EE4C5 Digital Signal Processing

Lecture 10 – FIR Filter Design

This lecture

- Based on Chapter 7 of O&S
- All images from O&S book unless otherwise stated

Recall…

- Difference equation
  - $\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k]$
- Or equivalently:
  - $H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^{M} b[k]z^{-k}}{\sum_{k=0}^{N} a[k]z^{-k}}$
- $N = 0$ FIR, so transfer function of our FIR filter:
  - $H(z) = \sum_{k=0}^{M} b[k]z^{-k} = \sum_{k=0}^{M} \mathrm{h}[k]z^{-k}$
- Frequency response:
  - $H\left(e^{j\omega}\right) = \sum_{\mathrm{n}=0}^{M} b[n]e^{-j\omega n} = \sum_{\mathrm{n}=0}^{M} \mathrm{h}[n]e^{-j\omega n}$

Why FIR filters?

- Inherently stable
- Can be designed to have linear phase or generalised linear phase
- Convenient/fast implementation

FIR Design Methods

- Windowing
  - Intuitive
- Optimisation approach
  - Define criterion to optimise
  - Iterative
  - E.g. Parks-McClellan method
- Exploit tools that implement algorithms for FIR filter design
  - E.g. MATLAB, Octave, NI, TI, Analog Devices etc.
  - Device providers

Windowing

The Window Method

- Ideal frequency response:
  - $H_d(e^{j\omega}) = \sum_{\mathrm{n}=-\infty}^{\infty} h_d[n] e^{-j\omega n}$
- With $h_d[n]$ the corresponding impulse response
- Can derive $h_d[n]$ from desired frequency response as:
  - $h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$
- ***<u>But</u>*** properties of $H_d(e^{j\omega})$ yield impulse response that is non-causal, infinitely long
  - Obtain an FIR by truncating the ideal response
  - => windowing method

Apply windowing

- Multiply desired impulse response by a finite-duration window $w[n]$
- $h[n] = h_d[n]w[n]$
- If the window is rectangular. i.e.
  - $w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$
- The operation is in effect then truncation
- Will see that choice of window has important implications

Explore window implications

- From convolution theorem (DTFT properties):
  - $H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$
- i.e. the periodic convolution of the desired ideal frequency response with the Fourier transform of the window.(fig 27)
- Ob#1: Short window duration for w[n] would minimise calculations
- Obs#2: Having $W(e^{j\omega})$ closer to an impulse will give closer approximation to the ideal response $H_d(e^{j\omega})$
- There is a trade-off between the *width of the main lobe* and the *magnitude of the side lobes* in the frequency domain from the window

