Linear Algebra Revision Class 3

Problem 1: Vector Spaces Through the Origin

Question 1.

(a) Prove that a line passing through the origin in $\mathbb{R}^3$ is a vector space under standard vector addition and scalar multiplication.

(b) Prove that a plane passing through the origin in $\mathbb{R}^3$ is a vector space under standard vector addition and scalar multiplication.

(c) Prove that a line *not* passing through the origin in $\mathbb{R}^3$ is *not* a vector space. Which vector space axiom(s) fail? Justify your answer with a specific example.

Problem 2: Polynomial Vector Spaces and Subspaces

Question 2A. Consider the following two sets of polynomials:

- $V_1 = \{p(x) \mid p(x) \text{ is a polynomial of degree exactly } 2\}$
- $V_2 = \{p(x) \mid p(x) \text{ is a polynomial of degree at most } 2\}$

(a) Prove that V_1 is *not* a subspace of the vector space of all polynomials. Which subspace axiom(s) fail? Provide specific counterexamples.

(b) Prove that V_2 is a vector space (subspace of all polynomials) under standard polynomial addition and scalar multiplication. Verify all three subspace conditions:

- Contains the zero vector
- Closed under addition
- Closed under scalar multiplication

Question 2B. Let P_2 be the vector space of all polynomials of degree at most 2. Consider the set $S = \{1, 1 + x^2, x + x^2\}$. Determine whether S forms a basis for P_2. Justify your answer by:

(a) Checking if S is linearly independent.

(b) Checking if S spans P_2.

(c) Stating your conclusion about whether S is a basis.

Question 2C. Consider the set of vectors in $\mathbb{R}^3$:

$$T = \{(1, 2, 1), (2, 1, 3), (1, -1, 2)\}.$$

Determine whether T forms a basis for $\mathbb{R}^3$. Show all work.

Problem 3: Elementary Matrices and Upper Triangularization

Question 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix}.$$

Find elementary matrices $E_1, E_2, E_3, \ldots$ such that the product $E_k \cdots E_2 E_1 A = U$ is an upper triangular matrix U.

(a) Write down each elementary matrix explicitly.

(b) Verify your answer by computing the final product.

(c) State what type of elementary row operation each matrix represents.

Problem 4: Four Fundamental Subspaces

Question 4. For the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix},$$

find the following:

(a) The **Column Space** $C(A)$: Find a basis and state its dimension. Where does $C(A)$ lie geometrically?

(b) The **Null Space** $N(A)$: Find a basis and state its dimension. Where does $N(A)$ lie geometrically?

(c) The **Row Space** $C(A^T)$: Find a basis and state its dimension. Where does $C(A^T)$ lie geometrically?

(d) The **Left Null Space** $N(A^T)$: Find a basis and state its dimension. Where does $N(A^T)$ lie geometrically?

(e) Verify the Fundamental Theorem of Linear Algebra:

- $\dim(C(A)) + \dim(N(A^T)) =$ number of rows
- $\dim(C(A^T)) + \dim(N(A)) =$ number of columns

(f) Discuss the geometric relationship between $C(A^T)$ and $N(A)$, and between $C(A)$ and $N(A^T)$.

Problem 5: Solution Set and Its Geometry

Question 5. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

(a) Determine whether the system is consistent. If consistent, find the complete solution set.

(b) Express the solution in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is the general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(c) Describe the geometry of the solution set. Is it a point, line, plane, or empty? Explain your answer in terms of the null space of A.

(d) If the system has infinitely many solutions, express the solution set in parametric vector form.

Note: The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{bmatrix}$ is used consistently across Problems 2B, 3, 4, and 5 to help you see the deep connections between different concepts in linear algebra.