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Dynamical Systems
Lecture 5.07

EEU45C09 / EEP55C09
Self Organising Technological Networks

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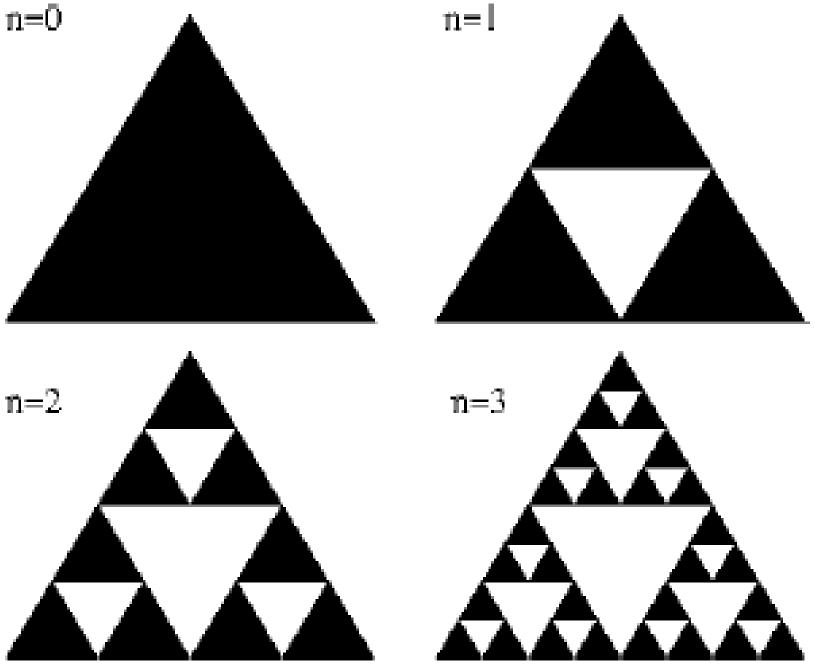
Self-Similarity

- Self-similar: small parts of the object are similar to the whole.
- This self-similarity extends over many scales

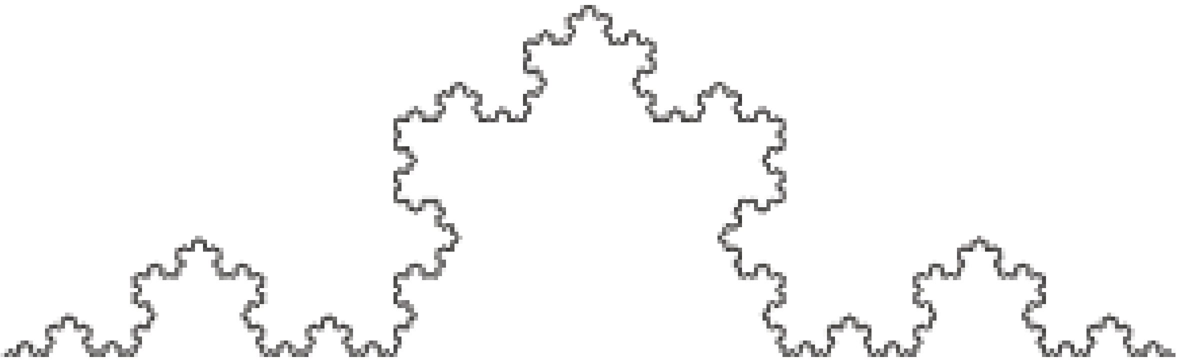
Self-Similarity Dimension

- **No. of small copies = (magnification factor)^D**
- Ex: No. of small copies = 3, mag factor = 2
- D = log(3)/log(2). (approx = 1.585.)



