From Robotics to Category Theory and Back

ACC Workshop:
Applied Category Theory for Compositional Decision Making
July 7th, 2025

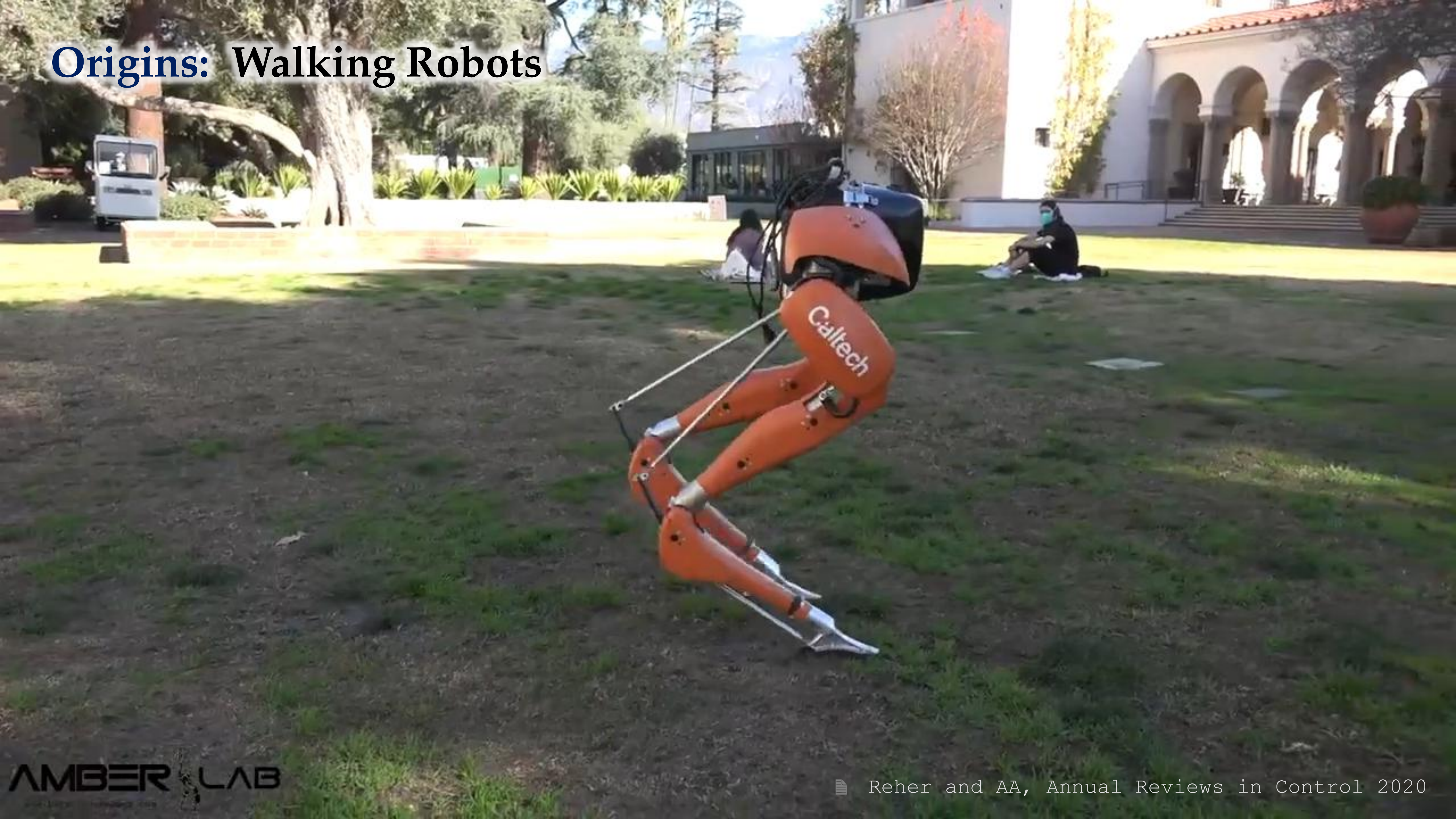
Aaron D. Ames
Bren Professor
Mechanical and Civil Engineering,
Aerospace,
Control and Dynamical Systems,
California Institute of Technology

