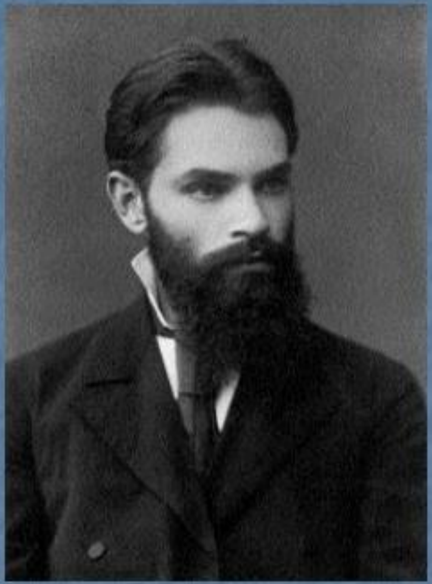
Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Dynamical Systems
Lecture 5.03

EEU45C09 / EEP55C09
Self Organising Technological Networks

Nicola Marchetti
nicola.marchetti@tcd.ie

Lyapunov A. M. (1857-1918)



Alexander Lyapunov was born 6 June 1857 in Yaroslavl, Russia in the family of the famous astronomer **M.V. Lyapunov who** played a great role in the education of Alexander and Sergey.

Aleksandr Lyapunov was a school friend of Markov and later a student of Chebyshev at Physics & Mathematics department of Petersburg University which he entered in **1876. He** attended the chemistry lectures of D.Mendeleev.

In 1885 he brilliantly defends his MSc diploma “On the equilibrium shape of rotating liquids”, which attracted the attention of physicists, mathematicians and astronomers of the world.

The same year he starts to work in Kharkov University at the Department of Mechanics. He gives lectures on Theoretical Mechanics, ODE, Probability.

In 1892 defends PhD. In 1902 was elected to Science Academy.

What is “chaos”?

Chaos – is a *aperiodic* long-time behavior arising in a *deterministic* dynamical system that exhibits a *sensitive dependence on initial conditions.*

The nearby trajectories separate exponentially fast

Lyapunov Exponent > 0

Trajectories which do not settle down to fixed points, periodic orbits or quasiperiodic orbits as $t \to \infty$

The system has no random or noisy inputs or parameters – the irregular behavior arises from system’s **nonlinearity**

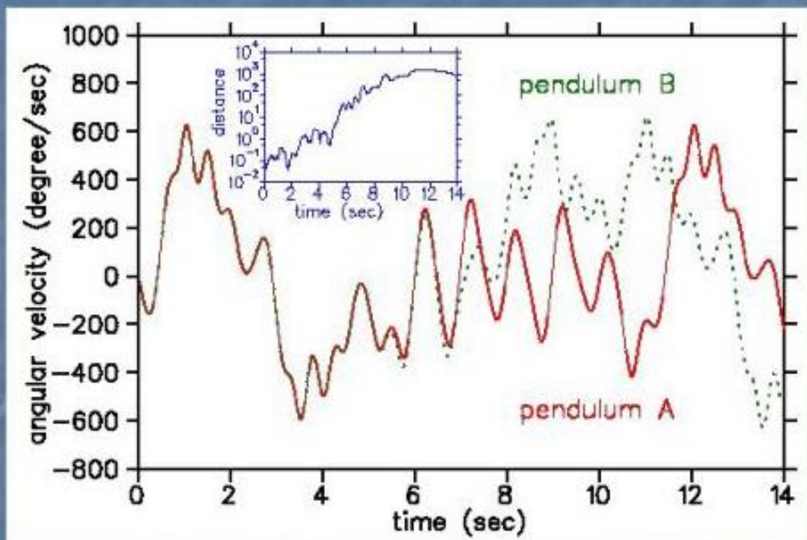
Non-wandering set

- a set of points in the phase space having the following property: All orbits starting from any point of this set come arbitrarily close and arbitrarily often to any point of the set.

- **Fixed points:** stationary solutions;
- **Limit cycles:** periodic solutions;
- **Quasiperiodic orbits:** periodic on a small scale but unpredictable at some larger scale.
- **Chaotic orbits:** bounded non-periodic solutions.

