A Compositional Framework for Non-Convex Sequential Decision Problems

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Georgia Tech

ACC '25 Workshop on Applied Category Theory for Compositional Decision Making

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Georgia Institute of Technology

This talk is about joint work

- Everything in this talk is joint with

James Fairbanks
(UF MAE)

Tyler Hanks
(UF CISE)

Hans Riess
(GT ECE)

Decision problems often have temporal coupling

- Example #1: State space control systems

$$\begin{aligned} \underset{u(0),\dots,u(T)}{\text{minimize}} \; & \sum_{t=0}^{T} \ell\big(x(t), u(t)\big) \\ \text{subject to } \; & x(t+1) = f\big(x(t), u(t)\big) \\ & x(0) = x_0 \end{aligned}$$

- Example #2: Markov decision processes

a	b	c	d	e
f	g	h	i	j
k	l	m	n	o

$$P_a(s, s') = P(s_{t+1} = s' \mid s_t = s, a_a = a)$$

- Example #3: Various classes of games

1 / 2	*L*	*M*	*R*
T	$8, 8$	$0, 0$	$1, 9^*$
M	$0, 0$	$5^*, 5^*$	$0, 0$
B	$9^*, 1$	$0, 0$	$3^*, 3^*$

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Today's talk will focus on this!

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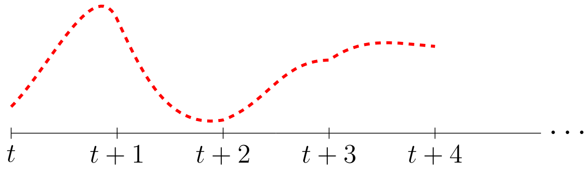
We will examine MPC for nonlinear systems

- Model-predictive control (MPC) optimizes inputs over a finite lookahead window



- A standard MPC problem formulation is then

$$
\begin{aligned}
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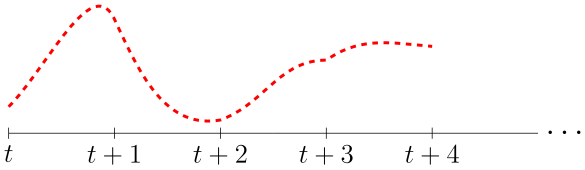

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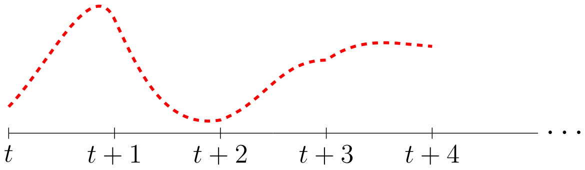

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- A decision at one time parameterizes the constraints at the next time
- Once $u^*(k)$ is computed and applied, we get the state $x(k+1) = f\big(x(k), u^*(k)\big)$
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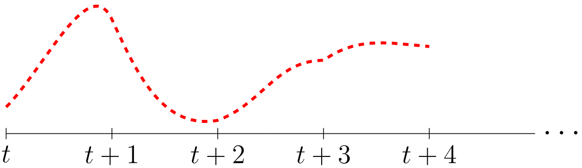

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Can be non-convex for non-linear systems!

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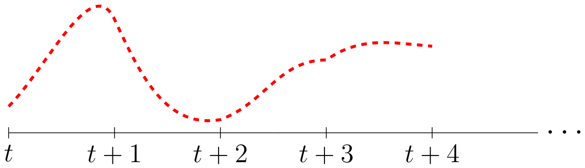

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Question for this talk

How can we use category theory to model this temporal coupling in MPC?

We can draw from classic work of Rockafellar

- In 1970, Rockafellar proposed "bifunctions" for convex problems[1]
- We replace

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & g(x) \le 0 \\ & h(x) = 0 \end{array} \qquad \text{with} \qquad \begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & g(x) \le y_1 \\ & h(x) = y_2 \end{array}$$

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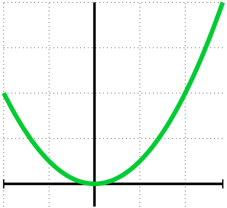
- Then with $y = (y_1^T, y_2^T)^T$ we form the bifunction

$$B(x, y) = \begin{cases} f(x) & g(x) \le y_1, h(x) = y_2 \\ \infty & \text{otherwise} \end{cases}$$



