Linear Algebra – Enhanced Revision Notes

Elementary Matrices, LU Decomposition, Vector Spaces, and More

Comprehensive Revision Guide

September 11, 2025

Contents

1 Elementary Matrices

Definition. An *elementary matrix* is obtained by applying a single elementary row operation to the identity matrix. Multiplying an elementary matrix E on the left of a matrix A performs the corresponding row operation on A.

Three Types of Elementary Matrices

1. **Type I (Row Swap):** E_{ij} swaps rows i and j
2. **Type II (Row Scaling):** $E_i(k)$ multiplies row i by nonzero scalar k
3. **Type III (Row Addition):** $E_{ij}(k)$ adds k times row j to row i

Key Properties

- Every elementary matrix is invertible
- $(E_{ij})^{-1} = E_{ij}$ (row swaps are self-inverse)
- $(E_i(k))^{-1} = E_i(1/k)$ for $k \neq 0$
- $(E_{ij}(k))^{-1} = E_{ij}(-k)$
- $\det(E_{ij}) = -1$, $\det(E_i(k)) = k$, $\det(E_{ij}(k)) = 1$

Example: Row swap $R_1 \leftrightarrow R_3$ in 4×4

$$E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For any 4×4 matrix A, $E_{13}A$ swaps the first and third rows of A.

Example: Row scaling $R_2 \to -3R_2$

$$E_2(-3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This multiplies the second row by -3.

Example: Row addition $R_3 \to R_3 + 5R_1$

$$E_{31}(5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$$

This adds 5 times the first row to the third row.

Example: Complex operation using multiple elementary matrices

To perform $R_1 \leftrightarrow R_2$, then $R_2 \to 2R_2$, then $R_3 \to R_3 - 4R_1$ on a 3×3 matrix A:

$$E = E_{31}(-4) \cdot E_2(2) \cdot E_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

Note: Right multiplication

Right multiplication AE performs column operations instead of row operations.

2 LU Decomposition

Definition. For a square matrix A, an LU decomposition is a factorization $A = LU$ where L is lower-triangular (often with unit diagonal) and U is upper-triangular.

Existence Conditions

- LU exists if all leading principal minors are nonzero
- If pivoting is needed, we get $PA = LU$ where P is a permutation matrix
- For any invertible matrix, PLU decomposition always exists

Example: Basic LU decomposition

Compute LU for $A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$.

Step 1: Use pivot $a_{11} = 2$. Multiplier: $\ell_{21} = 4/2 = 2$ **Step 2:** $R_2 \to R_2 - 2R_1$ gives $U = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ **Result:** $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

Verification: $LU = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} = A$

Example: LU with partial pivoting needed

For $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$, we need row swap since $a_{11} = 0$.

Step 1: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (swap rows) **Step 2:** $PA = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ **Result:** $PA = LU$ where $L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

Example: Larger matrix LU decomposition

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 1 & 3 & 4 \end{pmatrix}$:

Solution

Step 1: Eliminate below $a_{11} = 1$

$$\ell_{21} = 2/1 = 2, \quad R_2 \to R_2 - 2R_1 \tag{1}$$
$$\ell_{31} = 1/1 = 1, \quad R_3 \to R_3 - 1R_1 \tag{2}$$

After elimination: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

Step 2: Eliminate below $a_{22} = 1$

$$\ell_{32} = 1/1 = 1, \quad R_3 \to R_3 - 1R_2 \tag{3}$$

Final result:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

Example: LU applications: Solving systems

Given $A = LU$ and we want to solve $Ax = b$:

1. Solve $Ly = b$ (forward substitution)
2. Solve $Ux = y$ (backward substitution)

For $A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 18 \end{pmatrix}$:

$$Ly = b: \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 18 \end{pmatrix} \tag{4}$$
$$y_1 = 8, \quad 2y_1 + y_2 = 18 \Rightarrow y_2 = 2 \tag{5}$$
$$Ux = y: \quad \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \tag{6}$$
$$x_2 = 2, \quad 2x_1 + 3x_2 = 8 \Rightarrow x_1 = 1 \tag{7}$$

3 Vector Spaces

Definition. A set V with operations addition $(+)$ and scalar multiplication $(\cdot)$ is a vector space over field $\mathbb{F}$ if it satisfies these axioms:

Vector Space Axioms

1. **Closure:** $u + v \in V$ and $c \cdot v \in V$ for all $u, v \in V$, $c \in \mathbb{F}$
2. **Associativity:** $(u + v) + w = u + (v + w)$

3. **Commutativity:** $u + v = v + u$
4. **Zero element:** $\exists 0 \in V$ such that $v + 0 = v$ for all $v \in V$
5. **Additive inverse:** For each $v \in V$, $\exists(-v) \in V$ such that $v + (-v) = 0$
6. **Scalar multiplication axioms:** $1 \cdot v = v$, $c(dv) = (cd)v$
7. **Distributivity:** $c(u + v) = cu + cv$, $(c + d)v = cv + dv$

Example: Standard vector spaces

1. $\mathbb{R}^n$ with component-wise addition and scalar multiplication
2. $\mathcal{P}_n$ (polynomials of degree $\leq n$) with usual operations
3. $\mathcal{M}_{m \times n}$ (matrices) with matrix addition and scalar multiplication
4. $C[a, b]$ (continuous functions on $[a, b]$) with pointwise operations

Example: Verifying vector space: Polynomial space $\mathcal{P}_2$

Let $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$
Closure under addition: $(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in \mathcal{P}_2$
Closure under scalar multiplication: $c(a_0 + a_1x + a_2x^2) = ca_0 + ca_1x + ca_2x^2 \in \mathcal{P}_2$
Zero element: $0(x) = 0 + 0x + 0x^2$

Example: Non-vector space

Let $V = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ (first and third quadrants including axes).
This is **not** a vector space because it's not closed under addition: $(1, 1) \in V$ and $(-1, 1) \in V$, but $(1, 1) + (-1, 1) = (0, 2) \notin V$ since we need $0 \cdot 2 = 0 \geq 0$ but the set definition is stricter.
Actually, let me correct: $(0, 2)$ does satisfy $xy = 0 \cdot 2 = 0 \geq 0$. Better example: $(1, 1) \in V$ and $(1, -1) \notin V$ since $1 \cdot (-1) = -1 < 0$.

Example: Function spaces

$C^1[0, 1] = \{f : [0, 1] \to \mathbb{R} : f \text{ is differentiable and } f' \text{ is continuous}\}$
This is a vector space under pointwise addition and scalar multiplication: - $(f+g)'(x) = f'(x) + g'(x)$ (sum rule) - $(cf)'(x) = cf'(x)$ (constant rule) - Zero function: $0(x) = 0$ for all x

4 Column Space

Definition. The column space $\text{Col}(A)$ is the span of the columns of matrix A. Equivalently, $\text{Col}(A) = \{b : Ax = b \text{ is consistent}\}$.

Key Properties

- $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$ when A is $m \times n$
- $\dim(\text{Col}(A)) = \text{rank}(A)$

- Pivot columns of A form a basis for $\text{Col}(A)$

Example: Finding column space

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$
Notice that $c_2 = 2c_1$ and $c_3 = 3c_1$, so all columns are multiples of c_1. Therefore: $\text{Col}(A) = \text{span}\{(1,2,1)^T\}$ This is a 1-dimensional subspace (a line through origin) in $\mathbb{R}^3$.

Example: Column space via row reduction

For $A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 6 \\ 1 & 2 & 1 & 3 \end{pmatrix}$:
Row reduce to find pivot columns:

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 6 \\ 1 & 2 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns are 1 and 3, so: $\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Example: Geometric interpretation

For $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$:

$\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
This represents a plane in $\mathbb{R}^3$ passing through the origin with normal vector found by $(1,0,1) \times (0,1,1) = (-1,-1,1)$.

Warning: Common mistake

Don't confuse the columns of the original matrix with the columns of the row-reduced form! The pivot columns of the *original* matrix form the basis for $\text{Col}(A)$.

5 Null Space

Definition. The null space $\text{Nul}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ is the set of all solutions to the homogeneous system $Ax = 0$.

Key Properties

- $\text{Nul}(A)$ is always a subspace of $\mathbb{R}^n$ when A is $m \times n$
- $\dim(\text{Nul}(A)) = \text{nullity}(A) = n - \text{rank}(A)$

- Basis vectors correspond to free variables in the solution

Example: Basic null space calculation

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, solve $Ax = 0$:
From $x_1 + 2x_2 + 3x_3 = 0$, we get $x_1 = -2x_2 - 3x_3$. Let $x_2 = s$ and $x_3 = t$ be free variables.

