EE4C5 Digital Signal Processing

Lecture 9 – IIR Filter Design Methods

This lecture

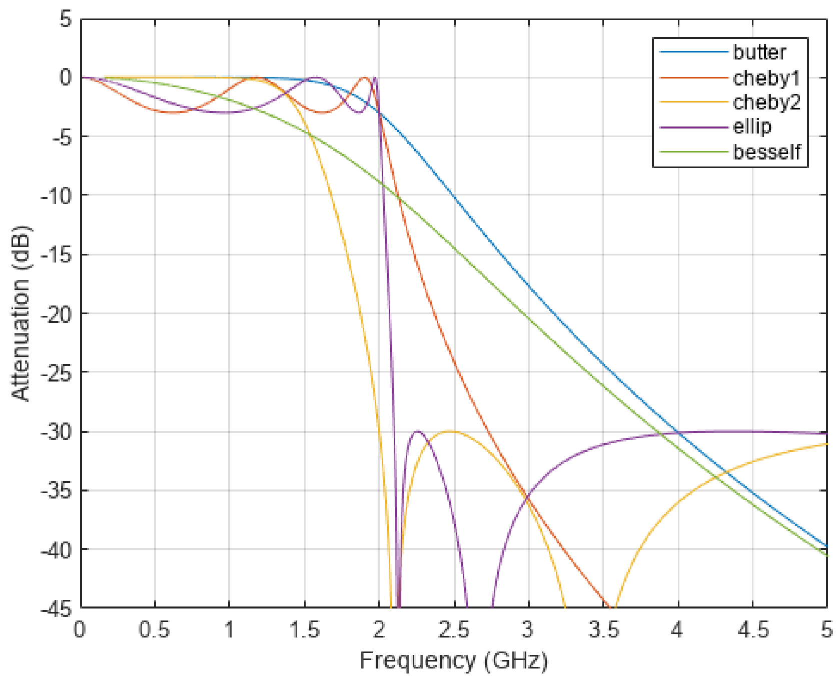
- Based on parts of Chapter 7 of O&S
- All images from O&S book unless otherwise stated

Compare FIR and IIR filters

Property	FIR	IIR
$h[n]$	Finite	Infinite
Stability	Inherently stable – all zeros	Depends on poles
Implementation	convolution	recursion
Output	Depends on current and previous inputs	Depends on current and previous inputs AND past outputs
Phase	Can have linear phase	Difficult to control phase - distortion
#Coefficients	Requires more coefficients	Fewer coefficients – faster, less memory
Causality	Can be made causal	Usually non-causal

Filter families

- **Butterworth Filters**
  - frequency response no ripples in the passband and the stopband -a maximally flat filter.
  - smooth, monotonically decreasing frequency response in transition region.
- **Chebyshev Filters**
  - narrower transition range than Butterworth for same filter - passband with more ripples.
  - equiripple magnitude response in passband, monotonically decreasing magnitude response in the stopband, and sharper roll-off in the transition region (compared to Butterworth of same order)
  - best approximation to the ideal response of any filter for a specified order and ripple
- **Elliptic Filters**
  - equiripple in both passband and stopband
  - For same filter order, fastest transition in gain between the passband and the stopband
- **Bessel Filters**
  - similar frequency response to the Butterworth - smooth in the passband and in the stopband.
  - For same filter order, stopband attenuation much lower than Butterworth.
  - Of all filter types, has the widest transition range if filter order is fixed.



MATLAB: 5th-order analog lowpass filter with a cutoff frequency of 2 GHz, 3 dB of passband ripple, 30 dB of stopband attenuation
(source: https://uk.mathworks.com/help/signal/ug/comparison-of-analog-iir-lowpass-filters.html)

Digital IIR filters

- Can derive from analog IIR filters
- Earliest design methods for digital filters mapped continuous-time designs to discrete-time designs
  - Impulse invariance
  - Bilinear transformation
  - Gives IIR filters
- Benefit from highly advanced methods developed for analog
- More interest in methods to design FIR filters directly once digital expanded
  - Windowing
  - Iterative algorithms e.g. Parks-McClellan

Digital versus discrete?

- Note - more exactly discrete-time here
- But we use term “digital” filter
- Refer to earlier discussion!

