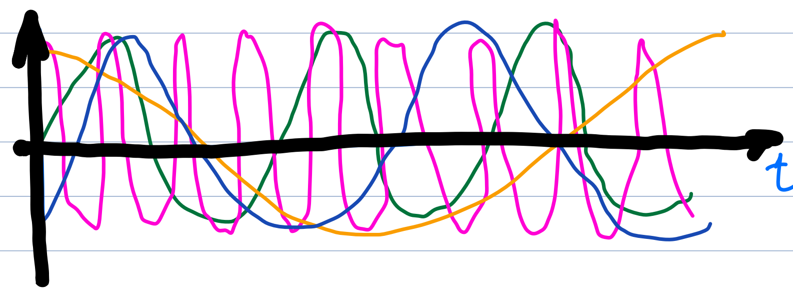
QUANTUM FOURIER TRANSFORM
(QFT)

Fourier Analysis :



$$\underset{\text{(time)}}{t} \xleftrightarrow{\mathcal{F}} \underset{\text{(frequency)}}{f}$$

This means that a Fourier Transform is a transformation of "Basis". The elementary blocks that can be used for write any element of the vetorial space.

In QUANTUM this would represent a change of basis as well!

You know one already!

$$\{|0\rangle, |1\rangle\} \xrightarrow{?} \{|+\rangle, |-\rangle\}$$

YES!

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$\boxed{\begin{aligned} H|0\rangle &\longrightarrow |+\rangle \\ H|1\rangle &\longrightarrow |-\rangle \end{aligned}}$$

In general case:

$$|\tilde{x}\rangle = QFT\,|x\rangle$$

Transformed basis (Fourier)

Original basis (Computational)

$$\boxed{|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i x y/N} |y\rangle}$$

for 1-qubit, $\boxed{N = 2^1 = 2}$

$$= \frac{1}{\sqrt{2}} \sum_{y=0}^{2-1} e^{\cancel{2}\pi i x y/\cancel{2}} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \left[\cancelto{1}{e^{\pi i x.0}} |0\rangle + \cancelto{e^{\pi i x}}{e^{\pi i x.1}} |1\rangle \right]$$

$$= \frac{1}{\sqrt{2}} \left[|0\rangle + e^{i\pi x} |1\rangle \right] = QFT|x\rangle$$

$\boxed{x = 0}$

$$QFT|0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + e^{i\pi.0} |1\rangle \right]$$

$$= \frac{1}{\sqrt{2}} \left[|0\rangle + |1\rangle \right] = |+\rangle$$

$\boxed{x=1}$

$$QFT|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + \overbrace{e^{i\pi\cdot 1}}^{-1}|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right] = |-\rangle$$

for 3-qubit

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2^3}}\sum_{y=0}^{8-1} e^{2\pi i xy/2^3}|y\rangle$$

remembering that $y = \sum_{k=1}^{n} y_k 2^{n-k}$ which is the binary notation:

$$= \frac{1}{\sqrt{8}}\sum_{y=0}^{7} e^{2\pi i \sum_{y=0}^{3} y^k/2^k}|y_1,\dots y_n\rangle$$

$$= \frac{1}{\ } \sum^{7} \prod^{3} e^{2\pi i\, x y_k/2^k}|y_1,\dots,y_n\rangle$$

$\sqrt{8}$ $\quad y=0 \;\; k=1$

$$= \frac{1}{\sqrt{8}}\left[\left(|0\rangle + \underbrace{e^{\frac{2\pi i x}{2^1}}}_{\text{phases}}|1\rangle\right)\otimes\left(|0\rangle + \underbrace{e^{\frac{2\pi i x}{2^2}}}|1\rangle\right)\otimes\left(|0\rangle + \underbrace{e^{\frac{2\pi i x}{2^3}}}|1\rangle\right)\right]$$

Similar to when you decompose numbers in binary:

$$\begin{array}{c} \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \\ n = [\,\_\;\_\;\_\;\_\;\ldots\;\_\;\_\;\_\,] \end{array}$$

$$2^{n-1}\cdot\alpha_1 + 2^{n-2}\cdot\alpha_2 + \cdots + 2^{n}\cdot\alpha_n$$