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Dynamical Systems
Lecture 5.04

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Self Organising Technological Networks

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The Lorenz attractor

The **Lorenz attractor** has the form

$$\begin{cases} \dot{x} = d\,(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where r, d, and b are the parameters. For $r = 28$, $d = 10$, and $b = 8/3$, the system has a chaotic solution. Other parameter values may generate other types of solutions.

The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis.

Properties of the Lorenz attractor

- There exists a symmetric pair of solutions. If $(x, y) \to (-x, -y)$ the system stays the same. If solution $(x(t), y(t), z(t))$ exists then solution $(-x(t), -y(t), z(t))$ is also a solution.
- The Lorenz system is **dissipative** : volumes V in phase space contract under the flow. As $t \to \infty$, $V \to 0$.

$$\dot{V} = \int_V \nabla \cdot \dot{\vec{x}} \, \mathrm{d}V \qquad (2)$$

$$\nabla \cdot \dot{\vec{x}} = \frac{\partial}{\partial x}[d(y - x)] + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) =$$
$$= -(d + 1 + b) < 0 = \text{const.} \qquad (3)$$

Since the divergence is constant, (2) reduces to

$$\dot{V} = -(d + 1 + b)\,V \quad \Rightarrow \quad V(t) = V(0)\mathrm{e}^{-(d+1+b)t}. \qquad (4)$$

Bifurcation analysis of the Lorenz attractor

For $r < 1$ ($d = 10$, $b = 8/3$) there is only one stable fixed point located at the origin. This point corresponds to no convection. All orbits converge to the origin — a global attractor.
A **supercritical pitchfork bifurcation** occurs at $r = 1$, and for $r > 1$ two additional fixed points appear at

$$(x^*, y^*, z^*) = C^{\pm} \tag{5}$$

These correspond to steady convection. Fixed points are stable only if

$$r < r_{\text{Hopf}}, \quad r_{\text{Hopf}} = d\,\frac{d+b+3}{d-b-1} = 24.74, \tag{6}$$

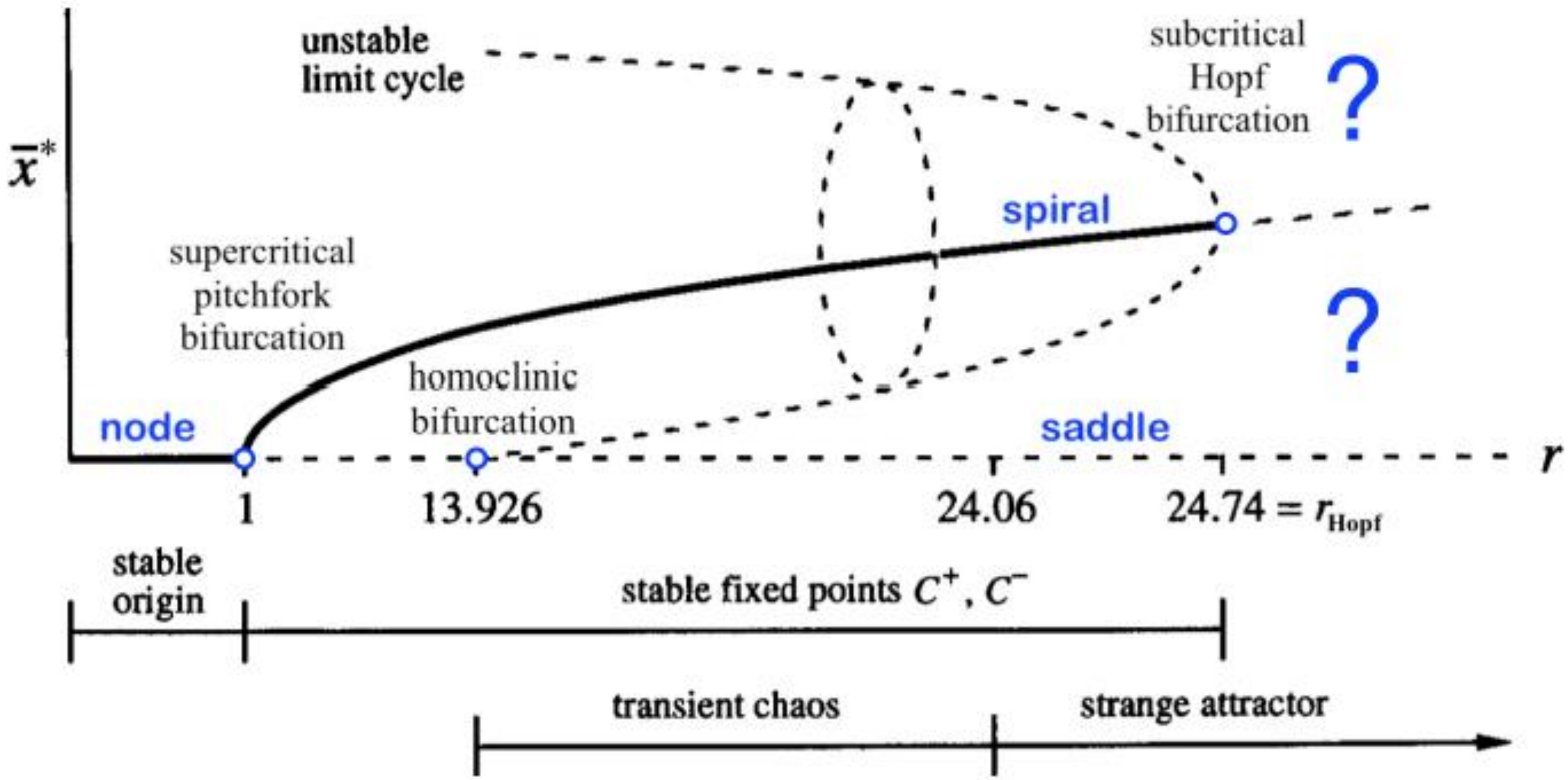
which can hold only for positive r and $d > b + 1$. At a critical value $r = r_{\text{Hopf}}$, both stable fixed points lose stability through a **subcritical Hopf** bifurcation.

