Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Dynamical Systems
Lecture 5.01

EEU45C09 / EEP55C09

Self Organising Technological Networks

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Pure Mathematics studies…

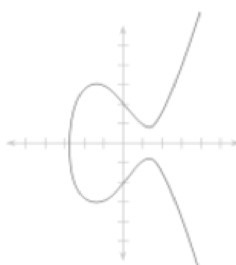
- **Quantity**

$1, 2, 3, \ldots$	$\ldots, -2, -1, 0, 1, 2 \ldots$	$-2, \frac{2}{3}, 1.21$	$-e, \sqrt{2}, 3, \pi$	$2, i, -2+3i, 2e^{i\frac{4\pi}{3}}$
Natural numbers	Integers	Rational numbers	Real numbers	Complex numbers

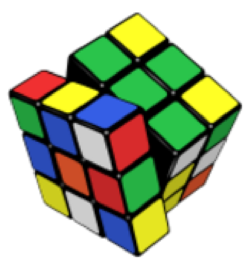
- **Structure**

$$\begin{matrix} (1,2,3) & (1,3,2) \\ (2,1,3) & (2,3,1) \\ (3,1,2) & (3,2,1) \end{matrix}$$

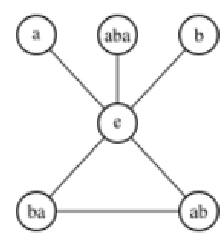
Combinatorics



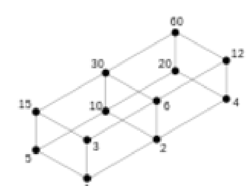
Number theory



Group theory



Graph theory

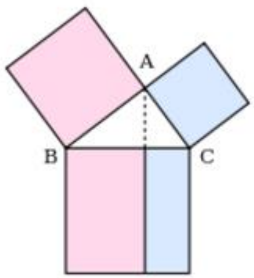


Order theory

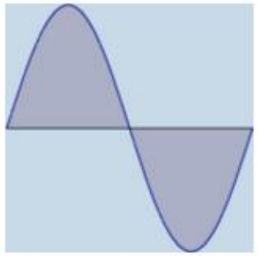
$$\begin{tikzcd} & B_3 \arrow[r] \arrow[d, hook] & \mathrm{PSL}_2(\mathbf{Z}) \arrow[d, hook] \\ \mathbf{Z} \arrow[ur, hook, bend left] \arrow[dr, hook, bend right] & & \\ & \widetilde{\mathrm{SL}_2(\mathbf{R})} \arrow[r] & \mathrm{PSL}_2(\mathbf{R}) \end{tikzcd}$$

