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Dynamical Systems
Lecture 5.05

(Tutorial #5)

EEU45C09 / EEP55C09

Self Organising Technological Networks

CONNECT
Networks of the Future

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Question 1

If a function $f(x)$ is determinisitc, then:

- A. Knowing the input determines the value of the output
- B. The same input always gives the same output
- C. For every input there is one and only one possibe output
- D. All of the above statements are true

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- C. For every input there is one and only one possibe output
- D. All of the above statements are true

Solution: The answer is **D**. A deterministic function means that its output is determined only by the input value (statement A). As a consequence of this, the same input always gives the same output (statement B). It then follows that there can be only one output for every input (statement C).

Question 2

True or False: If a function is deterministic, if you know the output you can always figure out the input.

- A. True
- B. False

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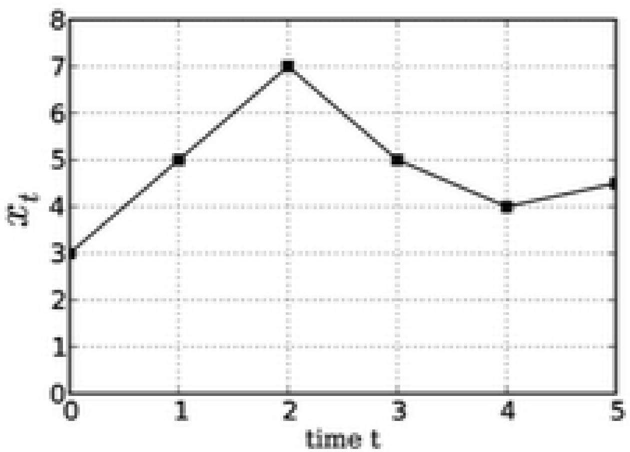
- A. True
- B. False

Solution: The statement is false, so the answer is **B**. For a function the input determines the output, but the opposite isn't necessarily the case. This is easiest to see with an example. Let $f(x) = x^2$. Suppose that the output is 4. That is, for some x, $f(x) = 4$. Can you determine x? Yes, but not uniquely. It could be that $x = 2$, but it also could be that $x = -2$. So knowing the output (in this case 4), does not let you figure out with certainty what the input was.

Question 3



A time series is plotted above.Could the graph of this time series be the orbit of a deterministic function?

- A. Yes
- B. No

Solution: The correct answer is ; the graph can **not** be a time series for the orbit of a deterministic function. The reason is as follows. Suppose the graph showed the time series obtained by iterating the function $f(x)$. At time 1, the orbit value is 5. At time 2, the value of the orbit is 7. This means that

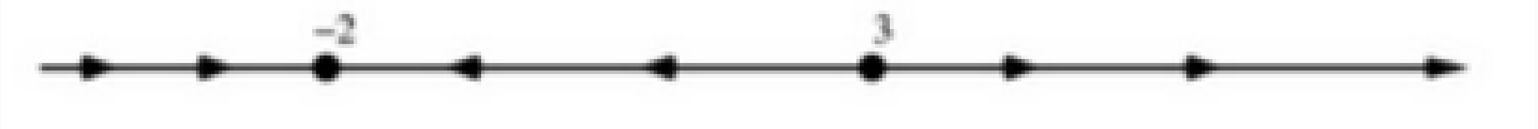
$$f(5) = 7\ . \tag{1}$$

At time 3 the orbit is again 5, and at time 4 it is 4. This means that

$$f(5) = 4\ . \tag{2}$$

The above two equations cannot both be true. If the function f is deterministic, then $f(5)$ must have only one value. It cannot be the case that $f(5)$ equals both 7 and 4. We are thus led to conclude that the time series plot shown above cannot be from the orbit of a deterministic function.

Instructions



The questions below refer to the phase line for a function shown above.

