Linear Algebra Solution

Question 15: The general solution to the system $Ax = b$ where:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

The solution set is:

A. A shifted line through the origin

B. A shifted plane not passing through the origin

C. A shifted line not passing through the origin

D. A single point

Solution:
We need to solve the system $Ax = b$ where:

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

First, we form the augmented matrix $[A|b]$ and perform row operations to find the reduced row echelon form:

$$[A|b] = \begin{bmatrix} 1 & 1 & 2 & 1 & | & 4 \\ 2 & 1 & 3 & 1 & | & 5 \\ 1 & 0 & 1 & 0 & | & 1 \end{bmatrix}$$

Performing row operations:

$$R_2 \leftarrow R_2 - 2R_1 \tag{1}$$
$$R_3 \leftarrow R_3 - R_1 \tag{2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & | & 4 \\ 0 & -1 & -1 & -1 & | & -3 \\ 0 & -1 & -1 & -1 & | & -3 \end{bmatrix}$$

Next, multiply R_2 by -1:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & | & 4 \\ 0 & 1 & 1 & 1 & | & 3 \\ 0 & -1 & -1 & -1 & | & -3 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 + R_2$:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & | & 4 \\ 0 & 1 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Finally, apply $R_1 \leftarrow R_1 - R_2$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this reduced row echelon form, we can identify:

- Leading variables: x_1 and x_2 (corresponding to pivot columns)
- Free variables: x_3 and x_4 (corresponding to non-pivot columns)

The system of equations becomes:

$$x_1 + x_3 = 1 \tag{3}$$
$$x_2 + x_3 + x_4 = 3 \tag{4}$$

Solving for the leading variables in terms of the free variables:

$$x_1 = 1 - x_3 \tag{5}$$
$$x_2 = 3 - x_3 - x_4 \tag{6}$$

Let $x_3 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$ are parameters.
The general solution is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - s \\ 3 - s - t \\ s \\ t \end{bmatrix}$$

This can be written as:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is of the form $\mathbf{x}_p + \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where:

- $\mathbf{x}_p = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution
- $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ span the null space of A

Since we have 2 free variables, the null space of A is 2-dimensional. The solution set is a particular solution plus a 2-dimensional subspace, which geometrically represents a plane that has been translated (shifted) away from the origin.

Answer: B. A shifted plane not passing through the origin

Question 16: Consider the non-homogeneous system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 4x_2 + 3x_3 &= 5 \\ x_1 + 2x_2 + 2x_3 &= 3 \end{aligned}$$

Which geometric object best describes the solution set?

A. A line parallel to the vector $(-2, 1, 0)$

B. A plane parallel to the x_2-axis

C. A line parallel to the vector $(2, -1, 1)$

D. A point in 3-dimensional space

Solution:

We need to solve the system by forming the augmented matrix and finding its reduced row echelon form:

$$[A|b] = \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 2 & 4 & 3 & | & 5 \\ 1 & 2 & 2 & | & 3 \end{bmatrix}$$

Performing row operations:

$$R_2 \leftarrow R_2 - 2R_1 \tag{7}$$
$$R_3 \leftarrow R_3 - R_1 \tag{8}$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 - R_2$:

$$\begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Finally, apply $R_1 \leftarrow R_1 - R_2$:

$$\begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this reduced row echelon form, we can identify:

- Leading variables: x_1 and x_3 (corresponding to pivot columns 1 and 3)
- Free variable: x_2 (corresponding to non-pivot column 2)

The system of equations becomes:

$$\begin{aligned} x_1 + 2x_2 &= 1 \quad (9) \\ x_3 &= 1 \quad (10) \end{aligned}$$

Solving for the leading variables in terms of the free variable:

$$\begin{aligned} x_1 &= 1 - 2x_2 \quad (11) \\ x_3 &= 1 \quad (12) \end{aligned}$$

Let $x_2 = t$ where $t \in \mathbb{R}$ is a parameter.
The general solution is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ t \\ 1 \end{bmatrix}$$

This can be written as:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

The solution set is of the form $\mathbf{x}_p + t\mathbf{v}$, where:

- $\mathbf{x}_p = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a particular solution
- $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is the direction vector of the line

Since we have exactly 1 free variable, the solution set is a line in 3-dimensional space. The direction vector of this line is $(-2, 1, 0)$.

Answer: A. A line parallel to the vector $(-2, 1, 0)$

Question 17: The system of equations:

$$\begin{aligned} 2x - y + 3z &= 1 \\ 4x - 2y + 6z &= 2 \\ x + y - z &= 0 \end{aligned}$$

has a solution set that is:

A. A line in $\mathbb{R}^3$

B. A plane in $\mathbb{R}^3$

C. The empty set

D. A single point

Solution:
We need to solve the system by forming the augmented matrix and finding its reduced row echelon form:

$$[A|b] = \begin{bmatrix} 2 & -1 & 3 & | & 1 \\ 4 & -2 & 6 & | & 2 \\ 1 & 1 & -1 & | & 0 \end{bmatrix}$$

First, let's swap R_1 and R_3 to get a leading 1:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 4 & -2 & 6 & | & 2 \\ 2 & -1 & 3 & | & 1 \end{bmatrix}$$

Performing row operations:

$$R_2 \leftarrow R_2 - 4R_1 \tag{13}$$
$$R_3 \leftarrow R_3 - 2R_1 \tag{14}$$

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -6 & 10 & | & 2 \\ 0 & -3 & 5 & | & 1 \end{bmatrix}$$

Divide R_2 by -6:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & -\frac{5}{3} & | & -\frac{1}{3} \\ 0 & -3 & 5 & | & 1 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 + 3R_2$:

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & -\frac{5}{3} & | & -\frac{1}{3} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Apply $R_1 \leftarrow R_1 - R_2$:

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & | & \frac{1}{3} \\ 0 & 1 & -\frac{5}{3} & | & -\frac{1}{3} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this reduced row echelon form, we can identify:

- Leading variables: x and y (corresponding to pivot columns 1 and 2)
- Free variable: z (corresponding to non-pivot column 3)

The system of equations becomes:

$$x + \frac{2}{3}z = \frac{1}{3} \tag{15}$$
$$y - \frac{5}{3}z = -\frac{1}{3} \tag{16}$$

Solving for the leading variables in terms of the free variable:

$$x = \frac{1}{3} - \frac{2}{3}z \tag{17}$$
$$y = -\frac{1}{3} + \frac{5}{3}z \tag{18}$$

Let $z = t$ where $t \in \mathbb{R}$ is a parameter.
The general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} - \frac{2}{3}t \\ -\frac{1}{3} + \frac{5}{3}t \\ t \end{bmatrix}$$

This can be written as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

The solution set is of the form $\mathbf{x}_p + t\mathbf{v}$, where:

- $\mathbf{x}_p = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$ is a particular solution
- $\mathbf{v} = \begin{bmatrix} -\frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$ is the direction vector of the line

Since we have exactly 1 free variable and the system is consistent (no row of the form $[0\ 0\ 0 \mid c]$ where $c \neq 0$), the solution set is a line in $\mathbb{R}^3$.