In-Between Dimensions

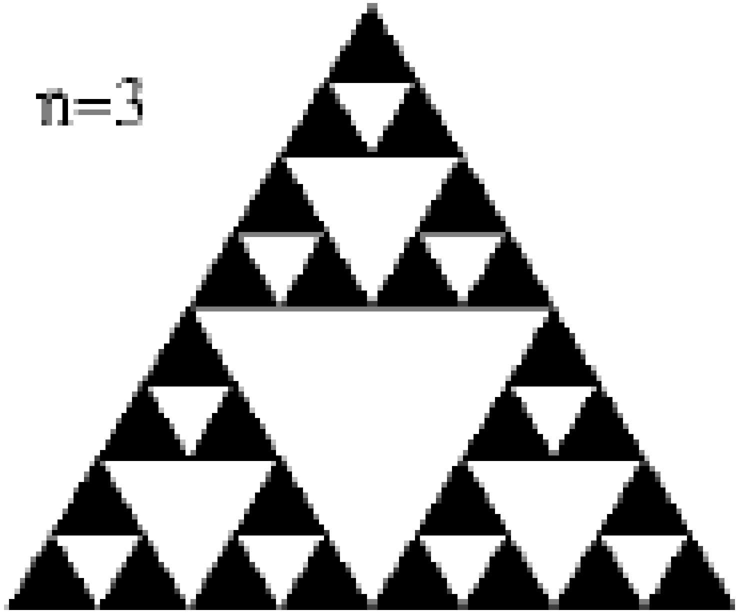
- The Koch curve has both 1-dimensional and 2-dimensional qualities.
- Infinite length in a finite area.



$$\text{Dimension } D = \log N / \log M = \log 4 / \log 3 \approx 1.26$$

In-Between Dimensions

- The Sierpiński triangle has zero area but infinite perimeter.



Dimension and Scaling

- **Increase in 'size' = (scale factor)^D**
- Ex: If sphere (3D) is stretched by a factor of two, it is now 8 times larger, since 2^3 = 8.
- The dimension tells you how the size of an object changes as it is scaled up.

Self-Similar and Scale-Free

- If an object is self-similar, it is **scale free.**
- Ex: There is no typical size of the bumps in a Koch curve that sets a scale.
- In contrast, there is a typical size to a tomato.
- In a fractal, if you were shrunk, you could not tell, because there are no objects that set a size scale.

(Real fractals are not self-similar forever, the way mathematical fractals are.)

Counting Boxes

Let N(s) = number of boxes of side s needed to cover an object.

$$N(s) = c\left(\frac{1}{s}\right)^{D} \quad \text{as s} \longrightarrow 0$$

Or $$N(s) = cs^{-D} \quad \text{as s} \longrightarrow 0$$

- Alternative notation: r or ϵ are often used instead of s.
- D is the **box-counting** dimension

Log-Log Plots

$$N(s) = c\left(\frac{1}{s}\right)^{D} \quad \text{as s} \longrightarrow 0$$

Equivalently (take log of both sides):

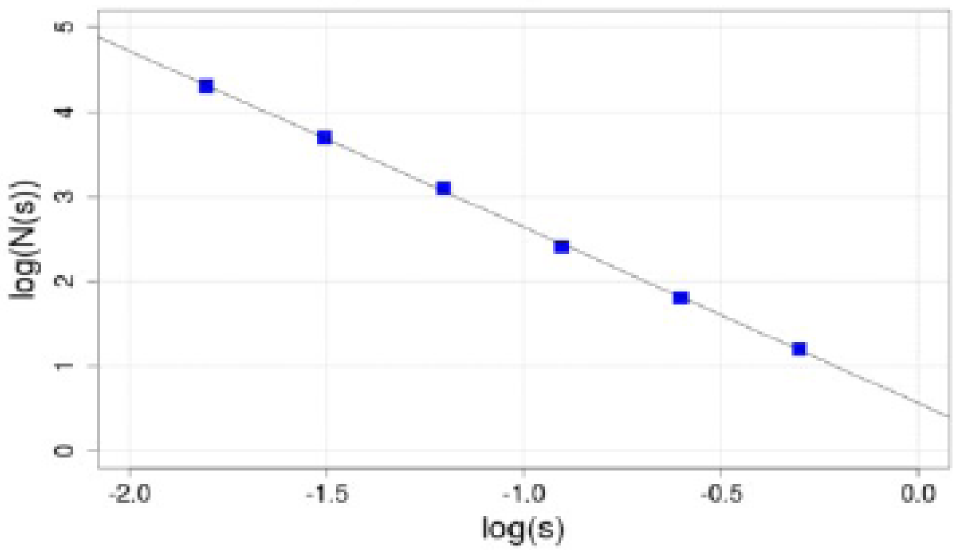
$$\log(N(s)) \;=\; -D\log(s) + \log(c)$$

This is the equation of a line: y = mx + b, where

y = log(N(s)), x = log(s), m = -D, b = log(c).

