Matrix Analysis: Pivot Columns, Free Variables, Rank, and Nullity

Linear Algebra Solution

Question 1: Given the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the number of pivot columns and free variables. Also find rank and nullity.

1 Solution

1.1 Step 1: Row Reduction to Row Echelon Form

We'll perform elementary row operations to reduce matrix $\mathbf{A}$ to row echelon form (REF).

Starting matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Operation 1: $R_2 \leftarrow R_2 - 2R_1$ (eliminate the first entry of row 2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 4-2(2) & 6-2(3) \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Operation 2: $R_3 \leftarrow R_3 - R_1$ (eliminate the first entry of row 3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1-1 & 1-2 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$$

Operation 3: Swap R_2 and R_3 to get proper row echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Operation 4: $R_2 \leftarrow -R_2$ (make the leading coefficient positive)

$$\text{REF}(\mathbf{A}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

1.2 Step 2: Continue to Reduced Row Echelon Form (RREF)

To get RREF, we eliminate above the pivots:

Operation 5: $R_1 \leftarrow R_1 - 2R_2$

$$\text{RREF}(\mathbf{A}) = \begin{bmatrix} 1-2(0) & 2-2(1) & 3-2(2) \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

1.3 Step 3: Identify Pivot Columns and Free Variables

From the RREF, we can identify:

Pivot Columns: The columns containing leading 1's (pivots) are:

- Column 1: contains pivot in row 1
- Column 2: contains pivot in row 2

Number of Pivot Columns: $\boxed{2}$

Free Variables: The columns without pivots correspond to free variables:

- Column 3: no pivot, so x_3 is a free variable

Number of Free Variables: $\boxed{1}$

1.4 Step 4: Calculate Rank and Nullity

Rank of A: The rank is the number of non-zero rows in the REF (or equivalently, the number of pivot columns).

From our REF:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2

Therefore: $\text{rank}(\mathbf{A}) = \boxed{2}$

Nullity of A: Using the Rank-Nullity Theorem:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$

where n is the number of columns.

Since $\mathbf{A}$ is a 3×3 matrix:

$$2 + \text{nullity}(\mathbf{A}) = 3$$

$$\text{nullity}(\mathbf{A}) = 3 - 2 = \boxed{1}$$

2 Summary of Results

Property	**Value**
Number of Pivot Columns	2
Number of Free Variables	1
Rank of **A**	2
Nullity of **A**	1

3 Verification

We can verify our results:

- Rank + Nullity $= 2 + 1 = 3 =$ number of columns
- Number of pivot columns = rank = 2
- Number of free variables = nullity = 1

The matrix **A** has rank 2, meaning its column space is 2-dimensional, and its null space is 1-dimensional.

Question 2: For the matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

perform Gaussian elimination and find the number of pivot columns and the rank of the matrix.

4 Solution for Question 2

4.1 Step 1: Gaussian Elimination (Forward Elimination)

We'll perform elementary row operations to reduce matrix $\mathbf{B}$ to row echelon form.

Starting matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

Operation 1: $R_2 \leftarrow R_2 - 2R_1$ (eliminate the first entry of row 2)

$$\begin{bmatrix} 1 & 3 & 2 \\ 2-2(1) & 6-2(3) & 4-2(2) \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

Operation 2: $R_3 \leftarrow R_3 - R_1$ (eliminate the first entry of row 3)

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 1-1 & 2-3 & 5-2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

Operation 3: Swap R_2 and R_3 to get proper row echelon form

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Operation 4: $R_2 \leftarrow -R_2$ (make the leading coefficient positive)

$$\text{REF}(\mathbf{B}) = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

4.2 Step 2: Analysis of Row Echelon Form

From the row echelon form:

$$\text{REF}(\mathbf{B}) = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can identify the following characteristics:

Non-zero rows: The first two rows are non-zero:

- Row 1: $[1 \quad 3 \quad 2]$
- Row 2: $[0 \quad 1 \quad -3]$
- Row 3: $[0 \quad 0 \quad 0]$ (zero row)

Leading entries (pivots):

- Row 1: Leading 1 in column 1
- Row 2: Leading 1 in column 2

4.3 Step 3: Identify Pivot Columns

The pivot columns are the columns that contain the leading entries:

Pivot Columns:

- Column 1: contains the pivot from row 1
- Column 2: contains the pivot from row 2

Column 3 does not contain a pivot.
Number of Pivot Columns: $\boxed{2}$

4.4 Step 4: Calculate the Rank

The rank of a matrix is equal to the number of non-zero rows in its row echelon form, which is also equal to the number of pivot columns.

From our REF: - Number of non-zero rows = 2 - Number of pivot columns = 2
Therefore: $\text{rank}(\mathbf{B}) = \boxed{2}$

4.5 Step 5: Additional Analysis

Verification of row 2 calculation: Let's verify that row 2 of the original matrix is indeed a multiple of row 1:

$$\text{Row } 2 = [2, 6, 4] = 2 \cdot [1, 3, 2] = 2 \cdot \text{Row } 1$$

This confirms that row 2 was linearly dependent on row 1, which is why it became a zero row after elimination.

Column space dimension: Since the rank is 2, the column space of $\mathbf{B}$ is 2-dimensional, meaning any two linearly independent columns span the entire column space.

5 Summary of Results for Question 2

Property	**Value**
Number of Pivot Columns	2
Rank of $\mathbf{B}$	2
Number of Non-zero Rows in REF	2

Final Row Echelon Form:

$$\text{REF}(\mathbf{B}) = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{B}$ has rank 2, indicating that its three columns span a 2-dimensional subspace of $\mathbb{R}^3$. The second row of the original matrix was a scalar multiple of the first row, resulting in linear dependence among the rows.

Question 3: If a matrix has n columns and r pivot columns, what is the dimension of the null space of the matrix?

