Aalto University
School of Electrical
Engineering

Lecture 5: Open-Loop Dynamics of a DC Motor

ELEC-E8405 Electric Drives (5 ECTS)

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Autumn 2020

Learning Outcomes

After this lecture and exercises you will be able to:

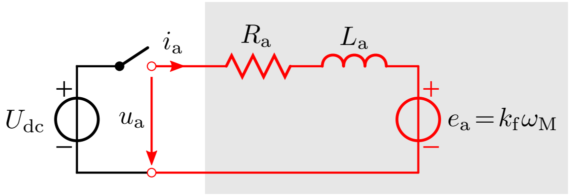
- Draw relevant block diagrams of the DC motor
- Derive transfer functions based on the block diagram
- Interpret the most essential properties of second-order systems
- Explain the concept of time-scale separation

Introduction

- Open-loop (plant) model of the DC motor
  - Combination of the electrical and mechanical models
  - Plant model is the starting point in the control design
- Brief recap on control theory tools in the context of the DC motor
  - Block diagram, transfer function, 2nd-order system, state-variable form
  - Basic knowledge of these tools is needed in the field of electric drives (and in many other fields as well)
- Transient response in open loop (speed and current)
- Time-scale separation (electrical and mechanical subsystems)

Note: Controllers will not be considered today

Example: Connection of a DC Voltage Source to the Terminals



- Assume that a DC voltage source is connected to the motor terminals
- How will the speed ω_M and the current i_a behave?
- How to model and analyse transient response in more general cases?

Outline

Dynamic Model of the DC Motor
- Model Equations
- Block Diagrams
- Transfer Functions and Their Properties
- Nice-to-Know: State-Variable Form

Simulation Examples

Time-Scale Separation

DC Motor Model

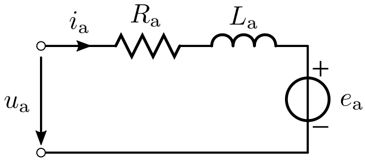
- Voltage equation

$$L_\mathrm{a}\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} = u_\mathrm{a} - R_\mathrm{a}i_\mathrm{a} - e_\mathrm{a}$$

  where $e_\mathrm{a} = k_\mathrm{f}\omega_\mathrm{M}$ is the back emf
- Motion equation

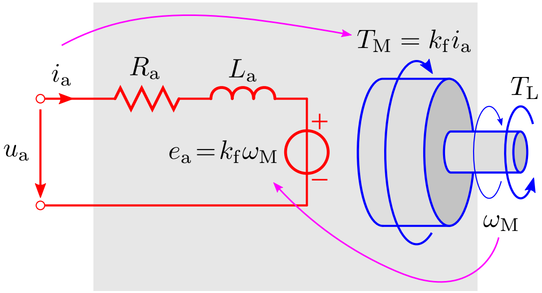
$$J\frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} = T_\mathrm{M} - T_\mathrm{L}$$

  where $T_\mathrm{M} = k_\mathrm{f}i_\mathrm{a}$ is the electromagnetic torque
- For simplicity, the flux factor k_f is assumed to be constant in the following



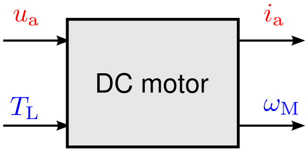
Electrical and Mechanical Dynamics Are Coupled

Physical model



$\Rightarrow$

Block diagram



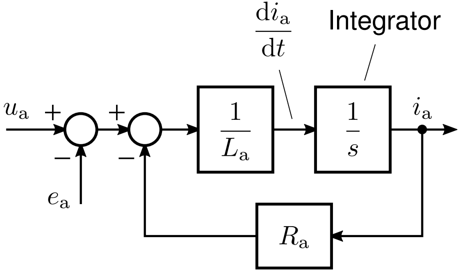
Electrical Dynamics in the Time Domain

- Differential equation

$$L_\mathrm{a}\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} = u_\mathrm{a} - e_\mathrm{a} - R_\mathrm{a}i_\mathrm{a}$$

- u_a and e_a are the inputs
- i_a is the output
- Integration of both sides gives

$$i_\mathrm{a} = \int \frac{1}{L_\mathrm{a}}\left(u_\mathrm{a} - e_\mathrm{a} - R_\mathrm{a}i_\mathrm{a}\right)\mathrm{d}t$$



- In the time domain, $s = \mathrm{d}/\mathrm{d}t$ refers to the differential operator

In some textbooks, the symbol $p = \mathrm{d}/\mathrm{d}t$ is used for the differential operator in the time domain.