Appears only in nonlinear systems

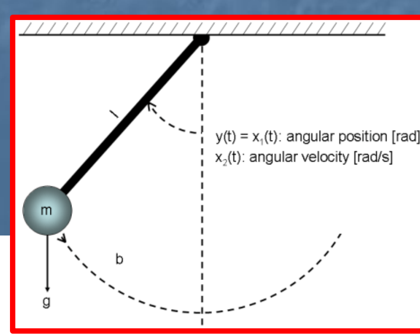
Sensitive dependence on the initial conditions



pendulum A: $\varphi = -140°$, $d\varphi/dt = 0$
pendulum B: $\varphi = -140°1'$, $d\varphi/dt = 0$

- Definition: A set S exhibits *sensitive dependence* if $\exists r>0$ s.t. $\forall \varepsilon>0$ and $\forall x \in S$ $\exists y$ s.t $|x-y|<\varepsilon$ and $|x_n-y_n|>r$ for some n.

The sensitive dependence of the trajectory on the initial conditions is a key element of deterministic chaos!



The Lyapunov Exponent

- A quantitative measure of the sensitive dependence on the initial conditions is **the Lyapunov exponent λ.** It is the averaged rate of divergence (or convergence) of two neighboring trajectories in the phase space.
- Actually there is a whole spectrum of Lyapunov exponents. Their number is equal to the dimension of the phase space. If one speaks about *the* Lyapunov exponent, the largest one is meant.

Definition of Lyapunov Exponents

- Given a continuous dynamical system in an *n*-dimensional phase space, we monitor the long-term evolution of an *infinitesimal n*-sphere of initial conditions.
- The sphere will become an n-ellipsoid due to the locally deforming nature of the flow.
- The *i*-th one-dimensional Lyapunov exponent is then defined as following:

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}}$$

- Order: $\lambda_1 > \lambda_2 > \ldots > \lambda_n$
- The linear extent of the ellipsoid grows as $2^{\lambda_1 t}$
- The area defined by the first 2 principal axes grows as $2^{(\lambda_1+\lambda_2)t}$
- The volume defined by the first 3 principal axes grows as $2^{(\lambda_1+\lambda_2+\lambda_3)t}$ and so on…
- The sum of the first *j* exponents is defined by the long-term exponential growth rate of a *j*-volume element.

DS : Dynamical System

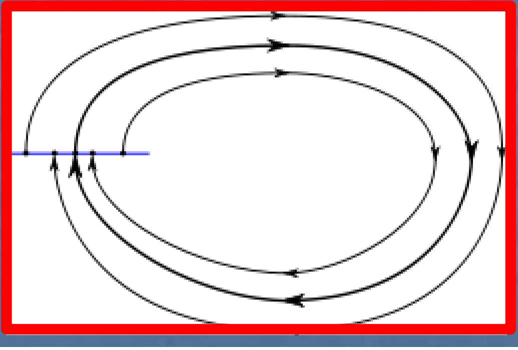
Signs of the Lyapunov exponents

- Any continuous time-dependent DS without a fixed point will have ≥ 1 zero exponents.
- The sum of the Lyapunov exponents must be negative in dissipative DS $\Rightarrow \exists$ at least one negative Lyapunov exponent.
- A positive Lyapunov exponent reflects a "direction" of *stretching* and *folding* and therefore determines chaos in the system.

The signs of the Lyapunov exponents provide a qualitative picture of a system's dynamics

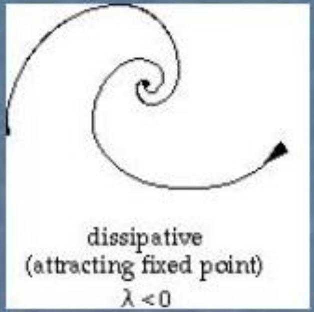
- 1D maps: $\exists!\ \lambda_1=\lambda$:
  - $\lambda=0$ – a marginally stable orbit;
  - $\lambda<0$ – a periodic orbit or a fixed point;
  - $\lambda>0$ – chaos.

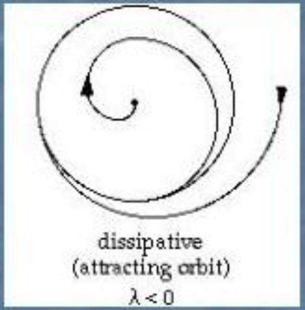
- 3D continuous dissipative DS: $(\lambda_1,\lambda_2,\lambda_3)$
  - (+,0,-) – a strange attractor;
  - (0,0,-) – a two-torus;
  - (0,-,-) – a limit cycle;
  - (-,-,-) – a fixed point.