Answer: A. A line in $\mathbb{R}^3$

Question 18: For the matrix:

$$H = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

find the column space of H.

A. The column space is spanned by $(2, 5, 8)$ and $(3, 6, 9)$

B. The column space is spanned by $(2, 5, 8)$ and $(4, 7, 10)$

C. The column space is spanned by $(1, 2, 3)$ and $(2, 3, 4)$

D. The column space is spanned by $(1, 3, 5)$ and $(2, 4, 6)$

Solution:

To find the column space of H, we need to find a basis for the span of the columns. We do this by finding the reduced row echelon form of H and identifying the pivot columns.

The matrix H has columns:

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}$$

Let's perform row operations to find the reduced row echelon form:

$$H = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

Divide R_1 by 2:

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

Apply $R_2 \leftarrow R_2 - 5R_1$ and $R_3 \leftarrow R_3 - 8R_1$:

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 \\ 0 & -\frac{3}{2} & -3 \\ 0 & -3 & -6 \end{bmatrix}$$

Divide R_2 by $-\frac{3}{2}$:

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 + 3R_2$:

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \leftarrow R_1 - \frac{3}{2}R_2$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

From the reduced row echelon form, we see that:

- Columns 1 and 2 are pivot columns
- Column 3 is not a pivot column

This means the first two columns of the original matrix H form a basis for the column space. Therefore, the column space of H is spanned by:

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

We can verify this: since the third column can be written as a linear combination of the first two columns. From the RREF, we have $\mathbf{c}_3 = -1 \cdot \mathbf{c}_1 + 2 \cdot \mathbf{c}_2$:

$$-1\begin{bmatrix}2\\5\\8\end{bmatrix}+2\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}-2+6\\-5+12\\-8+18\end{bmatrix}=\begin{bmatrix}4\\7\\10\end{bmatrix}=\mathbf{c}_3$$

Answer: A. The column space is spanned by $(2,5,8)$ **and** $(3,6,9)$

Question 19: Consider the matrix:

$$I=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

determine the rank and column space of I.

A. Rank = 3, Column space = $\mathbb{R}^3$

B. Rank = 2, Column space = $\mathbb{R}^2$

C. Rank = 2, Column space = $\mathbb{R}^1$

D. Rank = 1, Column space = $\mathbb{R}^2$

Solution:

To find the rank and column space of I, we need to find the reduced row echelon form and identify the pivot columns.

The matrix I has columns:

$$\mathbf{c}_1=\begin{bmatrix}1\\4\\7\end{bmatrix},\quad \mathbf{c}_2=\begin{bmatrix}2\\5\\8\end{bmatrix},\quad \mathbf{c}_3=\begin{bmatrix}3\\6\\9\end{bmatrix}$$

Let's perform row operations to find the reduced row echelon form:

$$I=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

Apply $R_2 \leftarrow R_2-4R_1$ and $R_3 \leftarrow R_3-7R_1$:

$$\begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix}$$

Divide R_2 by -3:

$$\begin{bmatrix}1&2&3\\0&1&2\\0&-6&-12\end{bmatrix}$$

Apply $R_3 \leftarrow R_3+6R_2$:

$$\begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix}$$

Apply $R_1 \leftarrow R_1 - 2R_2$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

From the reduced row echelon form, we can analyze:
Rank Analysis: The RREF has 2 non-zero rows, so the rank of I is 2.
Column Space Analysis: From the RREF, we see that:

- Columns 1 and 2 are pivot columns
- Column 3 is not a pivot column (it's a linear combination of the first two)

The column space is spanned by the pivot columns from the original matrix:

$$\text{Col}(I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$$

Since we have 2 linearly independent vectors in $\mathbb{R}^3$, the column space is a 2-dimensional subspace of $\mathbb{R}^3$. We can denote this as the column space being isomorphic to $\mathbb{R}^2$, but it's actually a 2-dimensional subspace (plane) within $\mathbb{R}^3$.

We can verify the linear dependence: from the RREF, $\mathbf{c}_3 = -1 \cdot \mathbf{c}_1 + 2 \cdot \mathbf{c}_2$:

$$-1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -1+4 \\ -4+10 \\ -7+16 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \mathbf{c}_3$$

Therefore:

- Rank = 2 (number of pivot columns)
- Column space dimension = 2 (isomorphic to $\mathbb{R}^2$)

Answer: B. Rank = 2, Column space = $\mathbb{R}^2$

Question 20: For the matrix:

$$J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$$

determine if the vector $v = (4, 5, 6)$ lies in the column space of J.

A. Yes, it lies in the column space.

B. No, it does not lie in the column space.

C. Yes, it can be expressed as a linear combination of the columns.

D. No, it cannot be expressed as a linear combination of the columns.

Solution:

To determine if the vector $\mathbf{v} = (4, 5, 6)$ lies in the column space of J, we need to check if the system $J\mathbf{x} = \mathbf{v}$ has a solution. This is equivalent to asking if $\mathbf{v}$ can be written as a linear combination of the columns of J.

Let's set up the augmented matrix $[J|\mathbf{v}]$ and find its reduced row echelon form:

$$[J|\mathbf{v}] = \begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 2 & 4 & 5 & | & 5 \\ 3 & 6 & 7 & | & 6 \end{bmatrix}$$

Performing row operations:

$$R_2 \leftarrow R_2 - 2R_1 \tag{19}$$
$$R_3 \leftarrow R_3 - 3R_1 \tag{20}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 0 & -1 & | & -3 \\ 0 & 0 & -2 & | & -6 \end{bmatrix}$$

Multiply R_2 by -1:

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & -2 & | & -6 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 + 2R_2$:

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Apply $R_1 \leftarrow R_1 - 3R_2$:

$$\begin{bmatrix} 1 & 2 & 0 & | & -5 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this reduced row echelon form, we can see that the system is consistent (there's no row of the form $[0\ 0\ 0 \mid c]$ where $c \neq 0$).

The system has a solution with:

- $x_3 = 3$ (from the second row)
- $x_1 + 2x_2 = -5$ (from the first row)
- x_2 is a free variable

Let $x_2 = t$, then $x_1 = -5 - 2t$.

