Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Dynamical Systems
Lecture 5.06

EEU45C09 / EEP55C09
Self Organising Technological Networks

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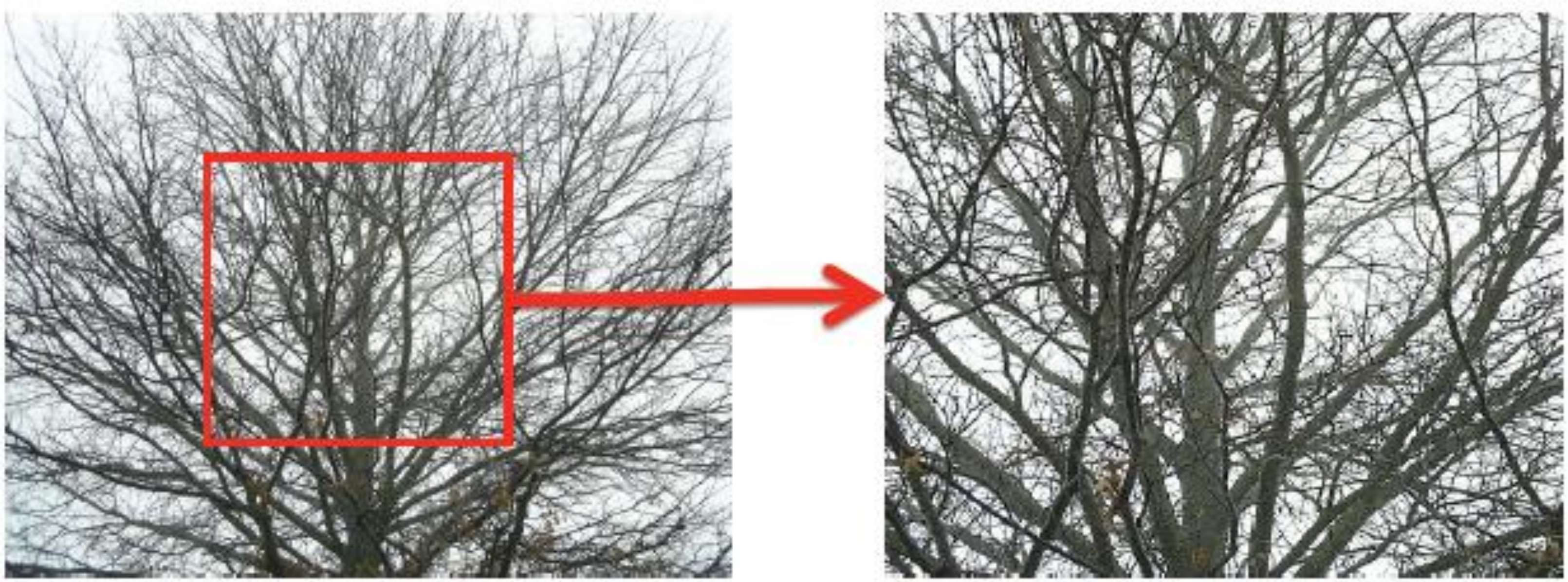
What is a Fractal?

