Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

EEU44C18 / EEP55C28
Digital Wireless Communications

Lecture 4: Orthogonal Signalling and Multi-tone Modulation

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Review of Lecture 3

- Digital modulation/demodulation.
    - signal space (constellation).
    - decision region.
- Nyquist criterion for distortionless channel
- Pulse shaping to avoid ISI.

Outline

- Determine if two given signals are orthogonal to each other or not.
- Design the necessary receiver demodulation structure to extract the information embedded into two orthogonal signals.
- Explain how the factorization of the pulse shaping filter between transmitter and receiver leads to self-orthogonality of the modulating signals.
- Demonstrate the necessary condition for two signals at two different carrier frequencies to be orthogonal.
- Describe the multi-tone modulation transmitter and receiver structure.
- Explain how the DFT might be used to perform multi-tone modulation and demodulation.
- Analyse the mission of the cyclic prefix in multi-tone modulation.

Orthogonal Functions

- In the previous lecture it was seen how the $\sin(2\pi f_C t)$ and $\cos(2\pi f_C t)$ functions are modulated to carry digital information.
- We also understand that $\sin(2\pi f_C t)$ and $\cos(2\pi f_C t)$ functions can be used because they are *orthogonal functions*. In fact, they form a basis for the 2-dimensional Euclidean signal space.
- Recall that two signals $x_1(t)$ and $x_2(t)$ are orthogonal if their inner product is equal to zero

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t) = 0 \qquad (1)$$

- However, $\sin(2\pi f_C t)$ and $\cos(2\pi f_C t)$ are not the only orthogonal functions that can be employed as a basis.

Orthogonal Functions

- Any set of orthogonal functions can be used for modulation. For example, we can use the following two signals
- Therefore, the each of these functions, $\phi_1(t)$ and $\phi_2(t)$, will be modulated by the projection of the transmitted signal over them.

$$s(t) = s_1\phi_1(t) + s_2\phi_2(t) \qquad (2)$$

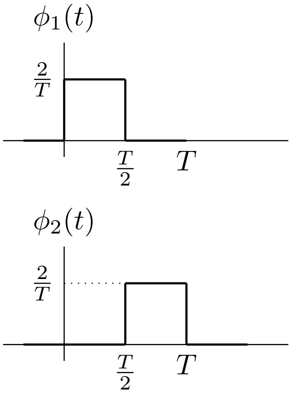


Figure: Orthogonal functions

Orthogonal Functions

- Data transmission corresponds to

$$s(t) = \sum_{k=-\infty}^{\infty} \left[s_{k_1} \phi_1(t - kT) + s_{k_2} \phi_2(t - kT) \right] \tag{3}$$

- In general, modulation and demodulation can be performed by direct implementation of the mathematical expression.

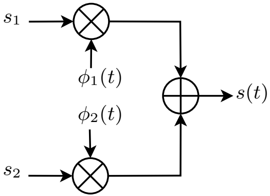


Figure: Orthogonal Modulation

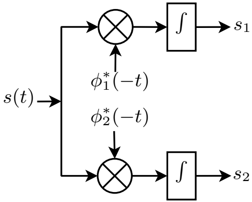


Figure: Orthogonal Demodulation

Self-orthogonality in time

- Recall that orthogonality implies the ability to uniquely resolve the constituent components (s_1, s_2) independently.
- To resolve data sent at different time instants, any function used for transmissions must be orthogonal to itself when delayed by an integer multiple of the sampling period T.
- This property of the basis functions is called **self-orthogonality in time**.
  - $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are *NOT* self-orthogonal in time.
  - Waveforms which satisfy the Nyquist criterion are self-orthogonal in time.
  - An appropriate pulse-shape multiplied to $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$, converts them to self-orthogonal signals in time. We will see this property soon.

Self-orthogonality in time

Theorem

Let $g(t)$ correspond to a filter which satisfies the Nyquist criterion. Let there exist a factorisation of $g(t)$ as $g(t) = p(t) * p^*(-t)$. Then $p(t)$ is orthogonal to $p(t - kT)$ $\forall k$.

- This factorisation is commonly known as pulse-shaping factorization.

Proof.

