CS 70 Discrete Mathematics and Probability Theory
Fall 2024 Rao, Hug
HW 01

Due: Saturday, 9/7, 4:00 PM
Grace period until Saturday, 9/7, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Solving a System of Equations Review

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

2 Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:
$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t\,.$$

(b) Compute the values of $x \in (-2,2)$ that correspond to local maxima and minima of the function
$$f(x) = \int_0^{x^2} t\cos\left(\sqrt{t}\right)\mathrm{d}t\,.$$
Classify which x correspond to local maxima and which to local minima.

(c) Compute the double integral
$$\iint_R 2x + y\,\mathrm{d}A\,,$$
where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

2.(a)

$$\int_0^\infty \sin(t)e^{-t}dt = ?$$

$$\int_0^\infty \cos(t)e^{-t}dt + i\int_0^\infty \sin(t)e^{-t}dt$$

$$= \int_0^\infty e^{it}e^{-t}dt = \left[\frac{e^{it}e^{-t}}{i-1}\right]_0^\infty = \frac{0-1}{i-1} = \frac{-(i+1)1}{-2} = \frac{1}{2}+\frac{i}{2}$$

As the imaginary part is $i\int_0^\infty \sin(t)e^{-t}dt \Rightarrow \int_0^\infty \sin(t)e^{-t}dt = \frac{i/i}{2} = \frac{1}{2}$

(b) $f(x) = \int_0^{x^2} t\cos(\sqrt{t})dt = F(x^2) - F(0)$

$$\frac{df}{dx} = 2xF'(x) = 2x\cdot|x|\cos(\sqrt{|x|}) = \begin{cases} 2x^2\cos\sqrt{x} & \text{for } x \geq 0 \\ -2x^2\cos\sqrt{|x|} & \text{for } x \leq 0 \end{cases}$$

As $\sqrt{2} = 1.414 < \frac{\pi}{2}$, thus $\cos(\sqrt{x})$ always positive

Thus $2x^2\cos\sqrt{x} > 0$, maxima occurs at boundary at $x=2$

While for $x<0$, $f'<0$, so ~~maxima~~ occur at boundary for $x=-2$. the minima is also at the other end of boundary at $x=0$

Thus maximas: $2, -2$

minimas: 0

(c) V

3 (a) Yes $\forall x(P(x)\wedge Q(x)) = (P(x_1)\wedge Q(x_1))\wedge(P(x_2)\wedge Q(x_2))\ldots$

$= (P(x_1)\wedge \ldots)\wedge(Q(x_1)\ldots)$

$= \forall x(P(x))\wedge \forall x Q(x)$

(b) No

$\forall x(P(x)\vee Q(x)) = (P(x_1)\vee Q(x_1))\wedge(P(x_2)\vee Q(x_2))\ldots$

$\neq (\forall x P(x))\vee(\forall x Q(x)) = (P(x_1)\wedge P(x_2)\ldots)\vee(Q(x_1)\wedge Q(x_2)\ldots)$

(c) Yes for similar reasons

(d) No for other reasons

3 Logical Equivalence?

Note 1

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \vee \forall x\, Q(x)$ No

(c) $\exists x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \vee \exists x\, Q(x)$ No

(d) $\exists x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \wedge \exists x\, Q(x)$ No

4 Equivalences with Quantifiers

Note 1

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a) $\forall x \exists y\, (P(x) \implies Q(x,y)) \stackrel{?}{\equiv} \forall x\, (P(x) \implies \exists y\, Q(x,y))$

(b) $\forall x\, ((\exists y\, Q(x,y)) \implies P(x)) \stackrel{?}{\equiv} \forall x \exists y\, (Q(x,y) \implies P(x))$

(c) $\neg \exists x \forall y\, (P(x,y) \implies \neg Q(x,y)) \stackrel{?}{\equiv} \forall x\, ((\exists y\, P(x,y)) \wedge (\exists y\, Q(x,y)))$

5 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd. ✓

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$. ✗

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational. ✓ if r ration then r^2 ration
then r iratn

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers) ✗

(e) The product of a non-zero rational number and an irrational number is irrational. ✓

(f) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.) ✓

6 Twin Primes

Note 2

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k + 1$ or $3k - 1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

4. (a) $\forall x \exists y (P(x) \Rightarrow Q(x, y)) = \forall x \exists y (\neg P(x) \vee Q(x, y))$

$= (\exists y (\neg P(x_1) \vee Q(x_1, y)) \wedge \ldots$

$\neg \quad = (\ (\neg P(x_1) \vee Q(x_1, y_1)) \vee \ldots) \wedge \ldots$

$= (\neg P(x_1) \vee \exists y Q(x_1, y)) \wedge \ldots$

$= \quad P(x_1) \Rightarrow \exists y Q(x_1, y)) \wedge \ldots$

$= \forall x (P(x) \Rightarrow \exists y Q(x, y)))$ Thus yes

(b) No, as $\forall x (\exists y Q \Rightarrow P) = \forall x \neg (\exists y) (Q \Rightarrow \neg P)$

(c) $\neg \exists x \forall y (\ldots) = \forall x \neg \forall y (\ ) = \forall x \exists y \neg (\ldots)$

while $\neg (\ldots) = \neg (\neg P \vee \neg Q) = (P \wedge Q)$

Thus $\neg \exists x \forall y (\ ) = \forall x \exists y (P \wedge Q)$

$\neq \forall x (\exists y P) \wedge (\exists y Q)$

6. (a) obvious for mod 3 reasons

(b) 3, 5, 7 satisfy condition

Say for example another set exists
$3k-1$, $3k+1$, $3k+3$, as they can
only be odd number due to 2
being prime and $3k+3 - (3k-1) = 4$
one of them must be a multiple
of 5 as if $3k-1$ isn't, then one
of $3k+1$ $3k+3$ is. However,
that would mean they're divisible
by 5, so they are not prime

Thus only double twin prime is 5