1 $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(x+y) - 2x(x^2-y^2)}{(x^2+y^2)^2}$

$= \frac{x^3+xy^2+x^2y-y^3-2x^3+2xy^2}{(x^2+y^2)^2}$

3 $\frac{dg(u,v)}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx}$

$= f(v)\frac{du}{dx} - f(u)\frac{dv}{dx}$

as ∇f, $\int f$, $\int \nabla f$, $\int \Delta f$

$\frac{\partial f}{\partial r}$ not continuous

$\frac{\partial}{\partial x}\left(r\frac{\partial f}{\partial x}\right) = \left(f'\frac{\partial r}{\partial x}\right)$

$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial f}{\partial x}\right)'\frac{\partial r}{\partial x} = f''\frac{\partial r}{\partial x} + f'\frac{\partial^2 r}{\partial x \partial r}\frac{\partial r}{\partial x} = \left(f''\frac{x}{r} + f'\left(\frac{1}{r} - \frac{x}{r^2}\right)\frac{x}{r}\right)$

r member x and r are related

$\Sigma = f'' + (f'\frac{1}{r}) = 0$

$= \frac{d(f'e^{\ln r})}{dr} = 0$

$\Rightarrow f' = Cr$

$f = Cr^2$

② $\frac{\partial f}{\partial x} = f'\left(\frac{\partial r}{\partial x}\right)' + f'\frac{\partial r}{\partial x} = f''\frac{x}{r} + f'\left(\frac{1}{r} - \frac{x}{r^2}\right)$

$\Sigma = 0 \Rightarrow f'' + \frac{1}{r}f' = 0$

$f'e^{\int \frac{1}{r}dr} = C$

$f = \frac{C}{r} + D$

5 $x\frac{\partial f}{\partial x} = x\frac{\partial f(tx,ty)}{\partial x}$

similarly $y\frac{\partial f}{\partial y} = \frac{y}{t}\frac{\partial f(tx,ty)}{\partial y}$

$\frac{\partial f(tx,ty)}{\partial t} = x\frac{\partial f(tx)}{\partial tx} + y\frac{\partial f(ty)}{\partial ty}$ bcs t const so derivative

$= f(x,y)$, $n f(x,y)\, t^{n-1}$

$\Rightarrow n t f(x,y) = x\frac{\partial f(tx)}{\partial x} + y\frac{\partial f(ty)}{\partial y}$

$= y\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x}$

b $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})f = nf(x,y)$

$(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^2 f = n(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})f$

$= (x^2\frac{\partial^2}{\partial x^2} + (y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + y^2\frac{\partial^2}{\partial y^2}$

$+ 2x\frac{\partial}{\partial x}y\frac{\partial}{\partial y})f = n^2 f$

nf

$\Rightarrow x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n^2-n)f = n(n-1)f$

6 a $\lim_{r\to 0} f(r,0) = \lim_{r\to 0}\frac{\sin r}{r}$ L'Hopital $= \lim_{r\to 0}\frac{\cos r}{1} = 1$

b since $\lim_{r\to 0} f(r,\theta) = f(0,0) = 1$

we try to differentiate

$\frac{\partial f}{\partial r} = \frac{\cos r}{r} - \frac{\sin r}{r^2} = \frac{r\cos r - \sin r}{r^2}$

$\lim_{r\to 0}\frac{\partial f(0,0)}{\partial r} = \lim_{r\to 0}\frac{r\cos r - \sin r}{r^2}$ L'Hopital $= \lim_{r\to 0}\frac{\cos r - r\cos r - \sin r}{2r}$