Truncation of ideal response

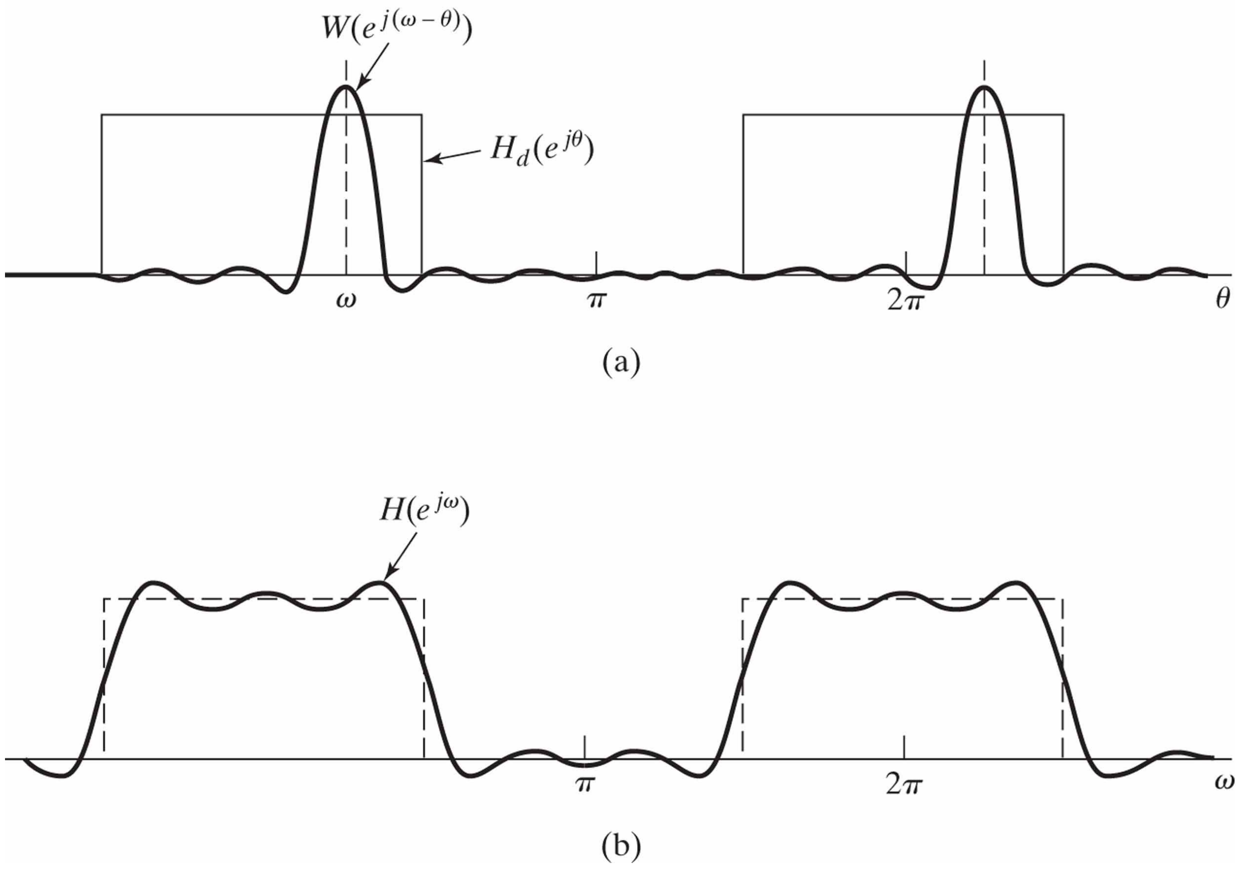


Figure 7.27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

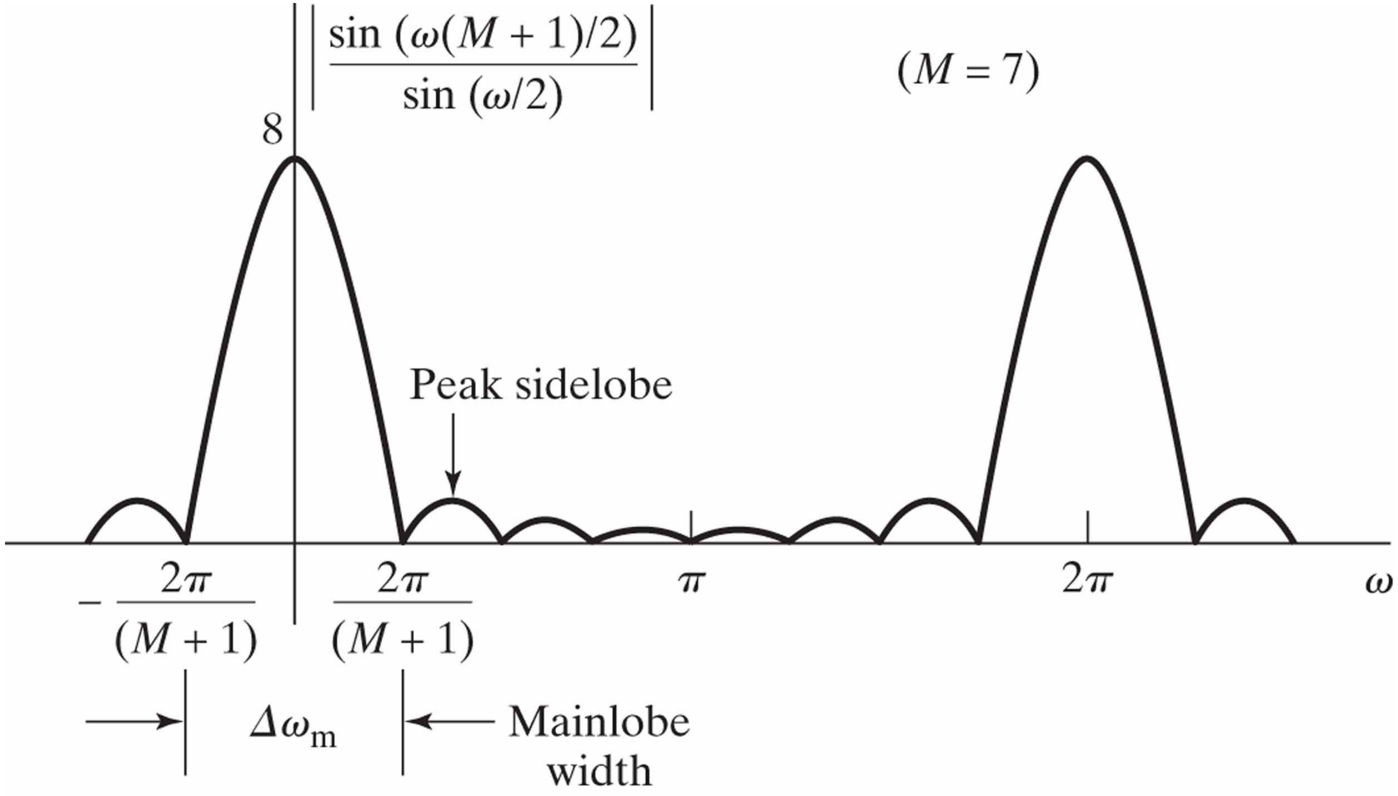
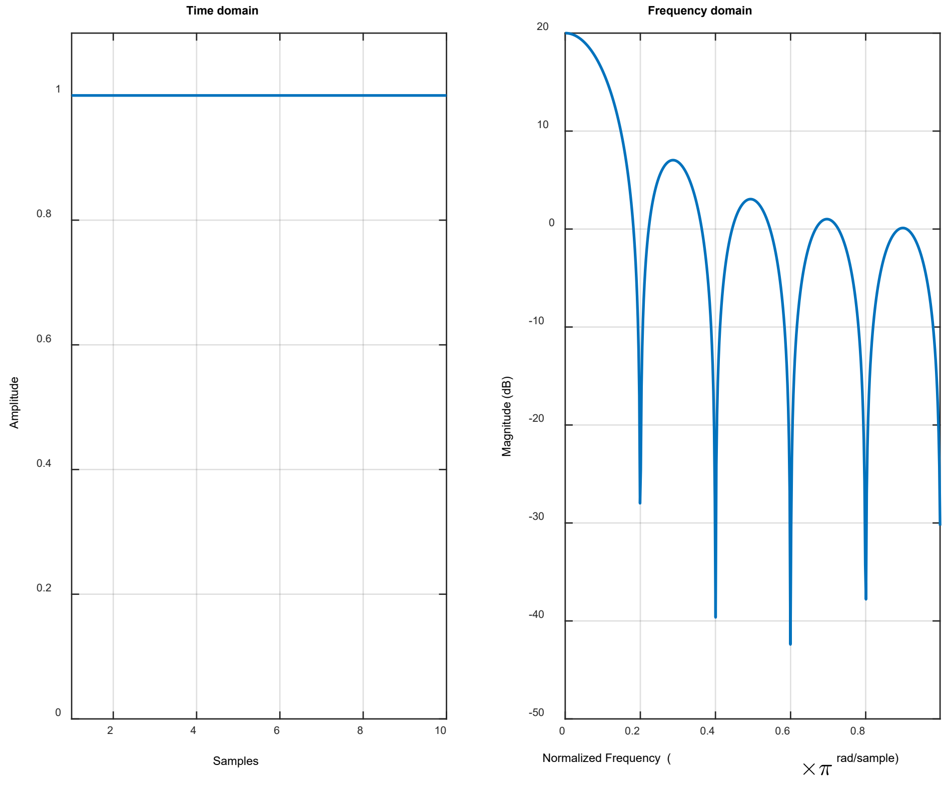


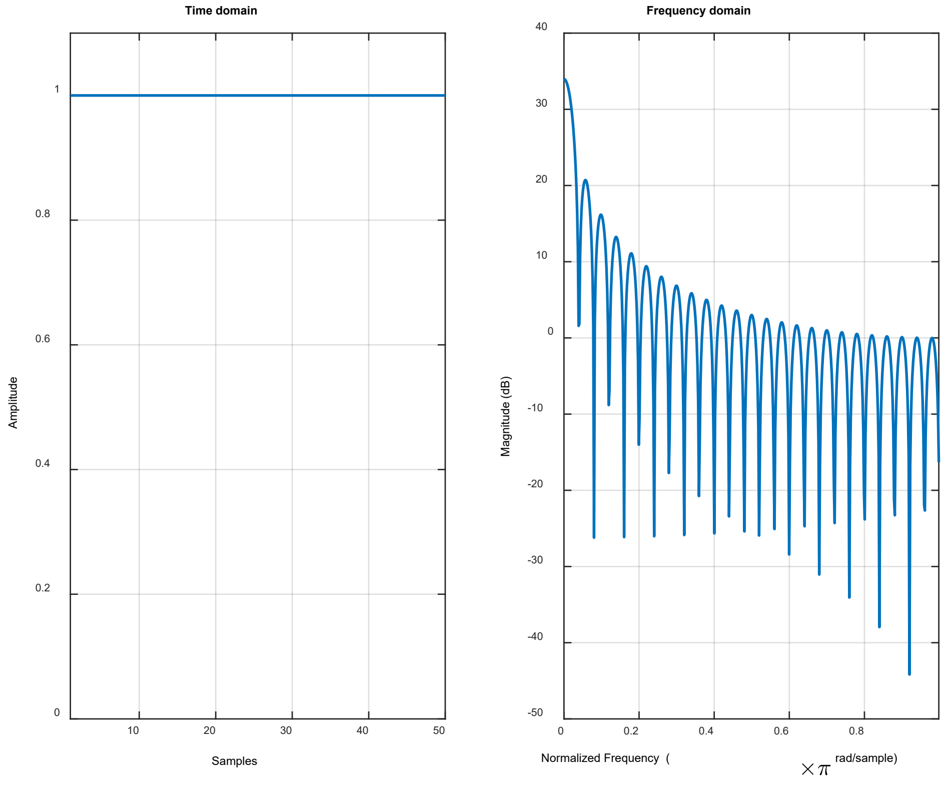
Figure 7.28 Magnitude of the Fourier transform of a rectangular window (M = 7).