Algebra

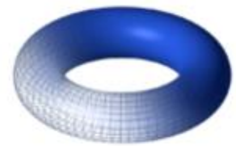
- Space



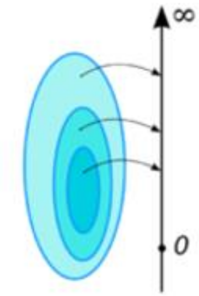
Geometry


Trigonometry


Differential geometry

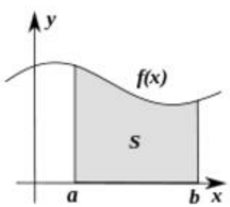

Topology


Fractal geometry

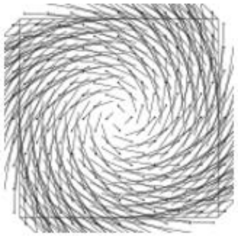


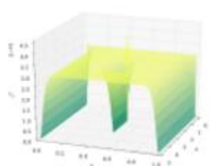
Measure theory

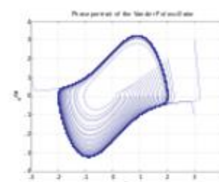
- Change



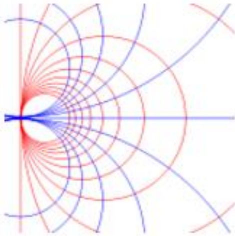
Calculus


Vector calculus


Differential equations

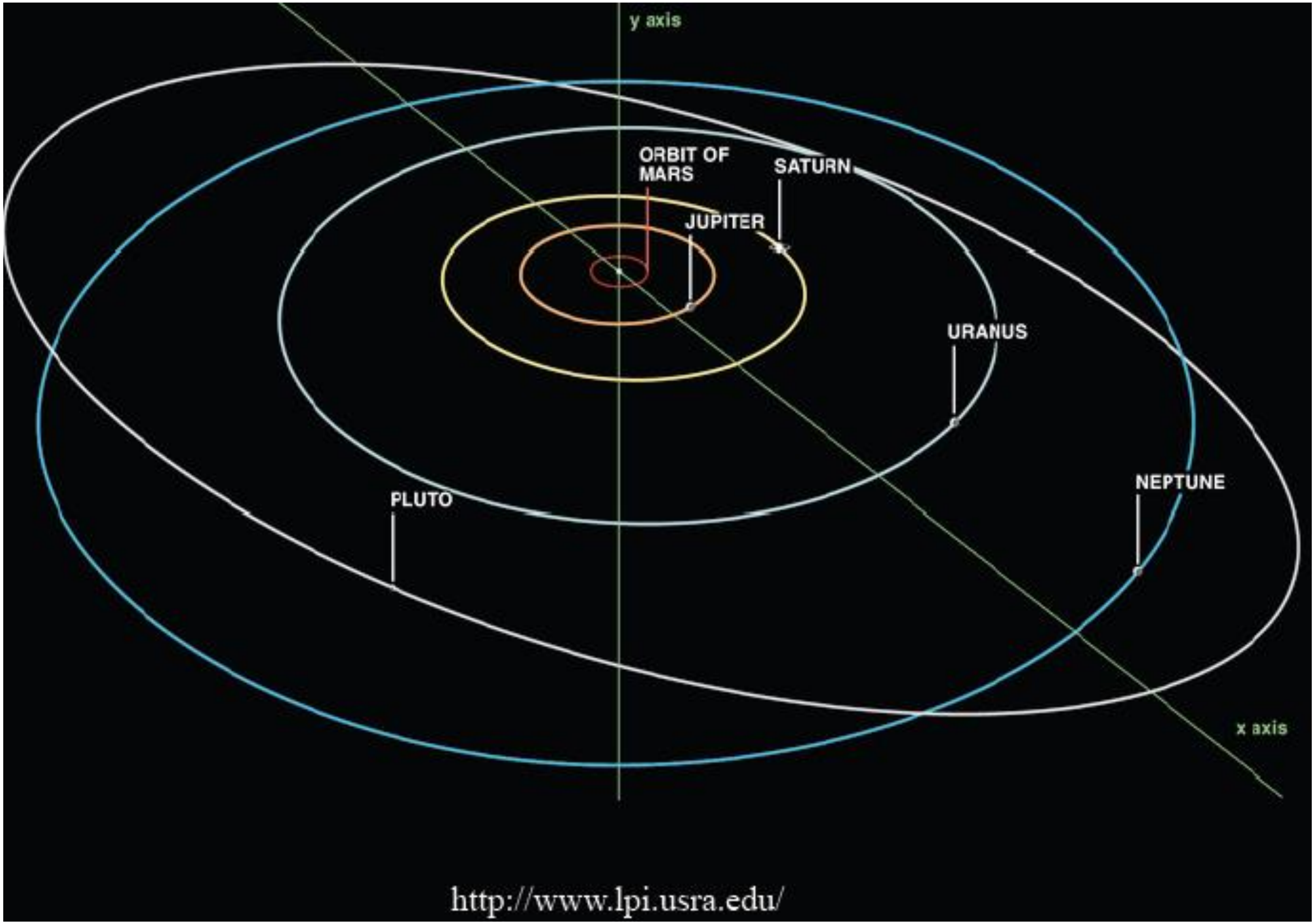

Dynamical systems


Chaos theory


Complex analysis

- **Dynamics ➔** the general study of how systems change over time

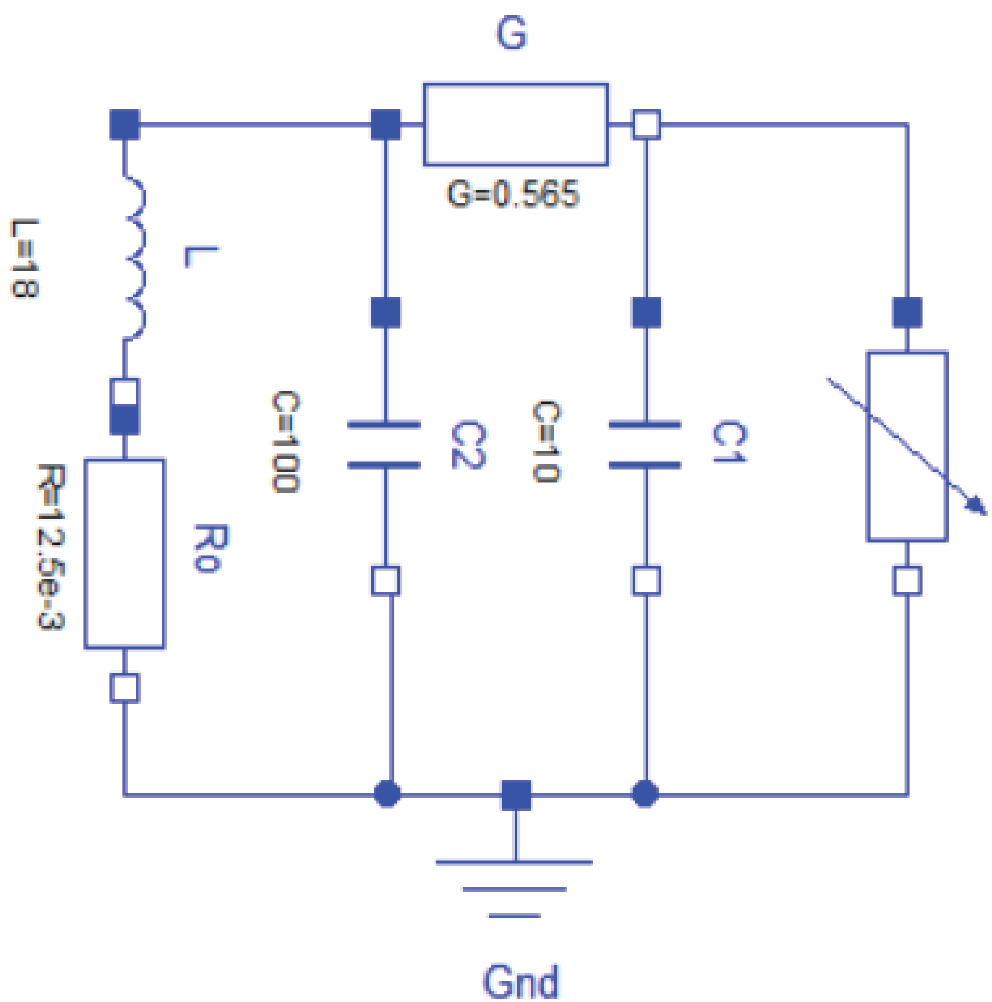
Planetary Dynamics



Fluid Dynamics

Dynamics of Turbulence

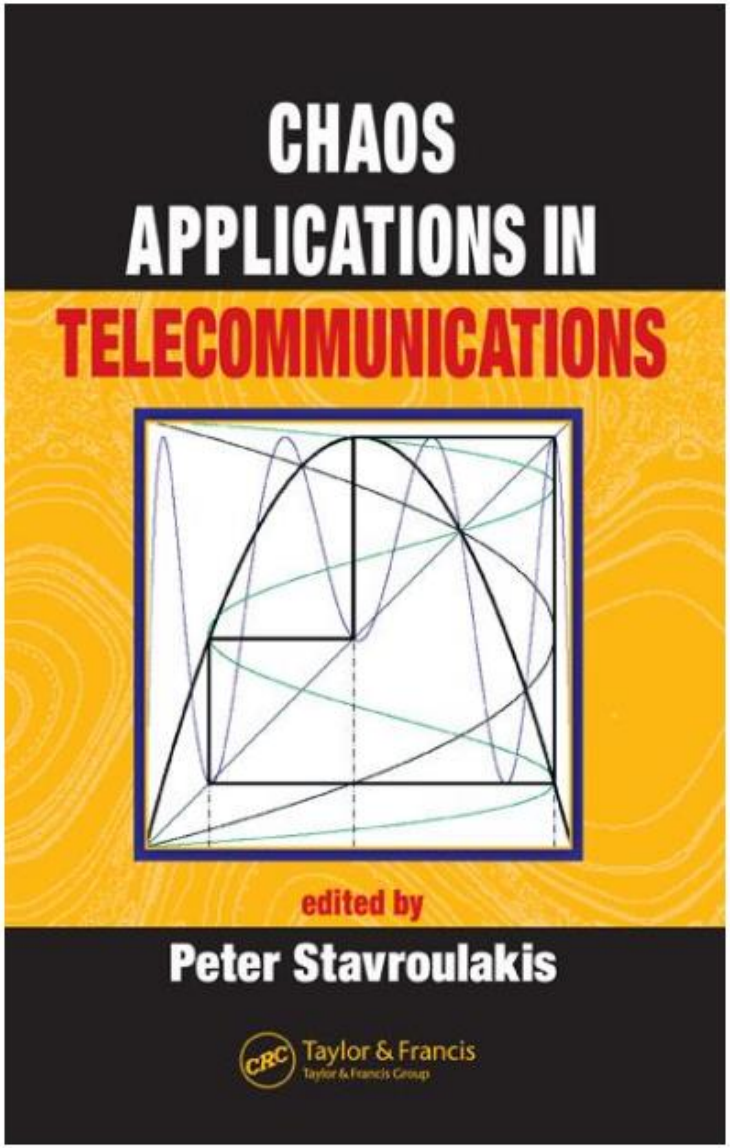
Chua Circuit



Non-periodic oscillator ➔ produces an oscillating waveform that, unlike an ordinary electronic oscillator, **never repeats**

https://www.youtube.com/watch?v=alqM1PWUNOE

Applications to Telecom



Chaotic **signal generation, modulation and demodulation** techniques