6 Solution for Question 3

6.1 The Rank-Nullity Theorem

The answer to this question is given directly by the fundamental **Rank-Nullity Theorem** (also known as the Fundamental Theorem of Linear Algebra).

Theorem: For any $m \times n$ matrix $\mathbf{A}$:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$

where:

- $\text{rank}(\mathbf{A})$ is the dimension of the column space (number of pivot columns)
- $\text{nullity}(\mathbf{A})$ is the dimension of the null space
- n is the number of columns in the matrix

6.2 Application to the Given Problem

Given:

- Matrix has n columns
- Matrix has r pivot columns

Since the number of pivot columns equals the rank of the matrix:

$$\text{rank}(\mathbf{A}) = r$$

Applying the Rank-Nullity Theorem:

$$r + \text{nullity}(\mathbf{A}) = n$$

Solving for the nullity (dimension of the null space):

$$\text{nullity}(\mathbf{A}) = n - r$$

6.3 Answer

$$\boxed{\text{Dimension of the null space} = n - r}$$

6.4 Summary

The dimension of the null space of a matrix with n columns and r pivot columns is:

$$\boxed{\dim(\text{null}(\mathbf{A})) = n - r}$$

This relationship is fundamental in linear algebra and connects the geometric concepts of column space and null space through the constraint that their dimensions must sum to the total number of columns.

Question 4: Given the matrix:

$$\mathbf{C} = \begin{bmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 4 & 6 & 8 & 3 \end{bmatrix}$$

find nullity and number of free variables after Gaussian elimination.

7 Solution for Question 4

7.1 Step 1: Gaussian Elimination

We'll perform elementary row operations to reduce matrix $\mathbf{C}$ to row echelon form.

Starting matrix:

$$\mathbf{C} = \begin{bmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 4 & 6 & 8 & 3 \end{bmatrix}$$

Operation 1: Swap R_1 and R_2 to get a leading 1

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 4 & 2 & 1 \\ 4 & 6 & 8 & 3 \end{bmatrix}$$

Operation 2: $R_2 \leftarrow R_2 - 3R_1$ (eliminate first entry of row 2)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3-3(1) & 4-3(2) & 2-3(3) & 1-3(2) \\ 4 & 6 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -2 & -7 & -5 \\ 4 & 6 & 8 & 3 \end{bmatrix}$$

Operation 3: $R_3 \leftarrow R_3 - 4R_1$ (eliminate first entry of row 3)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -2 & -7 & -5 \\ 4-4(1) & 6-4(2) & 8-4(3) & 3-4(2) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -2 & -7 & -5 \\ 0 & -2 & -4 & -5 \end{bmatrix}$$

Operation 4: $R_2 \leftarrow -\frac{1}{2}R_2$ (make leading coefficient 1)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & -2 & -4 & -5 \end{bmatrix}$$

Operation 5: $R_3 \leftarrow R_3 + 2R_2$ (eliminate second entry of row 3)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & -2+2(1) & -4+2(\frac{7}{2}) & -5+2(\frac{5}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Operation 6: $R_3 \leftarrow \frac{1}{3}R_3$ (make leading coefficient 1)

$$\text{REF}(\mathbf{C}) = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

7.2 Step 2: Analysis of Row Echelon Form

From the row echelon form:

$$\mathrm{REF}(\mathbf{C}) = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{7}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Pivot Analysis:

- Row 1: Leading 1 in column 1 (pivot in column 1)
- Row 2: Leading 1 in column 2 (pivot in column 2)
- Row 3: Leading 1 in column 3 (pivot in column 3)

Pivot Columns: Columns 1, 2, and 3 **Non-pivot Columns:** Column 4

7.3 Step 3: Calculate Rank and Nullity

Number of pivot columns: $r = 3$ **Total number of columns:** $n = 4$

Using the Rank-Nullity Theorem:

$$\mathrm{rank}(\mathbf{C}) = r = 3$$

$$\mathrm{nullity}(\mathbf{C}) = n - r = 4 - 3 = 1$$

7.4 Step 4: Identify Free Variables

From the row echelon form, column 4 does not contain a pivot. Therefore:

- x_1, x_2, x_3 are basic variables (correspond to pivot columns)
- x_4 is a free variable (corresponds to non-pivot column)

Number of free variables: 1

7.5 Answer

Property	**Value**
Nullity of $\mathbf{C}$	$\boxed{1}$
Number of Free Variables	$\boxed{1}$

Verification: The nullity equals the number of free variables, as expected from the theory. Both equal $n - r = 4 - 3 = 1$.

Question 5: Solve the system of linear equations using LU decomposition:

$$\begin{aligned} 2x + 3y + z &= 9, \\ 4x + 7y + 5z &= 23, \\ 6x + 18y + 19z &= 72. \end{aligned}$$

8 Solution for Question 5

8.1 Step 1: Write the System in Matrix Form

The system can be written as $\mathbf{Ax} = \mathbf{b}$ where:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 6 & 18 & 19 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 9 \\ 23 \\ 72 \end{bmatrix}$$

8.2 Step 2: LU Decomposition of Matrix A

We'll decompose $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular.