General solution: $x = \begin{pmatrix} -2s - 3t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

Therefore: $\mathrm{Nul}(A) = \mathrm{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Example: Null space of square matrix

For $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$:

Row reduce: $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

From $x_1 + 2x_2 = 0$: $x_1 = -2x_2$ Let $x_2 = t$: $\mathrm{Nul}(A) = \mathrm{span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

Check: $\mathrm{rank}(A) = 1$, $\mathrm{nullity}(A) = 2 - 1 = 1$

Example: Relationship with linear independence

If columns of A are linearly independent, then $\mathrm{Nul}(A) = \{0\}$.
Proof: If $Ax = 0$ and columns are linearly independent, then the only solution is $x = 0$.

Example: Computing fundamental matrix solutions

For $A = \begin{pmatrix} 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$:
RREF gives us: $x_1 - 3x_2 - x_4 = 0$ and $x_3 + 2x_4 = 0$ Free variables: x_2, x_4
Setting $x_2 = 1, x_4 = 0$: $x_1 = 3, x_3 = 0 \Rightarrow v_1 = (3, 1, 0, 0)^T$ Setting $x_2 = 0, x_4 = 1$: $x_1 = 1, x_3 = -2 \Rightarrow v_2 = (1, 0, -2, 1)^T$
$\mathrm{Nul}(A) = \mathrm{span}\{v_1, v_2\}$ with dimension 2.

6 Basis and Span

Basis. A set $\{v_1, v_2, \ldots, v_k\}$ is a basis for vector space V if:

1. The vectors are linearly independent
2. The vectors span V

Span. $\mathrm{span}\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k : c_i \in \mathbb{F}\}$

Example: Standard bases

- $\mathbb{R}^3$: $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$
- $\mathcal{P}_2$: $\{1, x, x^2\}$
- $\mathcal{M}_{2\times 2}$: $\left\{ \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \right\}$

Example: Testing for linear independence

Are vectors $v_1 = (1, 2, 1)$, $v_2 = (2, 1, 3)$, $v_3 = (1, -1, 2)$ linearly independent?
Set up: $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$$\begin{pmatrix}1 & 2 & 1\\2 & 1 & -1\\1 & 3 & 2\end{pmatrix} \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Row reduce to find if only trivial solution exists: $\begin{pmatrix}1 & 2 & 1\\2 & 1 & -1\\1 & 3 & 2\end{pmatrix} \to \begin{pmatrix}1 & 2 & 1\\0 & -3 & -3\\0 & 1 & 1\end{pmatrix} \to$ $\begin{pmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$
Since we get a zero row, the vectors are **linearly dependent**. We can express $v_3 = v_1 + v_2$ (verify this!).

Example: Finding a basis for span

Find a basis for $\text{span}\{(1, 2, 0, 1), (0, 1, 1, 0), (1, 3, 1, 1), (2, 5, 1, 2)\}$.
Create matrix with these as columns and row reduce:

$$\begin{pmatrix}1 & 0 & 1 & 2\\2 & 1 & 3 & 5\\0 & 1 & 1 & 1\\1 & 0 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & 0 & 1 & 2\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

Pivot columns 1 and 2, so basis is $\{(1, 2, 0, 1), (0, 1, 1, 0)\}$.

Example: Change of basis

Express $v = (5, 7)$ in the basis $\mathcal{B} = \{(1, 2), (3, 1)\}$ for $\mathbb{R}^2$.
Solve: $c_1(1, 2) + c_2(3, 1) = (5, 7)$

$$\begin{pmatrix}1 & 3\\2 & 1\end{pmatrix} \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix}$$

Solution: $c_1 = -2, c_2 = \frac{7}{3}$, so $[v]_{\mathcal{B}} = \begin{pmatrix}-2\\7/3\end{pmatrix}$.

7 Dimension

Definition. The dimension of a vector space V is the number of vectors in any basis of V. We write $\dim(V)$.

