

2011-2012 学年第一学期《高等数学 I》期末试卷

授课班号 _____ 年级专业 _____ 学号 _____ 姓名 _____

题型	选择题	填空题	计算题	综合题	总分	审核
得分						

一. 填空题 (8×4 分)

1. $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x-1} \right)^{x+1} = e^4$ $(1 + \frac{4}{x-1})^{\frac{x-1}{4} \cdot \frac{(x+1) \cdot 4}{x-1}}$

得分	阅卷人

2. 设 $f(x) = \lim_{n \rightarrow \infty} \frac{3nx}{1-nx}$, 则其连续区间是 $(-\infty, 0), (0, +\infty)$

3. 设 $y = y(x)$ 由方程 $x \sin y + y e^x = 0$ 所确定, 则 $y'(0) = 0$
 $x=0, y=0$
 $\sin y + x \cos y \cdot y' + y' \cdot e^x + y \cdot e^x = 0$

4. 设 $f(x)$ 在 $x=1$ 处可导, $\lim_{x \rightarrow 0} \frac{f(\cos x) - f(1)}{x^2} = 2$, 则

$\frac{f(\cos x) - f(1)}{\cos x - 1} \cdot \frac{\cos x - 1}{x^2}$

$f'(1) = -4$

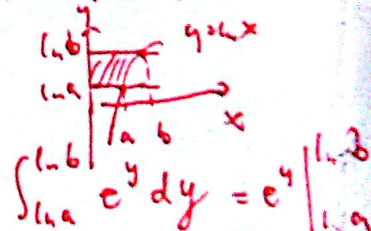
5. 曲线 $y = x e^{3x}$ 的拐点坐标是 $(-\frac{2}{3}, -\frac{2}{3} e^{-2})$
 $y' = e^{3x} + 3x e^{3x} = (3x+1)e^{3x}$
 $y'' = 3e^{3x} + 3(3x+1)e^{3x} = (9x+6)e^{3x}$
 $y'' = 0 \Rightarrow 9x+6=0 \Rightarrow x = -\frac{2}{3}$

6. 设 $f'(\sin^2 x) = \cos^2 x$ ($|x| < 1$), 则

$f(x) = x - \frac{1}{2} x^2 + C$

7. 设 $\varphi(x)$ 可导, 则 $\frac{d}{dx} \int_{\varphi(x)}^{\varphi(x^2)} \sin t^2 dt = \sin \varphi(x^2) \cdot \varphi'(x^2) \cdot 2x - \sin \varphi(x) \cdot \varphi'(x)$

8. 由曲线 $y = \ln x$ 与直线 $x=0, y=\ln a, y=\ln b$ ($b > a > 0$) 所围成图形的面积 $S = b - a$



二. 计算题 (6×6 分)

1. 求 $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n} \right)$

得分	阅卷人

$\left(\frac{1}{n^2+n+1} \right)$

$\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n}$ 记 $\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n}$

且 $\frac{1+2+\dots+n}{n^2+n+1} \leq \frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n} \leq \frac{1+2+\dots+n}{n^2+n+1}$

极限为 $\frac{1}{2} \lim_{n \rightarrow \infty} \frac{n(n+1)}{n^2+n+1} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n+1}{n+1} = \frac{1}{2}$

2.

求极限 $\lim_{x \rightarrow 1} \frac{(x-1)^2 \ln x}{x-1-\sin(x-1)}$.

$$\begin{aligned} \text{解: } & \text{令 } x-1=t, \text{ 则 } x=1+t, \text{ 当 } x \rightarrow 1 \text{ 时, } t \rightarrow 0. \\ & \lim_{t \rightarrow 0} \frac{t^2 \ln(1+t)}{t-\sin t} = \lim_{t \rightarrow 0} \frac{t^3}{t-\sin t} = \lim_{t \rightarrow 0} \frac{3t^2}{1-\cos t} \\ & = \lim_{t \rightarrow 0} \frac{3t^2}{\frac{1}{2}t^2} = 6. \end{aligned}$$

3.