We have

$$
\begin{aligned}
g(kT) &= \int_{-\infty}^{\infty} p(\tau) p^*(\tau - kT) d\tau && (4) \\
&= < p(t), p(t - kT) > && (5)
\end{aligned}
$$

Since $g(t)$ satisfies the Nyquist criterion, $g(t)$ is zero except at $k = 0$, thus $p(t)$ is orthogonal to $p(t - kT)$ $\forall k$. □

Self-orthogonality in time

- We are guaranteed that a filter which is a factorization of a Nyquist satisfying filter is self-orthogonal at delays of kT.
- Currently, the raised cosine filter is specified between the transmitter and receiver, and so, without upsetting our model, we may split this filter into a root-raised cosine (RRC) filter at the transmitter (pulse shaping filter) and a RRC filter at the receiver (matching filter).

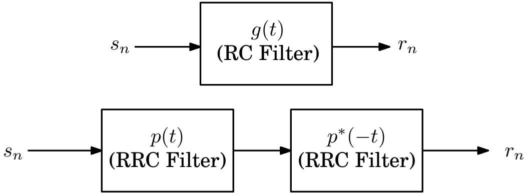


Figure: Pulse shaping factorisation

Self-orthogonality in time

- Orthogonal signalling using any set of orthogonal functions may be represented using a signal space diagram, with the pulse shape operation represented as a weighting of each orthogonal function.
- Consequently, the signalling pulses are:

$$\phi_1(t) = p(t)\cos(2\pi f_c t);\ \phi_2(t) = p(t)\sin(2\pi f_c t) \tag{6}$$

  where $p(t)$ is the RRC filter.
- We can easily see that $\phi_1(t)$ and $\phi_2(t)$ are also each self-orthogonal in time.

Multi-tone Modulation

- Multi-tone modulation means modulation onto carriers at different frequencies rather than just a single frequency, i.e.,$\cos(2\pi f_1 t)$, $\sin(2\pi f_1 t)$, $\cos(2\pi f_2 t)$, $\sin(2\pi f_2 t)$, ..., $\cos(2\pi f_N t)$, $\sin(2\pi f_N t)$.
- The problem is how to choose f_i so that these sinusoids are orthogonal.
- The condition for orthogonality

$$\begin{aligned} < \cos(2\pi f_m t), \cos(2\pi f_n t) > &= \int_{-\infty}^{\infty} \cos(2\pi f_m t)\cos(2\pi f_n t)dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big[\cos(2\pi(f_m - f_n)t) \\ &\qquad + \cos(2\pi(f_m + f_n)t)\big] dt \\ &= 0 \end{aligned} \tag{7}$$

Multi-tone Modulation

- However, it is not possible to implement this solution because the limits need to be $\pm\infty$ for orthogonality in the general case:
  - In reality, for two sinusoidal signals to be orthogonal, the limits of the inner product should be an integer number of periods for both signals.
  - For example, there is no non-infinite time over which two carriers of respective frequencies 1 Hz and $\sqrt{2}$ Hz each have an integer number of periods. Therefore, two sinusoidal signals with those frequencies, such as $\cos(2\pi t)$ and $\cos(2\pi\sqrt{2}t)$, are not orthogonal to each other.
- It is straightforward to place some constraints on the frequencies under consideration such that all carriers may be considered orthogonal over finite time.
- A good way is to choose a reference frequency f_r with corresponding period T, and let $f_m = k_m f_r$ and $f_n = k_n f_r$ for integer values of k_m and k_n. We see that $\cos(2\pi k_m f_r t)$ and $\sin(2\pi k_n f_r t)$ are orthogonal.
- Since the carriers are orthogonal to each other, they may be used to transmit simultaneously.

Multi-tone Modulation

- Discrete multi-tone (DMT) and Orthogonal Frequency Division Multiplexing (OFDM) are the main wired and wireless multi-tone modulation techniques, respectively.
- DMT and OFDM allow for spectrum overlap between signals at different frequencies, but ensure that the signals are orthogonal and thus may be uniquely recovered.
- DMT is the term commonly used for wired systems and is widely used in xDSL systems.
- OFDM is the term commonly used for wireless systems and is used in 802.11a and 802.11g wireless LAN standards, 802.16 WiMAX, Digital Audio Broadcasting (DAB) in the UK and throughout Europe, and has been proposed for the 802.15.3 Ultra-Wideband (UWB) standard and HiperMAN.