Forward Elimination to find U:

Starting matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 6 & 18 & 19 \end{bmatrix}$$

Operation 1: $R_2 \leftarrow R_2 - 2R_1$ (eliminate a_{21})

$$\begin{bmatrix} 2 & 3 & 1 \\ 4-2(2) & 7-2(3) & 5-2(1) \\ 6 & 18 & 19 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 6 & 18 & 19 \end{bmatrix}$$

Multiplier: $l_{21} = \frac{4}{2} = 2$

Operation 2: $R_3 \leftarrow R_3 - 3R_1$ (eliminate a_{31})

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 6-3(2) & 18-3(3) & 19-3(1) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 9 & 16 \end{bmatrix}$$

Multiplier: $l_{31} = \frac{6}{2} = 3$

Operation 3: $R_3 \leftarrow R_3 - 9R_2$ (eliminate a_{32})

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 9-9(1) & 16-9(3) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix}$$

Multiplier: $l_{32} = \frac{9}{1} = 9$

Therefore: $\mathbf{U} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix}$

Constructing L:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix}$$

8.3 Step 3: Verify LU Decomposition

Let's verify that $\mathbf{LU} = \mathbf{A}$:

$$\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 6 & 18 & 19 \end{bmatrix} = \mathbf{A}\checkmark$$

8.4 Step 4: Forward Substitution (Solve $\mathbf{Ly} = \mathbf{b}$)

We solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ 72 \end{bmatrix}$$

Forward substitution:

$$y_1 = 9 \tag{1}$$

$$2y_1 + y_2 = 23 \Rightarrow y_2 = 23 - 2(9) = 5 \tag{2}$$

$$3y_1 + 9y_2 + y_3 = 72 \Rightarrow y_3 = 72 - 3(9) - 9(5) = 72 - 27 - 45 = 0 \tag{3}$$

Therefore: $\mathbf{y} = \begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix}$

8.5 Step 5: Backward Substitution (Solve $\mathbf{Ux} = \mathbf{y}$)

We solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 0 \end{bmatrix}$$

Backward substitution:

$$-11z = 0 \Rightarrow z = 0 \tag{4}$$

$$y + 3z = 5 \Rightarrow y = 5 - 3(0) = 5 \tag{5}$$

$$2x + 3y + z = 9 \Rightarrow x = \frac{9 - 3(5) - 0}{2} = \frac{-6}{2} = -3 \tag{6}$$

8.6 Step 6: Solution

The solution to the system is:

$$\boxed{x = -3, \quad y = 5, \quad z = 0}$$

8.7 Step 7: Verification

Let's verify our solution by substituting back into the original equations:

Equation 1: $2(-3) + 3(5) + 0 = -6 + 15 = 9$

Equation 2: $4(-3) + 7(5) + 5(0) = -12 + 35 = 23$

Equation 3: $6(-3) + 18(5) + 19(0) = -18 + 90 = 72$

All equations are satisfied, confirming our solution is correct.

Question 6: For the matrix:

$$\mathbf{D} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

find the null space of $\mathbf{D}$ by solving $\mathbf{D}\mathbf{x} = \mathbf{0}$.

9 Solution for Question 6

9.1 Step 1: Set Up the Homogeneous System

We need to solve $\mathbf{D}\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the system:

$$x_1 + 2x_2 + 3x_3 = 0 \tag{7}$$
$$4x_1 + 5x_2 + 6x_3 = 0 \tag{8}$$
$$7x_1 + 8x_2 + 9x_3 = 0 \tag{9}$$

9.2 Step 2: Row Reduction to RREF

We'll reduce the augmented matrix $[\mathbf{D}|\mathbf{0}]$ to reduced row echelon form.

Starting matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Operation 1: $R_2 \leftarrow R_2 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4-4(1) & 5-4(2) & 6-4(3) & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

Operation 2: $R_3 \leftarrow R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 7-7(1) & 8-7(2) & 9-7(3) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

Operation 3: $R_2 \leftarrow -\frac{1}{3}R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

Operation 4: $R_3 \leftarrow R_3 + 6R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6+6(1) & -12+6(2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Operation 5: $R_1 \leftarrow R_1 - 2R_2$ (to get RREF)

$$\left[\begin{array}{ccc|c} 1 & 2-2(1) & 3-2(2) & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

9.3 Step 3: Identify Pivot and Free Variables

From the RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns: Columns 1 and 2 **Free variable:** x_3 (column 3 has no pivot)

9.4 Step 4: Express Basic Variables in Terms of Free Variable

From the RREF, we get the system:

$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3 \tag{10}$$

$$x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3 \tag{11}$$

Let $x_3 = t$ where t is a parameter. Then:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

9.5 Step 5: Null Space

The null space of $\mathbf{D}$ is:

$$\text{null}(\mathbf{D}) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Basis for the null space:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}}$$

General solution:

$$\boxed{\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}}$$

9.6 Step 6: Verification

Let's verify that our basis vector is indeed in the null space:

$$\mathbf{D} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + 2(-2) + 3(1) \\ 4(1) + 5(-2) + 6(1) \\ 7(1) + 8(-2) + 9(1) \end{bmatrix} = \begin{bmatrix} 1 - 4 + 3 \\ 4 - 10 + 6 \\ 7 - 16 + 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

9.7 Step 7: Properties of the Null Space

Dimension: The null space has dimension 1 (nullity = 1)

Rank-Nullity verification:

- Rank of $\mathbf{D} = 2$ (number of pivot columns)
- Nullity of $\mathbf{D} = 1$
- Total columns = 3
- Rank + Nullity = $2 + 1 = 3$

The null space is a 1-dimensional subspace (a line through the origin) in $\mathbb{R}^3$.

Question 7: Given the system of equations:

$$\mathbf{Ax} = \mathbf{0}, \quad \mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

find the dimension of the null space.

10 Solution for Question 7

10.1 Step 1: Row Reduction to Find Rank

To find the dimension of the null space, we first need to find the rank of matrix $\mathbf{A}$ using row reduction.