Impulse Invariance

Recall the design steps

- Specification of properties
  - We'll assume these desired specifications are in terms of the discrete-time frequency variable ω.
- Approximation of the specifications using a causal discrete-time system.
- Realisation of the system.

Preserve Continuous Time properties

- Essential properties of the continuous-time frequency response preserved in the frequency response of resulting discrete-time filter
  - => the imaginary axis of the s-plane to map onto the unit circle of the z -plane.
- Stable continuous-time filter should be transformed to a stable discrete-time filter
  - => if the continuous-time system has poles only in the left half of the s-plane, then the discrete-time filter must have poles only inside the unit circle in the z-plane.

Impulse Invariance

- Discrete-time system defined by sampling impulse response of continuous-time system.
- Impulse response of discrete-time filter chosen proportional to equally spaced samples of impulse response of the continuous-time filter:
  - $h[n] = T_d h_c(nT_d)$
  - With T_d the sampling interval – distinct from sampling period!
- Relationship*:
  - $H(e^{j\omega}) = \frac{1}{T_d}\sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} - \frac{2\pi k}{T_d}\right)\right)$
  - *see lecture 5

Impulse Invariance…

- Consider that the continuous-time filter is bandlimited:
  - $H_c(j\Omega) = 0, \quad |\Omega| \geq {}^{\pi}/_{T_d}$
- Then:
  - $H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), \quad |\omega| \leq \pi$
- Discrete-time and continuous-time frequency responses related by a linear scaling of the frequency axis:
  - $\omega = \Omega T_d$ for $|\omega| \leq \pi$
- Any practical continuous-time filter cannot be exactly bandlimited
  - => interference between successive terms i.e. aliasing