Objects with "self-similarity" at different scales

Trees are Fractal

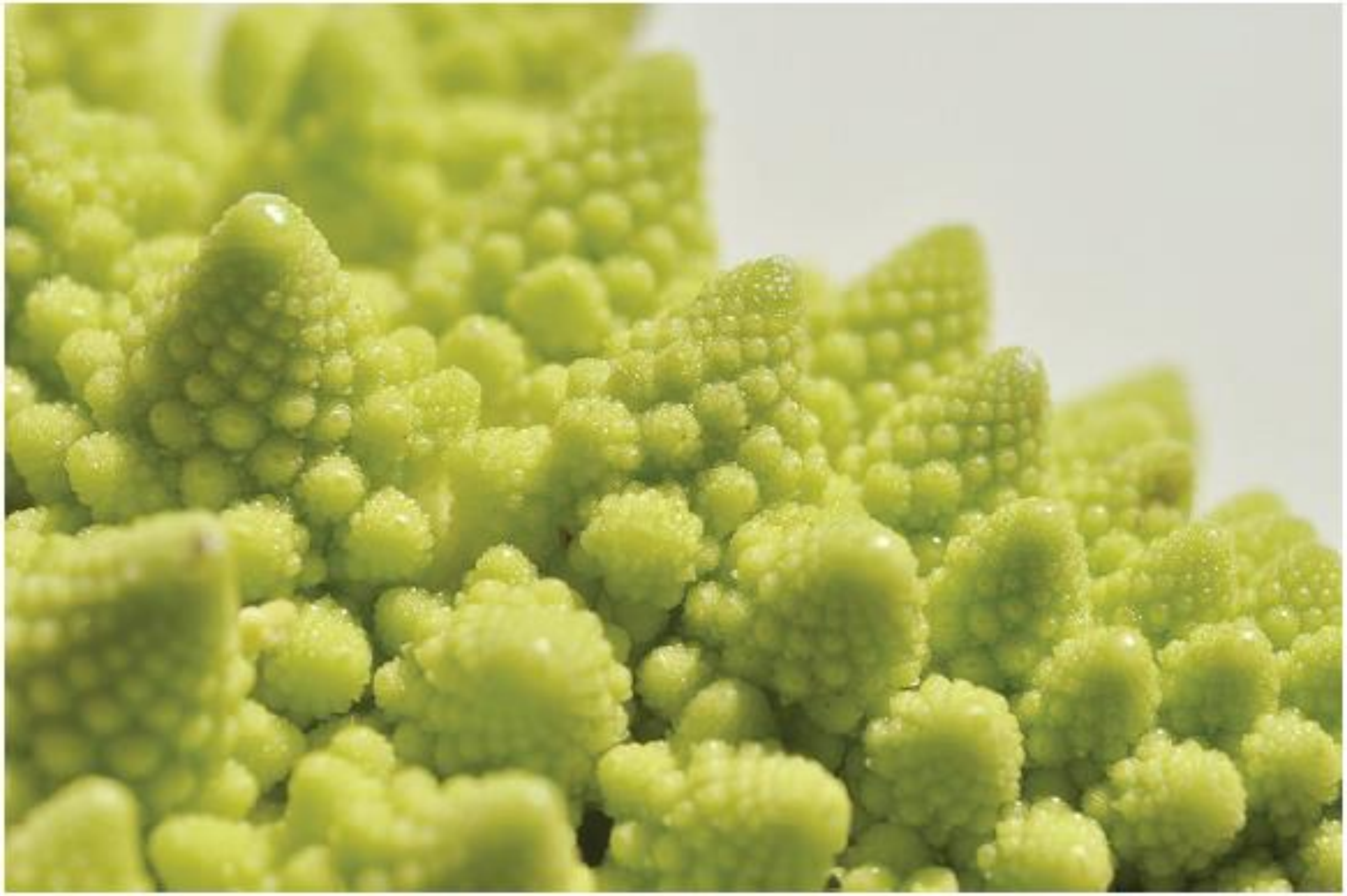
Self-Similarity



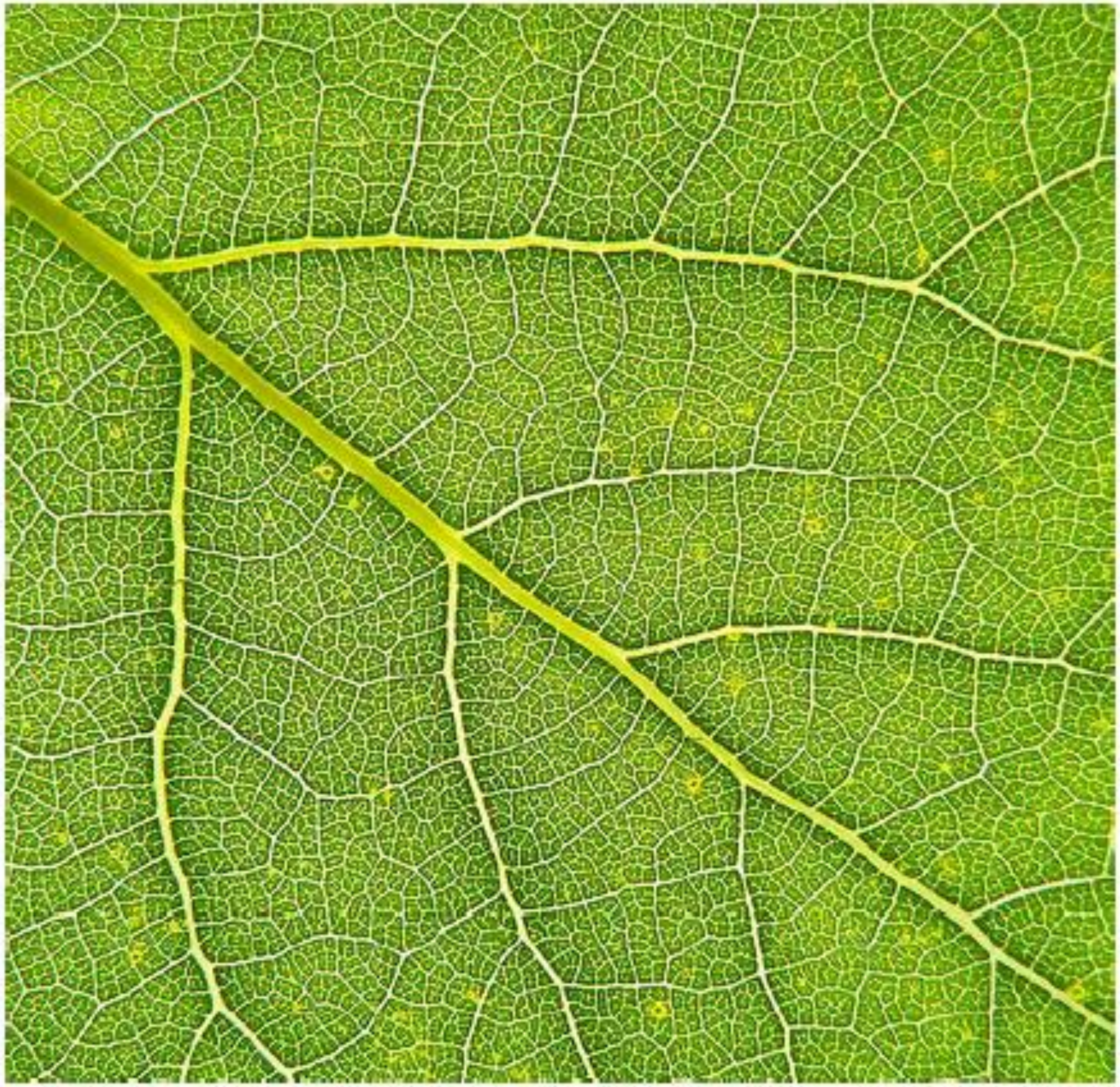
Self-Similarity

Self-Similarity

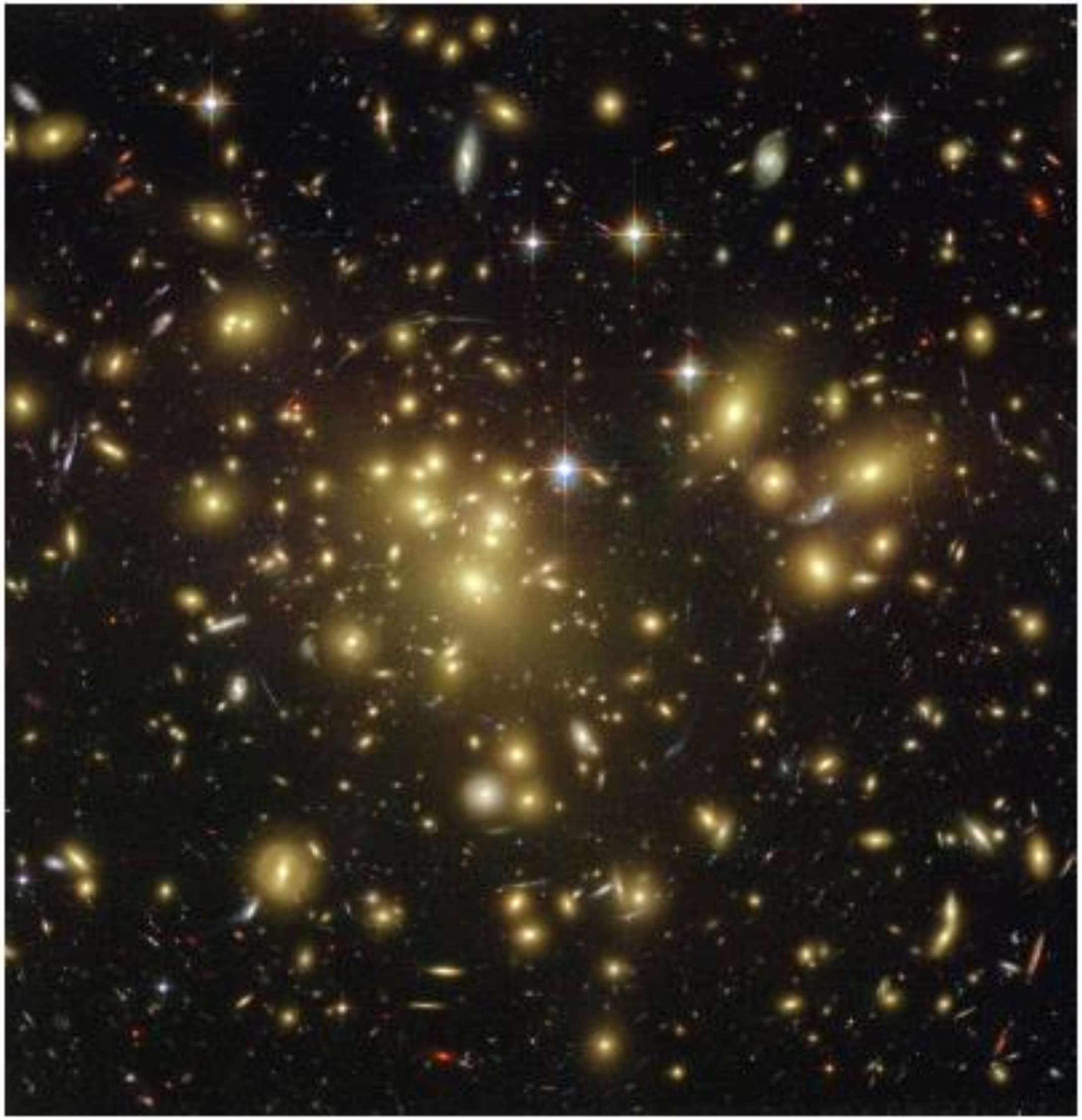
Broccoli is Fractal



Leaf Veins are Fractal

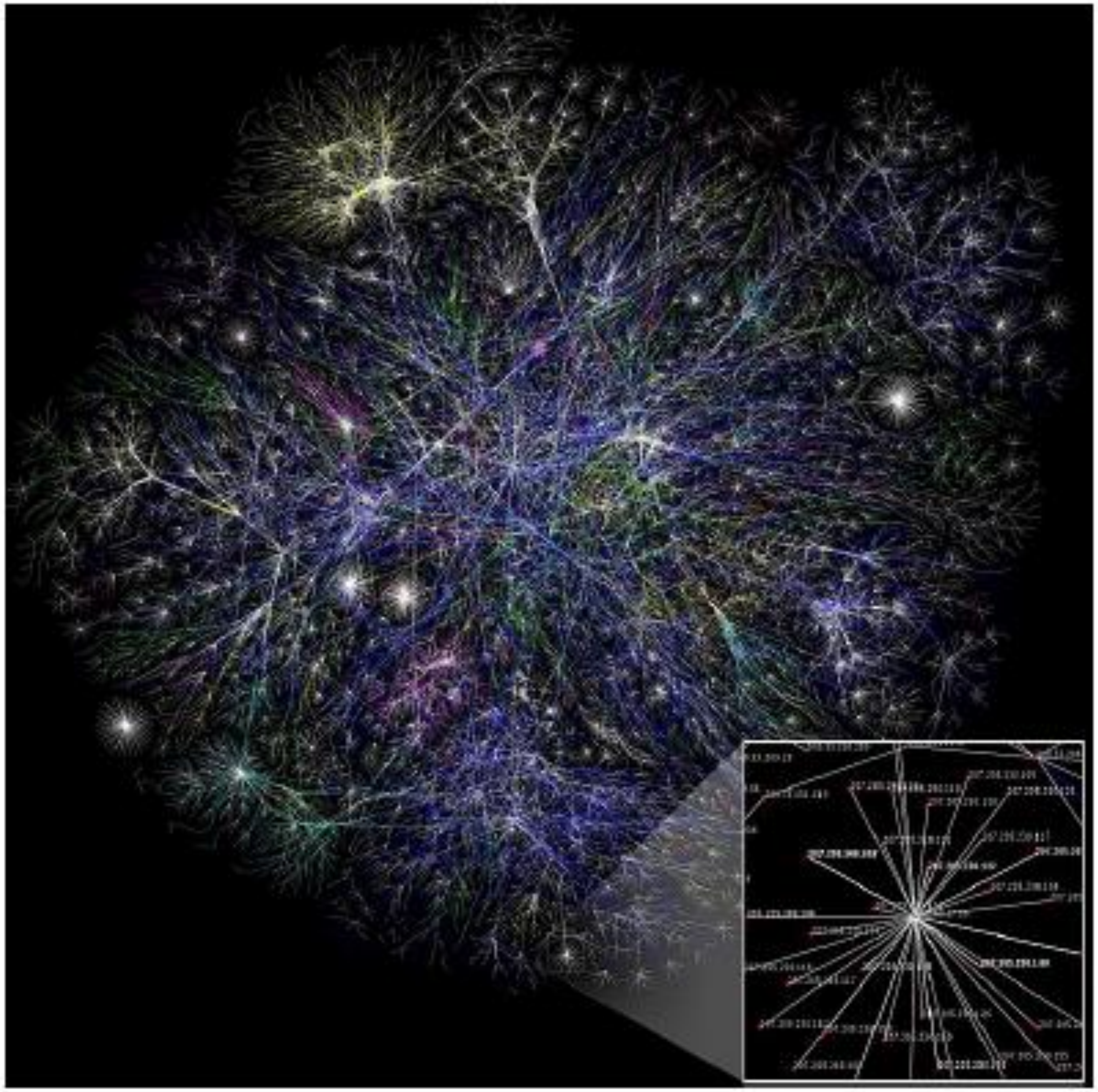


Galaxy Clusters are Fractal



Mountain Ranges are Fractal

The WWW is Fractal



The forefather

Benoit Mandelbrot, 1924–2010

Many mathematicians have studied the notions of self-similarity, and of “fractional dimension” and what an object with a fractional dimension would look like.