Log-Log Plots

- If we plot log(N(s)) vs. log(s), we expect a straight line:

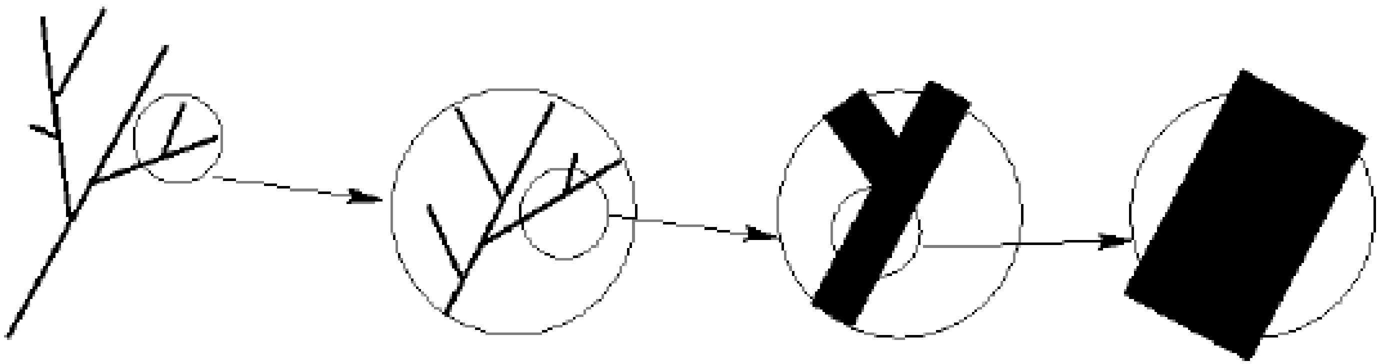


However, it's not always this simple...

1. The linear equation does not hold for large box size s.
2. For small s, the shape may no longer be self-similar, and/or we may run out of data.
3. The equation $\log(N(s)) = -D\log(s) + \log(c)$ is not true for everything. It is only true if the object is self-similar across scales. So a non-self similar shape will not have a linear log-log box-counting plot.

Problems with small s

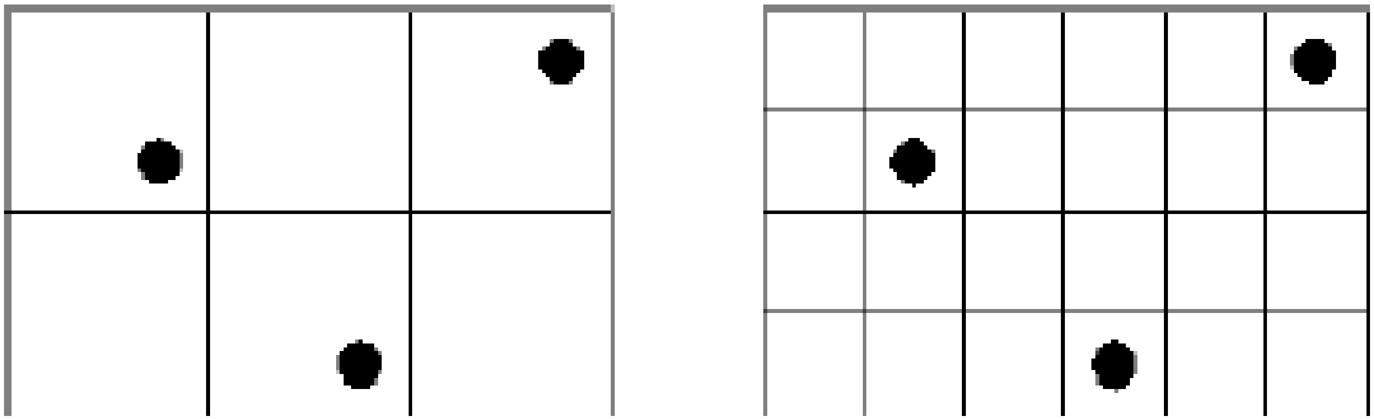
- In natural fractals, scaling does not continue forever:



Eventually branching stops and object looks two dimensional.

