Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

FACULTY OF ENGINEERING, MATHEMATICS & SCIENCE

SCHOOL OF ENGINEERING

Electronic & Electrical Engineering

Engineering
Senior Sophister
Annual Examinations

Hilary Term, 2016

Digital Signal Processing (4C5)

5th January 2016 **Venue: Exam Hall** **Time: 14.00 – 16.00**

Dr. W. Dowling

Instructions to Candidates:

Answer THREE questions. All questions carry equal marks.

Materials permitted for this examination:

Mathematical Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.

Please indicate the make and model of your calculator on each answer book used.

Q.1 (a) A continuous-time signal $x_a(t)$ has the Fourier transform $X_a(j\omega)$. The discrete-time signal $x[n]$ is derived from $x_a(t)$ by periodic sampling:

$$x[n] = x_a(nT), \text{where } T \text{ is a constant.}$$

Let $X(e^{j\Omega})$ denote the discrete-time Fourier transform of $x[n]$. Show that

$$X(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\frac{\Omega}{T} - \frac{2\pi k}{T}\right)\right).$$

[11 marks]

(b) A system for sampling rate reduction by a factor of 1.5 is shown in Fig. Q1-1.

$$x[n] \longrightarrow \boxed{\uparrow 2} \xrightarrow{r[n]} \boxed{h[n]} \xrightarrow{w[n]} \boxed{\downarrow 3} \longrightarrow y[n]$$

Fig. Q1-1

$$r[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \dots, \\ 0, & \text{otherwise} \end{cases}$$

$$w[n] = \sum_{k=-\infty}^{\infty} r[k]h[n-k]$$

$$y[n] = w[3n]$$

The ideal discrete-time low-pass filter has a unit sample response, $h[n]$, and a frequency response, $H(e^{j\Omega})$, given by

$$H(e^{j\Omega}) = \begin{cases} 2, & |\Omega| < \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |\Omega| \le \pi \end{cases}$$

Let $R(e^{j\Omega})$ and $Y(e^{j\theta})$ denote the discrete-time Fourier transforms of the sequences $r[n]$ and $y[n]$ respectively. A continuous-time signal $x_a(t)$ has the Fourier transform $X_a(j\omega)$ shown in Fig. Q1-2.

continued ...

[Q.1 ctd.]

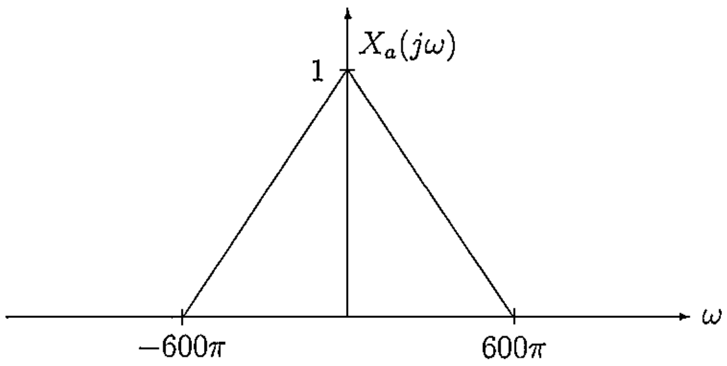

Fig. Q1-2

If $x[n] = x_a(nT)$, and the sampling period $T = 1$ millisecond,

(i) sketch $R(e^{j\Omega})$ for $-\pi \leq \Omega \leq \pi$, and **[5 marks]**

(ii) sketch $Y(e^{j\theta})$ for $-\pi \leq \theta \leq \pi$. **[4 marks]**

Q.2 (a) The sequence $x[n]$ is zero for $n < 0$ and for $n > N - 1$. Assume that $N = 2^M$, where M is a positive integer. Let $g[n] = x[2n]$, and $h[n] = x[2n + 1]$.
Show that the N-point discrete Fourier transform (DFT) of the sequence $x[n]$ can be obtained by appropriately combining the $N/2$-point DFTs of the sequences $g[n]$ and $h[n]$.