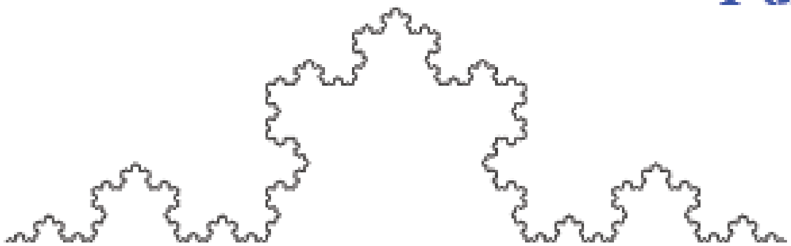
The term *fractal*, to describe such objects, was coined by the mathematician Benoit Mandelbrot, from the Latin root for “fractured”.

Mandelbrot’s goal was to develop a mathematical “theory of roughness” to better describe the natural world.

He brought together the work of different mathematicians in different fields to create the field of *Fractal Geometry*.

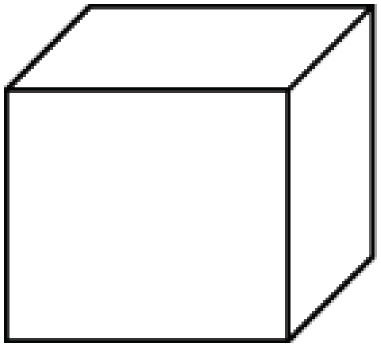
Fractal Dimension

1-dimensional



In between 1 and 2 dimensional!