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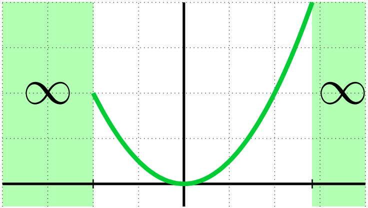

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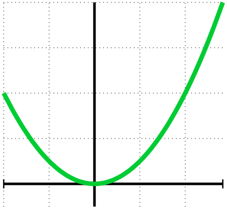

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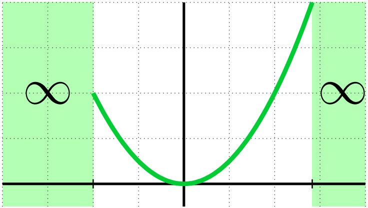

- Bifunctions have an associative composition law! It is inf-multiplication, i.e.,

$$\big(B_1 \circ B_2\big)(x, z) = \inf_{y} \big[B_1(x, y) + B_2(y, z)\big]$$

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Major changes are required to make this useful to us

- We can take objects as Euclidean spaces, e.g., $\mathbb{R}^m$ and $\mathbb{R}^n$
- A morphism is a bifunction $B : \mathbb{R}^m \nrightarrow \mathbb{R}^n$

$$\mathbb{R}^m \xrightarrow{\quad B \quad} \mathbb{R}^n$$

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- Composition via $\circ$ takes in $B_1 : \mathbb{R}^m \nrightarrow \mathbb{R}^n$ and $B_2 : \mathbb{R}^n \nrightarrow \mathbb{R}^p$ and gives back

$$\big(B_1 \circ B_2\big)(x, z) = \inf_y \left[B_1(x, y) + B_2(y, z)\right]$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "B_1"] \arrow[dr, "B_1 \circ B_2"'] & \mathbb{R}^n \arrow[d, "B_2"] \\ & \mathbb{R}^p \end{tikzcd}$$

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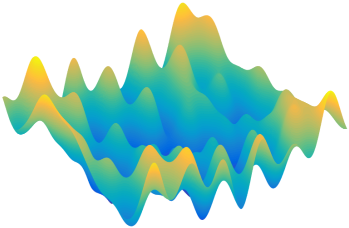
- We **will not** use $\inf$-multiplication for composition!

Optimality is non-convex problems is inherently local

- Non-convex problems are (almost always) solved to local optimality



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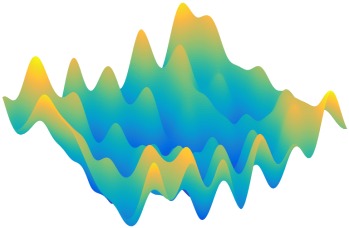

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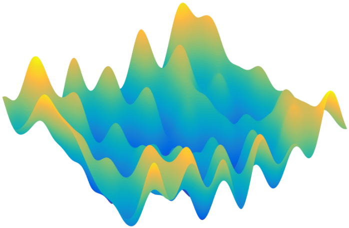

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- We will consider systems with polynomial dynamics and costs
  $\implies$ we have polynomial optimization problems!
- For polynomial f, g, and h:

$$\begin{aligned} \text{(locally) minimize } & f(x) \text{ (starting from } x_0\text{)} \\ \text{subject to } & h(x) = 0 \\ & g(x) \leq 0 \end{aligned}$$

- We want (i) local optimality and (ii) feasibility

Modeling question

Which local optimum?

We use negative gradient flows to model optimization algorithms

- The region $\mathscr{F} = \{x \in \mathbb{R}^n : h(x) = 0, g(x) \leq 0\}$ is a Nash manifold with corners

Assumption #1: Morse property

The objective f is a stratified Morse function on $\mathscr{F}$.

[2] D.G.C. Handron, "Generalized Billiard Paths and Morse Theory for Manifolds with Corners", *Topology and its Applications*, 126 (2002), pp. 83-118.