AMBER LAB

CAST

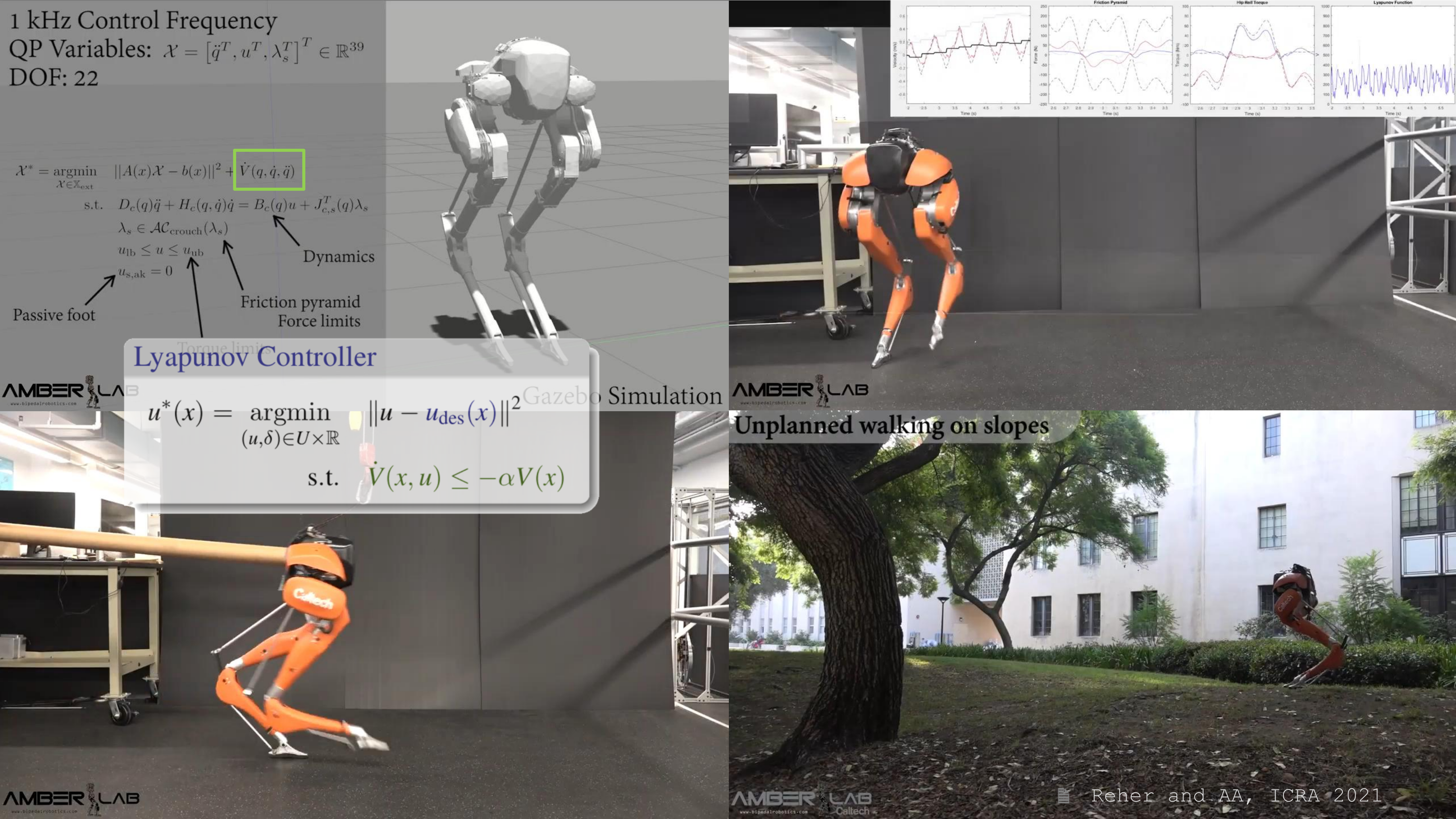
Caltech

Origins: Walking Robots

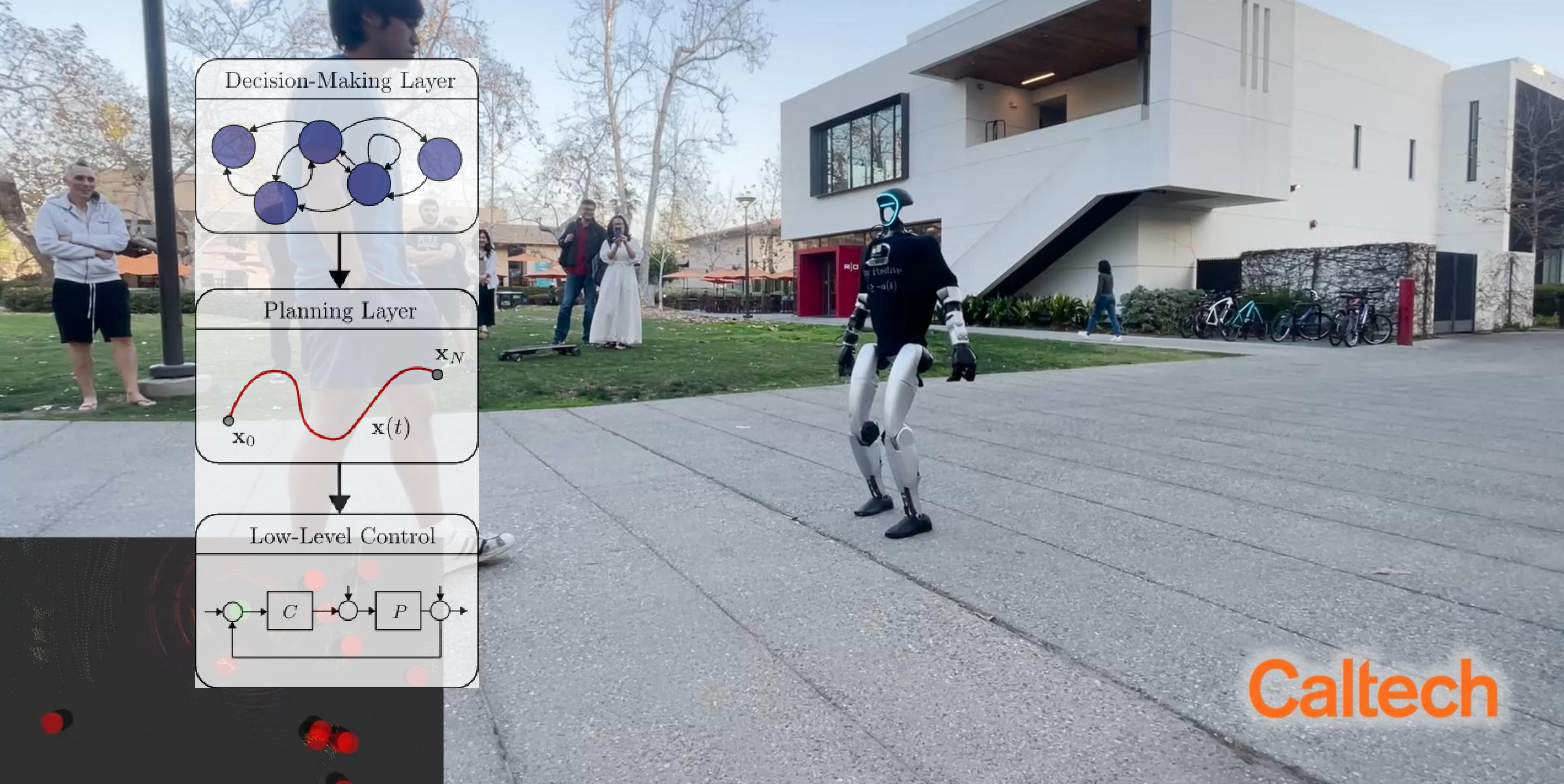


AMBER LAB

Reher and AA, Annual Reviews in Control 2020

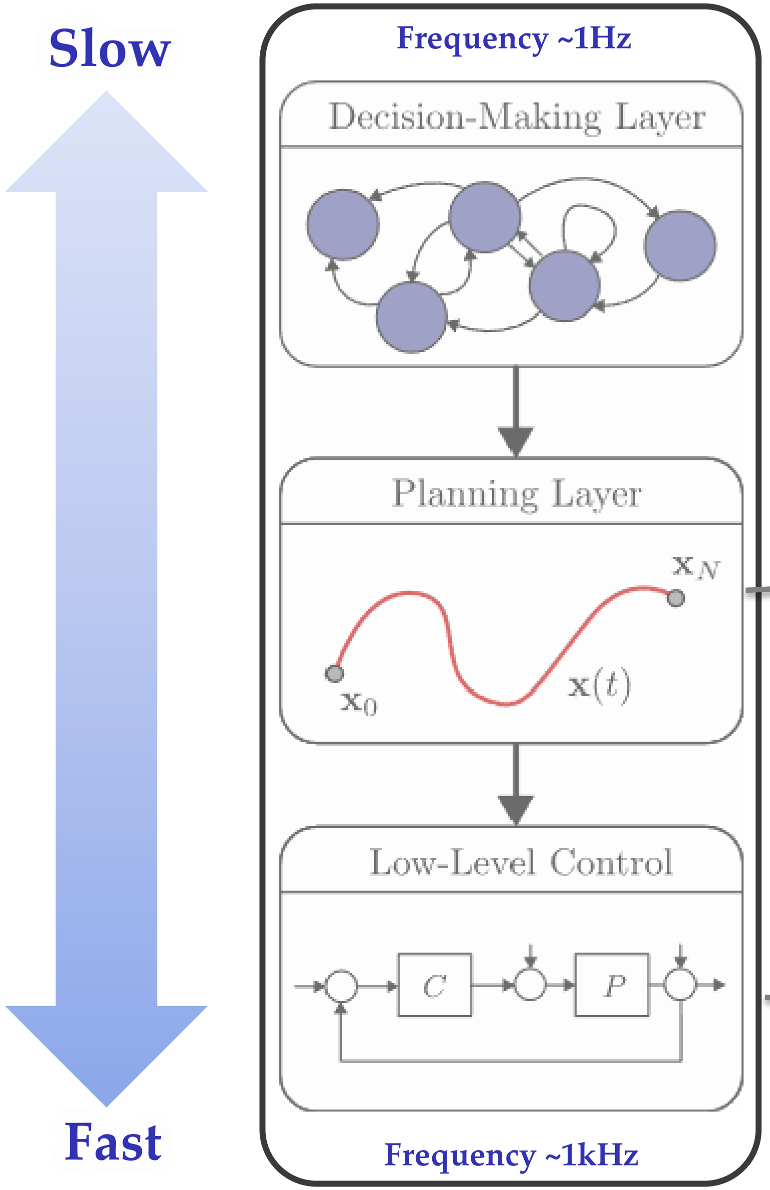

1 kHz Control Frequency
QP Variables: $\mathcal{X} = [\ddot{q}^T, u^T, \lambda_s^T]^T \in \mathbb{R}^{39}$
DOF: 22
$\mathcal{X}^* = \underset{\mathcal{X}\in\mathbb{X}_{\text{ext}}}{\operatorname{argmin}} \; ||A(x)\mathcal{X} - b(x)||^2 + \boxed{\dot{V}(q,\dot{q},\ddot{q})}$
s.t. $D_c(q)\ddot{q} + H_c(q,\dot{q})\dot{q} = B_c(q)u + J_{c,s}^T(q)\lambda_s$
$\lambda_s \in \mathcal{AC}_{\text{crouch}}(\lambda_s)$
$u_{\text{lb}} \le u \le u_{\text{ub}}$
$u_{\text{s,ak}} = 0$
Dynamics
Friction pyramid
Force limits
Passive foot
Torque limits
Gazebo Simulation
Lyapunov Controller
$u^*(x) = \underset{(u,\delta)\in U\times\mathbb{R}}{\operatorname{argmin}} \; \|u - u_{\text{des}}(x)\|^2$
s.t. $\dot{V}(x,u) \le -\alpha V(x)$
Friction Pyramid
Hip Roll Torque
Lyapunov Function
Unplanned walking on slopes
Caltech
Reher and AA, ICRA 2021

LCA: Layered Control Architecture



LCA: Historical Context

Origins of GNC Stack



Copyright, 195… …yright owner.

