Linear Algebra Practice Set: Linear Transformations, Diagonalization, and Basis

Medium Level Questions

October 10, 2025

Practice Questions

1. Linear Transformation and Matrix Representation

Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_3 \\ x_2 + 3x_1 \end{pmatrix}$$

(a) Show that T is a linear transformation.

(b) Find the standard matrix A for the linear transformation T.

(c) Compute $T(v)$ where $v = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$.

2. Diagonalization

Let A be the matrix:

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

(a) Find the eigenvalues of A.

(b) Find the corresponding eigenvectors for each eigenvalue.

(c) Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

3. Change of Basis for a Vector Space

Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$, where:

$$b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

Find the change-of-basis matrix $P_{C \leftarrow B}$ that converts coordinates from B-basis to C-basis.

4. Coordinates in a Non-Standard Basis

Consider the basis $B = \{p_1(t), p_2(t), p_3(t)\}$ for P_2 (the space of polynomials of degree at most 2), where:

$$p_1(t) = 1, \quad p_2(t) = 1 + t, \quad p_3(t) = 1 + t + t^2$$

Find the coordinate vector $[q]_B$ for the polynomial $q(t) = 5 - 2t + 4t^2$.

5. Powers of a Matrix via Diagonalization

Given the matrix A from Question 2:

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Use the result of its diagonalization to calculate the matrix A^5.

6. Matrix of Transformation Relative to Non-Standard Bases

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x_1, x_2) = (4x_1 + x_2, 3x_1 + 2x_2)$ for standard basis. Let $B = \{b_1, b_2\}$ be a non-standard basis, where:

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find the matrix $[T]_B$ for the transformation T relative to the basis B.

7. Similarity of Matrices and Change of Basis Transformation

Matrix $A = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix}$ is similar to a diagonal matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) What can be immediately concluded about the eigenvalues of A?

(b) Let $B = \{b_1, b_2\}$ be the basis of eigenvectors for A. If x is a vector such that its coordinate vector is $[x]_B = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, find the vector x in the standard basis, given that the change of basis matrix $P = \begin{pmatrix} 4 & 1 \\ 5 & 1 \end{pmatrix}$ converts B-coordinates to standard coordinates.

8. Rotation and Stretching in 3D Space

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that first rotates vectors by 90 counter-clockwise about the z-axis (when viewed from above), and then stretches the result by a factor of 2 in the x-direction, a factor of 3 in the y-direction, and leaves the z-direction unchanged.

(a) Write the matrix representation of the rotation about the z-axis.

(b) Write the matrix representation of the stretching transformation.

(c) Find the standard matrix A for the combined transformation T.

(d) Apply the transformation T to the vector $v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and describe geometrically what happens to this vector.

(e) Find the determinant of A and explain what it represents geometrically.

9. Combined Shearing and Rotation in 3D

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a composite linear transformation that first applies a shearing transformation, followed by a rotation about the x-axis.

The shearing transformation S changes coordinates as follows:

- x stays the same
- y stays the same
- z becomes $z + x$

The rotation transformation R rotates vectors by 45 about the x-axis (counterclockwise when looking from positive x-axis toward origin).

(a) Write the matrix representation S for the shearing transformation described above.

(b) Write the matrix representation R for rotation by 45 about the x-axis. (Hint: Rotation about x-axis keeps x-coordinate fixed)

(c) Find the standard matrix A for the combined transformation $T = R \circ S$ (first shear, then rotate).

(d) Apply the transformation T to the vector $v = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and describe the geometric effect step by step.

(e) Is the combined transformation T invertible? If yes, find the matrix representing T^{-1} and explain what transformation it represents.