Starting matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

Operation 1: Swap R_1 and R_2 to get a leading 1

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

Operation 2: $R_2 \leftarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2-2(1) & 4-2(3) & 6-2(5) \\ 2 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 2 & 6 & 8 \end{bmatrix}$$

Operation 3: $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 2-2(1) & 6-2(3) & 8-2(5) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix}$$

Operation 4: $R_2 \leftarrow -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Operation 5: $R_3 \leftarrow -\frac{1}{2}R_3$

$$\text{REF}(\mathbf{A}) = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

10.2 Step 2: Determine the Rank

From the row echelon form:

$$\text{REF}(\mathbf{A}) = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

We observe:

- Row 1: Leading 1 in column 1 (pivot)
- Row 2: Leading 1 in column 2 (pivot)
- Row 3: Leading 1 in column 3 (pivot)

Number of non-zero rows: 3 **Number of pivot columns:** 3

Therefore: $\text{rank}(\mathbf{A}) = 3$

10.3 Step 3: Apply the Rank-Nullity Theorem

For an $m \times n$ matrix, the Rank-Nullity Theorem states:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$

Given:

- Matrix $\mathbf{A}$ is 3×3, so $n = 3$ columns
- $\text{rank}(\mathbf{A}) = 3$

Substituting into the theorem:

$$3 + \text{nullity}(\mathbf{A}) = 3$$
$$\text{nullity}(\mathbf{A}) = 3 - 3 = 0$$

10.4 Step 4: Interpretation

Since the nullity is 0, this means:

- The null space contains only the zero vector
- The dimension of the null space is 0
- The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$
- Matrix $\mathbf{A}$ has full rank (rank = number of columns)
- Matrix $\mathbf{A}$ is invertible

10.5 Step 5: Verification

We can verify this by noting that since $\mathbf{A}$ has rank 3 (full rank for a 3×3 matrix), all columns are linearly independent. This means the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Alternatively, we can check that $\det(\mathbf{A}) \neq 0$:

$$\det(\mathbf{A}) = \det \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

Using cofactor expansion along the first row:

$$\det(\mathbf{A}) = 2 \det \begin{bmatrix} 3 & 5 \\ 6 & 8 \end{bmatrix} - 4 \det \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix} + 6 \det \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad (12)$$
$$= 2(3 \cdot 8 - 5 \cdot 6) - 4(1 \cdot 8 - 5 \cdot 2) + 6(1 \cdot 6 - 3 \cdot 2) \quad (13)$$
$$= 2(24 - 30) - 4(8 - 10) + 6(6 - 6) \quad (14)$$
$$= 2(-6) - 4(-2) + 6(0) \quad (15)$$
$$= -12 + 8 + 0 = -4 \neq 0 \quad (16)$$

Since $\det(\mathbf{A}) \neq 0$, matrix $\mathbf{A}$ is invertible, confirming that the null space contains only the zero vector.

10.6 Answer

$$\boxed{\text{Dimension of the null space} = 0}$$

Summary:

- Rank of $\mathbf{A} = 3$
- Nullity of $\mathbf{A} = 0$
- Null space $= \{\mathbf{0}\}$
- Matrix $\mathbf{A}$ is invertible

Question 8: For the matrix:

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$$

determine the basis for the null space of $\mathbf{G}$.

11 Solution for Question 8

11.1 Step 1: Set Up the Homogeneous System

We need to solve $\mathbf{Gx} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the system:

$$x_1 + x_2 + 2x_3 = 0 \tag{17}$$
$$2x_1 + 2x_2 + 4x_3 = 0 \tag{18}$$
$$3x_1 + 3x_2 + 6x_3 = 0 \tag{19}$$

11.2 Step 2: Row Reduction to RREF

We'll reduce the augmented matrix $[\mathbf{G}|\mathbf{0}]$ to reduced row echelon form.

Starting matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 3 & 3 & 6 & 0 \end{array}\right]$$

Operation 1: $R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2-2(1) & 2-2(1) & 4-2(2) & 0 \\ 3 & 3 & 6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 6 & 0 \end{array}\right]$$

Operation 2: $R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3-3(1) & 3-3(1) & 6-3(2) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The RREF is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

11.3 Step 3: Identify Pivot and Free Variables

From the RREF:

- **Pivot column:** Column 1 (contains the leading 1)
- **Free variables:** x_2 and x_3 (columns 2 and 3 have no pivots)

Rank of G: 1 (one pivot column) **Nullity of G:** $3 - 1 = 2$ (by Rank-Nullity Theorem)

11.4 Step 4: Express Basic Variable in Terms of Free Variables

From the RREF, we have:

$$x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = -x_2 - 2x_3$$

Let $x_2 = s$ and $x_3 = t$ where s, t are parameters. Then:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s - 2t \\ s \\ t \end{bmatrix}$$

11.5 Step 5: Write in Parametric Vector Form

We can separate the parameters:

$$\mathbf{x} = \begin{bmatrix} -s - 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

11.6 Step 6: Basis for the Null Space

The null space of $\mathbf{G}$ is spanned by the two vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for null(G):

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

11.7 Step 7: Verification

Let's verify that both basis vectors are in the null space:

For $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$**:**

$$\mathbf{G}\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(-1) + 1(1) + 2(0) \\ 2(-1) + 2(1) + 4(0) \\ 3(-1) + 3(1) + 6(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

For $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$**:**

$$\mathbf{G}\mathbf{v}_2 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(-2) + 1(0) + 2(1) \\ 2(-2) + 2(0) + 4(1) \\ 3(-2) + 3(0) + 6(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

11.8 Step 8: Linear Independence Check

To confirm our basis is valid, we verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Consider $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$:

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives:

$$-c_1 - 2c_2 = 0 \tag{20}$$
$$c_1 = 0 \tag{21}$$
$$c_2 = 0 \tag{22}$$

From equation 2: $c_1 = 0$ Substituting into equation 1: $-0 - 2c_2 = 0 \Rightarrow c_2 = 0$
Since the only solution is $c_1 = c_2 = 0$, the vectors are linearly independent.

11.9 Answer

Basis for the null space of G:

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

Properties:

- Dimension of null space: 2
- Rank of $\mathbf{G}$: 1
- General solution: $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ where $s, t \in \mathbb{R}$

The null space is a 2-dimensional subspace (a plane through the origin) in $\mathbb{R}^3$.

Question 9: If the null space of a matrix $\mathbf{A}$ has dimension 3 and the rank of $\mathbf{A}$ is 2, how many variables does $\mathbf{A}$ have?