Automatic N… …cket Vehicle

Daniel and Florence … …gy, Pasadena, Calif.

Trajectory Planning

The primary purpose or the "duty" of the computer is then to properly digest the dynamic and the aerodynamic information of the vehicle and thus to decide the right flight path correction that will insure landing at the chosen destination.

Feedback Control

The purpose or the duty of the elevator servo is now simply to follow the command of the computer. If the servo is considered to be

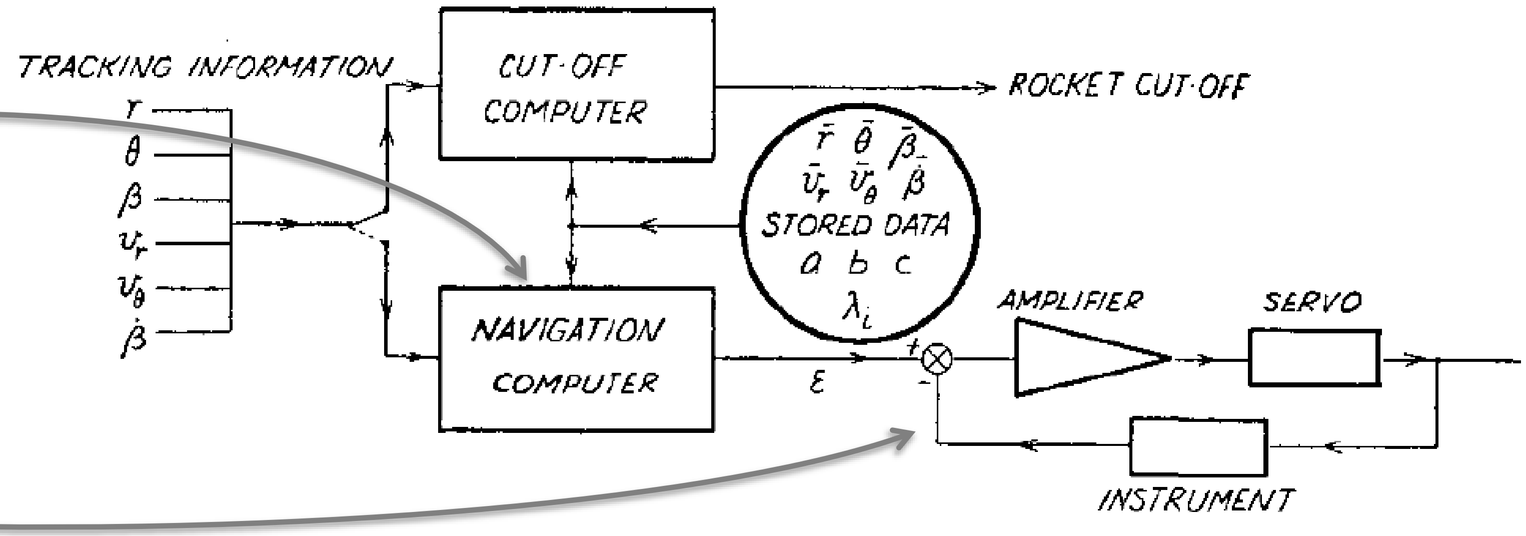
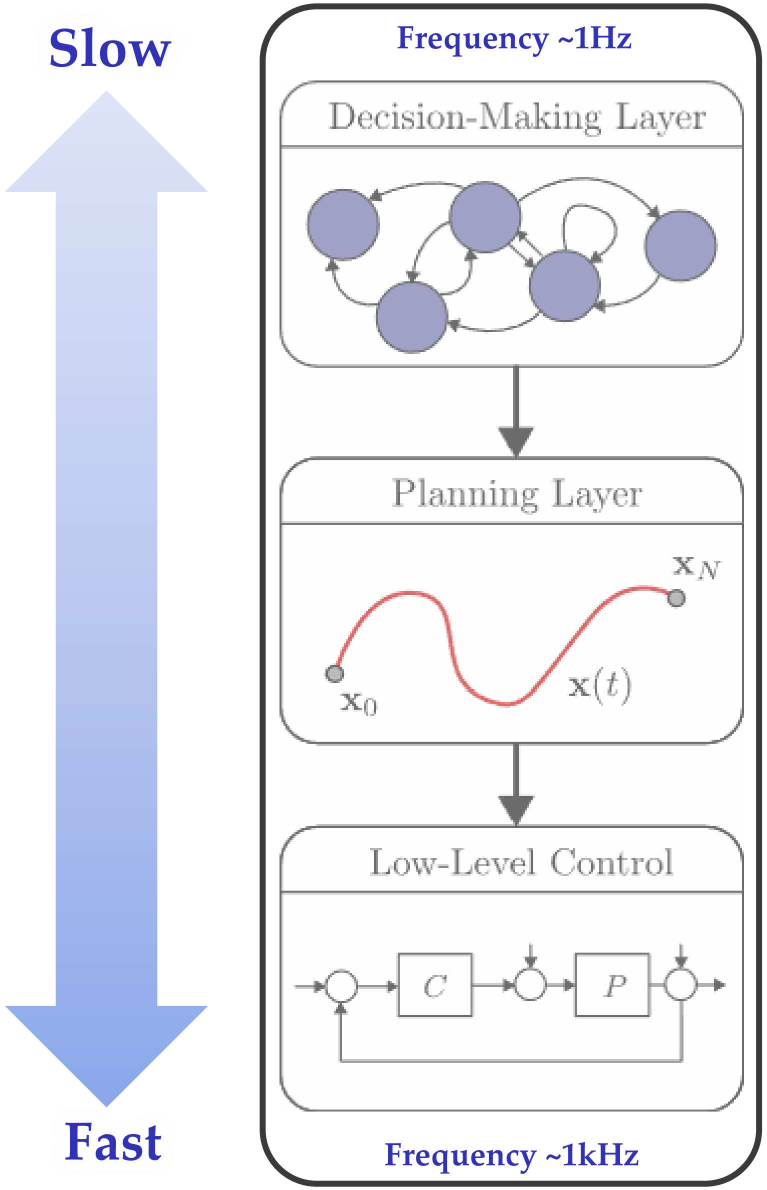