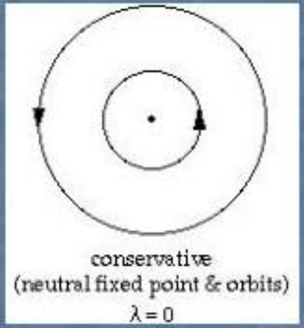
The sign of the Lyapunov Exponent

- **$\lambda<0$** - the system attracts to a fixed point or stable periodic orbit. These systems are non conservative (dissipative) and exhibit *asymptotic stability*.





- **$\lambda=0$** - the system is neutrally stable. Such systems are conservative and in a steady state mode. They exhibit *Lyapunov stability*.



- **$\lambda>0$** - the system is chaotic and unstable. Nearby points will diverge irrespective of how close they are.

Computation of Lyapunov Exponents

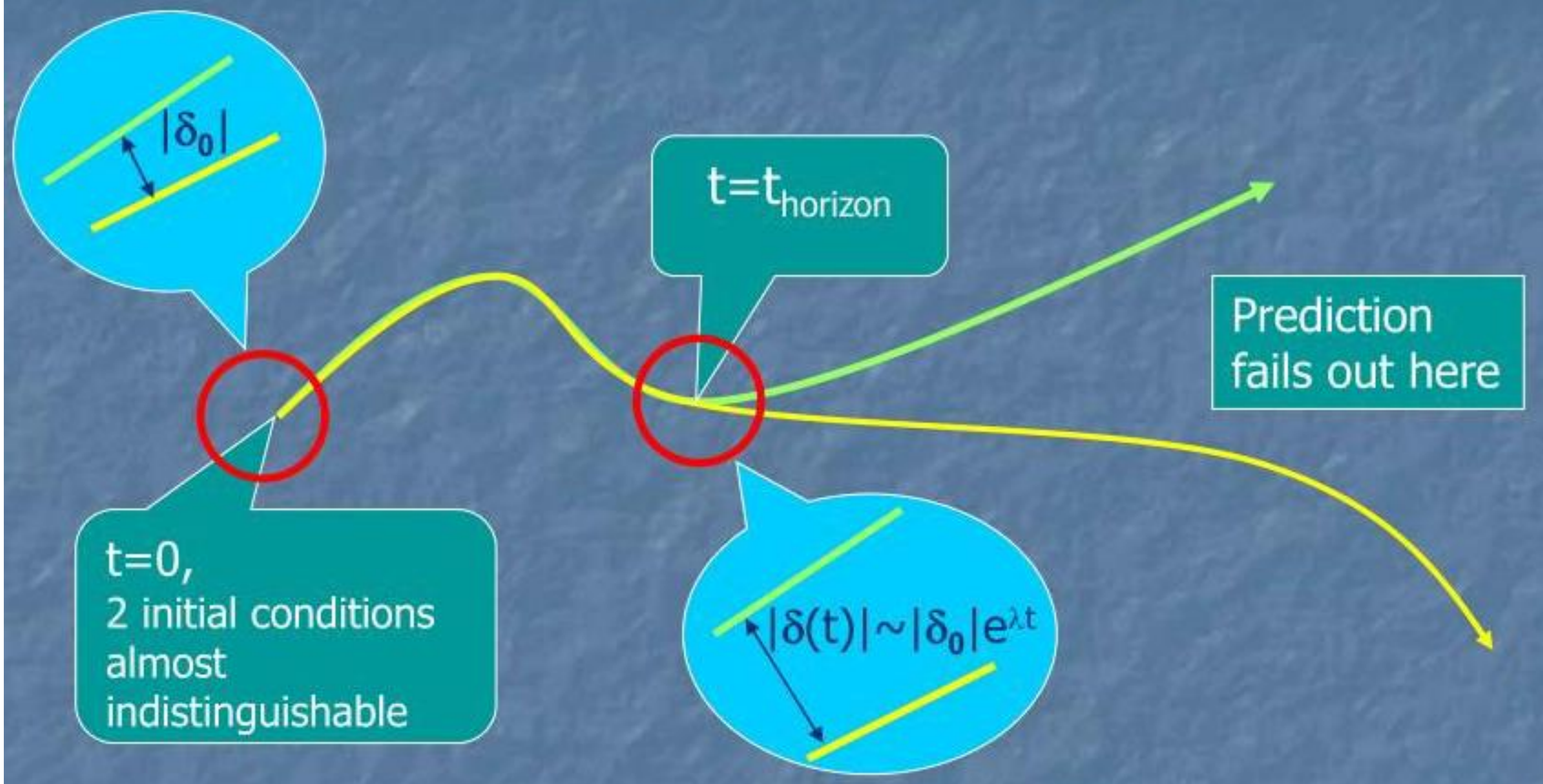
- Obtaining the Lyapunov exponents from a system with known differential equations is no real problem and was dealt with by Wolf.
- In most real world situations we do not know the differential equations and so we must calculate the exponents from a time series of experimental data. Extracting exponents from a time series is a complex problem and requires care in its application and the interpretation of its results.