Electrical Dynamics in the Laplace Domain

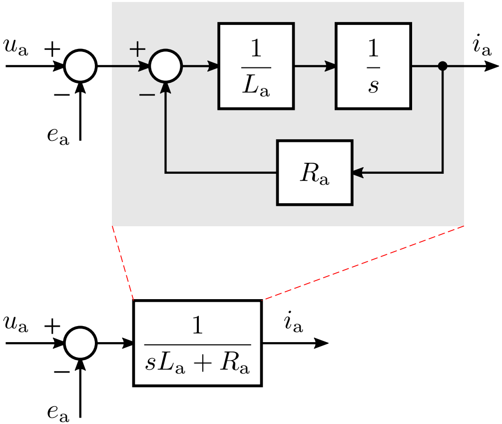
- Laplace transform: $\mathrm{d}/\mathrm{d}t \to s$
- Current can be solved

$$i_\mathrm{a}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}}[u_\mathrm{a}(s) - e_\mathrm{a}(s)]$$

- Transfer function (admittance)

$$Y_\mathrm{a}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}} = \frac{1/R_\mathrm{a}}{1 + \tau_\mathrm{a} s}$$

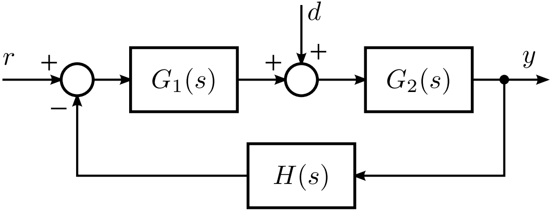
where $\tau_\mathrm{a} = L_\mathrm{a}/R_\mathrm{a}$



In the Laplace domain, $s = \sigma + \mathrm{j}\omega$ is a complex variable. However, the differential operator and the Laplace variable can be used interchangeably in many cases.

Useful Block Diagram Algebra

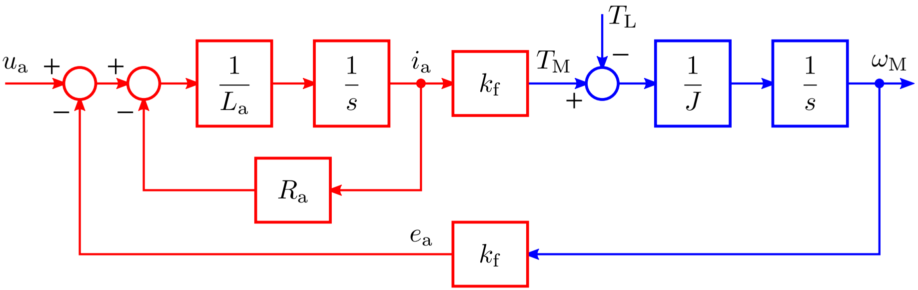
$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$
$$\frac{y(s)}{d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$



Block Diagram of the DC Motor

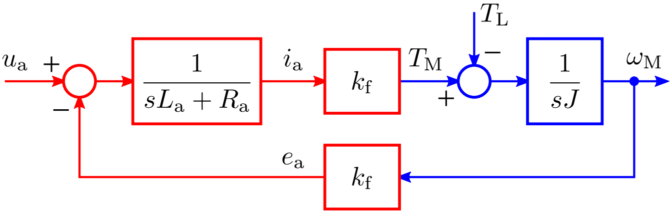
$$L_a \frac{di_a}{dt} = -R_a i_a - k_f \omega_M + u_a$$

$$J \frac{d\omega_M}{dt} = k_f i_a - T_L$$



- Flux factor k_f couples the electrical and mechanical dynamics

Block Diagram of the DC Motor



- Armature current depends on the armature voltage and the load torque

$$i_{\mathrm{a}}(s) = G_{iu}(s)u_{\mathrm{a}}(s) + G_{iT}(s)T_{\mathrm{L}}(s)$$

- Speed depends on the armature voltage and the load torque

$$\omega_{\mathrm{M}}(s) = G_{\omega u}(s)u_{\mathrm{a}}(s) + G_{\omega T}(s)T_{\mathrm{L}}(s)$$

- Could you derive the transfer functions based on the block diagram?