Bifurcation analysis of the Lorenz attractor

As we decrease r from r_{Hopf}, the unstable limit cycles expand and pass precariously close to the **saddle point** at the origin. At $r = 13.926$ the cycles touch the saddle point and **become homoclinic orbits**; hence we have a **homoclinic bifurcation** which is referred to as the *first homoclinic explosion*. Below $r = 13.926$ there are no limit cycles.

The region $13.926 < r < 24.06$ is referred to as **transient chaos**-region. Here, the chaotic trajectories eventually settle at C^{+} or C^{-}.

Bifurcation analysis of the Lorenz attractor



- Schematic above shows behaviour for small values of r , while keeping other parameters constant
- $\bar{x}^*$ is the distance from the origin

Exponential divergence of closely neighbouring trajectories

From numerical integration of the Lorenz system we know that the motion/flow on the attractor trajectories exhibits **sensitive dependence on initial conditions**. This means that two trajectories starting very close to each other will rapidly diverge, and thereafter have totally different futures, see Fig. 1.

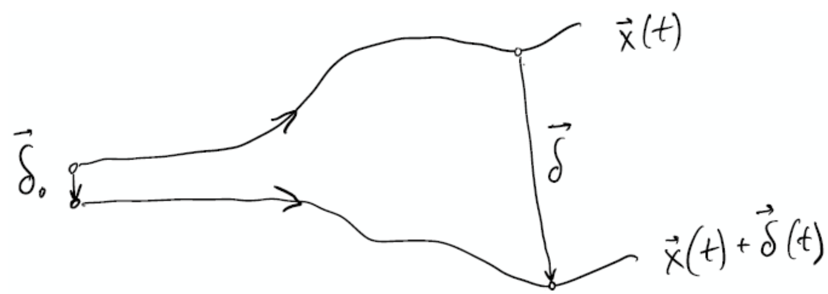
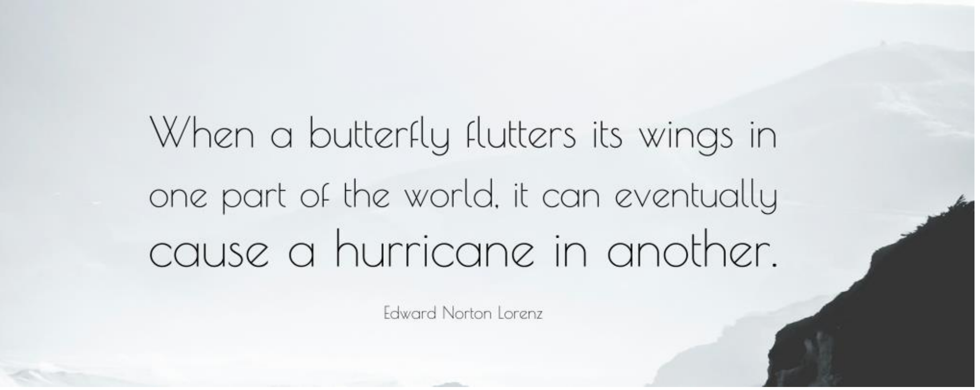


Figure 1: Rapidly diverging trajectories where the initial separation $|\vec{\delta}_0| = \delta_0 \approx 0$.