Time Horizon

- When the system has a positive λ, there is a time horizon beyond which prediction breaks down
- Suppose we measure the initial condition of an experimental system very accurately, but no measurement is perfect – let $|\delta_0|$ be the error.
- After time t: $|\delta(t)| \sim |\delta_0| e^{\lambda t}$, if a is our tolerance, then our prediction becomes intolerable when $|\delta(t)| \geq a$ and this occurs after a time

$$t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{||\delta_0||}\right)$$

Demonstration



- No matter how hard we work to reduce measurement error, we cannot predict longer than a few multiples of $1/\lambda$.

Example

Let $a = 10^{-3}$, initial error $||\delta_0|| = 10^{-7}$ then

$$t_{horizon} \approx \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-7}} = \frac{4 \ln 10}{\lambda}$$

If we improve initial error to $||\delta_0|| = 10^{-13}$ then

$$t_{horizon} \approx \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-13}} = \frac{10 \ln 10}{\lambda}$$

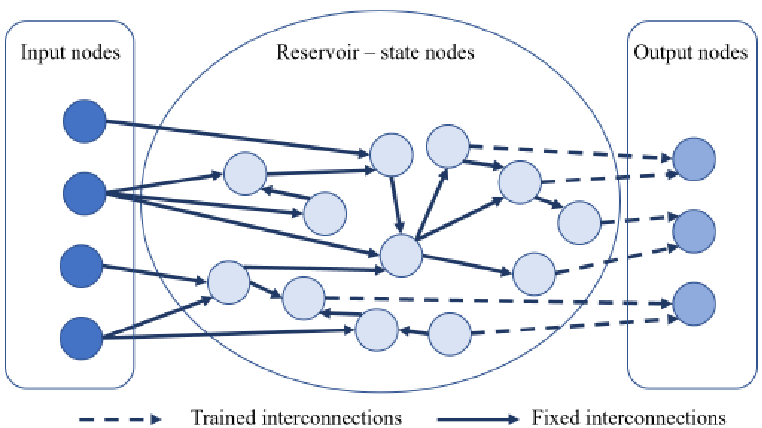
So, after a million fold improvement in the initial uncertainty, we can predict only 2.5 times longer!

Some philosophy

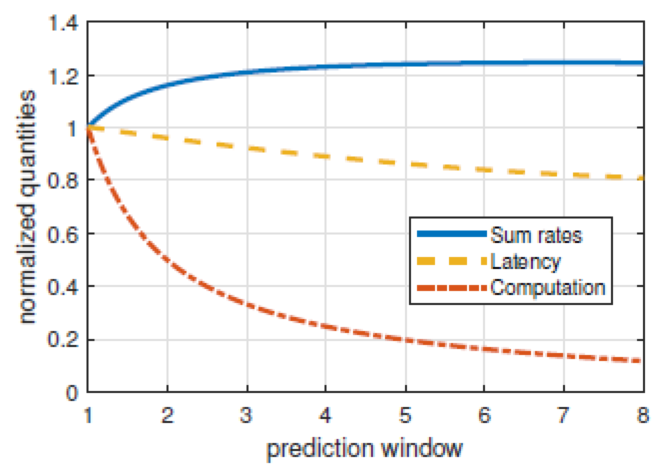
- The notion of chaos seems to conflict with that attributed to Laplace: given precise knowledge of the initial conditions, it should be possible to predict the future of the universe. However, Laplace's dictum is certainly true for any <u>deterministic</u> system.
- The main consequence of chaotic motion is that given imperfect knowledge, the predictability horizon in a deterministic system is much shorter than one might expect, due to the exponential growth of errors.