Important Theorems

- All bases of a vector space have the same number of elements
- If $\dim(V) = n$, then any set of n linearly independent vectors is a basis
- If $\dim(V) = n$, then any set of n vectors that spans V is a basis

Example: Computing dimensions

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathcal{P}_n) = n + 1$ (basis: $\{1, x, x^2, \ldots, x^n\}$)
- $\dim(\mathcal{M}_{m\times n}) = mn$
- $\dim(\{0\}) = 0$ (trivial vector space)

Example: Dimension of solution spaces

Consider the system: $\begin{cases} x + 2y - z = 0 \\ 2x + 4y - 2z = 0 \end{cases}$

Row reduce: $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

One pivot column, so rank $= 1$ and dim(solution space) $= 3 - 1 = 2$. Basis: $\{(-2, 1, 0), (1, 0, 1)\}$ (from free variables y and z).

Example: Dimension and subspaces

Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + 2z = 0\}$.
This is a plane through the origin (subspace of $\mathbb{R}^3$). To find dimension, solve $x+y+2z = 0$ for x: $x = -y - 2z$. Let $y = s, z = t$ be parameters.
General solution: $(x, y, z) = (-s - 2t, s, t) = s(-1, 1, 0) + t(-2, 0, 1)$ Therefore $\dim(W) = 2$ with basis $\{(-1, 1, 0), (-2, 0, 1)\}$.

Note: Dimension formula for subspaces

If W is a subspace of $\mathbb{R}^n$ defined by k linearly independent equations, then $\dim(W) = n - k$.

8 Rank and Nullity

Definitions.

- $\text{rank}(A) = \dim(\text{Col}(A))$ = number of pivot columns = number of linearly independent rows
- $\text{nullity}(A) = \dim(\text{Nul}(A))$ = number of free variables

Methods to Find Rank

1. Count pivot positions in row echelon form
2. Count linearly independent columns
3. Count linearly independent rows
4. Use determinants of submatrices (for square matrices)

Example: Rank calculation via row reduction

Find rank $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{pmatrix}$:
Row reduce:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two pivot positions $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 4 - 2 = 2$.

Example: Rank of matrix products

Properties of rank:

- $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$
- $\operatorname{rank}(A^T) = \operatorname{rank}(A)$
- $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$ when A has linearly independent columns

For $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: $\operatorname{rank}(A) = 1$, $\operatorname{rank}(B) = 2$, $\operatorname{rank}(AB) = \operatorname{rank}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1$ Indeed, $1 \leq \min(1, 2) = 1$

Example: Applications: Consistency of systems

System $Ax = b$ is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}([A|b])$.
For $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$:

$\operatorname{rank}(A) = 1$ and $\operatorname{rank}([A|b]) = \operatorname{rank}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} = 1$ Since ranks are equal, the system is consistent.

Example: Rank and invertibility

For $n \times n$ matrix A:

- A is invertible $\text{rank}(A) = n$
- A is invertible $\text{nullity}(A) = 0$
- A is invertible $\det(A) \neq 0$

Test: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ (upper triangular) $\text{rank}(A) = 3 = n$, so A is invertible. $\det(A) = 1 \cdot 1 \cdot 1 = 1 \neq 0$

9 Rank-Nullity Theorem

Fundamental Theorem. For any $m \times n$ matrix A:

$$\text{rank}(A) + \text{nullity}(A) = n$$

This connects the dimension of the column space with the dimension of the null space.

Example: Verifying rank-nullity theorem

For $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (3×4 matrix):

From row echelon form: - Pivots in columns 1 and 3 $\text{rank}(A) = 2$ - Free variables: x_2, x_4 $\text{nullity}(A) = 2$ - Check: $2 + 2 = 4 = n$

Example: Dimension counting in transformations

Linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with matrix representation A (3×5). If $\dim(\text{Nul}(T)) = 2$, what is $\dim(\text{Range}(T))$?
By rank-nullity: $\text{rank}(A) + \text{nullity}(A) = 5$ So $\text{rank}(A) = 5 - 2 = 3$ Since $\text{Range}(T) = \text{Col}(A)$, we have $\dim(\text{Range}(T)) = 3$.