Problems with small s

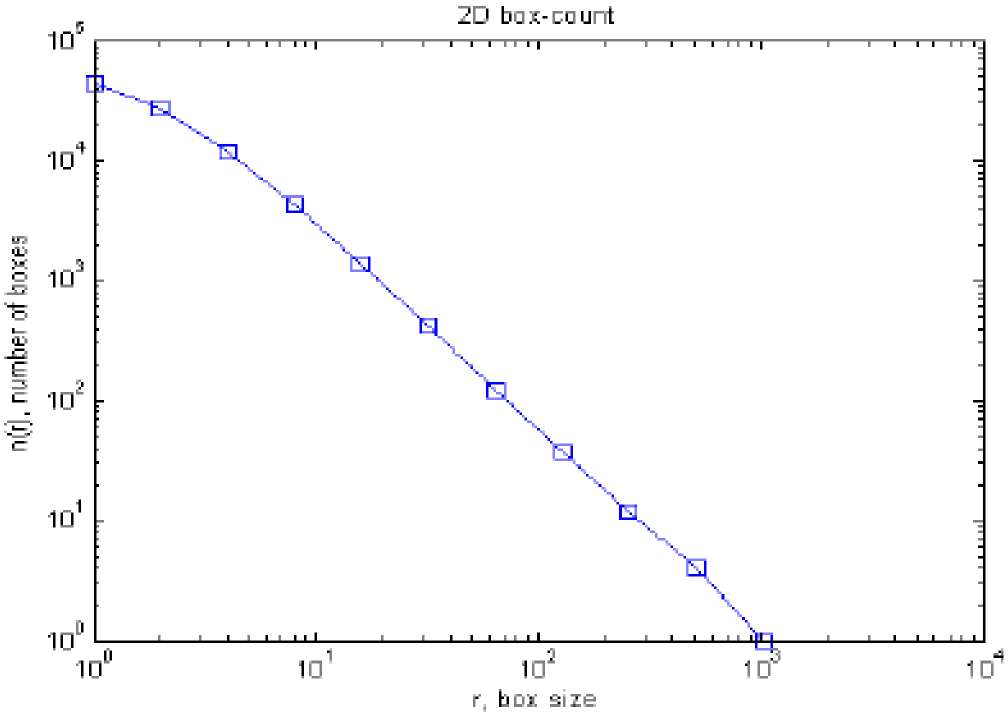
- Once one point is in each box, making boxes smaller will not increase N(s).



Log-Log Plots in Reality

So in practice we often see a log-log plot that is linear over a somewhat ambiguous middle region.

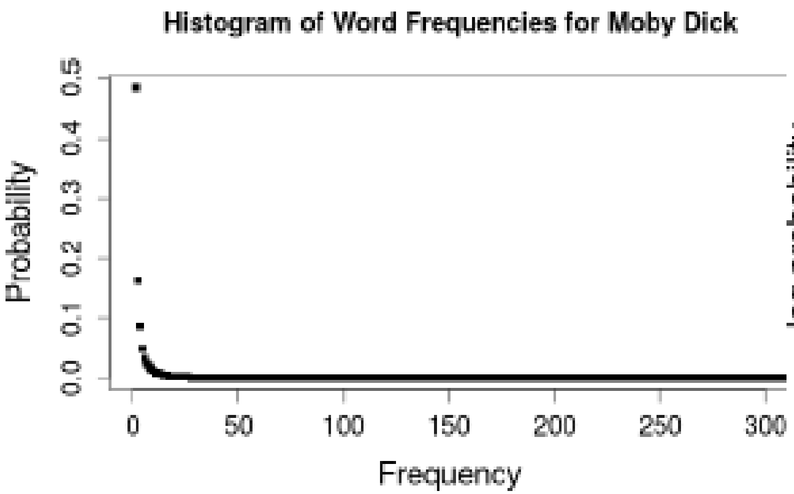
This poses challenges for estimating the slope D.