FIG. 2. SYSTEM OF AUTOMATIC NAVIGATION

H.S. Tsien et al., "Automatic navigation of a long range rocket vehicle," 1952

LCA: Historical Context

Apollo GNC Stack



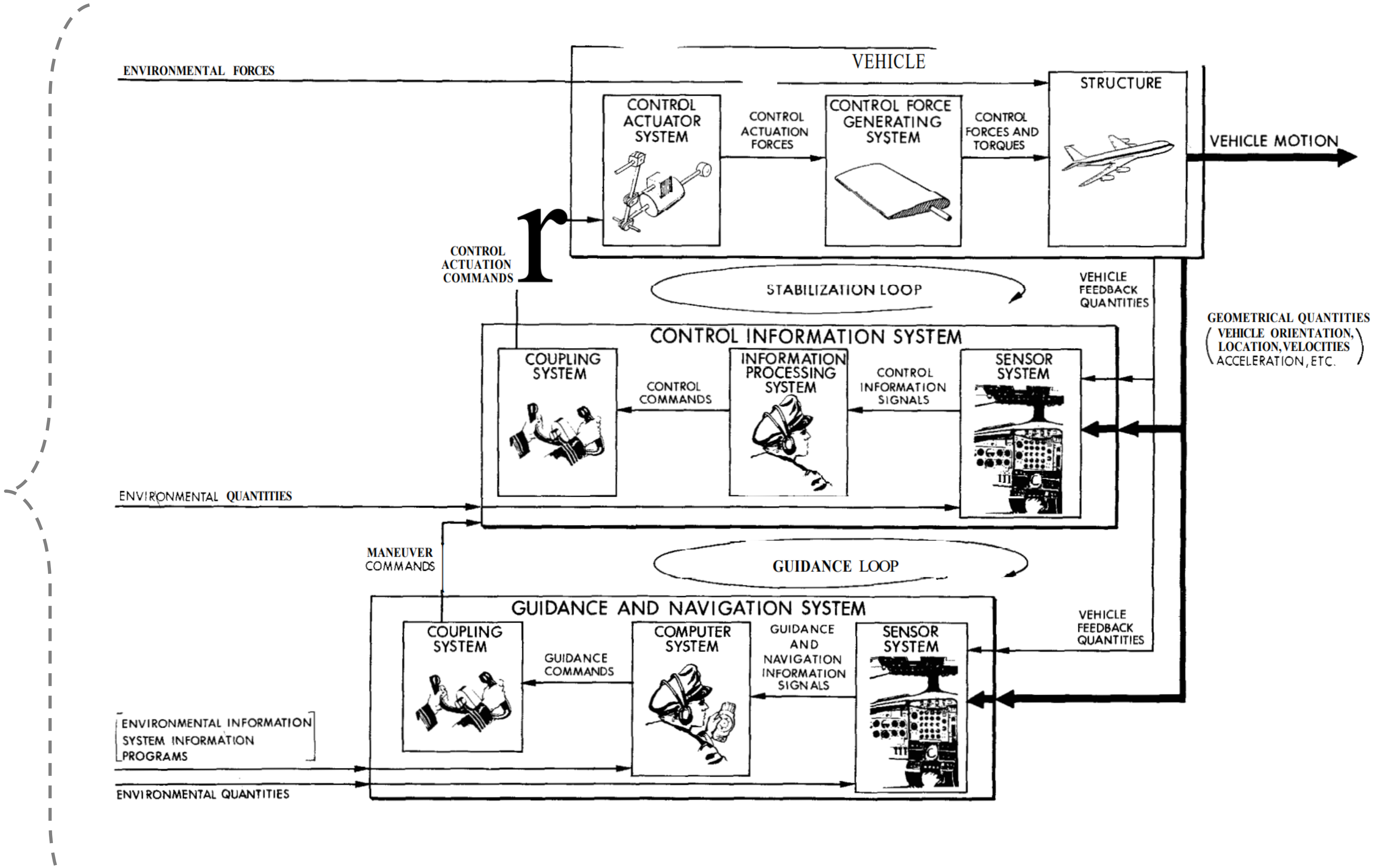
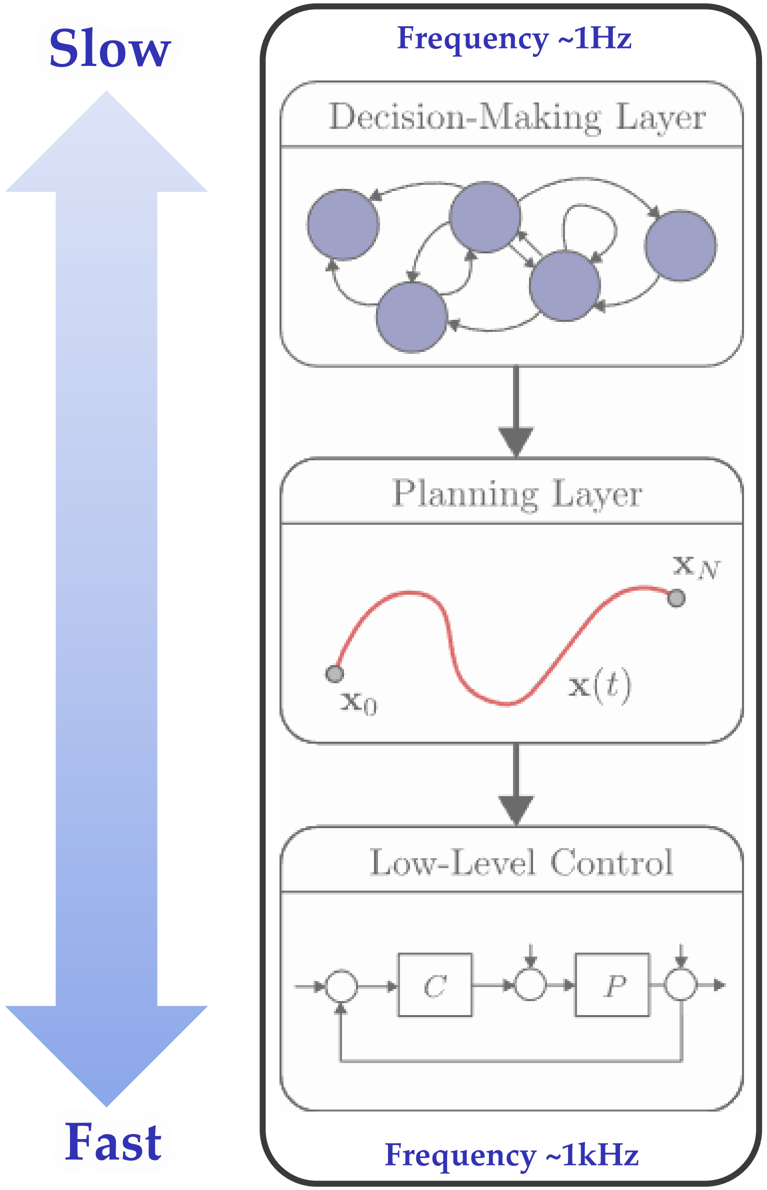
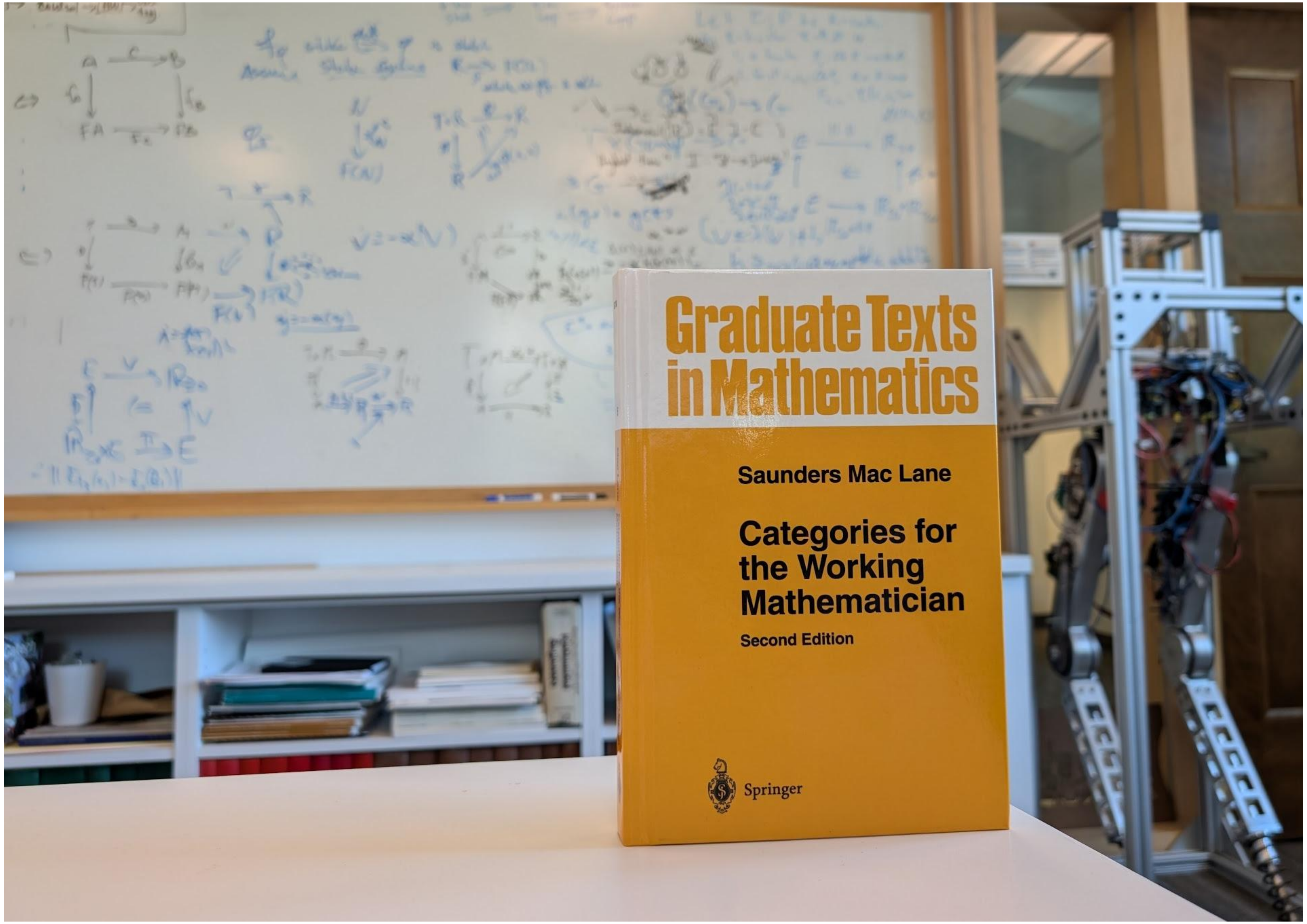


