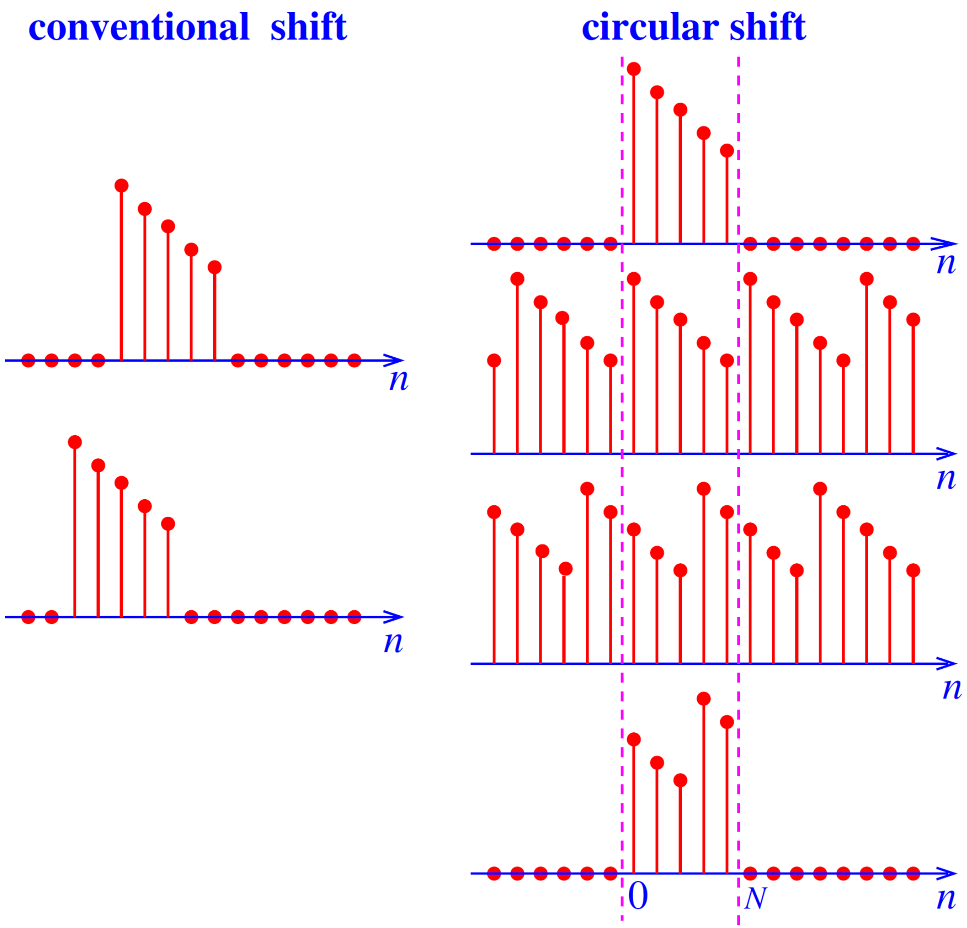
Linearity

- If
  - $x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k]$
- and
  - $x_2[n] \overset{DFT}{\longleftrightarrow} X_2[k]$
- Then
  - $ax_1[n] + bx_2[n] \overset{DFT}{\longleftrightarrow} aX_1[k] + bX_2[k]$
- Need to consider sequence length, N_1 and N_2
  - the lengths of the sequences and their DFTs are all equal to at least the maximum of the lengths of $x_1[n]$ and $x_2[n]$
- DFTs of greater length can be computed by augmenting both sequences with zero-valued samples
  - More on zero-padding later (really important)

Circular Shift

- For the DTFT $X(e^{j\omega})$ of $x[n]$, recall that $e^{j\omega m}X(e^{j\omega})$ is the DTFT of the sequence time-shifted, i.e. $x[n-m]$
- Is there some similar property for the DFT?
- For $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- What about $X[k]e^{-j(2\pi k/N)m}$?
- It's the DFT of $x[(n-m)\text{mod } N]$, i.e.
- $x[(n-m)\text{mod } N] \overset{DFT}{\longleftrightarrow} e^{-j(2\pi k/N)m}X[k]$
- This is a circular shift…

Demonstration of circular shift concept



This is the effect in the time domain of multiplying the DFT of the sequence by a linear-phase factor.