Example: Implications for system solutions

System $Ax = 0$ where A is $m \times n$:

- If $n > m$: Always has nontrivial solutions (more variables than equations)
- If $n = m$ and $\text{rank}(A) = n$: Only trivial solution
- If $\text{rank}(A) < n$: Infinitely many solutions with $\dim(\text{solution space}) = n - \text{rank}(A)$

10 Subspaces

Definition. A subset $W \subseteq V$ is a subspace if:

1. $0 \in W$ (contains zero vector)
2. $u, v \in W \Rightarrow u + v \in W$ (closed under addition)

3. $v \in W, c \in \mathbb{F} \Rightarrow cv \in W$ (closed under scalar multiplication)

Example: Standard subspaces of $\mathbb{R}^3$

- $\{(0,0,0)\}$ (trivial subspace)
- Lines through origin: $\{t(a,b,c) : t \in \mathbb{R}\}$ where $(a,b,c) \neq (0,0,0)$
- Planes through origin: $\{(x,y,z) : ax + by + cz = 0\}$
- All of $\mathbb{R}^3$

Example: Subspace verification

Is $W = \{(x,y,z) \in \mathbb{R}^3 : x + 2y - z = 0\}$ a subspace?
Zero test: $(0,0,0)$ satisfies $0 + 2(0) - 0 = 0$ Addition: If $(x_1,y_1,z_1),(x_2,y_2,z_2) \in W$, then: $(x_1+x_2)+2(y_1+y_2)-(z_1+z_2) = (x_1+2y_1-z_1)+(x_2+2y_2-z_2) = 0+0 = 0$ Scalar multiplication: If $(x,y,z) \in W$ and $c \in \mathbb{R}$: $c(x+2y-z) = cx + 2cy - cz = c(0) = 0$
Therefore, W is a subspace.

Example: Non-subspace examples

1. $W = \{(x,y) \in \mathbb{R}^2 : x+y = 1\}$ (line not through origin) Not a subspace: $(0,0) \notin W$
2. $W = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ (coordinate axes) Not a subspace: $(1,0),(0,1) \in W$ but $(1,0)+(0,1) = (1,1) \notin W$
3. $W = \{(x,y,z) \in \mathbb{R}^3 : |x| + |y| + |z| \leq 1\}$ (unit ball) Not a subspace: not closed under scalar multiplication (try $c = 2$)

Example: Intersection and sum of subspaces

Let $U = \text{span}\{(1,0,0),(0,1,0)\}$ and $V = \text{span}\{(1,1,0),(0,0,1)\}$
Intersection: $U \cap V = \{(x,y,z) : z = 0 \text{ and } (x,y,z) \in V\}$ Solving: $(x,y,0) = a(1,1,0) + b(0,0,1)$ gives $b = 0$, so $(x,y,0) = a(1,1,0)$ Therefore: $U \cap V = \text{span}\{(1,1,0)\}$ with $\dim(U \cap V) = 1$
Sum: $U + V = \{u + v : u \in U, v \in V\}$ Since any vector in $\mathbb{R}^3$ can be written as $(a,b,0) + (c,c,d) = (a+c, b+c, d)$ We can solve for any (x,y,z): $a = x - y$, $b = 0$, $c = y$, $d = z$ Therefore: $U + V = \mathbb{R}^3$ with $\dim(U+V) = 3$
Verification of dimension formula: $\dim(U) + \dim(V) = 2 + 2 = 4$ $\dim(U \cap V) + \dim(U+V) = 1 + 3 = 4$

11 Pivots, Free Variables, and Gaussian Elimination

Definitions:

- **Pivot:** First nonzero entry in each row of echelon form
- **Basic variables:** Variables corresponding to pivot columns
- **Free variables:** Variables not corresponding to pivot columns

Algorithm for Solving Systems

1. Form augmented matrix $[A|b]$

2. Row reduce to reduced row echelon form (RREF)
3. Identify pivot and free variables
4. Express basic variables in terms of free variables
5. Write general solution

Example: System with unique solution

$$\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 1 \\ x - y + 2z = -2 \end{cases}$$

Augmented matrix and reduction:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

All variables are basic (3 pivots for 3 variables). Unique solution: $(x, y, z) = (-1, 2, 0)$.

Example: System with infinitely many solutions

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 4y + 7z = 13 \\ 3x + 6y + 10z = 19 \end{cases}$$

Augmented matrix reduction:

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 4 & 7 & 13 \\ 3 & 6 & 10 & 19 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivots in columns 1 and 3. Basic variables: x, z. Free variable: $y = t$. From the RREF: - $z = 1$ (from row 2) - $x + 2t + 3(1) = 6 \Rightarrow x = 3 - 2t$ (from row 1)

General solution: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

This represents a line in $\mathbb{R}^3$.

