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Network Science
Lecture 4.05

(Tutorial #3)

EEU45C09 / EEP55C09
Self Organising Technological Networks

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The graph in Fig. Q.2, shows the interference topology for a cellular network, where an edge is present if the two base stations mutually interfere with each other.

(a) Compute the closeness centrality for each node.

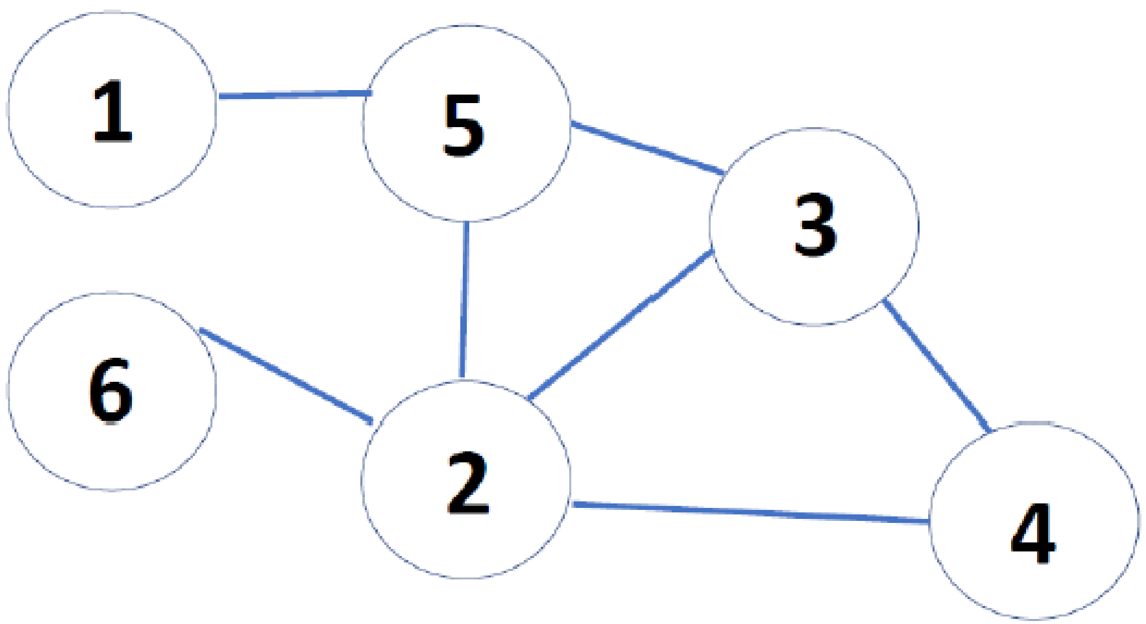


Fig. Q.2

(a)

$$CC_i = \frac{1}{l_i} = \frac{N}{\sum_j d_{ij}}$$

$CC_1 = N / (d_{12} + d_{13} + d_{14} + d_{15} + d_{16}) = 6 / (2+2+3+1+3) = 6/11$