Transfer Function From $u_\mathrm{a}(s)$ to $\omega_\mathrm{M}(s)$

- Transfer function from the voltage $u_\mathrm{a}(s)$ to the speed $\omega_\mathrm{M}(s)$

$$G_{\omega u}(s) = \frac{\dfrac{k_\mathrm{f}}{JL_\mathrm{a}}}{s^2 + \dfrac{R_\mathrm{a}}{L_\mathrm{a}}s + \dfrac{k_\mathrm{f}^2}{JL_\mathrm{a}}} = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Last form is a typical generic form of 2nd-order systems
- Undamped angular frequency, damping ratio, and DC gain

$$\omega_0 = \frac{k_\mathrm{f}}{\sqrt{JL_\mathrm{a}}} \qquad \zeta = \frac{R_\mathrm{a}}{2k_\mathrm{f}}\sqrt{\frac{J}{L_\mathrm{a}}} \qquad K = \frac{1}{k_\mathrm{f}}$$

You don't need to remember these more complex transfer functions, but practise deriving them based on the block diagram instead. However, you should remember the generic form used above.

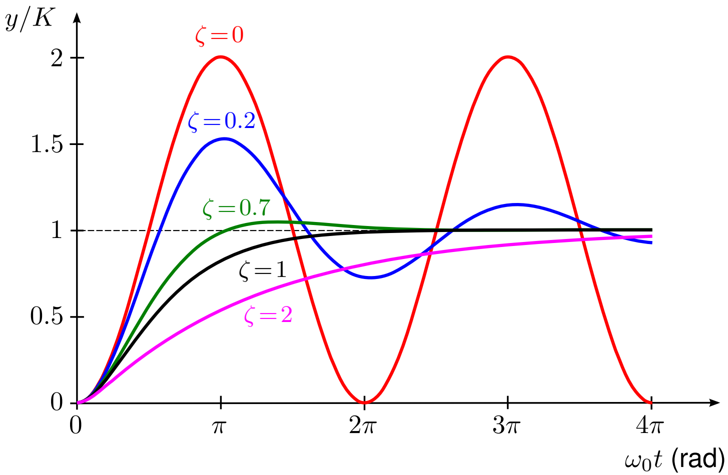
2nd-Order System in the Time Domain: Step Response

- 2nd-order system

$$G(s) = \frac{y(s)}{u(s)}$$

$$= \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Response $y(t)$ to the step input $u(t)$ is shown
- No overshoot if $\zeta \geq 1$



Step responses can be easily plotted using numerical simulation tools. If needed, an analytical solution could be obtained using the inverse Laplace transformation.

2nd-Order System in the Frequency Domain

- 2nd-order system

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Consider a sinusoidal input

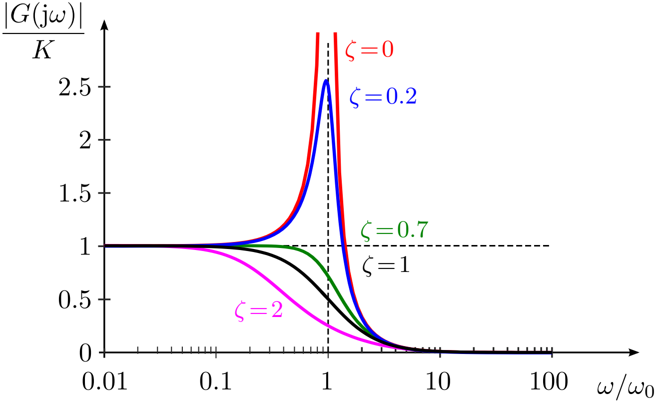
$$u(t) = U\sin(\omega t)$$

- For $\zeta > 0$, the output in steady state is

$$y(t) = AU\sin(\omega t + \phi)$$

where

$$A = |G(\mathrm{j}\omega)| \qquad \phi = \angle G(\mathrm{j}\omega)$$



Transfer Function From $u_\mathrm{a}(s)$ to $i_\mathrm{a}(s)$

- Transfer function from the voltage $u_\mathrm{a}(s)$ to the current $i_\mathrm{a}(s)$

$$G_{iu}(s) = \frac{s/L_\mathrm{a}}{s^2 + \dfrac{R_\mathrm{a}}{L_\mathrm{a}}s + \dfrac{k_\mathrm{f}^2}{JL_\mathrm{a}}}$$