as $\lim_{r\to 0}\frac{\sin r}{r} = 1$

$= \lim_{r\to 0}\frac{-3r\sin r}{r} = 0$

c $\frac{\partial f}{\partial \theta} = 0$ as the function doesn't contain theta

7 a obvious

b $|\vec r| = \sqrt{x^2+y^2+z^2}$

$\nabla(|\vec r|) = \frac{x^2+y^2+2x \cdot (x^2+y^2+z^2)^{\frac{1}{2}-1}}{}$

$= \vec r\left(\frac{1}{\ }\right)^{p-2} = \vec r\,|\vec r|^{p-2}$

c $\frac{r^2}{2}$ d obvious

e $\vec a = (c_1, c_2, c_3)$

$\nabla(\vec a \cdot \vec r) = \nabla(c_1x + c_2y + c_3z)$

$= (c_1, c_2, c_3) = \vec a$

8 Path tangent vector $\frac{d\vec r}{dt} = d(g(t), h(t))$

$= (g'(t), h'(t))$

$\nabla f = \frac{\partial f}{\partial x}\hat i + \frac{\partial f}{\partial y}\hat j$

$\frac{df}{dt} = \frac{\partial f}{\partial x}g'(t) + \frac{\partial f}{\partial y}h'(t) = 0$

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(g'(t), h'(t)) = 0$

So, normal to path tangent vector.

9. This point at when $t=1$. $\vec r(1) = (0,0,1)$

Surface $(x,y,z) = xz^2 - y - z\cos(\pi x) = 1$

has normal vector $\nabla f = (z^2 - z\sin(\pi x), -1, 2zx - y)$

If path is tangent to surface, then $\frac{d\vec r}{dt}$, the path tangent vector is normal to ∇f, and vice versa, as this is 3D.

$\frac{d\vec r}{dt} = \left(\frac{1}{t}, 1+\ln t, 1\right)$

At $t=1$: $\frac{d\vec r}{dt} = (1, 1, 1)$, at $(0,0,1)$ $\nabla f = (1, -1, 0)$

$\frac{d\vec r}{dt}\cdot\nabla f = 0$, ~~so opposite~~, thus, the path ~~tangent~~ tangent to surface

10 similar

11 Possible max/min occur at $\nabla f = 0$

$\nabla f = (3x^2 - 6y, 3y^2 - 6x, 0)$

$\Rightarrow x = y$ (as equations are symmetrical)

$\Rightarrow x = y = 0$, or $x = y = 3$

By the second derivative test

$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 6x \cdot 6y - (-6)^2$

$= 36xy - 36$

At $(0,0)$ $\Delta < 0$, thus it is saddle point

At $(3,3)$ $\Delta > 0$, and $f_{xx} > 0$, so minima

12 Point at which ∇f vanish

$\frac{\partial f}{\partial x} = 6y e^{-(x+y)} - 12xy e^{-(x+y)} = 0$

$\frac{\partial f}{\partial y} = 6x e^{-(x+y)} - 18xy e^{-(x+y)} = 0$

$\Rightarrow \begin{cases} 6y = 12xy \\ 6x = 18xy \end{cases} \Rightarrow \begin{cases} x = y = 0 \\ y = \frac{1}{3}, x = \frac{1}{3} \end{cases}$

② compare max with boundary

(1) along $x = 0$ or $y = 0$, $f(x,y) = 0$

(2) at $x = 0 = y$, $f(x,y) = 0$

(2.1) $f(\frac{1}{2}, \frac{1}{3}) = e^{-2}$

(2.2) max of all, thus max value is e^{-2}

13 Find equation for plane first

Plane tangent to ellipsoid so plane normal to ∇f, $\nabla f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$

$\vec r \cdot \nabla f = 0 \Rightarrow (x - x_0)\frac{2x_0}{a^2} + (y - y_0)\frac{2y_0}{b^2} + \frac{2z_0}{c^2}(z - z_0) = 0$

(x,y,z)

where x_0, y_0, z_0 is the point of contact

Volume would be

Volume for this structure would be $h \cdot S/3$, thus, we just find S (area of bottom of 3D triangle)

Set $z = 0$, $y = 0$ we have

$x_0\frac{(x - x_0)}{a^2} = \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1 - \frac{x_0^2}{a^2}$

$\Rightarrow \frac{x_0 x}{a^2} = 1 \Rightarrow x = \frac{a^2}{x_0}$ point intersect with x-axis