$CC_2 = N / (d_{21} + d_{23} + d_{24} + d_{25} + d_{26}) = 6 / (2+1+1+1+1) = 1$

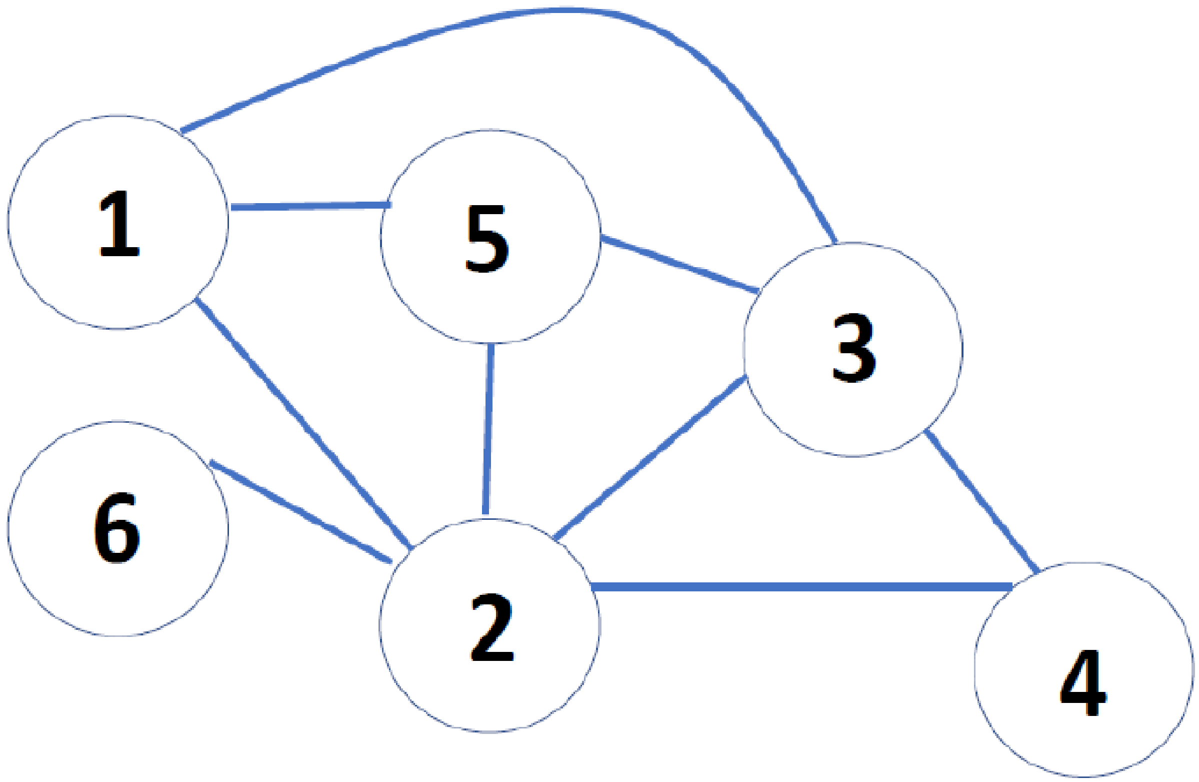
$CC_3 = N / (d_{31} + d_{32} + d_{34} + d_{35} + d_{36}) = 6 / (2+1+1+1+2) = 6/7$

$CC_4 = N / (d_{41} + d_{42} + d_{43} + d_{45} + d_{46}) = 6 / (3+1+1+2+2) = 6/9$

$CC_5 = N / (d_{51} + d_{52} + d_{53} + d_{54} + d_{56}) = 6 / (1+1+1+2+2) = 6/7$

$CC_6 = N / (d_{61} + d_{62} + d_{63} + d_{64} + d_{65}) = 6 / (3+1+2+2+2) = 6/10$

The following graph shows the communication network topology for a millimetre wave network, where an edge is present if the two radio entities establish a beamforming-type link between them.



Calculate the following quantities, providing an explanation for each answer.

(a) Calculate the degree of each vertex.

[3 marks]

(b) Calculate the degree distribution of the graph.

[3 marks]

(c) Calculate the average path length of the graph.

[6 marks]

(d) Calculate the clustering coefficient of the graph.

[6 marks]

(e) Calculate the degree matrix, the adjacency matrix of the graph.

[7 marks]

(a) Calculate the degree of each vertex.

[3 marks]

$k_1 = 3$, $k_2 = 5$, $k_3 = 4$, $k_4 = 2$, $k_5 = 3$, $k_6 = 1$.

(b) Calculate the degree distribution of the graph.

[3 marks]

$P(k = 1) = 1/6$, $P(k=2) = 1/6$, $P(k=3) = 2/6$, $P(k=4) = 1/6$, $P(k=5) = 1/6$.

(c) Calculate the average path length of the graph.

[6 marks]

Distance matrix:

$$L_{dist} = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

$$< l > = \frac{(1+1+2+1+2)+(1+1+1+1)+(1+1+2)+(2+2)+(2)}{15} = \frac{21}{15} = 1.4$$

(d) Calculate the clustering coefficient of the graph.

[6 marks]

$c_1 = 3 / (3 \cdot 2 / 2) = 1$, $c_2 = 4 / (5 \cdot 4 / 2) = 2/5$, $c_3 = 4 / (4 \cdot 3 / 2) = 2/3$, $c_4 = 1 / (2 \cdot 1 / 2) = 1$,
$c_5 = 3 / (3 \cdot 2 / 2) = 1$, $c_6 = 0$.

$$C = \frac{\left(1 + \frac{2}{5} + \frac{2}{3} + 1 + 1 + 0\right)}{6} = \frac{61/15}{6} = \frac{61}{90}$$

The clustering coefficient can be found by calculating the local clustering coefficient for each node,

$$c_i = \frac{2|e_{jk} : v_j, v_k \in N_i, e_{jk} \in E|}{k_i(k_i - 1)},$$

(e) Calculate the degree matrix, the adjacency matrix, of the graph.