Rectangular windows

Explore in MATLAB with:
wvtool(rectwin(10))
wvtool(rectwin(50))



M=10

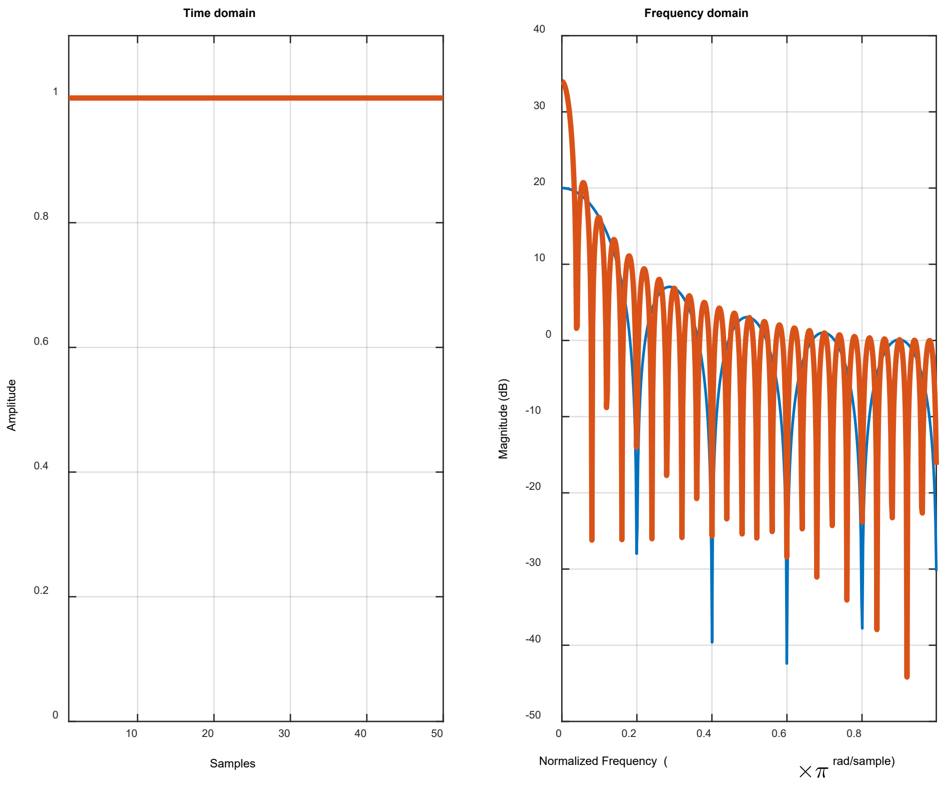


M=50

As M increases, width of main lobe decreases

On same axes..

w1 = rectwin(10);
w2 = rectwin(50);
wvtool(w1,w2)



Gibbs Phenomena

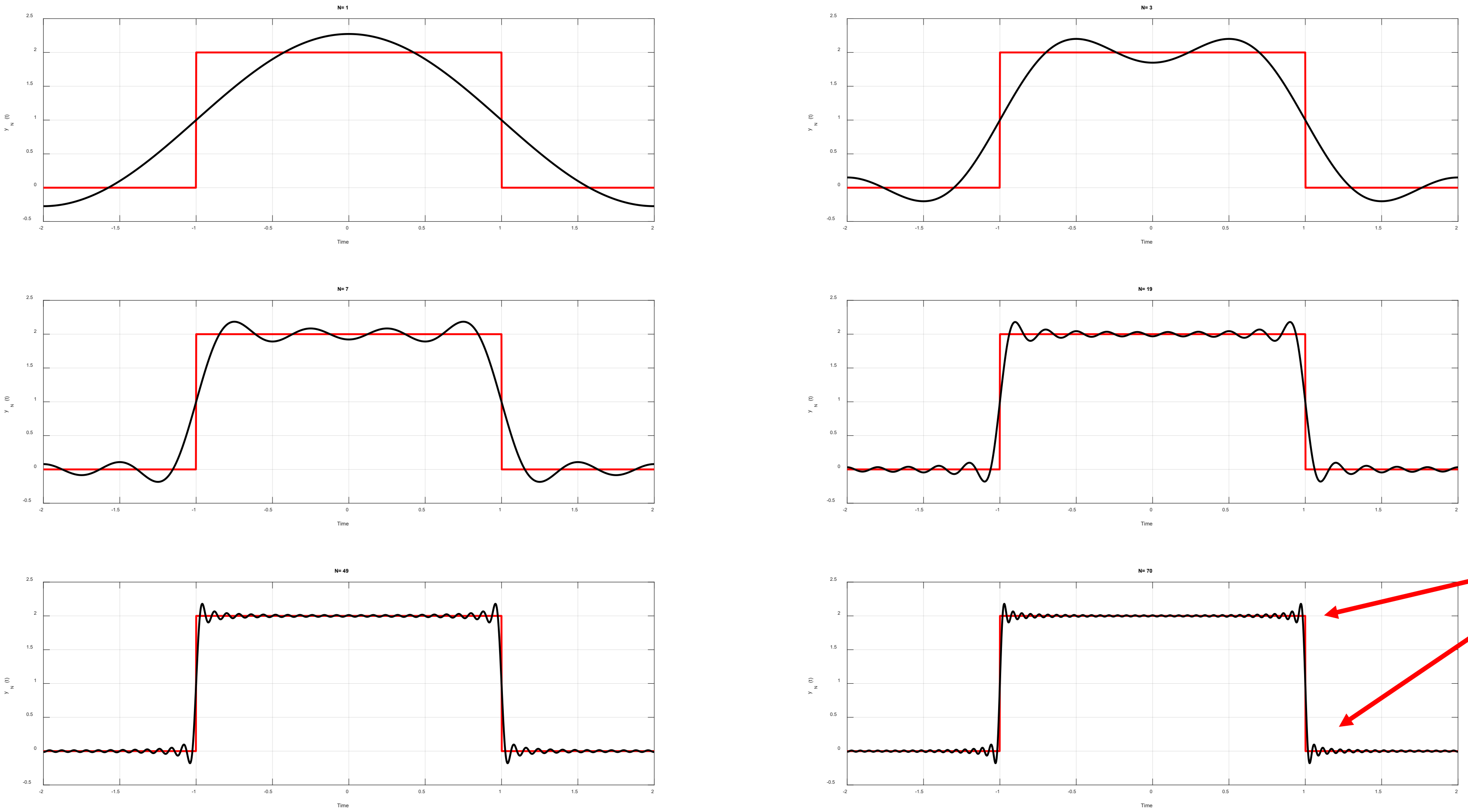
- Resultant oscillations in integral
- Oscillations increase as M increases but don’t change in amplitude
- Reduce effect with less abrupt truncation
- Taper window smoothly to zero at each end to reduce height of sidelobes
    - But that can cost you in termsof a wider main lobe
- Let’s examine a variety of commonly used windows…

An aside on Gibbs Phenomenon

Gibbs Phenomenon

- Truncation of the ideal impulse response is identical to the issue of convergence of the Fourier Series
- Fourier transform represents signals in frequency domain as summation of sinusoidal waves
- Abrupt discontinuities in the time domain signal
  - Requires infinite frequency content in the frequency domain
  - Think of a square wave or impulse in time domain
- In practice, the number of frequency terms is truncated
  - Truncation in frequency domain causes ringing in the time domain
- More generally, a discontinuity in one domain requires infinite number of components in the other domain

Increasing the number of Fourier components while synthesising a square waveform



Ringing artefact

Code based on:
https://uk.mathworks.com/matlabcentral/fileexchange/21291-demonstration-of-gibbs-phenomenon

Relation to FIR filter

- Ideal brick-wall frequency response e.g. LPF has sharp discontinuities
- A rectangular function with abrupt transition in frequency domain translates to a sinc function of infinite duration in time domain
- Truncation of this signal in the time-domain results in the Gibbs phenomena in the frequency domain
- Discarded samples in time-domain means missing frequencies in the frequency domain
    - => oscillations
- Can improve but not remove Gibbs phenomenon with better choice of time-domain window.