One particular solution is obtained by setting $t = 0$:

$$\mathbf{x} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$$

Let's verify:

$$J\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -5+0+9 \\ -10+0+15 \\ -15+0+21 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \mathbf{v}$$

Since the system $J\mathbf{x} = \mathbf{v}$ has a solution, the vector $\mathbf{v} = (4, 5, 6)$ lies in the column space of J. This means $\mathbf{v}$ can be expressed as a linear combination of the columns of J:

$$\mathbf{v} = -5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Answer: A. Yes, it lies in the column space.

Note: Options A and C are equivalent statements, as a vector lies in the column space if and only if it can be expressed as a linear combination of the columns.

Question 21: Consider the set $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$ in $\mathbb{R}^3$. Which of the following statements is true?

1. A: S forms a vector space.
2. B: S is not closed under addition.
3. C: S forms a vector space but is not closed under scalar multiplication.
4. D: S does not contain the zero vector.

Solution:

To determine if S forms a vector space, we need to check if it satisfies all the vector space axioms. The set S consists of all vectors in $\mathbb{R}^3$ whose components sum to zero:

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$$

Let's verify the key properties:

1. Contains the zero vector: The zero vector is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Check: $0 + 0 + 0 = 0$ So $\mathbf{0} \in S$.

2. Closed under addition: Let $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in S$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in S$.

This means $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$.

Consider $\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$.

Check if $\mathbf{u} + \mathbf{v} \in S$: $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0 + 0 = 0$

So $\mathbf{u} + \mathbf{v} \in S$.

3. Closed under scalar multiplication: Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ and $c \in \mathbb{R}$.

This means $x + y + z = 0$.

Consider $c\mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$.

Check if $c\mathbf{u} \in S$: $cx + cy + cz = c(x + y + z) = c \cdot 0 = 0$

So $c\mathbf{u} \in S$.

4. Other vector space axioms: Since S is a subset of $\mathbb{R}^3$ and inherits the vector operations from $\mathbb{R}^3$, all other vector space axioms (associativity, commutativity, distributivity, existence of additive inverses, etc.) are automatically satisfied.

Geometric interpretation: The set S represents all vectors in $\mathbb{R}^3$ that lie on the plane $x + y + z = 0$. This plane passes through the origin and is a 2-dimensional subspace of $\mathbb{R}^3$.

Conclusion: Since S satisfies all vector space axioms:

- Contains the zero vector
- Closed under addition
- Closed under scalar multiplication
- Inherits all other properties from $\mathbb{R}^3$

Therefore, S forms a vector space (specifically, it's a subspace of $\mathbb{R}^3$).

Answer: A. S forms a vector space.

Question 22: Which of the following is a valid example of a vector space?

1. A: The set of all polynomials of degree less than 5 with real coefficients, with addition and scalar multiplication.
2. B: The set of all non-negative real numbers with usual addition and multiplication.
3. C: The set of all integers under addition.
4. D: The set of all vectors in $\mathbb{R}^2$ where the first component is always positive.

Solution:

To determine which set forms a vector space, we need to check if each satisfies the vector space axioms. Let's examine each option:

Option A: Polynomials of degree less than 5 Let $P_4 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_i \in \mathbb{R}\}$

Check key properties:

- **Zero vector:** The zero polynomial $0(x) = 0$ has degree less than 5
- **Closure under addition:** If $p(x)$ and $q(x)$ have degree < 5, then $p(x) + q(x)$ has degree < 5

- **Closure under scalar multiplication:** If $p(x)$ has degree < 5 and $c \in \mathbb{R}$, then $c \cdot p(x)$ has degree < 5
- **Additive inverses:** For any polynomial $p(x)$, we have $-p(x)$ in the set

All other axioms are inherited from polynomial operations.

Option B: Non-negative real numbers Let $S = \{x \in \mathbb{R} : x \geq 0\}$ with usual addition and multiplication.

Check properties:

- **Zero vector:** $0 \geq 0$, so $0 \in S$
- **Closure under addition:** If $x, y \geq 0$, then $x + y \geq 0$
- **Scalar multiplication issue:** Consider $x = 2 \in S$ and scalar $c = -1$. Then $c \cdot x = -2 < 0$, so $-2 \notin S$

Not closed under scalar multiplication when the scalar is negative.

Option C: Integers under addition Let $\mathbb{Z}$ with addition (but we need scalar multiplication for a vector space).

Check properties:

- **Zero vector:** $0 \in \mathbb{Z}$
- **Closure under addition:** If $m, n \in \mathbb{Z}$, then $m + n \in \mathbb{Z}$
- **Scalar multiplication issue:** For scalar multiplication, we need $c \cdot n$ for $c \in \mathbb{R}$ and $n \in \mathbb{Z}$. But if $c = \frac{1}{2}$ and $n = 1$, then $c \cdot n = \frac{1}{2} \notin \mathbb{Z}$

Not closed under scalar multiplication by real numbers.

Option D: Vectors in $\mathbb{R}^2$ with positive first component Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x > 0, y \in \mathbb{R} \right\}$

Check properties:

- **Zero vector:** The zero vector is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but $0 \not> 0$, so $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$

Does not contain the zero vector.

Additionally, consider scalar multiplication: if $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S$ and $c = -1$, then $c\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, but $-1 \not> 0$, so $c\mathbf{v} \notin S$

Conclusion: Only Option A satisfies all vector space axioms. The set of polynomials of degree less than 5 forms a vector space (specifically, it's a 5-dimensional vector space with basis $\{1, x, x^2, x^3, x^4\}$).

Answer: A. The set of all polynomials of degree less than 5 with real coefficients, with addition and scalar multiplication.

Question 23: Consider the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ in $\mathbb{R}^3$. Which of the following is true regarding the span of S?

1. A: The span of S is $\mathbb{R}^3$.
2. B: The span of S is a one-dimensional subspace of $\mathbb{R}^3$.
3. C: The set S forms a basis for $\mathbb{R}^3$.
4. D: The span of S is the line through the origin in $\mathbb{R}^3$.

Solution:

To determine the span of S, we need to analyze the linear independence of the vectors and find what subspace they generate.

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

Step 1: Check linear independence

First, let's examine the relationship between $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2\mathbf{v}_1$$

Since $\mathbf{v}_2 = 2\mathbf{v}_1$, the vectors are linearly dependent. This means one vector is a scalar multiple of the other.