12 Solution for Question 9

12.1 Step 1: Apply the Rank-Nullity Theorem

The Rank-Nullity Theorem states that for any $m \times n$ matrix $\mathbf{A}$:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$$

where:

- $\text{rank}(\mathbf{A})$ is the dimension of the column space
- $\text{nullity}(\mathbf{A})$ is the dimension of the null space
- n is the number of columns (variables) in matrix $\mathbf{A}$

12.2 Step 2: Substitute Given Values

Given information:

- Dimension of null space = $\text{nullity}(\mathbf{A}) = 3$
- Rank of $\mathbf{A}$ = $\text{rank}(\mathbf{A}) = 2$
- Number of variables = n = ?

Substituting into the Rank-Nullity Theorem:

$$2 + 3 = n$$

$$n = 5$$

12.3 Step 3: Interpretation

The result tells us that matrix $\mathbf{A}$ has 5 columns, which means the system $\mathbf{Ax} = \mathbf{b}$ involves 5 variables: x_1, x_2, x_3, x_4, x_5.

Additional insights:

- **Rank = 2:** There are 2 pivot columns (2 leading variables)
- **Nullity = 3:** There are 3 free variables
- **Total variables = 5:** 2 leading + 3 free = 5 variables

12.4 Step 4: Verification

We can verify our answer:

$$\begin{aligned} \text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) &= n && (23) \\ 2 + 3 &= 5 && (24) \\ 5 &= 5\checkmark && (25) \end{aligned}$$

12.5 Step 5: Example Structure

A matrix with these properties might have a reduced row echelon form like:

$$\text{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

- Columns 1 and 2 are pivot columns (rank = 2)
- Columns 3, 4, and 5 are non-pivot columns (3 free variables)
- Total of 5 columns (5 variables)

12.6 Answer

$$\boxed{\text{Matrix } \mathbf{A} \text{ has 5 variables}}$$

Summary:

- Number of variables (columns): 5
- Number of pivot columns: 2
- Number of free variables: 3
- Rank-Nullity verification: $2 + 3 = 5$

Question 10: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$. Which of the following statements about the set of all affine combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ is true?

1. A: The set forms a vector space.
2. B: The set forms an affine space, but not a vector space.
3. C: The set is a subspace of $\mathbb{R}^3$.
4. D: The set forms a hyperplane.

13 Solution for Question 10

13.1 Step 1: Define Affine Combinations

An **affine combination** of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is a linear combination where the coefficients sum to 1:

$$\mathbf{w} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \quad \text{where } \alpha + \beta = 1$$

The set of all affine combinations is:

$$S = \{\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 : \alpha + \beta = 1, \alpha, \beta \in \mathbb{R}\}$$

13.2 Step 2: Analyze the Given Vectors

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Notice that:

$$\mathbf{v}_2 = 2\mathbf{v}_1$$

This means $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly dependent**.

13.3 Step 3: Express the Affine Combination

Since $\mathbf{v}_2 = 2\mathbf{v}_1$, we can write:

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \alpha \mathbf{v}_1 + \beta(2\mathbf{v}_1) = (\alpha + 2\beta)\mathbf{v}_1$$

With the constraint $\alpha + \beta = 1$, we have $\alpha = 1 - \beta$, so:

$$(\alpha + 2\beta) = (1 - \beta) + 2\beta = 1 + \beta$$

Therefore:

$$\mathbf{w} = (1 + \beta)\mathbf{v}_1 = (1 + \beta) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since β can be any real number, $(1 + \beta)$ can be any real number.

13.4 Step 4: Characterize the Set

The set S consists of all scalar multiples of $\mathbf{v}_1$:

$$S = \{t\mathbf{v}_1 : t \in \mathbb{R}\} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

This is the **line through the origin** in the direction of $\mathbf{v}_1$.

13.5 Step 5: Check Each Option

Option A: The set forms a vector space.

For a set to be a vector space, it must satisfy several properties, including: - Closure under addition - Closure under scalar multiplication - Contains the zero vector

Let's check: - Zero vector: $\mathbf{0} = 0 \cdot \mathbf{v}_1 \in S$ - Closure under addition: If $\mathbf{u} = s\mathbf{v}_1$ and $\mathbf{w} = t\mathbf{v}_1$, then $\mathbf{u} + \mathbf{w} = (s+t)\mathbf{v}_1 \in S$ - Closure under scalar multiplication: If $\mathbf{u} = s\mathbf{v}_1$ and $k \in \mathbb{R}$, then $k\mathbf{u} = (ks)\mathbf{v}_1 \in S$

Option A is TRUE.

Option B: The set forms an affine space, but not a vector space.

Since we proved that S is a vector space, this is **FALSE**.

Option C: The set is a subspace of $\mathbb{R}^3$.

Since S is a vector space contained in $\mathbb{R}^3$ and satisfies all subspace properties, this is **TRUE**.

Option D: The set forms a hyperplane.

A hyperplane in $\mathbb{R}^3$ is a 2-dimensional affine subspace (a plane). However, S is a 1-dimensional subspace (a line through the origin), so this is **FALSE**.

13.6 Step 6: Determine the Best Answer

Both options A and C are technically correct, but let's consider the context: - Option A states it's a vector space (correct) - Option C states it's a subspace of $\mathbb{R}^3$ (also correct, and more specific)

Since the question asks about affine combinations but the result is actually a vector subspace, **Option C** is the most precise answer.

13.7 Step 7: Geometric Interpretation

The set S is the line through the origin with direction vector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This line can be parameterized as:

$$\mathbf{r}(t) = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \quad t \in \mathbb{R}$$

This represents all points where $x = y = z$.

13.8 Answer

$$\boxed{\text{C: The set is a subspace of } \mathbb{R}^3}$$

Explanation:

- The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent ($\mathbf{v}_2 = 2\mathbf{v}_1$)
- All affine combinations reduce to scalar multiples of $\mathbf{v}_1$
- The resulting set is a 1-dimensional subspace (line through origin)
- It satisfies all vector space/subspace properties

Question 11: Which of the following is true about an affine space in $\mathbb{R}^3$?