设 $y=y(x)$ 由方程组 $\begin{cases} x=2te^{-t}+1 \\ y=t^3-3t \end{cases}$ 所确定, 求 $\frac{dy}{dx}\bigg|_{x=1}$ 和 $\frac{d^2y}{dx^2}\bigg|_{x=1}$.当 $x=1$ 时, $t=0$.

$$\frac{dy}{dx}\bigg|_{x=1} = \frac{3t^2-3}{2e^{-t}-2te^{-t}}\bigg|_{t=0} = \frac{-3(t+1)e^{-t}}{2} \bigg|_{t=0} = -\frac{3}{2}$$

$$\frac{d^2y}{dx^2}\bigg|_{x=1} = \frac{-\frac{3}{2}[e^{-t}+(t+1)e^{-t}]}{2(1-t)e^{-t}}\bigg|_{t=0} = -\frac{3}{4} \cdot \frac{t+2}{1-t} e^{2t} \bigg|_{t=0} = -\frac{3}{2}$$

4.

研究函数 $y = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right)e^{-x}$ 的极值 (n 为自然数).

$$\begin{aligned} y' &= \left[1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}\right]e^{-x} - \left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right) \cdot e^{-x} \\ &= -\frac{x^n}{n!}e^{-x} \Rightarrow \text{驻点 } x=0. \end{aligned}$$

1) n 为奇数时, y' 在 $x=0$ 两侧异号.

x	$(-\infty, 0)$	0	$(0, +\infty)$
y'	$+$	0	$-$
y	↗	极大值	↘

 极大值 $y(0)=1$.

2) n 为偶数时, y' 在 $x=0$ 两侧同号, 无极值.

5.

求不定积分 $\int \frac{\ln^2 x}{x^2} dx$.

$$\begin{aligned} \text{解: } & \int \ln^2 x d\left(-\frac{1}{x}\right) = -\frac{1}{x} \ln^2 x + \int \frac{1}{x} \cdot 2 \ln x \cdot \frac{1}{x} dx \\ &= -\frac{1}{x} \ln^2 x + 2 \int \ln x d\left(-\frac{1}{x}\right) \\ &= -\frac{1}{x} \ln^2 x - \frac{2}{x} \ln x + 2 \int \frac{1}{x^2} dx \\ &= -\frac{1}{x} \ln^2 x - \frac{2}{x} \ln x - \frac{2}{x} + C \end{aligned}$$

6. 设 $f(x) = \begin{cases} x^2, & 0 \leq x \leq 1, \\ 2-x, & 1 < x < 2, \end{cases}$ 求 $\int_0^2 f(x) dx$.

$$\text{解: } \int_0^2 f(x) dx = \int_0^1 x^2 dx + \int_1^2 (2-x) dx$$

$$= \frac{1}{3} + 2 - \frac{1}{2} x^2 \Big|_1^2 = \frac{1}{3} + 2 - \frac{3}{2} = \frac{5}{6}.$$

即可意义还原。

三. 综合题 (满分 32 分)

1. (7 分) 设不恒为零的奇函数 $f(x)$ 在 $x=0$ 处可导, 试说明 $x=0$ 为函数 $f(x)/x$ 的何种间断点.

得分	阅卷人

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x} \text{ 存在.}$$

2. 为第一类间断点.

2. (6 分) 若 $f(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 内有二阶导数, 且

$$f(a) = f(b) = 0, f''(x) \leq 0,$$

求证: 在 $[a, b]$ 上, $f(x) \geq 0$.

证: 由罗尔定理, $\exists \xi \in (a, b)$ 使得 $f'(\xi) = 0$.

$$f''(x) \leq 0 \Rightarrow f'(x) \downarrow. \quad \therefore f'(x) \geq 0, x \in [a, \xi].$$

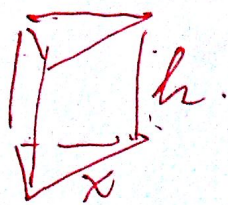
$$f'(x) \leq 0, x \in [\xi, b]$$

$$\Rightarrow f(x) \uparrow, x \in [a, \xi] \quad \because f(a) = 0, \therefore f(x) \geq 0, x \in [a, \xi].$$

$$f(x) \downarrow, x \in [\xi, b] \quad \because f(b) = 0, \therefore f(x) \geq 0, x \in [\xi, b].$$

证: 证毕.