The Lyapunov exponents

It can be demonstrated that after the **transient behaviour** has elapsed and a trajectory has settled on the **attracting set** (the Lorenz attractor) the norm (magnitude) of the separation vector

$$|\vec{\delta}(t)| \sim \delta_0 e^{\lambda t}, \quad \textbf{(1b)}$$

where in the case of Lorenz system $\lambda \approx 0.9$. Hence, neighbouring trajectories diverge exponentially fast. Equivalently, if we plot $\ln|\vec{\delta}|$ vs. t, a logarithmic plot, we find a curve that is close to a straight line with a positive slope of λ, see Fig. 2.

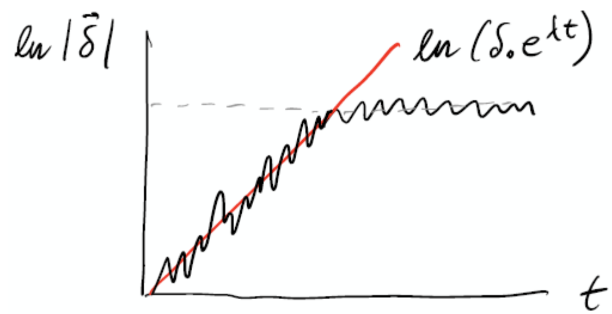


Figure 2: Norm of separation vector $|\vec{\delta}|$ as a function of t. Slope λ is shown with the straight red line.

The exponential divergence must stop when the separation is comparable to the *diameter* of the attractor (attracting set, think Lorenz butterfly)—the trajectories obviously can't get any farther apart than that. This explains the levelling off or **saturation** of the curve $\ln|\vec{\delta}(t)|$ shown with the dashed line in Fig. 2.

The exponent λ is often called the **Lyapunov exponent**, although this is a sloppy use of the term, for two reasons:

- First, there are actually n different Lyapunov exponents for each degree of freedom of an n-dimensional system (λ_x, λ_y, $\lambda_z, \ldots$).
- Second, λ depends slightly on which trajectory we study. We should average over many trajectories (many initial conditions) and over many different points on the same trajectory to get the true value.

The Lyapunov exponent λ of a system is usually close to the biggest of the exponents related to the degrees of freedom (λ_x, λ_y, $\lambda_z, \ldots$). The biggest exponent governs the resulting slope shown in Figs. 2 and 3.
Note: $\lambda > 0$ is a signature of chaos.

The Lyapunov exponent of 3-D chaotic systems, visual demonstration.

Plotted below is the quantitatively accurate $\ln|\vec{\delta}|$ vs. t graphs shown in Fig. 2 for the Lorenz attractor with the same parameters as used before.