Common Windows

Properties of suitable windows

- Real and symmetric for both odd and even M
  - $w[n] = w[M - n] \;\; n = 0, \dots, M$
  - If the FIR has linear phase, then a window with this property will preserve it
- Positive values => no change in sequence polarities
- Must satisfy the trade-off between width of mainlobe and magnitude of sidelobes in the frequency domain
- Smooth transition at the edges to reduce the Gibbs phenomenon
- Window length must match sequence length

Definitions for common choices of window

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \ \ M \text{ even} \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Hann

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

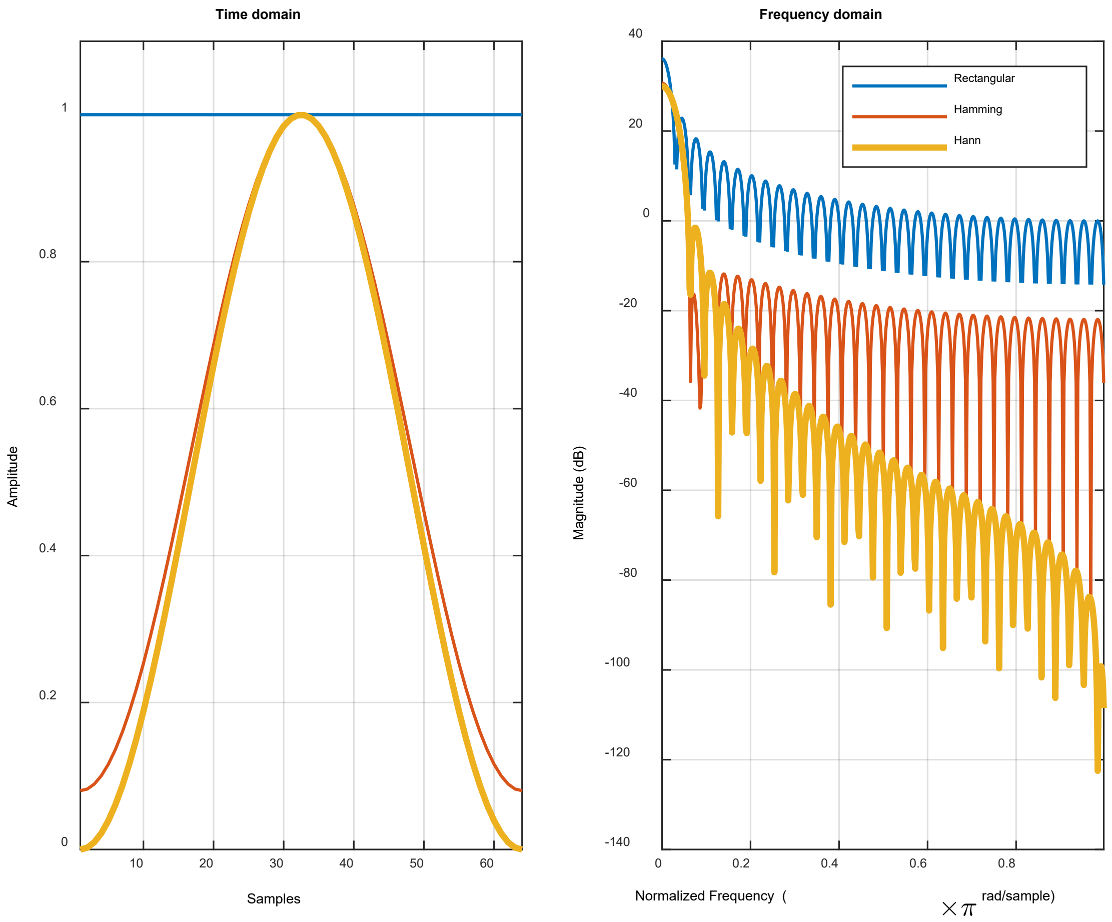
Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

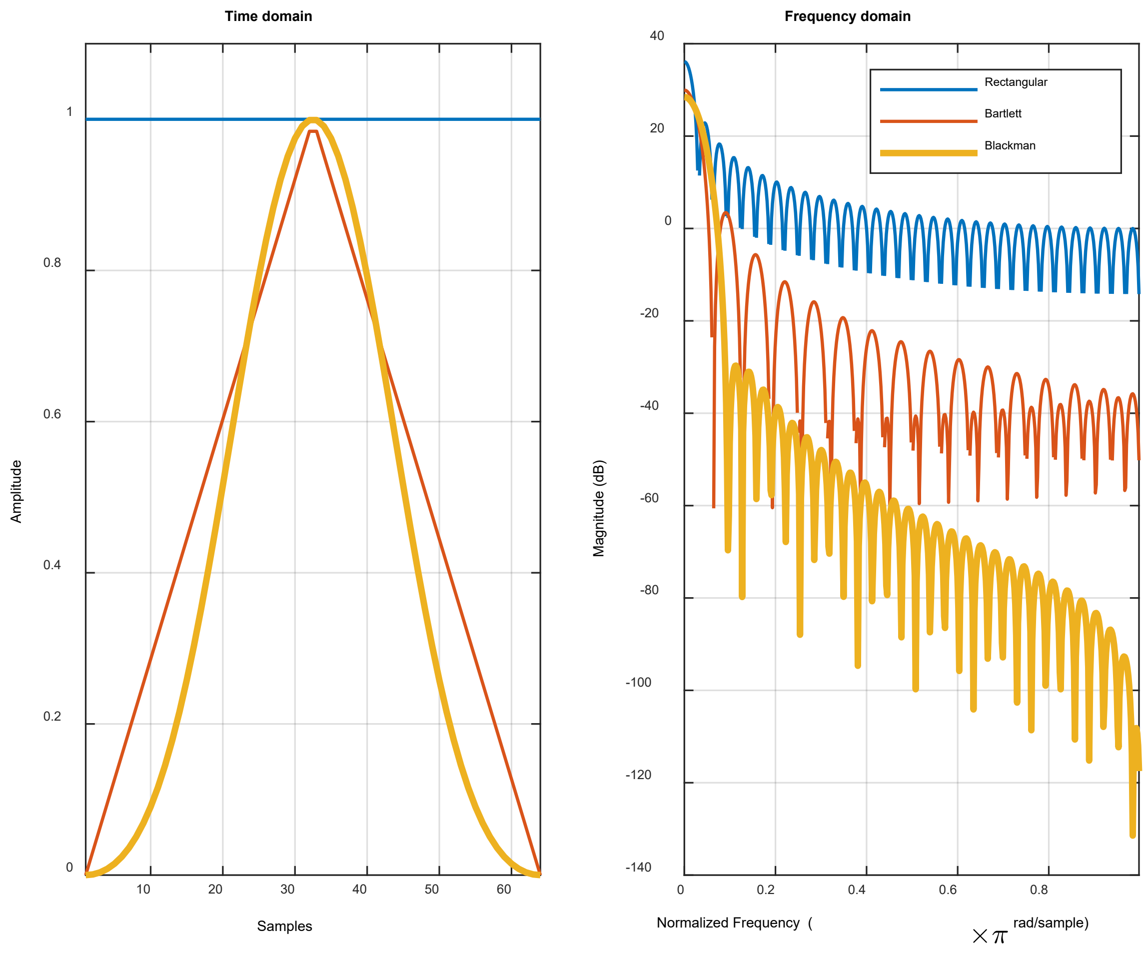
Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Hamming versus Hann Window M=64