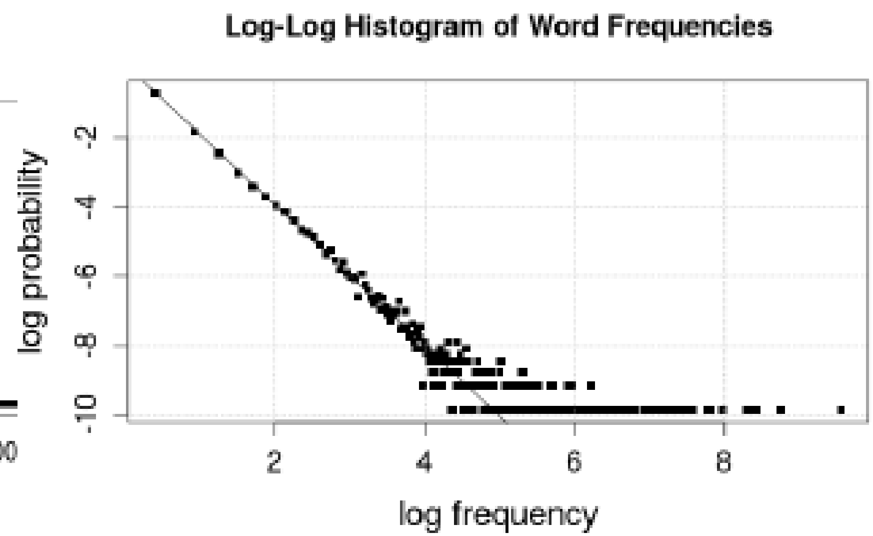


Box Counting and Power Laws

- Box-Counting: $N(s) = cs^{-D}$
- If equation is true, there is self-similarity, and we see a line on a log-log plot.
- Reverse logic: If we see linear behavior on log-log plot, there must be self-similarity.
- Power law: $p(x) = Ax^{-\alpha}$

Moby Dick

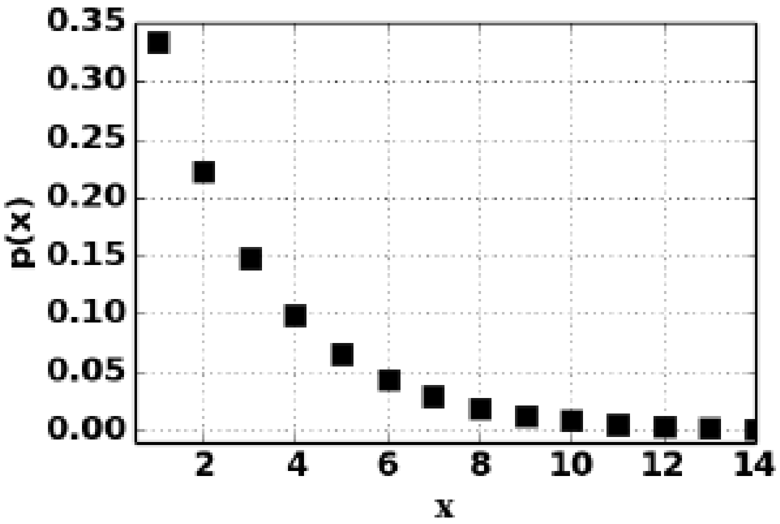




- Determine frequencies of all words.
- Plot histogram of frequencies.
- There are 18,855 different words. There is one word that appears 14,086 times. There are 9161 words that appear only once.

