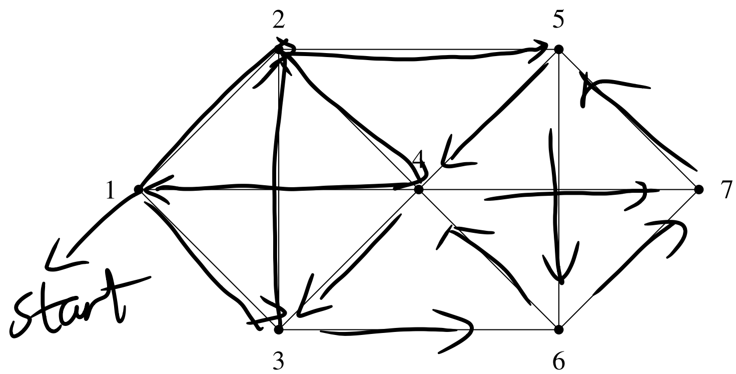
CS 70	Discrete Mathematics and Probability Theory	
Spring 2024	Seshia, Sinclair	DIS 2A

1 Eulerian Tour and Eulerian Walk

Note 5



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

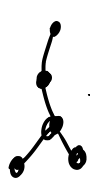
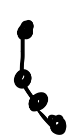
No. As degree(1) = 3 which is odd, imply no Eulerian tour bcs back and forth takes 2 edges.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

Yes, as shown above.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

Condition: ① connected ② 0, or 2 odd degree vertices

 → no walk, but  walkable.

We note that any Eulerian walk if not Eulerian tour after adding edge between start and end become Eulerian tour which is even & connected. Thus deleting that edge means Eulerian walk has 2 odd degree vertices.

Thus Eulerian walk ⇒ 0,2 odd degree & connected

0,2 odd & connected ⇒ Case 1: 0 ⇒ Eulerian tour

Case 2: 2 odd vertices: add edge between these 2 create Eulerian tour. thus if we choose one as start the other as end, we have Eulerian walk.

2 Coloring Trees

Note 5

(a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

Suppos vertex n, then edge $n-1$

As tree connected, degree(s) ≥ 1

If at most one leaf. $\therefore \sum^{2(n-1)}$ degree $= 1 + \sum^{n-1}$ degree $\geq 1 + 2(n-1)$

$\Rightarrow$ contradiction $\Rightarrow$ Thus at least two leaves.

(b) Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Color the vertices as blue or red.

A color contradiction occurs only if there is a cycle, but the tree is acyclic.

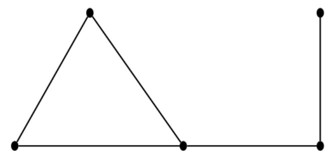
Thus. all red vertices partition into one group

while the other partition into the other.

3 Degree Sequences

Note 5

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3,2,2,2,1)$.



For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

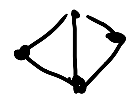
(a) $(3,3,2,2)$

(b) $(3,2,2,2,2,1,1)$

(c) $(6,2,2,2)$

(d) $(4,4,3,2,1)$

(a)



(b) Undirected means sum of degrees must be even, as a edge means two degree. While $\sum_i$ degree(i) = 13 not even. Thus no.

(c) No, as there are only 4 vertices. Each vertex at most 3 edges.

(d) No. As only 5 vertices. Any 4 edges means connected to every other edge. Thus, two 4 edges means every node at least 2 nodes connected while there exist degree of 1.