[7 marks]

Degree matrix:

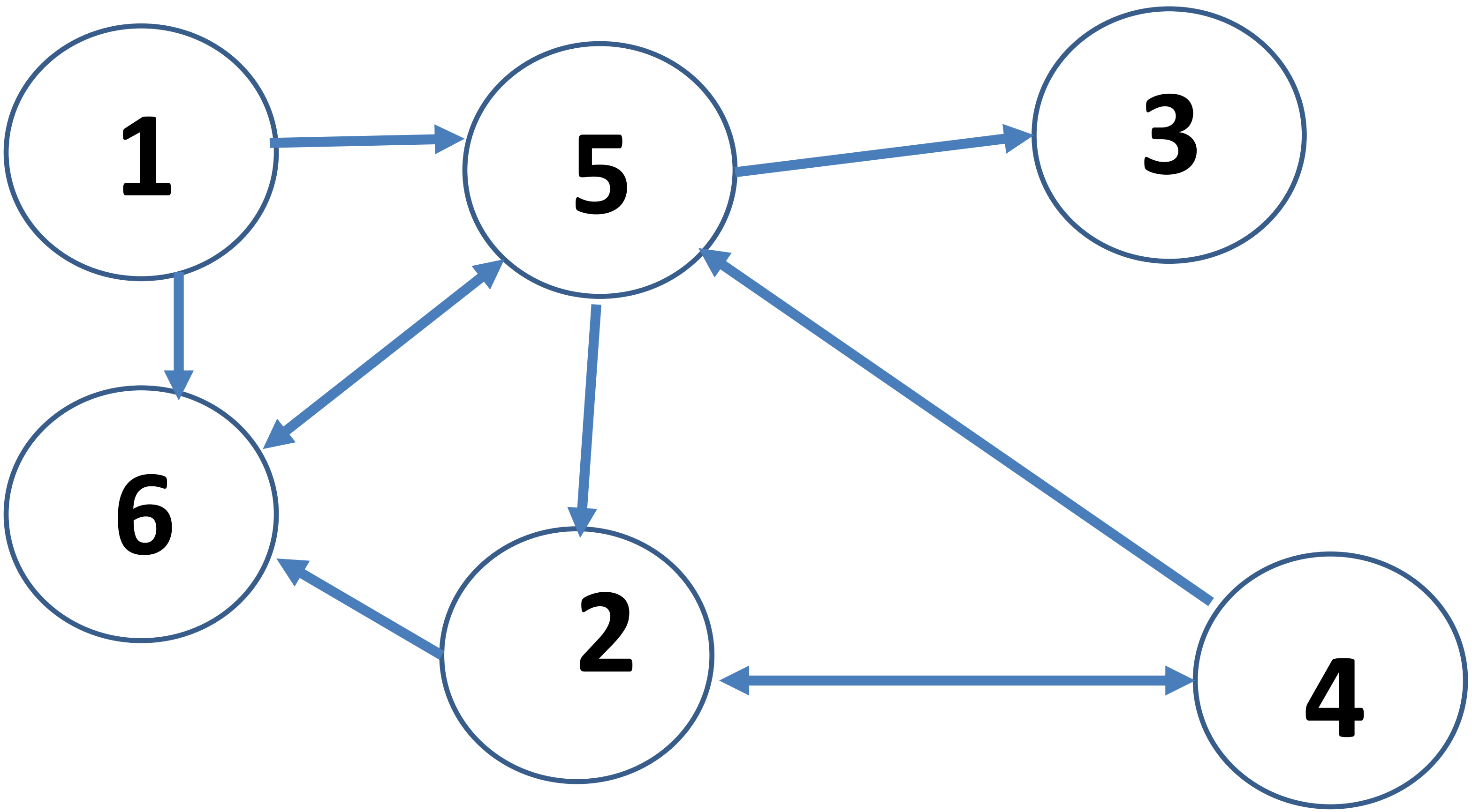
$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the graph in the next slide, calculate the following quantities, providing an explanation for each answer.

a) Calculate the in-degree and out-degree of each vertex.
b) Calculate the in-degree and out-degree distribution of the graph.
c) Calculate the adjacency matrix of the graph.


1
5
3
6
2
4

a) Calculate the in-degree and out-degree of each vertex.

$k_{in,1}$= 0; $k_{out,1}$= 2
$k_{in,2}$= 2; $k_{out,2}$= 2
$k_{in,3}$= 1; $k_{out,3}$= 0
$k_{in,4}$= 1; $k_{out,4}$= 2
$k_{in,5}$= 3; $k_{out,5}$= 3
$K_{in,6}$= 3; $k_{out,6}$= 1

b) Calculate the in-degree and out-degree distribution of the graph.