Chaos approach to asynchronous **DS-CDMA** systems

Channel equalization in chaotic communication systems

Dynamical Systems Theory

- The branch of mathematics of how systems change over time
  - Calculus
  - Differential equations
  - Iterated maps
  - Algebraic topology
  - etc.

- The *dynamics of a system*: the manner in which the system changes

- Dynamical systems theory gives us a ***vocabulary*** and ***set of tools*** for describing dynamics

Laplace vs Poincare

Pierre- Simon Laplace, 1749 – 1827

Henri Poincaré, 1854 – 1912

Laplace

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

—Pierre Simon Laplace, *A Philosophical Essay on Probabilities*

Poincare

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that **small differences in the initial conditions produce very great ones in the final phenomena**. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible..."

The Butterfly Effect

Chaos Theory

"You've never heard of Chaos theory? Non-linear equations? Strange attractors?"

https://www.youtube.com/watch?v=n-mpifTiPV4

Chaos in Nature

- Dripping faucets
- Electrical circuits
- Solar system orbits
- Weather and climate (the "butterfly effect")
- Brain activity (EEG)
- Heart activity (EKG)
- Computer networks
- Population growth and dynamics
- Financial data

Deterministic Chaos

"The fact that the simple and deterministic equation [i.e., the Logistic Map] can possess dynamical trajectories which look like some sort of random noise has disturbing practical implications. It means, for example, that apparently erratic fluctuations in the census data for an animal population need not necessarily betoken either the vagaries of an unpredictable environment or sampling errors; they may simply derive from a rigidly deterministic population growth relationship...Alternatively, it may be observed that in the chaotic regime, arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely. This means that, even if we have a simple model in which all the parameters are determined exactly, long-term prediction is nevertheless impossible"