Blackman and Bartlett M=64



Comparison summary (at a given length)

- Rectangular window narrowest mainlobe
- Hann, Hamming, Blackman smoother transition to 0
  - Reduces sidelobes
  - But wider mainlobe

Design Method

- E.g. LPF, choose
  - order M (filter length M+1)
  - cut-off frequency ω_c
  - window type
- Form the samples of the ideal low-pass filter of length $M+1$
  - $h_d[n] = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n)}{\omega_c n}\right) \;\; for \;\; {}^{-M}/_{2} \leq n \leq {}^{M}/_{2}$
- Form the chosen type of window $w[n]$ of length $M+1$
- Form the FIR filter impuse response $h[n]$ as:
  - $h[n] = h_d[n]\, w[n]$
- Shift all the samples to right by $M/2$
- But we'll usually use MATLAB ☺

Filter parameter estimation

TABLE 2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$4\pi/(M+1)$	-21	0	$1.81\pi/M$
Bartlett	-25	$8\pi/M$	-25	1.33	$2.37\pi/M$
Hann	-31	$8\pi/M$	-44	3.86	$5.01\pi/M$
Hamming	-41	$8\pi/M$	-53	4.86	$6.27\pi/M$
Blackman	-57	$12\pi/M$	-74	7.04	$9.19\pi/M$

Table 9-2 The Peak Side-Lobe Amplitude of Some Common Windows and the Approximate Transition Width and Stopband Attenuation of an Nth-Order Low-Pass Filter Designed Using the Given Window.

Window	Side-Lobe Amplitude (dB)	Transition Width (Δf)	Stopband Attenuation (dB)
Rectangular	-13	$0.9/N$	-21
Hanning	-31	$3.1/N$	-44
Hamming	-41	$3.3/N$	-53
Blackman	-57	$5.5/N$	-74

Kaiser Window

Kaiser Window

- More recently developed by James Kaiser in Bell Labs
- More optimal approach to achieving the desired trade-off between mainlobe width and sidelobe levels can be achieved

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On the Use of the I_0-Sinh Window for Spectrum Analysis

JAMES F. KAISER AND RONALD W. SCHAFER

Abstract—Closed form expressions for main-lobe width, modified main-lobe width, and relative sidelobe amplitude are given for the I_0-sinh window function. These formulas facilitate exploring the tradeoff between record length, spectral resolution, and leakage in digital spectrum analysis. An especially simple empirical approximation relating main-lobe width and relative sidelobe amplitude is given.

Time-limiting windows are used in the design of finite-duration impulse response (FIR) digital filters and in the estimation of Fourier spectra from finite length segments of data. In both cases, the window must be limited in duration in the time domain and must also have most of its energy concentrated at low frequencies in its Fourier transform. The prolate spheroidal wave functions have been shown to provide the greatest concentration of energy at low frequency [1]. Because of the difficulty of computing the prolate function, however, a much simpler approximation using the zeroth-order modified Bessel function of the first kind was suggested [2]. The resulting window, which is also known as the Kaiser window [4]-[6] is defined as

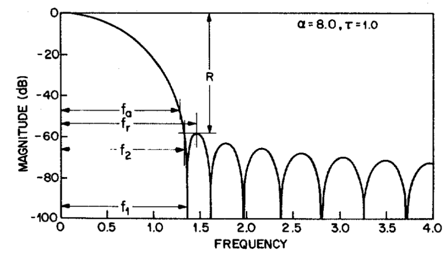

Fig. 1. A typical Fourier transform of an I_0-sinh window showing parameters defined in text (shown for $\alpha = 8.0$, $\tau = 1.0$).

sidelobe amplitude (instead of the approximation error). Since the I_0-sinh window provides great flexibility through the parameter α and since it is straightforward to compute using a simple Fortran subroutine [3], there is good reason to suggest that this window should be the window of choice for spectrum analysis as well. Therefore, the purpose of this correspondence is to provide formulas for the spectral parameters of the I_0-sinh window that will be useful for spectrum analysis applications.

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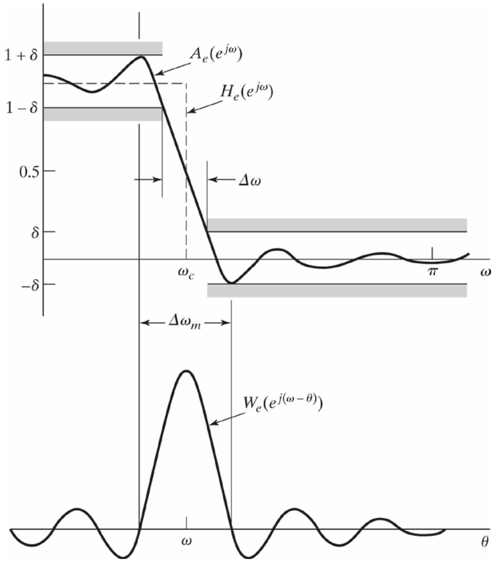
Definition

- $$w[n] = \begin{cases} \dfrac{I_0[\beta(1 - [(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$
- $\alpha = {}^{M}/_{2}$ and $I_0(\quad)$ represents the 0-th order modified Bessel Function of the first kind.
- Two parameters:
  - Window length $M + 1$
  - Shape parameter β
- By varying $(M + 1)$ and β , the window length and shape can be adjusted to trade side-lobe amplitude for mainlobe width
- Can work out required $(M + 1)$ and β to meet given specifications
- Deviation from target response very dependent on choice of β

Parameter choice



- Passband cut-off frequency ω_p
- Stopband cut-off frequency ω_s
- => transition region $\Delta\omega = \omega_s - \omega_p$
- Define $A = 20\log\delta$
- Kaiser showed that:

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases}$$

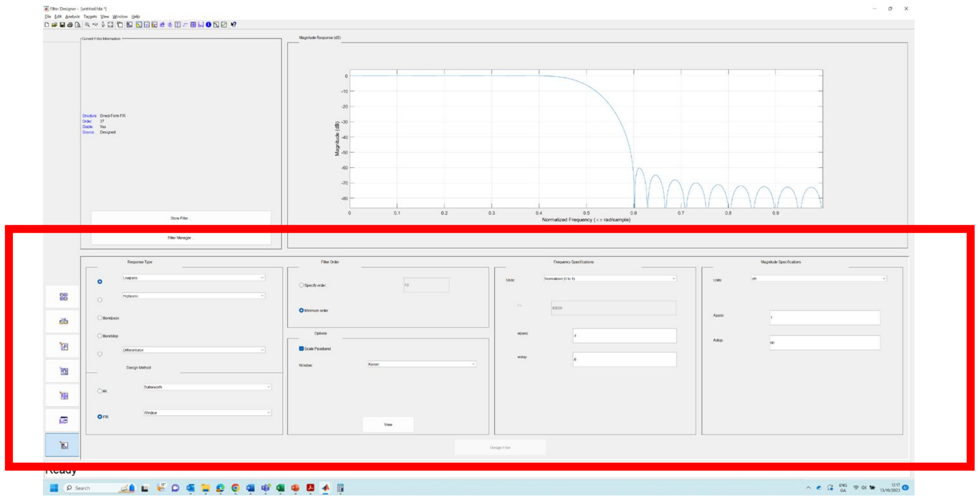
$$M = \frac{A-8}{2.285\Delta\omega}.$$

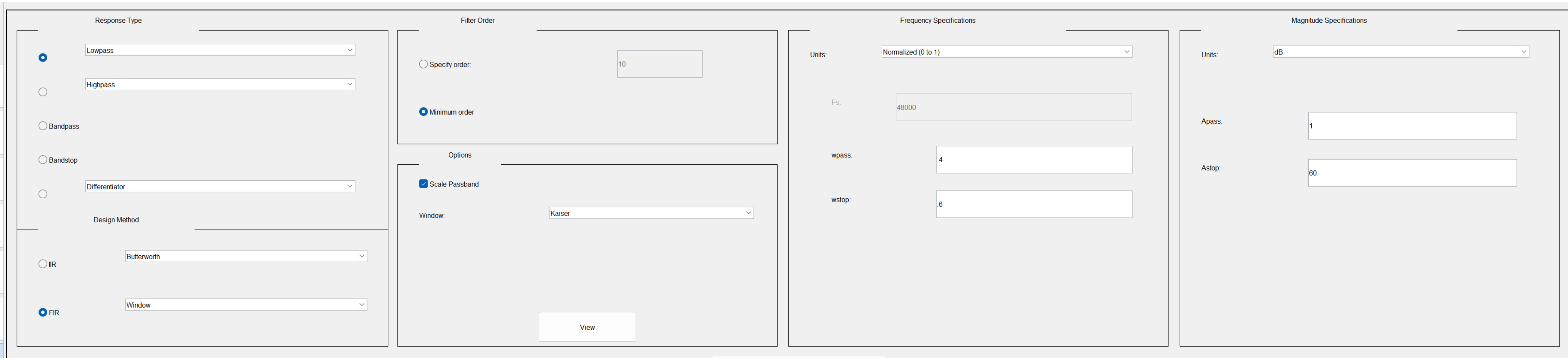
- Method straightforward

Lowpass FIR Filter using Kaiser window

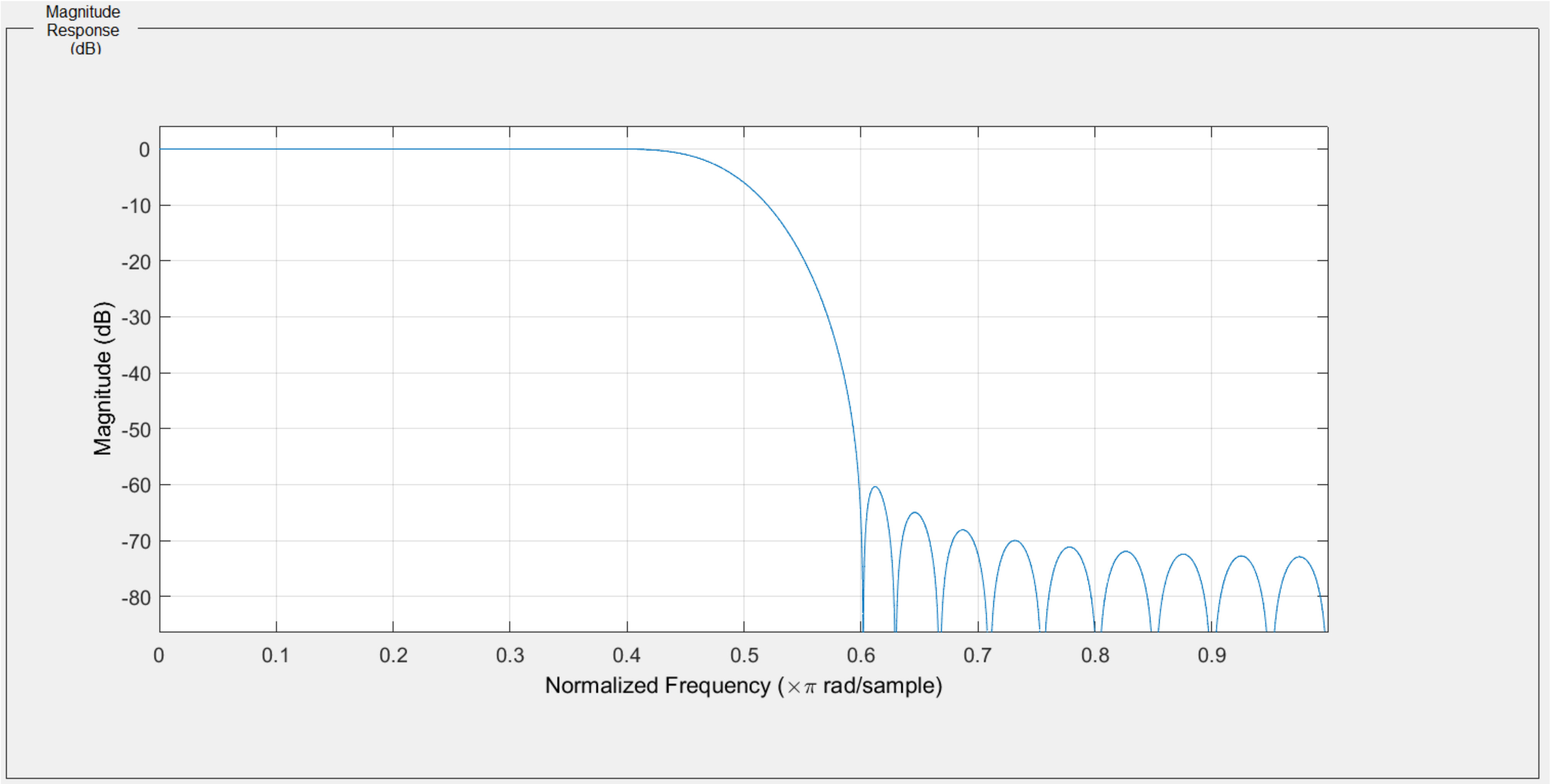
- Passband cut-off frequency $\omega_p = 0.4\pi$
- Stopband cut-off frequency $\omega_s = 0.6\pi$
- Stopband attenuation 0.001
- MATLAB – filter design tool