Aliasing

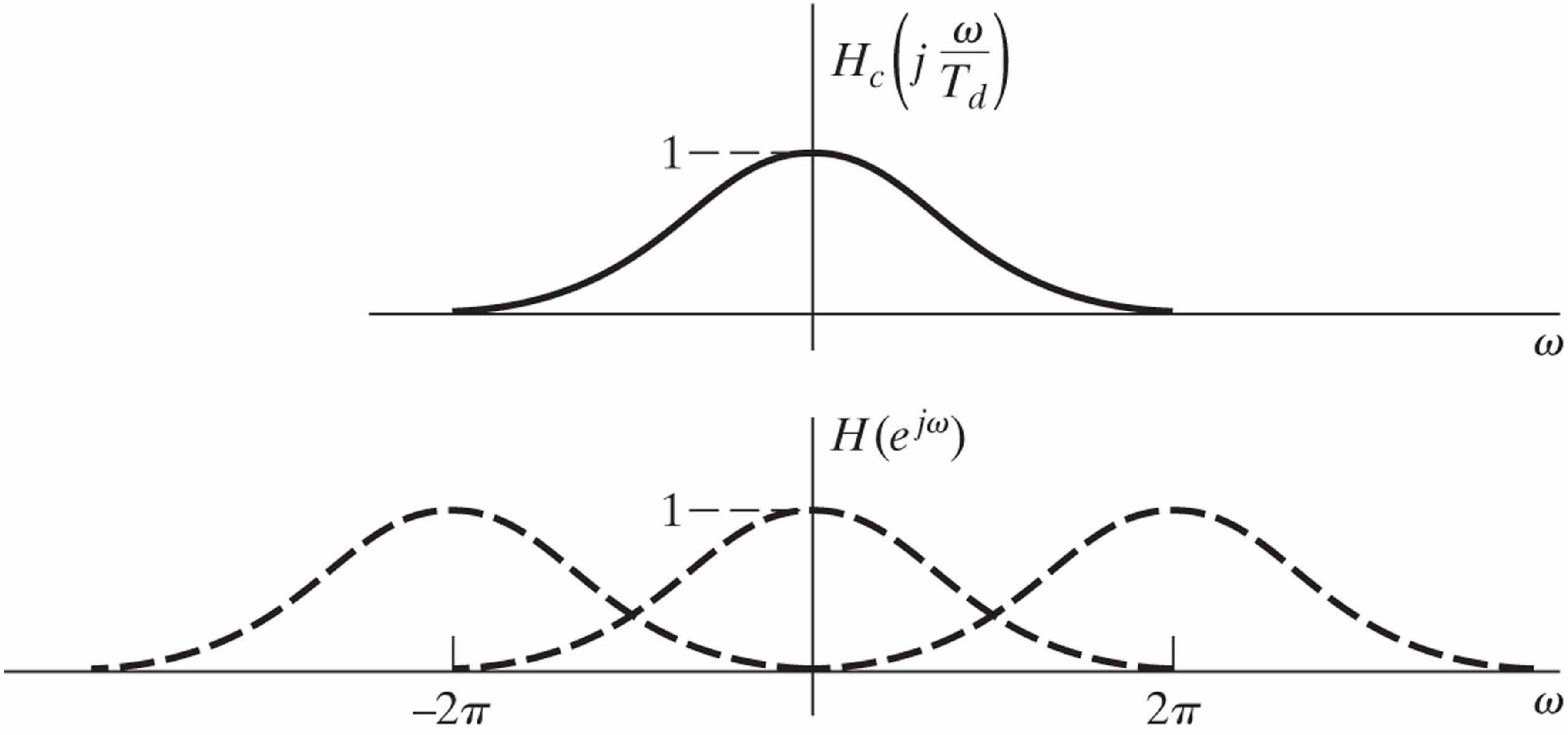


Figure 7.3 Illustration of aliasing in the impulse invariance design technique.

S-plane -> Z-plane

- Defined impulse invariance transformation from continuous time to discrete time in terms of time-domain sampling
- Easy to carry out the transformation from the s-domain to the z-domain:
  - by transformation of the system function $H_c(s)$

System function transformation

- System function for causal continuous time filter as a partial fraction expansion
- Assuming all poles are of single order
  - $H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$
- Corresponding impulse response given by:
  - $h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

System function transformation…

- The scaled and sampled impulse response is:
  - $h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n]$

    $= \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n]$
- Hence the system function of the discrete-time filter is:
  - $H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$
- Note that a pole at $s = s_k$ on the s-plane is mapped to a pole $z = e^{s_k T_d}$ on the z-plane
- If the continuous-time causal filter is stable, i.e. real part of s_k is < 0, then the magnitude of $e^{s_k T_d}$ will be less than unity,
  - =>corresponding pole in the discrete-time filter is inside the unit circle and filter is also stable

Impulse Invariance

- Technique appropriate only for bandlimited filters
- Aliasing distortion issues e.g. highpass or bandstop continuous-time filters would require additional bandlimiting
- This motivates use of bilinear transformation as alternative design method

Bilinear Transformation

Bilinear Transformation

- An algebraic transformation between the variables s and z that maps the entire $j\Omega$ -axis in the s -plane to one revolution of the unit circle in the z -plane.

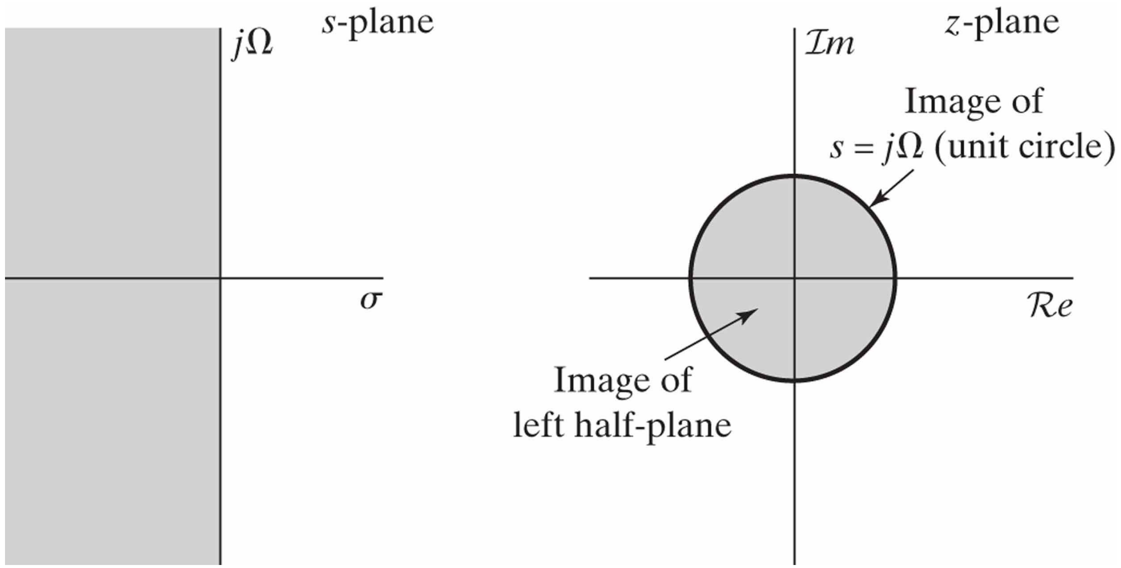


Figure 7.6 Mapping of the *s*-plane onto the *z* -plane using the bilinear transformation.