—— Robert May, 1976

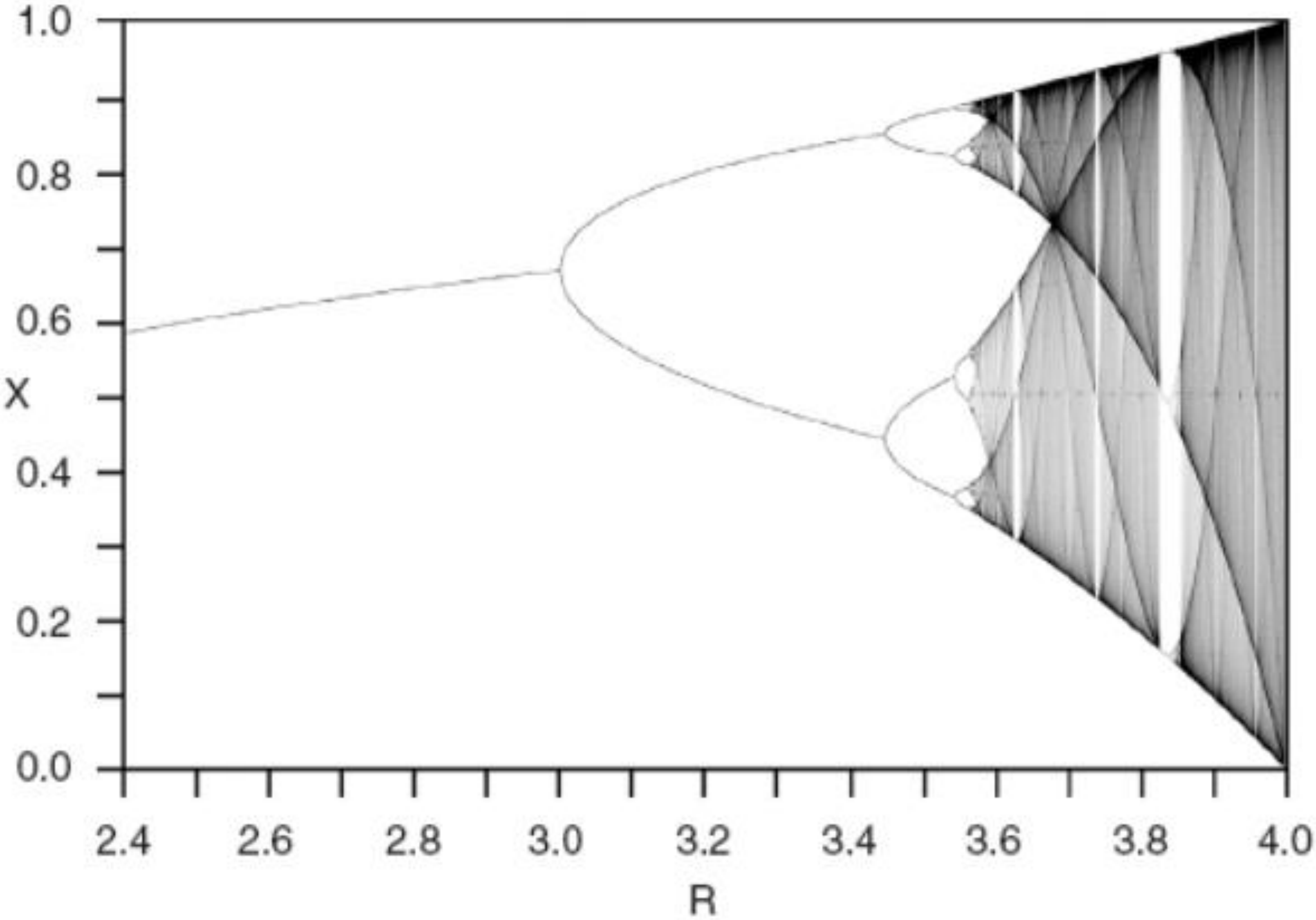
Deterministic Chaos

Chaos: Seemingly random behavior with *sensitive dependence on initial conditions*

Logistic map: A simple, completely deterministic equation that, when iterated, can display chaos (depending on the value of R).