Fig. 1-17 Guidance and Control System - Human Operator's Senses and Power Extended

C. Draper et al., "Guidance and navigation," Massachusetts Inst. Technol., Cambridge, USA, 1965

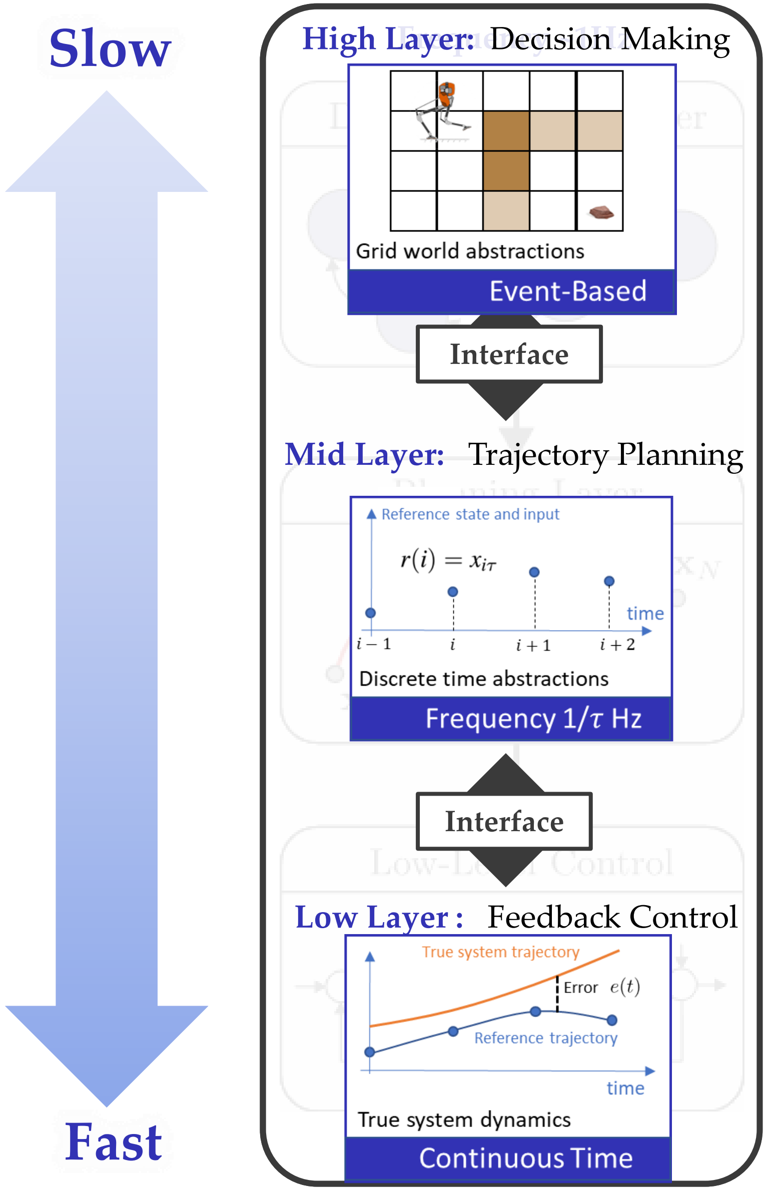
LCA: Historical Context





S. Mac Lane, "Categories for the Working Mathematician," Springer, 1971

LCA: Fractured Landscape



Formal Methods Worldview:

Specifications: $\xi(s) \models \varphi,$

Dynamics: $s' \in f_{\text{FTS}}(s, a)$

Model Predictive Control Worldview:

Dynamics: $z(i+1) = f_{\text{plan}}(z(i), u_{\text{ff}}(i))$

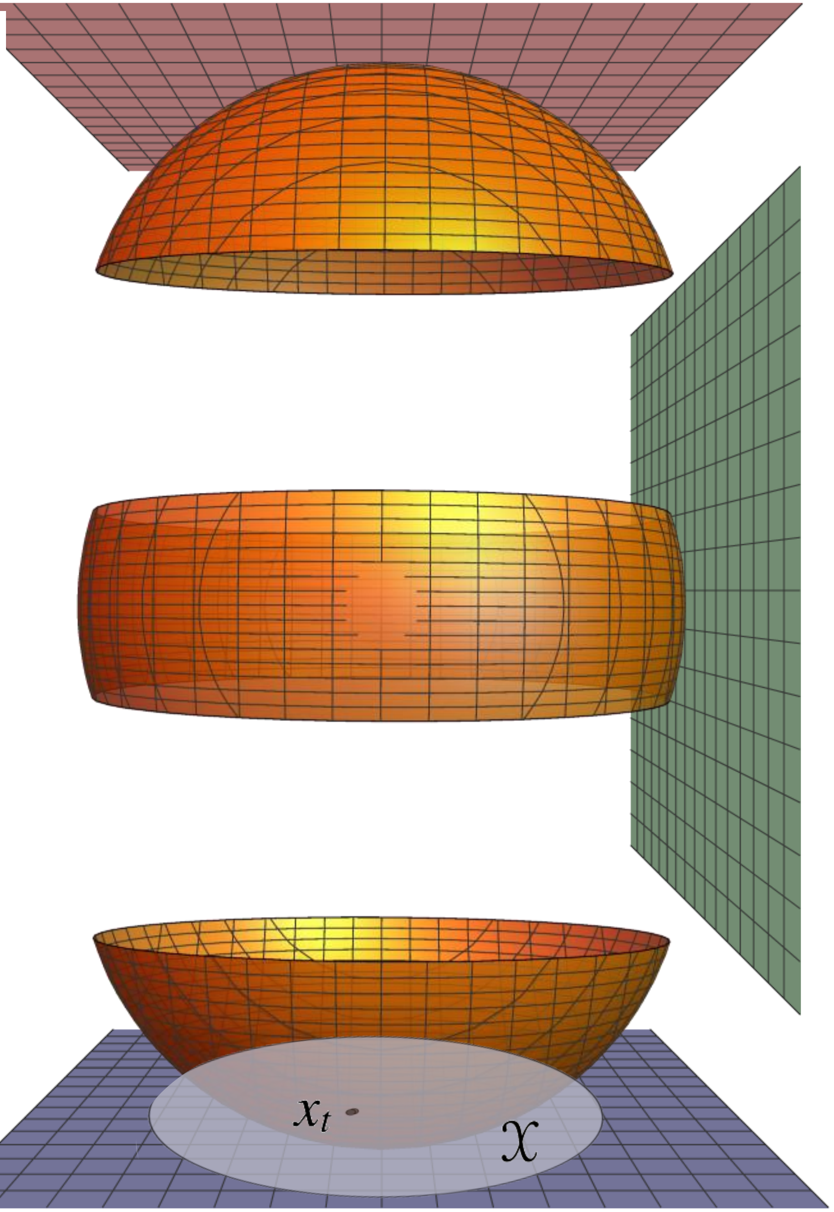
Constraints: $z(k) \in \Pi(\mathcal{X}), \forall k \geq 0$