3. (6分) 设有底为等边三角形的直柱体, 体积为 V , 要使其总面积为最小, 问底边的长应为多少?



设底边长为 x , 高为 h .

$$V = \frac{\sqrt{3}}{4} x^2 \cdot h \Rightarrow h = \frac{4V}{\sqrt{3} x^2}$$

$$S = \frac{\sqrt{3}}{4} x^2 \cdot 2 + 3xh = \frac{\sqrt{3}}{2} x^2 + \frac{4\sqrt{3}V}{x}$$

$$S' = \sqrt{3}x - \frac{4\sqrt{3}V}{x^2} \Rightarrow \sqrt{3}\sqrt{x} = \frac{4\sqrt{3}V}{x^2} \Rightarrow x = \sqrt[3]{4V}$$

$$S'' = \sqrt{3} + \frac{8\sqrt{3}V}{x^3} > 0 \Rightarrow S''(\sqrt[3]{4V}) > 0 \Rightarrow x = \sqrt[3]{4V} \text{ 为极小值点}$$

4. (6分) 设函数 $f(x)$ 在 $[0, 1]$ 上连续, 证明:

$$\int_0^{2\pi} f(|\cos x|) dx = 4 \int_0^{\frac{\pi}{2}} f(|\cos x|) dx.$$

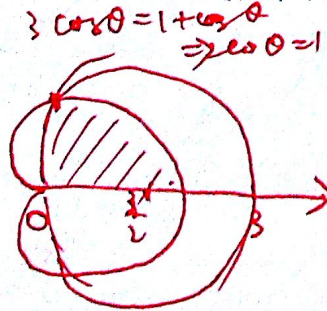
证: $T = \pi$. $\int_0^{2\pi} f(|\cos x|) dx = 2 \int_0^{\frac{\pi}{2}} f(|\cos x|) dx + 2 \int_{\frac{\pi}{2}}^{\pi} f(|\cos x|) dx$

$\int_0^{\frac{\pi}{2}} f(|\cos x|) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$
 $\int_{\frac{\pi}{2}}^{\pi} f(|\cos x|) dx = \int_{\frac{\pi}{2}}^{\pi} f(-\cos x) dx = \int_0^{\frac{\pi}{2}} f(\cos t) dt$

$\int_0^{\pi} f(|\cos x|) dx = 2 \int_0^{\frac{\pi}{2}} f(\cos x) dx$

$\int_0^{2\pi} f(|\cos x|) dx = 4 \int_0^{\frac{\pi}{2}} f(|\cos x|) dx$

5. (7分) 求由不等式 $r \leq 3 \cos \theta$ 和 $r \leq 1 + \cos \theta$ 所确定的公共部分的面积.



$A = 2 \left[\int_0^{\frac{\pi}{3}} \frac{1}{2} (1 + \cos \theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} (3 \cos \theta)^2 d\theta \right]$

$= 2 \left[\int_0^{\frac{\pi}{3}} \frac{1}{2} (1 + 2 \cos \theta + \cos^2 \theta) d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{9}{2} \cos^2 \theta d\theta \right]$

$= \int_0^{\frac{\pi}{3}} (1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9 \cdot \frac{1 + \cos 2\theta}{2} d\theta$

$= \frac{\pi}{3} + 2 \sin \theta \Big|_0^{\frac{\pi}{3}} + \frac{1}{2} \cdot \frac{\pi}{3} + \frac{1}{4} \sin 2\theta \Big|_0^{\frac{\pi}{3}} + \frac{9}{2} \left(\frac{\pi}{2} - \frac{\pi}{3} \right) + \frac{9}{4} \sin 2\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{5}{4} \pi$