Dynamic model of a DC motor with gear train

Paulo Loma Marconi — 2020-10-22 — 2 Comments

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Source code

Version PDF/HTML. LaTex source code on GitHub.

Introduction

The objective is to model the dynamics of a DC servo motor with gear train, Fig. 1, and to deduce two equilibrium points.

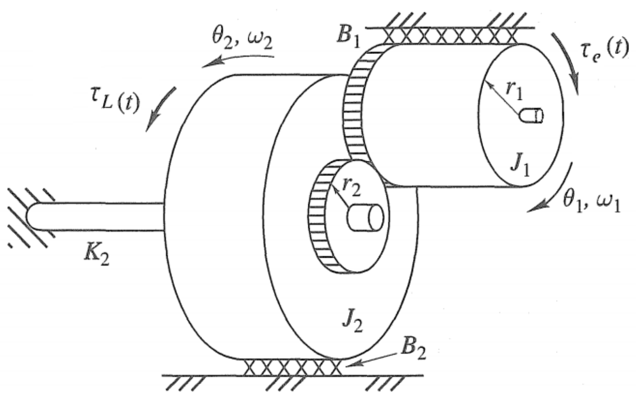

Fig.1 - DC servo motor with gear train.

Free-body diagram analysis

The system can be decomposed in two sections: a rotational mechanical, and an electro-mechanical. The rotational mechanical can be derived as follows,

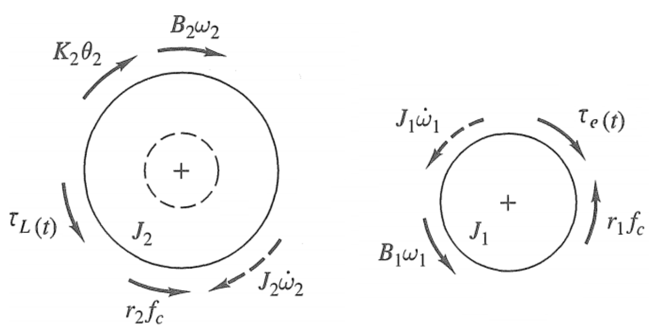

Fig.2 - Rotational mechanical free-body diagram.

where θ is the angular displacement, ω is the angular speed, B is the rotational viscous-damping coefficient, K is the stiffness coefficient, J is the moment of inertia, f_c is the contact force between two gears, and r is the gear radius.

The electromechanical section (DC motor) is

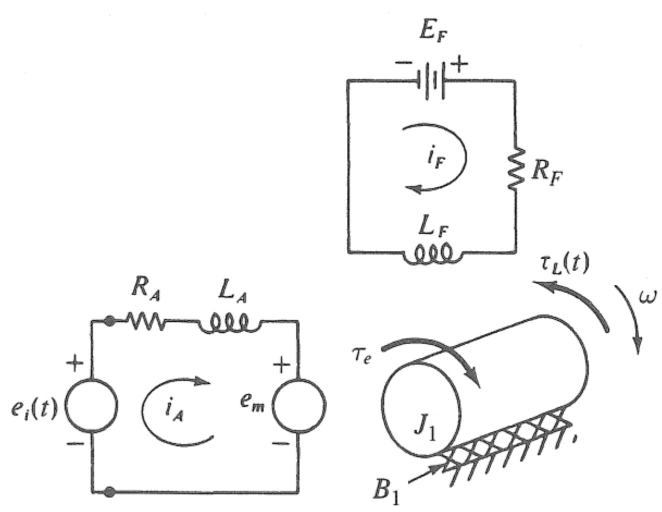

Fig.3 - Electro-mechanical free-body diagram.

where R_F is the field resistance, L_F is the field inductance, E_F is the applied constant field voltage, and i_F is the input field current. R_A is the stationary resistance, L_A is the stationary inductance, and e_m is the induced voltage, i_A is the input stationary current, and $e_i(t)$ is the applied armature voltage, and τ_e is the electro-mechanical driving torque exerted on the rotor.

If the flux density $\mathcal{B}$ is

$$\mathcal{B} = \frac{\phi(i_F)}{A} \tag{1}$$

the torque on the rotor is

s

$$\begin{aligned} \tau_e &= \mathcal{B} la \, i_A \\ \tau_e &= \frac{la}{A}\phi(i_F) i_A \end{aligned} \tag{2}$$

where $\phi(i_F)$ is the flux induced by i_F, A is the cross-sectional area of the flux path in the air gap between the rotor and stator, l is the total length of the armature conductors within the magnetic field, and a is the radius of the armature.

Also, the voltage induced in the armature e_m can be written as

$$e_m = \frac{la}{A}\phi(i_F)\omega \tag{3}$$

where both, τ_e and e_m, depend on the geometry of the DC motor.