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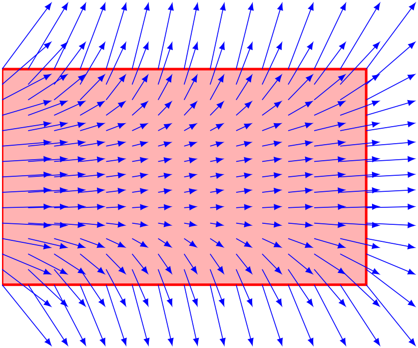

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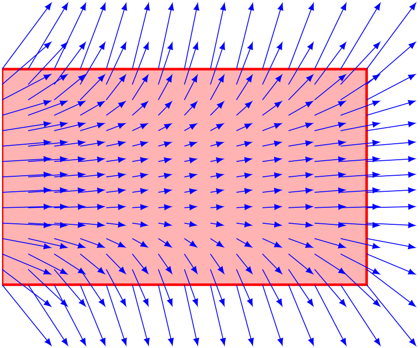

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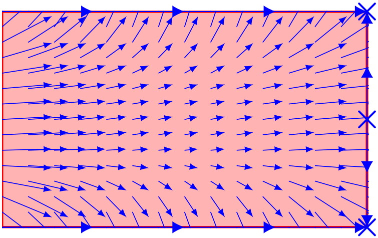

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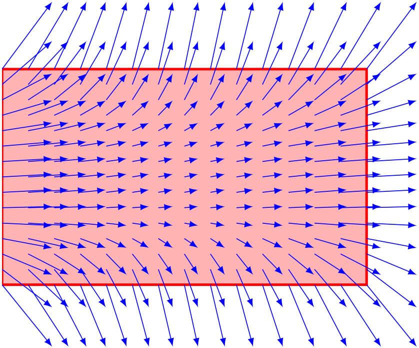

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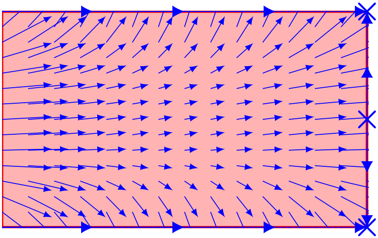

- Flows follow $-\nabla f$ as much as possible while keeping $\mathscr{F}$ forward-invariant.

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Now we can define local $\inf$-multiplication

- We need the stable foliation of $\mathscr{F}$ with respect to $-\tilde{\nabla} f$

Definition #3: Stable foliation[3]

Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be the stationary points of $-\tilde{\nabla} f$. Then $\mathscr{F} = \bigcup_{p_i \in \mathcal{P}} W^s(p_i)$

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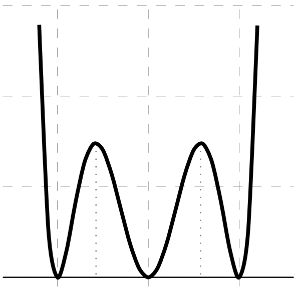

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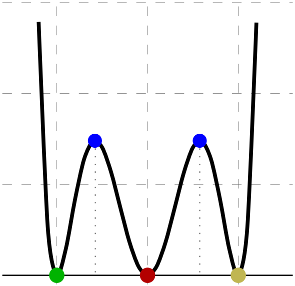

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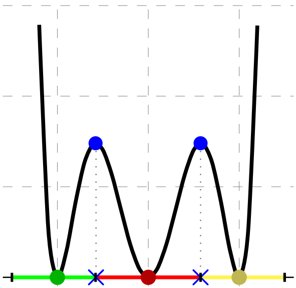

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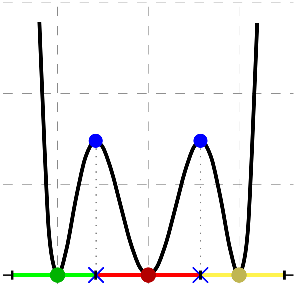

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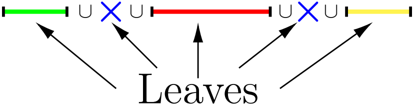

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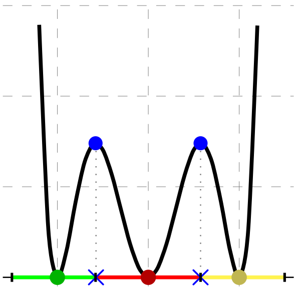

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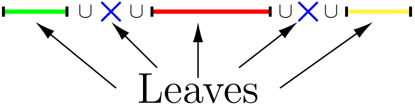

