Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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FACULTY OF SCIENCE, TECHNOLOGY, ENGINEERING AND MATHEMATICS

SCHOOL OF ENGINEERING

Electronic and Electrical Engineering

Engineering **Semester 1, 2021**

Digital Signal Processing

15th December 2021 **Venue: RDS Simmonscourt** **Time: 09.30 – 11.30**

Dr. W. Dowling

Instructions to Candidates:

Answer FOUR questions. All questions carry equal marks.

Materials permitted for this examination:

Mathematical Tables are available from the invigilators, if required.

Non programmable calculators are permitted for this examination

Please indicate the make and model of your calculator on each answer book used.

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Q.1 (a) The Nyquist rate is twice the highest frequency in a bandlimited signal.

If the Nyquist rate for a signal $x(t)$ is ω_0, find the Nyquist rate for the signal $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$. **[5 marks]**.

(b) An up-sampler with input $x[n]$ and output $r[n]$ is shown in Fig. Q1-1.

The up-sampler inserts 2 zeros between each sample of $x[n]$.

Is the up-sampler a linear system? Justify your answer. **[5 marks]**

(c) A system for increasing the sampling rate by a factor of 1.5 is shown in Fig. Q1-1.

$$x[n] \longrightarrow \boxed{\uparrow 3} \longrightarrow r[n] \longrightarrow \boxed{h[n]} \longrightarrow w[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow y[n]$$

Fig. Q1-1

$$r[n] = \begin{cases} x[n/3], & n = 0, \pm 3, \pm 6, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$w[n] = \sum_{k=-\infty}^{\infty} r[k]h[n-k]$$

$$y[n] = w[2n]$$

The ideal discrete-time low-pass filter has a unit sample response, $h[n]$, and a frequency response, $H(e^{j\Omega})$, given by

$$H(e^{j\Omega}) = \begin{cases} 3, & |\Omega| < \frac{\pi}{3}, \\ 0, & \frac{\pi}{3} < |\Omega| \leq \pi. \end{cases}$$

Let $R(e^{j\Omega})$ and $Y(e^{j\theta})$ denote the discrete-time Fourier transforms of the sequences $r[n]$ and $y[n]$ respectively. A continuous-time signal $x_a(t)$ has the Fourier transform $X_a(j\omega)$ shown in Fig. Q1-2.

continued ...

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[Q.1 continued]

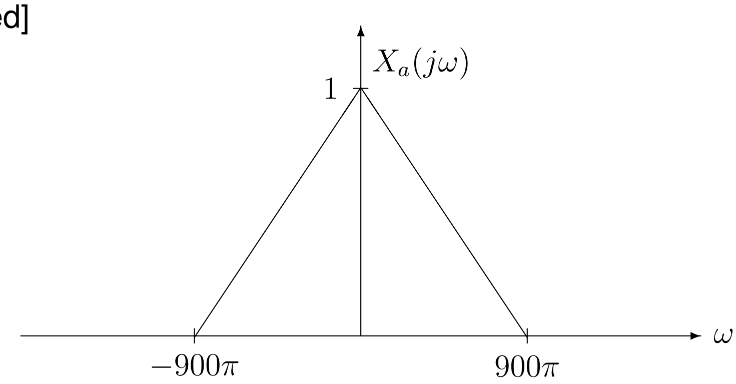

Fig. Q1-2

If $x[n] = x_a(nT)$, and the sampling period $T = 1$ millisecond,

(i) sketch $R(e^{j\Omega})$ for $-\pi \leq \Omega \leq \pi$, and **[5 marks]**

(ii) sketch $Y(e^{j\theta})$ for $-\pi \leq \theta \leq \pi$. **[5 marks]**

Q.2 (a) A discrete-time filter has a unit sample response, $h[n]$, that is zero for $n < 0$ and for $n > N - 1$. If $h[n] = -h[N - 1 - n]$ and N is odd, show that the filter has a frequency response with generalized linear phase. **[8 marks]**

(b) An ideal discrete-time high-pass filter has a frequency response, $H_{id}(e^{j\Omega})$, given by

$$H_{id}\left(e^{j\Omega}\right) = \begin{cases} 0, & |\Omega| < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < |\Omega| < \pi. \end{cases}$$

Obtain an expression for the unit-sample response of this filter. **[7 marks]**

(c) A 15-point Hamming window, $w_H[n]$, is given by

$$w_H[n] = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{7}\right), & -7 \leq n \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$

Using the Hamming window, design a causal, 15-point, generalised linear phase filter that approximates the magnitude response of the ideal high-pass filter in part (b). **[5 marks]**

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Q.3 (a) A continuous-time filter has an impulse response, $h_a(t)$, given by

$$h_a(t) = \begin{cases} e^{-t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

A discrete-time filter has the unit-sample response, $h[n]$, given by

$$h[n] = T h_a(nT),$$

where T is a positive constant.

