A categorical framework for Lyapunov stability

Joe Moeller

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A. Ames, J. Moeller, P. Tabuada, *Categorical Lyapunov Theory I: Stability of Flows*, arXiv:2502.15276

- systems are monoid actions
- notion of stability native to the setting
- Theorem: if a Lyapunov morphism exists for the flow, then the point is stable.

A. Ames, S. Mattenet, J. Moeller, *Categorical Lyapunov theory II: Stability of systems*, arXiv:2505.22968

- systems are $\mathcal{F}$-coalgebras
- borrows notion of stability from flows
- Theorem: if a Lyapunov morphism exists for the coalgebra, it is a Lyapunov morphism for the solution flow.

Lyapunov's Theorem

Theorem

Let x^* be an equilibrium point for a dynamical system $\dot{x} = \vec{f}(x)$, and $M \subset \mathbb{R}^n$ be a domain containing x^*. Let $V : M \to \mathbb{R}$ be a continuously differentiable function such that

- *(positive definite)* $V(x) \geq 0$ *and* $V(x^*) = 0$
- *(decrescent)* $\dot{V}(x) \leq 0$ *in* M.

Then x^* is a stable equilibrium point.

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- dynamical system
- solution curve
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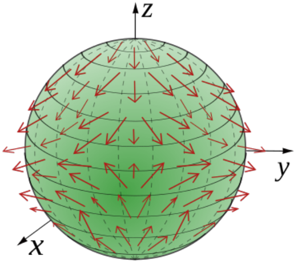
- stable equilibrium point
- Lyapunov function

Dynamical Systems, Categorically



- Vector field: smooth map $\vec{f}\colon M \to \mathcal{T}(M)$ with $p_M \circ \vec{f} = \mathrm{id}_M$

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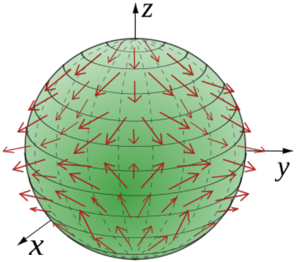

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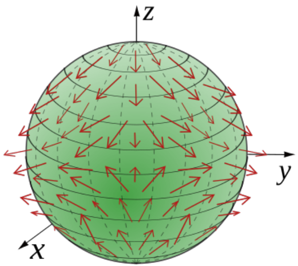
- M manifold, $\mathcal{T}(M)$ tangent bundle, $p_M\colon \mathcal{T}(M) \to M$ projection

Dynamical Systems, Categorically



As a commutative diagram:

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M \arrow[ur, "\vec{f}"] \arrow[rr, "\mathrm{id}_M"'] & & M
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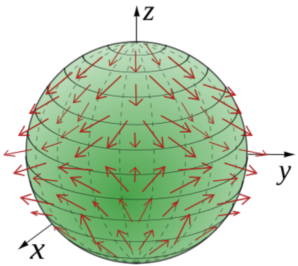

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Definition

*Let $\mathcal{C}$ be a category, $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ a functor. An **$\mathcal{F}$-system** is an object $X \in \mathcal{C}$ and a map $f\colon X \to \mathcal{F}(X)$.*

Examples of $\mathcal{F}$-Systems

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- $\mathcal{T}$-systems are (continuous-time) dynamical systems (not necessarily sections).
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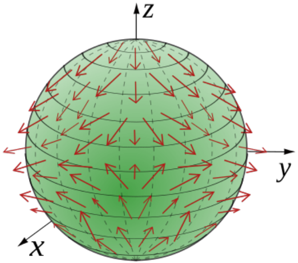

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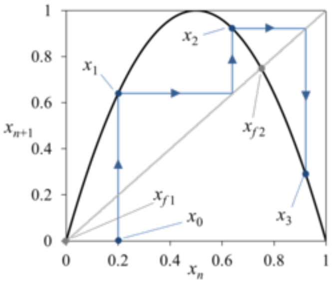

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- $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ the power set, $\mathcal{P}$-systems are graphs.
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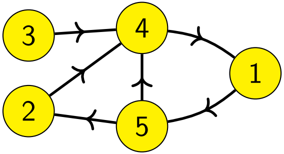
$$f\colon \{1,2,3,4,5\} \to \mathcal{P}(\{1,2,3,4,5\})$$



$$f(5) = \{2,4\}$$

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- $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$ the power set, $\mathcal{P}$-systems are graphs.
- Fix a set L of "labels", $\mathcal{P}(L \times -) \colon \mathsf{Set} \to \mathsf{Set}$, $\mathcal{P}(L \times -)$-systems are L-labeled transition systems.