Exponential Distribution

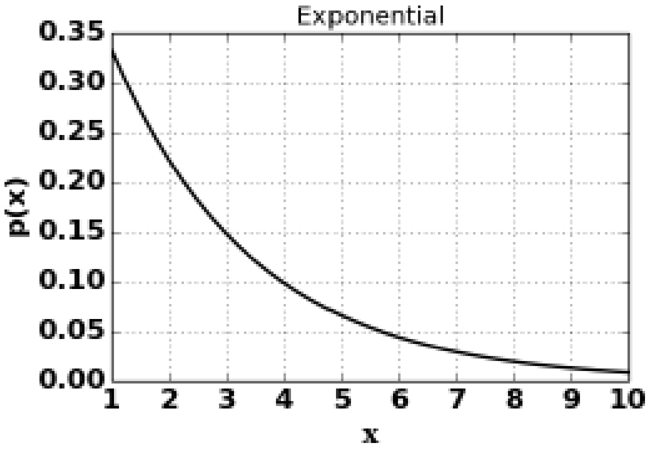
$$p(x) = (1/3)(2/3)^{x-1}$$

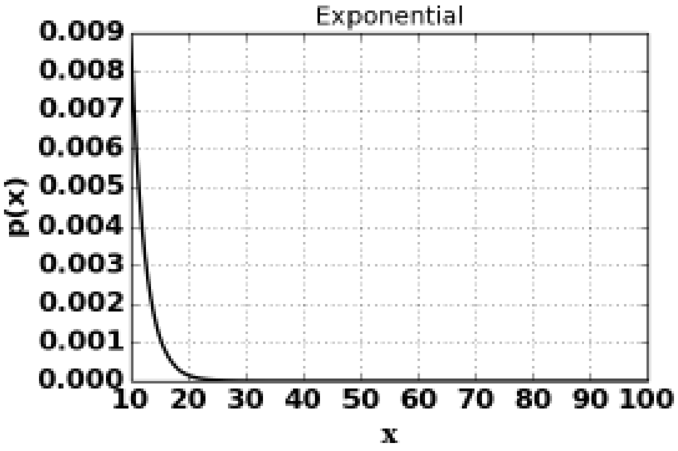


- Also called geometric distribution
- Large range of outcomes, but probability decreases very quickly
- Waiting times between events that happen with constant probability.

Exponentials are not Scale Free

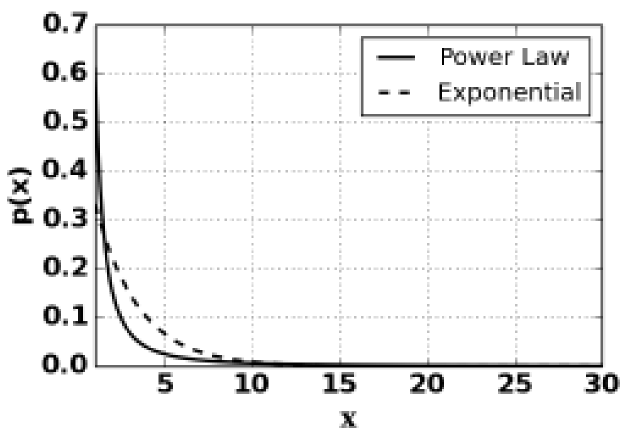
- Exponential functions do not look the same at all scales

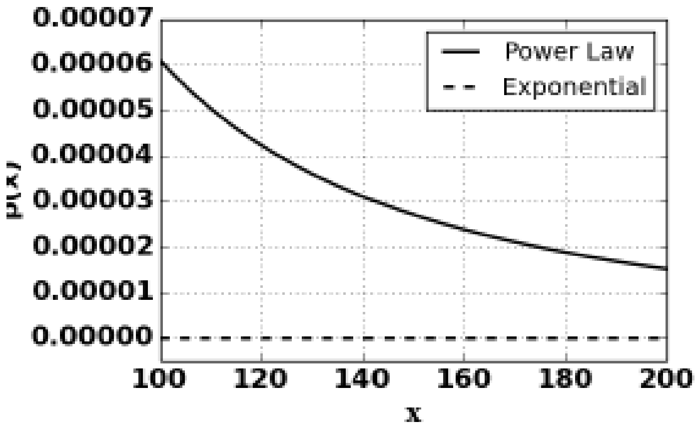




Power Laws have Long Tails

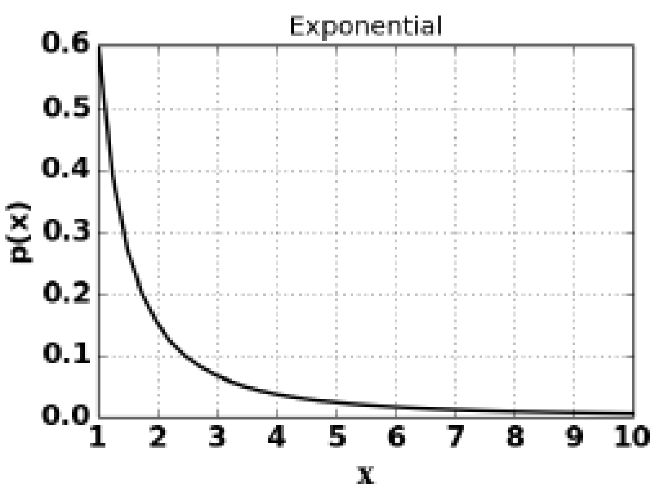
- Power laws decay much more slowly than exponentials.
- Very large x values, while rare, are still observed.
- Exponential: p(50) = 0.00000000078
- Power Law: p(50) = 0.000244