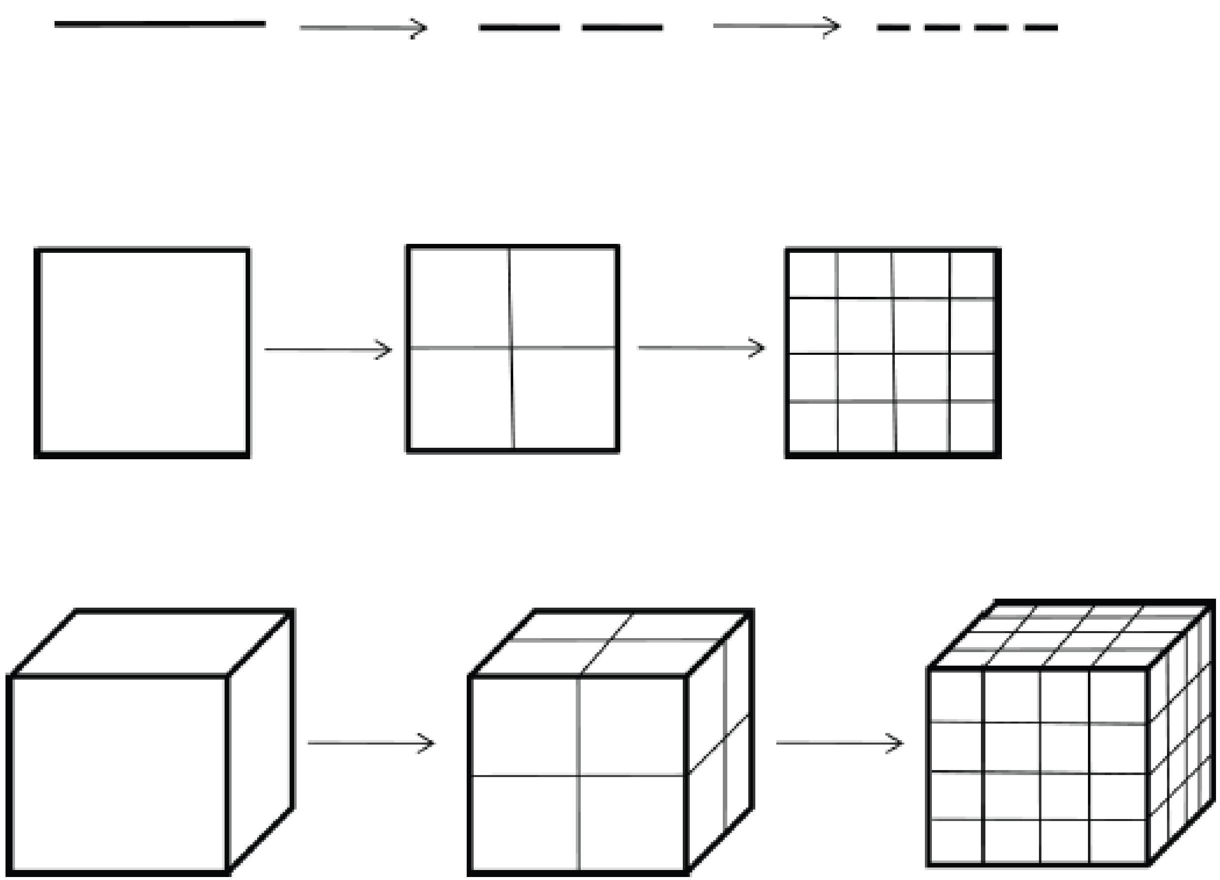
2-dimensional



3-dimensional

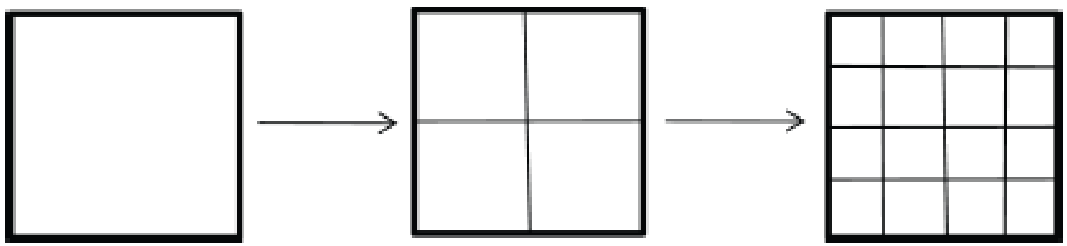
Bisecting Sides

What happens when you continually bisect (cut in two equal halves) the sides of lines, squares, cubes, etc.?

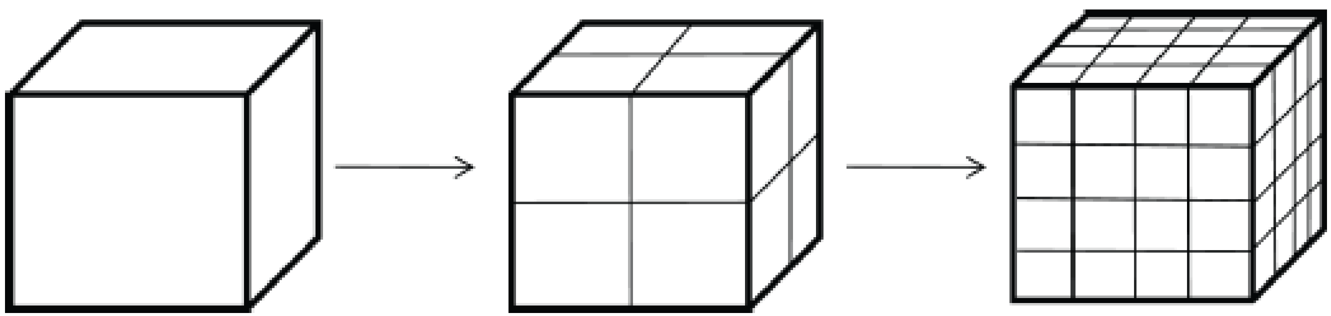


Dimension 1: Each level is made up of two 1/2-sized copies of previous level

Dimension 2: Each level is made up of four 1/4-sized copies of previous level



Dimension 3: Each level is made up of eight 1/8-sized copies of previous level

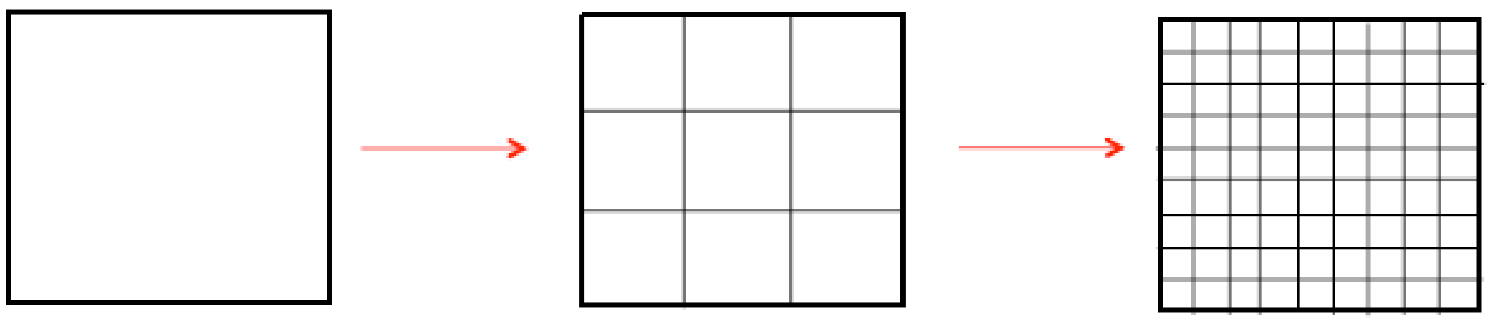


Dimension 4: Each level is made up of sixteen 1/16-sized copies of previous level