Nonlinear Control Worldview:

Dynamics: $\dot{x}_t = f(x_t, u_{\text{fb}}(x_t))$

Constraints: $x_t \in \mathcal{X}$

$u_{\text{fb}}(x_t) \in \mathcal{U}, \ \forall t \geq 0$



Rosolia, Singletary, AA, TAC 2022

LCA: Need to Unify

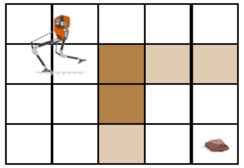
Overall LCA Problem

$\xi(x) \models \varphi,$
subject to: $\dot{x}_t = f(x_t, u(x_t))$.

Slow

Fast

High Layer: Decision Making

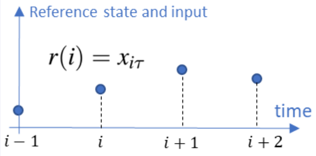


Grid world abstractions

Event-Based

Interface

Mid Layer: Trajectory Planning

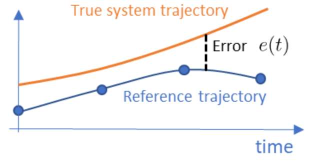


Discrete time abstractions

Frequency $1/\tau$ Hz

Interface

Low Layer: Feedback Control



True system dynamics

Continuous Time

Decision Making / Formal Methods Worldview:

Specifications: $\xi(s) \models \varphi,$
Dynamics: $s' \in f_{\mathrm{FTS}}(s, a)$

Top Interface
$z(k) \in s(\lfloor k/\delta \rfloor)$

Trajectory Planning / Model Predictive Control Worldview:

Dynamics: $z(i+1) = f_{\mathrm{plan}}(z(i), u_{\mathrm{ff}}(i))$
Constraints: $z(k) \in \Pi(\mathcal{X}), \forall k \geq 0$

Bottom Interface
$z(k) = \Pi(x_{k\tau})$

Feedback Control / Nonlinear Control Worldview:

Dynamics: $\dot{x}_t = f(x_t, u_{\mathrm{fb}}(x_t))$
Constraints: $x_t \in \mathcal{X}$
$u_{\mathrm{fb}}(x_t) \in \mathcal{U}, \ \forall t \geq 0$

Satisfy Specifications

$$\underset{(s,a)}{\text{satisfy}} \quad \xi(s,a) := (L(s(k), a(k)))_{k\in\mathbb{N}} \models \varphi$$
$$\text{s.t.} \quad z(k) \in s(\lfloor k/\delta \rfloor)$$

Interface

Optimize Trajectories

$$\underset{z(k), u_{\mathrm{ff}}(k)}{\text{minimize}} \quad \sum_{i=k}^{k+N-1} \mathrm{Cost}(z(i), u_{\mathrm{ff}}(i)) + \mathrm{Cost}_N(z(k+N))$$
$$\begin{aligned} \text{s.t.} \quad & z(i+1) = f_{\mathrm{DT}}(z(i), u_{\mathrm{ff}}(i)) \\ & z(k) = \Pi(x_{k\tau}) \\ & z(i) \in \Pi(\mathcal{X}) \cap s(\lfloor i/\delta \rfloor) \end{aligned}$$

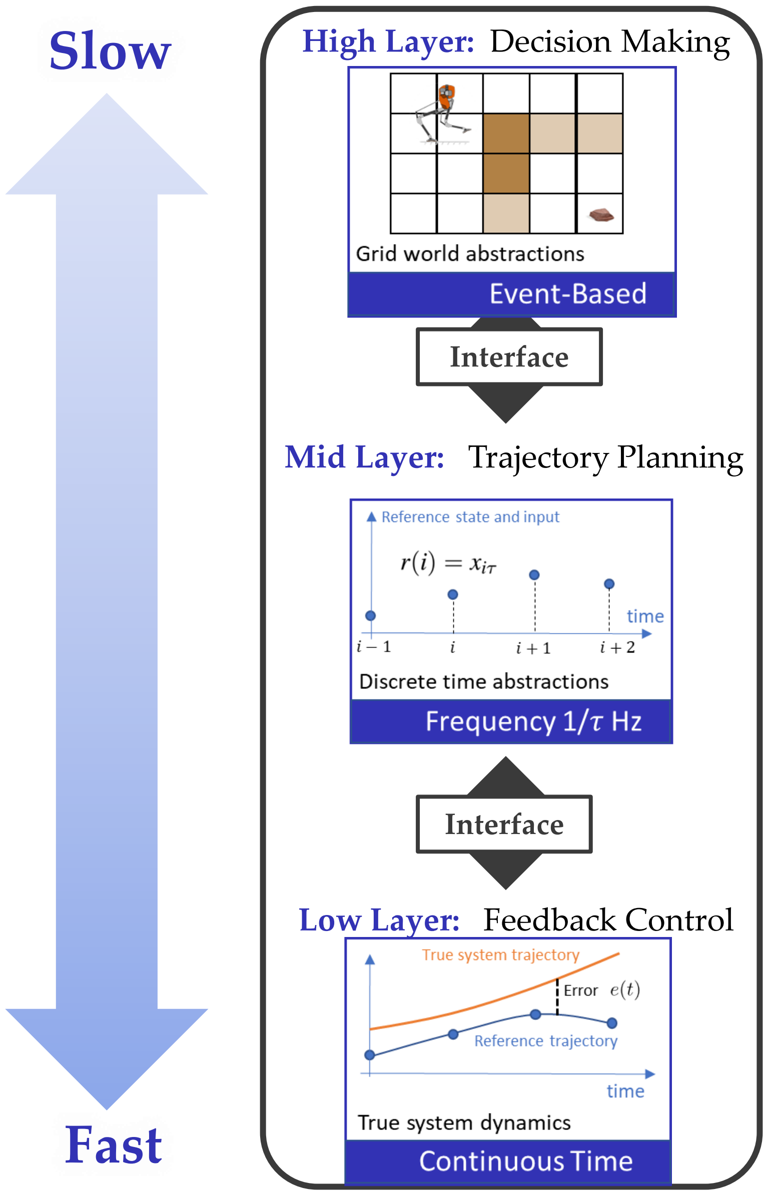
Interface

Optimize Real-Time Feedback

$$\underset{u = u_{\mathrm{ff}} + u_{\mathrm{fb}}}{\text{minimize}} \quad \int_{k\tau}^{(k+N)\tau} \left(\|x(s) - z(\lfloor s/\tau \rfloor)\|_Q^2 + \|u_{\mathrm{fb}}\|_R^2 \right) ds$$
$$\begin{aligned} \text{s.t.} \quad & \dot{x} = f(x, u_{\mathrm{ff}} + u_{\mathrm{fb}}) \\ & x \in \mathcal{X} \\ & u_{\mathrm{ff}} + u_{\mathrm{fb}} \in \mathcal{U} \end{aligned}$$

Matni, AA, Doyle, CSM 2024

Idea: Unify via Lyapunov



Labeled Transition System:

Dynamics: $s' \in f_{\mathrm{FTS}}(s, a)$

Constraints: $s \in \mathcal{S}$

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

Discrete-Time Dynamical System:

Dynamics: $z(i+1) = f_{\mathrm{plan}}(z(i))$

Constraints: $z(k) \in \mathcal{Z}$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

Continuous-Time Dynamical System:

Dynamics: $\dot{x}(t) = f(x(t))$

Constraints: $x(t) \in \mathcal{X}$

Lyapunov Condition (CT):

$$\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_{x} f(x(t)) \leq 0$$

Idea: Unify via Lyapunov

Caltech

Unified Lyapunov Theory:

What is the commonality?