1. A: It contains the zero vector.
2. B: It is closed under addition.
3. C: It is not closed under addition.
4. D: It is always a subspace.

14 Solution for Question 11

14.1 Step 1: Define Affine Space

An **affine space** (or affine subspace) in $\mathbb{R}^3$ is a translation of a vector subspace. It can be written as:

$$A = \mathbf{v}_0 + V = \{\mathbf{v}_0 + \mathbf{v} : \mathbf{v} \in V\}$$

where:

- $\mathbf{v}_0$ is a fixed point (translation vector)
- V is a vector subspace of $\mathbb{R}^3$

Examples of affine spaces in $\mathbb{R}^3$:

- Points: $\{\mathbf{v}_0\}$ (0-dimensional)
- Lines: $\mathbf{v}_0 + \text{span}\{\mathbf{d}\}$ (1-dimensional)
- Planes: $\mathbf{v}_0 + \text{span}\{\mathbf{u}, \mathbf{v}\}$ (2-dimensional)
- All of $\mathbb{R}^3$: $\mathbf{0} + \mathbb{R}^3$ (3-dimensional)

14.2 Step 2: Analyze Each Option

Option A: It contains the zero vector.

This is **FALSE** in general. An affine space contains the zero vector **if and only if** $\mathbf{v}_0 = \mathbf{0}$, which means the affine space is actually a vector subspace.

Counterexample: Consider the line $L = \{(1, 0, 0) + t(1, 1, 1) : t \in \mathbb{R}\}$. This line passes through $(1, 0, 0)$ and has direction $(1, 1, 1)$. The zero vector $(0, 0, 0)$ is not on this line since there's no value of t such that $(1, 0, 0) + t(1, 1, 1) = (0, 0, 0)$.

Option B: It is closed under addition.

This is **FALSE** in general. Let's check with an example.

Consider the line $L = \{(1, 0, 0) + t(1, 0, 0) : t \in \mathbb{R}\} = \{(1 + t, 0, 0) : t \in \mathbb{R}\}$.

Take two points: $\mathbf{p}_1 = (2, 0, 0)$ and $\mathbf{p}_2 = (3, 0, 0)$ (both in L).

Their sum is: $\mathbf{p}_1 + \mathbf{p}_2 = (5, 0, 0)$.

For $(5, 0, 0)$ to be in L, we need $5 = 1 + t$ for some t, so $t = 4$. While this works in this case, let's try a plane example:

Consider the plane $P = \{(1, 0, 0) + s(1, 0, 0) + t(0, 1, 0) : s, t \in \mathbb{R}\} = \{(1 + s, t, 0) : s, t \in \mathbb{R}\}$.

Take $\mathbf{p}_1 = (1, 1, 0)$ and $\mathbf{p}_2 = (2, 1, 0)$ (both in P). Their sum is $\mathbf{p}_1 + \mathbf{p}_2 = (3, 2, 0)$.

For this to be in P: $(3, 2, 0) = (1 + s, t, 0)$, so $s = 2, t = 2$. This point is in P.

Let's try another example: the line $L = \{(1, 1, 1)+t(1, 0, 0) : t \in \mathbb{R}\}$. Take $\mathbf{p}_1 = (1, 1, 1)$ and $\mathbf{p}_2 = (2, 1, 1)$. Sum: $\mathbf{p}_1 + \mathbf{p}_2 = (3, 2, 2)$.

For this to be in L: $(3, 2, 2) = (1+t, 1, 1)$, which requires $t = 2$, $2 = 1$, and $2 = 1$. The last two equations are impossible, so $(3, 2, 2) \notin L$.

Option C: It is not closed under addition.

This is **TRUE** in general. As shown above, affine spaces that are not vector subspaces (i.e., $\mathbf{v}_0 \neq \mathbf{0}$) are not closed under addition.

Option D: It is always a subspace.
This is **FALSE**. An affine space is a subspace if and only if it contains the zero vector (i.e., when $\mathbf{v}_0 = \mathbf{0}$).

14.3 Step 3: Rigorous Proof for Option C

Let $A = \mathbf{v}_0 + V$ be an affine space where $\mathbf{v}_0 \neq \mathbf{0}$ and V is a non-trivial subspace.
Take any two distinct points $\mathbf{p}_1, \mathbf{p}_2 \in A$:

$$\mathbf{p}_1 = \mathbf{v}_0 + \mathbf{u}_1, \quad \mathbf{p}_2 = \mathbf{v}_0 + \mathbf{u}_2$$

where $\mathbf{u}_1, \mathbf{u}_2 \in V$.
Their sum is:

$$\mathbf{p}_1 + \mathbf{p}_2 = (\mathbf{v}_0 + \mathbf{u}_1) + (\mathbf{v}_0 + \mathbf{u}_2) = 2\mathbf{v}_0 + (\mathbf{u}_1 + \mathbf{u}_2)$$

For this sum to be in A, we need:

$$2\mathbf{v}_0 + (\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{v}_0 + \mathbf{w}$$

for some $\mathbf{w} \in V$.
This gives us:

$$\mathbf{v}_0 + (\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{w}$$

Since V is a subspace, $\mathbf{u}_1 + \mathbf{u}_2 \in V$. For the equation to hold, we need $\mathbf{v}_0 \in V$. But if $\mathbf{v}_0 \in V$ and $\mathbf{v}_0 \neq \mathbf{0}$, then $A = \mathbf{v}_0 + V = V$ (since V contains $\mathbf{v}_0$ and is closed under addition), which contradicts our assumption that A is a proper affine space.
Therefore, when $\mathbf{v}_0 \neq \mathbf{0}$, the affine space is not closed under addition.