Setup

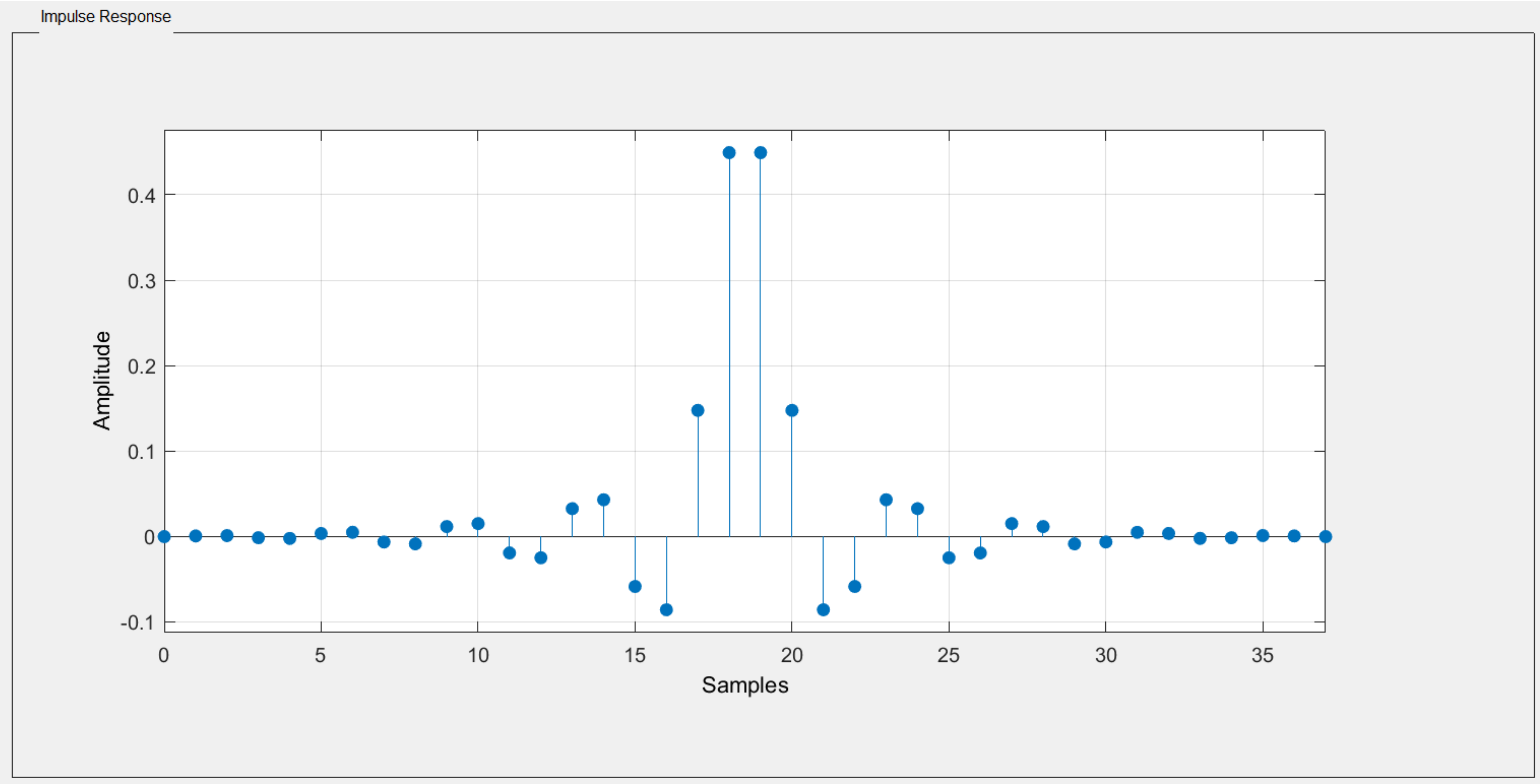



Response Type
Lowpass
Highpass
Bandpass
Bandstop
Differentiator
Design Method
IIR
Butterworth
FIR
Window
Filter Order
Specify order:
10
Minimum order
Options
Scale Passband
Window:
Kaiser
View
Frequency Specifications
Units:
Normalized (0 to 1)
Fs:
48000
wpass:
.4
wstop:
.6
Magnitude Specifications
Units:
dB
Apass:
1
Astop:
60