Bilinear Transformation

- An algebraic transformation between the variables s and z that maps the entire $j\Omega$ -axis in the s -plane to one revolution of the unit circle in the z -plane.
- Non-linear

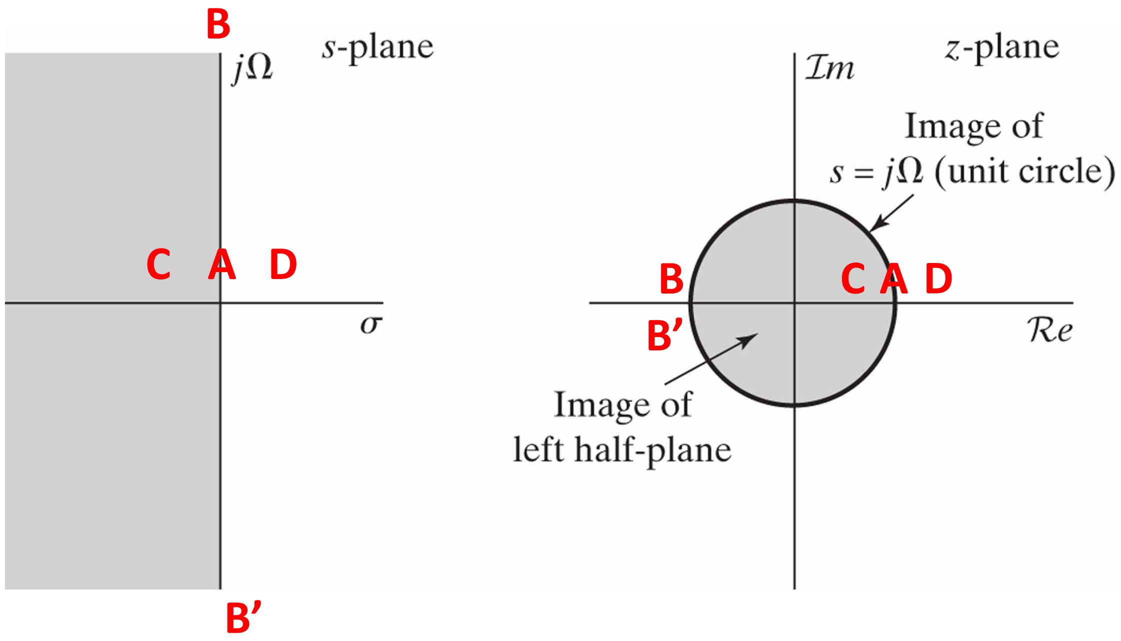


Figure 7.6 Mapping of the *s*-plane onto the *z* -plane using the bilinear transformation.

Definition

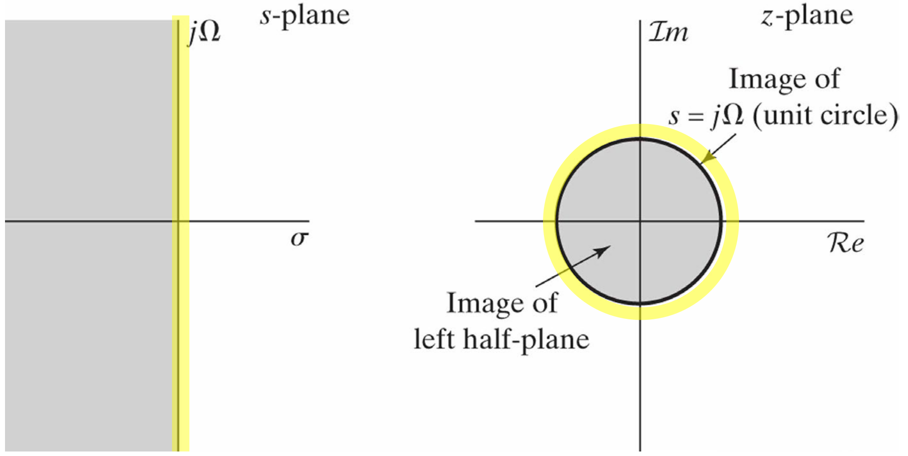
- Continuous-time system function $H_c(s)$
- Discrete-time transfer function $H(z)$
- Bilinear transform involves mapping s as:
  - $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$
- And equivalently:
  - $H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$
- (note that T_d disappears in the calculations later and is of no consequence. Same occurs in impulse invariance. So often ignore or set to 1 in reality)

