# 1 Short Answers

Note 5

In each part below, provide the number/equation and brief justification.

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

$v+f=e+2=v+7 \Rightarrow f=7$

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

~~Tree~~: one face. $\Rightarrow v+f=e+2=v+1$

$v=2^3=8$, while $e=\frac{3 \cdot 8}{2}=12$ (counting every edge twice)

But $v+1=e+2 \Rightarrow e=7$, thus 5 edges need be deleted.

(c) The Euler's formula $v-e+f=2$ requires the planar graph to be connected. What is the analogous formula for planar graphs wth $k$ connected components?

for $k=1$. same, $k=2$ apply to different compont

but overcounted a face (the big wide space face)

Thus, $k$. $v-e+f-(k-1)=2$

$\Rightarrow v+f=e+k+1$

# 2 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph $G$. Using only the information in the current part, say whether $G$ will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) $G$ can be vertex-colored with 4 colors.

$K_5$ or planar

(b) $G$ requires 7 colors to be vertex-colored.

$K_5$ & $K_{3,3}$ does not need 7 colors. Thus planar. ✗ Non-planar

5 color theorem states maximal color of planar, while $K_7$ requires 7 color.

(c) $e \leq 3v - 6$, where $e$ is the number of edges of $G$ and $v$ is the number of vertices of $G$.

K5 sat but also general planar graph sat

(d) $G$ is connected, and each vertex in $G$ has degree at most 2.

K5 or K3,3 both not sat this.
Due to theorem saying none plane graph must have K5 or K3,3, the graph must be planar.

(e) Each vertex in $G$ has degree at most 2.

Same as above.

## 3 Graph Coloring

Note 5

Prove that a graph with maximum degree at most $k$ is $(k+1)$-colorable.

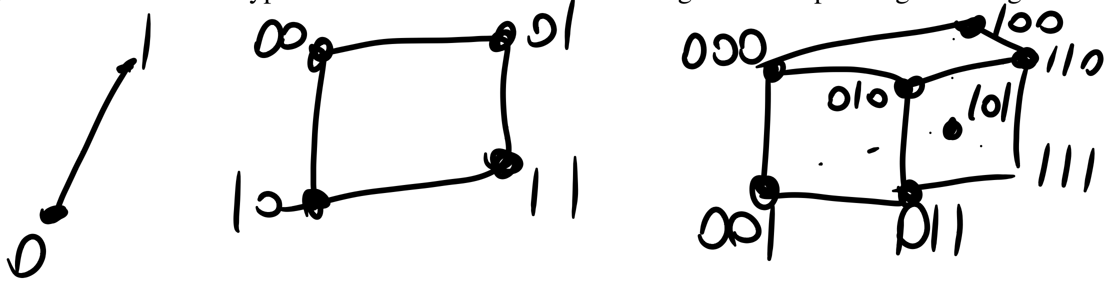
Color G, we see that any coloring scheme, no matter what color its vertices show, there is always a color left (the k+1), thus the remaing vertices can be colored.

## 4 Hypercubes

Note 5

The vertex set of the $n$-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all $n$-bit strings). There is an edge between two vertices $x$ and $y$ if and only if $x$ and $y$ differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.




(b) Show that the edges of an $n$-dimensional hypercube can be colored using $n$ colors so that no pair of edges sharing a common vertex have the same color.

Any tour of cube must take even steps as move flips one bit and we need to flip it back. Thus every tour is even and the cube is bipartite and thus n partite for $n \geq 2$. By using swap role of edge and vertex edge can be bipartite as well. One dim natural trivial.

(c) Show that for any $n \geq 1$, the $n$-dimensional hypercube is bipartite.

As above (b)