CS 70 Fall 2024	Discrete Mathematics and Probability Theory Rao, Hug	DIS 1B

1 Natural Induction on Inequality

Note 3

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

For $n=1$, $1+x \geq 1+x$ trivially true.

Induction on n, we want to prove $(1+x)^{n+1} \geq 1+(n+1)x$

then $(1+x)(1+x)^n \geq 1+nx+x$

$\Rightarrow (1+x)^n + x(1+x)^n \geq x+1+nx$

by induction hypothesis, $(1+x)^n \geq nx+1$, while $x(1+x)^n \geq x$ is true as $(1+x)^n \geq 1$ for $x>0$ (division by x). Thus $(1+x)^{n+1} \geq 1+(n+1)x$

by induction, the statement holds.

2 Make It Stronger

Note 3

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

Base $a_1 = 1 \leq 3^2 = 9$

Induction step: $1 \ldots n$ holds for $a_n \leq 3^{2^n}$. $a_{n+1} = 3a_n^2$ can't induct

as if $a_n = 3^{2^n}$. $3a_n^2 = 3^{2^{n+1}+1} > 3^{2^n}$

Thus induction hypothesis too weak.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

Base: $a_1 = 1 \leq 3^{2^1-1} = 3$

Induct: $a_n \leq 3^{(2^n-1)}$, $a_{n+1} = 3a_n^2 \leq 3 \cdot 3^{2^{n+1}-2} = 3^{2^{n+1}-1}$

while $3^{2^{n+1}-1} < 3^{2^{n+1}}$, $a_{n+1} \leq 3^{2^{n+1}}$. thus the statement true.

(c) Why does the hypothesis in part (b) imply the overall claim?

BCS it's a stronger claim that has stricter conditions.

3 Binary Numbers

Note 3

Prove that every positive integer n can be written in binary. In other words, prove that for any positive integer n, we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

for some $k \in \mathbb{N}$ and $c_i \in \{0, 1\}$ for all $i \leq k$.

4 Fibonacci for Home

Note 3

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

Base established.

As every third established, for $n \notin 3k$, $k \in 0 \ldots$

$F_{3k+1} = F_{3k} + F_{3k-1}$. $F_{3k+2} = F_{3k} + F_{3k+1}$

F_{3k+1} & F_{3k+2} has same odditivity.

Thus F_{3k+3} must be even

Conclude