Example: System with no solution (inconsistent)

$$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

Clearly inconsistent. Augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Last row represents $0 = 1$, which is impossible. Therefore, no solution exists.

Example: Parametric solutions with multiple free variables

Solve: $\begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$

From RREF: - Row 1: $x_1 + 2x_2 + 3x_4 = 5$ - Row 2: $x_3 + 2x_4 = -1$
Basic variables: x_1, x_3. Free variables: $x_2 = s, x_4 = t, x_5 = u$.
Solving: - $x_3 = -1 - 2t$ - $x_1 = 5 - 2s - 3t$
General solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

12 Consistent and Inconsistent Systems

Theorem (Consistency). The system $Ax = b$ is consistent if and only if $\text{rank}(A) = \text{rank}([A|b])$.

Cases for Solutions

Let A be $m \times n$ with $\text{rank}(A) = r$.

1. **Inconsistent:** $\text{rank}([A|b]) > \text{rank}(A)$
2. **Unique solution:** $\text{rank}([A|b]) = \text{rank}(A) = n$
3. **Infinitely many solutions:** $\text{rank}([A|b]) = \text{rank}(A) < n$

Example: Testing consistency

Test consistency of $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$:
Check ranks:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$\text{rank}(A) = 1$ (one pivot in coefficient matrix) $\text{rank}([A|b]) = 2$ (two pivots in augmented matrix)
Since $2 > 1$, the system is **inconsistent**.

Example: Geometric interpretation

System $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$:
This represents: - $x + y = 3$ (line in $\mathbb{R}^2$) - $2x + 2y = 6$, which simplifies to $x + y = 3$ (same line)
Since both equations represent the same line, there are **infinitely many solutions**. All points (x, y) on the line $x + y = 3$ are solutions.

Example: Parametric description of solution sets

For the consistent system with infinitely many solutions: $x + y = 3$
Solution set: $\{(3 - t, t) : t \in \mathbb{R}\} = \{(3, 0) + t(-1, 1) : t \in \mathbb{R}\}$
This is an affine set: a translation of the line $t(-1, 1)$ by the vector $(3, 0)$.

Warning: Common error in rank computation

When checking consistency, always row reduce the *augmented* matrix $[A|b]$, not just the coefficient matrix A. The appearance of a pivot in the last column indicates inconsistency.

13 Affine Spaces

Definition. An affine space (or affine set) is a translation of a linear subspace:

$$v_0 + W = \{v_0 + w : w \in W\}$$

where v_0 is a fixed vector and W is a subspace.

Properties of Affine Spaces

- Solution sets of consistent linear systems $Ax = b$ (when $b \neq 0$) are affine spaces
- Affine spaces are *not* subspaces (unless they contain the origin)
- The "direction" of an affine space is given by the associated subspace W

Example: Line in $\mathbb{R}^2$ not through origin

Consider $x + y = 2$. The solution set is:

$$\{(x, y) : x + y = 2\} = \{(2, 0) + t(-1, 1) : t \in \mathbb{R}\}$$

This is the affine space $(2, 0) + \text{span}\{(-1, 1)\}$. - Point: $(2, 0)$ (particular solution) - Direction: $\text{span}\{(-1, 1)\}$ (null space of coefficient matrix)

Example: Plane in $\mathbb{R}^3$ not through origin

System: $2x - y + z = 3$
Particular solution: $(0, 0, 3)$ (set $x = y = 0$, solve for z) Null space of $[2, -1, 1]$: solve $2x - y + z = 0$ Let $y = s, z = t$: then $x = \frac{s-t}{2}$
Direction space: $\text{span}\left\{\frac{1}{2}(1, 2, 0) + \frac{1}{2}(-1, 0, 2)\right\} = \text{span}\{(1, 2, 0), (-1, 0, 2)\}$
Solution set: $(0, 0, 3) + \text{span}\{(1, 2, 0), (-1, 0, 2)\}$

Example: Affine combinations

Points in affine space $v_0 + W$ can be written as:

$$v_0 + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k$$

where $\{w_1, w_2, \ldots, w_k\}$ is a basis for W.
Alternative characterization: Affine combinations of points $v_1, v_2, \ldots, v_k$:

$$t_1 v_1 + t_2 v_2 + \cdots + t_k v_k \quad \text{where } t_1 + t_2 + \cdots + t_k = 1$$

Note: Connection to linear algebra

If x_p is any particular solution to $Ax = b$ and $N = \mathrm{Nul}(A)$, then the complete solution set is the affine space $x_p + N$.