Discrete Multi-tone Modulation (DMT)

- In multi-tone modulation, the information bit stream is divided in N parallel bit streams by a serial-to-parallel conversion. Each of this bit streams have a data rate of R/N bps.
- Each bit stream is independently fed into a symbol mapper and filtered by a shaping filter. Then after the ADC, each signal is modulated by a sub-carrier at a particular orthogonal centre frequency.
- If linear digital modulation is employed, the bandwidth of each signal centred at each sub-carrier (considering a brickwall pulse shape filter) is equal to $W_N = R/(Nn)$, where n is the number of bits per symbol employed in the digital modulator.
- Effectively, the total signal bandwidth ($W = NW_N$) is very similar to the bandwidth of the original system which used a single tone modulation. However, this particular signal spectrum provides large benefits when used in wireless communications, such as a large reduction of the ISI. More details about the strengths of multi-tone modulation will be seen in Lecture 5.

Discrete Multi-tone Modulation (DMT)

- A direct implementation of multi-tone modulation involves simultaneous modulation of multiple streams using standard analogue modulation (with appropriate pulse shaping). This is a continuous time model and is shown in the figure below.

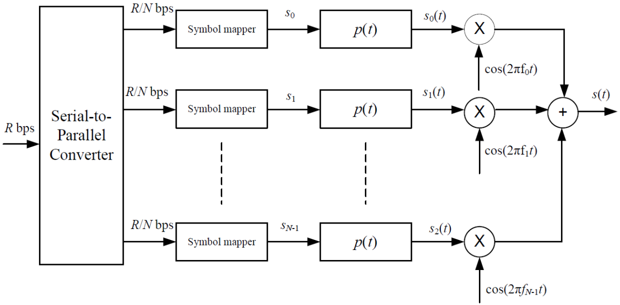


Figure: Direct DMT transmitter implementation

Discrete Multi-tone Modulation (DMT)

- A direct DMT receiver implementation is shown below.

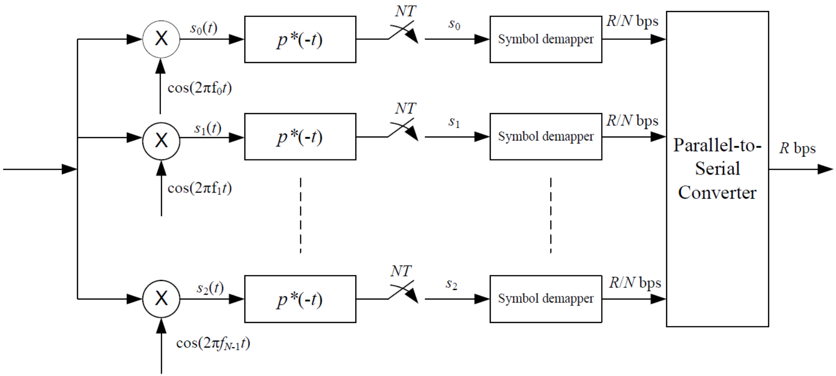


Figure: Direct DMT receiver implementation

Discrete Multi-tone Modulation (DMT)

Alternative DMT modulator – Use of the IDFT

- The structures shown in previous slides have a high computational complexity since an independent modulator and demodulator structure is required in both transmitter and receiver.
- Some DMT and OFDM systems use up to 2048 orthogonal tones or sub-carriers, therefore these structures do not provide a good scalable implementation.
- The bad scalability of the direct DMT transmitter and receiver was the main reason that this technology was underused for many years. However, the development of simple and cheap implementations of the Discrete Fourier Transform (DFT) and the inverse DFT (IDFT) and their direct use to implement DMT systems powered the use of multi-carrier communication systems.
- DFT is the discrete equivalent to the continuous time Fourier transform. The DFT of a discrete time sequence $x[n]$ characterises the frequency content associated with the original continuous time signal $x(t)$.

Discrete Multi-tone Modulation (DMT)

Alternative DMT modulator – Use of the IDFT

- Let $x[n]$ be a sequence with $0 \leq n \leq N-1$, The N-point DFT of $x[n]$ is equal to

$$\mathrm{DFT}\{x[n]\} = X[i] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi ni/N}, i = 0, 1, \ldots, N-1 \tag{8}$$

- Using the IDFT $x[n]$ can be recovered from $X[i]$ as

$$\mathrm{IDFT}\{X[i]\} = x[n] = \frac{1}{N} \sum_{i=0}^{N-1} X[i] e^{j2\pi ni/N}, n = 0, 1, \ldots, N-1 \tag{9}$$

Discrete Multi-tone Modulation (DMT)