- Characteristic polynomial remains the same (holds also for other transfer functions of the system)
- Zero at $s = 0$ in this transfer function
- If $J \to \infty$ (i.e. ω_M is constant)

$$G_{iu}(s) = \frac{1}{sL_\mathrm{a} + R_\mathrm{a}} = Y_\mathrm{a}(s)$$

State-Variable Form

- State-variable model consists of coupled 1st-order differential equations
- Derivatives $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ depend on the states $\boldsymbol{x}$ and the system input u

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x}$$

- States x depend on the history, but not on the present values of the inputs
- Output y depends only on the states (in physical systems)
- State variables are typically associated with the energy storage
  - Current i of an inductor (or its flux linkage $\psi = Li$)
  - Voltage u of a capacitor (or its charge $q = Cu$)
  - Speed v of a mass (or its momentum $p = mv$)
- Choice of state variables is not unique (as shown in the parenthesis above)

State-Variable Form of the DC Motor

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} i_\mathrm{a} \\ \omega_\mathrm{M} \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} -\frac{R_\mathrm{a}}{L_\mathrm{a}} & -\frac{k_\mathrm{f}}{L_\mathrm{a}} \\ \frac{k_\mathrm{f}}{J} & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} i_\mathrm{a} \\ \omega_\mathrm{M} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L_\mathrm{a}} \\ 0 \end{bmatrix}}_{\boldsymbol{B}_u} u_\mathrm{a} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{J} \end{bmatrix}}_{\boldsymbol{B}_T} T_\mathrm{L}$$

$$i_\mathrm{a} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}_i} \boldsymbol{x} \qquad \omega_\mathrm{M} = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\boldsymbol{C}_\omega} \boldsymbol{x}$$

- Transfer function from $u_\mathrm{a}(s)$ to $\omega_\mathrm{M}(s)$ as an example

$$G_{\omega u}(s) = \boldsymbol{C}_\omega (s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B}_u$$

- Transfer functions of the system are unique, i.e. the state-variable form leads to the previous transfer functions
- Poles of the transfer function are eigenvalues of the system matrix $\boldsymbol{A}$

Outline

Dynamic Model of the DC Motor

Simulation Examples

Time-Scale Separation

Time-Domain Simulation Examples

Rated values of a small PM DC motor

- Armature voltage $U_{\mathrm{N}} = 110$ V
- Armature current $I_{\mathrm{N}} = 10$ A
- Rotation speed $n_{\mathrm{N}} = 1200$ r/min

- Angular speed

$$\begin{aligned}\omega_{\mathrm{N}} &= 2\pi n_{\mathrm{N}} \\ &= 2\pi \cdot \frac{1200 \text{ r/min}}{60 \text{ s/min}} \\ &= 125.7 \text{ rad/s}\end{aligned}$$

Electrical parameters

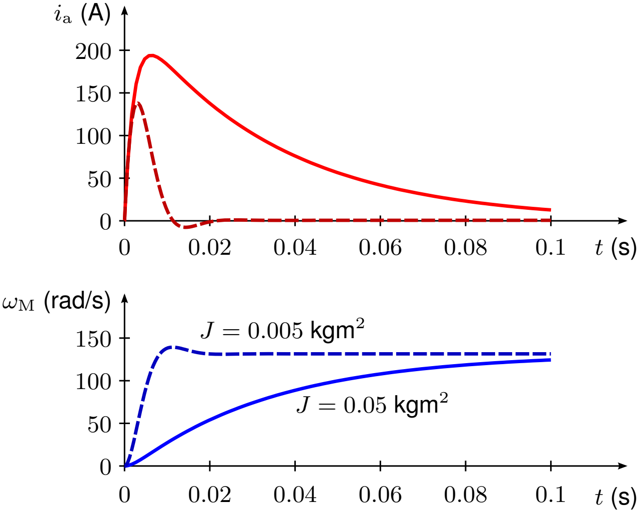
- $R_{\mathrm{a}} = 0.5\ \Omega$
- $L_{\mathrm{a}} = 1$ mH
- $k_{\mathrm{f}} = 0.836$ Vs