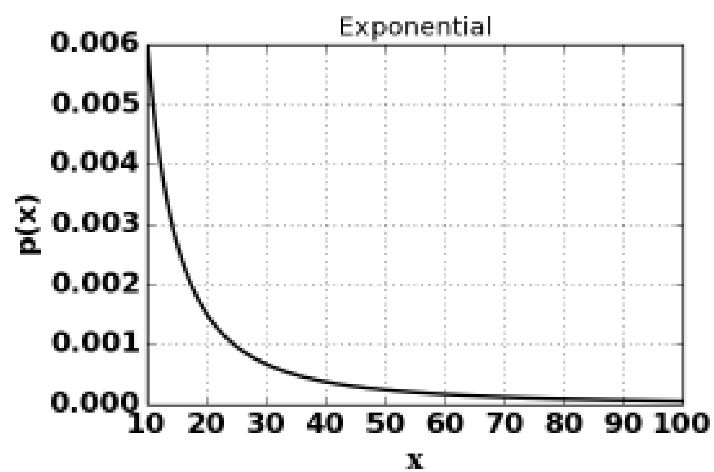




Power Laws are Scale Free

- Power laws look like the same at all scales.





Power Laws are Scale Free

- Power Law

$$p(x) = 0.61x^{-2}$$

$$\frac{p(x)}{p(2x)} = \frac{0.61x^{-2}}{0.61(2x)^{-2}} = 4$$

- Same ratio no matter what x is.

- Exponential

$$p(x) = (1/3)(2/3)^{x-1}$$

$$\frac{p(x)}{p(2x)} = \frac{(1/3)(2/3)^{x-1}}{(1/3)(2/3)^{2x-1}}$$

$$\frac{p(x)}{p(2x)} = (2/3)^{-x}$$

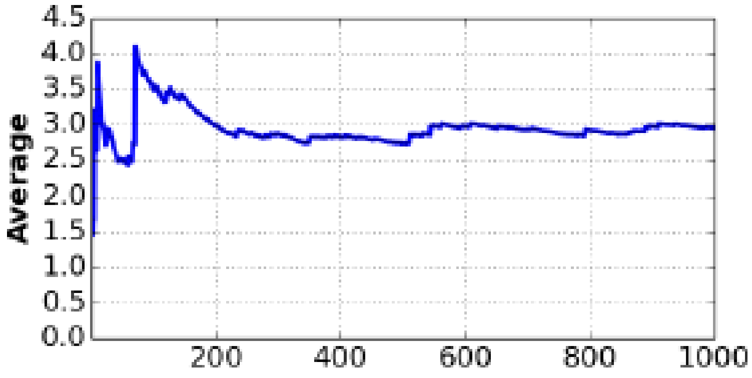
Ratio depends on x

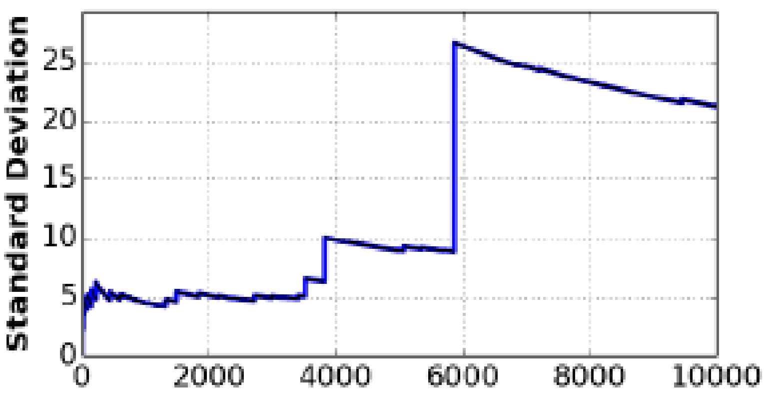
- Power laws are the **only** distribution that is scale free

Power Laws and Averages

$$p(x) = Ax^{-\alpha}$$

- $\alpha \leq 2$ the average does not exist.
- $\alpha \leq 3$ the standard deviation does not exist.
- Ex: $\alpha = 2.5$





Power Laws - Summary

- Long tails.
- Self-similar.
- Sometimes averages or standard deviation does not exist.
- Very different from most distributions we're used to.

Ackowledgement

- David Feldman, Santa Fe Institute