- Then local inf-multiplication is inf-multiplication over a leaf
- Our composition law is

$$\big(B_1 \circ_{y_0} B_2\big)(x,z) = \underset{y,y_0}{\text{local}\min} \big[B_1(x,y) + B_2(y,z)\big]$$
$$= \min_{y \in \mathscr{L}(y_0)} \big[B_1(x,y) + B_2(y,z)\big]$$

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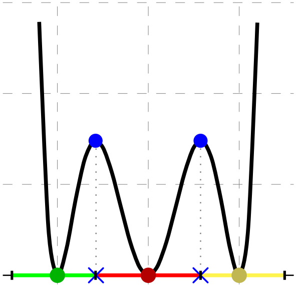
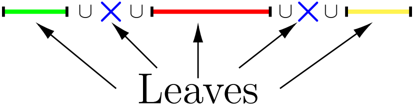


- The foliation is



- Then local inf-multiplication is inf-multiplication over a leaf
- Our composition law is

$$\big(B_1 \circ_{y_0} B_2\big)(x,z) = \operatorname*{local\,min}_{y,y_0} \big[B_1(x,y) + B_2(y,z)\big]$$
$$= \min_{y \in \mathscr{L}(y_0)} \big[B_1(x,y) + B_2(y,z)\big]$$

- We'd like to have

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "B_1"] \arrow[dr, "B_1 \circ_{y_0} B_2"'] & \mathbb{R}^n \arrow[d, "B_2"] \\
& \mathbb{R}^p
\end{tikzcd}$$

[3] R. Thom, Sur une partition en cellules associée à une fonction sur une variété, C. R. Acad. Sci. Paris 228 (1949), 973–975.

We now have a category!

Definition #4: The category C

We define C such that

- Ob(C): objects are pointed Euclidean spaces, e.g, $(\mathbb{R}^n, x_0)$
- Mor(C): morphisms are algebraic bifunctions, i.e., polynomial B where

$$B(x,y) = \begin{cases} f(x) & g(x) \le y_1, h(x) = y_2 \\ \infty & \text{otherwise} \end{cases}$$

- Composition uses local $\inf$-multiplication
- For $B_1 : (\mathbb{R}^m, x_0) \nrightarrow (\mathbb{R}^n, y_0)$ and $B_2 : (\mathbb{R}^n, y_0) \nrightarrow (\mathbb{R}^p, z_0)$, we have

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This construction of C satisfies all of the category axioms.

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This construction of C satisfies all of the category axioms.

- Fine, but where are the inputs?

We need to introduce external parameters

- We will use Para to introduce inputs[4]. To get there, we make a new category

[4] B. Fong, D. Spivak and R. Tuyéras, "Backprop as Functor: A compositional perspective on supervised learning," *34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2019

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- Define AlgBiFun $= \big(\mathsf{C}, \oplus, (\mathbb{R}^0, \bullet)\big)$ so that
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AlgBiFun is a strict symmetric monoidal category.

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This has two immediate outcomes:

1. We unlock "string diagrams". For
   $B_1 : (U, u_0) \nrightarrow (W, w_0) \oplus (X, x_0)$,
   $B_2 : (X, x_0) \nrightarrow (Y, y_0)$,
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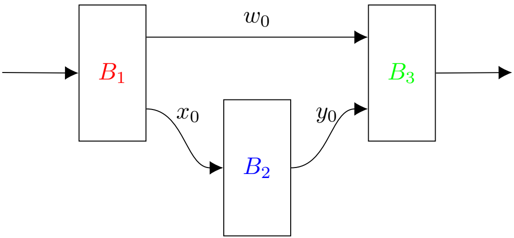

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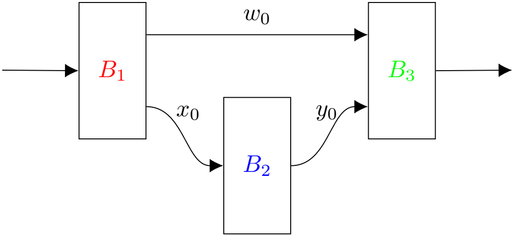
2. We can parameterize this category!