[8 marks]

(b) Draw the complete signal flow graph for an 8-point decimation-in-time fast Fourier transform (FFT) algorithm. **[12 marks]**

Q.3 (a) Let $H_c(s)$ denote the transfer function of a continuous-time filter. The transfer function of a discrete-time filter, $H(z)$, is obtained by applying the bilinear transformation to $H_c(s)$:

$$H(z) = H_c(s)\Big|_{s = (1-z^{-1})/(1+z^{-1})}$$

Show that the frequency responses of the discrete-time and continuous-time filters are related by

$$H\left(e^{j\Omega}\right) = H_c(j\omega)\Big|_{\omega = \tan(\Omega/2)}$$

[8 marks]

(b) A discrete-time low-pass filter with frequency response, $H\left(e^{j\Omega}\right)$, is to be designed to meet the following specifications:

$$\begin{aligned} 0.89 \le \left|H\left(e^{j\Omega}\right)\right| \le 1, &\qquad |\Omega| \le 0.2\pi \\ \left|H\left(e^{j\Omega}\right)\right| \le 0.18, &\qquad 0.6\pi \le |\Omega| \le \pi \end{aligned}$$

The filter is to be designed by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter.
Verify that a second order filter is sufficient to meet the specifications.
Determine the transfer function, $H(z)$, of the discrete-time filter.
Note that the transfer function of a second order Butterworth low-pass prototype filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

[12 marks]

Q.4 (a) Using a rectangular window sequence, design a causal, 15-point, discrete-time generalised linear phase filter with a magnitude response which approximates the ideal band-pass response, $\left|H_{id}\left(e^{j\Omega}\right)\right|$, given by

$$\left|H_{id}\left(e^{j\Omega}\right)\right| = \begin{cases} 0, & |\Omega| < \frac{\pi}{3} \\ 1, & \frac{\pi}{3} < |\Omega| < \frac{2\pi}{3} \\ 0, & \frac{2\pi}{3} < |\Omega| \leq \pi \end{cases}$$

[12 marks]

(b) Let $X\left(e^{j\Omega}\right)$ denote the discrete-time Fourier transform of the real sequence $x[n]$. If $r[n] = x[-n]$, show that $R\left(e^{j\Omega}\right)$, the discrete-time Fourier transform of $r[n]$, is given by

$$R\left(e^{j\Omega}\right) = X^{*}(e^{j\Omega})$$

where * denotes complex conjugation. **[2 marks]**

(c) Let $h[n]$ be the unit-sample response of a causal filter with an arbitrary phase characteristic. Assume that $h[n]$ is real and denote its Fourier transform by $H\left(e^{j\Omega}\right)$. Let $x[n]$ be a real finite duration sequence. The sequence $x[n]$ is first filtered to get $g[n]$:

$$g[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The sequence $r[n] = g[-n]$ is then filtered to get $w[n]$:

$$w[n] = \sum_{k=-\infty}^{\infty} h[k]r[n-k]$$

The sequence $y[n] = w[-n]$. Let $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$ denote the discrete-time Fourier transform of $x[n]$ and $y[n]$ respectively. Show that

$$Y\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right)\left|H\left(e^{j\Omega}\right)\right|^2$$

[6 marks]

Q.5 (a) Let $x[n]$ denote a finite-duration sequence of length M such that $x[n] = 0$ for $n < 0$ and $n \geq M$. Let $X(z)$ denote the z-transform of $x[n]$. If we sample $X(z)$ at $z = e^{j(2\pi/N)k}$, $\quad k = 0, 1, 2, \ldots, N-1$, we obtain

$$X_1[k] = X(z)\,|_{z=e^{j(2\pi/N)k}}\ , \quad k = 0, 1, 2, \ldots, N-1.$$

The number of samples N is *less than* the duration of the sequence M; i.e. $N < M$. The sequence $x_1[n]$ is obtained as the inverse DFT of $X_1[k]$. Determine the relation between $x_1[n]$ and $x[n]$. **[12 marks]**

(b) Consider a finite-duration sequence $x[n]$ of length M such that $x[n] = 0$ for $n < 0$ and $n \geq M$. We want to compute samples of the discrete-time Fourier transform of $x[n]$ at the N equally spaced frequencies

$$\Omega_k = \frac{2\pi k}{N}, \quad k = 0, 1, \ldots, N-1.$$

Determine and justify procedures for computing the N samples of the discrete-time Fourier transform using only one N-point DFT for the following cases: (i) $N > M$; and (ii) $N < M$.

[8 marks]