Let $H(z)$ denote the transfer function of the discrete-time filter. Show that

$$H(z) = \frac{T}{1 - e^{-T} z^{-1}}, \quad |z| > e^{-T}.$$

[4 marks]

Note that: $\displaystyle\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \quad |\alpha| < 1.$

(b) A discrete-time bandpass filter with frequency response, $H(e^{j\Omega})$, is to be designed to meet the following specifications:

$$\frac{1}{\sqrt{2}} \leq |H(e^{j\Omega})| \leq 1, \qquad 0.4\pi \leq |\Omega| \leq 0.6\pi,$$

$$|H(e^{j\Omega})| \leq 0.2, \quad 0.9\pi \leq |\Omega| \leq \pi,$$

and $|H(e^{j\Omega})| \leq 0.2, \quad |\Omega| \leq 0.1\pi.$

The filter is to be designed by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter.

Verify that a second order filter is sufficient to meet the specifications.

Determine the transfer function, $H(z)$, of the discrete-time filter.

Note that the transfer function of a first order Butterworth lowpass prototype filter is

$$H(s) = \frac{1}{s+1}$$

and the lowpass to bandpass transformation for a continuous-time filter is

$$s \to \frac{s^2 + \omega_1 \omega_2}{s(\omega_2 - \omega_1)}$$

where ω_1 and ω_2 are the lower and upper cut-off frequencies respectively.

[16 marks]

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Q.4 (a) Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N, and with discrete Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$ respectively. Let $\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n]$ with $\tilde{X}_3[k]$ denoting its discrete Fourier series coefficients. Show that $\tilde{X}_3[k]$ can be expressed in terms of $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$ as

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{m=0}^{N-1}\tilde{X}_1[m]\tilde{X}_2[k-m]]$$

[8 marks]

(b) Compute the 8-point DFT of the sequence

$$g[n] = \begin{cases} 1, & 0 \leq n \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$

[4 marks]

(c) Find the 8-point inverse DFT of

$$X[k] = \begin{cases} 3, & k = 0, \\ 1, & 1 \leq k \leq 7. \end{cases}$$

[4 marks]

(d) A 75,000 point sequence is to be linearly convolved with a sequence that is 128 points long. The convolution is to be implemented using DFTs and inverse DFTs of length 1024. If the overlap-add method is used, what is the minimum number of 1024-point DFTs and the minimum number of 1024-point inverse DFTs needed to implement the convolution for the entire 75,000 point sequence?

[4 marks]

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Q.5 (a) Let $\{X[n], n \in \mathbb{Z}\}$ be a discrete-time random process, defined by

$$X[n] = A\cos\left(\frac{\pi n}{4}\right)$$

where A is a random variable that is uniformly distributed over $[0, 2]$.

(i) Find the mean of the random process, $m_X[n]$. **[2 marks]**

(ii) Find the autocorrelation $\phi_{XX}[n_1, n_2]$ of $X[n]$. **[4 marks]**

(iii) Is $X[n]$ a wide-sense stationary process? **[2 marks]**

(b) Let $X[n]$ be a real, zero mean, white noise process with autocorrelation sequence $\phi_{XX}[m] = \sigma_X^2\delta[m]$, where $\delta[m]$ is the unit-sample sequence.
$X[n]$ is applied to the input the of a linear, shift-invariant filter.
The random process, $Y[n]$, at the output of the filter is given by

$$Y[n] = X[n] + 0.5X[n-1].$$

(i) Obtain an expression for the output autocorrelation sequence, $\phi_{YY}[m]$. **[6 marks]**

(ii) Determine the power spectral density, $S_{YY}(\Omega)$, of the output process. **[3 marks]**

(iii) Determine the mean, m_Y, and the variance, σ_Y^2, of the output process. **[3 marks]**

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