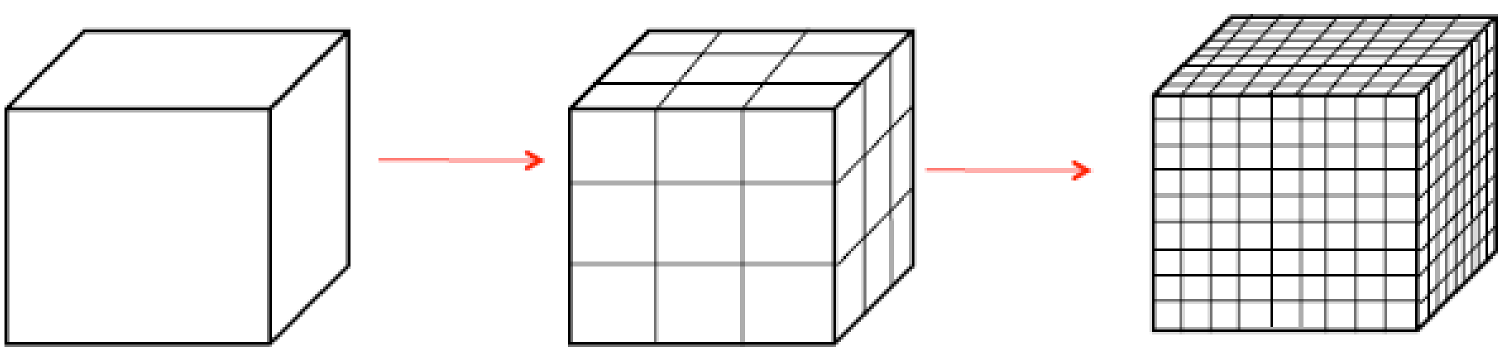
Dimension 20: Each level is made up of ??

Trisecting Sides

Dimension 1: Each level is made up of three 1/3-sized copies of previous level



Dimension 2: Each level is made up of nine 1/9-sized copies of previous level



Dimension 3: Each level is made up of 27 1/27-sized copies of previous level

M-secting Sides

Dimension 1: Each level is made up of ~~three 1/3-sized copies~~ of previous level

M $1/M$-sized copies

Dimension 2: Each level is made up of ~~nine 1/9-sized copies~~ of previous level

M^2 $1/M^2$-sized copies

Dimension D: Each level is made up of M^D $1/M^D$-sized copies of previous level

Dimension 3: Each level is made up of ~~27 1/27-sized copies~~ of previous level

M^3 $1/M^3$-sized copies

Definition of Dimension

Create a geometric structure from a given D-dimensional object (e.g., line, square, cube, etc) by repeatedly dividing the length of its sides by a number M.

Then each level is made up of M^D copies of the previous level.

Call the number of copies N.

Then $N = M^D$.

We have:

$$\log N = D \log M$$

$$D = \log N / \log M$$

log N =D log M

D = log N / log M

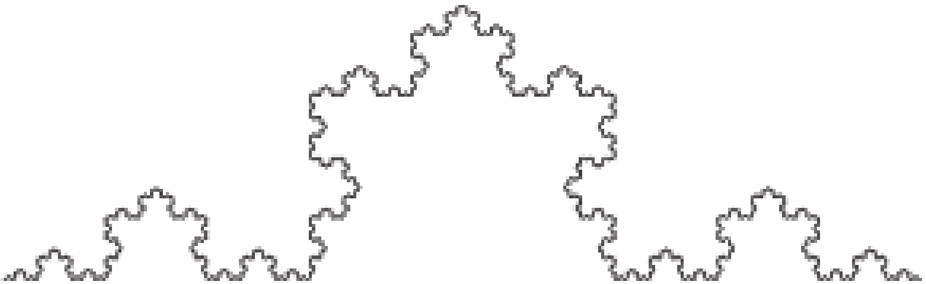
Dimension 1: N = 2, M = 2, D = log 2 / log 2 = 1

N = 3 , M = 3, D = log 3 / log 3 = 1

Dimension 2: N = 4, M = 2, D = log 4 / log 2 = 2

N = 9, M = 3, D = log 9 / log 3 = 2

Koch curve: Here, N = 4, M = 3



So Fractal Dimension = log 4 / log 3
≈ 1.26

A measure of how the increase in number of copies scales with the decrease in size of the segment -- Roughly – the density of the self-similarity.