Explore…

- Solving $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ for z:
  - $z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$
- Then substitute in $s = \sigma + j\Omega$ to give:
  - $z = \frac{1+\sigma T_d/2+j\Omega T_d/2}{1-\sigma T_d/2-j\Omega T_d/2}$
- Any $\sigma < 0$ (i.e. pole in LHS of s-plane) will map to $|z| < 1$ for any Ω (i.e. will fall inside unit circle)
- => Causal stable continuous-time filters map into causal stable discrete-time filters
- Any $\sigma > 0$ (i.e. pole in RHS of s-plane) will map to $|z| > 1$ for any Ω (i.e. will fall outside unit circle)

Unit circle?

- Show that $j\Omega$-axis of the s-plane maps onto the unit circle?
- Substitute in $s = j\Omega$ to give:
  - $z = \frac{1+j\Omega T_d/2}{1-j\Omega T_d/2}$
- Taking $|z|$ can see it will be $= 1$ for all values of s on the $j\Omega$ axis
  - => $j\Omega$ axis maps onto the unit circle in the z-plane



Mapping of frequency

- Relationship between Ω and ω?
- Examine $z = e^{j\omega}$. Sub in and yields:
  - $s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)$
  - $s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin(\omega/2))}{2e^{-j\omega/2}(\cos(\omega/2))}\right] = \frac{2j}{T_d}\tan(\omega/2)$
- Equate real and im parts
  - $\Omega = \frac{2}{T_d}\tan(\omega/2)$ or $\omega = 2\arctan(\Omega T_d/2)$