Question 1

The point -2 is:

- A. a stable fixed point
- B. an unstable fix point
- C. not a fiexed point

Question 2

The point 0 is:

- A. a stable fixed point
- B. an unstable fix point
- C. not a fixed point

Question 3

The point 3 is:

- A. a stable fixed point
- B. an unstable fix point
- C. not a fixed point

Solution: The answer is **A**. -2 is stable. Points near -2, when iterated, get closer to -2.

Solution: The answer is **C**. 0 is not a fixed point. Fixed points are indicated with dots on a phase line. In this case, 0 will move toward the stable fixed point at -2. But the point 0 is not fixed; it moves toward -2 when iterated with the function.

Solution: The answer is **B**. 3 is unstable. Points near 3, when iterated, get farther away from 3.

Question 1

For the function $f(x) = 2x - 4$, which of the following is a fixed point?

- A. 0
- B. 2
- C. 4
- D. 8

Solution: The answer is **C**. One way to see this is to try out the four possible answer and see if $f(x) = x$. For example, to test $x = 2$,

$$\begin{aligned} f(2) &= 2\ ? && (1)\\ 2 \times 2 - 4 &= 2\ ? && (2)\\ 4 - 4 &= 2\ ? && (3)\\ 0 &= 2\ ? && (4)\\ & && (5) \end{aligned}$$

Since 0 does not equal 2, we conclude that 2 is not a fixed point of $f(x)$. Performing a similar analysis with the other three possible answers will show that 4 is a fixed point.

Alternatively, one could use this fixed-point equation to solve for the fixed point.

$$\begin{aligned} f(x) &= x\ , && (6)\\ 2x - 4 &= x\ , && (7)\\ -4 &= -x\ , && (8)\\ 4 &= x\ . && (9) \end{aligned}$$

So $x = 4$ is a fixed point. (To go from Eq. (**7**) to Eq. (**8**) I subtracted $2x$ from both sides of the equation.)

Question 2

For the function $g(x) = x + 2$, which of the following statements is true?

- A. -2 is a fixed point
- B. 0 is a fixed point
- C. 2 is a fixed point
- D. $g(x)$ has no fixed points

Solution: The answer is **D**; $g(x)$ has no fixed points. There is no number that stays the same when 2 is added to it. If you try to solve the fixed point equation, one will see that it has no solution:

$$\begin{aligned} g(x) &= x && (10) \\ x+2 &= x && (11) \\ 2 &= 0\,. && (12) \end{aligned}$$

Since 2 does not equal 0, we conclude that the fixed point equation does not have a solution. There is no x that makes the equation $g(x) = x$ true.

Question 3

Which of the following numbers are fixed points for the square-root function, $f(x) = \sqrt{x}$?

- A. 0
- B. 1
- C. 0 and 1
- D. 0, 1, and 2

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- C. 0 and 1
- D. 0, 1, and 2

Solution: The answer is **C**; both 0 and 1 are fixed points. These numbers do not change when they are square-rooted:

$$\sqrt{1} = 1\,, \tag{13}$$

and

$$\sqrt{0} = 0\,, \tag{14}$$

1. The function $f(x) = 2x - 4$ has a fixed point at $x = 4$. Is this fixed point stable or unstable? Answer this by choosing initial conditions near the fixed point iterating (using a calculator if you wish) and seeing what happens to the orbits.