Step 2: Determine the span

Since $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$, any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be written as:

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(2\mathbf{v}_1) = (a + 2b)\mathbf{v}_1$$

This shows that every vector in the span of S is just a scalar multiple of $\mathbf{v}_1$.

Therefore:

$$\text{span}(S) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Step 3: Geometric interpretation

The span of a single non-zero vector is the set of all scalar multiples of that vector:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} : t \in \mathbb{R} \right\}$$

This represents a line passing through the origin in the direction of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Step 4: Analyze the options

- **Option A:** The span of S is $\mathbb{R}^3$ - FALSE. We only have one linearly independent vector, so we can't span all of $\mathbb{R}^3$.
- **Option B:** The span of S is a one-dimensional subspace of $\mathbb{R}^3$ - TRUE. The span is generated by one linearly independent vector.

- **Option C:** The set S forms a basis for $\mathbb{R}^3$ - FALSE. A basis for $\mathbb{R}^3$ requires 3 linearly independent vectors.
- **Option D:** The span of S is the line through the origin in $\mathbb{R}^3$ - TRUE. This is geometrically correct.

Both options B and D are mathematically correct, as they describe the same geometric object from different perspectives. However, option B is more precise in terms of linear algebra terminology (dimension and subspace), while option D gives the geometric interpretation.

Answer: B. The span of S is a one-dimensional subspace of $\mathbb{R}^3$.

Note: Option D is also correct geometrically, but option B uses more precise mathematical terminology.

Question 24: Consider the set

$$T = \{p(x) \in P_3 : p'(0) = 0 \text{ and } p''(1) = 0\}$$

where P_3 is the space of polynomials of degree at most 3. Is T a subspace of P_3?

A. Yes, T is a subspace with dimension 2

B. Yes, T is a subspace with dimension 3

C. No, T is not closed under addition

D. No, T does not contain the zero polynomial

Solution:

To determine if T is a subspace of P_3, we need to check the three subspace criteria and then find its dimension.

Let's start by understanding what the conditions mean. A general polynomial in P_3 has the form:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

The derivatives are:

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 \tag{21}$$

$$p''(x) = 2a_2 + 6a_3x \tag{22}$$

The conditions are:

$$p'(0) = a_1 = 0 \tag{23}$$

$$p''(1) = 2a_2 + 6a_3 = 0 \Rightarrow a_2 = -3a_3 \tag{24}$$

So polynomials in T have the form:

$$p(x) = a_0 + 0 \cdot x + (-3a_3)x^2 + a_3x^3 = a_0 - 3a_3x^2 + a_3x^3$$

This can be written as:

$$p(x) = a_0 \cdot 1 + a_3(-3x^2 + x^3)$$

Step 1: Check if T contains the zero polynomial The zero polynomial is $0(x) = 0$. Setting $a_0 = 0$ and $a_3 = 0$:

$$p(x) = 0 \cdot 1 + 0 \cdot (-3x^2 + x^3) = 0$$

Check conditions: $p'(0) = 0$ and $p''(1) = 0$ So the zero polynomial is in T.

Step 2: Check closure under addition Let $p_1(x) = a_0^{(1)} - 3a_3^{(1)}x^2 + a_3^{(1)}x^3 \in T$ and $p_2(x) = a_0^{(2)} - 3a_3^{(2)}x^2 + a_3^{(2)}x^3 \in T$.

Then:

$$p_1(x) + p_2(x) = (a_0^{(1)} + a_0^{(2)}) - 3(a_3^{(1)} + a_3^{(2)})x^2 + (a_3^{(1)} + a_3^{(2)})x^3$$

This has the same form as polynomials in T, with parameters $a_0^{(1)} + a_0^{(2)}$ and $a_3^{(1)} + a_3^{(2)}$. So $p_1(x) + p_2(x) \in T$.

Step 3: Check closure under scalar multiplication Let $p(x) = a_0 - 3a_3x^2 + a_3x^3 \in T$ and $c \in \mathbb{R}$.

Then:

$$c \cdot p(x) = ca_0 - 3(ca_3)x^2 + (ca_3)x^3$$

This has the same form as polynomials in T, with parameters ca_0 and ca_3. So $c \cdot p(x) \in T$.

Step 4: Find the dimension of T We showed that polynomials in T can be written as:

$$p(x) = a_0 \cdot 1 + a_3 \cdot (-3x^2 + x^3)$$

where $a_0, a_3 \in \mathbb{R}$ are free parameters.

This gives us a spanning set: $\{1, -3x^2 + x^3\}$.

To check linear independence, suppose:

$$c_1 \cdot 1 + c_2 \cdot (-3x^2 + x^3) = 0$$

This gives us:

$$c_1 - 3c_2x^2 + c_2x^3 = 0$$

Comparing coefficients:

$$\text{constant term:} \quad c_1 = 0 \tag{25}$$

$$\text{coefficient of } x^2: \quad -3c_2 = 0 \Rightarrow c_2 = 0 \tag{26}$$

$$\text{coefficient of } x^3: \quad c_2 = 0 \tag{27}$$

Since $c_1 = c_2 = 0$ is the only solution, the vectors are linearly independent.

Therefore, $\{1, -3x^2 + x^3\}$ forms a basis for T, and $\dim(T) = 2$.

Answer: A. Yes, T is a subspace with dimension 2

Question 25: Let $M_{3\times 2}$ be the space of 3×2 matrices. Consider the subset:

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} : a + d = 0,\ b - c = 0 \right\}$$

Which statement is true?

A. U is a subspace of dimension 4

B. U is a subspace of dimension 3

C. U is not a subspace because it's not closed under scalar multiplication

D. U is an affine space but not a vector space

Solution:

To determine if U is a subspace and find its dimension, we need to analyze the constraints and verify subspace properties.

Step 1: Understand the constraints

The constraints are:

$$a + d = 0 \quad \Rightarrow \quad d = -a \tag{28}$$

$$b - c = 0 \quad \Rightarrow \quad c = b \tag{29}$$

So matrices in U have the form:

$$\begin{bmatrix} a & b \\ b & -a \\ e & f \end{bmatrix}$$

where $a, b, e, f \in \mathbb{R}$ are free parameters.

Step 2: Check subspace properties

Contains zero matrix: The zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Setting $a = b = e = f = 0$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Check constraints: $0 + 0 = 0$ and $0 - 0 = 0$ So the zero matrix is in U.

Closure under addition: Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ b_1 & -a_1 \\ e_1 & f_1 \end{bmatrix} \in U$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ b_2 & -a_2 \\ e_2 & f_2 \end{bmatrix} \in U$.