Mapping of frequency

Note the non-linear mapping

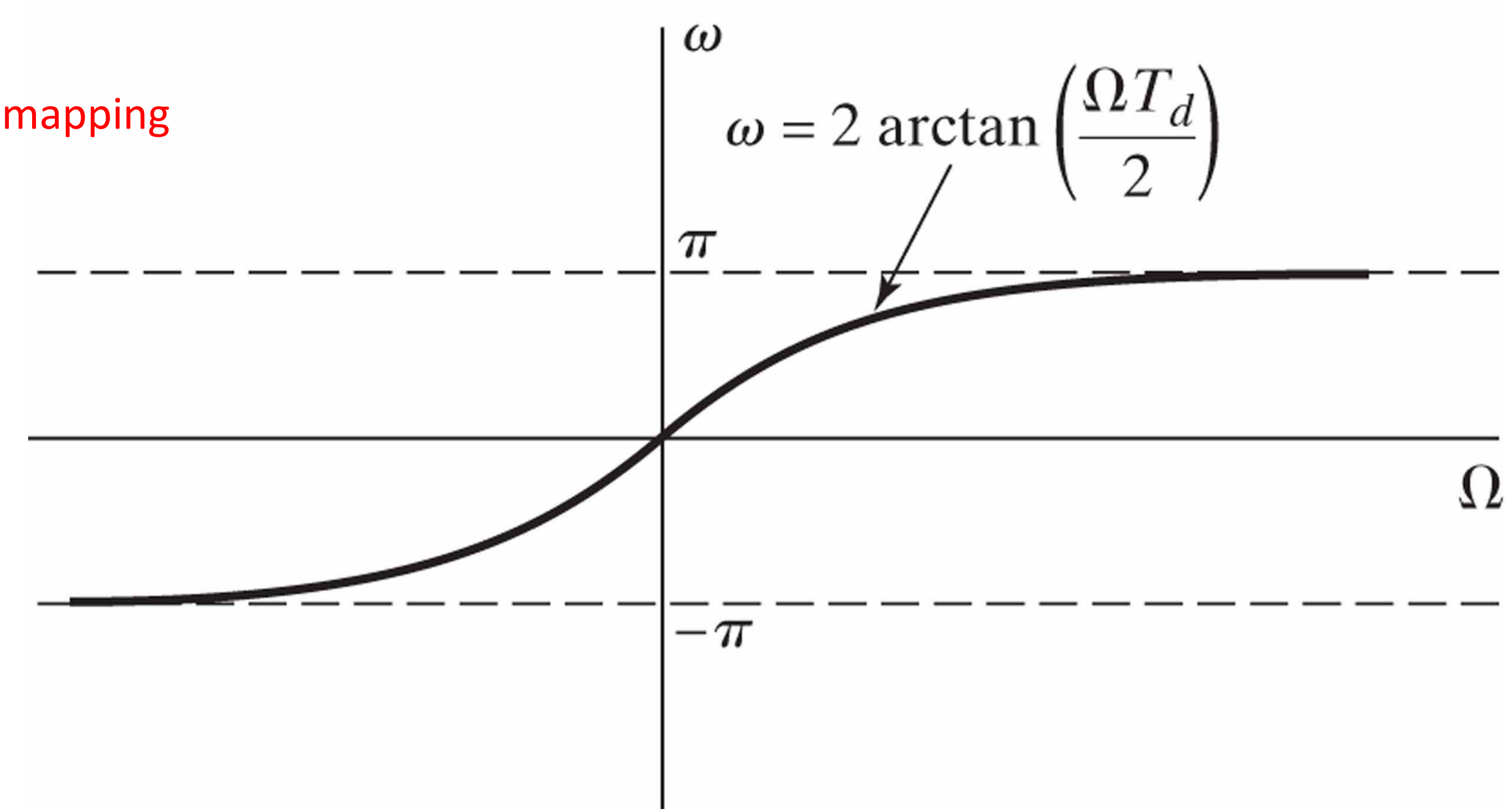


Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

Using the bilinear transformation method

- Step 1: Convert each specified edge-band (transition region) frequency of the desired digital filter to a corresponding edge-band frequency of an analog filter
- Step 2: Design an analog filter $H(s)$ of the desired type, according to the transformed specifications
- Step 3:Transform the analog filter $H(s)$ to a digital filter $H(z)$ using the bilinear transform

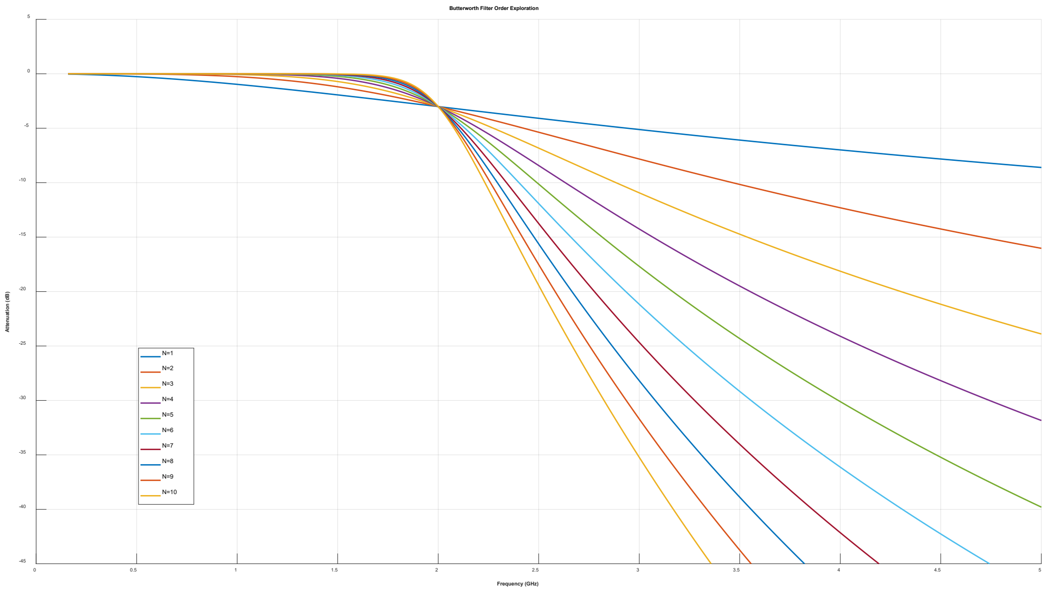
Worked examples

- Will be done in lectures next week

Butterworth Filters

Butterworth filter

- Characterized by a magnitude response that is maximally flat in the passband and monotonic overall.
- Squared magnitude response function is:
  - $|H_c(j\Omega)|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$
- With N the filter order, Ω_c the 3dB cut-off frequency