Deterministic chaos: Perfect prediction, *a la* Laplace's deterministic "clockwork universe", is impossible, even in principle, if we're looking at a chaotic system.

Logistic Map Bifurcation Diagram



Universality in Chaos

Bifurcations in the Logistic Map

$R_1 \approx 3.0$: period 2

$R_2 \approx 3.44949$ period 4

$R_3 \approx 3.54409$ period 8

$R_4 \approx 3.564407$ period 16

$R_5 \approx 3.568759$ period 32

$R_\infty \approx 3.569946$ period ∞
(chaos)

Rate at which distance between bifurcations is shrinking:

$$\frac{R_2 - R_1}{R_3 - R_2} = \frac{3.44949 - 3.0}{3.54409 - 3.44949} = 4.75147992$$

$$\frac{R_3 - R_2}{R_4 - R_3} = \frac{3.54409 - 3.44949}{3.564407 - 3.54409} = 4.65619924$$

$$\frac{R_4 - R_3}{R_5 - R_4} = \frac{3.564407 - 3.54409}{3.568759 - 3.564407} = 4.66842831$$

$$\vdots$$

$$\lim_{n\to\infty}\left(\frac{R_{n+1} - R_n}{R_{n+2} - R_{n+1}}\right) \approx 4.6692016....$$

Universality in Chaos

Bifurcations in the Logistic Map

$R_1 \approx 3.0$: period 2

$R_2 \approx 3.44949$ period 4

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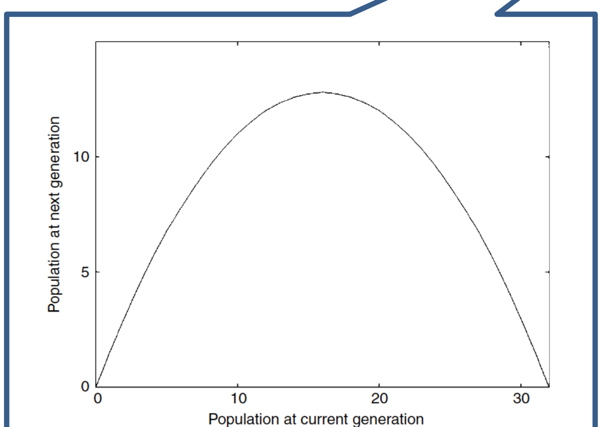


Rate at which distance between bifurcations is shrinking:

In other words, each new bifurcation appears about 4.6692016 times faster than the previous one.

Feigenbaum derived this constant mathematically!

He also showed that any unimodal (one-humped) map will have the same value for this rate.

A "universal" constant!!!

$$\lim_{n\to\infty}\left(\frac{R_{n+1}-R_n}{R_{n+2}-R_{n+1}}\right)\approx 4.6692016....$$

Feigenbaum's constant

Significance of Dynamics and Chaos for Complex Systems

- Complex, unpredictable behavior from simple, deterministic rules
- Dynamics gives us a vocabulary for describing complex behavior
- There are fundamental limits to detailed prediction
- At the same time there is universality: "Order in Chaos"

Relation between Chaos and Fractals

- Chaotic systems and fractals are both **iterated functions**
  - Each iteration takes the state of the previous iteration as an input to produce the next state

- Some chaotic systems like the bifurcation diagram show fractal repetition when you zoom in

- Although fractals can show beautiful patterns, you can't actually predict what their value is going to be for a specific iteration other than by calculating all the preceding iterations
  - So in a sense, **fractals are just beautiful chaos**

https://www.quora.com/What-is-the-connection-between-chaos-theory-and-fractals

Ways to think about Complex Systems

1. Ingredients: Dynamics, Information, Computation, Evolution and Learning, ...
2. Methods and Models: Statistical Physics, Agent-based models, Networks, Chaos and dynamics, ...
3. Phenomena: Immune system, ecosystems, economies, auction markets, evolutionary systems, the brain, natural computation, ...
4. Theoretical: General principles?
5. A particular interdisciplinary mix, style, and point of view.