$P(k_{in} = 0) = 1/6$; $P(k_{out} = 0) = 1/6$

$P(k_{in} = 1) = 2/6$; $P(k_{out} = 1) = 1/6$

$P(k_{in} = 2) = 1/6$; $P(k_{out} = 2) = 3/6$

$P(k_{in} = 3) = 2/6$; $P(k_{out} = 3) = 1/6$

$P(k_{in} = 4) = 0$; $P(k_{out} = 4) = 0$

$P(k_{in} = 5) = 0$; $P(k_{out} = 5) = 0$

$P(k_{in} = 6) = 0$; $P(k_{out} = 6) = 0$

c) Calculate the adjacency matrix of the graph.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For a random graph:

(a) Prove that the average degree is:

$$< k > = np$$

where n is the number of nodes in the graph, and p is the probability of an edge being created, and where we consider self-edges too.

(b) Prove that the degree distribution is:

$$P(k) = \exp(-< k >) \frac{< k >^k}{k!}$$

For a random graph:

(a) Prove that the average degree is:

$$< \boldsymbol{k} > = \boldsymbol{np}$$

where *n* is the number of nodes in the graph, and *p* is the probability of an edge being created, and where we consider self-edges too.

(b) Prove that the degree distribution is:

$$P(k) = \exp(-< k >)\frac{< k >^k}{k!}$$

(a)

$$< k > = \sum_{k=0}^{n} kP(k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k}.$$

Let q = 1-p , such that p+q = 1

$$= \sum_{k=0}^{n} k \binom{n}{k} p^{k} q^{n-k}$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^{k} q^{n-k}$$ since $k=0$, $k\binom{n}{k} p^{k} q^{n-k} = 0$

$$k\binom{n}{k} = \frac{k\, n!}{k!\,(n-k)!} = \frac{\cancel{k}\, n(n-1)!}{\cancel{k}(k-1)!\,(n-k)!}$$

$$= n\binom{n-1}{k-1}$$

$$\Rightarrow \sum_{k=1}^{n} n \binom{n-1}{k-1} p^{k} q^{n-k}$$

(a)

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k}$$

$$n-k = (n-1)-(k-1)$$

$$\Rightarrow np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$$

Let m = n-1 , j = k-1

$$= np \underbrace{\sum_{j=0}^{m} \binom{m}{j} p^{j} q^{m-j}}$$

By the Binomial Theorem $(x+a)^{\nu} = \sum_{k=0}^{\infty} \binom{\nu}{k} x^{k} a^{\nu-k},$

$$\sum_{j=0}^{m} \binom{m}{j} p^{j} q^{m-j} = (p+q)^{m}.$$

$$p+q=1 \quad \Rightarrow \quad (p+q)^{m} = 1$$

$$\Rightarrow \langle k \rangle = np(1) = np.$$

For a random graph:

(a) Prove that the average degree is:

$$< k > = np$$

where n is the number of nodes in the graph, and p is the probability of an edge being created, and where we consider self-edges too.

(b) Prove that the degree distribution is:

$$\boldsymbol{P(k) = \exp(-< k >)\frac{< k >^k}{k!}}$$

(b)

The Poisson distribution is a limiting case of the Binomial distribution which arises when the **number of trials n becomes very large** whilst the product μ = np remains constant.

The Binomial probability mass function is the Prob of k wins given n trials and success prob p :

$$b(k,n,p) = \binom{n}{k} p^k (1-p)^{n-k} =$$
$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \qquad ①$$

We can rewrite p as mean number of successes / number of trials, i.e., $p = \mu/n$ **(2)**

Substitute (2) into (1)

$$b(k,n,p) = \frac{n!}{k!\,(n-k)!} \frac{\mu^k}{n^k} \left(1 - \frac{\mu}{n}\right)^{n-k}$$
$$= \frac{\mu^k}{k!} \frac{n!}{(n-k)!\,n^k} \left(1 - \frac{\mu}{n}\right)^{n} \left(1 - \frac{\mu}{n}\right)^{-k} \qquad ③$$

(b)

Let's examine the three n-dependent terms of (3)

$$\frac{n!}{(n-k)!\,n^k} = \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)!}{(n-k)!\,n^k}$$

$$\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}$$

The leading term of n(n-1)(n-2)⋯(n-k+1) is n^k since there are k factors

$$\lim_{n\to\infty}\frac{n^k\left(1+\frac{1}{n}+\cdots\right)}{n^k} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$$

Result we know from Calculus.

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-k} = (1)^{-k} = 1$$

Reassembling the three terms we find

$$\lim_{n\to\infty} b(k,n,p) = \frac{\mu^k}{k!}e^{-\mu} = \frac{(np)^k}{k!}e^{-np} = \frac{<k>^k}{k!}e^{-<k>}$$