Butterworth filter

- To obtain the transfer function, use $s = j\Omega$ in the magnitude squared frequency response as
  - $H_c(s)\, H_c(-s) = \frac{1}{1+\left({s}/{\Omega_c}\right)^{2N}} = \frac{1}{1+(-1)^2(s/\Omega_c)^{2N}}$
- The $2N$ poles of this function are:
  - $s_k = \Omega_c e^{j\frac{(N+1+2k)\pi}{2N}}\ , 0 \le k \le 2\mathrm{N}-1$
- Poles uniformly distributed in angle of a circle of radius Ω_c
- Poles on LHS relate to $H_c(s)$

What’s actually happening…

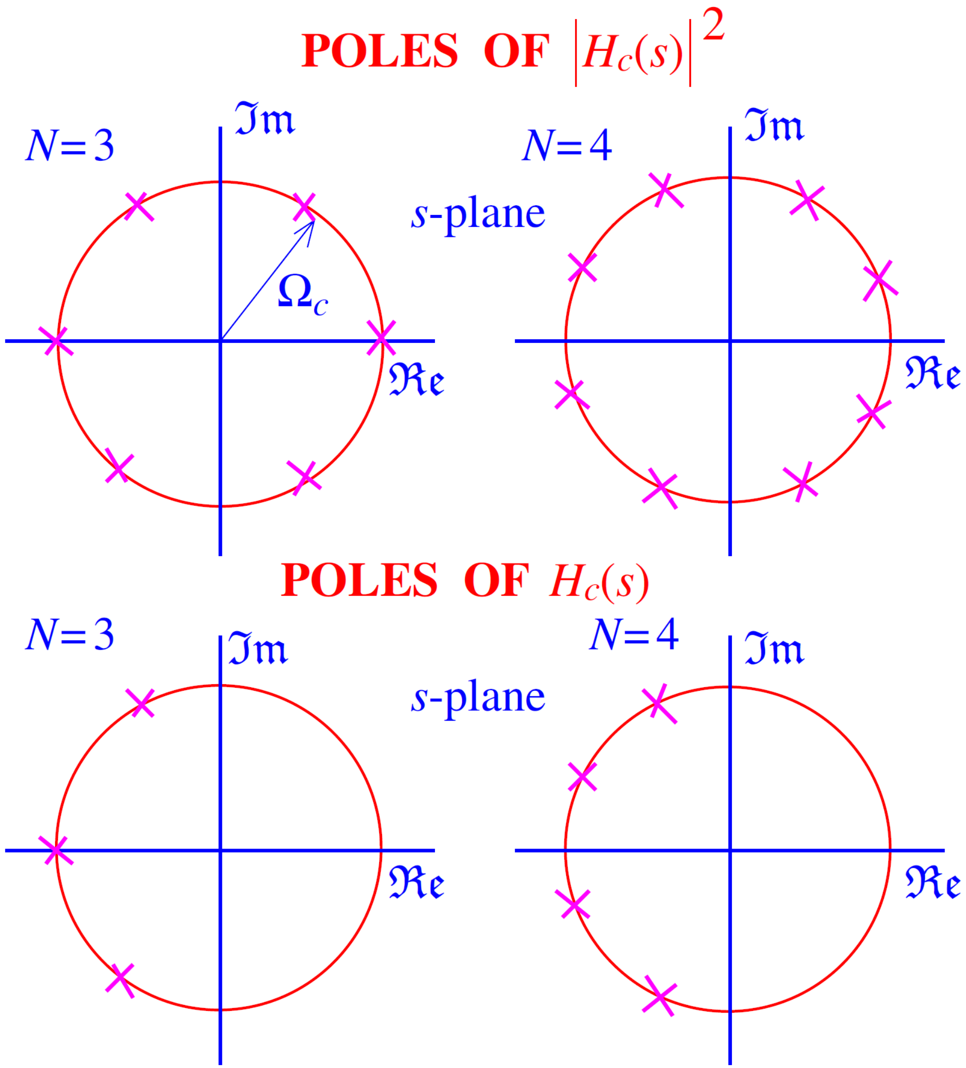


Diagram: Ian Bruce, McMaster

Prototype responses

N	$Q(s)$
1	$s+1$
2	$s^2+\sqrt{2s}+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+\sqrt{2s}+1)(s^2+1.9318s+1)$
7	$(s+1)(s^2+0.4450s+1)(s^2+1.2456s+1)(s^2+1.8022s+1)$