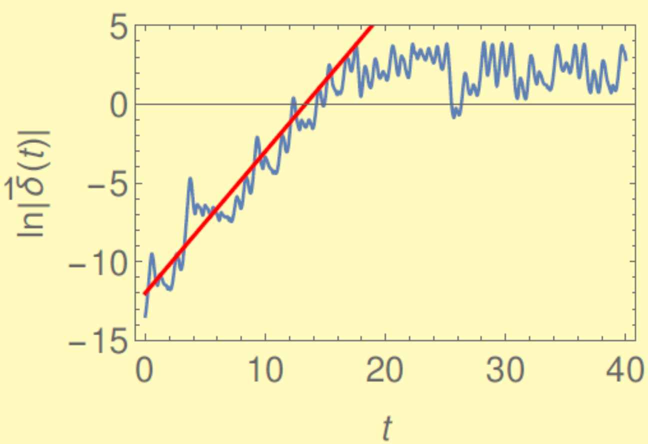


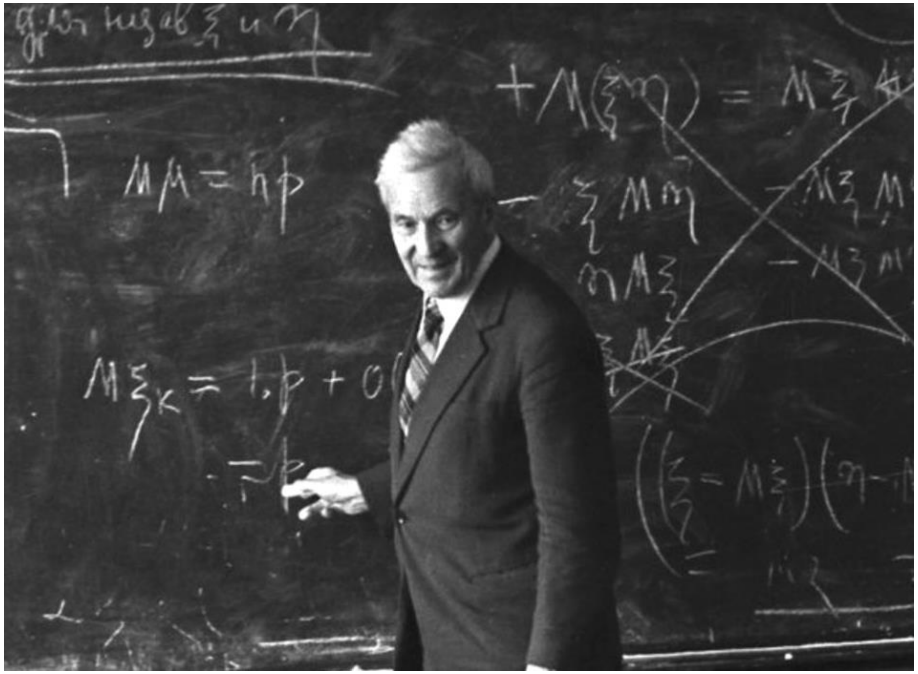
Figure 3: Exponential divergence of trajectories in the Lorenz system shown with the blue graph. The graph of $\ln(\delta_0 e^{\lambda t})$ where the Lyapunov exponent $\lambda = 0.9$ is shown with the red line.

Kolmogorov entropy

Also known as the **metric entropy**, the Kolmogorov-Sinai entropy, or KS entropy. The Kolmogorov entropy is zero for non-chaotic motion and positive for chaotic motion. For $\lambda_i > 0$ one can find the Kolmogorov entropy using the following formula:

$$K = \sum_i \lambda_i, \tag{2b}$$

where i spans the number of degrees of freedom of a given system. The Kolmogorov entropy is a measure of the growth of uncertainty due to the expansion rate given by the positive Lyapunov exponents λ_i.



Predictability horizon

The predictability horizon is also called the **Lyapunov time**. When a system has a positive Lyapunov exponent λ, there is a time horizon beyond which prediction breaks down, as shown schematically in Fig. 4.

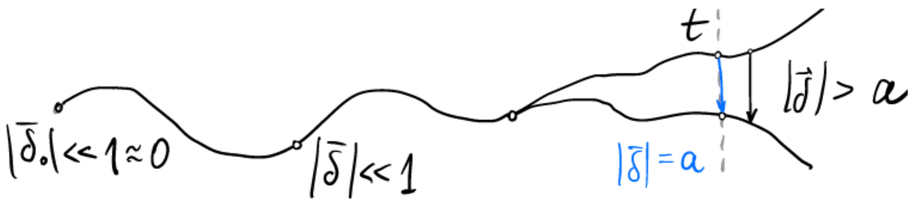


Figure 4: Deviation of two trajectories with close-by initial conditions, where a is the allowed tolerance representing a noticeable deviation from the true trajectory.

After a time period $t > 0$ the discrepancy, shown in Fig. 4, grows to

$$|\vec{\delta}(t)| = \delta(t) \approx \delta_0 e^{\lambda t}. \tag{3b}$$

Let a be a measure of our tolerance, i.e., if prediction is within a of the true/nominal state, we consider it acceptable. Then our prediction becomes intolerable when $|\vec{\delta}(t)| = \delta(t) > a$. How *long* will the system be predictable, i.e., evolve within selected tolerance a?