Then:

$$A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & -a_1 - a_2 \\ e_1 + e_2 & f_1 + f_2 \end{bmatrix}$$

Check constraints:

$$(a_1 + a_2) + (-a_1 - a_2) = 0 \tag{30}$$

$$(b_1 + b_2) - (b_1 + b_2) = 0 \tag{31}$$

So $A_1 + A_2 \in U$.

Closure under scalar multiplication: Let $A = \begin{bmatrix} a & b \\ b & -a \\ e & f \end{bmatrix} \in U$ and $k \in \mathbb{R}$.

Then:

$$kA = \begin{bmatrix} ka & kb \\ kb & -ka \\ ke & kf \end{bmatrix}$$

Check constraints:

$$ka + (-ka) = 0 \tag{32}$$
$$kb - kb = 0 \tag{33}$$

So $kA \in U$.
Step 3: Find the dimension of U
We can express any matrix in U as:

$$\begin{bmatrix} a & b \\ b & -a \\ e & f \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives us a spanning set:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

To check linear independence, suppose:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives us:

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & -c_1 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing entries: $c_1 = c_2 = c_3 = c_4 = 0$.
Since the only solution is the trivial one, the matrices are linearly independent.
Therefore, $\dim(U) = 4$.
Answer: A. U is a subspace of dimension 4

Question 26: Consider the set

$$V = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z = 0,\ 2x + 4y - 2z = 0\}.$$

What is the dimension of V?

A. 1

B. 2

C. 3

D. 4

Solution:

To find the dimension of V, we need to analyze the system of linear equations and determine how many free variables we have.

Step 1: Analyze the constraint equations

The constraints are:

$$x + 2y - z = 0 \quad \text{(equation 1)} \tag{34}$$

$$2x + 4y - 2z = 0 \quad \text{(equation 2)} \tag{35}$$

Let's examine the relationship between these equations:

$$\text{Equation 2} = 2 \times \text{Equation 1}$$

$$2x + 4y - 2z = 2(x + 2y - z)$$

Since equation 2 is just 2 times equation 1, these equations are not linearly independent. We effectively have only one constraint.

Step 2: Solve the system

From the single independent constraint $x + 2y - z = 0$, we can express one variable in terms of the others:

$$x = z - 2y$$

The variable w doesn't appear in any constraint, so it's completely free.

Therefore, vectors in V have the form:

$$(x, y, z, w) = (z - 2y, y, z, w)$$

where y, z, and w are free parameters.

Step 3: Express in parametric form

We can write any vector in V as:

$$(x, y, z, w) = (z - 2y, y, z, w) \tag{36}$$

$$= y(-2, 1, 0, 0) + z(1, 0, 1, 0) + w(0, 0, 0, 1) \tag{37}$$

This shows that V is spanned by the vectors:

$$\mathbf{v}_1 = (-2, 1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1, 0), \quad \mathbf{v}_3 = (0, 0, 0, 1)$$

Step 4: Check linear independence

To verify these vectors are linearly independent, suppose:

$$a_1(-2, 1, 0, 0) + a_2(1, 0, 1, 0) + a_3(0, 0, 0, 1) = (0, 0, 0, 0)$$

This gives us:

$$(-2a_1 + a_2, a_1, a_2, a_3) = (0, 0, 0, 0)$$

Comparing components:

$$a_3 = 0 \quad \text{(from 4th component)} \tag{38}$$

$$a_2 = 0 \quad \text{(from 3rd component)} \tag{39}$$

$$a_1 = 0 \quad \text{(from 2nd component)} \tag{40}$$

$$-2a_1 + a_2 = 0 \quad \text{(from 1st component, which is satisfied)} \tag{41}$$

Since $a_1 = a_2 = a_3 = 0$ is the only solution, the vectors are linearly independent.

Step 5: Verify using the dimension formula

We can also use the fact that for a subspace defined by homogeneous linear constraints:

$$\dim(V) = n - \text{rank of constraint matrix}$$

where $n = 4$ is the dimension of the ambient space.

The constraint matrix (coefficient matrix of the system) is:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \end{bmatrix}$$

Row reducing:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the constraint matrix is 1.

Therefore:

$$\dim(V) = 4 - 1 = 3$$

Answer: C. 3

Question 27: Consider the vector space P_4 of polynomials of degree at most 4. Let W be the subspace of polynomials $p(x)$ such that $p(1) = p(-1) = 0$. What is $\dim(W)$?

A. 2

B. 3

C. 4

D. 5

Solution:

To find the dimension of W, we need to determine how the conditions $p(1) = 0$ and $p(-1) = 0$ constrain the polynomial coefficients.

Step 1: Set up the general polynomial

A general polynomial in P_4 has the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Step 2: Apply the constraints

The conditions are:

$$p(1) = 0 \tag{42}$$
$$p(-1) = 0 \tag{43}$$

Let's evaluate these:

Constraint 1: $p(1) = 0$

$$a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 + a_4(1)^4 = 0$$

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0 \quad \text{(equation 1)}$$

Constraint 2: $p(-1) = 0$

$$a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 + a_4(-1)^4 = 0$$

$$a_0 - a_1 + a_2 - a_3 + a_4 = 0 \quad \text{(equation 2)}$$

Step 3: Solve the system of constraints
We have the system:

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0 \quad (1) \tag{44}$$
$$a_0 - a_1 + a_2 - a_3 + a_4 = 0 \quad (2) \tag{45}$$

Adding equations (1) and (2):

$$2a_0 + 2a_2 + 2a_4 = 0$$

$$a_0 + a_2 + a_4 = 0 \quad (3)$$

Subtracting equation (2) from equation (1):

$$2a_1 + 2a_3 = 0$$

$$a_1 + a_3 = 0 \quad (4)$$

From equations (3) and (4):

$$a_0 = -a_2 - a_4 \tag{46}$$
$$a_1 = -a_3 \tag{47}$$

Step 4: Express polynomials in W in parametric form
Substituting back, any polynomial in W has the form:

$$p(x) = (-a_2 - a_4) + (-a_3)x + a_2x^2 + a_3x^3 + a_4x^4$$

Rearranging:

$$p(x) = a_2(-1 + x^2) + a_3(-x + x^3) + a_4(-1 + x^4)$$

where a_2, a_3, and a_4 are free parameters.
Step 5: Find a basis for W
From the parametric form, W is spanned by:

$$p_1(x) = -1 + x^2 = x^2 - 1 \tag{48}$$
$$p_2(x) = -x + x^3 = x^3 - x \tag{49}$$
$$p_3(x) = -1 + x^4 = x^4 - 1 \tag{50}$$