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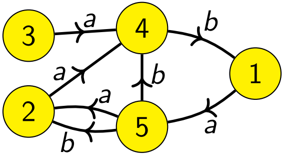
$$L = \{a, b\}$$

$$f \colon \{1,2,3,4,5\} \to \mathcal{P}(\{a,b\} \times \{1,2,3,4,5\})$$



$$f(5) = \{(a,2), (b,2), (b,4)\}$$

Lyapunov's Theorem

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Morphisms of $\mathcal{F}$-systems

A **map of vector fields** is a smooth function $\phi\colon M \to N$ such that

$$\frac{d\phi}{dx} \cdot \vec{f}(x) = \vec{g}(\phi(x))$$

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\mathcal{T}(M) \arrow[r, "d\phi"] & \mathcal{T}(N) \\
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$\vec{f}$ and $\vec{g}$ are "ϕ-related".

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Time, solutions

A **solution curve** of $\vec{f}\colon M \to \mathcal{T}(M)$ is a smooth map $c\colon \mathbb{R}_{\geq 0} \to M$ such that

$$\vec{g}(c(t)) = \dot{c}(t)$$

$$\begin{tikzcd}
\mathcal{T}(\mathbb{R}_{\geq 0}) \arrow[r, "dc"] & \mathcal{T}(M) \\
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Definition

Assume $\mathcal{C}$ has a **time object** *T with a* **unit clock** *system $1_T\colon T \to \mathcal{F}(T)$. A* **solution curve** *is a map:*

$$\begin{tikzcd}
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Equilibrium Points

Every manifold M has a zero vector field $0_M\colon M \to \mathcal{T}(M)$.

Definition

A **zero $\mathcal{F}$-system** *is a component of a natural transformation* $0_X\colon X \to \mathcal{F}(X)$*.*

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A point $x \in M$ is the same as a map $x\colon \{*\} \to M$.

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Assume $\mathcal{C}$ *has a* **terminal object** $1 = \{*\}$. *A* **point** *in category theory is a map* $x\colon 1 \to X$.

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equilibrium point:

$$\vec{f}(x) = \vec{0} \qquad \begin{tikzcd} \mathcal{T}(1) \arrow[r, "dx"] & \mathcal{T}(M) \\ 1 \arrow[u, "\vec{0}"] \arrow[r, "x"'] & M \arrow[u, "\vec{f}"'] \end{tikzcd}$$

Definition

Let $f \colon X \to \mathcal{F}(X)$ *be an* $\mathcal{F}$*-system. A point* $x \colon 1 \to X$ *is an* **equilibrium point** *if*

$$\vec{f}(x) = \vec{0} \qquad \begin{tikzcd} \mathcal{F}(1) \arrow[r, "\mathcal{F}(x)"] & \mathcal{F}(X) \\ 1 \arrow[u, "0_1"] \arrow[r, "x"'] & X \arrow[u, "f"'] \end{tikzcd}$$

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Measurement Object

Definition

An object $R \in \mathcal{C}$ is **posetal** *if each $\mathcal{C}(X, R)$ has a partial order such that for any $f : X \to Y$, if $g_1 \geq g_2$, then $g_1 \circ f \geq g_2 \circ f$.*

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$$\begin{tikzcd} T \arrow[r, "c"] \arrow[d, "1_T"'] \arrow[dr, phantom, "\not\leftrightharpoons"] & R \arrow[d, "0_R"] \\ \mathcal{F}(T) \arrow[r, "\mathcal{F}(c)"'] & \mathcal{F}(R) \end{tikzcd} \rightsquigarrow \begin{tikzcd} T \arrow[r, "!"] \arrow[d, "c"'] \arrow[dr, phantom, "\leq"] & 1 \arrow[d, "0_T \times \mathrm{id}"] \\ R & T \arrow[l, "c"] \end{tikzcd}$$

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Definition

A **semi-metric** *is a map $d\colon X \times X \to R$ such that*

- $d \Rightarrow 0$
- $\ker(d) \cong \Delta\colon X \to X \times X$

For a fixed $x_*\colon 1 \to X$, let $\|\cdot\|_{x^*}$ denote the composite

$$X \xrightarrow{\mathrm{id}_X \times x^*} X \times X \xrightarrow{d} X$$

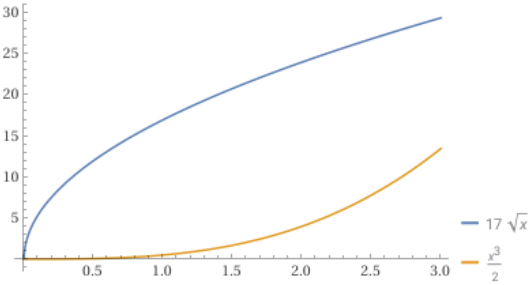
called the **semi-norm** *relative to x^*.*

Class K Morphisms

Definition

A morphism $\alpha\colon R \to R$ *is* **class** $\mathcal{K}$ *if:*

- α *is an order-preserving map*
- α *has an order-preserving inverse* α^{-1}
- $\alpha \circ 0_R = 0_R$.