Dynamic system

We begin applying D'Alembert's law (restatement of Newton's law) to the rotational mechanical section.

$$\sum \tau_{all} = 0$$

$$J_1\dot{\omega}_1 + B_1\omega_1 + r_1 f_c = \tau_e(t) \tag{4}$$

$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta - r_2 f_c = \tau_L(t) \tag{5}$$

where τ_{all} are the torques acting on a body, $K\theta$ is the stiffness torque, $B\omega$ is the viscous-frictional torque, $J\dot{\omega}$ is the inertial torque, $\tau_e(t)$ is the driving torque, $\tau_L(t)$ is the load torque, and rf_c is the contact torque.

Due to the relation between gears,

$$\begin{aligned} \theta_1 &= N\theta_2 \\ \omega_1 &= N\omega_2 \\ \dot{\omega}_1 &= N\dot{\omega}_2 \\ N &= \frac{r_2}{r_1} \end{aligned}$$

where N is the gear radius relation. We solve (4) and (5) in terms of ω_2 and θ_2,

$$(J_2 + N^2 J_1)\dot{\omega}_2 + (B_2 + N^2 B_1)\omega_1 + K_2\theta_2 - N\tau_e(t) - \tau_L(t) = 0$$

defining the relations

$$\begin{aligned} J_{eq} &= J_2 + N^2 J_1 \\ B_{eq} &= B_2 + N^2 B_1 \end{aligned}$$

it becomes in

$$J_{eq}\dot{\omega}_2 + B_{eq}\omega_2 + K_2\theta_2 - N\tau_e(t) - \tau_L(t) = 0 \tag{6}$$

Now, let us derive the equations of the electro-mechanical section using Kirchoff's law.

$$\sum V_{all} = 0$$

$$e_m + V_{L_A} + V_{R_A} = e_i(t) \tag{7}$$

where V_{all} are the induced voltages on the rotor and stator, V_{L_A} is the stationary resistance voltage, V_{R_A} is the stationary inductance voltage.

If i_F is defined as constant, then (2) is

$$\begin{aligned} \tau_e(t) &= \left(\frac{la}{A}\phi(i_F)\right) i_A(t) \\ \tau_e(t) &= \alpha i_A(t) \end{aligned} \tag{8}$$

where α is the internal parameters of the DC motor.

Then, simplifying and using (6) and (7) the dynamic system is,

$$J_{eq}\dot{\omega}_2 + B_{eq}\omega_2 + K_2\theta_2 - N\tau_e - \tau_L = 0 \tag{9}$$

$$L_A\dot{i}_A + R_A i_A + \alpha\omega_1 - e_i = 0 \tag{10}$$

State-space equations

Let us define the state-space equations for $x = \begin{bmatrix}\theta_2 \; \dot{\theta}_2 \; i_A\end{bmatrix}^\intercal$. From the dynamic system,

$$
\begin{aligned}
J_{eq}\ddot{\theta}_2 + B_{eq}\dot{\theta}_2 + K_2\theta_2 - N\alpha i_A - \tau_L = 0 \\
L_A\dot{i}_A + R_A i_A + \alpha\omega_1 - e_i = 0
\end{aligned}
$$

reordering,

$$
\begin{aligned}
\ddot{\theta}_2 &= -\frac{B_{eq}}{J_{eq}}\dot{\theta}_2 - \frac{K_2}{J_{eq}}\theta_2 + \frac{N\alpha}{J_{eq}}i_A - \frac{1}{J_{eq}}\tau_L \\
\dot{i}_A &= -\frac{R_A}{L_A}i_A - \frac{N\alpha}{L_A}\dot{\theta}_2 + \frac{1}{L_A}e_i
\end{aligned}
$$

defining the states as

$$
\begin{cases}
x_1 &= \theta_2, \quad \dot{x}_1 = \dot{\theta}_2 = x_2 \\
x_2 &= \dot{\theta}_2, \quad \dot{x}_2 = \ddot{\theta}_2 = -\frac{B_{eq}}{J_{eq}}x_2 - \frac{K_2}{J_{eq}}x_1 + \frac{N\alpha}{J_{eq}}x_3 - \frac{1}{J_{eq}}\tau_L \\
x_3 &= i_A, \quad \dot{x}_3 = \dot{i}_A = -\frac{R_A}{L_A}x_3 - \frac{N\alpha}{L_A}x_2 + \frac{1}{L_A}e_i
\end{cases}
$$

then

$$
\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_2}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A}\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_eq} & 0 \\ 0 & 0 & \frac{1}{L_A}\end{bmatrix}}_{B} \underbrace{\begin{bmatrix}0 \\ \tau_L \\ e_i\end{bmatrix}}_{\mathbf{u}} \tag{11}
$$