Let's verify these polynomials satisfy the conditions:
For $p_1(x) = x^2 - 1$:

$$p_1(1) = 1^2 - 1 = 0 \tag{51}$$
$$p_1(-1) = (-1)^2 - 1 = 0 \tag{52}$$

For $p_2(x) = x^3 - x$:

$$p_2(1) = 1^3 - 1 = 0 \tag{53}$$
$$p_2(-1) = (-1)^3 - (-1) = -1 + 1 = 0 \tag{54}$$

For $p_3(x) = x^4 - 1$:

$$p_3(1) = 1^4 - 1 = 0 \tag{55}$$
$$p_3(-1) = (-1)^4 - 1 = 1 - 1 = 0 \tag{56}$$

Step 6: Check linear independence
Suppose $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$ for all x.

$$c_1(x^2 - 1) + c_2(x^3 - x) + c_3(x^4 - 1) = 0$$
$$c_3x^4 + c_2x^3 + c_1x^2 - c_2x + (-c_1 - c_3) = 0$$

Comparing coefficients:

$$x^4: \quad c_3 = 0 \tag{57}$$
$$x^3: \quad c_2 = 0 \tag{58}$$
$$x^2: \quad c_1 = 0 \tag{59}$$
$$x^1: \quad -c_2 = 0 \quad \text{(satisfied)} \tag{60}$$
$$x^0: \quad -c_1 - c_3 = 0 \quad \text{(satisfied)} \tag{61}$$

Since $c_1 = c_2 = c_3 = 0$ is the only solution, the polynomials are linearly independent.
Step 7: Alternative verification using dimension formula
We can also use: $\dim(W) = \dim(P_4) -$ number of linearly independent constraints

P_4 has dimension 5, and we have 2 linearly independent constraints (the constraint matrix has rank 2), so:

$$\dim(W) = 5 - 2 = 3$$

Answer: B. 3

Question 28: Let P_2 be the set of all polynomials of degree at most 2. Which of the following sets is a subspace of P_2?

A: The set of all polynomials $p(x) = a + bx \quad$ where $a, b \in \mathbb{R}$.

B: The set of all polynomials $p(x) = a + bx^2 \quad$ where $a, b \in \mathbb{R}$.

C: The set of all polynomials of the form $p(x) = a + bx + cx^2 \quad$ where $a, b, c \in \mathbb{R}$.

D: The set of all polynomials of the form $p(x) = ax + bx^2 \quad$ where $a, b \in \mathbb{R}$.

Solution:

To determine which sets form subspaces of P_2, we need to check the three subspace criteria for each: 1. Contains the zero polynomial 2. Closed under addition 3. Closed under scalar multiplication

Let's examine each option:
Option A: $S_A = \{p(x) = a + bx : a, b \in \mathbb{R}\}$
This is the set of polynomials of degree at most 1 (linear polynomials).

- **Zero polynomial:** Setting $a = b = 0$ gives $p(x) = 0$, so $0 \in S_A$
- **Closure under addition:** If $p_1(x) = a_1 + b_1x$ and $p_2(x) = a_2 + b_2x$, then

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x$$

  This is still of the form $a + bx$, so $p_1 + p_2 \in S_A$

- **Closure under scalar multiplication:** If $p(x) = a + bx$ and $k \in \mathbb{R}$, then

$$kp(x) = ka + (kb)x$$

  This is still of the form $a + bx$, so $kp \in S_A$

S_A is a subspace of P_2.
Option B: $S_B = \{p(x) = a + bx^2 : a, b \in \mathbb{R}\}$
This is the set of polynomials with no x term.

- **Zero polynomial:** Setting $a = b = 0$ gives $p(x) = 0$, so $0 \in S_B$
- **Closure under addition:** If $p_1(x) = a_1 + b_1x^2$ and $p_2(x) = a_2 + b_2x^2$, then

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x^2$$

  This is still of the form $a + bx^2$, so $p_1 + p_2 \in S_B$

- **Closure under scalar multiplication:** If $p(x) = a + bx^2$ and $k \in \mathbb{R}$, then

$$kp(x) = ka + (kb)x^2$$

  This is still of the form $a + bx^2$, so $kp \in S_B$

S_B is a subspace of P_2.
Option C: $S_C = \{p(x) = a + bx + cx^2 : a, b, c \in \mathbb{R}\}$
This is the set of all polynomials of degree at most 2, which is exactly P_2 itself.
Since P_2 is a vector space, it's automatically a subspace of itself.
Option D: $S_D = \{p(x) = ax + bx^2 : a, b \in \mathbb{R}\}$
This is the set of polynomials with no constant term.

- **Zero polynomial:** Setting $a = b = 0$ gives $p(x) = 0$, so $0 \in S_D$
- **Closure under addition:** If $p_1(x) = a_1x + b_1x^2$ and $p_2(x) = a_2x + b_2x^2$, then

$$p_1(x) + p_2(x) = (a_1 + a_2)x + (b_1 + b_2)x^2$$

  This is still of the form $ax + bx^2$, so $p_1 + p_2 \in S_D$

- **Closure under scalar multiplication:** If $p(x) = ax + bx^2$ and $k \in \mathbb{R}$, then

$$kp(x) = (ka)x + (kb)x^2$$

  This is still of the form $ax + bx^2$, so $kp \in S_D$

S_D is a subspace of P_2.

Conclusion:

All four options represent subspaces of P_2:

- Option A: Polynomials of degree 1 (2-dimensional subspace)
- Option B: Polynomials with no linear term (2-dimensional subspace)
- Option C: All of P_2 (3-dimensional, the whole space)
- Option D: Polynomials with no constant term (2-dimensional subspace)

However, since the question asks "which of the following sets is a subspace" (singular), and all are valid, we should note that all options are correct. If forced to choose one, Option C represents the entire space P_2, which is the most complete answer.

Answer: All options A, B, C, and D are subspaces of P_2.

If only one answer is expected, then **C** is the most comprehensive as it represents the entire space.

Question 29: Consider the set of polynomials

$$S = \{p(x) \in P_2 : p(0) = 0\}.$$

Which of the following is true?

A: S is a subspace of P_2.

B: S is not closed under scalar multiplication.

C: S is not closed under addition.

D: S is a subset of P_1.

Solution:

To determine which statement is true, we need to analyze the set S and check if it satisfies the subspace properties.

Step 1: Understand the set S

S consists of all polynomials in P_2 that equal zero when $x = 0$. A general polynomial in P_2 has the form:

$$p(x) = a + bx + cx^2$$

The condition $p(0) = 0$ means:

$$p(0) = a + b(0) + c(0)^2 = a = 0$$

Therefore, polynomials in S have the form:

$$p(x) = 0 + bx + cx^2 = bx + cx^2$$

where $b, c \in \mathbb{R}$.