14.4 Answer

$$\boxed{\text{C: It is not closed under addition}}$$

Key Points:

- Affine spaces are translations of vector subspaces
- They contain the zero vector only when the translation vector is zero
- They are closed under addition only when they are actually vector subspaces
- In general, affine spaces are NOT closed under addition
- Only vector subspaces (special case where translation vector is zero) are always subspaces

Question 12: Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$ be a set in $\mathbb{R}^3$. Which of the following statements is true?

1. A: S forms a vector space.
2. B: S forms an affine space.
3. C: S forms a subspace.
4. D: S is not closed under scalar multiplication.

15 Solution for Question 12

15.1 Step 1: Characterize the Set S

The set S is defined by the constraint $x + y + z = 1$. This represents a **plane** in $\mathbb{R}^3$ that does not pass through the origin.

We can write this as:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \right\}$$

15.2 Step 2: Check if S Contains the Zero Vector

For the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to be in S, we need:

$$0 + 0 + 0 = 1$$

This is clearly false, so $\mathbf{0} \notin S$.

Since S does not contain the zero vector, it cannot be a vector space or a subspace.

15.3 Step 3: Analyze Each Option

Option A: S forms a vector space.

This is **FALSE**. A vector space must contain the zero vector, but we showed $\mathbf{0} \notin S$.

Option C: S forms a subspace.

This is **FALSE** for the same reason as Option A. A subspace is a special type of vector space.

Option D: S is not closed under scalar multiplication.

Let's verify this with an example. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S$ (since $1 + 0 + 0 = 1$).

Consider scalar multiplication by $k = 2$:

$$2\mathbf{v} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Check if $2\mathbf{v} \in S$: $2 + 0 + 0 = 2 \neq 1$

Therefore, $2\mathbf{v} \notin S$, confirming that S is not closed under scalar multiplication.

Option D is TRUE.

Option B: S forms an affine space.

Let's check this more carefully. An affine space can be written as $\mathbf{v}_0 + V$ where V is a vector subspace.

The constraint $x + y + z = 1$ can be rewritten as:

$$x + y + z - 1 = 0$$

This is equivalent to:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \text{null}\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right)$$

The null space of $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is:

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$$

This is a 2-dimensional subspace (a plane through the origin).
Therefore:

$$S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + V$$

This shows that S is indeed an affine space - it's a translation of the subspace V by the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Option B is also TRUE.

15.4 Step 4: Determine the Best Answer

Both Options B and D are correct statements about S. However, let's consider which is the more fundamental characterization:

- Option D states a negative property (what S is not) - Option B states a positive property (what S is)

In mathematical contexts, positive characterizations are generally preferred over negative ones. Moreover, identifying S as an affine space provides complete structural information about the set.

15.5 Step 5: Verification of Affine Space Properties

To confirm S is an affine space, let's verify it satisfies the key property: if $\mathbf{p}_1, \mathbf{p}_2 \in S$, then any affine combination $\alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2$ is also in S.

Let $\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ with $x_1 + y_1 + z_1 = 1$ and $x_2 + y_2 + z_2 = 1$.

The affine combination is:

$$\mathbf{w} = \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2 = \begin{bmatrix} \alpha x_1 + (1-\alpha)x_2 \\ \alpha y_1 + (1-\alpha)y_2 \\ \alpha z_1 + (1-\alpha)z_2 \end{bmatrix}$$

Check if $\mathbf{w} \in S$:

$$[\alpha x_1 + (1-\alpha)x_2] + [\alpha y_1 + (1-\alpha)y_2] + [\alpha z_1 + (1-\alpha)z_2] \tag{26}$$

$$= \alpha(x_1 + y_1 + z_1) + (1-\alpha)(x_2 + y_2 + z_2) \tag{27}$$

$$= \alpha(1) + (1-\alpha)(1) = \alpha + 1 - \alpha = 1 \tag{28}$$

Therefore, $\mathbf{w} \in S$, confirming that S is closed under affine combinations.

15.6 Answer

$$\boxed{\text{B: } S \text{ forms an affine space}}$$

Explanation:

- S is the plane $x + y + z = 1$ in $\mathbb{R}^3$
- It can be written as $S = \mathbf{v}_0 + V$ where $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and V is the plane $x + y + z = 0$
- S does not contain the zero vector, so it's not a vector space or subspace
- S is closed under affine combinations but not under addition or scalar multiplication
- While Option D is also true, Option B provides the most complete characterization

Question 13: Consider the set:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y = 0 \right\}.$$

Which of the following is true?

1. A: S is a subspace of $\mathbb{R}^3$.
2. B: S is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector.
3. C: S forms an affine space.
4. D: S is closed under scalar multiplication.

16 Solution for Question 13

16.1 Step 1: Characterize the Set S

The set S is defined by the constraint $x + y = 0$, or equivalently $y = -x$. This represents a **plane** in $\mathbb{R}^3$ that passes through the origin.

We can write this as:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y = 0 \right\} = \left\{ \begin{bmatrix} x \\ -x \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$$

16.2 Step 2: Check Subspace Properties

To determine if S is a subspace, we need to verify three properties: 1. Contains the zero vector 2. Closed under addition 3. Closed under scalar multiplication

Property 1: Contains the zero vector

Check if $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$:

We need $0 + 0 = 0$, which is true.

Therefore, $\mathbf{0} \in S$.

Property 2: Closed under addition

Let $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be arbitrary vectors in S.

This means $x_1 + y_1 = 0$ and $x_2 + y_2 = 0$.

Consider their sum:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

Check if $\mathbf{u} + \mathbf{v} \in S$:

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = 0 + 0 = 0$$

Therefore, $\mathbf{u} + \mathbf{v} \in S$.

Property 3: Closed under scalar multiplication

Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ (so $x + y = 0$) and $k \in \mathbb{R}$.