How do we generalize?

Answer:

Saunders Mac Lane

Categories for the Working Mathematician

"Good general theory does not search for the maximum generality, but for the right generality."

–Mac Lane

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

Lyapunov Condition (CT):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\Big|_x f(x(t)) \leq 0$$

Revisiting Classical Lyapunov Theory

Caltech

Solutions

- Expressed by commuting diagram:

$$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "Tc"] & \mathcal{X}\times\mathbb{R}^n \\ \mathbb{R} \arrow[u, "1_{\mathbb{R}} \triangleq (\mathrm{id}_{\mathbb{R}},1)"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "(\mathrm{id}_{\mathcal{X}},f) \triangleq f_{\mathcal{X}}"'] \end{tikzcd}$$

where $1(t)\equiv 1$, i.e., 1 is the unit clock: $\dot{t} = 1(t) = 1$ and

$$Tc(t,t') = (c(t),\dot{c}(t)t')$$

- The *diagram commutes* if $c(t)$ is a solution to $\dot{x} = f(x)$:

$$\begin{aligned} Tc\circ 1_{\mathbb{R}} = f_{\mathcal{X}}\circ c \quad &\Longrightarrow \quad (c(t),\dot{c}(t)1) = (c(t),f(c(t))) \\ &\Longrightarrow \quad \boxed{\dot{c}(t) = f(c(t))} \end{aligned}$$

Lyapunov

- Consider the following diagram that *commutes up to inequality*:

$$\begin{tikzcd} \mathcal{X}\times\mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0}\times\mathbb{R} \\ \mathcal{X} \arrow[u, "\vec{f}"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"'] \end{tikzcd}$$

where $0_{\mathbb{R}_{\geq 0}}$ is zero vector field $0_{\mathbb{R}_{\geq 0}}(r) = (r,0)$, and

$$TV(x,y) \triangleq (V(x),DV(x)y) = \left(V(x), \left.\frac{\partial V}{\partial x}\right|_x y\right).$$

- Saying"diagram commutes up to inequality" means:

$$\begin{aligned} TV\circ f_{\mathcal{X}} \leq 0_{\mathbb{R}_{\geq 0}}\circ V \quad &\Longrightarrow \quad TV(x,f(x)) \leq (V(x),0) \\ &\Longrightarrow \quad \boxed{\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x) \leq 0} \end{aligned}$$

Continuous-Time Dynamical System:

Dynamics:	$\dot{x}(t) = f(x(t))$
Constraints:	$x(t)\in\mathcal{X}$

Lyapunov Condition (CT):

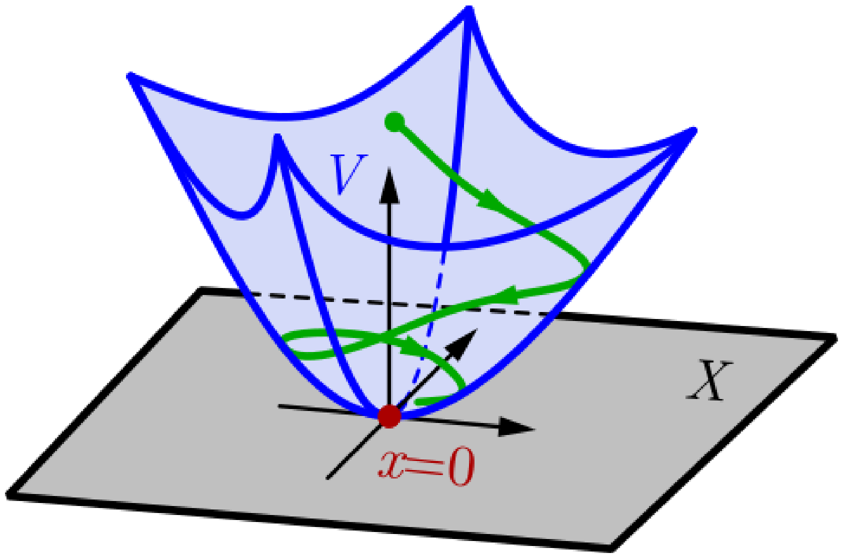
$$\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x(t)) \leq 0$$

Categorical Lyapunov Theory

Classical Conditions

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$
\begin{tikzcd}
\mathbb{R} \times \mathbb{R} \arrow[r, "Tc"] & \mathcal{X} \times \mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0} \times \mathbb{R} \\
\mathbb{R} \arrow[u, "1_{\mathbb{R}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "\vec{f}"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"']
\end{tikzcd}
$$



unit clock $1_{\mathbb{R}} = (\mathrm{id}_{\mathbb{R}}, 1)$, zero vector field $0_{\mathbb{R}_{\geq 0}}(r) = (r, 0)$.

Theorem

The system $\dot{x} = f(x)$ is stable iff the diagram (lax) commutes.

Lyapunov Condition (CT):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\bigg|_x f(x(t)) \leq 0$$

Categorical Lyapunov Theory

Coalgebras

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\mathcal{F}(c)"] & \mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
T \arrow[u, "1_T"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & E \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$

unit clock $1_T : T \to \mathcal{F}(R)$, zero system $0_R : R \to \mathcal{F}(R)$

Theorem

The system $f_E : E \to \mathcal{F}(E)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} : \mathsf{C} \to \mathsf{C}$ an endofunctor, $T =$ time and $R =$ measurement.

On the Stability of Zeno Equilibria*

Aaron D. Ames[1], Paulo Tabuada[2], and Shankar Sastry[1]

HSCC 2006

CATEGORICAL LYAPUNOV THEORY I: STABILITY OF FLOWS

AARON D. AMES, JOE MOELLER, AND PAULO TABUADA

CATEGORICAL LYAPUNOV STABILITY II: STABILITY OF SYSTEMS

AARON D. AMES, SÉBASTIEN MATTENET, AND JOE MOELLER

ArXiv 2025

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Categorical Lyapunov Theory

Caltech

Discrete Time

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\mathcal{F}(c)"] & Z \arrow[r, "\mathcal{F}(V)"] & \mathbb{R}_{\geq 0} \\
\mathbb{N} \arrow[u, "1_{\mathbb{N}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"']
\end{tikzcd}$$

unit clock $1_{\mathbb{N}}(k) = k + 1$, zero system $0_{\mathbb{R}_{\geq 0}}(r) = r$.

Theorem

The system $z_{k+1} = f_{\text{plan}}(z_k)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} = I$, $\mathsf{C} = \mathsf{Man}$, $T = \mathbb{N}$ and $R = \mathbb{R}_{\geq 0}$.

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\text{plan}}(z)) - V(z) \leq 0$$

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Categorical Lyapunov Theory

Transition Systems[1]

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathcal{P}(A \times A^*) \arrow[r, "\mathcal{P}(A \times c)"] & \mathcal{P}(A \times S) \arrow[r, "\mathcal{P}(A \times V)"] & \mathcal{P}(A \times \mathbb{R}_{\geq 0}) \\
A^* \arrow[u, "1_{A^*}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & S \arrow[u, "f_S"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"']
\end{tikzcd}$$

unit clock $1_{A^*}(a^*) = \{(a, a^* a)\}_{a \in A}$, zero system $0_{\mathbb{R}_{\geq 0}}(r) = A \times \{r\}$.