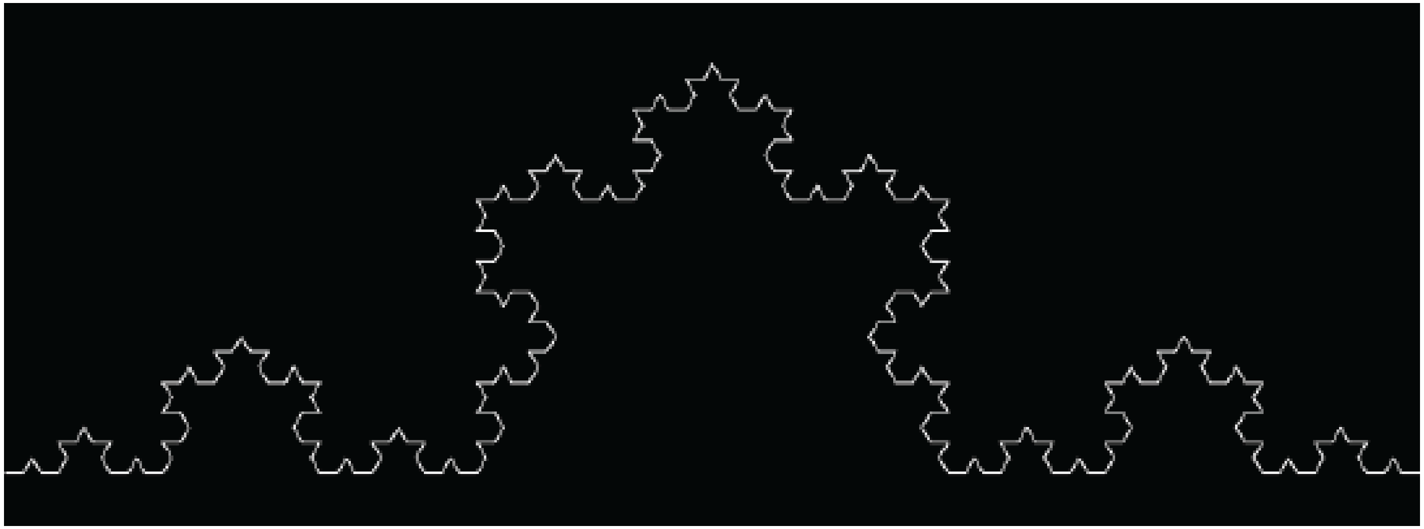
Hausdorff Dimension

N = number of copies of previous level = 4

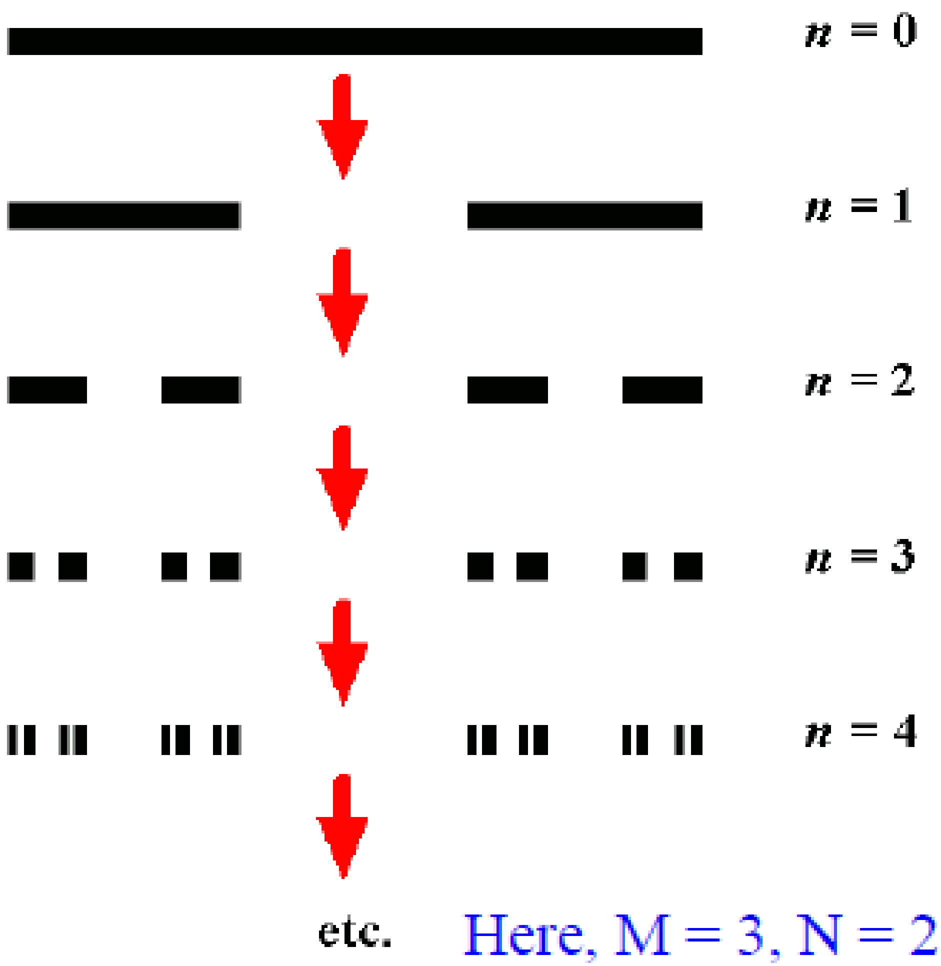
M = reduction factor from previous level = 3

Dimension $D = \log N / \log M = \log 4 / \log 3 \approx 1.26$

This version of fractal dimension is called *Hausdorff Dimension*, after the German mathematician Felix Hausdorff



Cantor Set



Here, M = 3, N = 2

So Fractal Dimension = log 2 / log 3
≈ .63

Cauliflower

Fractal Structure of a White Cauliflower

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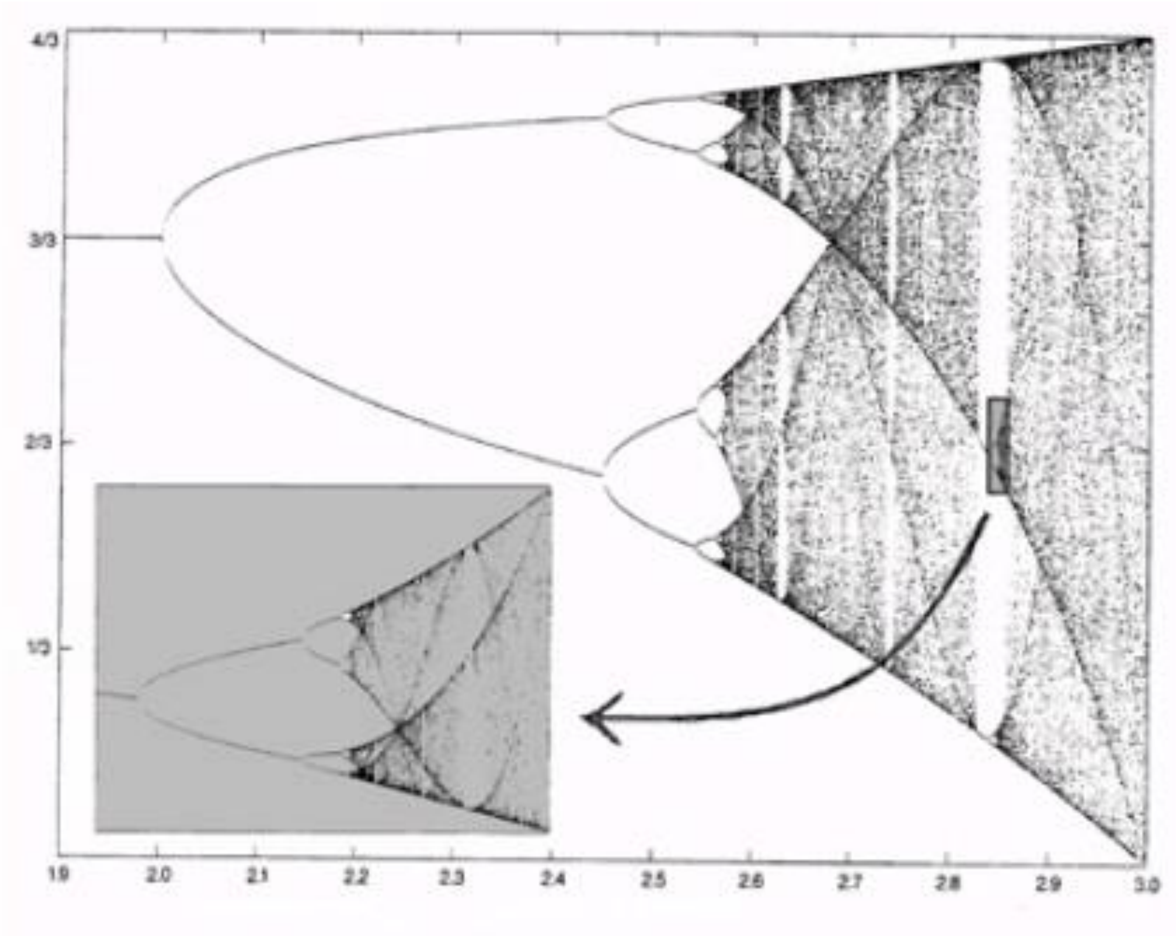
(Received 17 September 2004)