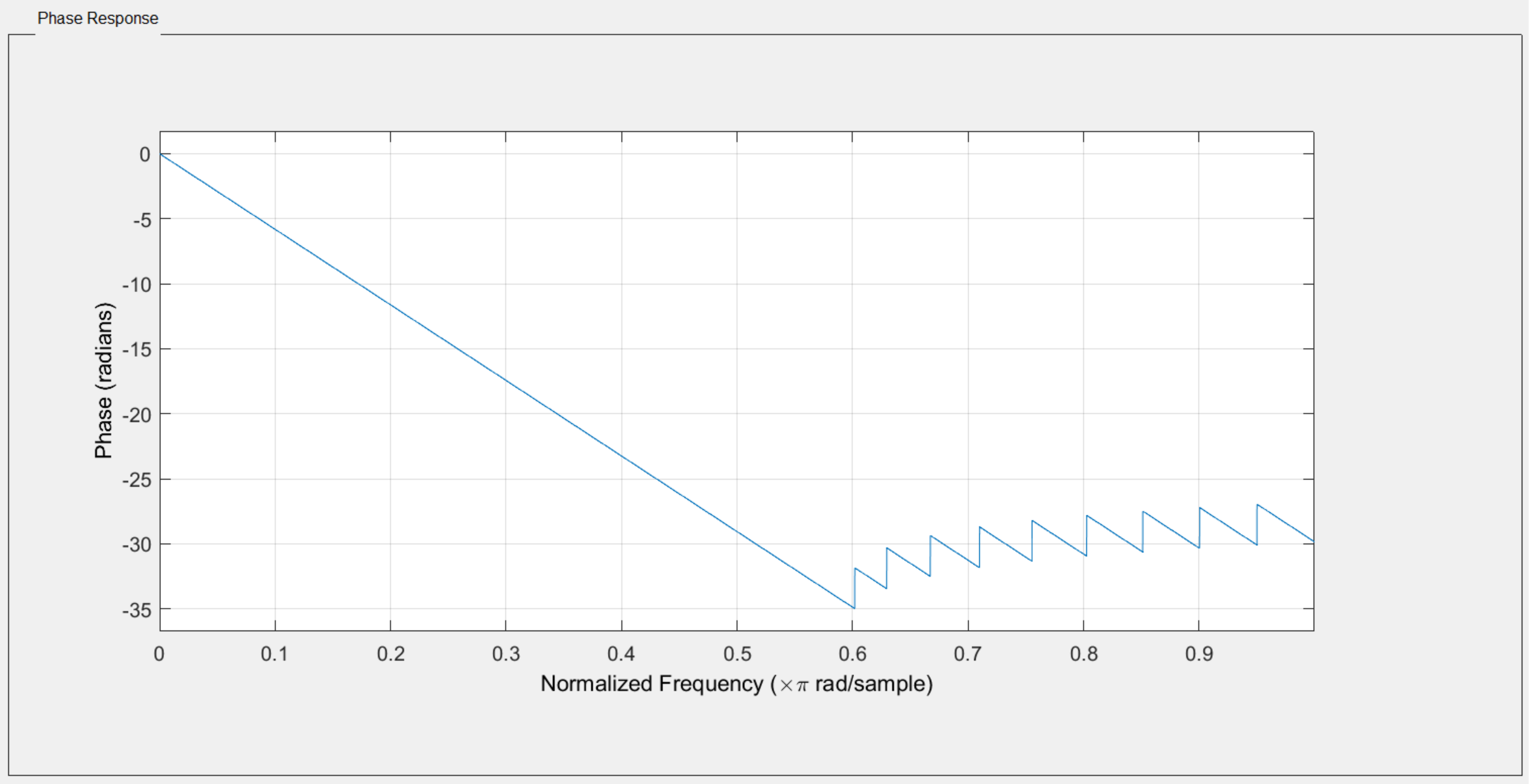
Magnitude Response, order 37



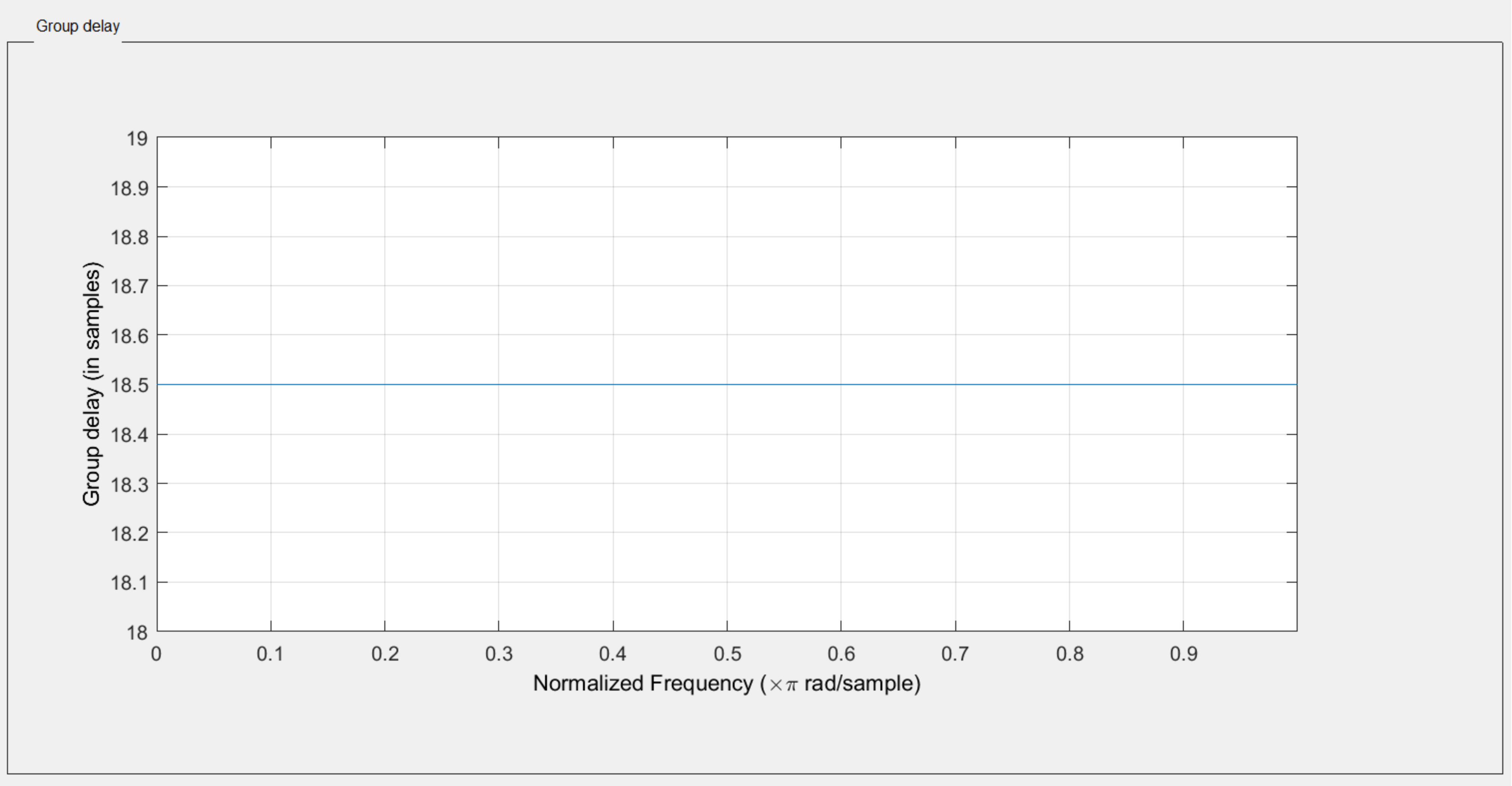
Impulse Response



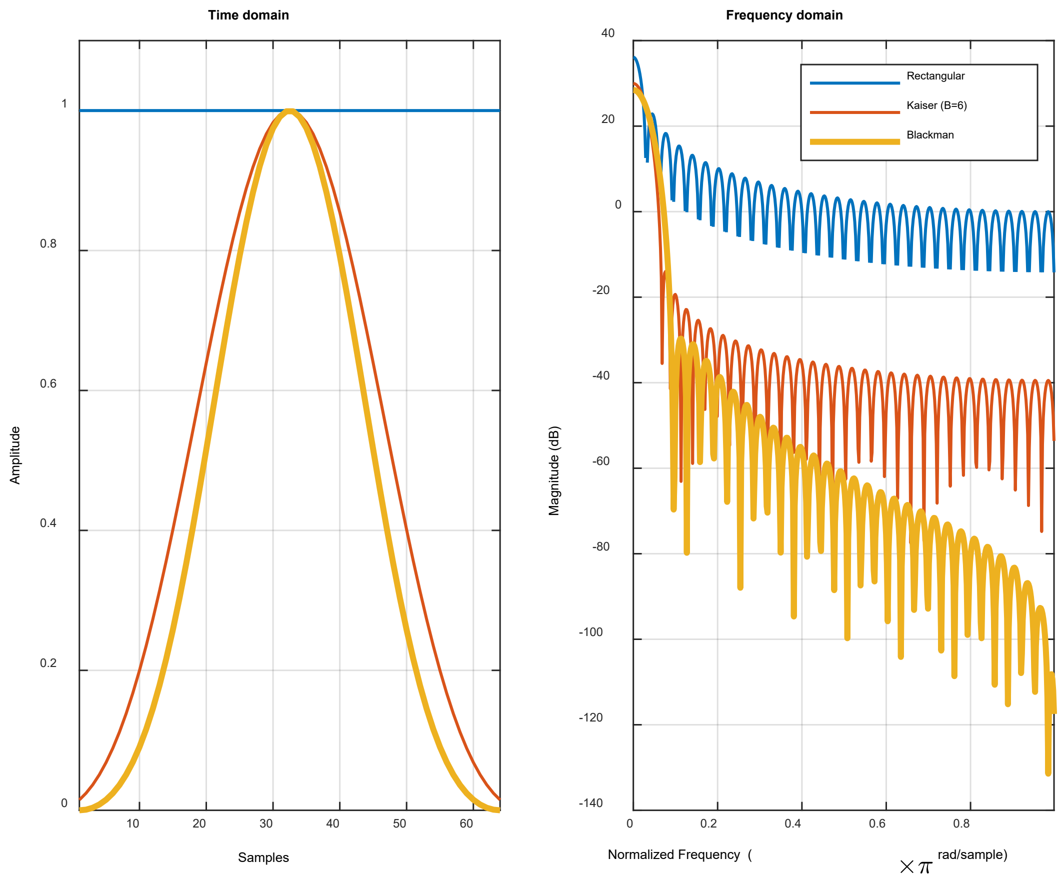
Phase Response



Group Delay is M/2! (why??)



Examine M=64



Time domain
Amplitude
Samples
Frequency domain
Magnitude (dB)
Normalized Frequency ($\times\pi$ rad/sample)
Rectangular
Kaiser (B=6)
Blackman

Required Reading & other material

- Oppenheim & Schafer, Chapter 7, Section 5
- Deeper insights on Gibbs Phenomena:
  - https://www.youtube.com/watch?v=Ol0uTeXoKaU