   A. The fixed point is unstable

   B. The fixed point is stable

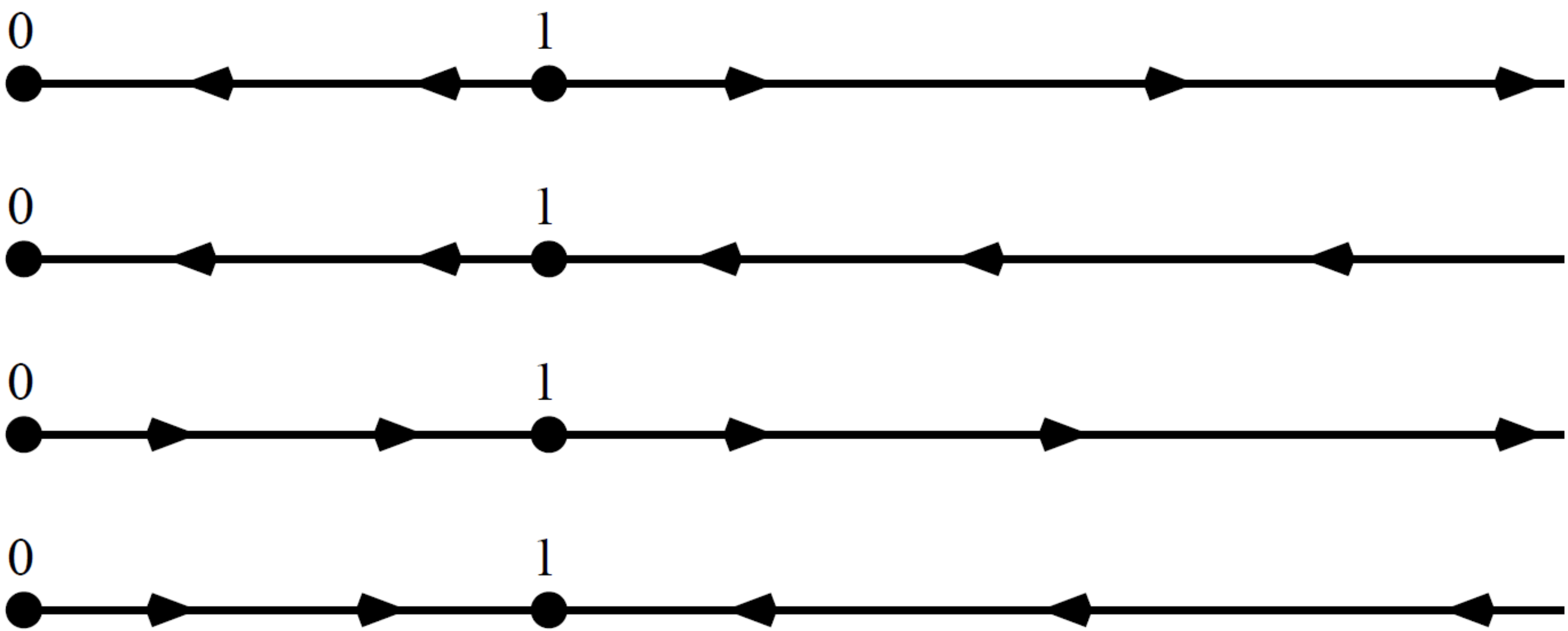
1. The function $f(x) = 2x - 4$ has a fixed point at $x = 4$. Is this fixed point stable or unstable? Answer this by choosing initial conditions near the fixed point iterating (using a calculator if you wish) and seeing what happens to the orbits.

   A. The fixed point is unstable

   B. The fixed point is stable

Solution: The answer is **A**. If you choose an initial condition near 4 you will find that its orbit moves away from 4. For example, the orbit of 3 is: 3 → 2 → 0 → -4 → -12 ... . The orbit moves away from the fixed point. Hence, the fixed point is unstable.

2. Consider the square root function $f(x) = \sqrt{x}$. What is its phase line (for positive x)? Choose from the four phase lines shown below. (To answer this question you might want to choose a few different initial conditions and use a calculate to determine their orbits.)



Solution: When a number larger than 1 is square rooted, it gets smaller. For example, $\sqrt{49} = 7$. And when a number between 0 and 1 is square rooted it gets larger. For example,

$$\sqrt{\frac{1}{4}} = \frac{1}{2}\,, \tag{1}$$

and 1/2 is larger than 1/4. Thus, he correct phase line is the one in the **bottom** of the figure, as it shows numbers between 0 and 1 increasing and approaching 1, and all numbers larger than 1 get smaller and approach 1.