Step 2: Check subspace properties

Contains the zero polynomial: The zero polynomial is $0(x) = 0$ for all x. Check: $0(0) = 0$ So the zero polynomial is in S.

Closure under addition: Let $p_1(x) = b_1x + c_1x^2 \in S$ and $p_2(x) = b_2x + c_2x^2 \in S$.

Then:

$$p_1(x) + p_2(x) = (b_1 + b_2)x + (c_1 + c_2)x^2$$

Check the condition:

$$(p_1 + p_2)(0) = (b_1 + b_2)(0) + (c_1 + c_2)(0)^2 = 0$$

So $p_1 + p_2 \in S$.

Closure under scalar multiplication: Let $p(x) = bx + cx^2 \in S$ and $k \in \mathbb{R}$.

Then:

$$kp(x) = k(bx + cx^2) = (kb)x + (kc)x^2$$

Check the condition:

$$(kp)(0) = (kb)(0) + (kc)(0)^2 = 0$$

So $kp \in S$.

Since S satisfies all three subspace criteria, S is a subspace of P_2.

Step 3: Analyze the other options

Option B: "S is not closed under scalar multiplication" We just showed that S is closed under scalar multiplication, so this is FALSE.

Option C: "S is not closed under addition" We just showed that S is closed under addition, so this is FALSE.

Option D: "S is a subset of P_1" This claims that all polynomials in S have degree at most 1.

Consider the polynomial $p(x) = x^2 \in S$ (since $p(0) = 0^2 = 0$). This polynomial has degree 2, which means it's not in P_1. Therefore, S is NOT a subset of P_1. This statement is FALSE.

Step 4: Geometric interpretation

The set S represents all polynomials in P_2 that pass through the origin (the point $(0, 0)$). This is indeed a subspace, and it can be expressed as:

$$S = \text{span}\{x, x^2\}$$

This is a 2-dimensional subspace of the 3-dimensional space P_2.

Conclusion:

Option A is the only true statement.

Answer: A. S is a subspace of P_2.

Question 30: Let P_2 be the set of all polynomials of degree at most 2. Which of the following sets is **not** a subspace of P_2?

A: The set of all polynomials $p(x) = a + bx + cx^2 \quad \text{where } a, b, c \in \mathbb{R}$.

B: The set of all polynomials $p(x) = a + bx \quad \text{where } a, b \in \mathbb{R}$.

C: The set of all polynomials $p(x) = 0 \quad \text{where } a \in \mathbb{R}$.

D: The set of all polynomials $p(x) = ax^2 \quad \text{where } a \in \mathbb{R}$.

E: None

Solution:

To find which set is NOT a subspace, we need to check each option for the three subspace criteria:
1. Contains the zero polynomial 2. Closed under addition 3. Closed under scalar multiplication

Let's examine each option:

Option A: $S_A = \{p(x) = a + bx + cx^2 : a, b, c \in \mathbb{R}\}$

This is exactly the definition of P_2 itself - all polynomials of degree at most 2.

Since P_2 is a vector space, it's automatically a subspace of itself.

Option B: $S_B = \{p(x) = a + bx : a, b \in \mathbb{R}\}$

This is the set of polynomials of degree at most 1 (linear polynomials).

- **Zero polynomial:** Setting $a = b = 0$ gives $p(x) = 0$, so $0 \in S_B$
- **Closure under addition:** If $p_1(x) = a_1 + b_1 x$ and $p_2(x) = a_2 + b_2 x$, then

$$p_1(x) + p_2(x) = (a_1 + a_2) + (b_1 + b_2)x$$

  This is still of the form $a + bx$, so $p_1 + p_2 \in S_B$
- **Closure under scalar multiplication:** If $p(x) = a + bx$ and $k \in \mathbb{R}$, then

$$kp(x) = ka + (kb)x$$

  This is still of the form $a + bx$, so $kp \in S_B$

S_B is a subspace of P_2.

Option C: $S_C = \{p(x) = 0\}$

This set contains only the zero polynomial.

Wait, let me read this more carefully. The option says "The set of all polynomials $p(x) = 0$ where $a \in \mathbb{R}$." This seems to be poorly written.

If it means the set containing only the zero polynomial, then:

- **Zero polynomial:** The zero polynomial is in the set by definition
- **Closure under addition:** $0 + 0 = 0$, which is still in the set
- **Closure under scalar multiplication:** $k \cdot 0 = 0$ for any k, which is still in the set

The set $\{0\}$ is a subspace (the trivial subspace).

Option D: $S_D = \{p(x) = ax^2 : a \in \mathbb{R}\}$

This is the set of polynomials that are pure quadratic terms (no constant or linear terms).

- **Zero polynomial:** Setting $a = 0$ gives $p(x) = 0$, so $0 \in S_D$
- **Closure under addition:** If $p_1(x) = a_1 x^2$ and $p_2(x) = a_2 x^2$, then

$$p_1(x) + p_2(x) = (a_1 + a_2)x^2$$

  This is still of the form ax^2, so $p_1 + p_2 \in S_D$
- **Closure under scalar multiplication:** If $p(x) = ax^2$ and $k \in \mathbb{R}$, then

$$kp(x) = (ka)x^2$$

  This is still of the form ax^2, so $kp \in S_D$

S_D is a subspace of P_2.
Conclusion:
All of the given sets are actually subspaces of P_2:

- Option A: The entire space P_2 (3-dimensional)
- Option B: Polynomials of degree 1 (2-dimensional subspace)
- Option C: The trivial subspace containing only the zero polynomial (0-dimensional)
- Option D: Pure quadratic polynomials (1-dimensional subspace)

Since all options represent valid subspaces, the answer is Option E.
Answer: E. None
(All of the given sets are subspaces of P_2.)

Question 31: Given the vector space P_2 and the set

$$S = \{1 + x,\ 1 + x + x^2\},$$

determine if S forms a basis for P_2.

A: Yes, S is linearly independent and spans P_2.

B: No, S is linearly dependent.

C: Yes, but S does not span P_2.

D: No, S does not span P_2.

Solution:
To determine if S forms a basis for P_2, we need to check two conditions: 1. The polynomials in S are linearly independent 2. S spans P_2

Since P_2 is a 3-dimensional vector space (with standard basis $\{1, x, x^2\}$), a basis must contain exactly 3 linearly independent vectors. The set S contains only 2 polynomials, so it cannot form a basis for P_2.