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We apply the Para construction to model inputs

- So far: made C (category), then AlgBiFun (strict symmetric monoidal category)
- Now: Para(AlgBiFun)

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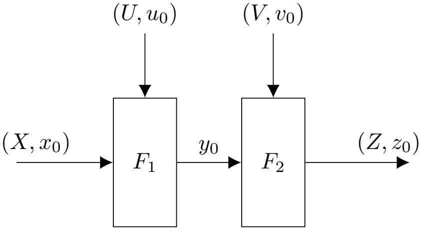

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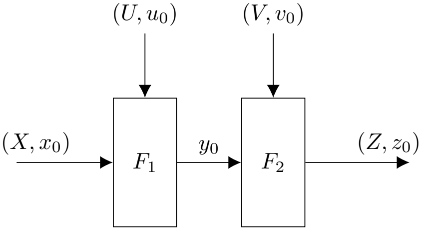

- Roughly, given $x(k)$, we have $x(k+1) = f\big(x(k), u\big)$
- Then $x(k+2) = f\big(x(k+1), v\big)$
- The figure essentially says

$$x(k+2) = f\Big(f\big(x(k), u\big), v\Big)$$

One-step MPC problems are morphisms

- Consider the one-step MPC problem

$$\begin{aligned} \underset{u(k)}{\text{minimize}} \quad & \ell\big(x(k), u(k)\big) \\ \text{subject to} \quad & x(k+1) = f\big(x(k), u(k)\big) \\ & g\big(x(k), u(k)\big) \leq 0 \end{aligned}$$

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- Its associated *one-step bifunction* is $G : (U, u_0) \oplus (X, x_0) \nrightarrow (X, \xi_0)$, i.e.,

$$G\Big(\big((u, u_0), (x, x_0)\big), (x', \xi_0)\Big) = \underbrace{\ell(x, u)}_{\text{Cost}} + \underbrace{\delta\big(x' \mid f(x, u)\big)}_{\text{Dynamics}} + \underbrace{\delta\big(x, u \mid g(x, u) \leq 0\big)}_{\text{Constraints}}$$

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Theorem #3: One-step problems are morphisms

The pair $\big((U, u_0), G\big)$ is a morphism from (X, x_0) to (X, ξ_0) in Para(AlgBiFun).

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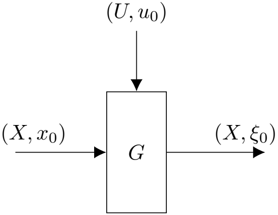
- Pictorially, a 1-step MPC problem is



N-step MPC problems are N-fold bifunction compositions

- Now consider the N-step MPC problem

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{k=t}^{t+N-1} \ell\big(x(k), u(k)\big) \\
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- Let $\big((U, u_0^i), G\big)$ be the morphism corresponding to time i

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Theorem #4: N-step problems are compositions of N morphisms

The N-step MPC problem can be represented as an N-fold composition of morphisms:

$$\big((U, u_0^t), G\big) \circ_{x_0} \big((U, u_0^{t+1}), G\big) \circ_{\xi_0} \cdots \circ_{\zeta_0} \big((U, u_0^{t+N-2}), G\big) \circ_{\psi_0} \big((U, u_0^{t+N-1}), G\big)$$

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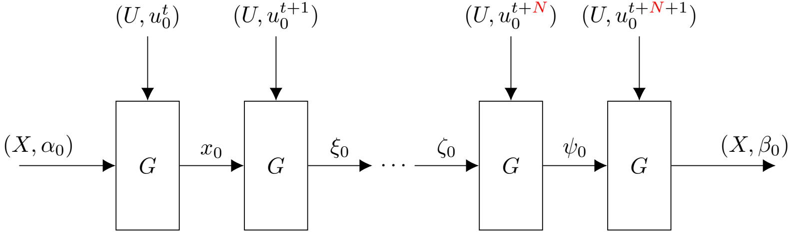

What's next?

- This talk generalized work from ACC '24 on compositional models for *convex* MPC problems[5]
- Extension (this talk) is in preparation for publication
- How can we generalize to non-convex (and non-MPC) decision problems on other spaces, e.g., manifolds?

[5] T. Hanks, B. She, M. Hale, E. Patterson, M. Klawonn, J. Fairbanks, "Modeling Model Predictive Control: A Category Theoretic Framework for Multistage Control Problems", *2024 American Control Conference*, 2024.

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Next steps: Can we prove stability of MPC in purely categorical terms?

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Thank you!

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