The DFT of an N-point signal

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eeweb.engineering.nyu.edu/iselesni/EL713/zoom/...

dFtprop.pdF

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$x[n] \quad n \in \{0, 1, \dots N-1\}$

is defined as

$$X[k] \stackrel{def}{=} \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi k n}{N}}}_{W_N^{-kn}} \quad , k \in \{0, 1 \dots N-1\}$$

where $W_N = e^{j\frac{2\pi}{N}}$ is the principal N-th root of unity.

$k \bmod N$ denotes the remainder when k is divided by N.

We can easily prove that $k \bmod N$ or $\langle k \rangle_N$ is a periodic function of k with period N:

$$\langle k + N \rangle_N = \langle k \rangle_N$$

. Periodicity of W_N

If we write k as $k = l.N + r$ with $r \in \{0, 1 \dots N-1\}$ then:

$$W_N^k = e^{j\frac{2\pi}{N}(lN + r)}$$

$$= e^{j\frac{2\pi l \cancel{N}}{\cancel{N}}} \cdot e^{j\frac{2\pi r}{N}}$$

$$= W_N^r$$

and : $r = \langle k \rangle_N$

so $W_N^k = W_N^{\langle k \rangle_N}$

. Useful identity

the geometric summation formula: $\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1 - a^N}{1 - a} & a \neq 1 \\ N & a = 1 \end{cases}$

for $a = W_N^k$, then we get:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} \frac{1 - W_N^{Nk} = 1}{1 - W_N^k} & W_N^k \neq 1 \\ N & W_N^k = 1 \end{cases}$$

Note that $W_N^k = 1$ if $k = 0, \pm N, \pm 2N \ldots$

In other words, $W_N^k = 1$ if $\langle k \rangle_N = 0$.

Therefore we simplify the summation further to obtain:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & \langle k \rangle_N \neq 0 \\ N & \langle k \rangle_N = 0 \end{cases}$$

Using the discrete-time delta function $\delta[n]$ defined as:

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

we get: $\displaystyle\sum_{n=0}^{N-1} W_N^{nk} = N . \delta[\langle k \rangle_N]$

Inverse DFT proof

To verify the inversion formula, we substitute the DFT into the inverse DFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{kn} \quad \text{inverse DFT}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{\ell=0}^{N-1} x[\ell] W_N^{-k\ell} \right) W_N^{kn}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] \sum_{k=0}^{N-1} W_N^{k(n-\ell)}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] N \delta[\langle n-\ell \rangle_N]$$

$$= x[\langle n \rangle_N] = x[n] \quad \text{provided } 0 \leq n \leq N-1$$

Circular Shifting

Given an N-point signal $\{x[n], n \in \{0,1 \ldots N-1\}\}$ the signal $g[n] = x[\langle n-m \rangle_N]$ represents a circular shift of $x[n]$ by m samples to the right. For example, if $g[n] = x[\langle n-1 \rangle_N]$, then:

$g[0] = x[\langle -1 \rangle_N] = x[N-1]$ $\quad -1 = (-1)N + (N-1)$ ↳ remainder

$g[1] = x[\langle 0 \rangle_N] = x[0]$

$g[2] = x[\langle 1 \rangle_N] = x[1]$ $\ldots\ldots$ $g[N-1] = x[\langle N-2 \rangle_N] = x[N-2]$

Periodicity property of the DFT

Given the N-point signal $\{x[n], n \in \{0,1..N-1\}\}$, we defined the DFT coefficients $X[k]$ for $k = 0 \dots N-1$. If k lies outside the range $0 \dots N-1$, then $\boxed{X[k] = X[\langle k \rangle_N]}$

$k = l.N + r \quad , r \in \{0,1 \dots N-1\} \quad . \langle k \rangle_N = r$

$$X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{-n(lN+r)}$$

$$= \sum_{n=0}^{N-1} x[n] \, W_N^{-nlN} \, W_N^{-nr}$$

$$= \sum_{n=0}^{N-1} x[n] \, W_N^{-nr}$$

$$= X[r] = X[\langle k \rangle_N]$$

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Likewise, given the N-point DFT vector $\{X[k], k \in \{0,1..N-1\}\}$, we define the Inverse DFT samples $x[n]$ for $n \in \{0,1 \dots N-1\}$. But if n lies outside the range $0 \dots N-1$,

$\boxed{x[n] = x[\langle n \rangle_N]}$

To derive this equation, $n = lN + r$ with $r \in \{0,1 \dots N-1\}$. Then $\langle n \rangle_N = r$

and
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{k(lN+r)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{klN} \, W_N^{kr}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{kr} = x[r] = x[\langle n \rangle_N]$$

Circular shift property of the DFT

If $G(k) = W_N^{-mk} X[k]$ then $g[n] = x[\langle n-m \rangle_N]$

$$g[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] \, W_N^{nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} \, X[k] \, W_N^{nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{k(n-m)} = x[n-m] = x[\langle n-m \rangle_N]$$