1. If a sequence is compressible it is:

    A. Random

    B. Not Random

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   A. Random

   B. Not Random

Solution: The answer is **B**. If a sequence is compressible, it possess some regularities, and thus is not random.

3. For $r = 4.0$, a symbol sequence generated by the logistic equation is:

   A. Generated by a deterministic process

   B. As random as a coin toss

   C. All of the above

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Solution: The answer is **C**. The logistic equation with $r = 4$ is a deterministic process that produces a random sequence.

Lotka-Volterra differential equations:

$$\frac{dR}{dt} = R - 0.25RF \,, \quad \frac{dF}{dt} = 0.2RF - 0.6F \,,$$

The initial conditions are $R(0) = 10$, $F(0) = 6$.

1. At $t = 0$, what is the rate of change of the rabbit population?

   A. 10

   B. 5

   C. 0

   D. -5

   E. -10

Solution: The correct answer is **D**. To see this, we just plug $R = 10$ and $F = 6$ into the LV equation for dR/dt:

$$\frac{dR}{dt} = R - 0.25RF = 10 - (0.25)(10)(6) = -5 .$$

2. If $R = 5$ and $F = 5$ the fox population F is

A. constant

B. increasing

C. decreasing

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A. constant

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C. decreasing

Solution: The correct answer is **B**. To see this, we just plug $R = 5$ and $F = 5$ into the LV equation for dF/dt:

$$\frac{dF}{dt} = 0.2RF - 0.6F = (0.2)(5)(5) - (0.6)(5) = 2\ . \tag{3}$$

Since the derivative is positive, the population is increasing.

Q.4

(a) Plot the phase lines of the following functions, indicating explicitly which are the stable and unstable fixed points (if any).

i. $f(x) = x$

ii. $f(x) = x + 1$

iii. $f(x) = \sqrt{x}$

iv. $f(x) = 1$

Phase line is made by dense fixed points

No fixed points

Unstable fixed point

Stable fixed point

Stable fixed point

Q.9

Which one among the following options is a fixed point of the function $f(x) = x^2 - 1$?

(a) $-0.5 - (\sqrt{5}/2)$

(b) $-0.5 + (\sqrt{5}/2)$

(c) $0.5 + (\sqrt{5}/2)$

(d) None of the above is a fixed point of f(x)

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Q.10

Which one among the following time series **cannot** be the itinerary of a wireless link's amplitude when no random phenomenon in the environment affects it?

(a) 10, 11, 12, 11.5
(b) 10, 11, 10, 11.5
(c) 10, 11, $11+\varepsilon$, $11-\varepsilon$ (where $\varepsilon>0$ arbitrarily small)
(d) All of the above could be possible itineraries

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(b) 10, 11, 10, 11.5
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Q.7

Which one among the following statements about the Chua's circuit is incorrect?

(a) It contains a non-linear component
(b) It necessarily displays the butterfly effect
(c) It can be applied as a pseudo-random number generator
(d) It is an example of complex system

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Q.13

For the channel gain $|h|^2$, which of the following statements is incorrect?

(a) h=0 is an attracting fixed point
(b) h=1 is a repelling fixed point
(c) h=2 is not a fixed point
(d) 1, 1.21, 1.46…, 2.14… is a possible orbit

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Q.14

For the Shannon capacity $\log_2(1+x)$ where x is the signal-to-noise ratio S/N, which of the following statements is incorrect?

(a) S=0 corresponds to a fixed point
(b) S=N corresponds to a fixed point
(c) S=1 corresponds to a fixed point
(d) S/N = 1 corresponds to a fixed point

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- (d) S/N = 1 corresponds to a fixed point

$$\log_2\left(1+\frac{S}{N}\right)=\frac{S}{N}\Rightarrow\log_2\left(1+\frac{1}{N}\right)=\frac{1}{N}\Leftrightarrow 1+\frac{1}{N}=2^{1/N}$$

$$\Leftrightarrow$$

$$N\rightarrow\infty\quad\text{or}\quad N=1$$

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