Dynamical systems, wireless comms + reservoir computing as the learning tool

Reservoir computer architecture



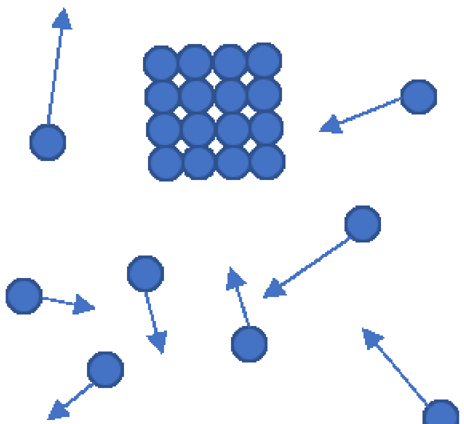
Reservoirs for massive MIMO and multi-hop

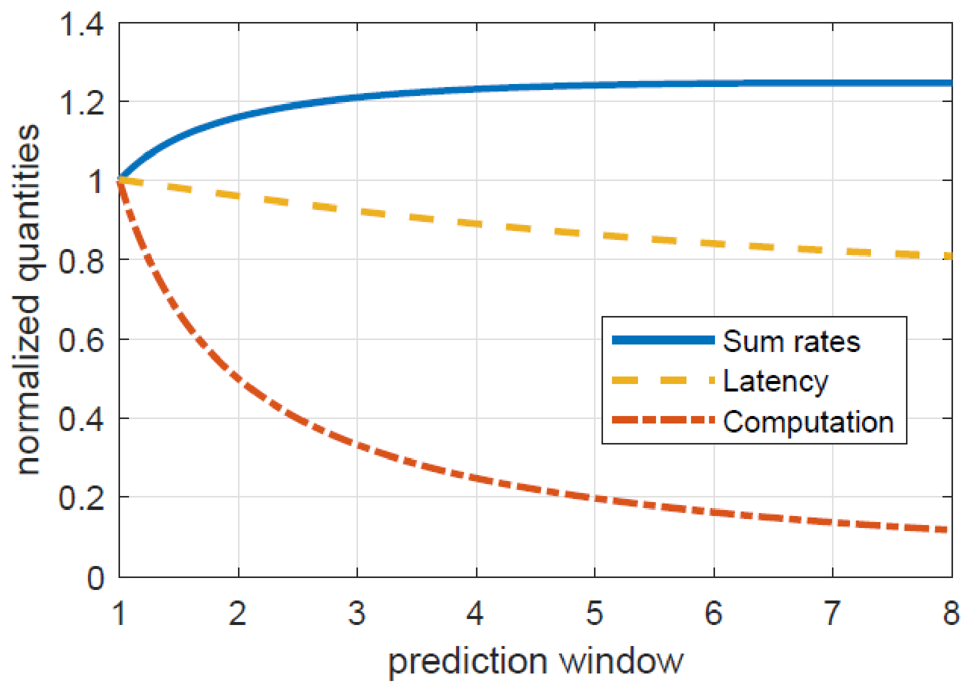


H. Siljak, I. Macaluso, N. Marchetti, "Artificial Intelligence for Dynamical Systems in Wireless Communications: Modeling for the Future", *IEEE Systems, Man, and Cybernetics Magazine*, vol. 7, no. 4, pp. 13-23, Oct 2021

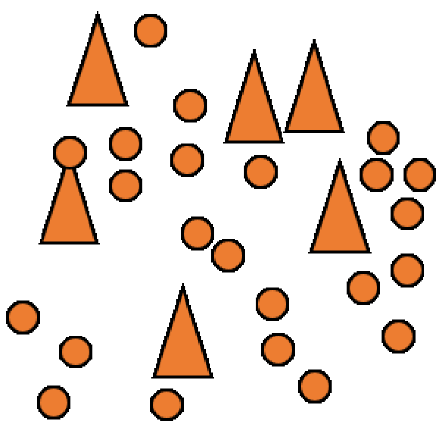
The Power of Prediction

Co-located Massive MIMO

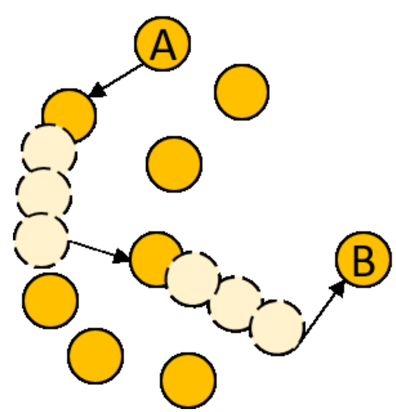




Distributed Massive MIMO



Multi-hop system



Ackowledgement

- Anna Rapoport
- Prof Harun Siljak, TCD