Consider the scalar multiple:

$$k\mathbf{u} = k \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$$

Check if $k\mathbf{u} \in S$:

$$kx + ky = k(x + y) = k \cdot 0 = 0$$

Therefore, $k\mathbf{u} \in S$.
Since all three properties are satisfied, S is a subspace of $\mathbb{R}^3$.

16.3 Step 3: Analyze Each Option

Option A: S is a subspace of $\mathbb{R}^3$.
This is **TRUE** based on our verification above.
Option B: S is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector.
This is **FALSE**. We showed that S does contain the zero vector and is indeed a subspace.
Option C: S forms an affine space.
This is technically **TRUE**, but not the best answer. Every vector subspace is also an affine space (it's an affine space with zero translation vector). However, this doesn't capture the full structure of S.
Option D: S is closed under scalar multiplication.
This is **TRUE** as we verified above. However, this is just one property of subspaces, not the complete characterization.

16.4 Step 4: Geometric Interpretation

The set S can be parameterized as:

$$S = \left\{ \begin{bmatrix} t \\ -t \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This represents a 2-dimensional subspace (plane) in $\mathbb{R}^3$ with basis vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The plane passes through the origin and has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (since $x + y = 0$ can be written as $\mathbf{n} \cdot \mathbf{v} = 0$).

16.5 Step 5: Determine the Best Answer

While options A, C, and D are all technically correct, option A provides the most complete and specific characterization of S. It identifies S as a subspace, which is more specific than just being an affine space, and more comprehensive than just noting closure under scalar multiplication.

16.6 Answer

$$\boxed{\text{A: } S \text{ is a subspace of } \mathbb{R}^3}$$

Explanation:

- S is the plane defined by $x + y = 0$ (or $y = -x$) in $\mathbb{R}^3$

- It contains the zero vector: $(0, 0, 0)$ satisfies $0 + 0 = 0$
- It is closed under addition and scalar multiplication
- It is a 2-dimensional subspace with basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
- The plane passes through the origin, making it a vector subspace rather than just an affine space

Question 14: Consider the system:

$$\begin{cases} x + 2y - z + w = 3 \\ 2x + 4y - 2z + 2w = 6 \\ x + 2y + z - w = 1 \end{cases}$$

The solution set forms:

A. A shifted line (1-dimensional affine space)

B. A shifted plane (2-dimensional affine space)

C. A single point

D. The empty set (inconsistent system)

17 Solution for Question 14

17.1 Step 1: Set Up the Augmented Matrix

We write the system in matrix form $\mathbf{Ax} = \mathbf{b}$ and form the augmented matrix:

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2 & 4 & -2 & 2 & 6 \\ 1 & 2 & 1 & -1 & 1 \end{array}\right]$$

17.2 Step 2: Row Reduction to RREF

We'll perform Gaussian elimination to reduce the augmented matrix to reduced row echelon form.

Operation 1: $R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2-2(1) & 4-2(2) & -2-2(-1) & 2-2(1) & 6-2(3) \\ 1 & 2 & 1 & -1 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 & 1 \end{array}\right]$$

Operation 2: $R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1-1 & 2-2 & 1-(-1) & -1-1 & 1-3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -2 \end{array}\right]$$

Operation 3: $R_3 \leftarrow \frac{1}{2}R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

Operation 4: Swap R_2 and R_3 to get proper echelon form

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Operation 5: $R_1 \leftarrow R_1 + R_2$ (eliminate above the pivot in column 3)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1+1 & 1+(-1) & 3+(-1) \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

17.3 Step 3: Analyze the RREF

From the reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can identify:

- **Pivot columns:** Columns 1 and 3 (contain leading 1's)
- **Free variables:** y and w (columns 2 and 4 have no pivots)
- **Basic variables:** x and z
- **Rank of coefficient matrix:** 2
- **Number of variables:** 4

17.4 Step 4: Check for Consistency

The system is consistent because there are no rows of the form $[0 \quad 0 \quad 0 \quad 0 \quad | \quad c]$ where $c \neq 0$.

17.5 Step 5: Express the General Solution

From the RREF, we get:

$$x + 2y = 2 \quad \Rightarrow \quad x = 2 - 2y \tag{29}$$

$$z - w = -1 \quad \Rightarrow \quad z = -1 + w \tag{30}$$

Let $y = s$ and $w = t$ where $s, t \in \mathbb{R}$ are parameters. The general solution is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 - 2s \\ s \\ -1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

17.6 Step 6: Determine the Geometric Structure

The solution set can be written as:

$$\mathbf{x}_p + \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

where:

- $\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is a particular solution
- $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the null space of $\mathbf{A}$

Since we have:

- 2 free variables (s and t)
- 2-dimensional null space
- Particular solution + 2-dimensional subspace

The solution set forms a **2-dimensional affine space** (shifted plane) in $\mathbb{R}^4$.

17.7 Step 7: Verification

Dimension check: Using the rank-nullity theorem:

- Rank of $\mathbf{A} = 2$
- Number of variables $= 4$
- Nullity $= 4 - 2 = 2$
- Solution space dimension $=$ nullity $= 2$

Particular solution verification: Let's check that $\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ satisfies the original system:

$$\text{Equation 1:} \quad 2 + 2(0) - (-1) + 0 = 2 + 1 = 3\checkmark \tag{31}$$

$$\text{Equation 2:} \quad 2(2) + 4(0) - 2(-1) + 2(0) = 4 + 2 = 6\checkmark \tag{32}$$

$$\text{Equation 3:} \quad 2 + 2(0) + (-1) - 0 = 2 - 1 = 1\checkmark \tag{33}$$

17.8 Answer

$$\boxed{\text{B. A shifted plane (2-dimensional affine space)}}$$

Summary:

- The system is consistent with rank 2
- There are 2 free variables, creating a 2-dimensional solution space
- The solution set is $\mathbf{x}_p + \text{null}(\mathbf{A})$ where $\text{null}(A)$ is 2-dimensional
- This forms a 2-dimensional affine space (shifted plane) in $\mathbb{R}^4$