Circular convolution property of the DFT

If $y[n] = \sum_{m=0}^{N-1} x[m] \, g[\langle n-m \rangle_N]$ then $Y[k] = X[k] \, G[k]$

$$Y[k] = \sum_{n=0}^{N-1} y[n] \, W_N^{-nk}$$

$$Y[k] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m]\, g[\langle n-m\rangle_N]\, W_N^{-nk}$$

$$= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} g[\langle n-m\rangle_N]\, W_N^{-nk} \qquad W_N^{-(n-m)k} \cdot W_N^{-mk} = W_N^{-nk}$$

$$= \sum_{m=0}^{N-1} x[m]\, W_N^{-mk} \cdot \cancel{G}[k]$$

$$= G[k] \cdot \sum_{m=0}^{N-1} x[m]\, W_N^{-mk}$$

$$= G[k] \cdot X[k]$$

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. Multiplication property

$$X[k] \circledast G[k] = \sum_{m=0}^{N-1} X[m]\, G[\langle n-m\rangle_N]$$

$$= \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} x[k]\, W_N^{-mk} \; g[k]\, W_N^{-\langle n-m\rangle_N k}$$

$$= \sum_{k=0}^{N-1} x[k]\, g[k] \sum_{m=0}^{N-1} W_N^{-(lN+r)k} \; W_N^{-\langle n-m\rangle_N k}$$

$$= \sum_{k=0}^{N-1} x[k]\, g[k] \sum_{m=0}^{N-1} W_N^{-lNk} \; W_N^{-rk} \; W_N^{-\langle n-m\rangle_N k}$$

$$= \sum_{k=0}^{N-1} x[k]\, g[k] \;\; N \;\; W_N^{-\langle m\rangle_N k} \;\; W_N^{-\langle n-m\rangle_N k}$$

$$= N \left(\sum_{k=0}^{N-1} x[k]\, g[k]\, W_N^{-\langle n\rangle_N k} \right)$$

$$= N \;\; DFT\{x[k]\, g[k]\}$$

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Q2 Compute the N-point DFT of:

a. $x_1[n] = \delta[n]$

from the definition of the DFT: $X_1[k] = \sum_{n=0}^{N-1} x_1[n] W_N^{-kn}$

$$= \sum_{n=0}^{N-1} \delta[n] W_N^{-kn}$$

$$= W_N^{-k0} = 1, \quad k = 0 \dots N-1$$

b. $x_2[n] = \delta[n - n_0]$ when $0 < n_0 < N$

$X_2[k] = W_N^{-n_0 k}$, $k = 0 \dots N-1$ from the definition of the DFT.

or we can use the z-transform

$X_2(z) = z^{-n_0}$

by sampling $X_2(z)$ at the points $z = W_N^{k}$ for $k = 0 \dots N-1$, we find

$X_2(k) = W_N^{-n_0 k}$, $k = 0 \dots N-1$

c. $x_3[n] = a^n$

$$X_3[k] = \sum_{n=0}^{N-1} x_3[n] W_N^{-kn} = \sum_{n=0}^{N-1} a^n W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} \left(a W_N^{-k}\right)^n$$

$$= \frac{1 - \left(a W_N^{-k}\right)^N}{1 - a W_N^{-k}} \quad k = 0 \dots N-1$$

d. $x_4[n] = u[n] - u[n - n_0]$ where $0 < n_0 < N$

$$X_4[k] = \sum_{n=0}^{n_0-1} W_N^{nk} = \frac{1 - W_N^{k n_0}}{1 - W_N^{k}} = \frac{W_N^{\frac{k n_0}{2}}}{W_N^{\frac{k}{2}}} \; \frac{W_N^{-\frac{k n_0}{2}} - W_N^{\frac{k n_0}{2}}}{W_N^{-\frac{k}{2}} - W_N^{\frac{k}{2}}}$$

$$= e^{-j \frac{2\pi k}{N} \frac{n_0 - 1}{2}} \; \frac{\sin(n_0 \pi k / N)}{\sin(\pi k / N)}, \quad k = 0 \dots N-1$$

Q3. Find the 10-point DFT inverse of

$$X[k] = \begin{cases} 3 & k=0 \\ 1 & 1 \le k \le 9 \end{cases}$$

Notice first that: $X[k] = 1 + 2\delta(k) \quad 0 \le k \le 9$ where $\begin{cases} \delta(k) = 1, & k=0 \\ \delta(k) = 0 & \text{otherwise} \end{cases}$

Now let's prove that $X[k] = N\delta(k) \Leftrightarrow x[n] = 1$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \cancel{N}\,\delta(k)\, e^{j\frac{2\pi k n}{N}}$$

$$= e^{j\frac{2\pi 0 n}{N}} = 1 \quad \text{as for } k \ne 0,\ \delta(k) = 0$$

Let's prove as well that $X[k] = 1 \Leftrightarrow x[n] = \delta(n)$

From the inverse DFT def: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} 1.\left(e^{j\frac{2\pi n}{N}}\right)^k$

$$\left.\begin{aligned} x[n] &= \frac{1}{N}\,\frac{1-\overbrace{e^{j\frac{2\pi n \cancel{N}}{\cancel{N}}}}^{=1}}{1-e^{j\frac{2\pi n}{N}}} = 0 \quad \text{for } n \ne lN,\ l \in \mathbb{Z} \\ &\text{or} \\ x[n] &= \frac{1}{N}.N = 1 \quad \text{for } n = lN,\ l \in \mathbb{Z} \end{aligned}\right\} \begin{aligned} &\underline{l=0} \\ &x[n] = \delta(n) \end{aligned}$$