Alternative DMT modulator – Use of the IDFT

- Let us consider frequency bins l and $N-l$ and assume that $X[l] = X^*[N-l] = s_l$ and $X[i] = 0$ for $i \neq l, N-l$. Then the IDFT may be written as

$$\begin{aligned} Nx[n] &= X[l]e^{j2\pi nl/N} + X[N-l]e^{j2\pi n(N-l)/N} \\ &= s_l e^{j2\pi nl/N} + s_l^* e^{-j2\pi nl/N} \\ &= 2\Re(s_l e^{j2\pi nl/N}) \end{aligned} \tag{10}$$

- Letting $s_l = x_l + jy_l$, (10) becomes

$$\frac{N}{2}x[n] = x_l \cos(\frac{2\pi nl}{N}) - y_l \sin(\frac{2\pi nl}{N}) \tag{11}$$

Discrete Multi-tone Modulation (DMT)

Alternative DMT modulator – Use of the IDFT

- $x[n]$ is the set of discrete samples corresponding to:

$$\frac{N}{2}x[n] = x_l \cos(\frac{2\pi nl}{N}) - y_l \sin(\frac{2\pi nl}{N}) \tag{12}$$

  and is periodic with period N/l samples.
- Using the Sampling Theorem, these time samples may be converted to the continuous time signal

$$x(t) = \frac{2}{N}x_l \cos(\frac{2\pi l}{N}f_s t) - \frac{2}{N}y_l \sin(\frac{2\pi l}{N}f_s t) \tag{13}$$

- This corresponds to modulation onto the orthogonal carriers.

Discrete Multi-tone Modulation (DMT)

Alternative DMT modulator – Use of the IDFT

- Note that in the above case, each digital value (i.e. x_l and y_l) is modulated onto carriers at $f_c = \frac{l}{N} f_s$ using a rectangular pulse shape.
- Since the carriers corresponding to each value of l are orthogonal to each other, the IDFT corresponds to the simultaneous modulation onto multiple orthogonal carriers, each at a different frequency.
- Note that $l = 0, N/2$ are not used in most occasions.
- The IDFT may be implemented using the IFFT, which is a computationally efficient method of computing the IDFT.
- The steps for modulation using the IFFT are detailed in the following slide.

Multi-tone Modulation – Implementation

1. Convert the input data into constellation points, e.g. group bits into groups of 3 bits each which are mapped to their respective 8-PSK values.
2. Take the serial data and convert it into blocks with the same length as the IDFT block size (excluding dc and the highest frequency).
3. Perform IFFT operation.
4. Convert the output block from the IFFT into serial data.
5. Pass the serial data through a DAC to convert to an analogue waveform.
6. Signal transmission over a physical channel or wireless channel:
   - **DMT**: The sub-carriers are allocated around DC and the data is transmitted in baseband or at a very low carrier frequency over a physical channel (i.e. copper wire).
   - **OFDM**: is mainly used for wireless communications at radio frequencies. After the DAC a single analogue modulation operation is performed to shift the set of sub-carriers to a carrier frequency in the radio frequency spectrum.

Multi-tone Modulation – Implementation

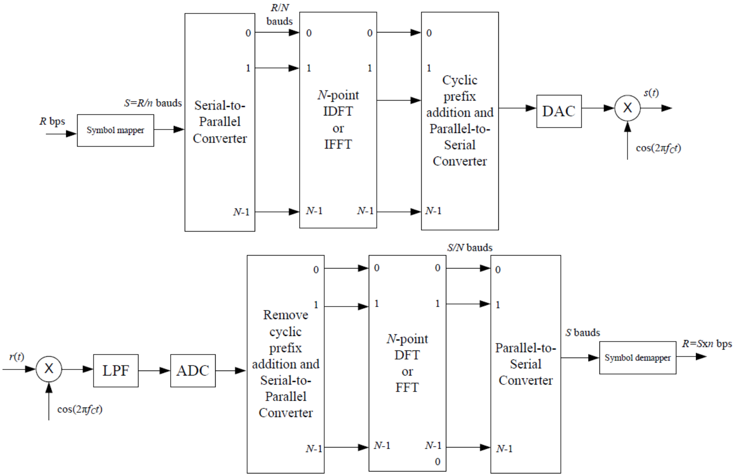


Figure: OFDM Modulation and Demodulation

Multi-tone Modulation
Cyclic Prefix

- In the figure in previous slide, we can see an important operation in OFDM/DMT modulator: the addition of a cyclic prefix.
- In Lecture 3 it was seen that a dispersive channel is a channel which spreads the transmitted symbols in time causing ISI.
  - The received symbols are sum of adjacent symbols (linear filtering).
  - One ISI source is the multipath components of the transmitted signal which arrive at different time instants at the receiver.