Theorem

The system $s' \in f_{\mathrm{FTS}}(s, a)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} = \mathcal{P}(A \times -)$, $\mathsf{C} = \mathsf{Set}$, $T = A^*$ (free monoid) and $R = \mathbb{R}_{\geq 0}$.

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Categorical Lyapunov Theory ↔ Ability to Generalize

Caltech

$$\boxed{system\, f_E : E \to \mathcal{F}(E)}$$

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\mathcal{F}(c)"] & \mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
T \arrow[u, "1_T"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & E \arrow[u, "f_E"'] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

$$system\ s' \in f_{\mathrm{FTS}}(s,a)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

$$system\ z_{k+1} = f_{\mathrm{plan}}(z_k)$$

Lyapunov Condition (CT):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\bigg|_x f(x(t)) \leq 0$$

$$system\ \dot{x} = f(x)$$

Categorical Lyapunov Theory – What is next?

Asymptotic Stability

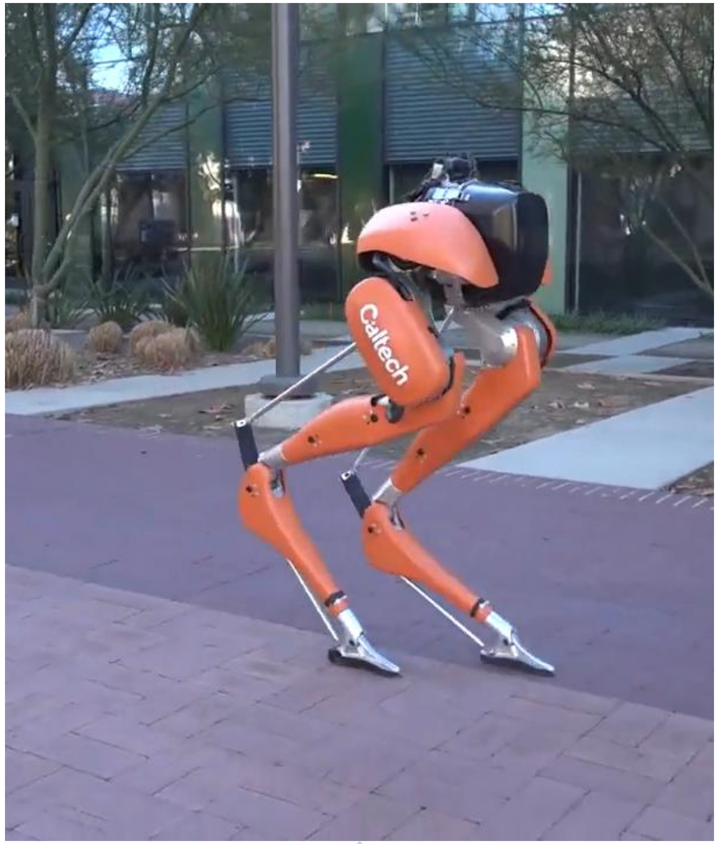
Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R} \arrow[r, "Tc"] & \mathcal{X} \times \mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0} \times \mathbb{R} \\
\mathbb{R} \arrow[u, "1_{\mathbb{R}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "\vec{f}"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "\sigma"']
\end{tikzcd}$$

unit clock $1_{\mathbb{R}} = (\mathrm{id}_{\mathbb{R}}, 1)$, "simplest" stable system $\sigma(r) = (r, -\alpha(r))$.

Theorem

The system $\dot{x} = f(x)$ is asy. stable iff the diagram (lax) commutes.

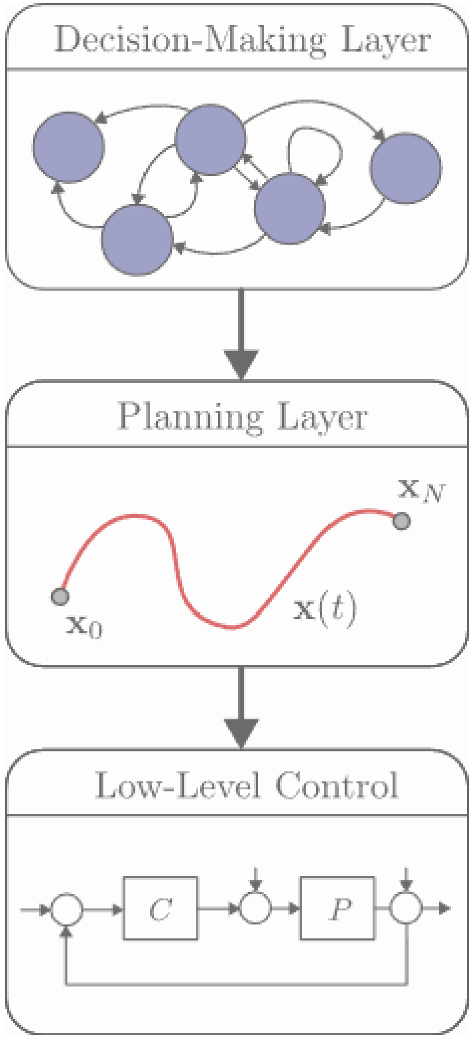


Lyapunov Condition (Asy.):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\bigg|_x f(x(t)) \leq -\alpha(V(x))$$

Categorical Theory ↔ Ability to Generalize

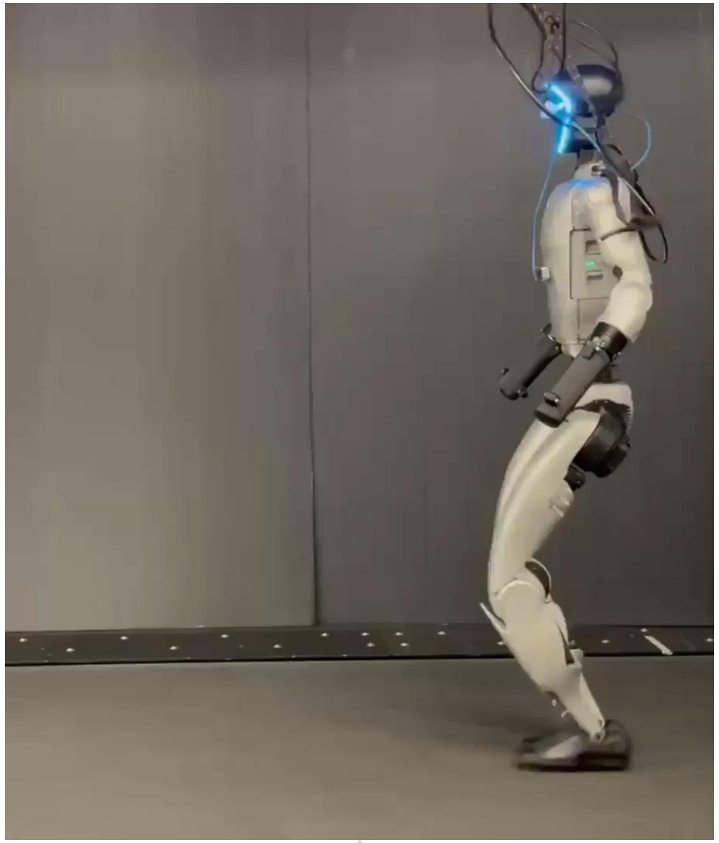
Caltech



Goal: Category theory on robots

$$
\begin{tikzcd}
\mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
E \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}
$$

- Robotics: ever increasing *complexity*
- Need for *formal guarantees* to deploy
- Category theory: *formal generalizability*



Lyapunov Condition ($\mathbb{C}$):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Goal: Category Theory on Robots

$$\begin{tikzcd}
\mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
E \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$