$$
\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{12}
$$

The output $y = \dot{\omega}_2$ can be defined as

$$
y = \underbrace{\begin{bmatrix}0 & 1 & 0\end{bmatrix}}_{C} \begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0 & 0\end{bmatrix}}_{D} e_i \tag{13}
$$

$$
y = C\dot{\mathbf{x}} \tag{14}
$$

Equilibrium point $\mathbf{x_0}$

Using $\dot{\mathbf{x}} = 0$ in (12), the equilibrium point $\mathbf{x_0}$ can be calculated as

$$
0 = A\mathbf{x_0} + B\mathbf{u} \tag{15}
$$

$$
\mathbf{x_0} = -A^{-1}B\mathbf{u} \tag{16}
$$

$$
\begin{bmatrix}x_{1_0} \\ x_{2_0} \\ x_{3_0}\end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_2}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A}\end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_eq} & 0 \\ 0 & 0 & \frac{1}{L_A}\end{bmatrix} \begin{bmatrix}0 \\ \tau_L \\ e_i\end{bmatrix} \tag{17}
$$

Solving for no external torque $\tau_L = 0$, constant applied armature voltage $e_i = E_0$, and $K_2 \neq 0$,

$$
\begin{aligned}
0 &= x_{2_0} \\
0 &= -\frac{K_2}{J_{eq}}x_{1_0} - \frac{B_{eq}}{J_{eq}}x_{2_0} + \frac{N\alpha}{J_{eq}}x_{3_0} \\
0 &= -\frac{N\alpha}{L_A}x_{2_0} - \frac{R_A}{L_A}x_{3_0} + \frac{1}{L_A}E_0
\end{aligned}
$$

due to $x_{2_0} = 0$, we have

$$
\begin{aligned}
0 &= -\frac{K_2}{J_{eq}}x_{1_0} + \frac{N\alpha}{J_{eq}}x_{3_0} \\
0 &= -\frac{R_A}{L_A}x_{3_0} + \frac{1}{L_A}E_0
\end{aligned}
$$

then

$$
\begin{aligned}
x_{1_0} &= \frac{N\alpha}{K_2 R_A}E_0 \\
x_{3_0} &= \frac{1}{R_A}E_0
\end{aligned}
$$

therefore the equilibrium point is

$$
\mathbf{x_0} = \begin{bmatrix}x_{1_0} \\ x_{2_0} \\ x_{3_0}\end{bmatrix} = \begin{bmatrix}\frac{N\alpha}{K_2 R_A} \\ 0 \\ \frac{1}{R_A}\end{bmatrix} E_0 \tag{18}
$$

This equilibrium point indicates that a **constant angular displacement (twist)** produced by $x_{1_0} = \theta_{2_0}$ is sufficient to balance the constant applied armature voltage $e_i = E_0$.

On the other hand, if we solve for no external torque $\tau_L = 0$, constant applied armature voltage $e_i = E_0$, and no stiffness $K_2 = 0$. The problem is,

$$\begin{bmatrix} x_{1_0} \\ x_{2_0} \\ x_{3_0} \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_e q} & 0 \\ 0 & 0 & \frac{1}{L_A} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_0 \end{bmatrix}$$

if we eliminate x_{1_0} because the first column of A^{-1} has zeros, the problem reduces to

$$\begin{bmatrix} x_{2_0} \\ x_{3_0} \end{bmatrix} = -\begin{bmatrix} -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{J_e q} & 0 \\ 0 & \frac{1}{L_A} \end{bmatrix} \begin{bmatrix} 0 \\ E_0 \end{bmatrix} \tag{19}$$

solving, we have

$$\begin{bmatrix} x_{2_0} \\ x_{3_0} \end{bmatrix} = \begin{bmatrix} \frac{N\alpha}{B_{eq}R_A + (N\alpha)^2} \\ \frac{-B_{eq}}{B_{eq}R_A + (N\alpha)^2} \end{bmatrix} E_0 \tag{20}$$

which indicates that a **constant angular speed** produced by $x_{2_0} = \dot{\theta}_{2_0}$ is needed to balance the constant applied armature voltage $e_i = E_0$.

References

[1] Close, Charles M. and Frederick, Dean K. and Newell, Jonathan C., *Modeling and Analysis of Dynamic Systems*, 2001, ISBN 0471394424.

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Comments

D **Daniel Sun**
a year ago
I think B_eq should be B_2 + N^2 B_1
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P **paulomarconi** Mod ➦ Daniel Sun
9 months ago
Solved! Thanks for spotting the typo.
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