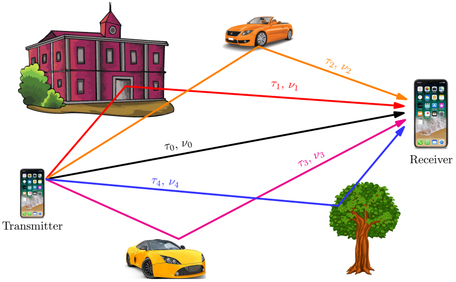


Figure: Multipath reception

Multi-tone Modulation
Cyclic Prefix

- Multi-carrier communications employ the cyclic prefix technique to fight ISI.
- Cyclic prefix addition corresponds to simple periodic extension of transmit signal.

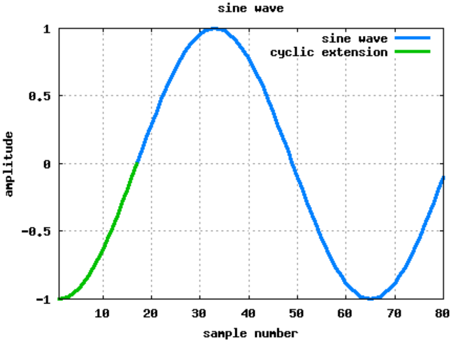


Figure: Example of cyclic extension of a periodic signal

Multi-tone Modulation

Cyclic Prefix

- In OFDM and DMT, addition of cyclic prefix is done by extracting a part of the information at the end of a symbol and adding it at the beginning. Therefore, the symbol becomes a periodic signal.
- The length of the cyclic prefix must be greater than the overall time-dispersion of the channel.
  - Otherwise, the received signal will suffer from inter-symbol interference.
- If $S(f)$ is the Fourier transform of a periodic transmit signal, sum of K delayed versions of this periodic signal, each one of them with a particular delay τ_k, is equal to:

$$\hat{S}(f) = \left(\sum_{k=1}^{K} e^{-j2\pi f \tau_k} \right) S(f) \tag{14}$$

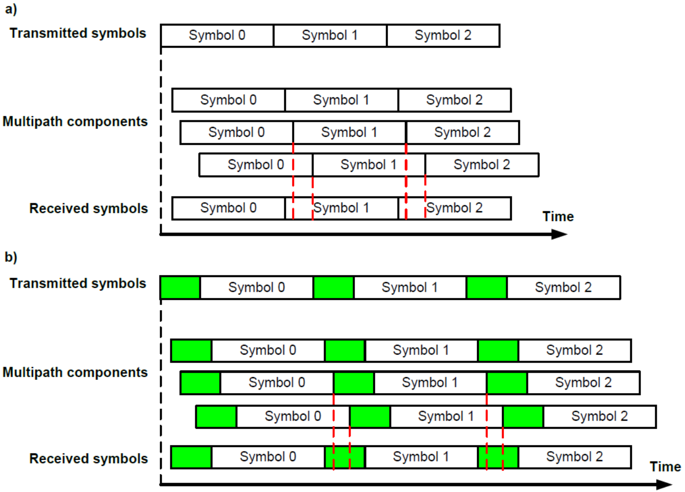
Multi-tone Modulation
Cyclic Prefix

- In other words, $\hat{S}(f)$ is a scaled/rotated version of the original signal $S(f)$ with no distortion.
- Therefore, the addition of the cyclic prefix allows the symbol information recovery without distortion of the data.
- The election of the cyclic prefix length is based on the channel delay spread.
- The benefits of the cyclic prefix come at a cost:
  - The signal overhead that is imposed by the cyclic prefix reduces the effective information data throughput of the system. The longer the cyclic prefix, the smaller the effective data transmission rate.
  - As an example, in the following figure, the reception of the three symbols occurs earlier in time when no cyclic prefix is used than when the cyclic prefix is used.

Multi-tone Modulation
Cyclic Prefix

a) OFDM transmission with no cyclic prefix and ISI in the receiver

b) OFDM transmission with cyclic prefix and no ISI in the receiver



Multi-tone Modulation
Cyclic Prefix

- Another main reason for the use of OFDM communication systems is that thanks to the reduced bandwidth used by each of the sub-carriers, it is more probable for them to experience flat fading channels instead of frequency-selective fading channel responses.
- We will go back to this when we study specifically the effects of the wireless channel.