14 General and Particular Solutions

For the non-homogeneous system $Ax = b$:

- **Particular solution** x_p: Any single solution satisfying $Ax_p = b$
- **Homogeneous solutions**: All solutions to $Ax = 0$ (the null space $\mathrm{Nul}(A)$)
- **General solution**: $x = x_p + x_h$ where $x_h \in \mathrm{Nul}(A)$

Theorem. If $Ax = b$ is consistent, then the solution set is:

$$\{x_p + n : n \in \mathrm{Nul}(A)\}$$

for any particular solution x_p.

Example: Complete solution structure

Solve $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Step 1: Find particular solution Set free variables $y = 0, z = 0$: then $x = 3$ Particular solution: $x_p = (3, 0, 0)$

Step 2: Find null space Solve $x + 2y - z = 0$: $x = -2y + z$ General null space vector: $(-2s + t, s, t) = s(-2, 1, 0) + t(1, 0, 1)$ So $\mathrm{Nul}(A) = \mathrm{span}\{(-2, 1, 0), (1, 0, 1)\}$

Step 3: General solution

$$x = (3, 0, 0) + s(-2, 1, 0) + t(1, 0, 1) = (3 - 2s + t, s, t)$$

Example: Verification of solution structure

Check that if x_1 and x_2 are two solutions to $Ax = b$, then $x_1 - x_2 \in \mathrm{Nul}(A)$:
$A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0$
This shows that any two particular solutions differ by an element of the null space.

Example: Finding particular solutions systematically

For system in RREF form, set all free variables to zero:

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix}$$

Set free variables $x_2 = x_4 = x_5 = 0$: - From row 2: $x_3 = 4$ - From row 1: $x_1 = 7$
Particular solution: $(7, 0, 4, 0, 0)$
General solution involves adding the null space:

$$x = (7, 0, 4, 0, 0) + s(-2, 1, 0, 0, 0) + t(-3, 0, -1, 1, 0) + u(-5, 0, 2, 0, 1)$$

15 Advanced Topics and Applications

Example: Matrix equations $AX = B$

Solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

This is equivalent to solving two systems: $AX_1 = (5,7)^T$ and $AX_2 = (6,8)^T$

Solution: $X = A^{-1}B$ where $A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

$X = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 1/2 & 1 \end{pmatrix}$

Example: Least squares and normal equations

For overdetermined system $Ax = b$ (more equations than unknowns), the least squares solution satisfies:

$$A^T Ax = A^T b$$

Example: Fit line $y = mx + c$ through points $(0,1), (1,2), (2,2)$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Normal equations: $A^T Ax = A^T b$

$$A^T A = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Solving: $x = (m, c) = (1/2, 3/2)$, so $y = \frac{1}{2}x + \frac{3}{2}$.

---

16 Quick Reference and Formulas

Key Theorems

- **Rank-Nullity:** $\text{rank}(A) + \text{nullity}(A) = n$ (for $m \times n$ matrix A)
- **Fundamental Subspaces:** $\text{Col}(A) \perp \text{Nul}(A^T)$, $\text{Nul}(A) \perp \text{Col}(A^T)$
- **Invertible Matrix Theorem:** A invertible $\text{rank}(A) = n$ $\det(A) \neq 0$
- **Dimension Formula:** $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$

Common Dimensions

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathcal{P}_n) = n + 1$
- $\dim(\mathcal{M}_{m\times n}) = mn$
- $\dim(\text{symmetric } n \times n \text{ matrices}) = \frac{n(n+1)}{2}$

Problem-Solving Checklist

1. Always check dimensions and compatibility
2. Use row reduction for systematic solutions
3. Verify answers by substitution
4. Remember geometric interpretations
5. Apply rank-nullity theorem for dimension checks

Study Tips: Practice with varied examples, visualize in low dimensions, and always verify theoretical results with concrete computations. Master the connections between algebraic and geometric viewpoints!