The fractal s … the box-counting method on its cross sectior … ± 0.02, independent of the direction. Fron … of a cauliflower is about 2.8. The vertical crc … set of a rectangular tree. We discuss the con … the vertical cross section has an angle of 67° in our model.

$D \approx 2.8$

Logistic Map’s Bifurcation Diagram



Fractal dimension ≈ 0.538

Coastlines

West Coast of Great Britain:
D ≈ 1.25

Coast of Australia:
D ≈ 1.13

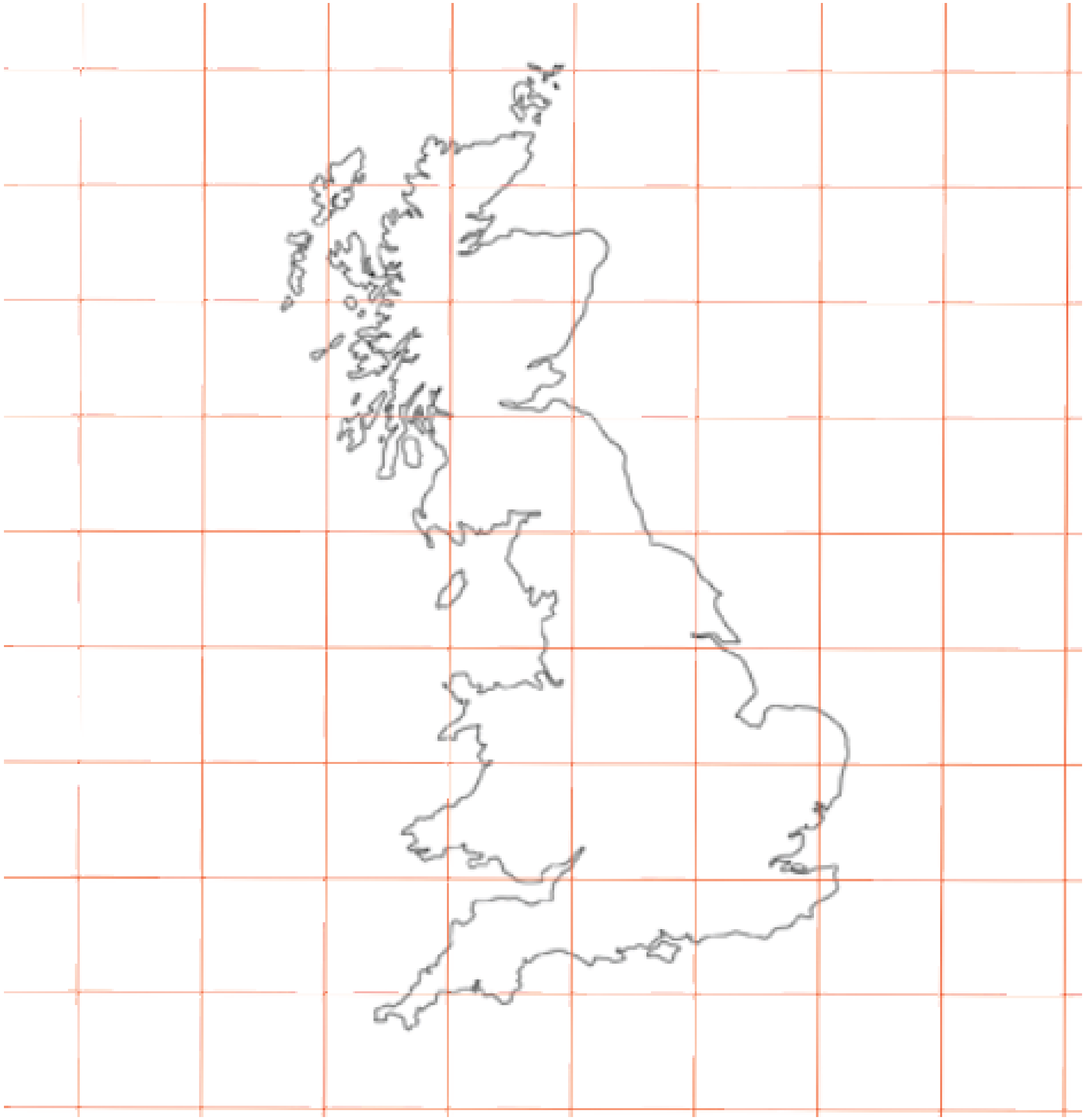
Coast of South Africa:
D ≈ 1.02

Box Counting Method

Box Counting Method

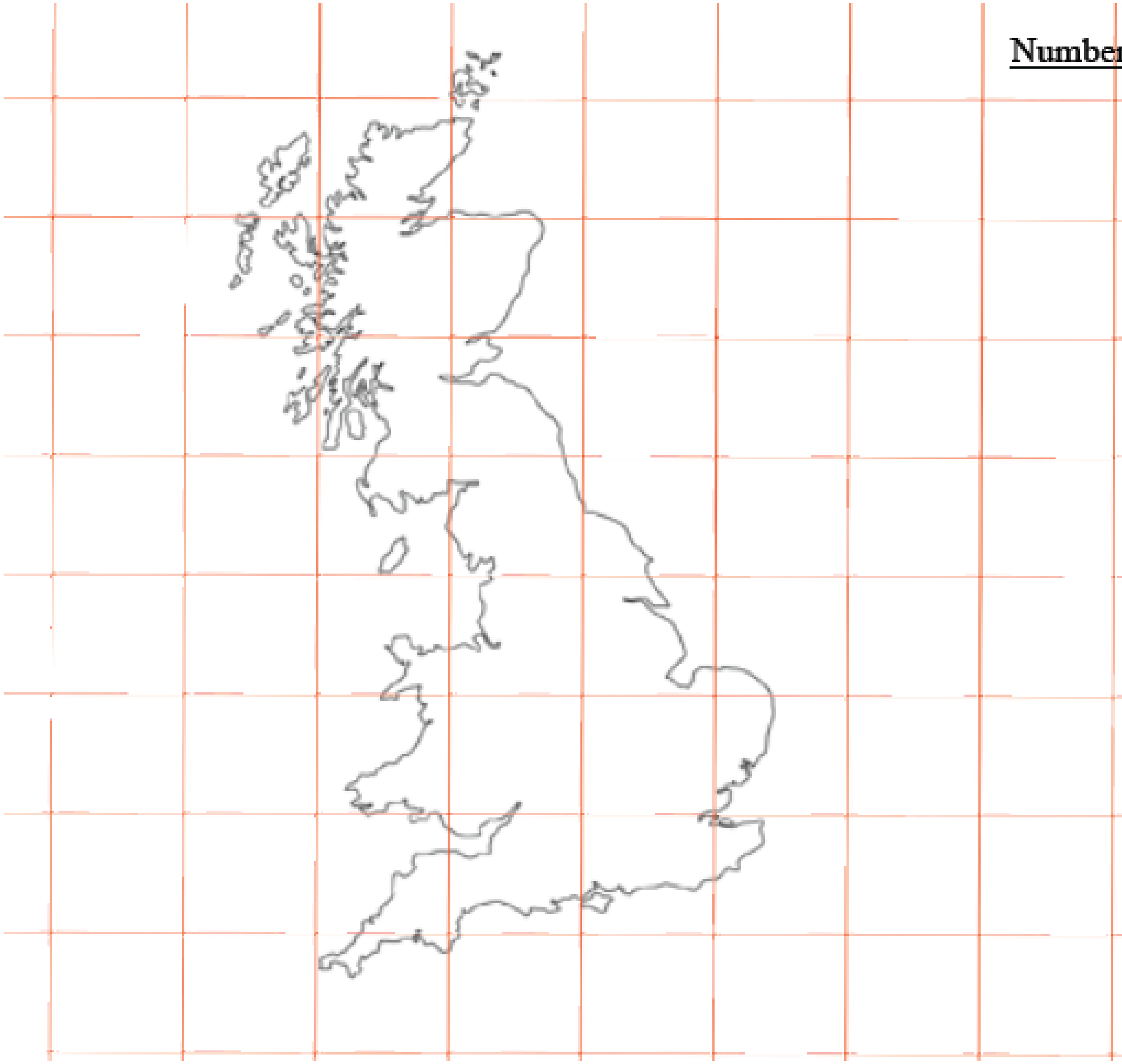
Number of boxes	Length of side
36	10

Box Counting Method