Predictability horizon

This question can be answered by solving

$$a \approx \delta_0 \mathrm{e}^{\lambda t}, \tag{4b}$$

for time t.

$$\mathrm{e}^{\lambda t} \approx \frac{a}{\delta_0} \quad \Big| \ln(\cdot), \tag{5b}$$

$$\ln \mathrm{e}^{\lambda t} \approx \ln \frac{a}{\delta_0}, \tag{6b}$$

$$\lambda t \approx \ln \frac{a}{\delta_0}, \tag{7b}$$

$$\boxed{t \approx \frac{1}{\lambda} \ln \frac{a}{\delta_0}.} \tag{8b}$$

This time interval is referred to as the **predictability horizon** or simply the **Lyapunov time**. The **characteristic time scale** (or order) of chaos

$$O(t) \approx O\left(\frac{1}{\lambda} \ln \frac{a}{\delta_0}\right) \approx O\left(\frac{1}{\lambda} \cdot 1\right) \approx O\left(\frac{1}{\lambda}\right). \tag{9b}$$

At time scales $1/\lambda$ chaos becomes noticeable. The trajectories deviate beyond acceptable tolerance.

Predictability horizon

Example: Suppose we're trying to predict the future state of a chaotic system within a tolerance of $a = 10^{-3}$. Given that our estimate of the initial state is uncertain to within $\delta_0 = 10^{-7}$, for about how long can we predict the state of the system, while remaining within the tolerance?

Now suppose we manage to measure the initial state a *million* times better, i.e., we improve our initial error to $\delta_0 = 10^{-13}$. How much longer can we predict?

Predictability horizon

Solution: The original prediction has

$$t = \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-7}} = \frac{1}{\lambda} \ln(10^4) = \frac{4 \ln 10}{\lambda}.$$

The improved prediction has

$$t = \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-13}} = \frac{1}{\lambda} \ln(10^{10}) = \frac{10 \ln 10}{\lambda}.$$

Thus, after a millionfold improvement in our initial uncertainty, we can predict only $10/4 = 2.5$ times longer (system's timeframe)!

Conclusions: If one wants to predict further into the future the task becomes exponentially *harder*. Since,

$$t \simeq \frac{1}{\lambda} \ln \frac{a}{\delta_0} \sim \ln \frac{a}{\delta_0} \quad \Rightarrow \quad \frac{a}{\delta_0} \sim e^t.$$

Predictability horizon

The logarithmic dependence on δ_0 is what hurts us. No matter how hard we work to reduce the initial measurement error, we can't predict longer than a **few multiples of** $1/\lambda$.

Figure 5: a/δ_0 dependence on the Lyapunov time t.

If you want to predict further into the future by increasing the Lyapunov time for given a the task becomes exponentially harder requiring ever-smaller δ_0 values because

$$t \simeq \frac{1}{\lambda} \ln \frac{a}{\delta_0} \sim \ln \frac{a}{\delta_0} \quad \Rightarrow \quad \frac{a}{\delta_0} \sim e^t. \tag{10b}$$

Figure 5 shows this exponential dependance.

Predictability horizon

Note: The above conclusions hold for all integration methods. In fact, a *bad* numerical method may worsen the situation significantly. Even if you have an analytic solution, it too will have the predictability horizon, because you are still required to input an initial condition a number with finite accuracy (significant figures in a decimal fraction) to generate a solution.

Question: Let's assume we are somehow able to eliminate the measurement error or/and are able to input initial conditions with absolute accuracy, i.e., $\delta_0 = 0$. Will this make a chaotic system predictable in practice?

The Lyapunov time given by (8b) for aforementioned assumption is

$$t = \lim_{\delta_0 \to 0} t = \lim_{\delta_0 \to 0} \left(\frac{1}{\lambda} \ln \frac{a}{\delta_0} \right) = \infty, \tag{11}$$

this suggest that the chaotic problem becomes predictable. The answer to the above question is "yes." Unfortunately, the above assumptions are utterly unrealistic.

Demonstration of the effect of calculation accuracy and precision on the numerical solution and the predictability horizon of a chaotic system. Transient and intermittent chaos.

By now it should be obvious that chaotic solutions are also **sensitive to numerical precision and accuracy of numerical integration**. The numerical accuracy is related to the number of significant figures or even to a specific floating point representation of real numbers themselves used in calculations by a computer. **Obtaining long-term true trajectories of chaotic dynamical systems by means of numerical methods is generally speaking not a trivial task.**

Ackowledgement

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