Frequency shift (modulation)

- For $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- $X[(k-m)\text{mod } N] \overset{DFT}{\longleftrightarrow} e^{j(2\pi n/N)m}\ x[n]$

Parseval's Theorem

- Idea that Fourier Transforms preserve "energy"
- $\sum_0^{N-1} |x[n]|^2 = \frac{1}{N} \sum_0^{N-1} |X[k]|^2$
- Important in function approximation

Conjugation

- For $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- Then:
- $X^*[(N-k) \text{mod } N] \overset{DFT}{\longleftrightarrow} x^*[n]$

Circular Convolution

- With
  - $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- and
  - $\mathrm{y}[n] \overset{DFT}{\longleftrightarrow} \mathrm{Y}[k]$
- Then
  - $X[k]\mathrm{Y}[k] \overset{DFT}{\longleftrightarrow} x[n] \circledast \mathrm{y}[n]$
- $\circledast$ denotes circular convolution
- (we'll do some examples in class)

Multiplication

- With
  - $x[n] \overset{DFT}{\longleftrightarrow} X[k]$
- and
  - $\mathrm{y}[n] \overset{DFT}{\longleftrightarrow} \mathrm{Y}[k]$
- Then
  - $\mathrm{x}[n]\mathrm{y}[n] \overset{DFT}{\longleftrightarrow} \frac{1}{N} X[k] \circledast \mathrm{Y}[k]$

Matrix formulation of DFT

Basic notation required

- Where $x[n] \overset{DFT}{\longleftrightarrow} X[k]$, introduce two Nx1 vectors
  - $\mathbf{x} = \left[x[0], x[1]\,, \ldots, x[N-1]\right]^T$
  - $\mathbf{X} = \left[X[0], X[1]\,, \ldots, X[N-1]\right]^T$
- And an $N\mathrm{x}N$ matrix $\mathbf{W}$
  - With $W = e^{-j\frac{2\pi}{N}}$

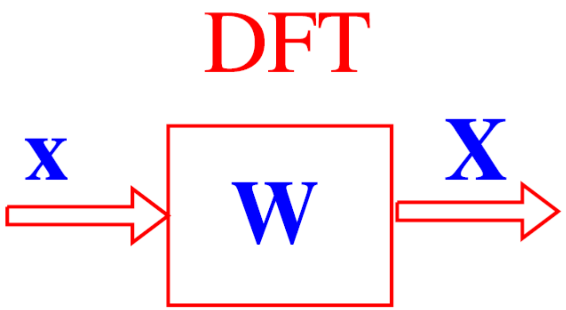
$$\mathbf{W} = \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ W^0 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix}$$

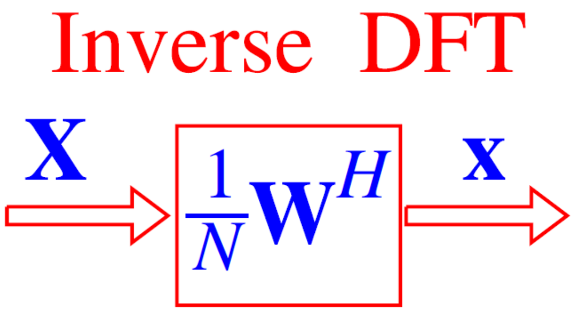
Reformulate DFT using matrix notation

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & e^{-j4\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ 1 & e^{-j4\pi/N} & e^{-j8\pi/N} & \cdots & e^{-j4(N-1)\pi/N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2(N-1)\pi/N} & e^{-j4(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{bmatrix}}_{\text{DFT matrix}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Signal vector

- DFT becomes:
  - $\mathbf{X} = \mathbf{W}\mathbf{x}$
- Inverse DFT becomes:
  - $\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$
- where the operator $\{\quad\}^H$ indicates the Hermitian(or complex conjugate) transpose





Number of operations

- Look at matrix
- Direct computation of all N samples in $\{X[k]\}$ requires N^2 complex multiplications and $N(N-1)$ complex additions
- Soon we'll look at efficient implementations of DFT

Suggested Reading & other material

- Oppenheim & Schafer, Chapter 9
  - Though they go into more detail on DFS
- Video (~20mins) about general concepts behind Fourier Transform:
  - But what is the Fourier Transform? A visual introduction.
  - https://www.youtube.com/watch?v=spUNpyF58BY
- How are the Fourier Series, Fourier Transform, DTFT, DFT, FFT, LT and ZT Related? (~20mins)
  - https://www.youtube.com/watch?v=2kMSLqAbLj4