Similarly, $y = \frac{b^2}{y_0}$ $z = \frac{c^2}{z_0}$

Thus Volume $= \frac{a^2b^2}{2x_0y_0} \cdot \frac{c^2}{2z_0 3} = \frac{a^2b^2c^2}{6}\frac{1}{x_0y_0z_0}$

We want to minimize volume, so maximize $x_0y_0z_0$

which is maximal when $x_0 = y_0 = z_0$

(verifiable by Lagrange multiplier)

$\Rightarrow 3a^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 1$

$\Rightarrow d = abc\sqrt{a^2b^2 + c^2b^2 + a^2c^2}$

Thus volume $= \frac{a^2b^2c^2}{6d^3} = \frac{(a^2b^2 + c^2b^2 + a^2c^2)\sqrt{a^2b^2 + c^2b^2 + a^2c^2}}{6abc}$

14 yes $f = (x - v^2)^2 + (x+1-v)^2$

minimize $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial v} = 0$

$\frac{\partial f}{\partial x} = 2(x - v^2) + 2(x+1-v)$

$\frac{\partial f}{\partial v} = -4v(x - v^2) - 2(x+1-v)$

$\begin{cases} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial v} = 0 \Rightarrow v = \frac{1}{2}, x = \frac{1}{8} \\ \frac{\partial f}{\partial z} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$ is the only point, as $x \to \infty$, distance become inf this has to be minimal point

$f = \frac{1}{64} + \frac{25}{64} = \frac{13}{32}$ min distance

15 Suppose $f(x,y)$ is not continuous at (x_0, y_0)

$\frac{\partial f}{\partial x} = \lim_{x\to x_0}\frac{f(x) - f(x_0)}{x - x_0}$ would approach inf

as discontinuity implies $\lim_{x\to x_0} f(x) - f(x_0) \neq 0$

Thus $\frac{\partial f}{\partial x} \to \infty$, so ∇f is not bounded

$\Rightarrow$ contradiction

16 $d\vec r = \nabla f\, d\vec r$

$\frac{d\vec r}{dt} = \nabla f \frac{d\vec r}{dt}$

extreme values $\Rightarrow \frac{df}{dt} = 0 \Rightarrow \nabla f$ orthogonal to $\frac{d\vec r}{dt}$

17 $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} = 0$, f independent of x, g independent of y, thus $\frac{df}{dy} = \frac{dg}{dx}$

We have $f(2) = 4$, $f(1) = 5$

$\frac{dg}{dx}(0,2) = \frac{df}{dy}(0,2) = \frac{df}{dy}(1,2) = \frac{dg}{dx}(1,2)$

Thus, g has const derivative

$g = Cx + C_2$, similarly $f = C_1y + C_3$

We, thus have: $f = \frac{1}{2}y + 4$, $g = \frac{1}{2}x + C_2$

18 Do $f(x,y) = (a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y})f$

$D_u f(x,y) = (a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, 0)\,\nabla f$

19 Maximal direction is in direction of ∇f

$\nabla f = (e^{2y}\sin x, e^{2y}\cos x)$

Thus, $\frac{dy}{dx}$ (direction of next motion) equals $\frac{e^{2y}\cos x\, dx}{e^{2y}\sin x} = \frac{1}{\tan x}$

$y = \int \frac{1}{\tan x}dx = \ln|\sin x| + C$

$y(\pi/4) = 0 \Rightarrow C = -\ln\frac{1}{\sqrt{2}}$

$y = \ln|\sin x| + \frac{1}{2}\ln 2$

23 $W_{xx} = \frac{W_t}{c^2}$

$= -\pi^2 e^{rt}\sin \pi x = re^{rt}\sin \pi x$

$\Rightarrow r = -\pi^2$

24 similarly $r = -k^2$

$\sin(0) = 0$, $\sin(kL) = 0$

$\Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L}$

As $t \to \infty$: $e^{rt} \to 0$, $w \to 0$