Two inertia values

- Case 1: $J = 0.05$ kgm^2 ($\zeta = 2.11$, $\omega_0 = 118$ rad/s)
- Case 2: $J = 0.005$ kgm^2 ($\zeta = 0.67$, $\omega_0 = 374$ rad/s)

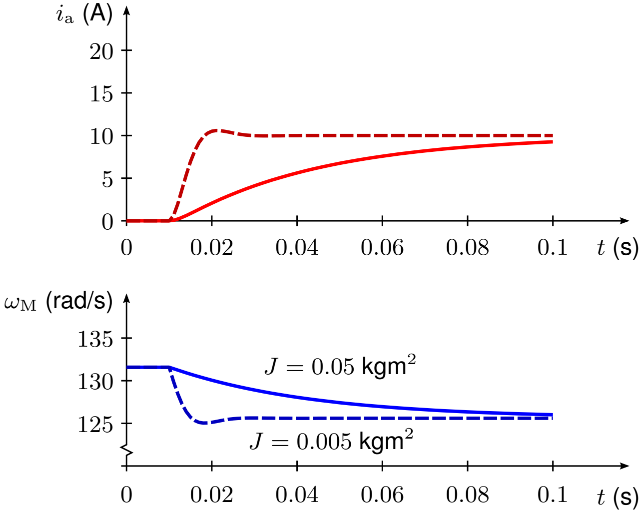
Voltage-Step Response

- Armature is connected to the rated voltage
- Load torque is zero
- Current rises quickly and then decreases as the back-emf $e_\mathrm{a} = k_\mathrm{f}\omega_\mathrm{M}$ increases
- Very large current peak is undesirable



Load-Torque-Step Response

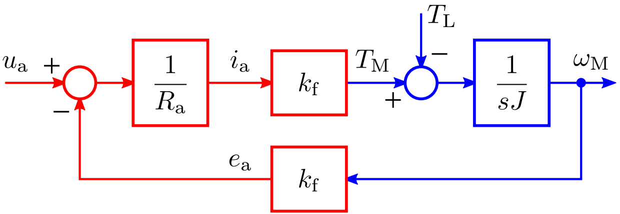
- Armature voltage is constant (rated)
- Initially no-load condition
- Rated load torque is applied at $t = 0.01$ s



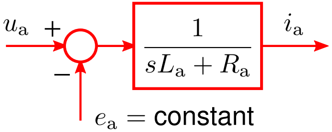
Outline

Time-Scale Separation

- When considering the slow mechanical dynamics, the quickly converging electrical dynamics may be approximated with the DC gain

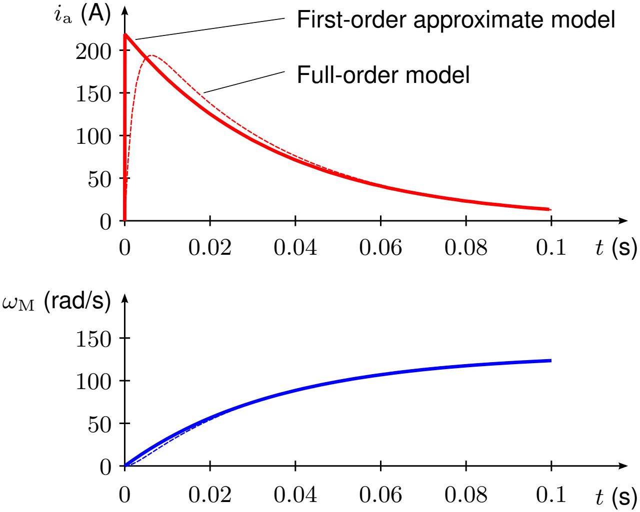


- When considering the fast electrical dynamics, the slowly varying rotor speed may be assumed to be constant



Reduced-Order Model for Slow Mechanical Dynamics

- Response to the rated voltage step
- Electrical dynamics are approximated with the steady-state gain
- Response of the reduced-order model is close to the full-order model



Reduced-Order Model for Fast Electrical Dynamics

- Response to the rated voltage step
- Speed is assumed to be constant
- Fast electrical transient is well modelled using the first-order model $Y_a(s)$

- Notice a different scale of the time axes compared to the previous case