Let's verify this systematically:
Step 1: Check linear independence
Let $p_1(x) = 1 + x$ and $p_2(x) = 1 + x + x^2$.
To check linear independence, we solve:

$$c_1 p_1(x) + c_2 p_2(x) = 0$$

Substituting:

$$c_1(1 + x) + c_2(1 + x + x^2) = 0$$

$$c_1 + c_1 x + c_2 + c_2 x + c_2 x^2 = 0$$

$$(c_1 + c_2) + (c_1 + c_2)x + c_2 x^2 = 0$$

For this to be the zero polynomial, all coefficients must be zero:

$$\text{Coefficient of } x^2: \quad c_2 = 0 \tag{62}$$
$$\text{Coefficient of } x: \quad c_1 + c_2 = 0 \tag{63}$$
$$\text{Coefficient of } 1: \quad c_1 + c_2 = 0 \tag{64}$$

From the first equation: $c_2 = 0$ Substituting into the second equation: $c_1 + 0 = 0 \Rightarrow c_1 = 0$
Since $c_1 = c_2 = 0$ is the only solution, the polynomials are linearly independent.

Step 2: Check if S spans P_2

For S to span P_2, every polynomial in P_2 must be expressible as a linear combination of the polynomials in S.

A general polynomial in P_2 has the form $p(x) = a + bx + cx^2$.

We need to determine if there exist constants α and β such that:

$$a + bx + cx^2 = \alpha(1 + x) + \beta(1 + x + x^2)$$

Expanding the right side:

$$a + bx + cx^2 = \alpha + \alpha x + \beta + \beta x + \beta x^2$$

$$a + bx + cx^2 = (\alpha + \beta) + (\alpha + \beta)x + \beta x^2$$

Comparing coefficients:

$$\text{Coefficient of } x^2: \quad c = \beta \tag{65}$$
$$\text{Coefficient of } x: \quad b = \alpha + \beta \tag{66}$$
$$\text{Coefficient of } 1: \quad a = \alpha + \beta \tag{67}$$

From equations 2 and 3: $b = a$

This means we can only represent polynomials where the coefficient of x equals the coefficient of the constant term. For example, we cannot represent the polynomial $p(x) = 1$ (where $a = 1, b = 0, c = 0$) because this would require $b = a$, i.e., $0 = 1$, which is impossible.

Therefore, S does not span P_2.

Step 3: Dimension argument

Since $\dim(P_2) = 3$ and $|S| = 2$, the set S cannot form a basis for P_2 regardless of whether it's linearly independent or spans the space. A basis for a 3-dimensional space must contain exactly 3 vectors.

Conclusion:

The set S is linearly independent but does not span P_2. Since both conditions are required for a basis, S does not form a basis for P_2.

The span of S is a 2-dimensional subspace of P_2 consisting of polynomials of the form $(\alpha+\beta) + (\alpha + \beta)x + \beta x^2$ where the coefficients of the constant and linear terms are equal.

Answer: D. No, S does not span P_2.

Question 32: Let $v_1 = \{1, 2, 3\},\ v_2 = \{4, 5, 6\},\ v_3 = \{7, 8, 9\}$ in $\mathbb{R}^3$. Which of the following sets forms a basis for the subspace spanned by these vectors?

A: $\{v_1, v_2\}$

B: $\{v_2, v_3\}$

C: $\{v_1, v_3\}$

D: $\{v_1, v_2, v_3\}$

Solution:

To find which set forms a basis for the subspace spanned by v_1, v_2, and v_3, we first need to determine the linear dependence relationships among these vectors.

Let's write the vectors as:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Step 1: Check if $\{v_1, v_2, v_3\}$ is linearly independent

We form the matrix with these vectors as columns and find its rank:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Let's row reduce this matrix:

Apply $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - 3R_1$:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

Divide R_2 by -3:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 + 6R_2$:

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \leftarrow R_1 - 4R_2$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of matrix A is 2, which means only 2 of the 3 vectors are linearly independent.

Step 2: Find the linear dependence relationship

From the reduced row echelon form, we see that columns 1 and 2 are pivot columns, while column 3 is not. This means v_3 can be expressed as a linear combination of v_1 and v_2.

From the RREF, we have: $v_3 = -1 \cdot v_1 + 2 \cdot v_2$

Let's verify:

$$-v_1 + 2v_2 = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1+8 \\ -2+10 \\ -3+12 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = v_3$$

Step 3: Determine which sets form a basis

Since the subspace spanned by $\{v_1, v_2, v_3\}$ has dimension 2, a basis must contain exactly 2 linearly independent vectors from this set.

From our analysis, we know: - v_1 and v_2 are linearly independent (they correspond to pivot columns) - $v_3 = -v_1 + 2v_2$, so v_3 is linearly dependent on v_1 and v_2

Let's check each option:

Option A: $\{v_1, v_2\}$ These are linearly independent (from our RREF analysis), and they span the same subspace as $\{v_1, v_2, v_3\}$ since v_3 is a linear combination of v_1 and v_2. This forms a basis.

Option B: $\{v_2, v_3\}$ We need to check if these are linearly independent:

$$c_1 v_2 + c_2 v_3 = 0$$

$$c_1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the system:

$$4c_1 + 7c_2 = 0 \tag{68}$$
$$5c_1 + 8c_2 = 0 \tag{69}$$
$$6c_1 + 9c_2 = 0 \tag{70}$$

From the first equation: $c_1 = -\frac{7}{4}c_2$ Substituting into the second equation: $5(-\frac{7}{4}c_2) + 8c_2 = 0$ $-\frac{35}{4}c_2 + 8c_2 = 0$ $(-\frac{35}{4} + \frac{32}{4})c_2 = 0$ $-\frac{3}{4}c_2 = 0$ Therefore, $c_2 = 0$, and thus $c_1 = 0$.

Since the only solution is trivial, $\{v_2, v_3\}$ is linearly independent and forms a basis.

Option C: $\{v_1, v_3\}$ Similar analysis shows these are also linearly independent and form a basis.

Option D: $\{v_1, v_2, v_3\}$ Since these vectors are linearly dependent (as we showed $v_3 = -v_1 + 2v_2$), they do not form a basis.

Conclusion:

Options A, B, and C all form bases for the subspace spanned by $\{v_1, v_2, v_3\}$. However, since the question asks which set forms "a basis" (singular), and typically in multiple choice questions we expect one answer, let's consider that Option A uses the first two vectors which correspond to the pivot columns in our RREF analysis.

Answer: A. $\{v_1, v_2\}$

Note: Options B and C are also correct answers, but A corresponds to the pivot columns from our row reduction.