Therefore it follows that: $\boxed{x[n] = \delta(n) + \frac{2}{10}}$

$X[k] = c.\delta(k) \Leftrightarrow x[n] = \frac{c}{N}$

Q4. $x[n] = \delta[n] + 2\delta[n-5]$

a. $X[k] = 1 + 2e^{-j\frac{2\pi}{10}5k}$

$= 1 + 2e^{-j\pi k}$

$= 1 + 2(-1)^k$

b. $X[k]W_N^{kn_0} = W_N^{kn_0} + 2W_N^{5k}W_N^{kn_0}$

$\underline{n_0 = 2}$ $= Y[k] + 2W_N^{5k}Y[k]$

$= Y[k]\underbrace{\left(1 + 2W_N^{5k}\right)}_{X[k]}$

$Y[k] = W_N^{2k} = e^{j2k\frac{2\pi}{10}}$ $(N=10)$

$y[n] = x[\langle n+2\rangle_{10}] = 2\delta(n-3) + \delta(n-8)$

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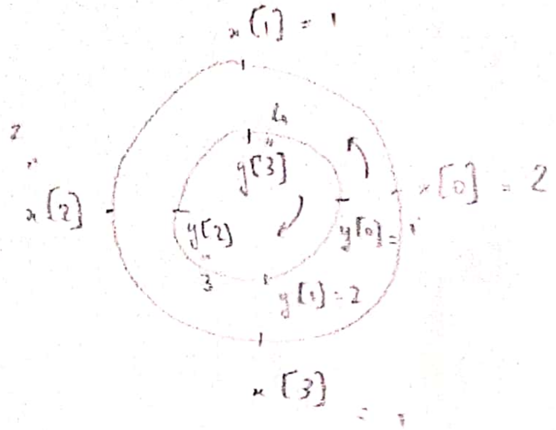
Let's define $z[n]$ the circular convolution of $x[n]$ and $y[n]$ Q5

$$z[n] = \sum_{m=0}^{N-1} x[m]\, y[\langle n-m \rangle_N]$$



$x[0]y[0] + x[1]y[3] + x[2]y[2] + x[3]y[1] = 2 + 4 + 6 + 2 = 14$

$x[0]y[3] + x[1]y[2] + x[2]y[1] + x[3]y[0] = 8 + 3 + 4 + 1 = 16$

$x[0]y[2] + x[1]y[1] + x[2]y[0] + x[3]y[3] = 14$

$x[0]y[1] + x[1]y[0] + x[2]y[3] + x[3]y[2] = 16$

$z[n] = \{14, 16, 14, 16\}$

Please follow tutorial from Youtube video: "Circular convolution - Concentric Circle Method - steps"

Q6 a. $x[n] = \{1, 0, 2, 1\}$

$$y[n] = \sum_{m=0}^{3} x[m]\, x[\langle n-m \rangle_4]$$

$\underline{n=0}$ $y[0] = x[0]x[0] + x[1]x[3] + x[2]x[2] + x[3]x[1] = 1 + 0 + 4 + 0 = 5$

$\underline{n=1}$ $y[1] = x[0]x[1] + x[1]x[0] + x[2]x[3] + x[3]x[2] = 0 + 0 + 2 + 2 = 4$

$\underline{n=2}$ $y[2] = x[0]x[2] + x[1]x[1] + x[2]x[0] + x[3]x[3] = 2 + 0 + 2 + 1 = 5$

$\underline{n=3}$ $y[3] = x[0]x[3] + x[1]x[2] + x[2]x[1] + x[3]x[0] = 1 + 0 + 0 + 1 = 2$

$$y[n] = 5\delta(n) + 4\delta(n-1) + 5\delta(n-2) + 2\delta(n-3)$$

b. $x[n] = \{1, 0, 2, 1\}$

$h[n] = \{1, 1, 0, 2\}$

$$z[n] = \sum_{m=0}^{3} x[m]\, h[\langle n-m \rangle_4]$$

$\underline{n=0}$ $z[0] = x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1] = 1 + 0 + 0 + 1 = 2$

$\underline{n=1}$ $z[1] = x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2] = 1 + 0 + 4 + 0 = 5$

$\underline{n=2}$ $z[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[3] = 0 + 0 + 2 + 2 = 4$

$\underline{n=3}$ $z[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = 2 + 0 + 2 + 1 = 5$

$$z[n] = 2\delta(n) + 5\delta(n-1) + 4\delta(n-2) + 5\delta(n-3)$$