Stable Equilibria

An equilibrium point $x^* \in M$ is **stable** if there is a class $\mathcal{K}$ function α such that for any solution curve c:

$$\|c(t) - x^*\| \leq \alpha(\|c(0) - x^*\|)$$

Definition

An equilibrium point $x^* : 1 \to X$ is **stable** *if there is a class $\mathcal{K}$ morphism α such that the following diagram lax commutes for any solution curve c:*

$$\begin{tikzcd}
T \arrow[r, "!"] \arrow[dd, "c"'] & 1 \arrow[r, "0_T"] & T \arrow[r, "c"] & X \arrow[d, "\|\cdot\|_{x^*}"] \\
& & & R \arrow[d, "\alpha"] \\
X \arrow[rrr, "\|\cdot\|_{x^*}"'] & & & R
\arrow[from=1-1, to=3-4, phantom, "\leq"]
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Lyapunov morphisms

$V\colon M \to R$ is a **Lyapunov morphism** for an $\mathcal{F}$-system $f\colon X \to \mathcal{F}(X)$ and equilibrium $x^*\colon 1 \to X$ if:

1. (positive definite) V is bounded by class $\mathcal{K}$ morphisms:

$$\begin{tikzcd}
& X \arrow[dl, "\|\cdot\|_{x^*}"'] \arrow[dd, "V"] \arrow[dr, "\|\cdot\|_{x^*}"] & \\
R \arrow[r, phantom, "\leq"] \arrow[dr, "\underline{\alpha}"'] & {} \arrow[r, phantom, "\leq"] & R \arrow[dl, "\overline{\alpha}"] \\
& R &
\end{tikzcd}$$

$V(x) \geq 0$, $V(x) = 0$ iff $x = x^*$.

2. (decrescent) the following diagram lax commutes:

$$\begin{tikzcd}
X \arrow[r, "V"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & R \arrow[d, "0_R"] \\
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$$\frac{\partial V}{\partial x} f(x) \leq 0$$

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Examples

- cts-time: $\frac{\partial V}{\partial x} f(x) \leq 0$
- discrete-time: $\nabla V(X) = V(f(x)) - V(x) \leq 0$
- transition system: $\max_{s' \in f(s)} V(s') \leq V(s)$

Lyapunov's Theorem, Categorically

Theorem (AMT + AMM)

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a functor. Let $x^*\colon 1 \to X$ be an equilibrium point for a $\mathcal{F}$-system $f\colon X \to \mathcal{F}(X)$. Let $V\colon M \to R$ be a Lyapunov morphism. Then x^* is a stable equilibrium point.

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Proof:

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\end{tikzcd}$$

positive definite:

$$\begin{tikzcd}
T \arrow[r, "!"] \arrow[d, "c"'] & 1 \arrow[r, "0_T"] & T \arrow[r, "c"] \arrow[d, phantom, "\leq"] & X \arrow[r, "\|\cdot\|_{x^*}"] \arrow[dr, "V"'] \arrow[r, phantom, shift right=4, "\leq"] & R \arrow[d, "\overline{\alpha}"] \\
X \arrow[rrrr, "V"] \arrow[drr, "\|\cdot\|_{x^*}"'] & & \leq & & R \arrow[d, "\underline{\alpha}^{-1}"] \\
& & R \arrow[urr, "\underline{\alpha}"'] \arrow[rr, "\mathrm{id}"'] & & R
\end{tikzcd}$$

Existence and Uniqueness Theorem

Theorem (Existence and Uniqueness)

Assume that the unit clock $1_T \colon T \to \mathcal{F}T$ is itself T-complete. If $\mathrm{D}\phi$ is T-complete for all T-flows ϕ, then

$$\phi = \int \mathrm{D}\phi, \qquad f = \mathrm{D}\int f$$

Therefore, $\int$ is an isomorphism of categories with inverse D:

$$\begin{tikzcd} T\text{-}\mathsf{Sys}_{\mathcal{F}} \arrow[r, bend left, "\int"] \arrow[r, phantom, "\simeq"] & T\text{-}\mathsf{Flow} \arrow[l, bend left, "\mathrm{D}"] \end{tikzcd}$$

Converse Lyapunov Theorem

Theorem (Converse Lyapunov Theorem)

Assume a converse setting such that R has local suprema commuting with whiskering.

$$\begin{tikzcd} & B \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow"] & \\ A \times B \arrow[ur, "\pi_B"] \arrow[rr, "f"'] & {} & R \end{tikzcd} \qquad \textit{implies} \qquad \begin{tikzcd} & B \arrow[dr, "\sup_A f"] \arrow[dr, bend left=60, "\beta"] \arrow[d, phantom, "\Downarrow"] & \\ A \times B \arrow[ur, "\pi_B"] \arrow[rr, "f"'] & {} & R \end{tikzcd}$$

$$f(a,b) \leq \beta(b) \qquad\qquad f(a,b) \leq \sup_A f(b) \leq \beta(b)$$

Let $x^* \colon 1 \to E$ be an equilibrium point of a T-complete system $f \colon E \to \mathcal{F}E$. If x^* is stable, then there exists a Lyapunov morphism $V \colon E \to R$.

Thanks!

Part I: flows

Part II: Coalgebras