However one thinks of complex systems, Chaos and Dynamical Systems play a role.

Chaos

Thoughts on how to think about chaos:

“We take the emergence of ’chaos’ as a science of nonlinear phenomena... as a vast process of sociodisciplinary convergence and conceptual reconfiguration.... In order to come up with an exhaustive historical analysis of these origins [of “nonlinear science”] one needs to be able to deal at once with domains as varied as fluid mechanics, parts of engineering, and population dynamics.”

Aubin and Dahan-Dalmedico, *Historia Mathematica* 29 (2002), 1-67. doi:10.1006/hmat.2002.2351

They refer to chaos as having an “**ample and bushy genealogy**.”

Chaos is not a sudden revolution.

Complex Systems and Chaos/Dynamical Systems

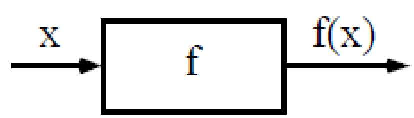
- much the same can be said about Complex Systems.
- There are many different streams of thought that flow together to form the study of Complex Systems: Chaos/Dynamical Systems, Genetic Algorithms/A-life, Economics, and so on.
- The confluence of these streams is not a unitary discipline or a coherent theory, but a “sociodisciplinary convergence and conceptual reconfiguration.”
- Complex Systems has a tangled genealogy. But one of the deepest roots is the study of chaos and dynamical systems.

An Overview of Dynamical Systems and Chaos

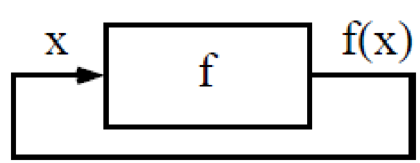
- A **Dynamical System** is any system that changes over time according to some rule:
  - A differential equation
  - A system of differential equations
  - Iterated functions
  - Cellular automata
- The goal of this brief introduction is to define a handful of terms, define chaos and sensitive dependence on initial conditions, and briefly discuss some of its implications.
- I will focus on iterated functions.
- Let's start with an example.

The Squaring Rule

- Consider the function $f(x) = x^2$. What happens if we start with a number and repeatedly apply this function to it?
- E.g., $3^2 = 9$, $9^2 = 81$, $81^2 = 6561$, etc.
- The iteration process can also be written $x_{n+1} = x_n^2$.
- In this example, the initial value 3 is the **seed**, often denoted x_0.
- The sequence $3, 9, 81, 6561, \cdots$ is the **orbit** or the **itinerary** of 3.
- Picture the function as a "box" that takes x as an input and outputs $f(x)$:



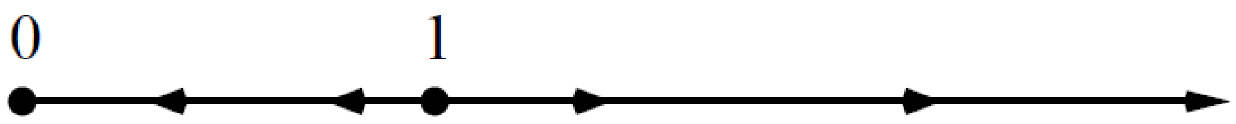
- Iterating the function is then achieved by feeding the output back to the function, making a feedback loop:



The Squaring Rule

In dynamics, we are usually interested in the long-term behavior of the orbit, not in the particulars of the orbit.

- The seed 3 tends toward infinity—it gets bigger and bigger.
- Any $x_0 > 1$ will tend toward infinity.
- If $x_0 = 1$ or $x_0 = 0$, then the point never changes. These are fixed points.
- If $0 \leq x_0 < 1$, then x_0 approaches 0.
- We can summarize this with the following diagram:



- 0 and 1 are both **fixed points**
- 0 is a **stable** or **attracting** fixed point
- 1 is an **unstable** or **repelling** fixed point

Ackowledgement

- Melanie Mitchell, Santa Fe Institute
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