Number of boxes	Length of side
36	10
34	11

Box Counting Method



Number of boxes	Length of side
36	10
34	11
27	12

Hausdorff and Box Counting

Hausdorff dimension:

$$\log N = D \log M,$$

where N = number of copies of figure from previous level, and M = size reduction factor of a side of the previous level.

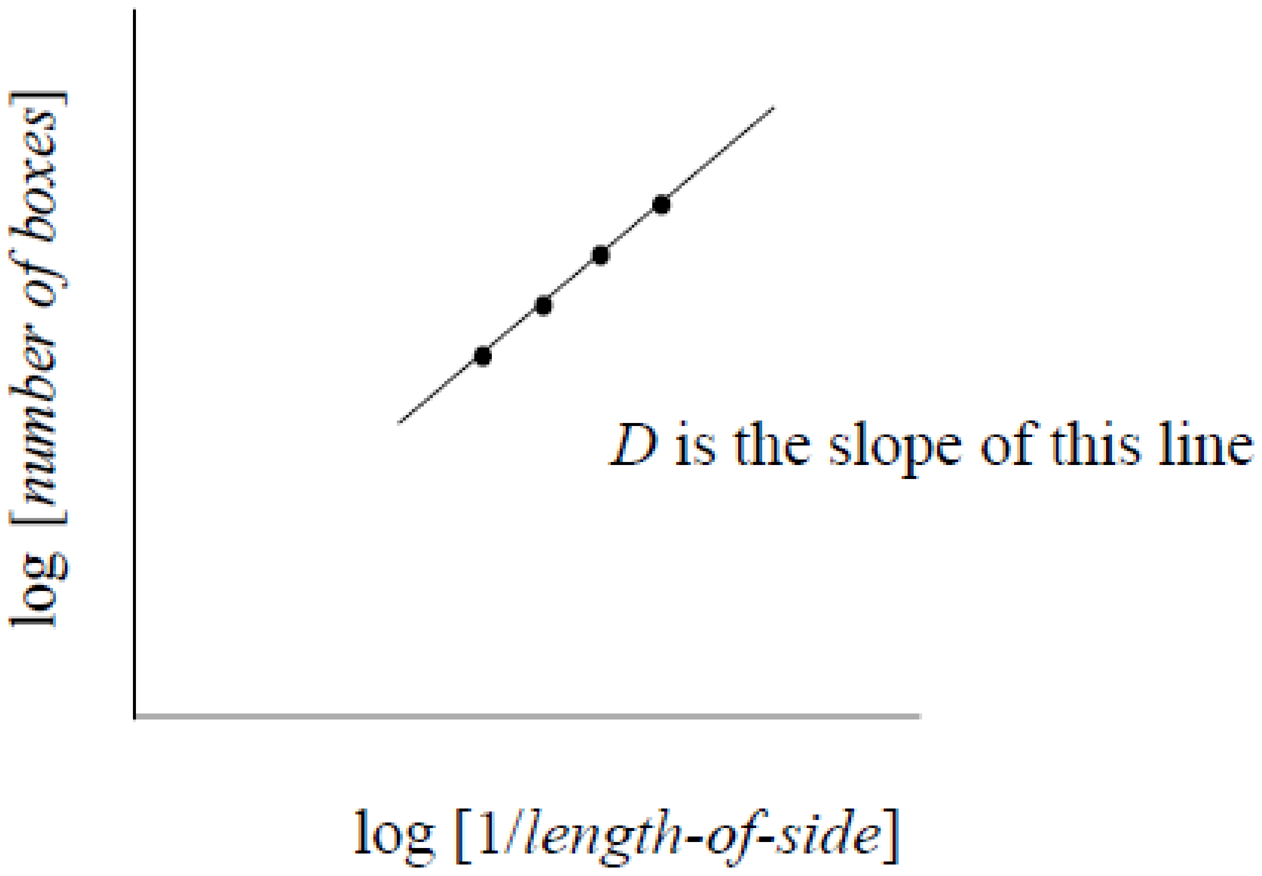
For box-counting, this can be approximated by

$$\log\ [\textit{number of boxes}] = D \log [1/\textit{length-of-side}]$$

D is called the ***Box-Counting Dimension***

Hausdorff and Box Counting

log [*number of boxes*] = *D* log [1/*box-size*]



Ackowledgement

- Melanie Mitchell, Santa Fe Institute