8	$(s^2+0.3986s+1)(s^2+1.1110s+1)(s^2+1.6630s+1)(s^2+1.9622\,s+1)$

Butterworth polynomials in factored form

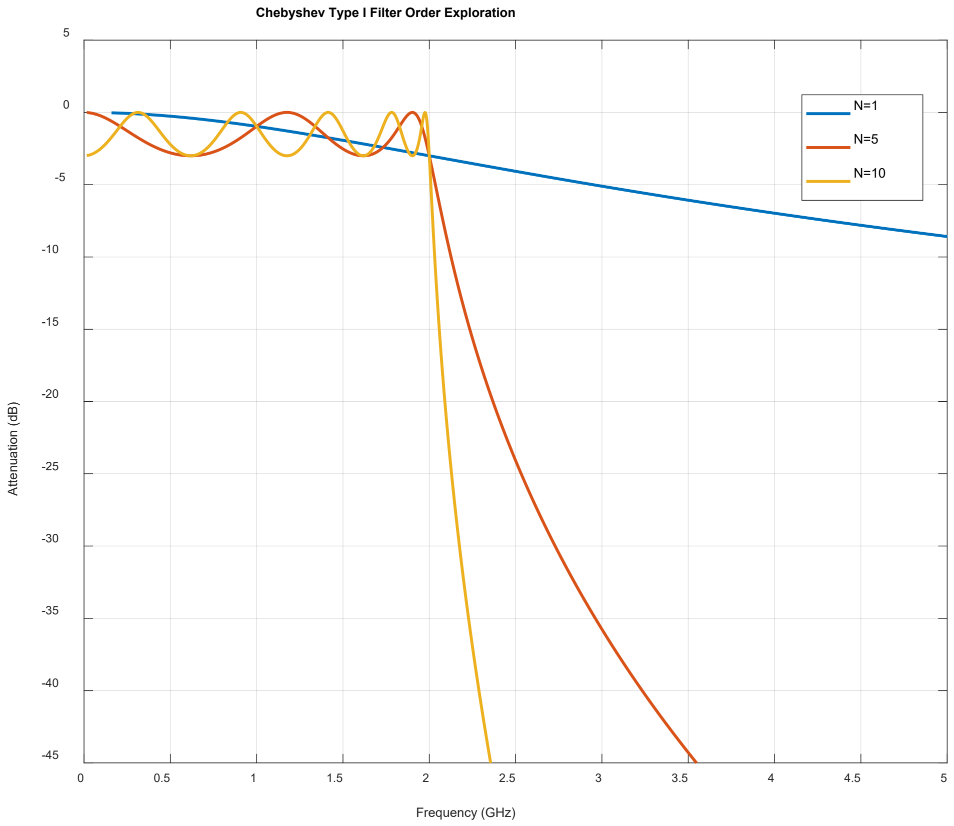
$$H_p(s) = \frac{1}{Q(s)}$$

Table adapted from: https://www.eeguide.com/butterworth-polynomials/

Chebyshev Filters

Chebyshev Type 1

- Lowpass Chebyshev Type I filter response:
  - $|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2\left({}^{\Omega}/_{\Omega_c}\right)}$
  - With ϵ ripple factor
  - T_N the Chebyshev polynomial of order N
  - $T_N(x) = \begin{cases} \cos(N \arccos x), |x| \leq 1 \\ \cosh(N \operatorname{arccosh} x), |x| > 1 \end{cases}$



Required Reading & other material

- Oppenheim & Schafer, Chapter 7
- Helpful video: https://www.youtube.com/watch?v=5RLMpdbt6B0&t=2s